Disclaimer

This summary is part of the lecture "ETH Communication Systems" (227-0121-00) by Prof. Dr. Armin Wittneben (FS19). It is based on the lecture.

Please report errors to huettern@student.ethz.ch such that others can benefit as well.

The upstream repository can be found at https://github.com/noah95/formulasheets

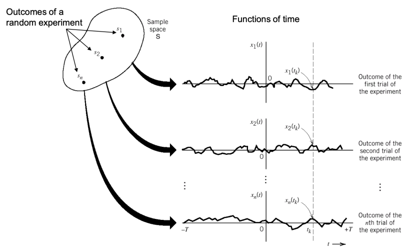
Contents

ETH Communication Systems 2019

Noah Huetter

October 24, 2019

1 Random Processes



A random process $X(t)$:

- is a sample space composed of (real valued) time functions: $\{x_1(t), x_2(t), \dots, x_n(t)\}$
- observed at a fixed t_k is a random variable $X(t_k) = \{x_1(t_k), x_2(t_k), \dots, x_n(t_k)\}$
- The time function $x_s(t)$ is a **realization** (sample function)
- $x_s(t_k)$ observed at t_k is a real number
- A stochastic process consists of infinitely many random variables, one for each t_k, with the CDF $F_{\{X(t_k)\}}(x) = P(X(t_k) \le x)$

1.1 Stationary processes

A process is **Strict Sense Stationary (SSS)** if:

- $X(t)$ and $X(t+\tau)$ have same statistics $\forall \tau$
- The joint distribution function of a set of r.v. observed at times $t_1, \dots, t_n$ is invariant to a time-shift.

$$\forall n, \tau, t_1, \dots, t_n : \\ F_{\{X(t_1+\tau), X(t_2+\tau), \dots, X(t_n+\tau)\}}(x_1, x_2, \dots, x_n) = \\ F_{\{X(t_1), X(t_2), \dots, X(t_n)\}}(x_1, x_2, \dots, x_n)$$

Properties:

$$\forall t_k : \mu_X(t_k) = \mu_X \\ \forall t_1, t_2 : R_X(t_1, t_2) = R_X(t_2 - t_1) = R_X(\tau) \\ C_X(t_1, t_2) = \mathrm{E}\left[(X(t_1) - \mu_X)(X(t_2) - \mu_X)\right] \\ = R_X(t_2 - t_1) - \mu_X^2$$

A process is **Wide Sense Stationary (WSS)** if a r.p. has a *constant* mean and the autocorrelation depends only on the *time difference*.

$$\forall t : \mu_X(t) = \mu_X \\ \forall t_1, t_2 : R_X(t_1, t_2) = R_X(t_2 - t_1) = R_X(\tau)$$

Strict sense stationary $\implies$ wide sense stationary.

1.2 Mean and correlation

Defined as expectation of r.v. $X(t_k)$ by observing process at time t_k.

$$\mu_X(t_k) = \mathrm{E}[X(t_k)] = \int_{-\infty}^{\infty} x f_{\{X(t_k)\}}(x)\,\mathrm{d}x$$

Autocorrelation function R_X and autovariance function C_X of a random process:

$$R_X(t_1, t_2) = \mathrm{E}[X(t_1)X(t_2)] \triangleq \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_{X_1,X_2}(x_1, x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2$$

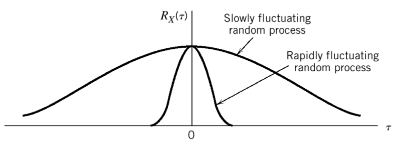
$$R_{XY}(x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x y f_{X,Y}(x, y) dx dy$$

$$C_X(t_1, t_2) = R_X(t_2 - t_1) - m_X^2$$

- The mean and autocorrelation function determine the autocovariance function
- The mean and autocorrelation function only describe the first two moments of the process

Properties of the autocorrelation function:

$$\mathrm{E}[X^2(t)] = R_X(0) \qquad R_X(\tau) = R_X(-\tau) \\ |R_X(\tau)| \le R_X(0)$$



The Cross-correlation function $R_{XY}(t, u)$ of two random processes:

$$R_{XY}(t, u) = \mathrm{E}[X(t)Y(u)] = \\ \int_{-\infty}^{\infty} xy \cdot f_{X,Y}(x, y)\,\mathrm{d}x\,\mathrm{d}y$$

- Stationariy menas $R_{XY}(t, u) = R_{XY}(\tau)$ for $\tau = t - u$
- Not generally an even function of t
- Not necessarily a maximum at $\tau = 0$
- Symmetry: $R_{XY}(\tau) = R_{XY}(-\tau)$

1.3 Ergodicity

Definition: A random process is *ergodic* in the mean if

- Time average approaches ensemble averages for increasing T
- The variance of the time average approaches zero for incr. T

$$\lim_{T\to\infty} \mu_X(T) = \mu_X \qquad \lim_{T\to\infty} \mathrm{Var}[\mu_X(T)] = 0$$

Or in other words: The same behavior averaged over time as averaged over the space of all the system's states.

1.4 Filtered processes

Stationary random process $X(t)$ is input to a linear timeinvariant (LTI) filter with impulse response $h(t)$.

$$Y(t) = \int_{-\infty}^{\infty} h(\tau_1) X(t - \tau_1)\,\mathrm{d}\tau_1 \\ S_Y(f) = |H(f)|^2 S_X(f)$$

Find mean and autocorrelation of $Y(t)$:

$$\mu_X = \mathrm{E}[X(t)] \quad R_X(\tau) = \mathrm{E}[X(t)X(t-\tau)]$$

$$\mu_Y = \mathrm{E}[Y(t)] = \mathrm{E}\left[\int_{-\infty}^{\infty} h(\tau_1) X(t - \tau_1)\,\mathrm{d}\tau_1\right]$$

Can interchange expectation and integration if stable $\int_{-\infty}^{\infty} |h(t)|\,\mathrm{d}t < \infty$ and finite mean $\mu_X < \infty$

$$\mu_Y = \int_{-\infty}^{\infty} h(\tau_1) \mathrm{E}[X(t - \tau_1)]\,\mathrm{d}\tau_1 = \mu_X \int_{-\infty}^{\infty} h(\tau_1)\,\mathrm{d}\tau_1$$

Autocorrelation:

$$R_Y(t, u) = \mathrm{E}[Y(t)Y(u)] = \\ \mathrm{E}\left[\int_{-\infty}^{\infty} h(\tau_1) X(t - \tau_1)\,\mathrm{d}\tau_1 \int_{-\infty}^{\infty} h(\tau_2) X(u - \tau_2)\,\mathrm{d}\tau_2\right]$$

Additional condition for interchange is finite mean-square value: $R_X(0) = \mathrm{E}[X^2(t)] < \infty$

$$R_Y(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1) h(\tau_2) R_X(\tau - \tau_1 + \tau_2)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2$$

(WS) stationary input process $X(t)$ to a stable LTI filter $\implies$ (WS) stationary output process $Y(t)$.

1.5 Power spectral density

$$S_X(f) = \mathscr{F}[R_X(\tau)](f) = \int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi f \tau}\,\mathrm{d}\tau$$

- $S_X(0) = \int_{-\infty}^{\infty} R_X(\tau) d\tau$
- $E[X^2(t)] = R_X(0) = \int_{-\infty}^{\infty} S_X(f) df$
- $S_X(f) \ge 0 \ \forall f$
- $S_X(f) = S_X(-f) \ \forall f$, iff $X(t) \in \mathbb{R}$

1.6 Gaussian process

Consider the r.v. $Y = \int_0^T g(t) X(t)\,\mathrm{d}t$ where $g(t)$ is in an arbitraty function. If Y is gaussian distributed, then the process $X(t)$ is a *Gaussian process*

- A filtered Gaussian process remains a Gaussian process
- If $X(t)$ is a GP, the arbitrary set of r.v. $\vec{X} = [X(t_1), \dots, X(t_n)]^T$ is jointly gaussian distributed for any n
- The joint cdf is of these r.v. is completely determined by the **means** $\mu_X(t_i) = \mathrm{E}[X(t_i)]$ and **covariances** $C_X(t_k, t_i) = \mathrm{E}[(X(t_k) - \mu_X(t_k))(X(t_i) - \mu_X(t_i))]$

Multivariative Guass distribution:

$$f(x) = \frac{\exp\left(-\frac{1}{2}(\vec{x} - \vec{m}_x)^T \underline{\Sigma}^{-1} (\vec{x} - \vec{m}_x)\right)}{(2\pi)^{\frac{n}{2}} \det(\underline{\Sigma})^{\frac{1}{2}}}$$

$$\underline{\Sigma} := \begin{bmatrix} \mathrm{Cov}(X_1, X_1) & \dots & \mathrm{Cov}(X_1, X_n) \\ \vdots & \vdots & \vdots \\ \mathrm{Cov}(X_n, X_1) & \dots & \mathrm{Cov}(X_n, X_n) \end{bmatrix}$$

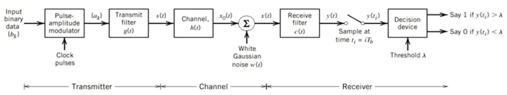
1.7 Noise

White noise is defined by its autocorrelation.

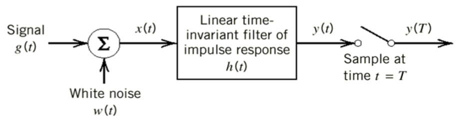
$$R_W(\tau) = \frac{N_0}{2}\delta(t) \qquad S_W(f) = \frac{N_0}{2}$$

2 Baseband Pulse Transmission

Digital Baseband Pulse Transmission System: Based on the sample $y(t_i)$ the receiver generates an estimate $\hat{a}_i$ of the amplitude a_i of the transmitted pulse $g(t-iT_b)$.



2.1 Matched Filter



$$y(t) = g_0(t) + n(t) = h(t)*g(t) + h(t)*w(t)$$

Maximize *pulse signal-to-noise ratio* η at sampling time $t = T$:

$$\eta = \frac{|g_0(T)|^2}{\mathrm{E}\left[n^2(t)\right]} = \frac{\left|\int_{-\infty}^{\infty} H(f)G(f)e^{j2\pi fT}\,\mathrm{d}f\right|^2}{\frac{N_0}{2}\int_{-\infty}^{\infty}|H(f)|^2\,\mathrm{d}f}$$

Using Schwarz's inequality:

$$\eta \le \frac{2}{N_0}\int_{-\infty}^{\infty}|G(f)|^2\,\mathrm{d}f$$

The quality sign (optimum) holds if $a(x) \propto b^*(x)$, i.e.

$$H_{\text{opt}}(f) = kG^*(f)e^{-j2\pi fT} \Rightarrow h_{\text{opt}}(t) = kg(T-t)$$

The impulse response of the optimum filter, except for the scaling factor k, is a time-reversed and delayed version of the input signal $g(t)$.

The pulse SNR of a machted filter depends only on the ratio of the signal energy E to the PSD of the white noise at the input filter.

$$\eta_{\max} = \frac{2}{N_0}\int_{-\infty}^{\infty}|G(f)|^2\,\mathrm{d}f = \frac{2E}{N_0}$$

$$E = \int_{-\infty}^{\infty}|g(t)|^2\,\mathrm{d}t = \int_{-\infty}^{\infty}|G(f)|^2\,\mathrm{d}f$$

2.2 Error Rate

Discussed for a binary bipolar non-return-to-zero (NRZ) signal with amplitude A, bit duration T_b.

$$x(t) = \begin{cases} +A + w(t), & \text{Symbol 1 transmitted} \\ -A + w(t), & \text{Symbol 0 transmitted} \end{cases}$$

$$p_{10} = \frac{1}{2}\operatorname{erfc}\left(\frac{A+\lambda}{\sqrt{N_0/T_b}}\right) = \mathrm{Q}\left(\sqrt{2}\frac{A+\lambda}{\sqrt{N_0/T_b}}\right)$$
$$= \mathbf{P}\left(y > \lambda \mid \text{symbol 0 was sent}\right)$$

The avg. prob. of symbol error P_e:

$$P_e = \frac{p_0}{2}\operatorname{erfc}\left(\frac{A+\lambda}{\sqrt{N_0/T_b}}\right) + \frac{p_1}{2}\operatorname{erfc}\left(\frac{A-\lambda}{\sqrt{N_0/T_b}}\right)$$

The error function:

$$\mathbf{P}\left(n > a\right) \equiv \mathrm{Q}\left(\frac{a}{\sigma_n}\right) = \frac{1}{2}\operatorname{erfc}\left(\frac{1}{\sqrt{2}}\frac{a}{\sigma_n}\right)$$

Optimum decision threshold λ that maximizes P_e:

$$\lambda_{\text{opt}} = \frac{N_0}{4AT_b}\log\left(\frac{p_0}{p_1}\right)$$

2.3 Intersymbol Interference

Arises when the channel is *dispersive*, the magn. freq. resp. is not constant over the range of interest.

$$s(t) = \sum_k a_k \cdot g(t-kT_b)$$

$$y(t) = \mu\sum_k a_k \cdot p(t-kT_b) + n(t)$$

$$t(t_i) = \underbrace{\mu a_i}_{i\text{-th bit}} + \underbrace{\sum_{\substack{k=-\infty \\ k\neq i}}^{\infty} a_k p(i-k)T_b}_{\text{ISI}} + n(t_i)$$

2.4 Nyquist's Criterion

In order to avoid ISI, we require $p(mT_b) = 0$ for $m \neq 0$ and obtain

$$\sum_{m=-\infty}^{\infty} p(mT_b)\delta(t-mT_b) = \delta(t) \circ\!\!-\!\!\bullet P_\delta(f) = 1$$

An the nyquist criterion ($R_b = 1/T_b$ symbol rate):

$$\sum_{n=-\infty}^{\infty} P(f-nR_b) = T_b$$

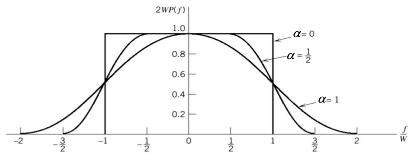
In words: The pulse function P in freq. domain copied with spacing R_b must be constant.

Ideal nyquist channel: The simplest function $P(f)$ that satisfies this is the rectangular function (ideal LPF) with $W = R_b/2$, R_b the nyquist rate.

$$P(f) = \frac{1}{2W}\operatorname{rect}\left(\frac{f}{2W}\right) = \begin{cases} \frac{1}{2W}, & -W \le f \le W \\ 0, & |f| > W \end{cases}$$

$$p(t) = \operatorname{sinc}(2Wt) \quad W = \frac{1}{2T_b} \quad E_b = \frac{A^2}{R_b}$$

Raised Cosine Spectrum: consists of flat portion and sinusodial rolloff.



$$P(f) = \begin{cases} \frac{1}{2W} & 0 \le |f| \le f_1 \\ \frac{1}{4W}\left(1 - \sin\left[\frac{\pi(|f|-W)}{2W-2f_1}\right]\right) & |f| \in [f_1, 2W-f_1] \\ 0 & |f| > 2W - f_1 \end{cases}$$

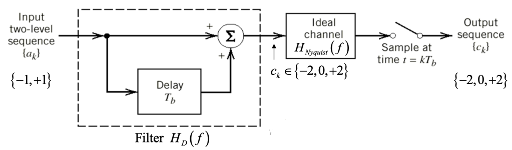
$$p(t) = \operatorname{sinc}(2Wt)\left(\frac{\cos(2\pi\alpha Wt)}{1-16\alpha^2W^2t^2}\right)$$

$$\alpha = 1 - \frac{f_1}{W} \in [0,1] \quad \text{Rollof factor}$$

Bandwidth is larger: $B_T = 2W - f_1 = W(1+\alpha)$.

2.5 Correlative-Level Coding

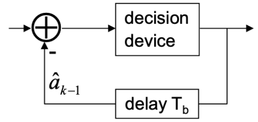
Use basepulses which introduce controlled ISI. Same BW but higher P_e



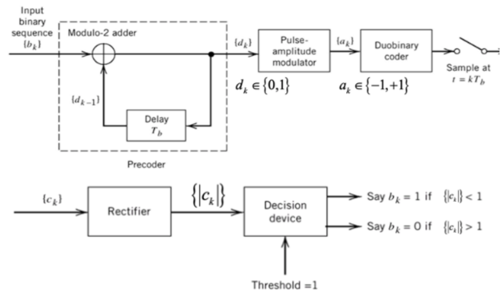
$$H_I(f) = \begin{cases} 2T_b\cos(\pi fT_b)e^{-i\pi fT_b}, & |f| < 1/2T_b \\ 0, \text{else} \end{cases}$$

$$h_I(t) = \frac{T_b^2\sin(\pi t/T_n)}{\pi t(T_b - t)}$$

Decoding



Precoding The decision feedback receiver is prone to propagating error. Using modulo-2 precoding, this can be omitted.



$$d_k = b_k \oplus d_{k-1} \Rightarrow b_k = d_k \oplus d_{k-1}$$
$$c_k = \begin{cases} 0, & b_k = 1 \\ \pm 2, & b_k = 0 \end{cases}$$

2.6 Baseband M-ary PAM Transmission

In a M-ary PAM system: M possible amplitude levels. One symbol encodes $\log_2 M$ bits. Thus the signal rate T is related to the bit duration T_b of a binary PAM as:

$$T = T_b \log_2 M$$

- For same avg. P_e, an M-ary PAM requires more Tx power
- If $M \gg 2$ the Tx energy per bit must be increased by $M^2/(3\log_2 M)$ for same P_e

3 Signal Space Analysis

Continuous AWGN (Additive white gaussian noise) channel.

- All symbols m_i from source are eually likely $p_i = p(m_i) = \frac{1}{M}$
- Transmitter codes each m_i into a signal $s_i(t) \in \{s_k(t) | 1 \le k \le M\}$

- Cahnnel adds AWGN $x(t) = s_i(t) + w(t)$ for $0 \leq t \leq T$
- The optimal receiver minimizes the avg. pob. of symbol error P_e

$$P_e = \sum_{i=1}^{M} p_i \mathbf{P}\left(\hat{m} \neq m_i \mid m_i\right)$$

3.1 Geometric Signal Representation

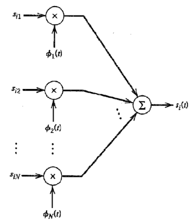
Let $\{\phi_i(t)\}_{i=1\dots N}$ be a set of othonormal basis fuctions of the signal set $\{s_i(t)\}_{i=1\dots M}$. All signals can be expressed as a finite sum. The coeff. s_{ij} are given by the projection onto $\{\phi_i(t)\}_{i=1\dots N}$.

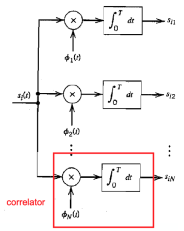
The orthonormal functions deine a N-dimensional Eucledian space - the signal space.

$$\int_0^T \phi_i(t)\phi_j(t)\,\mathrm{d}t = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & - \neq j \end{cases}$$

$$s_i(t) = \sum_{j=1}^{N} s_{ij}\phi_j(t) \quad s_{ij} = \int_0^T s_i(t)\phi_j(t)\,\mathrm{d}t$$

$$0 \leq t \leq T, \quad i = 1\dots M, \quad j = 1\dots N$$





$$\langle s_i(t), s_k(t)\rangle = \int_0^T s_i(t)s_k(d)\,\mathrm{d}t = \boldsymbol{s}_i^\top \cdot \boldsymbol{s}_k$$

$$\|\boldsymbol{s}_i\|^2 = \langle s_i(t), s_i(t)\rangle = \int_0^T s_i(t)^2$$

$$\|\boldsymbol{s}_i - \boldsymbol{s}_k\|^2 = \sum_{j=1}^{N}(s_{ij} - s_{kj})^2 = \int_0^T (s_i(t) - s_k(t))^2\,\mathrm{d}t$$

$$\cos\theta_{jk} = \frac{\boldsymbol{s}_i^\top \cdot \boldsymbol{s}_k}{\|\boldsymbol{s}_i\| \cdot \|\boldsymbol{s}_k\|} \quad E_i = \sum_{j=1}^{N} s_{ij}^2 = \|\boldsymbol{s}_i\|^2$$

Gram-Schmidt orthogonalization procedure: Start with a complete system $s_1(t), \dots, s_M(t)$ that generates the signal space. At each step generate a new basis function ϕ_i. The basis has only $N \leq M$ functions.

1. Build basis function ϕ_1 from s_1

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{\int_0^T s_1^2(t)\,\mathrm{d}t}}$$

2. Search for a basis function from $s_2(t)$

$$s_{21} = \langle s_2(t), \phi_1(t)\rangle = \int_0^T s_2(t)\phi_1(t)\,\mathrm{d}t$$
$$g_2(t) = s_2(t) - s_{21}\phi_1(t)$$

If $g_2 = 0$, s_2 is lin. dep. on ϕ_1 and does not lead to a new basis function. Otherwise:

$$\phi_2(t) = \frac{g_2(t)}{\sqrt{\int_0^T g_2^2(t)\,\mathrm{d}t}}$$

3. Search for a basis function from $s_3(t)$

$$s_{31} = \langle s_3(t), \phi_1(t)\rangle \quad s_{32} = \langle s_3(t), \phi_2(t)\rangle$$
$$g_3(t) = s_3(t) - s_{31}\phi_1(t) - s_{32}\phi_2(t)$$

If $g_3 = 0$, s_3 is lin. dep. on ϕ_1 and ϕ_2 and does not lead to a new basis function. Otherwise:

$$\phi_3(t) = \frac{g_3(t)}{\sqrt{\int_0^T g_3^2(t)\,\mathrm{d}t}}$$

4. Search for a basis function from $s_M(t)$. Project s_M on the already determined basis functions, decompose S_M into its projection and a difference term g_M. If $g_M \neq 0$:

$$\phi_N(t) = \frac{g_M(t)}{\sqrt{\int_0^T g_M^2(t)\,\mathrm{d}t}}$$

3.2 Discrete System Model

The signal vector $\boldsymbol{s}$, noise vector $\boldsymbol{w}$ and the received signal $\boldsymbol{x}$.

$$\boldsymbol{s}_i = \begin{bmatrix} s_{i1} & \dots & s_{iN}\end{bmatrix}^\top \quad \boldsymbol{w} = \begin{bmatrix} w_1 & \dots & w_N\end{bmatrix}^\top$$
$$\boldsymbol{x} = \begin{bmatrix} x_1 & \dots & x_N\end{bmatrix}^\top = \boldsymbol{s}_i + \boldsymbol{w}$$
$$\mathrm{E}\left[w_j\right] = 0 \quad \mathrm{E}\left[w_j \cdot w_k\right] = \delta_{jk} \quad \mathrm{Var}\left(w_j\right) = \frac{N_0}{2}$$

Theorem of Irrelevance For signal detection with AWGN, only the projection of the noise onto the basis functions of the signal set $\{s_i(t)\}_{i=1}^{M}$ affect the sufficient statistics of the detection problem. The remainder of the noise is irrelevant.

$$\mu_{X_j} = \mathrm{E}\left[X_j\right] = \mathrm{E}\left[s_{ij} + W_j\right] = s_{ij} + \mathrm{E}\left[W_j\right] = s_{ij}$$
$$\sigma_{X_j}^2 = \mathrm{Var}\left(X_j\right) = \mathrm{E}\left[(X_j - s_{ij})^2\right] = \mathrm{E}\left[W_j^2\right] = \frac{N_0}{2}$$
$$W_j = \int_0^T W(t)\phi_j(t)\,\mathrm{d}t$$

The elements X_j and X_k of the received signal vector have the covariance

$$\mathrm{Cov}\left(x_j, x_k\right) = \mathrm{E}\left[(x_j - \mu_{x_j})(x_k - \mu_{x_k})\right] = 0, \quad j \neq k$$

Thus the x_j are mutually uncorrelated. $\implies$ statistical independence.

Likelihood Function As the x_j are statistically indep. the conditional PDF of $\boldsymbol{x}$ given $\boldsymbol{s}$ (i.e. symbol m_i sent usign signal s_i) follows:

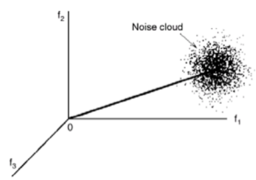
$$L(\boldsymbol{s}_i) := f_x(\boldsymbol{x}|\boldsymbol{s}_i) = f_W(\boldsymbol{w} = \boldsymbol{x} - \boldsymbol{s}_i) =$$
$$= \frac{1}{(\pi N_0)^{N/2}} \exp\left[-\frac{1}{N_0}\sum_{j=1}^{N}(x_j - s_{ij})^2\right]$$
$$l(\boldsymbol{s}_i) = \log L(\boldsymbol{s}_i) = -\frac{1}{N_0}\sum_{j=1}^{N}(x_j - s_{ij})^2 + c$$

$$c = -\frac{N}{2}\log(\pi N_0) \quad i \in \{1, \dots, M\}$$

L likelihood function, l log-likelihood function can be used because the pdf is always nonnegative and monot. incr. The constant c is indep. of hyp. $\boldsymbol{s}_i$ and can be discarded for the decision.

3.3 Detection and Decoding

Detection problem: Given the observation $\boldsymbol{x}$, determine an estimate $\hat{m}$ of the transmitted symbol m_i, s.t. the probability of error is minimized.



$$P_e(m_i|\boldsymbol{x}) = \mathbf{P}\left(m_i \text{not sent} \mid \boldsymbol{x}\right) = 1 - \mathbf{P}\left(m_i \text{sent} \mid \boldsymbol{x}\right)$$

The MAP (Maximum-A-Posteriori) decision rule is optimum in the minimum prob. of error sense. Set $\hat{m} = m_i$ if:

$$\mathbf{P}\left(m_i \text{sent} \mid \boldsymbol{x}\right) \geq \mathbf{P}\left(m_k \text{sent} \mid \boldsymbol{x}\right) \quad \forall k \neq i$$

Rephrased using Baye's rule, set $\hat{m} = m_i$ if (p_k a priori prob. of transmitting m_k, $f_x(\boldsymbol{x}|m_k)$ cond. pdf of $\boldsymbol{x}$ given m_k):

$$\hat{m} = \arg\max_{m_k} \frac{p_k \cdot f_x(\boldsymbol{x}|m_k)}{f_x(\boldsymbol{x})} \quad \forall k \neq i$$

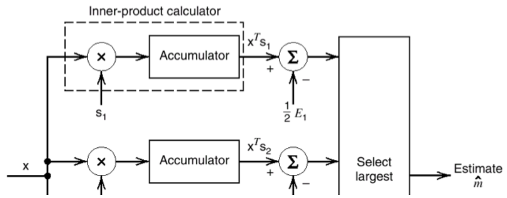
We can drop $f_x(\boldsymbol{x})$ as it is indep. of the symbol decision. For equiprobable source symbols, we obtain the ML decision rule: Set $\hat{m} = m_i$ if $l(m_k)$ max. for $k = i$.

Simplfied ML Rule: $\boldsymbol{x}$ lies in region Z_i if

$$\sum_{j=1}^{N} x_j s_{kj} - \frac{1}{2}E_k$$

is maximum for $k = i$.

Correlation receiver



3.4 Probability of Error

$\mathbf{P}\left(A_{ik}\right) = P_2(\boldsymbol{s}_i, \boldsymbol{s}_k)$ is the pairwise error prob. that observation $\boldsymbol{x}$ is closer to $\boldsymbol{s}_k$ than to $\boldsymbol{s}_i$:

$$P_2(\boldsymbol{s}_i, \boldsymbol{s}_k) = \mathbf{P}\left(\|\boldsymbol{x} - \boldsymbol{s}_k\|^2 < \|\boldsymbol{x} - \boldsymbol{s}_i\|^2\right)$$

With the eucledian distance $d_{14} := \|\boldsymbol{s}_1 - \boldsymbol{s}_4\|$:

$$P_2(\boldsymbol{s}_1, \boldsymbol{s}_2) = \mathbf{P}\left(z < \frac{1}{2}d_{14}\right) = Q\left(\frac{d_{14}}{\sqrt{2M_0}}\right)$$
$$= \frac{1}{2}\,\mathrm{erfc}\left(\frac{d_{14}}{2\sqrt{N_0}}\right)$$

The pairwise probability of error only depends on the Euclidean distance and is e.g. invariant to rotation and translation of the signal constellation

From the union bound we have

$$P_e(m_i) \leq \sum_{\substack{k=1 \\ k\neq i}}^{M} P_2(\boldsymbol{s}_i, \boldsymbol{s}_k)$$

P_e is the error prob. averaged over all symbols. An upper bound follows as

$$P_e = \sum_{i=1}^{M} p_i P_e(m_i) \leq \frac{1}{2} \sum_{i=1}^{M} \sum_{\substack{k=1 \\ k\neq i}}^{M} p_i \operatorname{erfc}\left(\frac{d_{ik}}{2\sqrt{N_0}}\right)$$

4 Passband Data Transmission

In bandpass data transmission, information modulates a carrier and occupies a restricted bandwidth in frequency. The carrier can be modulated by changing:

- Amplitude (ASK)
- Phase (PSK)
- Frequency (FSK)

Cherent modulation is when the receiver's local oscillator is phase-synchronous to the transmitter's local oscillator.

$M = s^n$ levels for signalling information (M-ary xSK). Using M levels, symbol duration $T = nT_b$ is changed whilek eeping the same datarate. Bandwidth shrinks accordingly ny $1/nT_b$.

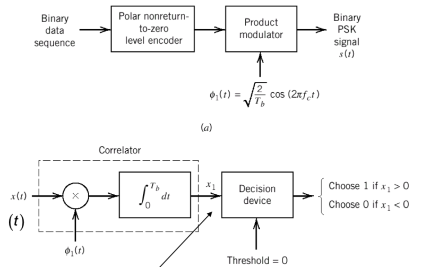
Figures of merit: Symbol error probability at given SNR, power spectral density, bandwidth efficiency $\rho = R_b/B$ [bit/s/Hz].

4.1 PSK: Coherent Phase Shift Keying

BPSK: Binary PSK

$$\phi_1(t) = \sqrt{\frac{2}{T_b}} \cos(2\pi f_c t)$$

$$s_1(t) = \sqrt{E_b}\phi_1(t) \quad s_2(t) = -\sqrt{E_b}\phi_1(t)$$



$$p_{10} = p_{01} = P_e = \frac{1}{2} \operatorname{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right)$$

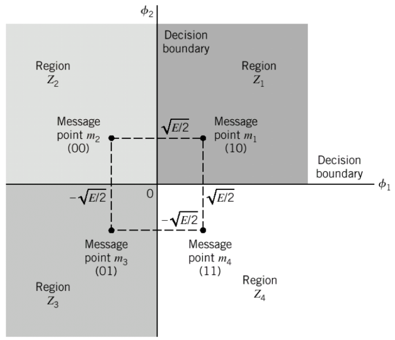
$$S(f - f_c) \approx 2E_b \operatorname{sinc}^2(T_b f)$$

QPSK: Quadriphase SK, use more than just two phase levels.

$$\phi_1(t) = \sqrt{\frac{2}{T_b}} \cos(2\pi f_c t) \quad \phi_2(t) = \sqrt{\frac{2}{T_b}} \sin(2\pi f_c t)$$

$$\boldsymbol{s}_{10} = \begin{bmatrix} +c \\ +c \end{bmatrix} \boldsymbol{s}_{00} = \begin{bmatrix} -c \\ +c \end{bmatrix} \boldsymbol{s}_{01} = \begin{bmatrix} -c \\ -c \end{bmatrix} \boldsymbol{s}_{11} = \begin{bmatrix} +c \\ -c \end{bmatrix}$$

$$c = \sqrt{E/2}$$



Every QPSK symbol carries 2 bits, hence the symbol energy is twice the energy per information bit: $E = 2E_b$. A QPSK system achieves same BER as a BPSK at same E_b/N_0 but at *twice the bit rate.*

$$\text{BER} = \frac{1}{2} \operatorname{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right)$$

$$S_B(f) = 4E_b \operatorname{sinc}^2(2T_b f)$$

4.2 QAM: Hybrid Amplitude/Phase Modulation

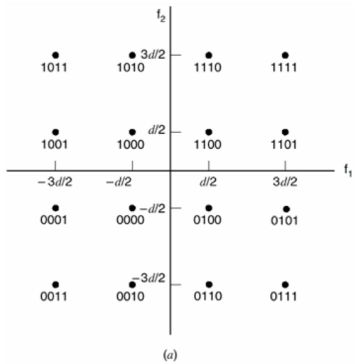
QAM: M-ary quadrature amplitude modulation, change phase and amplitude.

$d_{\min}$ is the distance between adjacent messages in the signal space.

$$\phi_1(t) = \sqrt{\frac{2}{T_b}} \cos(2\pi f_c t) \quad \phi_2(t) = \sqrt{\frac{2}{T_b}} \sin(2\pi f_c t)$$

$$\boldsymbol{s}_i = \frac{d_{\min}}{2} \begin{bmatrix} a_i \\ b_i \end{bmatrix} a_i, b_i \text{odd integers}, i = 1, \ldots, M$$

Mapping an even number f bits per symbol (e.g. 4bits → 16 symbolds), results in a quadratic $L \times L$ square constellation with $L = \sqrt{M}$. Gray coding is often used for mapipng the bits to the QAM symbols.



$$P_c = (1 - P_e')^2 \rightarrow P_e = 1 - P_c = 1 - (1 - P_e')^2 \approx 2P_e'$$

$$P_e' = \left(1 - \frac{1}{\sqrt{M}}\right) \operatorname{erfc}\left(\sqrt{\frac{d_{\min}^2}{4N_0}}\right)$$

$$E_{\text{av}} = \frac{(M-1)d_{\min}^2}{6}$$

$$P_e \approx 2P_e' = 2\left(1 - \frac{1}{\sqrt{M}}\right) \operatorname{erfc}\left(\sqrt{\frac{3E_{\text{av}}}{2(M-1)N_0}}\right)$$

E_{av} average symbol energy.

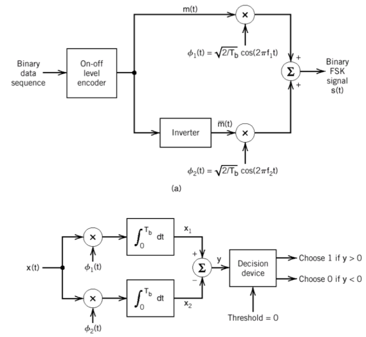
4.3 FSK: Coherent Frequency-Shoft Keying

$$\phi_i(t) = \begin{cases} \sqrt{\frac{2}{T_b}} \cos(2\pi f_i t), & 0 \leq t \leq T_b \\ 0, & \text{else.} \end{cases}$$

$$f_i = \frac{n_c + i}{T_b} \quad i = 1, 2, \quad n_c \in \mathbb{N}$$

$$\boldsymbol{s}_1 = \sqrt{E_b} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \boldsymbol{s}_2 = \sqrt{E_b} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

f_i chosen by rule to avoid phase discontinuities. The two frequencies f_1 and f_2 are $1/T_b$ Hz appart. The ϕ_i are orthogonal for $f_i = (n_c + i)/T_b$.



Distance between message points in signal space is $1/\sqrt{2}$ smaller compared to binary PSK.

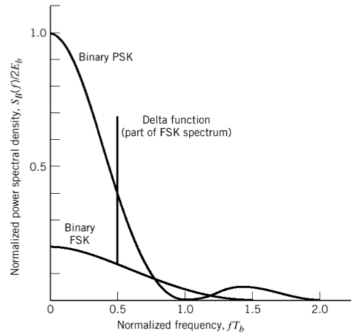
$$d_{\min} = \sqrt{2E_b}$$

$$P_e = \frac{1}{2} \operatorname{erfc}\left(\sqrt{\frac{(d_{\min}/2)^2}{N_0}}\right) = \frac{1}{2} \operatorname{erfc}\left(\sqrt{\frac{E_b}{2N_0}}\right)$$

$$S_B(f) = \frac{E_b}{2T_b}\left[\delta\left(f - \frac{1}{2T_b}\right) + \delta\left(f + \frac{1}{2T_b}\right)\right] + \ldots$$

$$\frac{8E_b \cos^2(\pi T_b f)}{\pi^2 (4T_b^2 f^2 - 1)^2}$$

PSD contains two delta pulses and decays much faster than BPSK due to continuous phase operation.



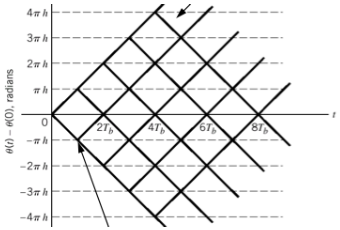
4.4 CPFSK Continuous Phase FSK

$$s(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t + \theta(t))$$

$$\theta(t) = \theta(0) \pm \frac{\pi h}{T_b} t, \quad 0 \leq t \leq T_b$$

$$h = T_b(f_1 - f_2), \quad f_c = \frac{1}{2}(f_1 + f_2)$$

h modulation index.



4.5 MSK Minimum Shift Keying

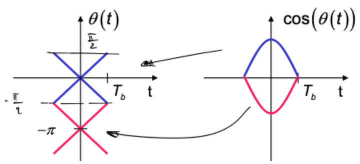
For integer valued h, the accumulated phase at end of symbol is independent of the previous and current symbol. $\implies$ no phase memory, each symbol can be decoded independently.

$$f_1 - f_2 = 0.5/T_b$$

The minimum difference, for which $s_1(t), s_2(t)$ orthogonal.

$\theta(0) = 0$	$\theta(T_b) = \pi/2$	symbol 1 transmitted
$\theta(0) = \pi$	$\theta(T_b) = \pi/2$	symbol 0 transmitted
$\theta(0) = -\pi$	$\theta(T_b) = -\pi/2$	symbol 1 transmitted
$\theta(0) = 0$	$\theta(T_b) = -\pi/2$	symbol 0 transmitted

Estimation of $\theta(0)$: Expanding $s(t)$ into two terms we get:



We can estimte $\theta(0)$ by observing

$$\sqrt{\frac{2E_b}{T_b}}\cos(\theta(t))\cos(2\pi f_c t)$$

MSK Signal-Space representation

$$\phi_1(t) = \sqrt{\frac{2}{T_b}}\cos\left(\frac{\pi}{2T_b}t\right)\cos(2\pi f_c t), \quad -T_b \le t \le T_b$$

$$\phi_2(t) = \sqrt{\frac{2}{T_b}}\sin\left(\frac{\pi}{2T_b}t\right)\sin(2\pi f_c t), \quad 0 \le t \le 2T_b$$

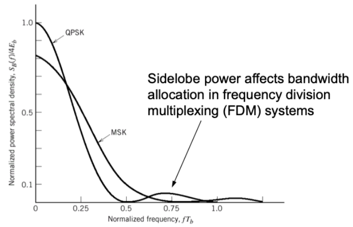
A coherent receiver has to integrate over two bit periods:

$$x_1 = \int_{-T_b}^{T_b} x(t)\phi_1(t)\,\mathrm{d}t = \sqrt{E_b}\cos(\theta(0)) + w_1$$

$$x_2 = \int_{0}^{2T_b} x(t)\phi_2(t)\,\mathrm{d}t = -\sqrt{E_b}\sin(\theta(T_b)) + w_2$$

Bit error rate The four points in the signal-space diagram correspond to two symbol, hence the BER is the same as with QPSK.

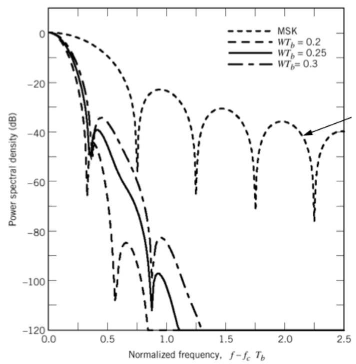
$$\mathrm{BER} = \frac{1}{2}\,\mathrm{erfc}\left(\frac{d_{\min}/2}{\sqrt{N_0}}\right) = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right)$$



4.6 GMSK

To make sidelobes of MSK smaller, filter the NRZ signal with pulse shaping function.

$$\boxed{H(f) = \exp\left[-\frac{\log 2}{2}\left(\frac{f}{W}\right)^2\right]}$$

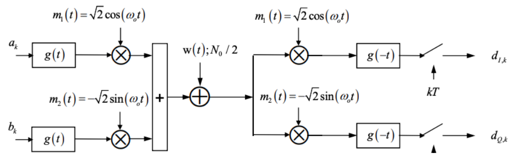


The parameter α depends on the time bandwidth product WT_b. The quantity $10\log(\alpha/2)$ expresses the degradation in dB of GMSK compared to MSK. MSK: $WT_b = \infty, \alpha = 0$

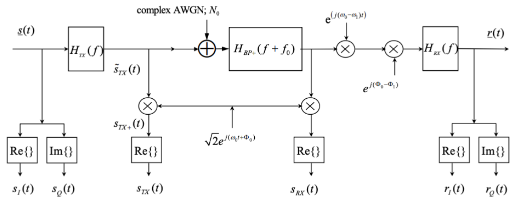
$$P_e = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{\frac{\alpha E_b}{2N_0}}\right)$$

4.7 Equivalent baseband representation

QAM: Two branches: inphase (I) and quadrature (Q)



By using complex valued signals, the transmission system can be written as an LTI system.



Important names and notation:

$\tilde{s}_{\mathrm{TX}}(t)$	compex envelope of s_{TX}
$s_{\mathrm{TX}+}(t)$	analytic signal (pre-envelope of s_{TX})
$s_{\mathrm{TX}}(t)$	physical passband signal

$$S_{TX+}(f) = \begin{cases} 2S_{\mathrm{TX}}(f) & f > 0 \\ 0 & \text{else} \end{cases}$$

$$H_{BP+}(f) = \begin{cases} H_{\mathrm{BP}}(f) & f > 0 \\ 0 & \text{else} \end{cases}$$

$$H_{\mathrm{TX}}(f) = 0 \forall |f| \ge f_0 \quad H_{\mathrm{RX}}(f) = 0 \forall |f| \ge B$$
$$B = \max(0, f_0 - |f_0 - f_1|)$$

Complex Envelope Inphase and quadrature components.

$$\tilde{s}_{\mathrm{TX}}(t) = \tilde{s}_{\mathrm{TX,I}}(t) + j\tilde{s}_{\mathrm{TX,Q}}(t) \quad \tilde{S}_{\mathrm{TX}}(f) = 0 \forall |f| > f_0$$

Analytic signal Pre-envelope. Has one-sided spectrum, scaling factor to preserve power values in passband and qeuivalent baseband.

$$s_{\mathrm{TX}+}(t) = \tilde{s}_{\mathrm{TX}}(t)\sqrt{2}\exp(j\omega_0 t)\exp(j\phi_0)$$
$$S_{\mathrm{TX}+}(f) = \sqrt{2}\exp(j\phi_0)\tilde{S}_{\mathrm{TX}}(f - f_0)$$
$$S_{\mathrm{TX}+}(f) = 0 \forall f < 0$$
$$\tilde{S}_{\mathrm{TX}}(f) = \frac{1}{\sqrt{2}}\exp(-j\phi_9 0)S_{\mathrm{TX}+}(f + f_0)$$

Physical passband signal

$$s_{\mathrm{TX}}(t) = \mathrm{Re}\{s_{\mathrm{TX}+}(t)\}$$
$$S_{\mathrm{TX}}(f) = \frac{1}{2}\left(S_{\mathrm{TX}+}(f) + S^*_{\mathrm{TX}+}(-f)\right)$$
$$s_{\mathrm{TX}}(t) = \sqrt{2}\tilde{s}_{\mathrm{TX,I}}(t)\cos(\omega_0 t + \phi_0) - \sqrt{2}\tilde{s}_{\mathrm{TX,Q}}(t)\sin(\omega_0 t + \phi_0)$$
$$s_{\mathrm{TX}}(t) = \left\{\sqrt{2}\sqrt{\tilde{s}^2_{\mathrm{TX,I}}(t) + \tilde{s}^2_{\mathrm{TX,Q}}(t)}\right\}\cos(\omega_0 t + \phi_0 + \phi(t))$$
$$\phi(t) = \mathrm{atan2}(\tilde{s}_{\mathrm{TX,Q}}(t), \tilde{s}_{\mathrm{TX,I}}(t))$$

Summary

$$\boxed{x(t) = \mathrm{Re}\{x_+(t)\} \quad x_+(t) = \tilde{x}(t)\sqrt{2}e^{j2\pi f t}}$$

$x(t)$	physical passband signal
$x_+(t)$	analytic signal (pre-envelope of $x(t)$)
$\tilde{x}(t)$	compex envelope of $x(t)$

4.8 Noncoherent Detection

Carrier phase θ at the receiver becomes a random variable.

4.9 ML detection with unknown phase shift

$$L(\boldsymbol{s}_i) \triangleq f_X(\boldsymbol{x}|\boldsymbol{s}_i) = \frac{1}{(\pi N_0)^{N/2}}\exp\left[-\frac{1}{N_0}\sum_{j=0}^{N}(x_j - s_{ij})^2\right]$$

The ML receiver selects the hypothesis, which maximizes the likelihood function

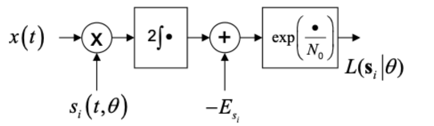
$$\hat{i} = \arg\max_i (L(\boldsymbol{s}_i))$$

Expanding the sum in the exponent, the likelihood functino can be calculated from the output of a correlator bank.

$$\boxed{L(\boldsymbol{s}_i) = c\exp\left[\frac{2}{N_0}\int x(t)s_i(t)\,\mathrm{d}t - \frac{1}{N_0}E_{s_i}\right]}$$

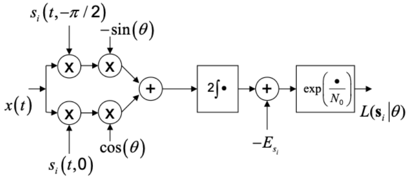
With $E_{s_i} = \sum_j s_{ij}^2 = \int s_i^2\,\mathrm{d}t$

For a known phase offset, the modified receiver correlates with a rotate versino of each hypothesis.



Two-branch correlator

$$s_i(t,\theta) = s_i(t, \theta = 0)\cos\theta - s_i(t, \theta = -\pi/2)\sin\theta$$



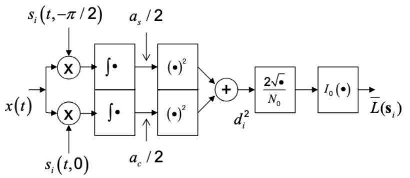
Equi-Energy Signals with unknown phase offset Shifting the integrator to each branch and obtain equi-energy signals with known phase offset:

$$L(\boldsymbol{s}_i|\theta) = \exp\left(\frac{1}{N_0}\left(a_c\cos\theta - a_s\sin\theta\right)\right)$$
$$= \exp\left(\frac{1}{N_0}\sqrt{a_c^2+a_s^2}\cos(\theta+\phi)\right)$$
$$\phi = \angle(a_c + ja_s)$$

With unknoen phase offset, we have to average the likelihood functino across all phase offsets θ.

$$\overline{L(\boldsymbol{s}_i)} = \frac{1}{2\pi}\int_{\pi}^{\pi}\exp\left(\frac{1}{N_0}\sqrt{a_c^2+a_s^2}\cos(\theta+\phi)\right)\,\mathrm{d}\theta$$
$$= I_0\left(\frac{1}{N_0}\sqrt{a_c^2+a_s^2}\right)$$

I_0 is the modified Bessel function of order zero.



As I_0 is monotonously increasing, a simplified decision rule follows as

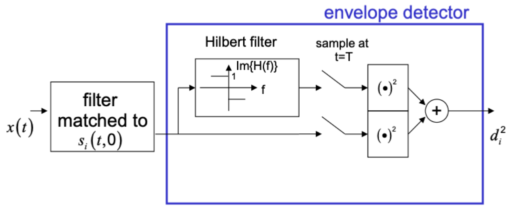
$$\hat{i} = \arg\max_i d_i^2$$

Note that we need a two-branch correlator for each hypothesis $s_i(t)$.

Instead of the two-branch correlator we can use two matched filter - sampler pairs to calculate the decision variable.

We can determine the decision variable woth one matched filter an a Hilbert tranformer. The matched filter - envelope detector pair is called a noncoherent matched filter.

$$s_i(t,\theta=-\pi/2) \circ\!\!-\!\!\bullet S_i(f,\theta=-\pi/2)$$
$$= -j\,\mathrm{sgn}(f) S_i(f,\theta=0)$$

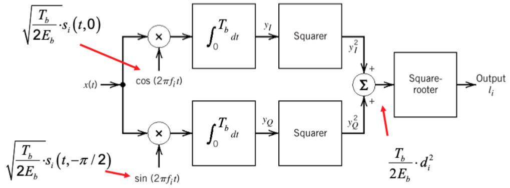


4.10 Noncoherent FSK

Signal $x(t)$ at the receiver with unknown carrier phase offset θ:

$$x(t) = sqrt\frac{2E_b}{T_b}\cos(2\pi f_i t+\theta)+w(t), \quad i=1,2,0\le t\le T$$

The signals s_1 and s_2 each require such a branch. A comparator subsequently compares the two outputs I_i to devide between the hypothesis s_1 and s_2.

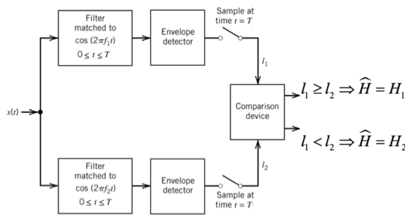


We have

$$P_e = \frac{1}{2}\exp\left(-\frac{E_b}{2N_0}\right)$$

This corresponds to a degradatino of at least 3dB compared to coherent MSK. Less degradation compared to BFSK.

Another implementation is with matched bandpass filters to f_1 and f_2 followed by envelope detectors, samplers and a comparison device.

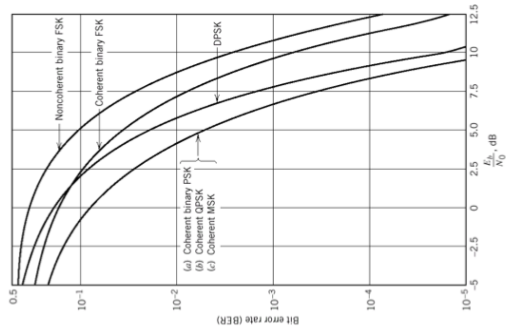


4.11 DPSK Differential PSK

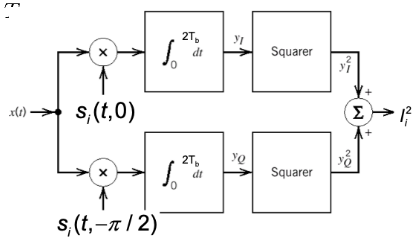
Differential precoding at transmitter: Symbol 0 $\Longrightarrow$ π phasejump, Symbol 1 $\Longrightarrow$ no phase-jump. Assumption: θ does not change significantly between two adjecent sampling instances.

$$s_1(t,\theta) = \begin{cases}\sqrt{\frac{2E_b}{T_b}}\cos(2\pi f_c t+\theta), & 0\le t\le T_b\\ \sqrt{\frac{2E_b}{T_b}}\cos(2\pi f_c t+\theta), & T_b\le t\le 2T_b\end{cases}$$

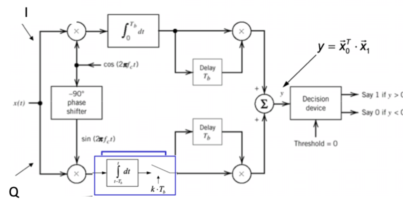
$$s_2(t,\theta) = \begin{cases}\sqrt{\frac{2E_b}{T_b}}\cos(2\pi f_c t+\theta), & 0\le t\le T_b\\ \sqrt{\frac{2E_b}{T_b}}\cos(2\pi f_c t+\pi+\theta), & T_b\le t\le 2T_b\end{cases}$$



Noncoherent detector for DPSK:



Quadrature implementation of simplified detector:



DPSK is a special case of noncoherent, orthogonal modulation with $T = 2T_b$ and $E = 2E_b$. Th bit error rate is given by:

$$P_e = \frac{1}{2}\exp\left(-\frac{E_b}{N_0}\right)$$

4.12 Performance comparison

Modulation	P_e
Coherent BPSK Coherent QPSK Coherent MSK	$\frac{1}{2}\,\mathrm{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right)$
Coherent binary FSK	$\frac{1}{2}\,\mathrm{erfc}\left(\sqrt{\frac{E_b}{2N_0}}\right)$
DPSK	$\frac{1}{2}\exp\left(-\frac{E_b}{N_0}\right)$
Noncoherent binary FSK	$\frac{1}{2}\exp\left(-\frac{E_b}{2N_0}\right)$

5 Math

5.1 General

$$\cos(a)\cos(b)+\sin(a)\sin(b) = \cos(a-b)$$

$$\cos(a+b)\cos(a-b) = \frac{1}{2}\left[\cos(2a)+\cos(2b)\right]$$

$$\cos(a)\cos(b) = \frac{1}{2}(\cos(a-b)+\cos(a+b))$$

$$\sin(a)\sin(b) = \frac{1}{2}(\cos(a-b)-\cos(a+b))$$

$$\cos(a)\sin(b) = \frac{1}{2}(\sin(a+b)-\sin(a-b))$$

$$\operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

$$\sin(x) = \frac{e^{ix}-e^{-ix}}{2i} \qquad \cos(x) = \frac{e^{ix}+e^{-ix}}{2}$$

5.2 Fourier Transform

Source: Haykin, Communication systems, 4th ed.

$\operatorname{rect}\left(\frac{t}{T}\right)$	$\circ\!\!-\!\!\bullet$	$T\operatorname{sinc}(fT)$
$\operatorname{sinc}(2Wt)$	$\circ\!\!-\!\!\bullet$	$\frac{1}{2W}\operatorname{rect}\left(\frac{f}{2W}\right)$
$\exp(-at)u(t), a>0$	$\circ\!\!-\!\!\bullet$	$\frac{1}{a+j2\pi f}$
$\exp(-a\lvert t\rvert), a>0$	$\circ\!\!-\!\!\bullet$	$\frac{2a}{a^2+(2\pi f)^2}$
$\exp(-\pi t^2)$	$\circ\!\!-\!\!\bullet$	$\exp(-\pi f^2)$
$\begin{cases}1-\frac{\lvert t\rvert}{T}, & \lvert t\rvert < T\\ 0, & \lvert t\rvert \geq T\end{cases}$	$\circ\!\!-\!\!\bullet$	$T\operatorname{sinc}^2(fT)$
$\delta(t)$	$\circ\!\!-\!\!\bullet$	1
1	$\circ\!\!-\!\!\bullet$	$\delta(f)$
$\delta(t-t_0)$	$\circ\!\!-\!\!\bullet$	$\exp(-j2\pi f t_0)$
$\exp(j2\pi f_c t)$	$\circ\!\!-\!\!\bullet$	$\delta(f-f_c)$
$\cos(2\pi f_c t)$	$\circ\!\!-\!\!\bullet$	$\frac{1}{2}[\delta(f-f_c)+\delta(f+f_c)]$
$\sin(2\pi f_c t)$	$\circ\!\!-\!\!\bullet$	$\frac{1}{2i}[\delta(f-f_c)+\delta(f+f_c)]$
$\operatorname{sgn}(t)$	$\circ\!\!-\!\!\bullet$	$\frac{1}{j\pi f}$
$\frac{1}{\pi t}$	$\circ\!\!-\!\!\bullet$	$-j\operatorname{sgn}(f)$
$u(t)$	$\circ\!\!-\!\!\bullet$	$\frac{1}{2}\delta(f)+\frac{1}{j2\pi f}$
$\sum_{i=-\infty}^{\infty}\delta(t-iT_0)$	$\circ\!\!-\!\!\bullet$	$\frac{1}{T_0}\sum_{n=-\infty}^{\infty}\delta\left(f-\frac{n}{T_0}\right)$

$u(t)$ unit step function
$\delta(t)$ delta function
$\operatorname{rect}(t)$ rectangular function of unit amplitude and unit duration centered on the origin
$\operatorname{sgn}(t)$ signum function
$\operatorname{sinc}(t)$ sinc function

Relations

$\alpha f(t)+\beta g(t)$	$\circ\!\!-\!\!\bullet$	$\alpha F(f)+\beta G(f)$
$f^*(t)$	$\circ\!\!-\!\!\bullet$	$F^*(-f)$
$f(at)$	$\circ\!\!-\!\!\bullet$	$\frac{1}{\lvert a\rvert}F\left(\frac{f}{a}\right)$
$f(t-a)$	$\circ\!\!-\!\!\bullet$	$\mathrm{e}^{-j2\pi f a}F(f)$
$e^{j2\pi f_0 f}$	$\circ\!\!-\!\!\bullet$	$F(f-f_0)$
$f^{(n)}$	$\circ\!\!-\!\!\bullet$	$(j2\pi f)^n F(f)$
$t^n f(t)$	$\circ\!\!-\!\!\bullet$	$j^n F^{(n)}(f)$
$\int_{-\infty}^{t} x(\tau)\,\mathrm{d}\tau$	$\circ\!\!-\!\!\bullet$	$\frac{1}{j2\pi f}F(f)+\pi F(0)\delta(f)$
$\frac{1}{t}x(t)+\pi x(0)\delta(t)$	$\circ\!\!-\!\!\bullet$	$\int_{-\infty}^{f} X(s)\,\mathrm{d}s$
$(f*g)(t)$	$\circ\!\!-\!\!\bullet$	$F(f)\cdot G(f)$
$f(t)\cdot g(t)$	$\circ\!\!-\!\!\bullet$	$\frac{1}{2\pi}F(f)*G(f)$

$f^{(n)}$ n^{th} derivation
f^* complex conjugate

5.3 Probability

$$\mathbf{P}(X>x) = \frac{1}{2}\operatorname{erfc}\left(\frac{x}{\sqrt{2}\sigma_X}\right)$$

$$\operatorname{erf}(x) = \frac{1}{\sqrt{\pi}}\int_{-x}^{x}\mathrm{e}^{-t^2}\,\mathrm{d}t = \frac{2}{\sqrt{\pi}}\int_{0}^{x}\mathrm{e}^{-t^2}\,\mathrm{d}t$$

$$\operatorname{erfc}(x) = 1-\operatorname{erf} x = \frac{2}{\sqrt{\pi}}\int_{x}^{\infty}\mathrm{e}^{-t^2}\,\mathrm{d}t$$