Disclaimer

This formulary was built for the Analysis III course taught by A. Figalli (401-5350-00L) in 18HS at ETHZ. I do not guarantee completeness and take no responsibility for content errors.

Contribution

If you make use of this formulary please help improve it by reporting errors or making pull requests with additions. The upstream is located at
`https://github.com/noah95/formulasheets`

Analysis III - PDE

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1 General

1.1 Derivative Rules

Product Rule	$\frac{d}{dx}f(x)g(x)$	$= f(x)g'(x) + f'(x)g(x)$
Chain Rule	$\frac{d}{dx}f(g(x))$	$= f'(g(x))g'(x)$

1.2 Some ODE

$x' + ax = c$	$x = c_1 e^{-ax} + \frac{c}{a}$		
$ax'' + bx' + cx = 0$	$b^2 > 4ac$	$x = a_1 e^{\lambda_1 x} + a_2 e^{\lambda_2 x}$	$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
$ax'' + bx' + cx = 0$	$b^2 = 4ac$	$x = (a_1 + a_2 x)e^{\lambda x}$	$\rho = \frac{b}{2a}$
$ax'' + bx' + cx = 0$	$b^2 < 4ac$	$x = a_1 e^{\rho t}\cos(\omega t) + a_2 e^{\rho t}\sin(\omega t)$	$\omega = \left\lvert \frac{\sqrt{b^2 - 4ac}}{2a} \right\rvert$

1.3 Some PDE

Using $\frac{\partial}{\partial t}x(t,s) = x'(t)$ partial derivatives. Further $x = x(t,s)$, $y = y(t,s)$

First order

$x' = x^2$ $\quad x = \frac{1}{c_1 - t}$

$x' - tx = 0$ $\quad x = c(s)e^{-t^2/2}$

Rules

$[\log(x)]_t \quad = \frac{x_t}{x}$

1.4 Solve linear ODE with Integrating Factor

Problem: $\boxed{y' + p(t)y = q(t)}$ the ODE can be solved using an integrating factor $\mu(t)$

1. Calculate integrating factor $\quad \mu(t) = e^{\int p(t)dt}$
2. Multiply both sides of the ODE by $\mu(t)$ $\quad \mu(t)\left[y' + p(t)y\right] = \mu(t)q(t)$
3. Calculate $\quad (\mu(t)y)' = \mu(t)q(t)$
4. Integrating each side with respect to t $\quad \mu(t)y = \int \mu(t)q(t)dt + C$
5. Final form $\quad y = \mu^{-1}(t)\left(\int \mu(t)q(t)dt + C\right)$

1.5 Inequalities

If $[a,b]$ is some interval, $f(a) < g(a)$, and $f'(x) \leq g'(x)$ on the interval, then $f(x) < g(x)$ on the interval. This implication cannot be reversed.

1.6 Integrals

Area of a disk B_R with radius R around (x_0, y_0)

$$\int_{B_R(x_0,y_0)} u(x,y)\mathrm{d}x\mathrm{d}y = \int_0^R \int_0^{2\pi} u(x_0 + r\cos(\theta), y_0 + r\sin(\theta) r \mathrm{d}\theta \mathrm{d}r$$

2 Notation of PDE

unknown function $u : D \subset \mathbb{R}^n \to \mathbb{R}$

$F(x_1, x_2, \ldots, x_n, u(x1, \ldots, x_n), u_{x_1}, \ldots, u_{x_n}, u_{x_1 x_1}, \ldots) = 0$

Short form: $\boxed{F(x) = 0}$

3 Classification of PDE

3.1 Order

Order = order of highest derrivative in equation

3.2 Linearity

PDE is linear if F is a linear function of the unknown function u

Semilinear All derivatives are linear (Nonlin. only in u)

$a(x,y,u)\frac{\partial u}{\partial x} + b(x,y,u)\frac{\partial u}{\partial y} + c(x,y)\frac{\partial^3 u}{\partial x^2 \partial y} = d(x,y,u)$

Quasilinear Highest-order derivative are linear

$a(x,y,u,u+x)\frac{\partial u}{\partial x} + a(x,y,u,u_y)\frac{\partial u}{\partial y} + c(x,y,u,u_x,u_y)\frac{\partial^2 u}{\partial x^2} = d(x,y,u)$

Linear linear in u $\quad a(x,y)u_x + b(x,y)u_y = c_0(x,y) + c_1(x,y)$

if u_1, u_2 are sol. then $\lambda u_1 + (1-\lambda)u_2$ is a sol.

3.3 Homogeneous

PDE is homogeneous if $F(x,y) = 0$.

4 First Order PDE: Method of Characteristics

4.1 Problem

$$\begin{cases} a(x,y,u)u_x + b(x,y,u)u_y & = c(x,y,u) \\ u(x_0,y_0) & = f(x) \text{or} f(y) \end{cases}$$

4.2 Solution

1. Bring to standard form

$$au_x + bu_y = c$$

2. Parametrize initial condition

$$\begin{aligned} x_t(t,s) &= a(x,y,u) & x(0,s) &= x_0(s) \\ y_t(t,s) &= b(x,y,u) & y(0,s) &= y_0(s) \\ u_t(t,s) &= c(x,y,u) & u(0,s) &= f(s) \end{aligned}$$

3. Solve characteristic equations

$$\begin{aligned} \frac{\partial}{\partial t}x(t,s) &= a(x,y,u) & x(0,s) &= x_0(s) \\ \frac{\partial}{\partial t}y(t,s) &= b(x,y,u) & y(0,s) &= y_0(s) \\ \frac{\partial}{\partial t}u(t,s) &= c(x,y,u) & u(0,s) &= f(s) \end{aligned}$$

If linked, parametrize: $x = s \cdot \cos(t)$, $y = s \cdot \sin(t)$

4. Find $s(x,y)$, $t(x,y)$ and put in $u(t,s)$. Solve for $u(x,y)$ if Jacobian $J \neq 0$

4.3 Jacobian

$J = \frac{\partial(x,y)}{\partial(t,s)} = \begin{vmatrix} \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{vmatrix} = \begin{vmatrix} a & b \\ \frac{\partial x_0}{\partial s} & \frac{\partial y_0}{\partial s} \end{vmatrix} = (y_0)_s a - (x_0)_s b$

Must be $\neq 0$ for x, y to be invertable and the method of characteristics to work.

4.4 Existence and uniqueness

Transversality condition

For a, b replace x, y by x_0, y_0. If transversality condition is fulfilled the PDE has a unique solution. Set $t = 0$.

$$J = \det\left(\begin{bmatrix} a_0 & b_0 \\ \frac{\partial x_0}{\partial s} & \frac{\partial y_0}{\partial s} \end{bmatrix}\right) = (y_0)_s a - (x_0)_s b \neq 0$$

4.5 Conservation law & shock waves

Considering the transport equation $\boxed{u_y + \frac{\partial}{\partial x} F(u) = 0}$

$$\begin{cases} u_y + \frac{\partial}{\partial x} F(u) = 0 \\ u(x,0) = h(s) = & \begin{cases} u^- & x \leq \alpha \\ u^+ & x > \alpha \end{cases} \end{cases}$$

If $u_0(s)$ is never decreasing, there will be no singularity (hence no shock wave).

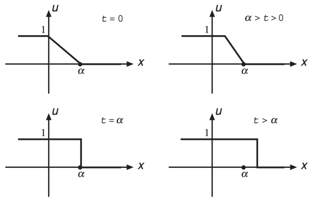

Figure 1: Several snapshots in the development of a shock wave.

Entropy Condition

Characteristics must enter the shock curve, and are not allowed to emanate from it.

$$F_u(u^-) > \gamma_y > F_u(u^+)$$

Applying this rule to the special case $F(u) = \frac{1}{2}u^2$ we obtain that the shock solution is valid only if $u^- > u^+$.

Case shock wave: $u^- > u^+$

The characteristics intersect and it results in a shock wave. The shock wave $\gamma(y)$ describes the curve along which $u(x,y)$ assumes different values. $\gamma_y(y)$ descirbes the speed at which the discontinuity is moving (Rankine-Hugoniot condition). The schock wave is given by

$$u(x,y) = \begin{cases} u^- & x < \gamma(y) \\ u^+ & x > \gamma(y) \end{cases} \qquad \gamma_y(y) = \frac{F(u^+) - F(u^-)}{u^+ - u^-} \qquad \begin{cases} \gamma(y) = \int \gamma_y(y) dy \\ \gamma(0) = \alpha \end{cases}$$

$$u^+(y) = \lim_{x \to \gamma_y(y)_+} u(x,y) \qquad u^-(y) = \lim_{x \to \gamma_y(y)_-} u(x,y)$$

α is the projection of the discontinuity point of the shock wave to $y = 0$.
y_c denotes the critical time where the solution becomes non-smooth. That is, the classical solution is not well defined for $y > y_c$.

$$y_c = \inf_{s \in \mathbb{R}} \left\{ -\frac{1}{h'(s)} : h'(s) < 0 \right\} \qquad h(s) = u(x,0)$$

If $h(s)$ (the initial value) has a discontinuity (e.g. a step), $y_c = 0$: the solution is weak immediately.

5 Second Order PDE

$$L[u] = a \cdot u_{xx} + 2b \cdot u_{xy} + c \cdot u_{yy} + d \cdot u_x + e \cdot u_y + f \cdot u = g$$

5.1 Classification

Discriminant δ:

$$\delta(L)(x,y) = b^2(x,y) - a(x,y) \cdot c(x,y) \qquad L[u] \begin{cases} \text{elliptic} & \delta(L) < 0 \\ \text{parabolic} & \delta(L) = 0 \\ \text{hyperbolic} & \delta(L) > 0 \end{cases}$$

6 1D Wave Equation

Cauchy problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & (x,t) \in \mathbb{R} \times (0, \infty), \\ u(x,0) = f(x) & x \in \mathbb{R}, \\ u_t(x,0) = g(x) & x \in \mathbb{R}. \end{cases}$$

Coordinate transformation to ξ, η. Solution $u(x,t)$ consists of forwards $F(x - ct)$ and backwards $G(c + xt)$ travelling wave.

$$\xi = x + ct \qquad \eta = x - ct \qquad \omega(\xi, \eta) = u(x(\xi,\eta), y(\xi,\eta))$$

$$-4c^2 \omega_{\xi\eta} = 0 \qquad \omega(\xi,\eta) = F(\xi) + G(\eta) \qquad u(x,t) = F(x - ct) + G(x + ct)$$

6.1 d'Alembert Formula

General solution to the 1D wave equation for $x \in \mathbb{R}$. (Not to be used for x in an interval, use seperation of variables.)

$$\boxed{u(x,t) = \frac{1}{2}\left(f(x+ct) + f(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\mathrm{d}s}$$

6.2 Nonhomogeneous case

$$\begin{cases} u_{tt} - c^2 u_{xx} = F(x,t) & (x,t) \in \mathbb{R} \times (0, \infty), \\ u(x,0) = f(x) & x \in \mathbb{R}, \\ u_t(x,0) = g(x) & x \in \mathbb{R}. \end{cases}$$

Using d'Alembert and extending with the integral on the triangle Δ_{x_0,t_0}:

$$u(x,t) = \frac{1}{2}\left(f(x+ct) + f(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\mathrm{d}s + \frac{1}{2c}\iint_{\Delta x_0, t_0} F(x,t)\mathrm{d}x\mathrm{d}t$$

Explicit form:

$$\boxed{u(x,t) = \frac{1}{2}\left(f(x+ct) + f(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\mathrm{d}s + \frac{1}{2c}\int_0^t \mathrm{d}\tau \int_{x-c(t-\tau)}^{x+c(t-\tau)} F(s,\tau)\mathrm{d}s}$$

6.3 Odd initial data

If the problem is stated for $x > 0$ instead of $x \in (R)$, the initial data $f(x)$ and $g(x)$ have to be extended oddly around zero.

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & (x,t) \in (0,\infty) \times (0,\infty), \\ u(0,t) = 0 & t \in (0,\infty), \\ u(x,0) = f(x) & x \in [0,\infty), \\ u_t(x,0) = g(x) & x \in [0,\infty). \end{cases} \qquad f(-x) = f(x) \qquad g(-x) = g(x)$$

For example:

$$x^2 \mapsto x|x| \qquad x^4 \mapsto x^3|x| \qquad \sin(x) \mapsto \sin(x)$$

6.4 Domain of dependance

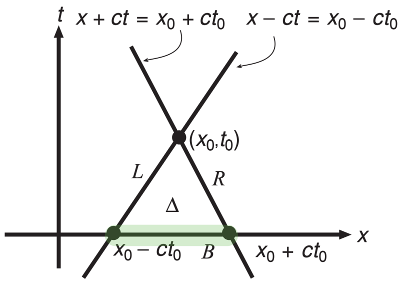

Figure 2: Domain of dependence.

The values at (x_0, t_0) depend only on the initial data at

$$[a = x_0 - ct, b = x_0 + ct]$$

Therefore if the initial data live in $[a, b]$ then a point (x_0, y_0) will feel them only if

$$x_0 - ct, x_0 + ct] \cap [a, b] \neq \emptyset$$

6.5 Domain of influence

$$\text{DOI} = [x_0 - ct, x_0 + ct]$$

6.6 Wave equation in interval

If the problem is stated in an interval $x \in (a, b)$ with zero boundary conditions, a global problem must be found whose solution $\tilde{u}(x, t)$ must coincide with u in the interval.

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & (x,t) \in (a,b) \times (0,\infty), \\ u(a,t) = 0 & t \in (0,\infty), \\ u(b,t) = 0 & t \in (0,\infty), \\ u(x,0) = f(x) & x \in (a,b), \\ u_t(x,0) = g(x) & x \in (a,b). \end{cases}$$

Therefore extend $f(x)$ to be odd with respect to a and b. That is

$$\tilde{u}(-(x-a), t) = -\tilde{u}((x-a), t) \qquad \tilde{u}(-(x-b), t) = -\tilde{u}((x-b), t)$$

Then solve the new Cauchy problem:

$$\begin{cases} \tilde{u}_{tt} - c^2 \tilde{u}_{xx} = 0 & (x,t) \in \mathbb{R} \times (0,\infty), \\ \tilde{u}(x,0) = \tilde{f}(x) & x \in \mathbb{R}, \\ \tilde{u}_t(x,0) = \tilde{g}(x) & x \in \mathbb{R}. \end{cases} \qquad u(x,t) = \tilde{u}(x,t) \quad \text{for} \quad x \in (a,b)$$

7 Separation of variables

7.1 Ansatz

1. Write solution as product

$$u(x,t) = X(x)T(t) \qquad u_t(x,t) = X(x)T'(t) \qquad u_{tt}(x,t) = X(x)T''(t)$$
$$u_x(x,t) = X(x)'T(t) \qquad u_{xx}(x,t) = X(x)''T(t)$$

2. Substitute into problem

$$u_t - u_{xx} = 0 \qquad \frac{T'}{T} = \frac{X''}{X} = -\lambda = \text{const} \qquad u_{tt} - u_{xx} = 0 \qquad \frac{T''}{T} = \frac{X''}{X} = -\lambda = \text{const}$$

3. Solve ODE

$$\begin{array}{lll} X'' = -\lambda X & \lambda > 0 & X(x) = \alpha \sin(\sqrt{\lambda} x) + \beta \cos(\sqrt{\lambda} x) \\ & \lambda = 0 & X(x) = \alpha + \beta x \\ & \lambda < 0 & X(x) = \alpha \sinh(\sqrt{-\lambda} x) + \beta \cosh(\sqrt{-\lambda} x) \\ & & X(x) = \alpha \sinh(\sqrt{-\lambda} x) + \beta \sinh(\sqrt{-\lambda}(x - \pi)) \\ T' = -\lambda T & & T(t) = e^{-\lambda t} \end{array}$$

4. Write $u(x,t)$ as sum and impose initial condition

$$u(x,t) = \sum_{n=0}^{\infty} X_n(x) T_n(t) \qquad u(x,0) = \sum_{n=0}^{\infty} X_n(0) T_n(t)$$

7.2 Application: Heat equation

$$\begin{cases} u_t - \kappa u_{xx} = 0, & 0 < x < L, t > 0, \\ u(0,t) = u(L,t) = 0, & t \geq 0, \\ u(x,0) = f(x), & 0 \leq x \leq L. \end{cases}$$

There exists only a solution for $\lambda > 0$. And hence $u(0,t) = u(L,t) = 0$ the boundary conditions are zero, the sin is chosen.

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad u(x,t) = \sum_{n=0}^{\infty} B_n \cdot \sin\left(\frac{n\pi}{L} x\right) e^{-\kappa \left(\frac{n\pi}{L}\right)^2 t}$$

7.3 Boundary conditions

$$\begin{array}{llll} u(0,t) = u(L,t) & \rightarrow \text{Dirichlet} & X_n(x) = \alpha_n \sin(\sqrt{\lambda} x) & \sqrt{\lambda} = \frac{n\pi}{L} \\ u_x(0,t) = u_x(L,t) & \rightarrow \text{Neumann} & X_n(x) = \alpha_n \cos(\sqrt{\lambda} x) & \sqrt{\lambda} = \frac{n\pi}{L} \\ \text{combination} & \rightarrow \text{Mixed/Robin} & X_n(x) = \alpha_n \sin((n + \frac{1}{2}) \frac{\pi}{L} x) & \sqrt{\lambda} = \left(n + \frac{1}{2}\right) \frac{\pi}{L} \end{array}$$

7.4 Nonhomogeneous case

$$\begin{cases} u_t - u_{xx} = h(x,t), & 0 < x < L, t > 0, \\ u_x(0,t) = u_x(L,t) = 0, & t \geq 0, \\ u(x,0) = f(x), & 0 \leq x \leq L. \end{cases}$$

1. Check boundary conditions and decide which $X_n(x)$ to use (See 7.3). Here Dirichlet is used.
2. Look for linear combination and make derivatives of $u(x, y)$.

$$u(x,t) = \sum_{n \geq 0} T_n(t) cos(\sqrt{\lambda} x) \qquad u_t(x,t) = \sum_{n \geq 0} T_n'(t) cos(\sqrt{\lambda} x) \qquad u_{xx}(x,t) = \sum_{n \geq 0} -\lambda T_n(t) cos(\sqrt{\lambda} x)$$

3. From the initial condition $u(x,0) = f(x)$ the initial values of $T_n(0)$ can be derived

$$u(x,0) = \sum_{n \geq 0} T_n(0) cos(\sqrt{\lambda} x) = f(x) \qquad \longrightarrow T_n(0) = \ldots$$

4. Imposing $u_t - u_{xx} = h(x,t)$ using the ansatz we get a set of ODE:

$$\sum_{n \geq 0} (T_n'(t) + \lambda T_n(t)) cos(\sqrt{\lambda} x) = h(x,t) \qquad \longrightarrow T_n'(t) = \ldots$$

5. Use $T_n'(t)$ and $T_n(0)$ to solve for all $T_n(t)$
6. Put solution together
7. Check solution

7.5 Nonhomogeneous boundary conditions

$$\begin{cases} u_t - u_{xx} = h(x,t), & 0 < x < L, t > 0, \\ u_x(0,t) = a(t), & t \geq 0, \\ u_x(L,t) = b(t), & t \geq 0, \\ u(x,0) = f(x), & 0 \leq x \leq L. \end{cases} \qquad \begin{cases} B_a[u] = \alpha u(a,t) + \beta u_x(a,t) & = a(t),\ t \geq 0 \\ B_b[u] = \gamma u(b,t) + \delta u_x(b,t) & = b(t),\ t \geq 0 \end{cases}$$

Find new functino $w(x,t)$ satisfying such non-homogeneous boundary conditions and study the problem being satisfied by $v(x,t) = u(x,t) - w(x,t)$. If the boundary condition is of the following form, table 1 provides $w(x,t)$. ⚠Resubstitute to $u(x,t) = v(x,t) + w(x,t)$

	Boundary condition		$w(x,t)$
Dirichlet	$u(0,t) = a(t)$	$u(L,t) = b(t)$	$w(x,t) = a(t) + \frac{x}{L}[b(t) - a(t)]$
Neumann	$u_x(0,t) = a(t)$	$u_x(L,t) = b(t)$	$w(x,t) = xa(t) + \frac{x^2}{2L}[b(t) - a(t)]$
Mixed	$u(0,t) = a(t)$	$u_x(L,t) = b(t)$	$w(x,t) = a(t) + xb(t)$
Mixed	$u_x(0,t) = a(t)$	$u(L,t) = b(t)$	$w(x,t) = (x-L)a(t) + b(t)$

Table 1: Boundary conditions

8 Elliptic Equation

8.1 Harmonic functions

Let $D \subset \mathbb{R}^n$. A function $f : D \to \mathbb{R}^n$ is harmonic in D if it is twice differenciable for all $x \in D$. It holds:

$$\Delta f(x) = 0$$

8.2 Poisson equation

$$\Delta u = F(x,y) \qquad \begin{cases} u(x,y) = g(x,y) & \text{on}\partial D & \text{Dirichlet problem} \\ \dfrac{\partial u}{\partial v}(x,y) = g(x,y) & \text{on}\partial D & \text{Neumann problem} \\ U(x,y) + \alpha\dfrac{\partial u}{\partial v}(x,y) = g(x,y) & \text{on}\partial D & \text{Robin problem} \end{cases}$$

Exists a solution?: A necessary condition for the existence of a solution to the Neumann problem is

$$\int_{\partial D} g(x(s))$$

8.3 Definition

$$\Delta u = u_{xx} + u_{yy} + u_{zz} = 0 \qquad \Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

8.4 Maximum Principle

weak maximum principle: Let D be a bounded domain, and let $u(x,y) \in C^2(D) \cap C(\bar{D})$ be a harmonic function in D. Then the maximum of u in $\bar{D}$ is achieved on the boundary ∂D. If u is harmonic in D, then $-u$ is harmonic there too. Therefore the minimum of a harmonic function u is also obtained on the boundary ∂D.

$$\max_{\bar{D}} u = \max_{\partial D} u \qquad \min_{\bar{D}} u = \min_{\partial D} u$$

strong maximum principle: Let u be a harmonic function in a domain D. If u attains its maximum (minimum) at an interior point of D, then u is constant.

8.5 Mean value Principle

Let (x_0, y_0) be a point in D. Assume that B_R is a disk of radius R centered at (x_0, y_0), fully contained in D. For any $r > 0$ set $C_r = \partial B_r$. Then the value of u at (x_0, y_0) is the average of the values of u on the circle C_R:

$$\begin{aligned} u(x_0, y_0) &= \frac{1}{2\pi R}\oint_{C_R} u(x(s), y(s))\mathrm{d}s \\ &= \frac{1}{2\pi}\int_0^{2\pi} u(x_0 + R\cos(\theta), y_0 + R\sin\theta)\mathrm{d}\theta \end{aligned}$$

8.6 General uniqueness for Poisson eq.

From Theorem (7.12):

1. Let $v = u_1 - u_2$ (u_1 and u_2 are soltutions to the poisson eq.)
2. Take a point (x_0, y_0) and propose $v(x_0, y_0) = M > 0$
3. Show by writing $0 = \Delta v(x_0, y_0) - kv(x_0, y_0) \leq -kM$ a contradiction $\to v(x_0, y_0) \not< 0$
4. Take a point (x_1, y_1) and propose $v(x_1, y_1) = M < 0$
5. Show by writing $0 = \Delta v(x_1, y_1) - kv(x_1, y_1) \geq -kM$ a contradiction $\to v(x_1, y_1) \not> 0$
6. Using *weak max principle* it holds that $v = 0 \iff u_1 = u_2$

8.7 Laplace equation in rectangular domain

$$\begin{cases} \Delta u = 0, & (x,y) \in (a,b) \times (c,d), \\ u(a,y), = f(y) & c < y < d, \\ u(b,y), = g(y) & c < y < d, \\ u(x,c), = h(x) & a < x < b, \\ u(x,d), = k(x) & a < x < b. \end{cases}$$

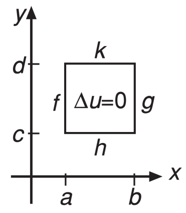

Figure 3: Separation of variables in rectangles.

Neumann ∃ sol. if: For $u_x()/u_y()$. The neccessary condition must hold that the integral over the boundary is zero. ⚠Mind the integral limits/direction

$$0 = -\int_a^b h\mathrm{d}x + \int_c^d g\mathrm{d}y + \int_a^b k\mathrm{d}x - \int_c^d f\mathrm{d}y$$

Ansatz: $u(x,y) = X(x)Y(y)$ For $X(x)$ proceed as in 7.3. For $Y(y)$ The following are equivalent:

$$\begin{aligned} Y_n(y) &= A_n\sinh(\sqrt{\lambda}y) + B_n\sinh(\sqrt{\lambda}y \\ &= C_n\sinh(\sqrt{\lambda}(y-c)) + D_n\sinh(\sqrt{\lambda}(y-d)) \\ &= E_n\cosh(\sqrt{\lambda}(y-c)) + F_n\cosh(\sqrt{\lambda}(y-d)) \\ &= G_n\cosh(\sqrt{\lambda}(y-c)) + H_n\sinh(\sqrt{\lambda}(y-d)) \\ &= J_n\cosh(\sqrt{\lambda}y) + K_n\cosh(\sqrt{\lambda}y) && \leftarrow \text{ use this if } u_x/u_y \text{ is given} \end{aligned}$$

Most common:

$$\boxed{Y_0(y) = \alpha_0 y + \beta_0, \quad Y_n(y) = \alpha_n\sinh(\sqrt{\lambda}(y-c)) + \beta_n\sinh(\sqrt{\lambda}(y-d)) \quad n \geq 1}$$

Then use the following and introduce boundaries $u(x,c)$ and $u(x,d)$.

$$\boxed{u(x,y) = \alpha_0 y + \beta_0 + \sum_{n\leq 1}\cos(\sqrt{\lambda}x)\left[\alpha_n\sinh(\sqrt{\lambda}(y-c)) + \beta_n\sinh(\sqrt{\lambda}(y-d))\right]}$$

8.8 Laplace eq. in circular domain

Problem:

$$\begin{cases} \Delta u = 0, & \text{in} \quad D, \\ u(r_a, \theta) = f(\theta), & \text{for} \quad 0 \le \theta \le 2\pi, \\ u(r_b, \theta) = g(\theta), & \text{for} \quad 0 \le \theta \le 2\pi. \end{cases} \qquad \Delta = \frac{\partial^2}{\partial r^2} \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}$$

Ansatz: If the boundary datum depends only on sine, the sine only ansatz can be chosen. Provide reason! Then introduce boundary conditions and solve for $A_n, B_n, \ldots$

General
$$u(t,\theta) = \sum_{n>0} \left(A_n r^n + B_n r^{-n}\right) \sin(n\theta) + C_0 + D_0 \log(r) + \sum_{n>0} \left(C_n r^n + D_n r^{-n}\right) \cos(n\theta)$$

sin only
$$u(t,\theta) = \sum_{n>0} \left(A_n r^n + B_n r^{-n}\right) \sin(n\theta)$$

cos only
$$u(t,\theta) = \sum_{n>0} \left(C_n r^n + D_n r^{-n}\right) \cos(n\theta) + C_0 + D_0 \log(r)$$

If origin $(r = 0)$ is inside domain D then $D_n = D_0 = 0 \quad \forall n$ because r^{-n} and $\log(r)$ blow up for $r \to 0$.

9 Common Tables

9.1 Derivatives and Integrals

$[c]' = 0$	$\int 0 \mathrm{d}x = c$
$[x]' = 1$	$\int 1 \mathrm{d}x = x + c$
$[x^{n+1}]' = (n+1)x^n, n \neq -1$	$\int x^n \mathrm{d}x = \frac{1}{n+1} x^{n+1} + x, n \neq -1$
$[\ln x]' = \frac{1}{x}, x > 0$	$\int \frac{1}{x} \mathrm{d}x = ln\lvert x\rvert + c$
$[\ln(-x)]' = \frac{-1}{-x} = \frac{1}{x}, x < 0$	$\int \frac{1}{x} \mathrm{d}x = ln\lvert x\rvert + c$
$[e^x]' = e^x$	$\int e^x \mathrm{d}x = e^x + c$
$[a^x]' = a^x \ln a$	$\int a^x \mathrm{d}x = \frac{1}{\ln a} a^x + c, a \neq 1$
$[\sin x]' = \cos x$	$\int \cos x \mathrm{d}x = \sin x + x$
$[\cos x]' = -\sin x$	$\int \sin x \mathrm{d}x = -\cos x + x$
$[\tan x]' = \frac{1}{\cos^2 x} = 1 + \tan^2 x$	$\int \frac{1}{\cos^2 x} \mathrm{d}x = \tan x + c$
	$\int \tan^2 x \mathrm{d}x = \tan x - x + c$
$[\cot x]' = -\frac{1}{\sin^2 x} = -1 - \cot^2 x$	$\int \frac{1}{\sin^2 x} \mathrm{d}x = -\cot x + c$
	$\int \cot^2 x \mathrm{d}x = -\cot x - x + c$
$[\arcsin x]' = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}} \mathrm{d}x = \arcsin x + c_1$
$[\arccos x]' = -\frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}} \mathrm{d}x = -\arccos x + c_2$
$[\arctan x]' = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2} \mathrm{d}x = \arctan x + c_1$
$[\mathrm{arccot} x]' = -\frac{1}{1+x^2}$	$\int \frac{1}{1+x^2} \mathrm{d}x = \mathrm{arccot} x + c_2$
$[\sinh x]' = \cosh x$	$\int \cosh x \mathrm{d}x = \sinh x + c$
$[\cosh x]' = \sinh x$	$\int \sinh x \mathrm{d}x = \cosh x + c$
$[\tanh x]' = \frac{1}{\cosh^2 x} = 1 - \tanh^2 x$	$\int \frac{1}{\cosh^2 x} \mathrm{d}x = \tanh x + c$
$[\coth x]' = -\frac{1}{\sinh^2 x} = 1 - \coth^2 x$	$\int \frac{1}{\sinh^2 x} \mathrm{d}x = -\coth x + c$

$$[\mathrm{arsinh}x]' = \frac{1}{\sqrt{x^2+1}} \qquad \int \frac{1}{\sqrt{x^2+1}}\mathrm{d}x = \mathrm{arsinh}x + c = \ln(x+\sqrt{x^2+1})$$

$$[\mathrm{arcosh}x]' = \frac{1}{\sqrt{x^2-1}} \qquad \int \frac{1}{\sqrt{x^2-1}}\mathrm{d}x = \mathrm{arcosh}x + c = \ln(x+\sqrt{x^2-1})$$

$$[\mathrm{artanh}x]' = \frac{1}{\sqrt{1-x^2}}, |x| < 1 \qquad \int \frac{1}{\sqrt{1-x^2}}, |x| < 1\mathrm{d}x = \mathrm{artanh}x + c = \frac{1}{2}\ln\frac{1+x}{1-x} + c, |x| < 1 + c$$

$$[\mathrm{arcoth}x]' = \frac{1}{\sqrt{1-x^2}}, |x| > 1 \qquad \int \frac{1}{\sqrt{1-x^2}}\mathrm{d}x = \mathrm{arcoth}x + c = \frac{1}{2}\ln\frac{x+1}{x-1} + c, |x| > 1 + c$$

$$\int \sin^2(x)\mathrm{d}x = -\frac{1}{4}\sin(2x) + \frac{1}{2}x + c \qquad \int \cos^2(x)\mathrm{d}x = \frac{1}{4}\sin(2x) + \frac{1}{2}x + c$$

9.2 Derivative rules

$$(f+g)' = f' + g' \qquad (cf)' = cf' \qquad (fg)' = f'g + g'f$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

9.3 Integral rules

substitution $\int_a^b f[u(x)]u'(x)\mathrm{d}x = \int_{u(a)}^{u(b)} f(z)\mathrm{d}z \quad z = u(x)$

partial integral $\int_a^b u(x)v'(x)\mathrm{d}x = u(x)v(x)|_a^b - \int_a^b u'(x)v(x)\mathrm{d}x$

9.4 Trigonometric identities

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i} \qquad \cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$

$$\sin^2(x) + \cos^2(x) = 1 \qquad \sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \sin\beta\cos(\alpha)$$

$$\cos(2x) = \cos^2(x) - \sin^2(x) \qquad \sin(\alpha-\beta) = \sin(\alpha)\cos(\beta) - \sin\beta\cos(\alpha)$$

$$\sin(2x) = 2\sin(x)\cos(x) \qquad \cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin\alpha\sin(\beta)$$

$$c^2 = a^2 + b^2 - 2ab\cos(\gamma) \qquad \cos(\alpha-\beta) = \cos(\alpha)\cos(\beta) + \sin\alpha\sin(\beta)$$

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} \qquad \cos^2(x) = \frac{1+\cos(2x)}{2}$$

9.5 Hyperbolic identities

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$\cosh^2(x) - \sinh^2(x) = 1 \qquad \sinh(x \pm y) = \sinh(x)\cosh(y) \pm \sinh y\cosh(x)$$

$$\sinh(2x) = 2\sinh(x)\cosh(x) \qquad \cosh(x \pm y) = \cosh(x)\cosh(y) \pm \sinh x\sinh(y)$$

$$\cosh(2x) = \cosh^2(x) + \sinh^2(x)$$

$$\sinh(-x) = -\sinh x$$

$$\cosh(-x) = \cosh x$$

9.6 Sums

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \qquad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

$$\sum_{k=0}^{n-1} q^k = \frac{1-q^n}{1-q} \qquad \sum_{k=0}^{\infty} q^k = \frac{1}{1-q}, \text{for}|q| < 1 \qquad \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Good Luck!