Problem Set 4

Due May 8, 2018

Turn in your solutions to the box marked AA273 on the second floor of Durand, south wing, on the date they are due.

1. In this problem you will derive an equivalent form of the Kalman filter called the information filter, which is advantageous for some problems. Consider the quantities $\Omega_{t|t} := \Sigma_{t|t}^{-1}$, $\Omega_{t|t-1} := \Sigma_{t|t-1}^{-1}$, called the information matrix, and $\xi_{t|t} := \Sigma_{t|t}^{-1}\mu_{t|t}$, $\xi_{t|t-1} := \Sigma_{t|t-1}^{-1}\mu_{t|t-1}$ called the information state.

   (a) Derive recursive expressions for $\Omega_{t|t}$ and $\Omega_{t|t-1}$ from the Kalman filter equations. Your expressions should each require, at most, a single matrix inversion (aside from R_t^{-1} and Q_t^{-1}, which can be precomputed). The following matrix inversion identity will be useful: $(A + UCV)^{-1} = A^{-1} - A^{-1}U(VA^{-1}U + C^{-1})^{-1}VA^{-1}$ for matrices A, U, C, and V.

   (b) Similarly, derive recursive expressions for $\xi_{t|t}$ and $\xi_{t|t-1}$ from the Kalman filter equations. Again, your expressions should require no more than a single matrix inverse except for R_t^{-1} and Q_t^{-1}. The following matrix identity will be useful: $(A + UCV)^{-1}UC = A^{-1}U(C^{-1} + VA^{-1}U)^{-1}$.

   (c) Matrix inversion dominates the computational complexity of the Kalman filter equations (with larger matrices taking longer to invert). With this in mind, when might one prefer the standard form or the information form of the Kalman filter?

2. Consider the rotational motion of a satellite, which can be modeled by Euler's equations of rigid body motion,

$$\begin{aligned} J_x\dot{\omega}_x &= (J_y - J_z)\omega_y\omega_z + \tau_x \\ J_y\dot{\omega}_y &= (J_z - J_x)\omega_z\omega_x + \tau_y \\ J_z\dot{\omega}_z &= (J_x - J_y)\omega_x\omega_y + \tau_z, \end{aligned}$$

   where $[\omega_x \quad \omega_y \quad \omega_z]^T$ is the angular velocity vector, $J = \text{diag}([J_x \quad J_y \quad J_z])$ is the inertia tensor in principal axes, and $\tau = [\tau_x \quad \tau_y \quad \tau_z]^T$ is the input torque from thrusters.

   (a) Use a first order discrete time approximation to find a difference equation model for the satellite. Use the time step $\delta t = 0.001$s.

   (b) Find the Jacobian of the resulting nonlinear discrete-time dynamics.

   (c) Suppose the satellite gets angular velocity measurements from on-board gyroscopes with the measurement equation

$$y_t = \text{sat}(\omega_t) + v_t,$$

   where $\text{sat}(\cdot)$ is the element-wise saturation function defined as follows. Let $\omega_t = [\omega_t^1 \quad \omega_t^2 \quad \omega_t^3]^T$. Then $\text{sat}(\cdot)$ returns a vector with ith element ω_t^i if $|\omega_t^i| \leq c$ for some threshold value c, and $\text{sign}(\omega_t^i)c$ if $|\omega_t^i| > c$ is above the threshold (where $\text{sign}(\cdot)$ returns 1 or -1 depending on the sign of its argument). Find the Jacobian of the measurement function.

3. Simulate the above nonlinear discrete time system in Matlab, and implement an extended Kalman filter to estimate its state. Let $J = \text{diag}([1 \quad 5 \quad 5])$, $\omega_0 = [10 \quad 0.1 \quad 0.1]^T$, $c = 10$, the process noise has covariance $Q = 0.004I$, and the measurement noise has covariance $R = 0.1I$. Let your initial state estimate be $\mu_0 = [10 \quad 0 \quad 0]^T$, $\Sigma_0 = \text{diag}([1 \quad 1 \quad 1])$. Assume the input torque is zero. Attach plots showing the satellite motion, and the performance of the filter.