A PDE-Based Jump Estimation for Phase Field Regularized Cracks

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Overview

- The jump over a discontinuity is an important quantity in many physics:
  - **Cohesive fracture**: traction-separation law;
  - **Hydraulic fracturing**: crack-open-displacement(COD)-dependent anisotropic permeability;
  - **Thermal contact**: COD-dependent thermal conductivity, temperature-jump-dependent heat flux;
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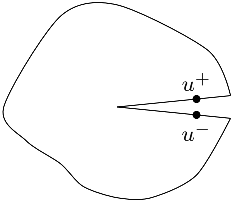

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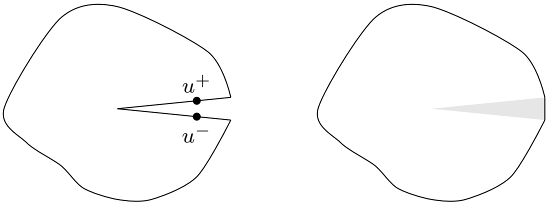

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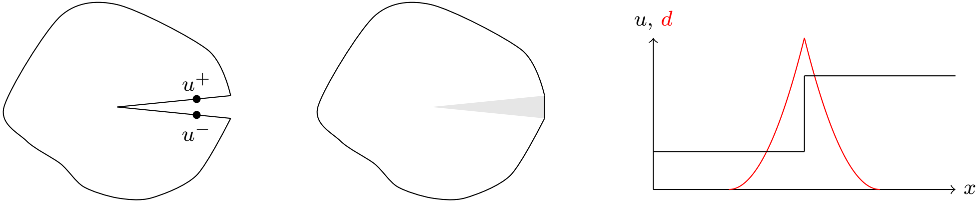
  - Broadcasting the jump estimation across the phase field band is inherently a nonlocal process.



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Our proposed method is a direct extension of [Verhoosel and de Borst, 2013].

- Assumptions:
  - The phase field d has a compact support whose width W is known a priori.
  - The approximated crack normal $\boldsymbol{n}_d \approx \boldsymbol{\nabla} d / \left\| \boldsymbol{\nabla} d \right\|$ satisfies $\boldsymbol{\nabla} \cdot \boldsymbol{n}_d \approx 0$.
- In 1D, for any integrable function $f : [a, b] \mapsto \mathbb{R}$, there exists a constant f^* s.t.

$$I = \int_a^b f \; \mathrm{d}s = f^*(b-a).$$

- Next, we introduce an auxliary integrable function $I^* : [a, b] \mapsto \mathbb{R}$ that is continuously differentiable on (a, b), and the constant $I = f^*(b-a)$ can be solved as

$$\int_a^b \left[(b-a)f - I^* \right] \; \mathrm{d}s = 0, \quad \text{subject to } I^*_{,s}(x) = 0 \quad \forall x \in (a, b),$$
$$I^* = I, \quad \forall x \in [a, b].$$

- In 2D and 3D, the problem can be generalized as

$$\boxed{\int_\Omega \left(W f \boldsymbol{n} - \boldsymbol{I}^* \right) \; \mathrm{d}V = \boldsymbol{0}, \quad \text{subject to } \boldsymbol{\nabla} \boldsymbol{I}^* \boldsymbol{n} = \boldsymbol{0} \quad \forall \boldsymbol{x} \in \Omega.}$$

- The jump at a discontinuity can be expressed using the dirac delta function in the integral form, and can be further regularized by a finite characteristic length.

$$[\![u]\!] = u^+ - u^- = \int_{0+}^{\infty} u\delta_\Gamma \, \mathrm{d}x - \int_{-\infty}^{0^-} u\delta_\Gamma \, \mathrm{d}x = \lim_{\epsilon \to 0} \left(\int_{0+}^{\infty} u\delta_\Gamma^\epsilon \, \mathrm{d}x - \int_{-\infty}^{0^-} u\delta_\Gamma^\epsilon \, \mathrm{d}x \right).$$

- In the context of phase field, a family of indicator functions $I(d)$ can be used as such regularization:

$$[\![u]\!] \approx \int_{0+}^{\infty} u \left\| \boldsymbol{\nabla} I \right\| \, \mathrm{d}x - \int_{-\infty}^{0^-} u \left\| \boldsymbol{\nabla} I \right\| \, \mathrm{d}x.$$

- Following the remark on the previous page, this directional jump be solved as

$$\boxed{\int_\Omega \left(W u \boldsymbol{n} - \boldsymbol{J}_u \right) \, \mathrm{d}V = \boldsymbol{0}, \quad \text{subject to } \boldsymbol{\nabla} \boldsymbol{J}_u \boldsymbol{n} = \boldsymbol{0} \quad \forall \boldsymbol{x} \in \Omega,}$$

which is the integral statement of

$$\boldsymbol{J}_u = [\![u]\!] \, \boldsymbol{n}, \quad \forall \boldsymbol{x} \in \Omega.$$

- The directional jump is solved using FEM. The pathwise-constant constraint can be enforced with a penalty approach:

$$r = \left(Wu\boldsymbol{n} - \boldsymbol{J}_u, \delta\boldsymbol{J}_u\right) + \alpha\left(\boldsymbol{\nabla}\boldsymbol{J}_u\boldsymbol{n}, \boldsymbol{\nabla}\delta\boldsymbol{J}_u\boldsymbol{n}\right).$$

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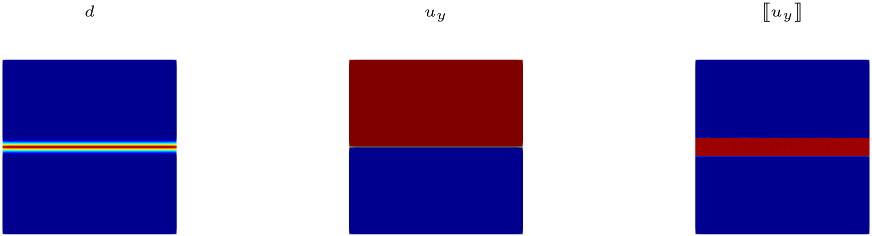

It works well for a fully developed crack.

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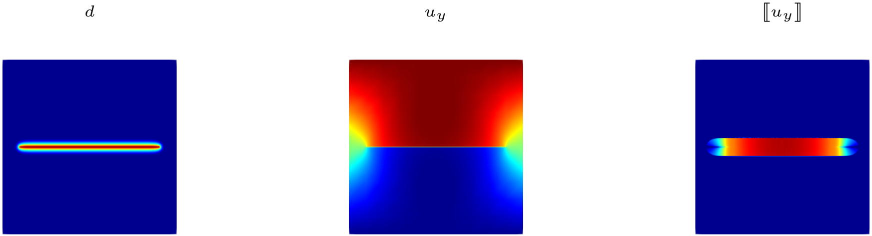

But it doesn't work well for an embedded crack. Recall our assumption $\boldsymbol{\nabla}\cdot\boldsymbol{n} \approx 0$.

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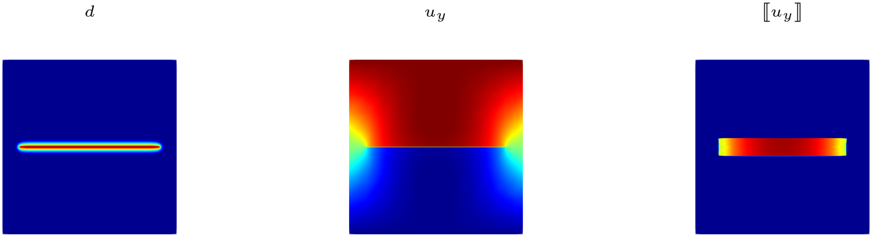
- The crack tips can be "ignored" using the so-called crack tip indicator obtained by projecting the phase field onto an anisotropic subspace.



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- Some remarks on the computational cost:
  - Computing/broadcasting the jump across the phase field crack is now fully "automated", at the cost of solving one (optionally up to three) additional system of equations.
  - The additional auxiliary problems are all linear.
  - The level-set and the crack tip indicator can be solved on a confined domain where $d > 0$. The directional jump can be solved on an even smaller domain excluding the crack tips.

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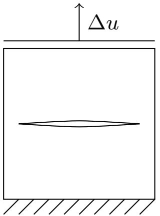
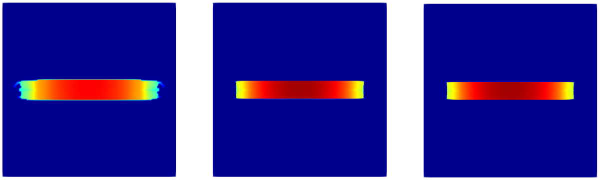
- The proposed method allows for monolithic coupling between the jump and other physics.



Parameters

- Domain $\Omega = [0,1]^2$
- Young's modulus $E = 1$
- Poisson's ratio $\nu = 0.3$
- Crack length $a = 0.8$
- Phase field reg. length $l = 0.025$
- Displacement $\Delta u = 0.01$
- Penalty parameter $\alpha = 1$



(a) $l/h = 4$ (b) $l/h = 8$ (c) $l/h = 16$

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