Coding the Matrix — Written Questions

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June 5, 2019

Here we will have the written answers to selected problems that do not require code or drawings to solve.

1 The Field

1.6 Review

1. The complex numbers, the reals, and the integers.
2. z.real − z.imag, and the formula for the absolute value of a complex number is $z * z_c$
3. Adding the real and imaginary components seperately.
4. Putting them in an equation and using distributive property.
5. Adding two complex numbers together.
6. Multiplying a real number by a complex number.
7. Multiplying by -1.
8. Multiplying by $e^{\frac{\pi i}{2}}$.
9. Adding the two bits and then applying modulo 2.
10. Setting the result to 0 if one of the bits is 0 and 1 otherwise.

1.7 Problems

10. (a) $5 + 3i$
    (b) i
    (c) $-1 + 0.001i$
    (d) $0.001 + 9i$
11. (a) e^{3i}
    (b) $e^{(\frac{11\pi}{12})i}$

(c) $e^{(\frac{5\pi}{12})i}$

12. (a) $a = (2)(e^{(\frac{\pi}{2})i})$, $b = 1 + 1i$

    (b) Not possible to scale the real part by two and imaginary part by three in only one multiplication.

13. (a) $1 + 1 + 1 + 0 = (1 + 1) + (1 + 0) = (0) + (1) = 1$

    (b) 0

    (c) 0

2 The Vector

2.6 Combining Vector Addition and Scalar Multiplication

1. $[3, 4]$, and the translation vector is $[2, 3]$.

2. $\{\alpha[5, -1] + [1, 4] : \alpha \in \mathbb{R}, 0 \leq \alpha \leq 1\}$

6. Same approach as in 2.5.5. For each element k of the domain D, entry k of $(\alpha + \beta)u$ is $(\alpha + \beta)u[k]$. By the distributive law for fields, this is equal to $\alpha u[k] + \beta u[k]$, which is the kth element of $\alpha u + \beta u$.

2.8 Vectors over $GF(2)$

3. Have two n-bit keys chosen randomly and uniformly over $GF(2)$ this time, v_a and v_b, given to Alice and Bob. Give a third TA the key $v_c := v - v_a - v_b$.

2.9 Dot Product

23. Suppose $u = [u_1, u_2 \dots u_n]$ and $v = [v_1, v_2 \dots v_n]$.
    $(\alpha u) * v$
    $= (\alpha u_1)(v_1) + (\alpha u_2)(v_2) \dots (\alpha u_n)(v_n)$
    $= \alpha(u_1)(v_1) + \alpha(u_2)(v_2) \dots \alpha(u_n)(v_n)$
    $= \alpha((u_1)(v_1) + (u_2)(v_2) \dots (u_n)(v_n))$
    $= \alpha(u * v)$

24. $\alpha = 2, u = [2, 2], v = [2, 2]$
    $(2 * [2, 2]) * (2 * [2, 2]) = [4, 4] * [4, 4] = 32$
    $2 * ([2, 2] * [2, 2]) = 2 * (4 + 4) = 2 * 8 = 16$

26. $u, v, w, x = [2, 2]$
    $(u + v) * (w + x) = [4, 4] * [4, 4] = 32$
    $(u * v) + (w * x) = 8 + 8 = 16$

29. For the first number, sum the last three challenges, $101010 + 111011 + 001100 = 011101$.
    The sum of their responses is 0.

For the second number, sum the first, third, and last challenge, $110011 + 111011 + 001100 = 000100$.
The sum of their responses is also 0.

2.11 Solving a triangular system of linear equations

3. $x_3 = -\frac{6}{5}$
$x_2 = \frac{4-4x_3}{2} = 2 - 2x_3 = 2 - 2(-\frac{6}{5}) = 2 + \frac{12}{5} = \frac{22}{5}$
$x_1 = 7 + 3x_2 + 2x_3 = 7 + 3(-\frac{6}{5}) + 2(\frac{22}{5}) = 7 - \frac{18}{5} + \frac{44}{5}$
$= 7 + \frac{26}{5} = \frac{61}{5}$

2.13 Review Questions

- Vector addition is, as the name suggests, the summation of two or more vectors. If you consider, as the book does, vectors to be special types of functions, then vector addition is the act of creating a new function equal to the sum of two "vector functions".

- Points in an X-D space can be modeled as arrows from the origin in that X-D space. Similarly, the sum of vectors that can be represented as points can be interpreted as "summing" the arrows, or laying their displacements from the origin on top of each other.

- Scalar-Vector multiplication is the act of multiplying a "scalar", that's within a field, and a vector together, in the manner that the above definition of vector addition would imply.

- $(\alpha + \beta)u = \alpha u + \beta u$

- $\alpha(u + v) = \alpha u + \alpha v$

- The set of all possible solutions to the equation $\alpha[x, y]$, which is analogous to the line connecting and moving beyond the points $(0, 0)$ and (x, y).

- The similar set $\{\alpha[w, x] + \beta[y, z] : \alpha, \beta \in \mathbb{F}, \alpha + \beta = 1\}$, where $\mathbb{F}$ is your field of choice.

- The sum of the product of the corresponding entries of two vectors. We defined vectors as functions earlier and so I don't know if this is a sane way to go about talking about dot products, but here is the book's definition. Two vectors must have the same domain for this dot product definition to make sense.

- $(\alpha u) * v = \alpha(u * v)$

- $(u + v) * w = u * w + v * w$

- An equation of the form $a * x = \beta$, where a is a static vector, b is a vector variable, and β is a scalar.

- A set of linear equations.
- A linear system in upper-triangular form.
- Solving for the variable at the bottom, and progressing orderly to the top, solving for one variable each equation, until you have reached the top of the triangle and solved for all x_k.

2.14 Problems

1. $[-1, 7]$,
   $[-1, -1]$, and
   $[-3, 1]$.

2. $[1, 0, 6]$,
   $[3, -2, -4]$,
   $[5, -3, 9]$,
   $[0, 1, 7]$

3. $[one, 0, 0]$
   $[0, one, one]$

4. (a) $c + d + e$
   (b) $b + c + d + e$

5. (a) $c + d$
   (b) Could not find one.

6.
   - 1011
   - 1101
   - 1000
   - 1011 + 1111 = 0100
     0100 * 1100 = 0 + 1 + 0 + 0 = 1
     1101 + 1111 = 0010
     1010 + 0010 = 0 + 0 + 1 + 0 = 1
     1000 + 1111 = 0111
     1111 * 0111 = 0 + 1 + 1 + 1 = 1 + 0 = 1

7. (a) $v_1 = [2, 3, 4, 3]$
   (b) $v_2 = [1, -5, 2, 0]$
   (c) $v_3 = [4, 1, -1, -1]$

9. (a) 5
   (b) 6
   (c) 16
   (d) -1

3 The Vector Space

3.2 Span

15. (a) $[3,4]+(-2)[1,2]=[3,4]-[2,4]=[1,0]$
    $(-\frac{1}{2})[3,4]+(\frac{3}{2})[1,2]=[-\frac{3}{2},-2]+[\frac{3}{2},3]=[0,1]$
    It is possible to write each of the standard generators, so this is a generator.

    (b) This is not a generator and it is impossible to write either of the standard generators. We add $[0,2]$:
    $(\frac{1}{2})[0,2]+(0)[1,1]+(0)[2,2]+(0)[3,3]=[0,1]$
    $(-\frac{1}{2})[0,2]+[1,1]+(0)[2,2]+(0)[3,3]=[1,0]$
    So this is now a generator.

    (c) We have one of the standard generators already, $[0,1]$.
    $[1,1]+(0)[1,-1]+(-1)[0,1]=[1,1]-[0,1]=[1,0]$, so this is a generator.

16. Our new vector is $[0,0,1]$, one of the standard generators.

    $(-\frac{4}{9})[1,1,1]+(\frac{10}{9})[0.4,1.3,-2.2]+(\frac{22}{9})[0,0,1]=$
    $[-\frac{4}{9},-\frac{4}{9},-\frac{4}{9}]+[\frac{4}{9},\frac{13}{9},-\frac{22}{9}]+[0,0,\frac{26}{9}]=$
    $[-\frac{4}{9},-\frac{4}{9},-\frac{4}{9}]+[\frac{4}{9},\frac{13}{9},\frac{4}{9}]=[0,\frac{9}{9},0]=[0,1,0]$ is #2, and:

    $[1,1,1]+(-1)[0,1,0]+(-1)[0,0,1]=[1,0,0]$ is #3.

17.

3.7 Review Questions

- A linear combination is a solution to an equation of the form $\alpha_1(u_1)+\ldots+\alpha_n(u_n)$, where $\alpha_1\ldots\alpha_n$ is any given set of scalars and $u_1\ldots u_n$ are a particular assortment of vectors. We say a vector x is a *linear combination of* the vectors $u_1\ldots u_n$ if there exists such a sum, with those vectors, that equals x.
- Coefficients are multiplicative factors in a mathematical expression.
- The set of all linear combinations of those vectors.
- The standard generators for a field of dimension n are the n vectors that are each a list of zeros excepting a single one in different and distinct positions, such that their span is equal to the field.
- $\{Span\{[1,1]\}\}$ and $\{Span\{[1,1,1],[1,2,3]\}\}$
- A linear equation the "right hand side" of which is equal to zero.
- A concurrent set of homogenous linear equations.

- The span of a set of vectors, or the solution set to a homogenous linear system.
- A vector space "over $\mathbb{F}$", where $\mathbb{F}$ is some field, is a set of objects such that the following three properties hold:
  1. The set contains the zero vector.
  2. The sum of two members of the set is also inside the set (It is *"closed under vector addition"*).
  3. The product of a scalar in $\mathbb{F}$ and a member of the set is also inside the set (It is *"closed under scalar multiplication"*).
- A subspace is a subset of a vector space that is also, itself, a vector space.
- An affine combination is a linear combination of some set of vectors where the scalar coefficients all add up to one.
- An affine hull of vectors is the set of all affine combinations of some sert of vectors.
- An affine space is the set of all possible solutions to the sum of a specified vector and all the vectors in some vector space.
- Two kinds of representations are an affine hull, and a translation of flat containing the origin.

3.8 Problems

4. - If $v_1 = [x_1, y_1, z_1] = [1, 0, a]$, $ax_1 + by_1 = a(1) + b(0) = a = z_1$, for all choice of a. Since a = z, for any coefficient α, $\alpha a = \alpha z$, and so $\forall v \in Span v_1 : z = ax + by$.
   - Similarly, if $v_2 = [x_2, y_2, z_2] = [0, 1, b] \implies ax_2 + by_2 = a(0) + b(1) = b = z_2$. Since a = z, for any coefficient β, $\beta a = \beta z$, and so $\forall v \in Span v_2 : z = ax + by$.
   - $Span\{v_1, v_2\}$ is equal to $\{u+w, u \in Span\{v_1\}, w \in Span\{v_2\}\}$. Since $\forall u, w : z = ax+by$, $\forall (u+w) : z_1+z_2 = z = a(x_1+x_2)+b(y_1+y_2) = ax + by$. So $\forall v \in Span v_1, v_2 : z = ax + by$.
   - Insert other backwards proof here.
5. This proof is nearly identical to the one above.
6. (a) $\{\alpha[1, 3], \alpha \in \mathbb{R}, 0 < \alpha < 1\}$
   (b) $\{\alpha[2, 2, 2] + \beta[2, 2, 0], \alpha, \beta \in mathbbR, 0 < \alpha, \beta < 1, \alpha + \beta = 1\}$
7. (a) It does not contain the zero vector, so it is not a vector space.
   (b) i. It contains the zero vector $[0, 0, 0]$

ii. It is closed under vector addition, because $x_1 + y_1 + z_1 = 0 \wedge x_2 + y_2 + z_2 = 0 \implies (x_1 + x_2) + (y_1 + y_2) + (z_1 + z_2) = (x_1 + y_1 + z_1) + (x_2 + y_2 + z_2) = 0 + 0 = 0$

iii. It is also cloed under scalar multiplication, because $x_1 + y_1 + z_1 = 0 \implies \alpha(x_1 + y_1 + z_1) = \alpha(0) = 0$

So, this is a vector space.

(c) i. It contains the zero vector $[0, 0, 0, 0, 0]$

ii. It is closed under vector addition, because $x_{21} = 0 \wedge x_{22} = 0 \implies (x_{21} + x_{22}) = 0$, and likewise for x_{51} and x_{52}

iii. It is closed under scalar multiplication, because $x_2 = 0 \implies \alpha(x_2) = \alpha(0) = 0$, and likewise for $\alpha(x_5)$

So, this is a vector space.

8. (a) As far as I can tell, yes, but I have not proved it.

(b) No, firstly because it does not contain the zero vector, $[0, 0, 0, 0, 0]$. This contains an even number of ones, specifically 0.

4 The Matrix

4.6 Matrix-Vector Multiplication in terms of dot products

14. (a) Entry r of the left hand side equals the dot product of row r of M with $(u + v)$

(b) The first entry r of the right hand side equals the dot product of row r of M with (u)

(c) The second entry r of the right hand side equals the dot product of row r of M with (v)

(d) The entry r of the sum of those last two vectors equals the sum of those two dot products.

(e) $r * (u + v) = r * u + r * v$, becuse of the distributive property of the dot product

4.7 Null space

3. (a) $[-1, 0, 1]$

(b) $[0, -1, 1]$

(c) $[0, 1, 0]$

4.9 The Matrix meets the function

2. (a) $\mathbb{R}^{\{a,b\}}$

(b) $\mathbb{R}^{\{\#,@,?\}}$

(c) $\{a : 0, b : 0\}$

4.10 Linear Functions

7. No, because $f([1,1])+f([1,1]) = [2,2,2]$ whereas $f([1,1]+[1,1]) = [2,2,1]$, and it defies property two.

8. (a) $h([x,y]) = [x, (-1) * y]$.

   (b) Yes. Multiplication of one part of the vector by -1 satisfies both properties one and two.

9. Example 4.9.6. The image of $[1,0]$ is $[2,2]$, but the image of $[2,0]$ and $[3,0]$ is $[3,2]$ and $[4,2]$, respectively, according to the definition of the function. Since $f([1,0]+[2,0]) = [4,2] \neq f([1,0]) + f([2,0]) = [5,4]$, this is not a linear function.

13. (a) If f is a linear map and $f(x) = 0$, then $\forall \alpha \in \mathbb{F} : f(\alpha x) = \alpha f(x) = 0$ by property one of linear maps. As a result, $\forall \alpha \in \mathbb{F}, x \in \text{Ker } f : \alpha x \; in$ Ker f.

    (b) Similarly, if f is a linear map and $f(x) = 0$, then $f(y) = 0 \implies f(x+y) = f(x) + f(y) = 0 + 0 = 0$ by property two of linear maps. This means that $\forall x, y \in \text{Ker } f : x + y \in \text{Ker } f$.

    (c) It contains the zero vector, because of Lemma 4.10.10

4.13 From function inverse to matrix inverse

2. Let $x = g(y)$. $g(\alpha y) = g(\alpha f(x))$
   $g(\alpha f(x)) = g(f(\alpha x))$
   $g(f(\alpha x)) = \alpha x$, so:
   $g(\alpha y) = \alpha g(y) = \alpha x$

4.16 Review Questions

- The transpose of an R x C matrix is the C x R matrix for which the position of all entries at r,c have been flipped to appear at c,r

- The sparsity of a matrix is the extent to which values tend to be zero, which, when combined with the standard that all row/label pairs that do not appear are assumed zero, can speed up many matrix operations and make matrices easier to store in memory.

- The linear combination definition of matrix-vector multiplication is a linear combination of the matrix's columns, the coefficients of which are the column-label-matching entries of the vector.

- The linear combination definition of vector-matrix multiplication is a linear combination of the matrix's rows, the coefficients of which are the row-label-matching entries of the vector.

- The dot product definition of matrix-vector multiplication is a series of dot products between the columns of the matrix and the vector, each consisting of an entry of the resulting vector.

- The dot product definition of vector-matrix multiplication is a series of dot products between the rows of the matrix and the vector, each consisting of an entry of the resulting vector.

- An identity matrix I for a given matrix M is the simplest diagonal matrix such that IM = M.

- An upper triangular matrix is a square matrix such that, within each column i, all rows greater than i contain the entry zero.

- A matrix that only has nonzero entries in the locations where the row and column have the same index.

- A linear function is a function whose domain is a vector space, whose codomain is a vector space, and which satisfies the following two properties:

  1. $f(x + y) = f(x) + f(y)$
  2. $f(\alpha x) = \alpha f(x)$

  where 0, x, and y are vectors, and α is a scalar.

- It can be represented as the product of a m x n matrix M and a vector, namely, $f(x) = M * x$, or the transpose of that matrix, an n $\times$ m one we can name M_T and thus define the function by $f(x) = x * M_T$.

- The kernel of a linear function is the set of all vectors in its co-domain that produce the zero vector. The image of a linear function is the set of all outputs of its domain.

- The null space of a matrix A, analogous to the kernel of a linear function, is the set of vectors such that $A * v$ equals the zero vector. The row space of A is the vector space spanned by the rows of A, and the column space is the vector space spanned by the columns of A.

- The matrix where, for each row-label r of A, row r of AB = (row r of A) * B.

- The matrix where, for each column-label c of A, column c of AB = A * (column c of B).

- The matrix where entry entry rc of AB is the dot product of row r of A with column c of B.

- The associativity property of matrix-matrix multiplication says that the order in which multiplication is performed does not change the results of the equation.

- By defining the vector as a single-column or single-row matrix, you can generalize matrix-matrix multiplication to include matrix-vector multiplication.
- The outer-product of two vectors u, v, represented as single column matrices, is the product uv^T.
- It can be represented as the product of two matrices u, v, u being a $1xn$ vector and v being an $n \times 1$ vector.
- The inverse of a matrix M is the matrix N whose linear map is the functional inverse of the linear map of M.
- If the matrix N is to be an inverse of the matrix M, NM must be a valid product and NM must equal the identity matrix for M.

4.17 Problems

1. (a) $[1, 0]$
   (b) $[0, 4.44]$
   (c) $[14, 20, 26]$
2. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
3. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$
4. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$
5. (i) Invalid
   (ii) Invalid
   (iii) 1×2
   (iv) 2×1
   (v) Invalid
   (vi) 1×1
   (vii) 3×3
6. (i) $\begin{bmatrix} 8 & 13 \\ 8 & 14 \end{bmatrix}$
   (ii) $\begin{bmatrix} 24 & 11 & 4 \\ 3 & 3 & 0 \end{bmatrix}$
   (iii) $\begin{bmatrix} 3 & 13 \end{bmatrix}$

(iv) $[14]$

(v) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$

(vi) $\begin{bmatrix} -2 & 4 \\ -1 & 1 \\ -1 & -3 \end{bmatrix}$

7. (i) $AB = \begin{bmatrix} 5 & 2 & 0 & 1 \\ 2 & 1 & -4 & 6 \\ 2 & 3 & 0 & -4 \\ -2 & 3 & 4 & 0 \end{bmatrix}$

$BA = \begin{bmatrix} 1 & -4 & 6 & 2 \\ 3 & 0 & -4 & 2 \\ 3 & 4 & 0 & -2 \\ 2 & 0 & 1 & 5 \end{bmatrix}$

(ii) $AB = \begin{bmatrix} 5 & 1 & 0 & 2 \\ 2 & 6 & -4 & 1 \\ 2 & -4 & 0 & 3 \\ -2 & 0 & 4 & 3 \end{bmatrix}$

$BA = \begin{bmatrix} 3 & 4 & 0 & -2 \\ 3 & 0 & -4 & 2 \\ 1 & -4 & 6 & 2 \\ 2 & 0 & 1 & 5 \end{bmatrix}$

(iii) $AB = \begin{bmatrix} 1 & 0 & 5 & 2 \\ 6 & -4 & 2 & 1 \\ -4 & 0 & 2 & 3 \\ 0 & 4 & -2 & 3 \end{bmatrix}$

$BA = \begin{bmatrix} 3 & 4 & 0 & -2 \\ 1 & -4 & 6 & 2 \\ 2 & 0 & 1 & 5 \\ 3 & 0 & -4 & 2 \end{bmatrix}$

8. (i) $\begin{bmatrix} 1 & b+a \\ 0 & 1 \end{bmatrix}$

(ii) $A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

$A^3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

$A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$

9. (a) $AB = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 6 & 0 \end{bmatrix}$

$BA = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 4 & 4 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(b) $AB = \begin{bmatrix} 0 & 2 & -1 & 0 \\ 0 & 5 & 3 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 6 & 5 & 0 \end{bmatrix}$

$BA = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 5 & -2 & 3 \\ 0 & 0 & 0 & 0 \\ 4 & 4 & 4 & 0 \end{bmatrix}$

(c) $AB = \begin{bmatrix} 6 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 \\ 8 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 \end{bmatrix}$

$BA = \begin{bmatrix} 4 & 2 & 1 & -1 \\ 4 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(d) $AB = \begin{bmatrix} 0 & 3 & 0 & 4 \\ 0 & 4 & 0 & 1 \\ 0 & 4 & 0 & 4 \\ 0 & -6 & 0 & -1 \end{bmatrix}$

$BA = \begin{bmatrix} 0 & 11 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 5 & -2 & 3 \end{bmatrix}$

(e) $AB = \begin{bmatrix} 0 & 3 & 0 & 8 \\ 0 & -9 & 0 & 2 \\ 0 & 0 & 0 & 8 \\ 0 & 15 & 0 & -2 \end{bmatrix}$

$BA = \begin{bmatrix} -2 & 12 & 4 & -10 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -3 & -15 & 6 & -9 \end{bmatrix}$

(f) $AB = \begin{bmatrix} -4 & 4 & 2 & -3 \\ -1 & 10 & -4 & 9 \\ -4 & 8 & 8 & 0 \\ 1 & 12 & 4 & -15 \end{bmatrix}$

$$BA = \begin{bmatrix} -4 & -2 & -1 & 1 \\ 2 & 10 & -4 & 6 \\ 8 & 8 & 8 & 0 \\ -3 & 18 & 6 & -15 \end{bmatrix}$$

10. (i) Invalid.
    (ii) Invalid.
    (iii) 1×2
    (iv) 2×1
    (v) 1×1
    (vi) Invalid.