1. Conditional independence and factorizations

1. The implied factorization of a distribution $p \in \mathcal{L}(G)$ is $p(x, y, z, t) = p(t|z)p(z|x, y)p(x)p(y)$. For two real valued independent r.v X and Y that follow two distribution $\mathcal{U}(0, 2)$, and $Z = X + Y$, $T = \mathbf{1}_{(Z\geq 3)}$, the statement $X \perp\!\!\!\perp Y|T$ is not true. Indeed, knowing that $X + Y \geq 3$ makes the two random variables not independent ($p(x|y = 2, z = 1) = p(x \geq 1) \neq p(x|z = 1)$).

2.a. If Z is a binary variable, let $p = p(z = 1) = 1 - p(z = 0)$. We have

$$\begin{aligned} p(x, y) = p(x)p(y) &= p(x, y|z = 1)p + p(x, y|z = 0)(1 - p) && (X \perp\!\!\!\perp Y) \\ &= p(x|z = 1)p(y|z = 1)p + p(x|z = 0)p(y|z = 0)(1 - p) && (X \perp\!\!\!\perp Y|Z) \\ &= p(x)p(y) \left(\frac{1}{p}p(z = 1|x)p(z = 1|y) + \frac{1}{1-p}p(z = 0|x)p(z = 0|y)\right) \end{aligned}$$

If we write $p_x = p(z = 1|x) = 1 - p(z = 0|x)$ and $p_y = p(z = 1|y)$, the equality becomes $p^2 - (p_x + p_y)p + p_x p_y = 0$ which yields ($p(z = 1) =$) $p = p_x$ ($= p(z = 1|x)$) or $p = p_y$. Therefore either $X \perp\!\!\!\perp Z$ or $Y \perp\!\!\!\perp Z$.

2.b. Let $\mathcal{A}$ be a finite space, X_0 and Y_0 two independent random variables on $\mathcal{A}$, we define $X = \begin{bmatrix} X_0 \\ 0 \end{bmatrix}$, $Y = \begin{bmatrix} 0 \\ Y_0 \end{bmatrix}$ and $Z = \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix}$.

X, Y and Z are three random variables with $X \perp\!\!\!\perp Y$ and $X \perp\!\!\!\perp Y|Z$

2. Distributions factorizing in a graph

1. Let $p \in \mathcal{L}(G)$, $p(x) = \prod_{s\in V} p(x_s|x_{\pi_s})$. Since $\{i \to j\}$ is a covered edge, $\pi_j = \pi_i \cup \{i\}$. Therefore,

$$p(x) = p(x_i|x_{\pi_i}) \cdot p(x_j|x_{\pi_i}, x_i) \cdot \prod_{s\in V-\{i,j\}} p(x_s|x_{\pi_s})$$

And

$$\begin{aligned} p(x_i|x_{\pi_i}) \cdot p(x_j|x_{\pi_i}, x_i) &= \frac{p(x_i, x_{\pi_i})}{p(x_{\pi_i})} \frac{p(x_i, x_{\pi_i}, x_j)}{p(x_{\pi_i}, x_i)} = p(x_i, x_j|x_{\pi_i}) \cdot \frac{p(x_j, x_{\pi_i})}{p(x_j, x_{\pi_i})} \\ &= p(x_i|x_{\pi_i}, x_j) \cdot p(x_j|x_{\pi_i}) \end{aligned}$$

In G', $\{i \to j\}$ has been replaced by $\{j \to i\}$, thus $\pi'_i = \pi_i \cup \{j\}$ and $\pi'_j = \pi_i$, the equation above shows that p can be factorized in this new graph, hence $p \in \mathcal{L}(G')$. From the same equality above, if $p \in \mathcal{L}(G')$, also $p \in \mathcal{L}(G)$. Therefore $\boxed{\mathcal{L}(G) = \mathcal{L}(G')}$.

2. Since G is a directed tree, it doesn't contain any v-structure and all nodes have either 0 or 1 parent (only the root of the tree doesn't have a parent). Moreover, all cliques or G' contain at most 2 elements because any clique of 3 or more elements would contain a 3-clique. That 3-clique in the directed graph must contain either a v-structure or a cycle, which is not possible in a directed tree.

Hence if $p \in \mathcal{L}(G)$, $p(x) = \prod_i p(x_i|x_{\pi_i}) = \prod_i \Psi_i(x_i, x_{\pi_i})$, where Ψ_i is a function of an element and its parent (a 2-clique of the graph), or of the root of the tree only (a 1-clique). We have $\boxed{\mathcal{L}(G) \subset \mathcal{L}(G')}$