Turbulence Modelling – Theoretical Background

$$[\mu\nabla^2 u] - \frac{\partial P}{\partial x} + \rho\, g_x \;= \rho\left(\frac{D}{Dt}u\right)$$

$$[\mu\nabla^2 v] - \frac{\partial P}{\partial y} + \rho\, g_y \;= \rho\left(\frac{D}{Dt}v\right)$$

$$[\mu\nabla^2 w] - \frac{\partial P}{\partial z} + \rho\, g_z \;= \rho\left(\frac{D}{Dt}w\right)$$

$$\nabla\cdot(\vec{v}) = 0$$

The above are the navier stokes equations (momentum and mass transport or balance)

$$\frac{\partial\vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\vec{\mathrm{u}} - \nu\nabla^2\vec{u} = -\nabla w + \vec{g}$$

$$w = \frac{Pressure}{density}$$

https://en.wikipedia.org/wiki/Navier%E2%80%93Stokes_equations

$$\frac{\partial\vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\vec{\mathrm{u}} - \nu\nabla^2\vec{u} = -\frac{1}{\rho}\,\nabla \mathrm{P} \;+ \vec{g}$$

Let's begin…

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} = \frac{\partial}{\partial t} + \vec{v}\cdot\nabla$$

$$u = \bar{u} + u'$$

$$v = \bar{v} + v'$$

$$w = \overline{w} + w'$$

$$[\mu\nabla^2 u] - \frac{\partial P}{\partial x} + \rho\, g_x \;= \rho\left(\frac{D}{Dt}u\right)$$

$$[\mu\nabla^2 v] - \frac{\partial P}{\partial y} + \rho\, g_y \;= \rho\left(\frac{D}{Dt}v\right)$$

$$[\mu\nabla^2 w] - \frac{\partial P}{\partial z} + \rho\, g_z \;= \rho\left(\frac{D}{Dt}w\right)$$

$$\nabla\cdot(\vec{v}) = 0$$

Let's expand in x direction:

$$[\mu\nabla^2 u] - \frac{\partial P}{\partial x} + \rho\, g_x \;= \rho\left(\frac{D}{Dt}u\right)$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} = \frac{\partial}{\partial t} + \vec{v}\cdot\nabla$$

$$[\mu\nabla^2 u] - \frac{\partial P}{\partial x} + \rho\, g_x \ = \rho\left(\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u + w\frac{\partial}{\partial z}u\right)$$

$$P = \bar{P} + P'$$

$$[\mu\nabla^2 u] - \frac{\partial}{\partial x}(\bar{P} + P') + \rho\, g_x \ = \rho\left(\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u + w\frac{\partial}{\partial z}u\right)$$

Substituting:

$$u = \bar{u} + u'$$

$$v = \bar{v} + v'$$

$$w = \bar{w} + w'$$

$$[\mu\nabla^2(\bar{u} + u')] - \frac{\partial}{\partial x}(\bar{P} + P') + \rho\, g_x$$
$$= \rho\left(\frac{\partial}{\partial t}(\bar{u} + u') + (\bar{u} + u')\frac{\partial}{\partial x}(\bar{u} + u') + (\bar{v} + v')\frac{\partial}{\partial y}(\bar{u} + u') + (\bar{w} + w')\frac{\partial}{\partial z}(\bar{u} + u')\right)$$

Continuity Equation

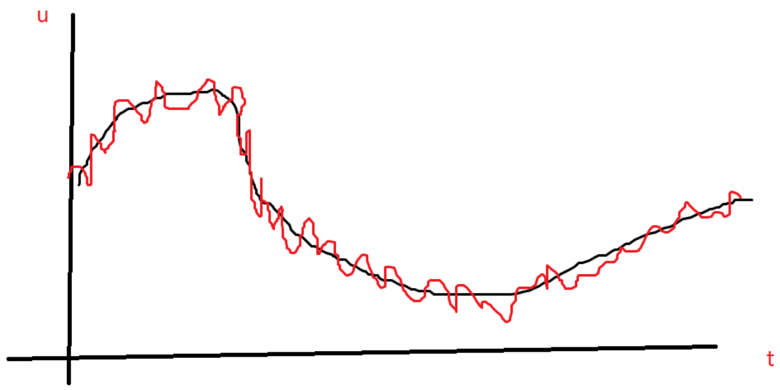
$$\nabla \cdot (\vec{v}) = 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial}{\partial x}(\bar{u} + u') + \frac{\partial}{\partial y}(\bar{v} + v') + \frac{\partial}{\partial z}(\bar{w} + w') = 0$$

$$\frac{\partial}{\partial x}(\bar{u}) + \frac{\partial}{\partial y}(\bar{v}) + \frac{\partial}{\partial z}(\bar{w}) + \frac{\partial}{\partial x}(u') + \frac{\partial}{\partial y}(v') + \frac{\partial}{\partial z}(w') = 0$$

Time averaging



$$\bar{u} = \frac{1}{T}\int_t^{t+T} d\tau\, u$$

After time averaging, the continuity equation looks like so:

$$\frac{\partial}{\partial x}(\bar{u}) + \frac{\partial}{\partial y}(\bar{v}) + \frac{\partial}{\partial z}(\bar{w}) = 0$$

We also have to time average our momentum balance:

$$[\mu\nabla^2(\bar{u}+u')] - \frac{\partial}{\partial x}(\bar{P}+P') + \rho\, g_x = \rho\left(\frac{\partial}{\partial t}(\bar{u}+u') + (\bar{u}+u')\frac{\partial}{\partial x}(\bar{u}+u') + (\bar{v}+v')\frac{\partial}{\partial y}(\bar{u}+u') + (\bar{w}+w')\frac{\partial}{\partial z}(\bar{u}+u')\right)$$

$$[\mu\nabla^2(\bar{u})] - \frac{\partial}{\partial x}(\bar{P}) + \rho\, g_x$$

RHS

$$\rho\left(\frac{\partial}{\partial t}(\bar{u}) + (\bar{u}+u')\frac{\partial}{\partial x}(\bar{u}+u') + (\bar{v}+v')\frac{\partial}{\partial y}(\bar{u}+u') + (\bar{w}+w')\frac{\partial}{\partial z}(\bar{u}+u')\right)$$

$$(\bar{u}+u')\frac{\partial}{\partial x}(\bar{u}+u') = (\bar{u})\frac{\partial}{\partial x}(\bar{u}+u') + (u')\frac{\partial}{\partial x}(\bar{u}+u')$$

$$(\bar{u})\frac{\partial}{\partial x}(\bar{u}+u') + (u')\frac{\partial}{\partial x}(\bar{u}+u') = \bar{u}\frac{\partial}{\partial x}\,\bar{u} + \bar{u}\frac{\partial}{\partial x}\,u' + u'\frac{\partial}{\partial x}\bar{u} + u'\frac{\partial}{\partial x}\,u'$$

Time average

$$\bar{u}\frac{\partial}{\partial x}\,\bar{u} \rightarrow \bar{u}\frac{\partial}{\partial x}\,\bar{u}$$

$$\bar{u}\frac{\partial}{\partial x}\,u' \rightarrow \frac{1}{T}\int_t^{t+T} d\tau\,\bar{u}\frac{\partial}{\partial x}\,u' = \bar{u}\frac{1}{T}\int_t^{t+T} d\tau\,\frac{\partial}{\partial x}\,u' = \frac{\bar{u}}{T}\frac{\partial}{\partial x}\int_t^{t+T} d\tau\;u' = 0$$

$$u'\frac{\partial}{\partial x}\bar{u} \rightarrow \frac{1}{T}\int_t^{t+T} d\tau\,u'\frac{\partial}{\partial x}\bar{u} = \frac{\partial}{\partial x}\bar{u}\,\frac{1}{T}\int_t^{t+T} d\tau\,u' = 0$$

$$(\bar{u})\frac{\partial}{\partial x}(\bar{u}+u') + (u')\frac{\partial}{\partial x}(\bar{u}+u') = \bar{u}\frac{\partial}{\partial x}\,\bar{u} + \overline{u'\frac{\partial}{\partial x}\,u'}$$

Now for the v and w direction…

$$(\bar{v}+v')\frac{\partial}{\partial y}(\bar{u}+u') \rightarrow \bar{v}\frac{\partial}{\partial y}\,\bar{u} + \overline{v'\frac{\partial}{\partial y}\,u'}$$

$$(\bar{v})\frac{\partial}{\partial y}(\bar{u}+u') + (v')\frac{\partial}{\partial y}(\bar{u}+u') = \bar{v}\frac{\partial}{\partial y}\,\bar{u} + \bar{v}\frac{\partial}{\partial y}\,u' + v'\frac{\partial}{\partial y}\bar{u} + v'\frac{\partial}{\partial y}\,u'$$

$$(\bar{w}+w')\frac{\partial}{\partial z}(\bar{u}+u') \rightarrow \bar{w}\frac{\partial}{\partial z}\,\bar{u} + \overline{w'\frac{\partial}{\partial z}\,u'}$$

$$(\bar{w})\frac{\partial}{\partial z}(\bar{u}+u') + (w')\frac{\partial}{\partial z}(\bar{u}+u') = \bar{w}\frac{\partial}{\partial z}\,\bar{u} + \,\bar{w}\frac{\partial}{\partial z}\,u' + w'\frac{\partial}{\partial z}\bar{u} + w'\frac{\partial}{\partial z}\,u'$$

$$(\bar{u}+u')\frac{\partial}{\partial x}(\bar{u}+u') + (\bar{v}+v')\frac{\partial}{\partial y}(\bar{u}+u') + (\bar{w}+w')\frac{\partial}{\partial z}(\bar{u}+u') =$$

$$= \bar{u}\frac{\partial}{\partial x}\,\bar{u} + \,u'\frac{\partial}{\partial x}\,u' \; + \,\bar{v}\frac{\partial}{\partial y}\,\bar{u} + v'\frac{\partial}{\partial y}\,u' \; + \bar{w}\frac{\partial}{\partial z}\,\bar{u} + w'\frac{\partial}{\partial z}\,u'$$

http://daad.wb.tu-harburg.de/fileadmin/BackUsersResources/Flood_Probability/2D/Steffi-2D/pdf/Reynolds_average_Navier-Stokes_equation.pdf

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \frac{\partial(u^2)}{\partial x} + \frac{\partial(uv)}{\partial y} + \frac{\partial(uw)}{\partial z} - u\underbrace{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)}_{=0}$$

$$= \frac{\partial(u^2)}{\partial x} + \frac{\partial(uv)}{\partial y} + \frac{\partial(uw)}{\partial z}$$

$$\bar{u}\frac{\partial}{\partial x}\,\bar{u} + \bar{v}\frac{\partial}{\partial y}\,\bar{u} + \bar{w}\frac{\partial}{\partial z}\,\bar{u} = \frac{\partial(\bar{u}^2)}{\partial x} + \frac{\partial(\bar{u}\bar{v})}{\partial y} + \frac{\partial(\bar{u}\bar{w})}{\partial z} - \bar{u}\left(\frac{\partial\bar{u}}{\partial x} + \frac{\partial\bar{v}}{\partial y} + \frac{\partial\bar{w}}{\partial z}\right)$$

$\frac{\partial}{\partial x}(\bar{u}) + \frac{\partial}{\partial y}(\bar{v}) + \frac{\partial}{\partial z}(\bar{w}) = 0$ (after time averaging)

$$\bar{u}\frac{\partial}{\partial x}\,\bar{u} + \bar{v}\frac{\partial}{\partial y}\,\bar{u} + \bar{w}\frac{\partial}{\partial z}\,\bar{u} = \frac{\partial(\bar{u}^2)}{\partial x} + \frac{\partial(\bar{u}\bar{v})}{\partial y} + \frac{\partial(\bar{u}\bar{w})}{\partial z}$$

$$= \bar{u}\frac{\partial}{\partial x}\,\bar{u} + \,u'\frac{\partial}{\partial x}\,u' \; + \,\bar{v}\frac{\partial}{\partial y}\,\bar{u} + v'\frac{\partial}{\partial y}\,u' \; + \bar{w}\frac{\partial}{\partial z}\,\bar{u} + w'\frac{\partial}{\partial z}\,u'$$

$$= \frac{\partial(\bar{u}^2)}{\partial x} + \frac{\partial(\bar{u}\bar{v})}{\partial y} + \frac{\partial(\bar{u}\bar{w})}{\partial z} + u'\frac{\partial}{\partial x}\,u' \; + \,v'\frac{\partial}{\partial y}\,u' + w'\frac{\partial}{\partial z}\,u'$$

$$u'\frac{\partial}{\partial x}\,u' \; + \,v'\frac{\partial}{\partial y}\,u' + w'\frac{\partial}{\partial z}\,u' = \frac{\partial(u'^2)}{\partial x} + \frac{\partial(u'v')}{\partial y} + \frac{\partial(u'w')}{\partial z} - u'\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z}\right)$$

After time averaging

$$\overline{u'\frac{\partial}{\partial x}\,u'} + \overline{v'\frac{\partial}{\partial y}\,u'} + \overline{w'\frac{\partial}{\partial z}\,u'} = \frac{\partial\overline{(u'^2)}}{\partial x} + \frac{\partial\overline{(u'v')}}{\partial y} + \frac{\partial\overline{(u'w')}}{\partial z}$$

$$= \bar{u}\frac{\partial}{\partial x}\,\bar{u} + \,u'\frac{\partial}{\partial x}\,u' \; + \,\bar{v}\frac{\partial}{\partial y}\,\bar{u} + v'\frac{\partial}{\partial y}\,u' \; + \bar{w}\frac{\partial}{\partial z}\,\bar{u} + w'\frac{\partial}{\partial z}\,u'$$

$$= \frac{\partial(\bar{u}^2)}{\partial x} + \frac{\partial(\bar{u}\bar{v})}{\partial y} + \frac{\partial(\bar{u}\bar{w})}{\partial z} + \frac{\partial\overline{(u'^2)}}{\partial x} + \frac{\partial\overline{(u'v')}}{\partial y} + \frac{\partial\overline{(u'w')}}{\partial z}$$

$$[\mu\nabla^2(\bar{u})] - \frac{\partial}{\partial x}(\bar{P}) + \rho\,g_x$$

$$[\mu\nabla^2(\bar{u})] - \frac{\partial}{\partial x}(\bar{P}) + \rho\, g_x$$

$$= \rho\left(\frac{\partial}{\partial t}(\bar{u}) + \frac{\partial(\bar{u}^2)}{\partial x} + \frac{\partial(\bar{u}\bar{v})}{\partial y} + \frac{\partial(\bar{u}\bar{w})}{\partial z} + \overline{\frac{\partial(u'^2)}{\partial x}} + \overline{\frac{\partial(u'v')}{\partial y}} + \overline{\frac{\partial(u'w')}{\partial z}}\right)$$

$$\overline{\frac{\partial(u'^2)}{\partial x}} + \overline{\frac{\partial(u'v')}{\partial y}} + \overline{\frac{\partial(u'w')}{\partial z}} \rightarrow reynold'stress\ terms$$

How do we determine

$$-\rho\left(\overline{\frac{\partial(u'^2)}{\partial x}} + \overline{\frac{\partial(u'v')}{\partial y}} + \overline{\frac{\partial(u'w')}{\partial z}}\right)$$

$$-\rho\left(\overline{\frac{\partial(v'u')}{\partial x}} + \overline{\frac{\partial(v'v')}{\partial y}} + \overline{\frac{\partial(v'w')}{\partial z}}\right)$$

$$-\rho\left(\overline{\frac{\partial(u'w')}{\partial x}} + \overline{\frac{\partial(v'w')}{\partial y}} + \overline{\frac{\partial(w'w')}{\partial z}}\right)$$

Closure Problem…

$$-\rho\overline{u'v'}?$$

Boussinesq Relation

Reynold's stresses ($-\rho\overline{u'v'}$) can be linked to shear rate/mean deformation rate ($eg\,\frac{\partial\bar{u}}{\partial y}$) with the use of a constant ie, eddy diffisuvity/viscosity or turbulent diffusivity

$$-\rho\overline{u'u'} = 2\mu_T\frac{\partial\bar{u}}{\partial x} - \frac{2}{3}\rho k$$

$$-\rho\overline{v'v'} = 2\mu_T\frac{\partial\bar{v}}{\partial y} - \frac{2}{3}\rho k$$

$$-\rho\overline{w'w'} = 2\mu_T\frac{\partial\bar{w}}{\partial z} - \frac{2}{3}\rho k$$

$$-\rho\,\overline{u'v'} = \mu_T\left(\frac{\partial\bar{v}}{\partial x} + \frac{\partial\bar{u}}{\partial y}\right)$$

$$-\rho\,\overline{u'w'} = \mu_T\left(\frac{\partial\bar{w}}{\partial x} + \frac{\partial\bar{u}}{\partial z}\right)$$

$$\tau = \mu\left(\frac{\partial u}{\partial y}\right)$$

Isotropic assumption:

μ_T is the same regardless of direction

K is also the same regardless of direction

$$-\overline{v_i'v_j'} = \nu_T \left(\frac{\partial \overline{v_i}}{\partial x_j} + \frac{\partial \overline{v_j}}{\partial x_i}\right) - \frac{2}{3}\left(k\delta_{ij}\right)$$

$$-\rho\overline{v_i'v_j'} = \mu_T \left(\frac{\partial \overline{v_i}}{\partial x_j} + \frac{\partial \overline{v_j}}{\partial x_i}\right) - \frac{2}{3}\left(\rho k\delta_{ij}\right)$$

$$-\rho\overline{u'u'} = 2\mu_T \frac{\partial \bar{u}}{\partial x} - \frac{2}{3}\rho k$$

$$-\rho\,\overline{u'w'} = \mu_T \left(\frac{\partial \bar{w}}{\partial x} + \frac{\partial \bar{u}}{\partial z}\right)$$

$k - \varepsilon$ model is a subset of Boussinesq Relation:

Turbulent kinetic energy

$$k = \frac{1}{2} u_i' u_i'$$

Turbulent KE dissipation rate

$$\varepsilon = \nu_t \overline{\left(\left(\frac{\partial u_i'}{\partial x_j}\right)\left(\frac{\partial u_i'}{\partial x_j}\right)\right)}$$

Note that this is in Einstein notation

$$\nu_t = C_\mu \frac{k^2}{\varepsilon}, C_\mu = 0.09\ (empirical)$$

What is k and ε? And why develop such a model?

$$-\overline{v_i'v_j'} = \nu_T \left(\frac{\partial \overline{v_i}}{\partial x_j} + \frac{\partial \overline{v_j}}{\partial x_i}\right) - \frac{2}{3}\left(k\delta_{ij}\right)$$

$$-\rho\overline{v_i'v_j'} = \mu_T \left(\frac{\partial \overline{v_i}}{\partial x_j} + \frac{\partial \overline{v_j}}{\partial x_i}\right) - \frac{2}{3}\left(\rho k\delta_{ij}\right)$$

$$-\rho\overline{u'u'} = 2\mu_T \frac{\partial \bar{u}}{\partial x} - \frac{2}{3}\rho k$$

$$-\rho\,\overline{u'w'} = \mu_T \left(\frac{\partial \bar{w}}{\partial x} + \frac{\partial \bar{u}}{\partial z}\right)$$

K is in the units of u^2

$$KE = \frac{1}{2}mv^2, \frac{1}{2}\rho u^2, \frac{1}{2}u^2$$

$$k = \frac{1}{2}\left(u'^2 + v'^2 + w'^2\right)$$

$$\mu_T = \rho C_\mu \frac{k^2}{\varepsilon}$$

$$\varepsilon = \nu_T \overline{\left(\frac{\partial u_i'}{\partial x_j}\right)\left(\frac{\partial u_i'}{\partial x_j}\right)}$$

Units of Kinematic viscosity: $\nu_T \rightarrow \frac{m^2}{s}$

$$\varepsilon \rightarrow \frac{m^2}{s}\left(\frac{1}{s}\right)^2 \rightarrow \frac{m^2}{s^2}\left(\frac{1}{s}\right)$$

ε is in units of some sort of specific power

$$\varepsilon \rightarrow rate\ of\ KE\ dissipation$$

$$\frac{\partial k}{\partial t} + u\frac{\partial k}{\partial x} + v\frac{\partial k}{\partial y} + w\frac{\partial k}{\partial z} = \frac{\partial}{\partial x}\left(\frac{\nu_T}{\sigma_k}\frac{\partial k}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\nu_T}{\sigma_k}\frac{\partial k}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\nu_T}{\sigma_k}\frac{\partial k}{\partial z}\right) + P - D$$

$$\frac{\partial \varepsilon}{\partial t} + u\frac{\partial \varepsilon}{\partial x} + v\frac{\partial \varepsilon}{\partial y} + w\frac{\partial \varepsilon}{\partial z} = \frac{\partial}{\partial x}\left(\frac{\nu_T}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\nu_T}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\nu_T}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial z}\right) + \frac{\varepsilon}{k}\left(C_{\varepsilon 1}P - C_{\varepsilon 2}D\right)$$

So what are P&D?

Tu, J., Yeoh, G. H., & Liu, C. (2018). *Computational fluid dynamics: a practical approach*. Butterworth-Heinemann. (Tu, Yeoh, & **Liu**, 2018) → chapter 3

(note that these refer to average quantities!)

$$P = 2\nu_T\left[\left(\frac{\partial \bar{u}}{\partial x}\right)^2 + \left(\frac{\partial \bar{v}}{\partial y}\right)^2 + \left(\frac{\partial \bar{w}}{\partial z}\right)^2\right] + \nu_T\left[\left(\frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x}\right)^2 + \left(\frac{\partial \bar{v}}{\partial z} + \frac{\partial \bar{w}}{\partial y}\right)^2 + \left(\frac{\partial \bar{w}}{\partial x} + \frac{\partial \bar{u}}{\partial z}\right)^2\right]$$

$$D = \varepsilon$$

Refer to the turbulent KE equation on Wikipedia page

https://en.wikipedia.org/wiki/K-epsilon_turbulence_model

What's E_{ij}?

This is rate of deformation, or equivalently rate of strain, since strain describes deformation in material science, or materials in general…

$$S_{ij} = E_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)$$

We need to get used to Einstein notation in tensor algebra…

$$S_{ij} = \sum_{i=1}^{3}\sum_{j=1}^{3} S_{ij}$$

A lot of textbooks use this notation, so just be aware. Learning tensors is good but complicated. If you want to go in depth in turbulence studies, this will come in very handy.

$$S_{ij} = E_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)$$

$$\sum_{i=1}^{3}\sum_{j=1}^{3} S_{ij} = \sum_{i=1}^{3}\sum_{j=1}^{3} \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)$$

$$\sum_{i=1}^{3}\sum_{j=1}^{3} \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) = \frac{1}{2}\sum_{i=1}^{3}\left[\left(\frac{\partial U_i}{\partial x_1} + \frac{\partial U_1}{\partial x_i}\right) + \left(\frac{\partial U_i}{\partial x_2} + \frac{\partial U_2}{\partial x_i}\right) + \left(\frac{\partial U_i}{\partial x_3} + \frac{\partial U_3}{\partial x_i}\right)\right]$$

$$= \frac{1}{2}\left\{\left[\left(\frac{\partial U_1}{\partial x_1} + \frac{\partial U_1}{\partial x_1}\right) + \left(\frac{\partial U_1}{\partial x_2} + \frac{\partial U_2}{\partial x_1}\right) + \left(\frac{\partial U_1}{\partial x_3} + \frac{\partial U_3}{\partial x_1}\right)\right] + \left[\left(\frac{\partial U_2}{\partial x_1} + \frac{\partial U_1}{\partial x_2}\right) + \left(\frac{\partial U_2}{\partial x_2} + \frac{\partial U_2}{\partial x_2}\right) + \left(\frac{\partial U_2}{\partial x_3} + \frac{\partial U_3}{\partial x_2}\right)\right] + \left[\left(\frac{\partial U_3}{\partial x_1} + \frac{\partial U_1}{\partial x_3}\right) + \left(\frac{\partial U_3}{\partial x_2} + \frac{\partial U_2}{\partial x_3}\right) + \left(\frac{\partial U_3}{\partial x_3} + \frac{\partial U_3}{\partial x_3}\right)\right]\right\}$$

$$= \frac{1}{2}\left\{\left[\left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{u}}{\partial x}\right) + \left(\frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x}\right) + \left(\frac{\partial \bar{u}}{\partial z} + \frac{\partial \bar{w}}{\partial x}\right)\right] + \left[\left(\frac{\partial \bar{v}}{\partial x} + \frac{\partial \bar{u}}{\partial y}\right) + \left(\frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{v}}{\partial y}\right) + \left(\frac{\partial \bar{v}}{\partial z} + \frac{\partial \bar{w}}{\partial y}\right)\right] + \left[\left(\frac{\partial \bar{w}}{\partial x} + \frac{\partial \bar{u}}{\partial z}\right) + \left(\frac{\partial \bar{w}}{\partial y} + \frac{\partial \bar{v}}{\partial z}\right) + \left(\frac{\partial \bar{w}}{\partial z} + \frac{\partial \bar{w}}{\partial z}\right)\right]\right\}$$

$$= \frac{1}{2}\left\{\left[\left(2\frac{\partial \bar{u}}{\partial x}\right) + \left(\frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x}\right) + \left(\frac{\partial \bar{u}}{\partial z} + \frac{\partial \bar{w}}{\partial x}\right)\right] + \left[\left(\frac{\partial \bar{v}}{\partial x} + \frac{\partial \bar{u}}{\partial y}\right) + \left(2\frac{\partial \bar{v}}{\partial y}\right) + \left(\frac{\partial \bar{v}}{\partial z} + \frac{\partial \bar{w}}{\partial y}\right)\right] + \left[\left(\frac{\partial \bar{w}}{\partial x} + \frac{\partial \bar{u}}{\partial z}\right) + \left(\frac{\partial \bar{w}}{\partial y} + \frac{\partial \bar{v}}{\partial z}\right) + \left(2\frac{\partial \bar{w}}{\partial z}\right)\right]\right\}$$

Advantages/Disadvantages of K-epsilon model

Good for predicting thin shear layers, boundary layers, duct/pipe flow,

- Model constants fit these

Boundary Conditions for K epsilon model

What about actual boundary conditions??

It's not a uniform k or ε...

So correlations have been found for **inlet boundary conditions**.

https://www.simscale.com/docs/content/simulation/model/turbulenceModel/kEpsilon.html

$$\varepsilon(boundary) = C_\mu^{0.75}\frac{k^{1.5}}{l}$$

$$C_\mu = 0.09$$

Where l is some length scale, (length scale of largest eddies)

$$l = 0.07d_h$$

$$d_h = hydraulic\ diameter$$

https://www.cfd-online.com/Wiki/Turbulence_intensity

This only applies for pipe flow:

Find k (at the inlet)

$$k = \frac{3}{2}(IU)^2$$

$$I \sim 5\%\ for\ pipeflow$$

$$U = mean\ velocity = \frac{flowrate}{area}$$

$$\varepsilon = C_\mu^{\frac{3}{4}} * \frac{k^{\frac{3}{2}}}{l}$$

$$\varepsilon = 0.09^{\frac{3}{4}} * \frac{k^{\frac{3}{2}}}{0.07 * d_h}$$

K-Omega Model

While k-epsilon describes turbulent viscosity in terms of turbulent KE and dissipation rate,

We also have other choices for the second part of the transport eqn.

K-epsilon is sufficiently accurate for simple flows (page 381 of turbulent flows) but inaccurate for complex flows.

So an alternative to this given the weakness is the $k - \omega$ model where $\omega \equiv \frac{\varepsilon}{k}$

k-omega is good for near wall regions and effects of streamwise pressure gradient, but performs poorly in non-turbulent freestream boundaries. Where $k-\varepsilon$ excels

Standard RANS equation

$$[\mu\nabla^2(\bar{u})]-\frac{\partial}{\partial x}(\bar{P})+\rho\, g_x = \rho\left(\frac{\partial}{\partial t}(\bar{u})+\frac{\partial(\bar{u}^2)}{\partial x}+\frac{\partial(\bar{u}\bar{v})}{\partial y}+\frac{\partial(\bar{u}\bar{w})}{\partial z}+\overline{\frac{\partial(u'^2)}{\partial x}}+\overline{\frac{\partial(u'v')}{\partial y}}+\overline{\frac{\partial(u'w')}{\partial z}}\right)$$

k-epsilon (for reference)

$$-\overline{v_i'v_j'}=\nu_T\left(\frac{\partial\bar{v}_i}{\partial x_j}+\frac{\partial\bar{v}_j}{\partial x_i}\right)-\frac{2}{3}(k\delta_{ij})$$

$$\nu_t=C_\mu\frac{k^2}{\varepsilon}$$

$$\frac{\partial k}{\partial t}+u\frac{\partial k}{\partial x}+v\frac{\partial k}{\partial y}+w\frac{\partial k}{\partial z}=\frac{\partial}{\partial x}\left(\frac{\nu_T}{\sigma_k}\frac{\partial k}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\nu_T}{\sigma_k}\frac{\partial k}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{\nu_T}{\sigma_k}\frac{\partial k}{\partial z}\right)+P-D$$

$$\frac{\partial \varepsilon}{\partial t}+u\frac{\partial \varepsilon}{\partial x}+v\frac{\partial \varepsilon}{\partial y}+w\frac{\partial \varepsilon}{\partial z}=\frac{\partial}{\partial x}\left(\frac{\nu_T}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\nu_T}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{\nu_T}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial z}\right)+\frac{\varepsilon}{k}\left(C_{\varepsilon 1}P-C_{\varepsilon 2}D\right)$$

$$\sigma_k=1.0$$

$$\sigma_\varepsilon=1.3$$

$$P=\nu_t S^2=2\nu_t S_{ij}S_{ij}$$

$$D=\varepsilon$$

$$C_{\varepsilon 1}=1.44, C_\varepsilon=1.92$$

$$C_\mu=0.09$$

What about the $k-\omega$ model?

Closure model and viscosity model are the same….

$$-\overline{v_i'v_j'}=\nu_T\left(\frac{\partial\bar{v}_i}{\partial x_j}+\frac{\partial\bar{v}_j}{\partial x_i}\right)-\frac{2}{3}(k\delta_{ij})$$

$$\nu_t=C_\mu\frac{k^2}{\varepsilon}$$

$$\omega[s^{-1}]\equiv\frac{\varepsilon}{k}\rightarrow textbook\ definition$$

From the cfd-online website,

$$\nu_t=\frac{k}{\omega},\qquad \varepsilon=\beta^*\omega k$$

$$\beta^*=C_\mu=0.09\ (if\ k-\omega\ model\ is\ same\ as\ k-\varepsilon)$$

$$\omega^* = \frac{\varepsilon}{\beta^* k} \to cfd\ website$$

$$\nu_t = \frac{k}{\frac{\varepsilon}{\beta^* k}} = \beta^* \frac{k^2}{\varepsilon} = C_\mu \frac{k^2}{\varepsilon} = 0.09 * \frac{k^2}{\varepsilon}$$

Let's compare the k equations first

$$\frac{\partial k}{\partial t} + u\frac{\partial k}{\partial x} + v\frac{\partial k}{\partial y} + w\frac{\partial k}{\partial z} = \frac{\partial}{\partial x}\left(\frac{\nu_T}{\sigma_k}\frac{\partial k}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\nu_T}{\sigma_k}\frac{\partial k}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\nu_T}{\sigma_k}\frac{\partial k}{\partial z}\right) + P - D$$

From the CFD online website,

$$\frac{\partial}{\partial t}(\rho k) + \frac{\partial}{\partial x_i}(\rho k u_i) = \frac{\partial}{\partial x_j}\left(\left(\mu + \frac{\mu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right) + P_k + P_b - \rho\varepsilon - Y_M + S_K$$

$$P_b = buoyancy\ term\ (natural\ conv, ignore\ for\ now)$$

$$Y_M = ????\ (ignore\ for\ now)$$

$$S_K = source\ terms$$

$$\frac{\partial}{\partial t}(k) + \frac{\partial}{\partial x_i}(k u_i) = \frac{\partial}{\partial x_j}\left(\left(\nu + \frac{\nu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right) + \frac{P_k}{\rho} - \varepsilon$$

$$\frac{\partial}{\partial x_i}(k u_i) = k\frac{\partial u_i}{\partial x_i} + u_i\frac{\partial}{\partial x_i}k$$

From continuity equation (incompressible fluid)

$$\frac{\partial u_i}{\partial x_i} = 0$$

$$\frac{\partial}{\partial x_i}(k u_i) = u\frac{\partial k}{\partial x} + v\frac{\partial k}{\partial y} + w\frac{\partial k}{\partial z}$$

$$\frac{\partial}{\partial x_j}\left(\left(\nu + \frac{\nu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right) = \frac{\partial}{\partial x}\left(\frac{\nu_T}{\sigma_k}\frac{\partial k}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\nu_T}{\sigma_k}\frac{\partial k}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\nu_T}{\sigma_k}\frac{\partial k}{\partial z}\right)$$

In turbulence regime: $\frac{\nu_t}{\sigma_k} \gg \nu$, so $\nu + \frac{\nu_t}{\sigma_k} \approx \frac{\nu_t}{\sigma_k}$

Scott-Pomerantz, C. D. (2005). *The k-epsilon model in the theory of turbulence* (Doctoral dissertation, University of Pittsburgh).

To derive k equation:

1) Have our NS equations
2) Multiply NS by u_i , do averaging (1)
3) Average NS equation, multiply by $\bar{u}_i$ (2)
4) To derive k, subtract (2) from (1), multiply by ½ to get Turbulent KE

http://d-scholarship.pitt.edu/10241/1/cdsp-new.pdf we look in the k equation for $k-\omega$,

$$\frac{\partial k}{\partial t}+U_j\frac{\partial k}{\partial x_j}=\tau_{ij}\frac{\partial k}{\partial x_j}-\beta^* k\omega^*+\frac{\partial}{\partial x_j}\left(\left(\nu+\frac{\nu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right)$$

From $k-\varepsilon$

$$\frac{\partial}{\partial t}(k)+u_j\frac{\partial}{\partial x_j}k=\frac{\partial}{\partial x_j}\left(\left(\nu+\frac{\nu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right)+\frac{P_k}{\rho}-\varepsilon$$

$$\varepsilon=\beta^* k\omega^*$$

$$\frac{P_k}{\rho}=\tau_{ij}\frac{\partial k}{\partial x_j}=\frac{\mu_t}{\rho}S^2$$

$$S^2=2S_{ij}S_{ij}$$

$$\tau_{ij}=\overline{u_i'u_j'}\ldots$$

Now let's move on to compare the ε and ω transport equations

$$\frac{\partial \varepsilon}{\partial t}+u\frac{\partial \varepsilon}{\partial x}+v\frac{\partial \varepsilon}{\partial y}+w\frac{\partial \varepsilon}{\partial z}=\frac{\partial}{\partial x}\left(\frac{\nu_T}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\nu_T}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{\nu_T}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial z}\right)+\frac{\varepsilon}{k}\left(C_{\varepsilon 1}P-C_{\varepsilon 2}D\right)$$

$$\frac{\partial \varepsilon}{\partial t}+u_i\frac{\partial \varepsilon}{\partial x_i}=\frac{\partial}{\partial x_j}\left(\left(\nu+\frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_j}\right)+\frac{\varepsilon}{k}\left(C_{\varepsilon 1}P-C_{\varepsilon 2}D\right)$$

We need to make some substitutions….

$$\omega\equiv\frac{\varepsilon}{k}$$

$$\varepsilon=k\omega$$

$$\frac{\partial k\omega}{\partial t}+u_i\frac{\partial k\omega}{\partial x_i}=\frac{\partial}{\partial x_j}\left(\left(\nu+\frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial k\omega}{\partial x_j}\right)+\frac{k\omega}{k}\left(C_{\varepsilon 1}P-C_{\varepsilon 2}D\right)$$

$$\frac{\partial k\omega}{\partial t}+u_i\frac{\partial k\omega}{\partial x_i}=\frac{\partial}{\partial x_j}\left(\left(\nu+\frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial k\omega}{\partial x_j}\right)+\omega\left(C_{\varepsilon 1}P-C_{\varepsilon 2}D\right)$$

Now we go term by term

$$\frac{\partial k\omega}{\partial t}=k\frac{\partial \omega}{\partial t}+\omega\frac{\partial k}{\partial t}$$

$$u_i\frac{\partial k\omega}{\partial x_i}=u_i\,k\frac{\partial \omega}{\partial x_i}+u_i\,\omega\frac{\partial k}{\partial x_i}$$

$$\frac{\partial k\omega}{\partial x_j}=k\frac{\partial \omega}{\partial x_j}+\omega\frac{\partial k}{\partial x_j}$$

$$k\frac{\partial \omega}{\partial t}+\omega\frac{\partial k}{\partial t}+u_i\,k\frac{\partial \omega}{\partial x_i}+u_i\,\omega\frac{\partial k}{\partial x_i}=\frac{\partial}{\partial x_j}\left(\left(\nu+\frac{\nu_T}{\sigma_\varepsilon}\right)\left(k\frac{\partial \omega}{\partial x_j}+\omega\frac{\partial k}{\partial x_j}\right)\right)+\omega\left(C_{\varepsilon 1}P-C_{\varepsilon 2}D\right)$$

$$k\left[\frac{\partial\omega}{\partial t}+u_i\frac{\partial\omega}{\partial x_i}\right]+\omega\left[\frac{\partial k}{\partial t}+u_i\frac{\partial k}{\partial x_i}\right]=\frac{\partial}{\partial x_j}\left(\left(\nu+\frac{\nu_T}{\sigma_\varepsilon}\right)\left(k\frac{\partial\omega}{\partial x_j}+\omega\frac{\partial k}{\partial x_j}\right)\right)+\omega\left(C_{\varepsilon1}P-C_{\varepsilon2}D\right)$$

$$k\left[\frac{\partial\omega}{\partial t}+u_i\frac{\partial\omega}{\partial x_i}\right]+\omega\left[\frac{\partial k}{\partial t}+u_i\frac{\partial k}{\partial x_i}\right]$$
$$=\frac{\partial}{\partial x_j}\left(\left(\nu+\frac{\nu_T}{\sigma_\varepsilon}\right)\left(\omega\frac{\partial k}{\partial x_j}\right)\right)+\frac{\partial}{\partial x_j}\left(\nu+\frac{\nu_T}{\sigma_\varepsilon}\right)k\frac{\partial\omega}{\partial x_j}+\omega\left(C_{\varepsilon1}P-C_{\varepsilon2}D\right)$$

$$\frac{\partial}{\partial x_j}\left(\nu+\frac{\nu_T}{\sigma_\varepsilon}\right)k\frac{\partial\omega}{\partial x_j}=k\frac{\partial}{\partial x_j}\left(\nu+\frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial\omega}{\partial x_j}+\left(\nu+\frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial\omega}{\partial x_j}\frac{\partial k}{\partial x_j}$$

$$\frac{\partial}{\partial x_j}\left(\left(\nu+\frac{\nu_T}{\sigma_\varepsilon}\right)\left(\omega\frac{\partial k}{\partial x_j}\right)\right)=\omega\frac{\partial}{\partial x_j}\left(\left(\nu+\frac{\nu_T}{\sigma_\varepsilon}\right)\left(\frac{\partial k}{\partial x_j}\right)\right)+\frac{\partial\omega}{\partial x_j}\left(\nu+\frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial k}{\partial x_j}$$

$$k\left[\frac{\partial\omega}{\partial t}+u_i\frac{\partial\omega}{\partial x_i}-\frac{\partial}{\partial x_j}\left(\nu+\frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial\omega}{\partial x_j}\right]+\omega\left[\frac{\partial k}{\partial t}+u_i\frac{\partial k}{\partial x_i}-\frac{\partial}{\partial x_j}\left(\left(\nu+\frac{\nu_T}{\sigma_\varepsilon}\right)\left(\frac{\partial k}{\partial x_j}\right)\right)\right]$$
$$=2\left(\nu+\frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial\omega}{\partial x_j}\frac{\partial k}{\partial x_j}+\omega\left(C_{\varepsilon1}P-C_{\varepsilon2}D\right)$$

From our k equation,

$$\frac{\partial}{\partial t}(k)+u_j\frac{\partial}{\partial x_j}k-\frac{\partial}{\partial x_j}\left(\left(\nu+\frac{\nu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right)=\frac{P_k}{\rho}-\varepsilon$$

$$k\left[\frac{\partial\omega}{\partial t}+u_i\frac{\partial\omega}{\partial x_i}-\frac{\partial}{\partial x_j}\left(\nu+\frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial\omega}{\partial x_j}\right]+\omega[P-\varepsilon]=2\left(\nu+\frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial\omega}{\partial x_j}\frac{\partial k}{\partial x_j}+\omega\left(C_{\varepsilon1}P-C_{\varepsilon2}\varepsilon\right)$$

$$k\left[\frac{\partial\omega}{\partial t}+u_i\frac{\partial\omega}{\partial x_i}-\frac{\partial}{\partial x_j}\left(\nu+\frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial\omega}{\partial x_j}\right]=2\left(\nu+\frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial\omega}{\partial x_j}\frac{\partial k}{\partial x_j}+\omega\left(C_{\varepsilon1}P-P-C_{\varepsilon2}\varepsilon+\varepsilon\right)$$

$$k\left[\frac{\partial\omega}{\partial t}+u_i\frac{\partial\omega}{\partial x_i}-\frac{\partial}{\partial x_j}\left(\nu+\frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial\omega}{\partial x_j}\right]=2\left(\nu+\frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial\omega}{\partial x_j}\frac{\partial k}{\partial x_j}+\omega\left(P[C_{\varepsilon1}-1]-\varepsilon[C_{\varepsilon2}-1]\right)$$

$$\frac{\partial\omega}{\partial t}+u_i\frac{\partial\omega}{\partial x_i}-\frac{\partial}{\partial x_j}\left(\nu+\frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial\omega}{\partial x_j}=\frac{2}{k}\left(\nu+\frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial\omega}{\partial x_j}\frac{\partial k}{\partial x_j}+\frac{\omega}{k}\left(P[C_{\varepsilon1}-1]-\varepsilon[C_{\varepsilon2}-1]\right)$$

We remember…

$$\omega\equiv\frac{\varepsilon}{k}$$

$$\frac{\partial\omega}{\partial t}+u_i\frac{\partial\omega}{\partial x_i}-\frac{\partial}{\partial x_j}\left(\nu+\frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial\omega}{\partial x_j}=\frac{2}{k}\left(\nu+\frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial\omega}{\partial x_j}\frac{\partial k}{\partial x_j}+\frac{\omega}{k}\left(P[C_{\varepsilon1}-1]-\omega k[C_{\varepsilon2}-1]\right)$$

$$\frac{\partial\omega}{\partial t}+u_i\frac{\partial\omega}{\partial x_i}-\frac{\partial}{\partial x_j}\left(\nu+\frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial\omega}{\partial x_j}=\frac{2}{k}\left(\nu+\frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial\omega}{\partial x_j}\frac{\partial k}{\partial x_j}+P\frac{\omega}{k}[C_{\varepsilon1}-1]-\omega k\frac{\omega}{k}[C_{\varepsilon2}-1]$$

$$\frac{\partial\omega}{\partial t}+u_i\frac{\partial\omega}{\partial x_i}-\frac{\partial}{\partial x_j}\left(\nu+\frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial\omega}{\partial x_j}=\frac{2}{k}\left(\nu+\frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial\omega}{\partial x_j}\frac{\partial k}{\partial x_j}+P\frac{\omega}{k}[C_{\varepsilon1}-1]-\omega^2[C_{\varepsilon2}-1]$$

$$\frac{\partial\omega}{\partial t}+u_i\frac{\partial\omega}{\partial x_i}=\frac{\partial}{\partial x_j}\left(\nu+\frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial\omega}{\partial x_j}+\frac{2}{k}\left(\nu+\frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial\omega}{\partial x_j}\frac{\partial k}{\partial x_j}+P\frac{\omega}{k}[C_{\varepsilon1}-1]-\omega^2[C_{\varepsilon2}-1]$$

This is the omega equation we derived from the $k-\varepsilon$ model

$$\frac{\partial\omega}{\partial t}+u_i\frac{\partial\omega}{\partial x_i}=\frac{\partial}{\partial x_j}\left(\nu+\frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial\omega}{\partial x_j}+\frac{2}{k}\left(\nu+\frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial\omega}{\partial x_j}\frac{\partial k}{\partial x_j}+P\frac{\omega}{k}[C_{\varepsilon1}-1]-\omega^2[C_{\varepsilon2}-1]$$

Let's copy from cfd online…

$$\frac{\partial\omega^*}{\partial t}+U_j\frac{\partial\omega^*}{\partial x_j}=\frac{\partial}{\partial x_j}\left[(\nu+\sigma\nu_t)\frac{\partial\omega^*}{\partial x_j}\right]+\alpha\frac{\omega^*}{k}\tau_{ij}\frac{\partial U_i}{\partial x_j}-\beta\omega^{*2}$$

Let's change ω^* to ω using, $\omega^*=\frac{\varepsilon}{\beta^*k}=\frac{1}{\beta^*}\omega$

$$\frac{1}{\beta^*}\frac{\partial\omega}{\partial t}+\frac{1}{\beta^*}U_j\frac{\partial\omega}{\partial x_j}=\frac{1}{\beta^*}\frac{\partial}{\partial x_j}\left[(\nu+\sigma\nu_t)\frac{\partial\omega}{\partial x_j}\right]+\alpha\frac{1}{\beta^*}\frac{\omega}{k}\tau_{ij}\frac{\partial U_i}{\partial x_j}-\beta\frac{1}{\beta^{*2}}\omega^2$$

$$\frac{\partial\omega}{\partial t}+U_j\frac{\partial\omega}{\partial x_j}=\frac{\partial}{\partial x_j}\left[(\nu+\sigma\nu_t)\frac{\partial\omega}{\partial x_j}\right]+\alpha\frac{\omega}{k}\tau_{ij}\frac{\partial U_i}{\partial x_j}-\frac{\beta}{\beta^*}\omega^2$$

For the diffusion term

If $\frac{1}{\sigma_\varepsilon}=\sigma$, the models are the same,

So we have production of ω

$$\alpha\frac{\omega}{k}\tau_{ij}\frac{\partial U_i}{\partial x_j}$$

From $k-\varepsilon$ model

$$[C_{\varepsilon1}-1]\frac{\omega}{k}P$$

If

$$\tau_{ij}\frac{\partial U_i}{\partial x_j}=P$$

$$\alpha=C_{\varepsilon1}-1$$

The production term of ω is the same…

In the models,

$$C_{\varepsilon1}=1.44,\alpha=1.44-1=0.44,$$

From $k-\omega$,

$$\alpha=\frac{5}{9}=0.556$$

We compare the destruction term for ω

From $k-\varepsilon$

$$\omega^2[C_{\varepsilon 2} - 1]$$

From $k - \omega$

$$\frac{\beta}{\beta^*}\omega^2$$

If

$$\frac{\beta}{\beta^*} = [C_{\varepsilon 2} - 1]$$

From the k Omega model to equal $k - \varepsilon$ model,

$$\frac{\beta}{\beta^*} = 0.8333$$

$$C_{\varepsilon 2} - 1 = 1.92 - 1 = 0.92$$

From $k - \varepsilon$ model,

And additional destruction term for ε pops out…

$$\frac{2}{k}\left(\nu + \frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial \omega}{\partial x_j}\frac{\partial k}{\partial x_j}$$

K-Omega SST Model

To have the best of both worlds, we have a blend of k-Omega and k-Epsilon in the k-Omega SST model.

So the models switches to k-Omega near the wall and k-Epsilon near the free stream.

Dr Aiden (Fluid Mechanics 101)

https://www.youtube.com/watch?v=myv-ityFnS4

$$\frac{\partial k}{\partial t} + u_j\frac{\partial k}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right)\frac{\partial k}{\partial x_j}\right) + \frac{P_{k(dynamic)}}{\rho} - \varepsilon$$

$$\varepsilon = \beta^*\omega^* k;\ \omega = \omega^*\beta^* = \omega^* C_\mu;\ \beta^* = 0.09$$

$$\frac{\partial k}{\partial t} + u_j\frac{\partial k}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right)\frac{\partial k}{\partial x_j}\right) + \frac{P_{k(dynamic)}}{\rho} - \beta^*\omega^* k$$

$$\frac{\partial k}{\partial t} + u_j\frac{\partial k}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right)\frac{\partial k}{\partial x_j}\right) + \frac{P_{k(dynamic)}}{\rho} - \omega k$$

(note: $k - \varepsilon, k - \omega$ models, k equation remains the same-ish, so $k - \omega\ SST$ is also the same)

From $k - \varepsilon$

$$\frac{\partial\omega}{\partial t}+u_i\frac{\partial\omega}{\partial x_i}=\frac{\partial}{\partial x_j}\left(\nu+\frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial\omega}{\partial x_j}+\frac{2}{k}\left(\nu+\frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial\omega}{\partial x_j}\frac{\partial k}{\partial x_j}+P[C_{\varepsilon1}-1]\frac{\omega}{k}-\omega^2[C_{\varepsilon2}-1]$$

From $k-\omega$

$$\frac{\partial\omega}{\partial t}+u_i\frac{\partial\omega}{\partial x_i}=\frac{\partial}{\partial x_j}\left(\nu+\frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial\omega}{\partial x_j}+P\alpha\frac{\omega}{k}-\omega^2\frac{\beta}{\beta^*}$$

$$\omega\equiv\frac{\varepsilon}{k}$$

Let's say we want to blend…

In deriving the $k-\omega\ SST$ formulation,

$$\frac{\partial\omega}{\partial t}+u_i\frac{\partial\omega}{\partial x_i}=\frac{\partial}{\partial x_j}\left(\nu+\frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial\omega}{\partial x_j}+(f_{blending})\frac{2}{k}\left(\nu+\frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial\omega}{\partial x_j}\frac{\partial k}{\partial x_j}+P[C_{\varepsilon1}-1]\frac{\omega}{k}-\omega^2[C_{\varepsilon2}-1]$$

$$f_{blending}=1-F_1, 0<F_1<1$$

At the wall, $F_1\to1$, in bulk region $F_1\to0$

$$\frac{\partial\omega}{\partial t}+u_i\frac{\partial\omega}{\partial x_i}=\frac{\partial}{\partial x_j}\left(\nu+\frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial\omega}{\partial x_j}+(1-F_1)\frac{2}{k}\left(\nu+\frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial\omega}{\partial x_j}\frac{\partial k}{\partial x_j}+P[C_{\varepsilon1}-1]\frac{\omega}{k}-\omega^2[C_{\varepsilon2}-1]$$

$$\left(\nu+\frac{\nu_T}{\sigma_\varepsilon}\right)\approx\frac{\nu_T}{\sigma_\varepsilon}\ in\ turbulent\ bulk\ fluid\ region$$

$$\nu_T\approx C_\mu\frac{k^2}{\varepsilon}=\beta^*\frac{k^2}{\varepsilon}$$

$$\left(\nu+\frac{\nu_T}{\sigma_\varepsilon}\right)*\frac{1}{k}\approx\frac{\beta^*}{\sigma_\varepsilon}\frac{k^2}{\varepsilon}\frac{1}{k}=\frac{\beta^*}{\sigma_\varepsilon}\frac{k}{\varepsilon}=\frac{\beta^*}{\sigma_\varepsilon}\frac{1}{\omega}=\frac{1}{\sigma_\varepsilon}\frac{1}{\omega}$$

$$\omega=\omega^*\beta^*\to\omega^*=\frac{\omega}{\beta^*}$$

To make it look like the

$$\frac{2}{k}\left(\nu+\frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial\omega}{\partial x_j}\frac{\partial k}{\partial x_j}\approx2\frac{1}{\sigma_\varepsilon}\frac{1}{\omega}\frac{\partial\omega}{\partial x_j}\frac{\partial k}{\partial x_j}\ (only\ applies\ when\frac{\nu_t}{\sigma_\varepsilon}\gg\nu, ie\ far\ from\ wall)$$

With blending function

$$2(1-F_1)\frac{1}{\sigma_\varepsilon}\frac{1}{\omega}\frac{\partial\omega}{\partial x_j}\frac{\partial k}{\partial x_j}\to0\ at\ the\ wall, otherwise\ finite\ far\ from\ wall$$

The blending function F_1. F_1 is the degree to which $k-\omega$ model activates

$$F_1=\tanh\left\{\left\{\min\left[\max\left(\frac{\sqrt{k}}{\omega y},\frac{500\nu C_\mu}{\omega y^2}\right),\frac{4\sigma_{\omega2}k}{CD_{k\omega}y^2}\right]\right\}^4\right\}$$

So what do $\frac{\sqrt{k}}{\omega y}$, $\frac{500\nu C_\mu}{\omega y^2}$ and $\frac{4\sigma_{\omega2}k}{CD_{k\omega}y^2}$

We want to discuss near the wall. Why?

Because we want $k - \omega$ to activate near the wall. $F_1 \to 1$

How does F_1 change in Viscous Sublayer (VSL) and then the log law region (turbulent sublayer), see how this velocity profile goes for large y^+ values.

Of course, this is for flat plate, BL flow, which we can assume applies to most flow types, **except convection and flow with strong pressure gradient→ this negates the constant shear stress hypothesis**

$$\frac{\tau_{app}}{\rho} = (\nu + \nu_t)\frac{\partial \bar{u}}{\partial y}$$

Eventually we want to see how F_1 varies with y or y^+...

For that, we need wall fns for k and ω

$\frac{\sqrt{k}}{\omega y}$, $\frac{500\nu C_\mu}{\omega y^2}$ and $\frac{4\sigma_{\omega 2} k}{CD_{k\omega} y^2}$

ω should vary with y^+ and k also...

How did this k equation come about

$$\frac{\partial k}{\partial t} + u_j \frac{\partial k}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right)\frac{\partial k}{\partial x_j}\right) + \frac{P_{k(dynamic)}}{\rho} - \omega k$$

Nondimensionalising it...

$$k^+ = \frac{k}{u_*^2}, \omega^+ = \frac{\omega \nu}{u_*^2}$$

$$u_* = \sqrt{\frac{\tau_{wall}}{\rho}}$$

We can substitute:

$$k = k^+ u_*^2\ ; \omega = \frac{\omega^+ u_*^2}{\nu}$$

$$\frac{\partial k}{\partial t} + u_j \frac{\partial k}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right)\frac{\partial k}{\partial x_j}\right) + \frac{P_{k(dynamic)}}{\rho} - \omega k$$

$$u_*^2\left(\frac{\partial k^+}{\partial t} + u_j \frac{\partial k^+}{\partial x_j}\right) = u_*^2 \frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right)\frac{\partial k^+}{\partial x_j}\right) + \frac{P_{k(dynamic)}}{\rho} - \frac{\omega^+ u_*^2}{\nu} k^+ u_*^2$$

$$u_*^2\left(\frac{\partial k^+}{\partial t} + u_j \frac{\partial k^+}{\partial x_j}\right) = u_*^2 \frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right)\frac{\partial k^+}{\partial x_j}\right) + \frac{P_{k(dynamic)}}{\rho} - \frac{\omega^+ k^+ u_*^4}{\nu}$$

$$\left(\frac{\partial k^+}{\partial t} + u_j \frac{\partial k^+}{\partial x_j}\right) = \frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right)\frac{\partial k^+}{\partial x_j}\right) + \frac{P_{k(dynamic)}}{\rho} - \frac{\omega^+ k^+ u_*^2}{\nu}$$

$$u_j = u_j^+ u_*$$

$$\left(\frac{\partial k^+}{\partial t} + u_j^+ u_* \frac{\partial k^+}{\partial x_j}\right) = \frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right)\frac{\partial k^+}{\partial x_j}\right) + \frac{P_{k(dynamic)}}{\rho} - \frac{\omega^+ k^+ u_*^2}{\nu}$$

So in BL, we assume

$$\frac{\partial k^+}{\partial t} + u_j^+ u_* \frac{\partial k^+}{\partial x_j} \approx 0$$

In BL, the production of turbulent KE is negligible (more so in VSL)

$$\frac{P_{k(dynamic)}}{\rho} \approx 0$$

So we get:

$$0 = \frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right)\frac{\partial k^+}{\partial x_j}\right) - \frac{\omega^+ k^+ u_*^2}{\nu}$$

$$0 = \frac{\partial}{\partial x_j}\left(\left(1 + \sigma_{K(k\omega SST)}\frac{\nu_t}{\nu}\right)\frac{\partial k^+}{\partial x_j}\right) - \frac{\omega^+ k^+ u_*^2}{\nu^2}$$

In BL, we only consider y direction, so we simplify

$$0 = \frac{\partial}{\partial y}\left(\left(1 + \sigma_{K(k\omega SST)}\frac{\nu_t}{\nu}\right)\frac{\partial k^+}{\partial y}\right) - \frac{\omega^+ k^+ u_*^2}{\nu^2}$$

$$y = \frac{y^+ \nu}{u_*}$$

$$0 = \frac{\partial}{\partial \frac{y^+ \nu}{u_*}}\left(\left(1 + \sigma_{K(k\omega SST)}\frac{\nu_t}{\nu}\right)\frac{\partial k^+}{\partial \frac{y^+ \nu}{u_*}}\right) - \frac{\omega^+ k^+ u_*^2}{\nu^2}$$

$$0 = \frac{u_*^2}{\nu^2}\frac{\partial}{\partial y^+}\left(\left(1 + \sigma_{K(k\omega SST)}\frac{\nu_t}{\nu}\right)\frac{\partial k^+}{\partial y^+}\right) - \frac{\omega^+ k^+ u_*^2}{\nu^2}$$

$$0 = \frac{u_*^2}{\nu^2}\frac{\partial}{\partial y^+}\left(\left(1 + \sigma_{K(k\omega SST)}\frac{\nu_t}{\nu}\right)\frac{\partial k^+}{\partial y^+}\right) - \frac{\omega^+ k^+ u_*^2}{\nu^2}$$

$$0 = \frac{\partial}{\partial y^+}\left(\left(1 + \sigma_{K(k\omega SST)}\frac{\nu_t}{\nu}\right)\frac{\partial k^+}{\partial y^+}\right) - \omega^+ k^+$$

In VSL, we assume $\nu_t \ll \nu$

$$0 = \frac{\partial}{\partial y^+}\left(\frac{\partial k^+}{\partial y^+}\right) - \omega^+ k^+$$

I think im missing a C_μ

$$\omega = \omega^* \beta^* \to \omega^* = \frac{\omega}{\beta^*}, \beta^* = C_\mu$$

If we use ω_*

$$0 = \frac{\partial}{\partial y^+}\left(\frac{\partial k^+}{\partial y^+}\right) - C_\mu {\omega^*}^+ k^+$$

I'll skip the ${\omega^*}^+$ derivation…

$$0 = \frac{\partial}{\partial y^+}\left(\frac{\partial {\omega^*}^+}{\partial y^+}\right) - \beta_1 \left({\omega^*}^+\right)^2$$

We solve the ODE's in the VSL,

$${\omega^*}^+ = \frac{6}{\beta_1 (y^+)^2}$$

$$\beta_1 = \frac{3}{40} \; (k - \omega \; equation \; constant)$$

Subs this into:

$$0 = \frac{\partial}{\partial y^+}\left(\frac{\partial k^+}{\partial y^+}\right) - C_\mu \frac{6}{\beta_1 (y^+)^2} k^+$$

And solve the ODE here, we get:

$$k^+ = C_k (y^+)^{3.23}$$

By intuition, there is no turbulence in the VSL,

So $C_k = 0$

We also have a log law region:

$${\omega^*}^+ = \frac{1}{\kappa \sqrt{C_\mu} y^+}$$

$${k^*}^+ = \frac{1}{\sqrt{C_\mu}}$$

To derive the above,

recall

$$\left(\frac{\partial k^+}{\partial t} + u_j^+ u_* \frac{\partial k^+}{\partial x_j}\right) = \frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)} \nu_t\right) \frac{\partial k^+}{\partial x_j}\right) + \frac{P_{k(dynamic)}}{\rho} - \frac{\omega^+ k^+ u_*^2}{\nu}$$

$$\left(\frac{\partial k^+}{\partial t} + u_j^+ u_* \frac{\partial k^+}{\partial x_j}\right) = 0$$

But for turbulent sublayer

$$\frac{P_{k(dynamic)}}{\rho} \neq 0$$

$$0 = \frac{\partial}{\partial y^+}\left(\left(1 + \sigma_{K(k\omega SST)}\frac{\nu_t}{\nu}\right)\frac{\partial k^+}{\partial y^+}\right) + \nu_t^+\left|\frac{dU^+}{dy^+}\right|^2 - C_\mu \omega^{*+} k^+$$

$$\frac{P_{k(dynamic)}}{\rho} = \overline{u_i' u_j'}\frac{\partial U}{\partial y}$$

$$\frac{P_{k(dynamic)}}{\rho} = \nu_t \frac{\partial U}{\partial y}\frac{\partial U}{\partial y}$$

$$0 = \frac{\partial}{\partial y^+}\left(\left(1 + \sigma_{K(k\omega SST)}\frac{\nu_t}{\nu}\right)\frac{\partial k^+}{\partial y^+}\right) + \nu_t^+\left|\frac{dU^+}{dy^+}\right|^2 - C_\mu \omega^{*+} k^+$$

In VSL, we assume $\nu_t \ll \nu$

But in turbulent sublayer, we assume $\nu_t \gg \nu$

$$0 = \frac{\partial}{\partial y^+}\left(\left(\sigma_{K(k\omega SST)}\frac{\nu_t}{\nu}\right)\frac{\partial k^+}{\partial y^+}\right) + \nu_t^+\left|\frac{dU^+}{dy^+}\right|^2 - C_\mu \omega^{*+} k^+$$

We assume here in turbulent sublayer:

$$\frac{\partial}{\partial y^+}\left(\left(\sigma_{K(k\omega SST)}\frac{\nu_t}{\nu}\right)\frac{\partial k^+}{\partial y^+}\right) \ll \nu_t^+\left|\frac{dU^+}{dy^+}\right|^2$$

$$\frac{\partial}{\partial y^+}\left(\left(\sigma_{K(k\omega SST)}\frac{\nu_t}{\nu}\right)\frac{\partial k^+}{\partial y^+}\right) \approx 0$$

$$\left(\sigma_{K(k\omega SST)}\frac{\nu_t}{\nu}\right)\frac{\partial k^+}{\partial y^+} = C_k$$

$$\nu_t^+\left|\frac{dU^+}{dy^+}\right|^2 - C_\mu \omega^{*+} k^+ = 0$$

$$\nu_t^+ = \frac{k^+}{\omega^{*+}}$$

$$\omega^{*+} = \frac{k^+}{\nu_t^+}$$

$$\nu_t^+\left|\frac{dU^+}{dy^+}\right|^2 - C_\mu \frac{k^+}{\nu_t^+} k^+ = 0$$

$$\nu_t^{+2}\left|\frac{dU^+}{dy^+}\right|^2 - C_\mu k^{+2} = 0$$

Or to eliminate k^+

$$k^+ = \omega^{*+}\nu_t^+$$

$$\nu_t^+ \left|\frac{dU^+}{dy^+}\right|^2 - C_\mu \omega^{*+}\omega^{*+}\nu_t^+ = 0$$

$$\left|\frac{dU^+}{dy^+}\right|^2 - C_\mu {\omega^{*+}}^2 = 0$$

$$\omega^{*+} = \frac{\left|\frac{dU^+}{dy^+}\right|}{\sqrt{C_\mu}}$$

What is $\frac{dU^+}{dy^+}$ in log law region

$$u^+ = 2.5 \ln y^+ + 5.5 \; (\log \; law\; of\; the\; wall)$$

$$u^+ = \frac{1}{\kappa}\ln y^+ + 5.5, \kappa \approx 0.4$$

$$\frac{du^+}{dy^+} = \frac{1}{\kappa}\frac{1}{y^+}$$

Subs into the ω^{*+} and we get

$$\omega^{*+} = \frac{1}{\kappa y^+ \sqrt{C_\mu}}$$

Now we want to get back our k equation…

Recall:

$$k^+ = \omega^{*+}\nu_t^+$$

$$k^+ = \frac{1}{\kappa y^+ \sqrt{C_\mu}}\nu_t^+$$

What is ν_t^+

$$\tau_{wall} = \rho(\nu + \nu_t)\frac{\partial u}{\partial y}$$

Nondimensionalise,

$$1 = \left(1 + \frac{\nu_t}{\nu}\right)\frac{du^+}{dy^+}$$

In log law region:

$$\nu_t \gg \nu$$

So we get

$$\frac{\nu_t}{\nu}\frac{du^+}{dy^+} = 1$$

$$\nu_t^+ = \frac{1}{\frac{du^+}{dy^+}}$$

Substitute this back:

$$k^+ = \frac{1}{\kappa y^+ \sqrt{C_\mu}} \frac{1}{\frac{du^+}{dy^+}}$$

$$k^+ = \frac{1}{\kappa y^+ \sqrt{C_\mu}} \kappa y^+$$

$$k^+ = \frac{1}{\sqrt{C_\mu}}$$

This is k^+ for log law region…

How about k^+ in VSL

Recall:

$$k^+ = C_k (y^+)^{3.23}$$

In TSL,

$$\left(\sigma_{K(k\omega SST)} \frac{\nu_t}{\nu}\right) \frac{\partial k^+}{\partial y^+} = C_k$$

$$k^+ = \frac{1}{\sqrt{C_\mu}}, so\ \frac{\partial k^+}{\partial y^+} = 0$$

Therefore

$$C_k = 0$$

In intermediate region, we take root mean sqr of VSL and TSL ω^*

	VSL	Intermediate (buffer) layer	Turbulent Sublayer
k^+	$k^+ = C_k (y^+)^{3.23} = 0$ As $C_k = 0$	$k^+ = \omega^{*+} \nu_t^+$ $1 = \left(1 + \frac{\nu_t}{\nu}\right) \frac{du^+}{dy^+}$ In log law: $\frac{du^+}{dy^+} = \frac{1}{\kappa} \frac{1}{y^+}$ Or in VSL $\frac{du^+}{dy^+} = 1$ Or you can use Van Driest Model	$k^+ = \frac{1}{\sqrt{C_\mu}}$ $C_\mu = 0.09$
ω^{*+}	$\omega^{*+} = \frac{6}{\beta_1 (y^+)^2}$ $\beta_1 = \frac{3}{40}$	ω^{*+} $= \sqrt{{\omega^{*+}}^2_{VSL} + {\omega^{*}_{TSL}}^{+2}}$	$\omega^{*+} = \frac{1}{\kappa y^+ \sqrt{C_\mu}}$ $\kappa = 0.41$ $C_\mu = 0.09$

$\frac{\sqrt{k}}{C_\mu \omega^* y}$, $\frac{500\nu}{\omega^* y^2}$ and $\frac{4\sigma_{\omega 2} k}{CD_{k\omega} y^2}$

So first we want to nondimensionalise them…. So we can get these functions as a function of y^+

$$k^+ = \frac{k}{u_*^2}$$

$$\omega^{*+} = \frac{\omega^* \nu}{u_*^2}$$

$$y^+ = \frac{y u_*}{\nu}$$

We can nondimensionalise

$$\frac{\sqrt{k}}{C_\mu \omega^* y} = \frac{\sqrt{u_*^2 k^+}}{C_\mu \frac{\omega^{*+} u_*^2}{\nu} \frac{y^+ \nu}{u_*}} = \frac{u_* \sqrt{k^+}}{C_\mu \omega^{*+} u_* y^+} = \frac{\sqrt{k^+}}{C_\mu \omega^{*+} y^+}$$

Now we can substitute in…

$$y^+ < 11.6$$

$$u^+ = y^+ \ (viscous\ sublyaer\ equation)$$

$$y^+ > 11.6$$

$$u^+ = 2.5 \ln y^+ + 5.5 \ (\log\ \ law\ of\ the\ wall)$$

The next function is this:

$$\frac{500\nu}{\omega^* y^2}$$

$$y = \frac{y^+ \nu}{u_*}$$

$$\omega^* = \frac{\omega^{*+} u_*^2}{\nu}$$

Substitute in…

$$\frac{500\nu}{\omega^* y^2} = \frac{500\nu}{\frac{\omega^{*+} u_*^2}{\nu} \left(\frac{y^+ \nu}{u_*}\right)^2} = \frac{500\nu}{\frac{\omega^{*+} u_*^2}{\nu} \frac{{y^+}^2 \nu^2}{u_*^2}} = \frac{500\nu}{\omega^{*+} {y^+}^2 \nu} = \frac{500}{\omega^{*+} {y^+}^2}$$

One last function…

$$\frac{4\sigma_{\omega 2}k}{CD_{k\omega}y^2}$$

Let’s deal with the stuff outside…

$$\frac{4\sigma_{\omega 2}k}{y^2} = 4\sigma_{\omega 2}\frac{u_*^2 k^+}{\left(\frac{y^+\nu}{u_*}\right)^2} = 4\sigma_{\omega 2}\frac{k^+}{y^{+2}}\frac{u_*^4}{\nu^2}$$

$$CD_{k\omega} = \max\left(2\rho\sigma_{\omega 2}\frac{1}{\omega^*}\frac{\partial\omega^*}{\partial x_j}\frac{\partial k}{\partial x_j}, 10^{-10}\right)$$

$$\omega^* = \frac{\omega^{*+}u_*^2}{\nu}$$

$$y = \frac{y^+\nu}{u_*}$$

We can cheat to find an estimate in the boundary layer…

We can assume in the BL

$$\frac{\partial\omega^*}{\partial x_j}\frac{\partial k}{\partial x_j} = \frac{\partial\omega^*}{\partial y}\frac{\partial k}{\partial y}$$

In BL

$$CD_{k\omega} = \max\left(2\rho\sigma_{\omega 2}\frac{1}{\omega^*}\frac{\partial\omega^*}{\partial y}\frac{\partial k}{\partial y}, 10^{-10}\right)$$

$$\omega^* = \frac{\omega^{*+}u_*^2}{\nu}$$

$$y = \frac{y^+\nu}{u_*}$$

$$k = k^+u_*^2$$

Nondimensionalising

$$CD_{k\omega} = \max\left(2\rho\sigma_{\omega 2}\frac{1}{\frac{\omega^{*+}u_*^2}{\nu}}\frac{\partial\left(\frac{\omega^{*+}u_*^2}{\nu}\right)}{\partial\left(\frac{y^+\nu}{u_*}\right)}\frac{\partial(k^+u_*^2)}{\partial\left(\frac{y^+\nu}{u_*}\right)}, 10^{-10}\right)$$

$$CD_{k\omega} = \max\left(2\rho\sigma_{\omega 2}\frac{1}{\frac{\omega^{*+}u_*^2}{\nu}}\frac{\frac{u_*^2}{\nu}\,\partial(\omega^{*+})}{\frac{\nu}{u_*}\,\partial(y^+)}\frac{u_*^2\partial(k^+)}{\frac{\nu}{u_*}\,\partial(y^+)}, 10^{-10}\right)$$

$$CD_{k\omega} = \max\left(2\rho\sigma_{\omega 2}\frac{1}{\omega^{*+}\nu}\frac{\frac{u_*^2}{\nu}\partial(\omega^{*+})}{\partial(y^+)}\frac{u_*^2\partial(k^+)}{\partial(y^+)}, 10^{-10}\right)$$

$$CD_{k\omega} = \max\left(2\rho\sigma_{\omega 2}\frac{u_*^4}{\nu^2}\frac{1}{\omega^{*+}}\frac{\partial(\omega^{*+})}{\partial(y^+)}\frac{\partial(k^+)}{\partial(y^+)}, 10^{-10}\right)$$

So for,

$$\frac{4\sigma_{\omega 2}k}{CD_{k\omega}y^2} = 4\sigma_{\omega 2}\frac{k^+}{y^{+2}}\frac{u_*^4}{\nu^2} * \frac{1}{CD_{k\omega}}$$

$$\frac{4\sigma_{\omega 2}k}{CD_{k\omega}y^2} = 4\sigma_{\omega 2}\frac{k^+}{y^{+2}}\frac{u_*^4}{\nu^2} * \frac{1}{\frac{u_*^4}{\nu^2}\max\left(2\rho\sigma_{\omega 2}\frac{1}{\omega^{*+}}\frac{\partial(\omega^{*+})}{\partial(y^+)}\frac{\partial(k^+)}{\partial(y^+)}, \frac{\nu^2}{u_*^4}10^{-10}\right)}$$

$$\frac{4\sigma_{\omega 2}k}{CD_{k\omega}y^2} = 4\sigma_{\omega 2}\frac{k^+}{y^{+2}} * \frac{1}{\max\left(2\rho\sigma_{\omega 2}\frac{1}{\omega^{*+}}\frac{\partial\omega^{*+}}{\partial y^+}\frac{\partial k^+}{\partial y^+}, \frac{\nu^2}{u_*^4}10^{-10}\right)}$$

Can we nondimensionalise $\frac{\nu^2}{u_*^4}$?

$$u_* = \sqrt{\frac{\tau_{wall}}{\rho}}$$

At the wall

$$\tau = \rho\nu\frac{\partial u}{\partial y}$$

$$u_* = \sqrt{\nu\frac{\partial u}{\partial y}}$$

Hang on, why is there an extra density term in the denominator?

According to literature, there should be a density term in the numerator

$$F_1 = \tanh\left\{\left\{\min\left[\max\left(\frac{\sqrt{k}}{\omega y}, \frac{500\nu C_\mu}{\omega y^2}\right), \frac{4\rho\sigma_{\omega 2}k}{CD_{k\omega}y^2}\right]\right\}^4\right\}$$

Hellsten, A. (1998). Some improvements in Menter's k-omega SST turbulence model. In 29th AIAA, Fluid Dynamics Conference (p. 2554).

http://cfd.mace.manchester.ac.uk/twiki/pub/Main/CDAdapcoMeetingsM4/AIAA_98-2554-CP.PS.pdf

This is from an earlier paper, and there **is** a density term in the numerator…

Okay so let's correct this:

$$\frac{4\rho\sigma_{\omega 2}k}{CD_{k\omega}y^2} = 4\sigma_{\omega 2}\frac{k^+}{y^{+2}}\frac{\rho u_*^4}{\nu^2} * \frac{1}{CD_{k\omega}}$$

$$CD_{k\omega} = \max\left(2\sigma_{\omega 2}\frac{\rho u_*^4}{\nu^2}\frac{1}{\omega^{*+}}\frac{\partial\omega^{*+}}{\partial y^+}\frac{\partial k^+}{\partial y^+}, 10^{-10}\right)$$

Looks like this term is quite important

$$\frac{\rho u_*^4}{\nu^2}$$

$$u_* = \sqrt{\frac{\tau_{wall}}{\rho}}$$

$$\frac{\rho\left(\frac{\tau_{wall}}{\rho}\right)^2}{\nu^2} = \frac{\tau_{wall}^2}{\rho\nu^2}$$

At the wall

$$\tau_{wall} = \rho(\nu + \nu_t)\frac{\partial u}{\partial y}$$

At the wall, $\nu_t = 0$

$$\tau_{wall} = \rho\nu\frac{\partial u}{\partial y}$$

$$y = \frac{y^+\nu}{u_*}$$

$$u = u^+u_*$$

$$\tau_{wall} = \rho\nu\frac{\partial u^+u_*}{\partial\frac{y^+\nu}{u_*}}$$

$$\tau_{wall} = \rho u_*^2\frac{\partial u^+}{\partial y^+}$$

In VSL region

$$\frac{\partial u^+}{\partial y^+} = 1$$

$$\tau_{wall} = \rho u_*^2$$

$$\frac{\rho\left(\frac{\tau_{wall}}{\rho}\right)^2}{\nu^2} = \frac{\tau_{wall}^2}{\rho\nu^2} = \frac{(\rho u_*^2)^2}{\rho\nu^2}$$

Looks like we can’t cancel out all the terms here…

$$\frac{4\rho\sigma_{\omega 2}k}{CD_{k\omega}y^2} = 4\sigma_{\omega 2}\frac{k^+}{{y^+}^2}\frac{\rho u_*^4}{\nu^2} * \frac{1}{CD_{k\omega}}$$

$$CD_{k\omega} = \max\left(2\sigma_{\omega 2}\frac{\rho u_*^4}{\nu^2}\frac{1}{\omega^{*+}}\frac{\partial \omega^{*+}}{\partial y^+}\frac{\partial k^+}{\partial y^+}, 10^{-10}\right)$$

$$\frac{4\rho\sigma_{\omega 2}k}{CD_{k\omega}y^2} = \frac{4\sigma_{\omega 2}\frac{k^+}{{y^+}^2}}{\max\left(2\sigma_{\omega 2}\frac{1}{\omega^{*+}}\frac{\partial \omega^{*+}}{\partial y^+}\frac{\partial k^+}{\partial y^+}, \frac{\nu^2}{\rho u_*^4}10^{-10}\right)}$$

We can do parametric analysis to see the effect of the value of $\frac{\nu^2}{\rho u_*^4(Re, surface\ roughness)}$

So k^+ we already have in excel… (plotted)

$$\frac{\partial \omega^{*+}}{\partial y^+} = \frac{-12}{\beta_1 (y^+)^3}$$

ω^{*+}	$\omega^{*+} = \frac{6}{\beta_1 (y^+)^2}$ $\beta_1 = \frac{3}{40}$	ω^{*+} $= \sqrt{{\omega^{*+}}_{VSL}^2 + {\omega_{TSL}^*}^{+2}}$	$\omega^{*+} = \frac{1}{\kappa y^+ \sqrt{C_\mu}}$ $\kappa = 0.41$ $C_\mu = 0.09$
$\frac{\partial \omega^{*+}}{\partial y^+}$	$\frac{-12}{\beta_1 (y^+)^3}$	-	$\omega^{*+} = \frac{-1}{\kappa (y^+)^2 \sqrt{C_\mu}}$

$$\frac{\partial k^+}{\partial y^+}$$

For $\frac{\partial k^+}{\partial y^+}$, it is equal to 0 in the BL

k^+	$k^+ = C_k (y^+)^{3.23} = 0$ As $C_k = 0$	$k^+ = \omega^{*+}\nu_t^+$ $1 = \left(1 + \frac{\nu_t}{\nu}\right)\frac{du^+}{dy^+}$ In log law: $\frac{du^+}{dy^+} = \frac{1}{\kappa}\frac{1}{y^+}$ Or in VSL $\frac{du^+}{dy^+} = 1$ Or you can use Van Driest Model	$k^+ = \frac{1}{\sqrt{C_\mu}}$ $C_\mu = 0.09$

In BL

$$\frac{4\rho\sigma_{\omega 2}k}{CD_{k\omega}y^2} = \frac{4\sigma_{\omega 2}\frac{k^+}{y^{+2}}}{\max\left(2\sigma_{\omega 2}\frac{1}{\omega^{*+}}\frac{\partial\omega^{*+}}{\partial y^+}\frac{\partial k^+}{\partial y^+}, \frac{\nu^2}{\rho u_*^4}10^{-10}\right)} = \frac{4\sigma_{\omega 2}\frac{k^+}{y^{+2}}}{\frac{\nu^2}{\rho u_*^4}10^{-10}}$$

The first term may be negligible…

$$\frac{4\sigma_{\omega 2}\frac{k^+}{y^{+2}}}{\frac{\nu^2}{\rho u_*^4}10^{-10}} = 0\ in\ VSL$$

$$\frac{\frac{4\sigma_{\omega 2}}{\sqrt{C_\mu}}}{\frac{\nu^2}{\rho u_*^4}10^{-10}}\frac{1}{y^{+2}} = 0\ in\ VSL$$

$$\sigma_{\omega 2} = 0.856$$

$$\sqrt{C_\mu} = \sqrt{0.09} = 0.3$$

$$\frac{4\sigma_{\omega 2}}{\sqrt{C_\mu}10^{-10}} = 1.14133E + 11$$

$$Z = \frac{4\rho\sigma_{\omega 2}k}{CD_{k\omega}y^2} \approx \frac{1.14133E + 11}{\frac{\nu^2}{\rho u_*^4}y^{+2}}$$

We show that for this, typical Z values with zeta =0.01 , 1 and 100,

$F_1 = 1$ in the BL → this turns on $k-\omega$ in BL always.

But as $2\sigma_{\omega 2}\frac{1}{\omega^{*+}}\frac{\partial\omega^{*+}}{\partial y^+}\frac{\partial k^+}{\partial y^+}$ increases in turbulent region

Then F_1 decreases to reach 0, and thus turn on the $k-\varepsilon$ model.

$$\zeta = \frac{\nu^2}{\rho u_*^4}$$

Typical value for ζ

https://www.aiche.org/ccps/resources/glossary/process-safety-glossary/friction-velocity-u

for air flow

$$u_* \approx 0.05\frac{m}{s}\ (light\ wind)$$

$$u_* \approx 1\frac{m}{s}\ (strong\ wind)$$

$$\rho = 1\frac{kg}{m^3}$$

$$\nu_{air} = 1.48e - 5\frac{m^2}{s}$$

https://www.engineersedge.com/physics/viscosity_of_air_dynamic_and_kinematic_14483.htm

$$\zeta = \frac{\left(1.48e - 5\frac{m^2}{s}\right)^2}{1\ (1^4)} = 2.1904E - 10$$

From baseline model (with just the blending function only), also called BSL, to $\boldsymbol{k - \omega}$ SST

Problem 1: expt data shows overprediction of reynold's shear stresses in adverse pressure gradient flows…

(Menter, F. R. (1994). Two-equation eddy-viscosity turbulence models for engineering applications. AIAA journal, 32(8), 1598-1605.) → menter's paper for $k - \omega$ SST

What are adverse pressure gradient flows?

https://en.wikipedia.org/wiki/Adverse_pressure_gradient

when $\frac{dp}{dx}$ does not favour flow direction, that's adverse pressure gradient…

- ➔ We haven't quite dealt with that in the previous flows, those wall functions are based on flat plate flow…
- ➔ But for curved surfaces, eg. Airfoil, spheres, this doesn't work as well anymore…

How does one sort this out??

In that same paper,

We find a key: the bradshaw assumption

In the turbulent BL:

$$\tau_{turbulent} = \rho a_1 k$$

Where k is the turbulent kinetic energy

But how do we calculate reynold's shear stress normally?

$$\tau_{turbulent} = \rho \nu_t \frac{\partial u}{\partial y}$$

Or else we know $\frac{\partial U}{\partial y}$ is known as the rate of strain S, sometimes known as Ω

If we were to rewrite this in terms of turbulent KE (not going to show it here)

$$\tau_{turbulent} = \sqrt{\frac{production\ of\ k}{dissipation\ of\ k}}\rho a_1 k$$

For turbulent BL flows, for the Bradshaw assumption ie,

$$\tau_{turbulent} = \rho a_1 k$$

To be true,

$$\frac{production\ of\ k}{dissipation\ of\ k} = 1$$

Unfortunately in k-omega baseline model, this doesn't hold true.

So we need to artificially force:

$$\tau_{turb} = \rho a_1 k$$

In the turbulent BL and

$$\tau_{turb} = \rho \nu_t \left(\frac{\partial u}{\partial y}\right)$$

In the rest of the flow…

In other words, we resort to blending functions again!

Let's call the new blending function Z

$$\tau_{turb} = Z\rho a_1 k + (1 - Z)\rho \nu_t \left(\frac{\partial u}{\partial y}\right)$$

Similar to F_1 we have $Z = 1$ in turbulent BL, $Z = 0$ in the main fluid. (I wont use F2 yet, because it is used in the model differently)

This is the general idea, but instead of writing things like this, Menter uses a different form:

$$\tau_{turb} = \rho \nu_t \left(\frac{\partial u}{\partial y}\right)$$

But we redefine ν_t such that in the turbulent bulk/freestream

$$\nu_t = \frac{k}{\omega^*} = C_\mu \frac{k}{\omega}$$

In BL

ν_t is such that $\tau_{turb} = \rho a_1 k$

So to do that,

We equate

$$\rho a_1 k = \rho \nu_t \left(\frac{\partial u}{\partial y}\right)$$

$$\nu_t = \frac{a_1 k}{\left(\frac{\partial u}{\partial y}\right)} \ in\ the\ BL$$

This forces $\tau_{turb} = \rho a_1 k$ in the BL

$$\tau_{turb} = \rho \nu_t \left(\frac{\partial u}{\partial y}\right) = \frac{a_1 k}{\left(\frac{\partial u}{\partial y}\right)} \rho \left(\frac{\partial u}{\partial y}\right) = \rho a_1 k$$

The way this is done:

$$\nu_t = \frac{a_1 k}{\max\left(a_1 \omega^*, F_2 \left(\frac{\partial u}{\partial y}\right)\right)}$$

$$F_2 = \tanh\left[\left[\max\left(\frac{\sqrt{k}}{\omega y}, \frac{500 \nu C_\mu}{\omega y^2}\right)\right]^2\right]$$

$$F_2 = \tanh\left[\left[\max\left(\frac{\sqrt{k}}{C_\mu \omega^* y}, \frac{500 \nu}{\omega^* y^2}\right)\right]^2\right]$$

Again, $F_2 = 1$ in BL, $F_2 = 0$ in turbulent bulk (freestream or centre of pipeflow) so that

$$F\left(\frac{\partial u}{\partial y}\right) > a_1 \omega \ in\ turbulent\ sublayer$$

As this limits production of turbulent shear stress or turbulent viscosity in the turbulent sublayer, this modification is known as the **production limiter**

$$\nu_t = \frac{a_1 k}{\max\left(a_1 \omega^*, F_2 \left(\frac{\partial u}{\partial y}\right)\right)}$$

in 3D, we don't just have u velocity and y, hence we use strain rate S instead.

$$S = \left(\frac{\partial u}{\partial y}\right) \ in\ 3D\ (i'm\ oversimplifying\ but\ yeah\ \ldots)$$

$$\nu_t = \frac{a_1 k}{\max(a_1 \omega^*, F_2 S)}$$

But producing this change now poses a problem in the sublayer, since we artificially suppress turbulent viscosity, and this kind of upsets the balance of transport equations for k and ω

So we have to modify them…

What are the changes?

1) Some of the constants have to be tweaked from the original $k - \omega$ model
2) The production term of ω must be written a certain way
    a. In original $k - \omega$ / baseline model, $P_\omega = \alpha \frac{1}{\nu_t} \tau_{ij} \frac{\partial u_i}{\partial x_j} = \alpha \frac{\omega^*}{k} \tau_{ij} \frac{\partial u_i}{\partial x_j}$
    b. In SST model, $\nu_t \neq \frac{k}{\omega^*}$, so we can't use this same formulation

i. To remove ambiguity, $P_\omega = \alpha S^2 = \alpha \frac{\omega^*}{k} \tau_{ij} \frac{\partial u_i}{\partial x_j}$

In most CFD guides eg. Cfd online, we won't see this comparison being made,

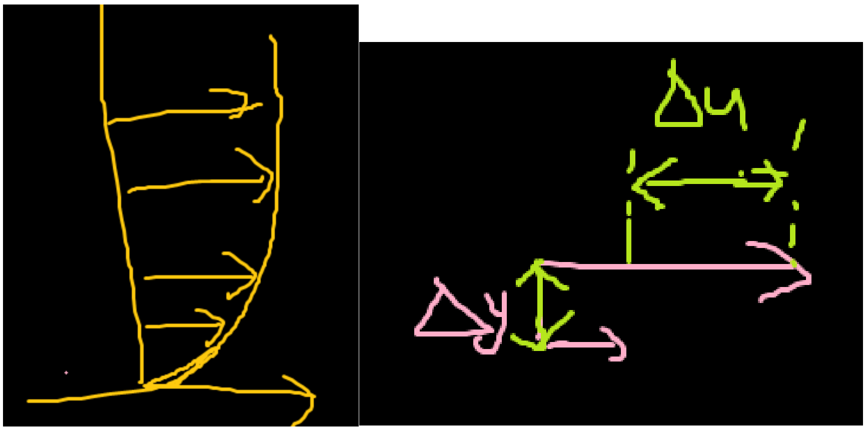
ν_t is removed from the expression even in $k - \omega$ model

This is the 1994 version of the $k - \omega$ SST model.

<u>Problem 2: in stagnation regions</u>

Large normal strain produces excessive turbulent kinetic energy…

Normal strain $= \frac{\partial U}{\partial y}$



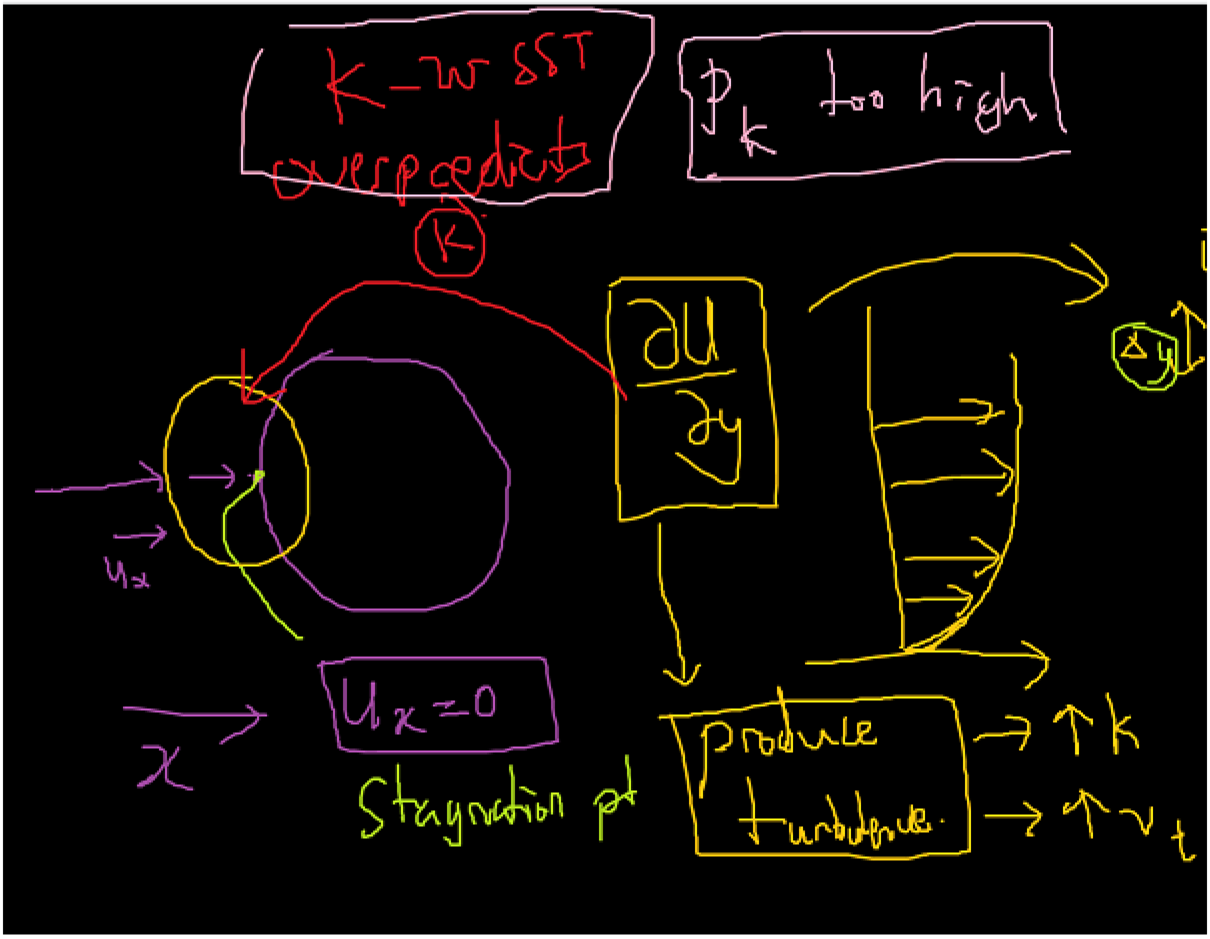
Characteristic length scale $= \Delta y$

Characteristic velocity scale $= \Delta u = \frac{\partial u}{\partial y}(strain\,rate)\Delta y$

$$Re = \frac{Ux}{\nu} = \frac{\Delta u \Delta y}{\nu} = \frac{\Delta y^2 \frac{\partial u}{\partial y}(strain\,rate)}{\nu}$$

Increasing strain rate increases Re

Eg in stagnation regions… take a sphere for example.



To compensate for this, a limiter is used on P_k in the turbulent kinetic energy equation

$$P_k = \min\left(\tau_{ij}\frac{\partial u_i}{\partial x_j}, 10\,\beta^* k\omega^*\right)$$

$$\beta^*\omega^*k = \varepsilon$$

We can P_k at 10ε= 10x dissipation rate.

https://www.openfoam.com/documentation/guides/latest/doc/guide-turbulence-ras-k-omega-sst.html

https://turbmodels.larc.nasa.gov/sst.html

http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.460.2814&rep=rep1&type=pdf

Menter, F. R., Kuntz, M., & Langtry, R. (2003). Ten years of industrial experience with the SST turbulence model. Turbulence, heat and mass transfer, 4(1), 625-632.

This is known as the 2003 update, as of 2020, OpenFOAM uses this model…

All right so this is $k - \omega$ SST for you!

Summary of key features

- Note that a lot of it was data fitting, empirical testing…
    - K-omega in BL, k-epsilon in bulk region
    - Shear stress production limiter in adverse pressure gradient BL region (redefines ν_t to reduce it artificially)
    - Turbulent KE production limiter in stagnation regions (limit P_k)

SpalartAllmaras Model

Why study this when we have $k-\omega\ SST$?

1) Sometimes if you like simplicity, this is good, easy to understand/intuitive model
2) Developers of the model also developed Detached Eddy Simulation
    a. Hybrid between RANS and LES

https://www.researchgate.net/publication/236888804_A_One-Equation_Turbulence_Model_for_Aerodynamic_Flows

Spalart, P., & Allmaras, S. (1992, January). A one-equation turbulence model for aerodynamic flows. In 30th aerospace sciences meeting and exhibit (p. 439).

S-A model Fluid Mechanics 101

https://www.youtube.com/watch?v=Xivc0EIGFQw

Follow up paper

http://ae.metu.edu.tr/tuncer/ae546/docs/ICCFD7-1902.pdf

from the paper:

Development of transport equations…

Previously, $k-\omega$ and $k-\varepsilon$ are used

Here

It's just the kinematic viscosity → easy to use and low computation cost (1 less transport equation)

Derviation/Thought Process in development:

Closure model

$$-\overline{u_i'u_j'} = 2\nu_t S_{ij}$$

So we want a transport equation for turbulent kinematic viscosity, Remember?

$$\nu_t = C_\mu \frac{k^2}{\varepsilon} = C_\mu \frac{k}{\omega} = \frac{k}{\omega^*}$$

$$\omega \equiv \frac{k}{\varepsilon}, \omega = \omega^* C_\mu = \omega^* \beta^*$$

Yes this is our friend, turbulent kinematic viscosity or eddy diffusivity

So we'll have our standard terms:

$$\frac{\partial \nu_t}{\partial t} + u_j \frac{\partial \nu_t}{\partial x_j} = RHS$$

What's in the right hand side?

We'll have a source of turbulent viscosity

Confused viscosity $\nu_{LOL}^{?}$

Remember our turbulence viscosity has its source in the strain rate $\frac{\partial U}{\partial y}$, remember the dimensional argument?

- Just a note though, I said S and Ω were similar in meaning in the last video, but they're NOT
  - $S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$, this is strain rate tensor, rate of deformation of fluid
    - $S = \sqrt{2 S_{ij} S_{ij}}$ (strain rate), velocity gradients…
  - $\Omega_{ij} = \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$, this is not the same, and it's more closely related to vorticity (curl of fluid)
    - $\vec{\omega} = \vec{\nabla} \times \vec{u} \equiv \sqrt{\Omega_{ij}\Omega_{ij}}$ (vorticity), tendency of fluid to twist/curl

Source term:

$$source = c_{b1}\, \nu_t\, S$$

Diffusion terms: (see k Omega SST/ or other RANS models)

$$diffusion = \frac{1}{\sigma}\{\nabla \cdot (\nu_t \nabla \nu_t)\}$$

But there are extra terms that pop up… why?

Recall our k and omega equations…

$$\frac{\partial k}{\partial t} + u_j \frac{\partial k}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right)\frac{\partial k}{\partial x_j}\right) + \frac{P_{k(dynamic)}}{\rho} - \beta^* \omega^* k$$

$$\frac{\partial \omega^*}{\partial t} + u_i \frac{\partial \omega^*}{\partial x_i} = \frac{\partial}{\partial x_j}\left(\nu + \frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial \omega^*}{\partial x_j} + (1 - F_1) 2\left(\frac{1}{\omega^*}\right)\frac{\partial \omega^*}{\partial x_j}\frac{\partial k}{\partial x_j} + \alpha S^2 - \beta \omega^{*2}$$

Remember $\omega = \beta^* \omega^*$

Try combining both into one long equation for ν_t

$$\nu_t = \frac{k}{\omega^*}$$

$$k = \omega^* \nu_t$$

Substitute into the k equation

$$\frac{\partial \omega^* \nu_t}{\partial t} + u_j \frac{\partial \omega^* \nu_t}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right) \frac{\partial \omega^* \nu_t}{\partial x_j} \right) + \frac{P_{k(dynamic)}}{\rho} - \beta^* \omega^* \omega^* \nu_t$$

Try differentiating it all the way and subtract the ω^* equation from it…

What happens to the viscosity term?

$$\frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right) \frac{\partial \omega^* \nu_t}{\partial x_j} \right)$$

noting

$$\frac{\partial \omega^* \nu_t}{\partial x_j} = \nu_t \frac{\partial \omega^*}{\partial x_j} + \omega^* \frac{\partial \nu_t}{\partial x_j}$$

$$\frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right) \left\{ \nu_t \frac{\partial \omega^*}{\partial x_j} + \omega^* \frac{\partial \nu_t}{\partial x_j} \right\} \right)$$

$$= \frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right) \omega^* \frac{\partial \nu_t}{\partial x_j} \right) + \frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right) \nu_t \frac{\partial \omega^*}{\partial x_j} \right)$$

$$\frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right) \omega^* \frac{\partial \nu_t}{\partial x_j} \right)$$
$$= \omega^* \frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right) \frac{\partial \nu_t}{\partial x_j} \right) + \frac{\partial \nu_t}{\partial x_j} \frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right) \frac{\partial \nu_t}{\partial x_j} \right)$$

$$\frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right) \nu_t \frac{\partial \omega^*}{\partial x_j} \right)$$
$$= \nu_t \frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right) \frac{\partial \omega^*}{\partial x_j} \right) + \frac{\partial \omega^*}{\partial x_j} \frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right) \nu_t \right)$$

$$\frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right) \frac{\partial \omega^* \nu_t}{\partial x_j} \right)$$

$$= \omega^* \frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right) \frac{\partial \nu_t}{\partial x_j} \right) + \frac{\partial \nu_t}{\partial x_j} \frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right) \frac{\partial \nu_t}{\partial x_j} \right)$$
$$+ \nu_t \frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right) \frac{\partial \omega^*}{\partial x_j} \right) + \frac{\partial \omega^*}{\partial x_j} \frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right) \nu_t \right)$$

$$\frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right) \nu_t \right) = constant * \frac{\partial}{\partial x_j}\left(\nu_t^2\right) = constant * \nabla\left(\nu_t^2\right)$$

Near wall treatment

<u>Log law layer</u>

Destruction of turbulence near wall

After some dimensional analysis…

$$-c_{w1}\left(\frac{\nu_t}{d}\right)^2$$

$$w = wall$$

In free flows, $d \to \infty$ so doesn't really matter

After some testing: show a great prediction for velocity profile, log law

$$\frac{\partial \nu_t}{\partial t} + u_j \frac{\partial \nu_t}{\partial x_j} = -c_{w1}\left(\frac{\nu_t}{d}\right)^2 + c_{b1}\,\nu_t\,S + \frac{1}{\sigma}\{\nabla \cdot (\nu_t \nabla \nu_t) + c_{b2}(\nabla \nu_t^2)\}$$

Produces a good $u^+ = f(y^+)$ in log law region

Provided have some suitable wall function for VSL region

Problem: skin friction coefficient too low (underpredict wall shear stress)

- ➔ Destruction term "decays too slowly" in outer BL, - in the paper
- ➔ Need to decrease destruction of turbulent viscosity in this area

Introduce a nondimensional coefficient to compensate for this…

- ➔ In log law region, $f_w = 1$
- ➔ It should help dampen the destruction term in the outer BL

Idea:

- Take inspiration from mixing length model
- Decay it with distance from the wall relation

Nondimnesional group

$$r = \left(\frac{mixing\ length\ scale}{characterisc\ distance\ from\ the\ wall}\right)^2 = \left(\frac{\sqrt{\frac{\nu_t}{S}}}{\kappa d}\right)^2 = \frac{\nu_t}{S\kappa^2 d^2}$$

$$f_w = g = \left(r + c_{w2}(r^6 - r)\right)$$

<u>Problem: f_w gets too high in bulk fluid! (upsets numerical stability)</u>

Dampen it again…

$$g = \left(r + c_{w2}(r^6 - r)\right)$$

Damping function for numerical stability= $\left[\frac{1+c_{w3}^6}{g^6+c_{w3}^6}\right]^{\frac{1}{6}}$

$$f_w = g\left[\frac{1 + c_{w3}^6}{g^6 + c_{w3}^6}\right]^{\frac{1}{6}}$$

$\left[\frac{1+c_{w3}^6}{g^6+c_{w3}^6}\right]^{\frac{1}{6}}$ prevents f_w from getting too big and thus upsetting numerical stability

<u>Viscous sublayer</u>

<u>Problem: How can we force</u>

$$\underline{u^+ = y^+}$$

<u>Here? And also correct predictions in the buffer layer?</u>

In log law region

$$\nu_t = \kappa y u_*$$

But in the VSL, this doesn't hold true!

How can we get it to work…

Can we take the turbulent kinematic viscosity above (log law region correct turbulent kinematic viscosity) and dampen it?

In the VSL, buffer reigon, we have this:

$$\nu_t = \kappa y u_* * damping\ function$$

Damping function for VSL/buffer layer

$$f_{v1} = \frac{\chi^3}{\chi^3 + c_{v1}^3}$$

$$v = viscous$$

$$\chi = \kappa y^+$$

$$\frac{v_t\ (disclaimer)}{v} = \frac{\kappa y u_*}{v} = \kappa y^+ = \chi$$

Disclaimer:

The turbulence viscosity, v_t in the log law region and the bulk region of the fluid is called:

$$v_t = \tilde{v}\ (in\ bulk\ region\ and\ log - law\ region)$$

$$\chi = \frac{\tilde{v}}{v} = \kappa y^+$$

(definition starts in the log law region)

$$\frac{\partial \tilde{v}}{\partial t} + u_j \frac{\partial \tilde{v}}{\partial x_j} = -c_{w1}\left(\frac{\tilde{v}}{d}\right)^2 + c_{b1}\,\tilde{v}\,S + \frac{1}{\sigma}\{\nabla \cdot (\tilde{v}\nabla\tilde{v}) + c_{b2}(\nabla\tilde{v}^2)\}$$

So that’s how we introduce the <u>spalart allmaras variable ($\tilde{v}$)</u>

$$v_t = \tilde{v}\frac{\chi^3}{\chi^3 + c_{v1}^3}$$

$$f_{v1} = \frac{\chi^3}{\chi^3 + c_{v1}^3}$$

<u>Problem: too much production of turbulence in VSL</u>

Dampen it again!

We replace S with $\tilde{S}$

$$\tilde{S} = f(S)$$

$$\tilde{S} = S + \frac{\tilde{v}}{\kappa^2 d^2} f_{v2}$$

$$f_{v2} = 1 - \frac{\chi}{1 + \chi f_{v1}}$$

$$f_{v1} = \frac{\chi^3}{\chi^3 + c_{v1}^3}$$

Now our equation looks like this:

$$\frac{\partial \tilde{v}}{\partial t} + u_j \frac{\partial \tilde{v}}{\partial x_j} = -c_{w1} f_w \left(\frac{\tilde{v}}{d}\right)^2 + c_{b1}\,\tilde{v}\,\tilde{S} + \frac{1}{\sigma}\{\nabla \cdot (\tilde{v}\nabla\tilde{v}) + c_{b2}(\nabla\tilde{v}^2)\}$$

So far, we have dealt with:

- Log law region (with f_w)

- VSL ($\tilde{\nu}, \tilde{S}$)
- Bulk region

Problem: numerical instabilities present in this formulation

Long story short,

$\tilde{\nu} = 0$ causes the solvers to blow up… (skipping explanation)

Laminar to turbulent transition for laminar region shear layers

What is a shear layer?

http://thermopedia.com/content/1118/

we need to set a "**trip**" in order to prevent instability, allow $\tilde{\nu} = 0$ in VSL to be a stable solution and not cause blow-ups.

To do so, artificially dampen production term to zero in the BL

Now that we artificially reduce production of $\tilde{\nu}$ in BL, we also need to artificially reduce the destruction of $\tilde{\nu}$ in BL and transition region.

To artificially reduce production term:

$$\frac{\partial \tilde{\nu}}{\partial t} + u_j \frac{\partial \tilde{\nu}}{\partial x_j} = -c_{w1} f_w \left(\frac{\tilde{\nu}}{d}\right)^2 + c_{b1}[1 - f_{t2}]\, \tilde{\nu}\, \tilde{S} + \frac{1}{\sigma}\{\nabla \cdot (\tilde{\nu} \nabla \tilde{\nu}) + c_{b2}(\nabla \tilde{\nu}^2)\}$$

To artificially reduce the destruction term:

$$\frac{\partial \tilde{\nu}}{\partial t} + u_j \frac{\partial \tilde{\nu}}{\partial x_j} = -[c_{w1} f_w - \frac{c_{b1}}{\kappa^2} f_{t2}] \left(\frac{\tilde{\nu}}{d}\right)^2 + c_{b1}[1 - f_{t2}]\, \tilde{\nu}\, \tilde{S} + \frac{1}{\sigma}\{\nabla \cdot (\tilde{\nu} \nabla \tilde{\nu}) + c_{b2}(\nabla \tilde{\nu}^2)\}$$

See f_{t2}

$$f_{t2} = c_{t3} * \exp(-c_{t4} \chi^2)$$

Remember

$$\chi = \frac{\tilde{\nu}}{\nu} = \kappa y^+$$

T stands for trip

Problem: transition from laminar to turbulent layer not smooth

We sort of kaboomed our BL $\tilde{\nu}$ source and destruction terms by doing the above (trip terms, f_{t2})…

How can we generate turbulence in the BL?

See f_{t1} and ΔU

$$\frac{\partial \tilde{v}}{\partial t} + u_j \frac{\partial \tilde{v}}{\partial x_j} = -\left[c_{w1} f_w - \frac{c_{b1}}{\kappa^2} f_{t2}\right]\left(\frac{\tilde{v}}{d}\right)^2 + c_{b1}[1 - f_{t2}]\, \tilde{v}\, \tilde{S} + \frac{1}{\sigma}\{\nabla \cdot (\tilde{v} \nabla \tilde{v}) + c_{b2}(\nabla \tilde{v}^2)\} + f_{t1} \Delta U^2$$

$$\Delta U = norm\ of\ difference\ between\ trip\ point, and\ point\ where\ we\ are\ calculating$$

Trip point→ remember we wanted to trip our production and destruction terms in the turbulent/laminar transition part of BL…

http://brennen.caltech.edu/fluidbook/basicfluiddynamics/turbulence/lawofthewall.pdf

now what does f_{t1} look like?

$$f_{t1} = c_{t1} g_t \exp\left(-c_{t2} \frac{\omega_t^2}{\Delta U^2}[d^2 + g_t^2 d_t^2]\right)$$

$$g_t \equiv \min\left(0.1, \frac{\Delta U}{\omega_t \Delta x_t}\right)$$

$$\Delta x_t = grid\ spacing\ along\ wall\ at\ trip$$

$$\vec{\omega}_t = vorticity\ of\ wall\ at\ trip\ point \equiv \nabla \times \vec{u}_{trip}$$

log law region → turbulent BL

VSL → laminar BL

Buffer zone → transition region…

https://mathworld.wolfram.com/Norm.html

what is a norm?

well in this context, this is…

if $\vec{x}$ is a vector

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$norm\ (\vec{x}) = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

So it means like for a position, it is the length between two points

Technically correct way: L2 norm

In this case, it's the magnitude difference in velocity,

So it's somewhat like ΔU we are used to

So this serves as the term to build up turbulence somewhat in the transition zone….

That's it!

The end product:

$$\frac{\partial \tilde{\nu}}{\partial t} + u_j \frac{\partial \tilde{\nu}}{\partial x_j} = -\left[c_{w1} f_w - \frac{c_{b1}}{\kappa^2} f_{t2}\right] \left(\frac{\tilde{\nu}}{d}\right)^2 + c_{b1}[1 - f_{t2}]\, \tilde{\nu}\, \tilde{S} + \frac{1}{\sigma}\{\nabla \cdot (\tilde{\nu} \nabla \tilde{\nu}) + c_{b2} (\nabla \tilde{\nu}^2)\} + f_{t1} \Delta U^2$$

https://www.cfd-online.com/Wiki/Spalart-Allmaras_model

Wall Functions

So, what about near the wall region, is it truly turbulent?

Let's consider the near wall region… (Pope, 2001)

Are the inertial forces at the wall really so powerful as to cause turbulence at the wall?

We know there is a viscous sublayer(VSL) near the wall…

What are the equations applicable here?

Let's use the shear stress

$$\frac{\tau_{app}}{\rho} = (\nu + \nu_t) \frac{\partial \bar{u}}{\partial y}$$

For the model, we need to do some data fitting (experimental).

To prepare the equation for data fitting, we need to nondimensionalise.

Recall how we calculate friction factor in Moody chart?

We can use similar methods for wall region modelling of viscous sublayer…

$$\frac{\tau_{app}}{\rho} = (\nu + \nu_t)\frac{\partial \bar{u}}{\partial y}$$

We want to nondimensionalise u and y.

What is a good velocity scale, or characteristic velocity?

$$\frac{\tau_{app}}{\rho}\left[\frac{m^2}{s^2}\right] = (\nu + \nu_t)\left[\frac{m^2}{s}\right]\frac{\partial \bar{u}}{\partial y}\left[\frac{1}{s}\right]$$

$$\left(\frac{\tau_{app}}{\rho}\right)^{\frac{1}{2}}$$

For sake of having a good reference point, we take τ_{app} at y=0… (at the wall)

At the wall, there can be **no** turbulence, so $\nu_t = 0$

$$\left(\frac{\tau_{wall}}{\rho}\right)^{\frac{1}{2}} = \left(\nu\frac{\partial \bar{u}}{\partial y}\right)^{\frac{1}{2}} = friction\ velocity = u_*$$

So for $\bar{u}$ the nondimensional form is:

$$u^+ = \frac{\bar{u}}{u_*} = \frac{\bar{u}}{\left(\frac{\tau_{wall}}{\rho}\right)^{\frac{1}{2}}}$$

We also want to nondimensionalise y…

So what is the appropriate length scale?

Let's make use of what we have already:

We have a characteristic friction velocity:

$$u_* = \left(\frac{\tau_{wall}}{\rho}\right)^{\frac{1}{2}}\left[\frac{m}{s}\right]$$

We need some length scale…

One good length scale is this:

$$length\ scale\ [m] = \frac{\nu\left[\frac{m^2}{s}\right]}{\left(\frac{\tau_{wall}}{\rho}\right)^{\frac{1}{2}}\left[\frac{m}{s}\right]} = \frac{\nu}{u_*}[m]$$

We can substitute this back to get a nondimensional form of this equation…

$$\frac{\tau_{app}}{\rho} = (\nu + \nu_t)\frac{\partial \bar{u}}{\partial y}$$

$$u_* = \left(\frac{\tau_{wall}}{\rho}\right)^{\frac{1}{2}}\left[\frac{m}{s}\right]$$

$$y_* = \frac{\nu}{u_*}[m]$$

$$\frac{\tau_{app}}{\rho}\frac{1}{u_*} = (\nu + \nu_t)\frac{\partial \bar{u}}{\partial y}\frac{1}{u_*}$$

$$\frac{\tau_{app}}{\rho}\frac{1}{u_*} = (\nu + \nu_t)\frac{\partial u^+}{\partial y}$$

$$u^+ = \frac{\bar{u}}{u_*} \to du^+ = d\bar{u} * \frac{1}{u_*}$$

$$\frac{\tau_{app}}{\rho}\frac{y_*}{u_*} = (\nu + \nu_t)\frac{\partial u^+}{\partial y}y_*$$

$$\frac{\tau_{app}}{\rho}\frac{y_*}{u_*} = (\nu + \nu_t)\frac{\partial u^+}{\partial y^+}$$

$$\frac{\tau_{app}}{\rho}\frac{\nu}{u_*^2} = (\nu + \nu_t)\frac{\partial u^+}{\partial y^+}$$

$$\frac{\tau_{app}}{\rho}\frac{1}{u_*^2} = \left(1 + \frac{\nu_t}{\nu}\right)\frac{\partial u^+}{\partial y^+}$$

$$noting\ u_* = \left(\frac{\tau_{wall}}{\rho}\right)^{\frac{1}{2}}$$

$$\frac{\tau_{app}}{\rho}\frac{1}{\left(\frac{\tau_{wall}}{\rho}\right)} = \left(1 + \frac{\nu_t}{\nu}\right)\frac{\partial u^+}{\partial y^+}$$

$$\frac{\tau_{app}}{\tau_{wall}} = \left(1 + \frac{\nu_t}{\nu}\right)\frac{\partial u^+}{\partial y^+}$$

Let’s make an assumption considering the boundary layer….

We can assume

$\tau_{app} \approx constant \approx \tau_{wall}$ in the BL region close to the wall (Prandtl’s assumptions)

$$1 = \left(1 + \frac{\nu_t}{\nu}\right)\frac{\partial u^+}{\partial y^+}$$

So we have a simpler kind of equation, and now we can separate into two regimes in the BL

$$1 = \left(1 + \frac{\nu_t}{\nu}\right)\frac{\partial u^+}{\partial y^+}$$

Regime 1: $\nu \gg \nu_t$ (viscous sublayer)

$$1 = \frac{\partial u^+}{\partial y^+}$$

$$u^+ = y^+ + C$$

We need a boundary condition…

$$at\ y^+ = 0, u^+ = 0\ (no\ slip)$$

$$u^+ = y^+\ (viscous\ sublyaer\ equation)$$

Regime 2: $\nu_t \gg \nu$ (fully turbulent sublayer)

$$1 = \left(\frac{\nu_t}{\nu}\right)\frac{\partial u^+}{\partial y^+}$$

$$-\overline{u'v'} = \nu_t \frac{\partial \bar{u}}{\partial y}$$

We can do order of magnitude estimate for $\overline{u'v'}$ → prandtl's mixing length model

Let's visualise fluid motion in turbulence, usually they come in the form of eddies (we can think of eddies being balls of fluid going round in circles)

The maximum distance this ball travels on average without disintegrating is the **mixing length** l,

What is the biggest velocity fluctuation within this mixing length?

$$\bar{u}_2 - \bar{u}_1 \approx u'$$

$$\frac{\partial \bar{u}}{\partial y} = \frac{\bar{u}_2 - \bar{u}_1}{l} \approx \frac{u'}{l}$$

$$\frac{\partial \bar{u}}{\partial y} \approx \frac{v'}{l}$$

$$\frac{u'v'}{l^2} = \left(\frac{\partial \bar{u}}{\partial y}\right)^2$$

$$-\overline{u'v'} = l^2\left(\frac{\partial \bar{u}}{\partial y}\right)^2 = \nu_t\left(\frac{\partial \bar{u}}{\partial y}\right)$$

$$\nu_t = l^2\left(\frac{\partial \bar{u}}{\partial y}\right)$$

So for l

We need this in terms of y^+... How?

$$l = \kappa y$$

What is κ?

- Experiments show: $\kappa \approx 0.4$

Let's substitute back...

$$1 = \left(\frac{l^2\left(\frac{\partial \bar{u}}{\partial y}\right)}{\nu}\right)\frac{\partial u^+}{\partial y^+}$$

$$1 = \left(\frac{(\kappa y)^2 \left(\frac{\partial \bar{u}}{\partial y} \right)}{\nu} \right) \frac{\partial u^+}{\partial y^+}$$

$$\bar{u} = u^+ u_*$$

$$y = y^+ y_*$$

$$1 = \left(\frac{(\kappa y^+ y_*)^2 \frac{u_*}{y_*} \left(\frac{\partial u^+}{\partial y^+} \right)}{\nu} \right) \frac{\partial u^+}{\partial y^+}$$

$$1 = \left(\frac{(\kappa y^+)^2 y_* u_* \left(\frac{\partial u^+}{\partial y^+} \right)}{\nu} \right) \frac{\partial u^+}{\partial y^+}$$

$$u_* = \left(\frac{\tau_{wall}}{\rho} \right)^{\frac{1}{2}} \left[\frac{m}{s} \right]$$

$$y_* = \frac{\nu}{u_*} [m]$$

$$1 = \left(\frac{(\kappa y^+)^2 \nu \left(\frac{\partial u^+}{\partial y^+} \right)}{\nu} \right) \frac{\partial u^+}{\partial y^+}$$

$$1 = \left(\frac{(\kappa y^+)^2 \left(\frac{\partial u^+}{\partial y^+} \right)}{1} \right) \frac{\partial u^+}{\partial y^+}$$

$$1 = (\kappa y^+)^2 \left(\frac{\partial u^+}{\partial y^+} \right)^2$$

- ➔ Integrate the equation directly

$$u^+ = \frac{1}{\kappa} \ln y^+ + y^+_{VSL} - \frac{1}{\kappa} \ln y^+_{VSL}$$

$$y^+_{VSL} = y^+ \; where\ we\ transition\ from\ VSL\ regime\ to\ the\ turbulent\ sublayer\ regime$$

$$\pm 1 = (\kappa y^+) \left(\frac{\partial u^+}{\partial y^+} \right)$$

We take positive root because u^+ should increase with y^+

$$1 = (\kappa y^+) \left(\frac{\partial u^+}{\partial y^+} \right)$$

$$u^+ = B + \int \frac{1}{\kappa y^+} dy^+$$

$$u^+ = A\ln y^+ + B$$

What are the limits of integration?

Remember the applicable region is the turbulent sublayer→ so that $\nu_t \gg \nu$

At which y^+ does this condition hold?

- ➔ $y^+ > 11.6\,, \kappa = 0.4$
- ➔ $A \approx 2.5, B \approx 5.5$

Based on expts:

$$y^+ < 11.6$$

$$u^+ = y^+\ (viscous\ sublyaer\ equation)$$

$$y^+ > 11.6$$

$$u^+ = 2.5\ln y^+ + 5.5\ (\log\ \ law\ of\ the\ wall)$$

This is how we fit the expt data to find velocity profile inside the near wall region → here, turbulence isn't the same as the bulk fluid.

Van Driest model for near wall…

$$1 = \left(1 + \frac{\nu_t}{\nu}\right)\frac{\partial u^+}{\partial y^+}$$

$$\frac{\partial u^+}{\partial y^+} = \frac{1}{\left(1 + (\kappa y^+)^2\left(\frac{\partial u^+}{\partial y^+}\right)\right)}$$

If we take a look at the Van Driest model…

$$\frac{du^+}{dy^+} = \frac{2}{1 + \left(1 + 4\kappa^2 y^{+2}\left[1 - \exp\left(-\frac{y^+}{A^+}\right)\right]^2\right)^{\frac{1}{2}}}$$

Where $A^+ = 26$

Van driest mixing length

$$l \sim \kappa y\left(1 - \exp\left(-\frac{y^+}{A^+}\right)\right)$$

The exponential term accounts for the viscous effects on mixing length

https://en.wikipedia.org/wiki/Law_of_the_wall

Potential confusion:

$$1 = \left(1 + \frac{\nu_t}{\nu}\right)\frac{\partial u^+}{\partial y^+}$$

$$\frac{\tau_{app}}{\tau_{wall}} = \left(1 + \frac{\nu_t}{\nu}\right)\frac{\partial u^+}{\partial y^+}$$

Let's make an assumption considering the boundary layer….

We can assume

$\tau_{app} \approx constant \approx \tau_{wall}$ in the BL region close to the wall (Prandtl's assumptions)

As the apparent viscosity changes, $\frac{\partial u^+}{\partial y^+}$ changes to keep τ_{app} constant.

$$\frac{du^+}{dy^+} = \frac{2}{1 + \left(1 + 4\kappa^2 y^{+2}\left[1 - \exp\left(-\frac{y^+}{A^+}\right)\right]^2\right)^{\frac{1}{2}}}$$

$$\kappa = 0.4, A^+ = 26$$

Once we have u^+ and y^+ we still need our ultimate velocity profile in BL, u vs y,

$$u_* = \left(\frac{\tau_{wall}}{\rho}\right)^{\frac{1}{2}}\left[\frac{m}{s}\right]$$

$$u^+ = \frac{\bar{u}}{u_*}$$

$$u_* = \left(\frac{\tau_{wall}}{\rho}\right)^{\frac{1}{2}}$$

Question is what is the wall shear stress?

We can find τ_{wall} experimentally…

For flat plate case:

$$C_{f,x} = 0.37 * \left[\log_{10}\left(\frac{U_\infty x}{\nu}\right)\right]^{-2.584}$$

Valid: $Re = 5 * 10^5\ to\ 10^9$

Schultz-Grunow correlation

https://www.ecourses.ou.edu/cgi-bin/eBook.cgi?doc=&topic=fl&chap_sec=09.3&page=theory

turbulent transition → $Re_{Cr} = 5 * 10^5$

From experiment: we fit a curve to the data

$$C_{f,x} = \frac{\tau_{wall,x}}{\frac{1}{2}\rho U_\infty^2}$$

If let's say we have a pipe/duct

We use the moody chart:

https://en.wikipedia.org/wiki/Moody_chart

$$f = \frac{\tau_{wall}}{\frac{1}{2}\rho U_{ave}^2}$$

$$U_{ave} = \frac{\dot{m}}{\rho A} = \frac{\dot{V}}{A_{crossSectional}}$$

What about other kinds of flow?

How can we estimate this friction coefficient?

We think of BL flow:

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} = -\frac{1}{\rho}\frac{\partial \bar{P}}{\partial x} + \frac{\partial}{\partial y}\left[(\nu + \nu_t)\frac{\partial \bar{u}}{\partial y}\right]$$

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} = -\frac{1}{\rho}\frac{\partial \bar{P}}{\partial x} + \frac{\partial}{\partial y}\left[\frac{\tau_{app}}{\rho}\right]$$

To make things simple, we assume: $\frac{\partial \bar{P}}{\partial x} = 0$, $\bar{v} \approx negligble$

$$\bar{u}\frac{\partial \bar{u}}{\partial x} = \frac{\partial}{\partial y}\left[\frac{\tau_{app}}{\rho}\right]$$

For pipe flow, slightly different, fully developed flow…

$$\bar{v}\frac{\partial \bar{u}}{\partial y} = -\frac{1}{\rho}\frac{\partial \bar{P}}{\partial x} + \frac{\partial}{\partial y}\left[\frac{\tau_{app}}{\rho}\right]$$

$$\bar{v} = 0$$

$$\frac{1}{\rho}\frac{\partial \bar{P}}{\partial x} = \frac{\partial}{\partial y}\left[\frac{\tau_{app}}{\rho}\right]$$

Back to BL equation,

$$\int_0^\infty \bar{u}\frac{\partial \bar{u}}{\partial x}\, dy = \left[\frac{\tau_{app}}{\rho}\right]$$

As long as you integrate over the entire BL, that's enough.

$$\int_0^\infty \bar{u}\frac{\partial \bar{u}}{\partial x}\, dy = \left[\frac{\tau_{app}}{\rho}\right]$$

From laminar BL calculations…

$$\frac{\partial}{\partial x}\int_0^\infty \bar{u}(U_\infty - \bar{u})\, dy = \left[\frac{\tau_{app}}{\rho}\right]$$

https://ocw.mit.edu/courses/mechanical-engineering/2-20-marine-hydrodynamics-13-021-spring-2005/lecture-notes/lecture17.pdf

$$\frac{\partial}{\partial x}\int_0^\infty \bar{u}(U_\infty - \bar{u})\, dy = \left[\frac{\tau_{app}}{\rho}\right]$$

http://web.mit.edu/13.021/demos/lectures/lecture17.pdf

$$\frac{u}{U_\infty} = \left(\frac{y}{\delta}\right)^{\frac{1}{7}}$$

We can use the 1/7th power law to fit our experimental data in the BL,

$$u^+ = 8.7(y^+)^{\frac{1}{7}}$$

Substitute back in here:

$$\frac{\partial}{\partial x}\int_0^\infty \bar{u}(U_\infty - \bar{u})\, dy = \left[\frac{\tau_{app}}{\rho}\right]$$

After substitution, we realise that…

$$\frac{\tau_{app}}{\rho U_\infty^2} = 0.0225\left(\frac{U_\infty \delta}{\nu}\right)^{-\frac{1}{4}}$$

$$\frac{\delta}{x} = 0.37 * \left(\frac{U_\infty x}{\nu}\right)^{-\frac{1}{5}}$$

$$\frac{\tau_{app}}{\rho U_\infty^2} = \frac{1}{2}C_{f,x} = 0.0296\left(\frac{U_\infty x}{\nu}\right)^{-\frac{1}{5}}$$

➔ This expression works for Re from turbulent transition (5e5 to 1e8)

The useful bit is that we derived this expression without performing additional experimental work to measure $C_{f,x}$ directly, the only thing is to have a function which describes

In summary, if we have:

$$u^+ = f(y^+)$$

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} = -\frac{1}{\rho}\frac{\partial \bar{P}}{\partial x} + \frac{\partial}{\partial y}\left[\frac{\tau_{app}}{\rho}\right]$$

And we substitute this into our N-S equation (RANS), we can estimate $C_{f,x}$ rather than measure it from experiment. As long as those experiments span the Re which you want.

Rough Wall Treatments

Correction to Previous video:

And $u^+ = f(y^+)$, is this a universal law, this law of the wall?

Experimental data shows so for channels, flat plate (BL) flow, and pipes, so as a universal law, we can use that as a good guestimate. (Adrian Bejan)

What about rough pipes?

Recall: Moody's Chart

Eg. Engineering toolbox websites

Moody Chart

https://www.engineeringtoolbox.com/moody-diagram-d_618.html

Typical Roughness constants

https://www.engineeringtoolbox.com/surface-roughness-ventilation-ducts-d_209.html

Roughness does not affect the laminar regime, but affects the turbulent regime!!!

What does this mean?

Usually in laminar sublayer, roughness does not play much of a role

But if roughness protrudes into the turbulent layer, it increases friction factor.

How does this play a role in the wall function?

Conversion Chart

https://www.mactechonsite.com/wp-content/uploads/Surface-Roughness-Conversion-Chart.pdf

Source code for OpenFOAM rough wall Boundary Condition

https://develop.openfoam.com/Development/openfoam/blob/master/src/TurbulenceModels/turbulenceModels/derivedFvPatchFields/wallFunctions/nutWallFunctions/nutkRoughWallFunction/nutkRoughWallFunctionFvPatchScalarField.C

Video by Fluid Mechanics 101

https://www.youtube.com/watch?v=vYbRUmVTmGM

Accounting for wall roughness effects in turbulence models : a wall function approach

https://www.eucass.eu/doi/EUCASS2017-372.pdf

In that paper sand grain height h_s is an estimate of surface roughness.

How does this relate to the surface roughness in moody chart?

- The moody chart is **absolute roughness**
- In CFD models, for wall functions, eg OpenFOAM Ks is used
  - https://www.symscape.com/node/1350
  - https://www.youtube.com/watch?v=vYbRUmVTmGM
- To correlate both, we can see the Nikuradse paper
  - https://ntrs.nasa.gov/archive/nasa/casi.ntrs.nasa.gov/19930093938.pdf

in the paper,

λ is the friction factor discussed (Darcy friction factor)

Note that there are **several friction factors** as you would know

Darcy friction factor $f_{Darcy} = \frac{64}{Re}$ in the laminar regime, that's how you can tell it's Darcy,

Fanning friction factor $f_{fanning} = 4f_{Darcy} = \frac{16}{Re}$

https://neutrium.net/fluid_flow/pressure-loss-in-pipe/

So just check the number on top if you're not sure.

In Nikuradse's paper, k is the surface roughness, otherwise known as ε in the chart, r is the radius, but Moody's chart converted it to diameter so its easier for engineers to use.

Surface roughness is usually listen in mm, eg. For moody chart in Perry's chemical engineering handbook.

https://en.wikipedia.org/wiki/Moody_chart#/media/File:Moody_EN.svg

Get this (trivia):

Nikuradse was Ludwig Prandtl's PhD student, both were Germans

And Prandtl was involved in Nazi Germany's Regime, and there was quite some political involvement. Yes the same guy with which we get our Prandtl number. He was involved with Nazi Germany's air force (Luftwaffe) research, can u believe it???

Here's the paper in case you're interested

https://www.sciencedirect.com/science/article/pii/S1631072117300815

anyway,

important findings from Nikuradse's paper

BL thickness if greater than average projection (surface roughness), then can be treated like smooth pipe in terms of frictional losses.

If BL thickness is approximately height of projects (ie surface roughness height), then we get increase in pressure drop as Re increases, this is shown by the initial part of flattening out of the curve.

Moody chart for reference

https://www.engineeringtoolbox.com/moody-diagram-d_618.html

In Nikuradse's paper

$$v_* = \sqrt{\frac{\tau_0}{\rho}} = friction\ velocity$$

The dimensionless surface roughness using friction velocity and viscosity to normalise is:

$$\eta = \frac{v_* k}{\nu};$$

Otherwise

$$\varepsilon^+ = \frac{\varepsilon v_*}{\nu}$$

This is a terminology we're more familiar with…

So now the juicy bit, how all this stuff we're familiar with relates to the wall function

In Nikuradse's paper

$$\frac{2.83}{\sqrt{\lambda}} - \left(5.75 \log_{10}\frac{r}{k} - \beta\right) = A$$

$$\frac{u}{v_*} - 5.75 \log_{10}\frac{r}{k} = \frac{2.83}{\sqrt{\lambda}} - \left(5.75 \log_{10}\frac{r}{k} - \beta\right)$$

Smooth pipe Region ($\delta_{VSL} > \varepsilon$)

$$0 \leq \log_{10}\frac{v_* k}{\nu} \leq 0.55$$

$$A = 5.5 + 5.75 \log_{10}\frac{v_* k}{\nu}$$

Transition Region

$$I.\, 0.55 \leq \log_{10}\frac{v_* k}{\nu} \leq 0.85\,;\, A = 6.59 + 3.5 \log_{10}\frac{v_* k}{\nu}$$

$$II. 0.85 \leq \log_{10} \frac{v_* k}{\nu} \leq 1.15; A = 9.58$$

$$III. 1.15 \leq \log_{10} \frac{v_* k}{\nu} \leq 1.83 \,; A = 11.5 - 1.62 \log_{10} \frac{v_* k}{\nu}$$

Fully protruding region / fully rough…

$$\log_{10} \frac{v_* k}{\nu} > 1.83 \;\; ; \quad A = 8.48$$

comparing these two papers, the logarithms used in Nikuradse's paper are **log base 10** not natural logs.

$$10^{1.83} = 67.6, \exp 1.83 = 6.233$$

Of course what you see in the more recent paper is using natural logarithms, that's why the numbers change slightly

The above is called the Karman Nikuradse equation

http://www.thermopedia.com/content/789/

litmus test as to whether this k (or ε) is the same as the one in the moody chart,

$$\lambda = \frac{1}{\left(1.74 + 2 \log_{10} \frac{r}{k}\right)^2}$$

Compare with the flat part of the Moody Chart

In moody diagram, we use $\frac{\varepsilon}{D_h} = \frac{\varepsilon}{2r} = \frac{k}{2r}$

$$\lambda = \frac{1}{\left(1.74 + 2 \log_{10} \frac{r}{k}\right)^2}$$

$$\frac{\varepsilon}{D_h} = \frac{\varepsilon}{2r} = \frac{k}{2r}; \frac{\varepsilon}{D_h} = relative\ roughness$$

$$relative\ roughness = \frac{k}{2r} = \frac{1}{2}\frac{k}{r}$$

$$\frac{k}{r} = 2\ \frac{\varepsilon}{D_h} = 2 * relative\ roughness$$

$$\lambda = \frac{1}{\left(1.74 + 2 \log_{10} \frac{r}{k}\right)^2}$$

$$\lambda = \frac{1}{\left(1.74 - 2\log_{10}\frac{k}{r}\right)^2}$$

$$\lambda = \frac{1}{\left(1.74 - 2\log_{10}\left[2\,\frac{\varepsilon}{D_h}\right]\right)^2}$$

relative roughness	Darcy/Moody friction factor (**Moody Chart) - FLAT PART**	Darcy/Moody Friction Factor (**equation)**
0.002	~0.026	0.023409485
0.004	~0.029	0.028400845
0.006	~0.032	0.032097909
0.008	~0.036	0.035177001
0.01	~0.038	0.037881044
0.015	~0.044	0.043661497
0.02	~0.05	0.048604545
0.03	~0.059	0.057132003
0.04	~0.065	0.064620606
0.05	~0.0715	0.071491893

So based on the readings,

k in the Darcy friction factor plot

https://www.engineeringtoolbox.com/moody-diagram-d_618.html

https://www.engineeringtoolbox.com/surface-roughness-ventilation-ducts-d_209.html

is the same as the ε, and is the same as h_s

https://www.eucass.eu/doi/EUCASS2017-372.pdf

and both are the same in here:

https://ntrs.nasa.gov/archive/nasa/casi.ntrs.nasa.gov/19930093938.pdf

$$\Delta u^+ = \frac{1}{0.41} * \ln\left(1 + \frac{h_s^+}{\exp(3.25 * 0.41)}\right)$$

where

$$h_s^+ = \frac{k * u_*}{\nu} = \frac{\varepsilon * u_*}{\nu}$$

https://www.engineeringtoolbox.com/surface-roughness-ventilation-ducts-d_209.html

So yeah, that accounts for surface roughness, but nevertheless

The following correlation by Van Driest still useful for smooth surface turbulence modelling.

$$\frac{du^+}{dy^+} = \frac{2}{1 + \left\{1 + 4\kappa^2 {y^+}^2 \left[1 - \exp\left(-\frac{y^+}{A^+}\right)\right]^2\right\}^{0.5}}$$

$$A^+ = 26, \kappa = 0.4$$

There are other correlations as well but Van Driest is one of the more well known ones.

Using these correlations, we can develop a model for the velocities in the viscous sublayer near the wall…

Most of these are based on experimental data.

https://www.simscale.com/docs/content/simulation/model/OF_thermalModelsBoussinesq.html

Surface Roughness in OpenFoam

https://www.openfoam.com/documentation/guides/latest/doc/guide-turbulence-ras-wall-modelling.html

Wall function source code

https://develop.openfoam.com/Development/openfoam/tree/master/src/TurbulenceModels/turbulenceModels/derivedFvPatchFields/wallFunctions

from the C code file in the above link,

code looks complicated so let's start with smooth bit first

$$yp = \frac{kappaRe + yp(last)}{1.0 + \ln\big(E * yp(last)\big)}$$

So, technically we iterate such that $yp = yp(last)$ to some level of tolerance

$$1 + \ln(E * yp) = \frac{kappaRe}{yp} + 1$$

$$kappaRe = \kappa * Re$$

$$Re = u * \frac{y}{\nu_w}$$

Note this also:

$$y^+ = \frac{y u_*}{\nu}$$

$$y = y^+ \frac{\nu}{u_*}$$

$$\nu_{wall} = \nu\ (no\ turbulence\ there!)$$

So this Re is

$$Re = u * \frac{y^+ \frac{\nu}{u_*}}{\nu_w} = \frac{u}{u_*} * y^+$$

$$kappaRe = \kappa * \frac{u}{u_*} * y^+$$

Sub back in:

$$1 + \ln(E * yp) = \frac{kappaRe}{yp} + 1$$

$$\ln(E * yp) = \frac{kappaRe}{yp}$$

$$\ln(E * yp) = \frac{\kappa * \frac{u}{u_*} * y^+}{yp} = \kappa u^+$$

$$u^+ = \frac{1}{\kappa} \ln(E * yp)$$

$$u^+ = \frac{1}{\kappa} \ln(E * y^+)$$

This is an iterative formula, not timestep based

Here, E carries a value of 9.8

How does this compare to:

$$u^+ = \frac{1}{\kappa} \ln y^+ + C - \Delta u^+; \quad u^+ = \frac{u}{u_\tau} \quad y^+ = \frac{y u_\tau}{\nu} \quad C = 5.5 \quad \kappa = 0.40$$

$$u^+ = \frac{1}{\kappa} \ln \frac{y^+}{h_s^+} + B$$

Where:

$$1 < h_s^+ < 3.5 \quad B = 5.5 + \frac{1}{\kappa} \ln h_s^+$$

$$3.5 < h_s^+ < 7 \quad B = 6.59 + 1.52 \ln h_s^+$$

$$7 < h_s^+ < 14 \quad B = 9.58$$

$$14 < h_s^+ < 68 \quad B = 11.5 - 0.7 \ln h_s^+$$

$$68 < h_s^+ \quad B = 8.48$$

Let's start comparing…

$$u^+ = \frac{1}{\kappa} \ln y^+ + C$$

$$u^+ = \frac{1}{\kappa} \ln y^+ + \ln(\exp C)$$

However we need to bring in the κ

$$u^+ = \frac{1}{\kappa} \ln y^+ + \frac{1}{\kappa} \ln(\exp C)^\kappa$$

$$u^+ = \frac{1}{\kappa} \ln[y^+ * (\exp C)^\kappa]$$

$$(\exp C)^\kappa = (\exp 5.5)^{0.41} = 9.53529331$$

Compare with:

$$E = 9.8$$

Close enough I guess!

$$u^+ = \frac{1}{\kappa} \ln[y^+ * 9.53529]$$

From the paper for smooth walls

From equation

I look for nutU wall function

https://develop.openfoam.com/Development/openfoam/tree/master/src/TurbulenceModels/turbulenceModels/derivedFvPatchFields/wallFunctions/nutWallFunctions

deals with velocity and also roughness!

Stuff is here:

https://develop.openfoam.com/Development/openfoam/tree/master/src/TurbulenceModels/turbulenceModels/derivedFvPatchFields/wallFunctions/nutWallFunctions/nutURoughWallFunction

most of the good stuff happens on line 101 onwards…

in the above one..

Okay now for the roughness bit

Look at line 115 onwards

What we know:

For smooth walls

$$u^+ = \frac{1}{\kappa} \ln y^+ + C$$

Becomes

$$u^+ = \frac{1}{\kappa} \ln(Ey^+)$$

$$E = 9.8, \kappa = 0.41, C = 5.566$$

For rough walls:

$$u^+ = \frac{1}{\kappa} \ln(Ey^+) - \Delta u^+$$

$$\Delta u^+ = \frac{1}{\kappa} * \ln\left(1 + \frac{h_s^+}{\exp(3.25\kappa)}\right)$$

What does OpenFOAM use?

$$\frac{dK_s^+}{dy^+} = \frac{h_s}{y} = \frac{\varepsilon}{y} = \frac{\varepsilon^+}{y^+}$$

In C++ code

$$A *= B$$

This means

$$A_{new}\ is\ defined\ as\ A_{old} * B$$

So we have:

$$\frac{dK_s^+}{dy^+} = \frac{\varepsilon}{y} * F_{roughness}$$

By default $F_{roughness} = 1$

Line 134

$$K_s^+ = y^+ \frac{dK_s^+}{dy^+} = y^+ \left(\frac{\varepsilon}{y} F_{roughness}\right)$$

$$K_s^+ = F_{roughness}\, y^+ \left(\frac{\varepsilon^+}{y^+}\right) = F_{roughness}\, \varepsilon^+ = F_{roughness}\, h_s^+$$

1st regime $K_s^+ > 90$ (line 141)

$$t_1 = 1 + K_{roughness} K_s^+$$

You can also adjust $K_{roughness}$, but default value is 0.5

$$G = \ln(1 + K_{roughness} K_s^+)$$

$$y^+ G' = \frac{K_{roughness} K_s^+}{1 + K_{roughness} K_s^+}$$

What relevance are G and G' I don't know yet, let's keep looking

OpenFOAM defines a denominator: (line 158)

$$denom = 1.0 + \ln(Ey^+) - G - y^+ G'$$

At line 161

$$y^+ = \frac{\kappa Re + y^+ * (1 - y^+ G')}{1.0 + \ln(Ey^+) - G - y^+ G'}$$

$$u^+ = \frac{1}{\kappa} \ln(Ey^+) - \Delta u^+$$

$$\Delta u^+ = \frac{1}{\kappa} * \ln\left(1 + \frac{h_s^+}{\exp(3.25\kappa)}\right)$$

Let's break it down…

$$y^+ = \frac{\kappa Re + y^+ * (1 - y^+ G')}{1.0 + \ln(Ey^+) - G - y^+ G'}$$

$$1.0 + \ln(Ey^+) - G - y^+ G' = \frac{\kappa Re + y^+ * (1 - y^+ G')}{y^+}$$

$$1.0 + \ln(Ey^+) - G - y^+ G' = \frac{\kappa Re + y^+ * (1 - y^+ G')}{y^+}$$

$$1.0 + \ln(Ey^+) - G - y^+ G' = \kappa u^+ + 1 - y^+ G'$$

$$\ln(Ey^+) - G - y^+ G' = \kappa u^+ - y^+ G'$$

$$\ln(Ey^+) - G = \kappa u^+$$

$$u^+ = \frac{1}{\kappa}(\ln(Ey^+) - G)$$

$$u^+ = \frac{1}{\kappa} \ln(Ey^+) - \frac{1}{\kappa} G$$

Compare to:

$$u^+ = \frac{1}{\kappa} \ln(Ey^+) - \Delta u^+$$

$$\Delta u^+ = \frac{1}{\kappa} * \ln\left(1 + \frac{h_s^+}{\exp(3.25\kappa)}\right)$$

So in effect,

$$\frac{1}{\kappa}G = \frac{1}{\kappa} * \ln\left(1 + \frac{h_s^+}{\exp(3.25\kappa)}\right)$$

$$G = \ln\left(1 + \frac{h_s^+}{\exp(3.25\kappa)}\right)$$

What is G in openFoam nut wallfunction?

$$G = \ln(1 + K_{roughness}K_s^+)$$

$$if\ K_{roughness} = \frac{1}{\exp(3.25 * 0.41)} = 0.263816894$$

Not exactly 0.5, but we can adjust as so if we wish…

We can also use:

$$K_s^+ = F_{roughness}\ y^+\left(\frac{\varepsilon^+}{y^+}\right) = F_{roughness}\ \varepsilon^+ = F_{roughness}\ h_s^+$$

$$K_s^+ = F_{roughness}\ h_s^+$$

We can adjust

$$K_{roughness}K_s^+ = 0.5K_s^+$$

As long as

$$0.5 * F_{roughness}h_s^+ = \frac{1}{\exp(3.25 * 0.41)}h_s^+ = 0.2638 * h_s^+$$

So we roughly understand the first regime, let's see the second regime…

$$F_{roughness} \approx 0.5276$$

$$K_s^+ = F_{roughness}h_s^+$$

$$h_s^+ = \frac{K_s^+}{F_{roughness}}$$

2nd regime $2.25 < K_s^+ < 90$

Let's find out what G is…

From comparison, G is supposed to be:

$$G = \ln\left(1 + \frac{h_s^+}{\exp(3.25\kappa)}\right)$$

This is common among 1st and 2nd regime, but

G is defined differently here

From line 153

$$G = (\ln t_1)(\sin t_2)$$

$$t_1 = C_1 K_s^+ - C_2$$

$$t_2 = C_3 \ln K_s^+ - C_4$$

From line 104 onwards

$$C_1 = \frac{1}{90 - 2.25} + K_{roughness}$$

$$C_2 = \frac{2.25}{90 - 2.25}$$

$$C_3 = 2 * \frac{\arctan 1}{\ln \frac{90}{2.25}}$$

$$C_4 = C_3 * \ln 2.25 = 2 * \frac{\arctan 1}{\ln \frac{90}{2.25}} * \ln 2.25$$

3rd regime $K_s^+ < 2.25$, smooth wall treatment

At the end of the day,

If we use $F_{roughness} = 0.5276, K_{roughness} = 0.5$, we can sort of get reasonably good results.

These constants match colebrook's data reasonably well.

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