Turbulence Modelling – LES and Hybrid Methods

This document outlines some LES and Hybrid RANS/LES methods such as DES.

K-Omega SST Model

To have the best of both worlds, we have a blend of k-Omega and k-Epsilon in the k-Omega SST model.

So the models switches to k-Omega near the wall and k-Epsilon near the free stream.

Dr Aiden (Fluid Mechanics 101)

https://www.youtube.com/watch?v=myv-ityFnS4

$$\frac{\partial k}{\partial t} + u_j \frac{\partial k}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right)\frac{\partial k}{\partial x_j}\right) + \frac{P_{k(dynamic)}}{\rho} - \varepsilon$$

$$\varepsilon = \beta^* \omega^* k;\ \omega = \omega^* \beta^* = \omega^* C_\mu;\ \beta^* = 0.09$$

$$\frac{\partial k}{\partial t} + u_j \frac{\partial k}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right)\frac{\partial k}{\partial x_j}\right) + \frac{P_{k(dynamic)}}{\rho} - \beta^* \omega^* k$$

$$\frac{\partial k}{\partial t} + u_j \frac{\partial k}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right)\frac{\partial k}{\partial x_j}\right) + \frac{P_{k(dynamic)}}{\rho} - \omega k$$

(note: $k - \varepsilon, k - \omega$ models, k equation remains the same-ish, so $k - \omega\ SST$ is also the same)

From $k - \varepsilon$

$$\frac{\partial \omega}{\partial t} + u_i \frac{\partial \omega}{\partial x_i} = \frac{\partial}{\partial x_j}\left(\nu + \frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial \omega}{\partial x_j} + \frac{2}{k}\left(\nu + \frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial \omega}{\partial x_j}\frac{\partial k}{\partial x_j} + P[C_{\varepsilon 1} - 1]\frac{\omega}{k} - \omega^2[C_{\varepsilon 2} - 1]$$

From $k - \omega$

$$\frac{\partial \omega}{\partial t} + u_i \frac{\partial \omega}{\partial x_i} = \frac{\partial}{\partial x_j}\left(\nu + \frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial \omega}{\partial x_j} + P\alpha\frac{\omega}{k} - \omega^2\frac{\beta}{\beta^*}$$

$$\omega \equiv \frac{\varepsilon}{k}$$

Let's say we want to blend…

In deriving the $k - \omega\ SST$ formulation,

$$\frac{\partial \omega}{\partial t} + u_i \frac{\partial \omega}{\partial x_i} = \frac{\partial}{\partial x_j}\left(\nu + \frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial \omega}{\partial x_j} + (f_{blending})\frac{2}{k}\left(\nu + \frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial \omega}{\partial x_j}\frac{\partial k}{\partial x_j} + P[C_{\varepsilon 1} - 1]\frac{\omega}{k} - \omega^2[C_{\varepsilon 2} - 1]$$

$$f_{blending} = 1 - F_1, 0 < F_1 < 1$$

At the wall, $F_1 \to 1$, in bulk region $F_1 \to 0$

$$\frac{\partial\omega}{\partial t}+u_i\frac{\partial\omega}{\partial x_i}=\frac{\partial}{\partial x_j}\left(\nu+\frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial\omega}{\partial x_j}+(1-F_1)\frac{2}{k}\left(\nu+\frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial\omega}{\partial x_j}\frac{\partial k}{\partial x_j}+\ P[C_{\varepsilon 1}-1]\frac{\omega}{k}-\omega^2[C_{\varepsilon 2}-1]$$

$$\left(\nu+\frac{\nu_T}{\sigma_\varepsilon}\right)\approx\frac{\nu_T}{\sigma_\varepsilon}\ in\ turbulent\ bulk\ fluid\ region$$

$$\nu_T\approx C_\mu\frac{k^2}{\varepsilon}=\beta^*\frac{k^2}{\varepsilon}$$

$$\left(\nu+\frac{\nu_T}{\sigma_\varepsilon}\right)*\frac{1}{k}\approx\frac{\beta^*}{\sigma_\varepsilon}\frac{k^2}{\varepsilon}\frac{1}{k}=\frac{\beta^*}{\sigma_\varepsilon}\frac{k}{\varepsilon}=\frac{\beta^*}{\sigma_\varepsilon}\frac{1}{\omega}=\frac{1}{\sigma_\varepsilon}\frac{1}{\omega}$$

$$\omega=\omega^*\beta^*\rightarrow\omega^*=\frac{\omega}{\beta^*}$$

To make it look like the

$$\frac{2}{k}\left(\nu+\frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial\omega}{\partial x_j}\frac{\partial k}{\partial x_j}\approx 2\frac{1}{\sigma_\varepsilon}\frac{1}{\omega}\frac{\partial\omega}{\partial x_j}\frac{\partial k}{\partial x_j}\ (only\ applies\ when\frac{\nu_t}{\sigma_\varepsilon}\gg\nu, ie\ far\ from\ wall)$$

With blending function

$$2(1-F_1)\frac{1}{\sigma_\varepsilon}\frac{1}{\omega}\frac{\partial\omega}{\partial x_j}\frac{\partial k}{\partial x_j}\rightarrow 0\ at\ the\ wall, otherwise\ finite\ far\ from\ wall$$

The blending function F_1. F_1 is the degree to which $k-\omega$ model activates

$$F_1=\tanh\left\{\left\{\min\left[\max\left(\frac{\sqrt{k}}{\omega y},\frac{500\nu C_\mu}{\omega y^2}\right),\frac{4\sigma_{\omega 2}k}{CD_{k\omega}y^2}\right]\right\}^4\right\}$$

So what do $\frac{\sqrt{k}}{\omega y}$, $\frac{500\nu C_\mu}{\omega y^2}$ and $\frac{4\sigma_{\omega 2}k}{CD_{k\omega}y^2}$

We want to discuss near the wall. Why?

Because we want $k-\omega$ to activate near the wall. $F_1\rightarrow 1$

How does F_1 change in Viscous Sublayer (VSL) and then the log law region (turbulent sublayer), see how this velocity profile goes for large y^+ values.

Of course, this is for flat plate, BL flow, which we can assume applies to most flow types, **except convection and flow with strong pressure gradient→ this negates the constant shear stress hypothesis**

$$\frac{\tau_{app}}{\rho}=(\nu+\nu_t)\frac{\partial\bar{u}}{\partial y}$$

Eventually we want to see how F_1 varies with y or y^+...

For that, we need wall fns for k and ω

$\frac{\sqrt{k}}{\omega y}$, $\frac{500\nu C_\mu}{\omega y^2}$ and $\frac{4\sigma_{\omega 2}k}{CD_{k\omega}y^2}$

ω should vary with y^+ and k also...

How did this k equation come about

$$\frac{\partial k}{\partial t}+u_j\frac{\partial k}{\partial x_j}=\frac{\partial}{\partial x_j}\left(\left(\nu+\sigma_{K(k\omega SST)}\nu_t\right)\frac{\partial k}{\partial x_j}\right)+\frac{P_{k(dynamic)}}{\rho}-\omega k$$

Nondimensionalising it…

$$k^+=\frac{k}{u_*^2},\omega^+=\frac{\omega\nu}{u_*^2}$$

$$u_*=\sqrt{\frac{\tau_{wall}}{\rho}}$$

We can substitute:

$$k=k^+u_*^2\,;\omega=\frac{\omega^+u_*^2}{\nu}$$

$$\frac{\partial k}{\partial t}+u_j\frac{\partial k}{\partial x_j}=\frac{\partial}{\partial x_j}\left(\left(\nu+\sigma_{K(k\omega SST)}\nu_t\right)\frac{\partial k}{\partial x_j}\right)+\frac{P_{k(dynamic)}}{\rho}-\omega k$$

$$u_*^2\left(\frac{\partial k^+}{\partial t}+u_j\frac{\partial k^+}{\partial x_j}\right)=u_*^2\frac{\partial}{\partial x_j}\left(\left(\nu+\sigma_{K(k\omega SST)}\nu_t\right)\frac{\partial k^+}{\partial x_j}\right)+\frac{P_{k(dynamic)}}{\rho}-\frac{\omega^+u_*^2}{\nu}k^+u_*^2$$

$$u_*^2\left(\frac{\partial k^+}{\partial t}+u_j\frac{\partial k^+}{\partial x_j}\right)=u_*^2\frac{\partial}{\partial x_j}\left(\left(\nu+\sigma_{K(k\omega SST)}\nu_t\right)\frac{\partial k^+}{\partial x_j}\right)+\frac{P_{k(dynamic)}}{\rho}-\frac{\omega^+k^+u_*^4}{\nu}$$

$$\left(\frac{\partial k^+}{\partial t}+u_j\frac{\partial k^+}{\partial x_j}\right)=\frac{\partial}{\partial x_j}\left(\left(\nu+\sigma_{K(k\omega SST)}\nu_t\right)\frac{\partial k^+}{\partial x_j}\right)+\frac{P_{k(dynamic)}}{\rho}-\frac{\omega^+k^+u_*^2}{\nu}$$

$$u_j=u_j^+u_*$$

$$\left(\frac{\partial k^+}{\partial t}+u_j^+u_*\frac{\partial k^+}{\partial x_j}\right)=\frac{\partial}{\partial x_j}\left(\left(\nu+\sigma_{K(k\omega SST)}\nu_t\right)\frac{\partial k^+}{\partial x_j}\right)+\frac{P_{k(dynamic)}}{\rho}-\frac{\omega^+k^+u_*^2}{\nu}$$

So in BL, we assume

$$\frac{\partial k^+}{\partial t}+u_j^+u_*\frac{\partial k^+}{\partial x_j}\approx 0$$

In BL, the production of turbulent KE is negligible (more so in VSL)

$$\frac{P_{k(dynamic)}}{\rho}\approx 0$$

So we get:

$$0=\frac{\partial}{\partial x_j}\left(\left(\nu+\sigma_{K(k\omega SST)}\nu_t\right)\frac{\partial k^+}{\partial x_j}\right)-\frac{\omega^+k^+u_*^2}{\nu}$$

$$0 = \frac{\partial}{\partial x_j}\left(\left(1 + \sigma_{K(k\omega SST)}\frac{\nu_t}{\nu}\right)\frac{\partial k^+}{\partial x_j}\right) - \frac{\omega^+ k^+ u_*^2}{\nu^2}$$

In BL, we only consider y direction, so we simplify

$$0 = \frac{\partial}{\partial y}\left(\left(1 + \sigma_{K(k\omega SST)}\frac{\nu_t}{\nu}\right)\frac{\partial k^+}{\partial y}\right) - \frac{\omega^+ k^+ u_*^2}{\nu^2}$$

$$y = \frac{y^+ \nu}{u_*}$$

$$0 = \frac{\partial}{\partial \frac{y^+ \nu}{u_*}}\left(\left(1 + \sigma_{K(k\omega SST)}\frac{\nu_t}{\nu}\right)\frac{\partial k^+}{\partial \frac{y^+ \nu}{u_*}}\right) - \frac{\omega^+ k^+ u_*^2}{\nu^2}$$

$$0 = \frac{u_*^2}{\nu^2}\frac{\partial}{\partial y^+}\left(\left(1 + \sigma_{K(k\omega SST)}\frac{\nu_t}{\nu}\right)\frac{\partial k^+}{\partial y^+}\right) - \frac{\omega^+ k^+ u_*^2}{\nu^2}$$

$$0 = \frac{u_*^2}{\nu^2}\frac{\partial}{\partial y^+}\left(\left(1 + \sigma_{K(k\omega SST)}\frac{\nu_t}{\nu}\right)\frac{\partial k^+}{\partial y^+}\right) - \frac{\omega^+ k^+ u_*^2}{\nu^2}$$

$$0 = \frac{\partial}{\partial y^+}\left(\left(1 + \sigma_{K(k\omega SST)}\frac{\nu_t}{\nu}\right)\frac{\partial k^+}{\partial y^+}\right) - \omega^+ k^+$$

In VSL, we assume $\nu_t \ll \nu$

$$0 = \frac{\partial}{\partial y^+}\left(\frac{\partial k^+}{\partial y^+}\right) - \omega^+ k^+$$

I think im missing a C_μ

$$\omega = \omega^* \beta^* \to \omega^* = \frac{\omega}{\beta^*}, \beta^* = C_\mu$$

If we use ω_*

$$0 = \frac{\partial}{\partial y^+}\left(\frac{\partial k^+}{\partial y^+}\right) - C_\mu \omega^{*+} k^+$$

I'll skip the ω^{*+} derivation…

$$0 = \frac{\partial}{\partial y^+}\left(\frac{\partial \omega^{*+}}{\partial y^+}\right) - \beta_1 \left(\omega^{*+}\right)^2$$

We solve the ODE's in the VSL,

$$\omega^{*+} = \frac{6}{\beta_1 (y^+)^2}$$

$$\beta_1 = \frac{3}{40}\ (k - \omega\ equation\ constant)$$

Subs this into:

$$0 = \frac{\partial}{\partial y^+}\left(\frac{\partial k^+}{\partial y^+}\right) - C_\mu \frac{6}{\beta_1 (y^+)^2} k^+$$

And solve the ODE here, we get:

$$k^+ = C_k (y^+)^{3.23}$$

By intuition, there is no turbulence in the VSL,

So $C_k = 0$

We also have a log law region:

$$\omega^{*+} = \frac{1}{\kappa\sqrt{C_\mu} y^+}$$

$$k^{*+} = \frac{1}{\sqrt{C_\mu}}$$

To derive the above,

recall

$$\left(\frac{\partial k^+}{\partial t} + u_j^+ u_* \frac{\partial k^+}{\partial x_j}\right) = \frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)} \nu_t\right)\frac{\partial k^+}{\partial x_j}\right) + \frac{P_{k(dynamic)}}{\rho} - \frac{\omega^+ k^+ u_*^2}{\nu}$$

$$\left(\frac{\partial k^+}{\partial t} + u_j^+ u_* \frac{\partial k^+}{\partial x_j}\right) = 0$$

But for turbulent sublayer

$$\frac{P_{k(dynamic)}}{\rho} \neq 0$$

$$0 = \frac{\partial}{\partial y^+}\left(\left(1 + \sigma_{K(k\omega SST)} \frac{\nu_t}{\nu}\right)\frac{\partial k^+}{\partial y^+}\right) + \nu_t^+ \left|\frac{dU^+}{dy^+}\right|^2 - C_\mu \omega^{*+} k^+$$

$$\frac{P_{k(dynamic)}}{\rho} = \overline{u_i' u_j'} \frac{\partial U}{\partial y}$$

$$\frac{P_{k(dynamic)}}{\rho} = \nu_t \frac{\partial U}{\partial y}\frac{\partial U}{\partial y}$$

$$0 = \frac{\partial}{\partial y^+}\left(\left(1 + \sigma_{K(k\omega SST)} \frac{\nu_t}{\nu}\right)\frac{\partial k^+}{\partial y^+}\right) + \nu_t^+ \left|\frac{dU^+}{dy^+}\right|^2 - C_\mu \omega^{*+} k^+$$

In VSL, we assume $\nu_t \ll \nu$

But in turbulent sublayer, we assume $\nu_t \gg \nu$

$$0 = \frac{\partial}{\partial y^+}\left(\left(\sigma_{K(k\omega SST)}\frac{\nu_t}{\nu}\right)\frac{\partial k^+}{\partial y^+}\right) + \nu_t^+\left|\frac{dU^+}{dy^+}\right|^2 - C_\mu \omega^{*+} k^+$$

We assume here in turbulent sublayer:

$$\frac{\partial}{\partial y^+}\left(\left(\sigma_{K(k\omega SST)}\frac{\nu_t}{\nu}\right)\frac{\partial k^+}{\partial y^+}\right) \ll \nu_t^+\left|\frac{dU^+}{dy^+}\right|^2$$

$$\frac{\partial}{\partial y^+}\left(\left(\sigma_{K(k\omega SST)}\frac{\nu_t}{\nu}\right)\frac{\partial k^+}{\partial y^+}\right) \approx 0$$

$$\left(\sigma_{K(k\omega SST)}\frac{\nu_t}{\nu}\right)\frac{\partial k^+}{\partial y^+} = C_k$$

$$\nu_t^+\left|\frac{dU^+}{dy^+}\right|^2 - C_\mu \omega^{*+} k^+ = 0$$

$$\nu_t^+ = \frac{k^+}{\omega^{*+}}$$

$$\omega^{*+} = \frac{k^+}{\nu_t^+}$$

$$\nu_t^+\left|\frac{dU^+}{dy^+}\right|^2 - C_\mu \frac{k^+}{\nu_t^+} k^+ = 0$$

$$\nu_t^{+2}\left|\frac{dU^+}{dy^+}\right|^2 - C_\mu k^{+2} = 0$$

Or to eliminate k^+

$$k^+ = \omega^{*+}\nu_t^+$$

$$\nu_t^+\left|\frac{dU^+}{dy^+}\right|^2 - C_\mu \omega^{*+}\omega^{*+}\nu_t^+ = 0$$

$$\left|\frac{dU^+}{dy^+}\right|^2 - C_\mu \omega^{*+2} = 0$$

$$\omega^{*+} = \frac{\left|\frac{dU^+}{dy^+}\right|}{\sqrt{C_\mu}}$$

What is $\frac{dU^+}{dy^+}$ in log law region

$$u^+ = 2.5\ln y^+ + 5.5\ (\log\ \ law\ of\ the\ wall)$$

$$u^+ = \frac{1}{\kappa}\ln y^+ + 5.5, \kappa \approx 0.4$$

$$\frac{du^+}{dy^+} = \frac{1}{\kappa}\frac{1}{y^+}$$

Subs into the ω^{*+} and we get

$$\omega^{*+} = \frac{1}{\kappa y^+ \sqrt{C_\mu}}$$

Now we want to get back our k equation…

Recall:

$$k^+ = \omega^{*+} \nu_t^+$$

$$k^+ = \frac{1}{\kappa y^+ \sqrt{C_\mu}} \nu_t^+$$

What is ν_t^+

$$\tau_{wall} = \rho(\nu + \nu_t)\frac{\partial u}{\partial y}$$

Nondimensionalise,

$$1 = \left(1 + \frac{\nu_t}{\nu}\right)\frac{du^+}{dy^+}$$

In log law region:

$$\nu_t \gg \nu$$

So we get

$$\frac{\nu_t}{\nu}\frac{du^+}{dy^+} = 1$$

$$\nu_t^+ = \frac{1}{\frac{du^+}{dy^+}}$$

Substitute this back:

$$k^+ = \frac{1}{\kappa y^+ \sqrt{C_\mu}} \frac{1}{\frac{du^+}{dy^+}}$$

$$k^+ = \frac{1}{\kappa y^+ \sqrt{C_\mu}} \kappa y^+$$

$$k^+ = \frac{1}{\sqrt{C_\mu}}$$

This is k^+ for log law region…

How about k^+ in VSL

Recall:

$$k^+ = C_k (y^+)^{3.23}$$

In TSL,

$$\left(\sigma_{K(k\omega SST)}\frac{\nu_t}{\nu}\right)\frac{\partial k^+}{\partial y^+} = C_k$$

$$k^+ = \frac{1}{\sqrt{C_\mu}}, so\ \frac{\partial k^+}{\partial y^+} = 0$$

Therefore

$$C_k = 0$$

In intermediate region, we take root mean sqr of VSL and TSL ω^*

	VSL	Intermediate (buffer) layer	Turbulent Sublayer
k^+	$k^+ = C_k(y^+)^{3.23} = 0$ As $C_k = 0$	$k^+ = \omega^{*+}\nu_t^+$ $1 = \left(1 + \frac{\nu_t}{\nu}\right)\frac{du^+}{dy^+}$ In log law: $\frac{du^+}{dy^+} = \frac{1}{\kappa}\frac{1}{y^+}$ Or in VSL $\frac{du^+}{dy^+} = 1$ Or you can use Van Driest Model	$k^+ = \frac{1}{\sqrt{C_\mu}}$ $C_\mu = 0.09$
ω^{*+}	$\omega^{*+} = \frac{6}{\beta_1(y^+)^2}$ $\beta_1 = \frac{3}{40}$	ω^{*+} $= \sqrt{\omega^{*+2}_{VSL} + \omega^{*+2}_{TSL}}$	$\omega^{*+} = \frac{1}{\kappa y^+\sqrt{C_\mu}}$ $\kappa = 0.41$ $C_\mu = 0.09$

$\frac{\sqrt{k}}{C_\mu\omega^* y}, \frac{500\nu}{\omega^* y^2}$ and $\frac{4\sigma_{\omega 2}k}{CD_{k\omega}y^2}$

So first we want to nondimensionalise them…. So we can get these functions as a function of y^+

$$k^+ = \frac{k}{u_*^2}$$

$$\omega^{*+} = \frac{\omega^*\nu}{u_*^2}$$

$$y^+ = \frac{yu_*}{\nu}$$

We can nondimensionalise

$$\frac{\sqrt{k}}{C_\mu \omega^* y} = \frac{\sqrt{u_*^2 k^+}}{C_\mu \frac{\omega^{*+} u_*^2}{\nu} \frac{y^+ \nu}{u_*}} = \frac{u_* \sqrt{k^+}}{C_\mu \omega^{*+} u_* y^+} = \frac{\sqrt{k^+}}{C_\mu \omega^{*+} y^+}$$

Now we can substitute in…

$$y^+ < 11.6$$

$$u^+ = y^+ \ (viscous\ sublyaer\ equation)$$

$$y^+ > 11.6$$

$$u^+ = 2.5 \ln y^+ + 5.5 \ (\log\ \ law\ of\ the\ wall)$$

The next function is this:

$$\frac{500\nu}{\omega^* y^2}$$

$$y = \frac{y^+ \nu}{u_*}$$

$$\omega^* = \frac{\omega^{*+} u_*^2}{\nu}$$

Substitute in…

$$\frac{500\nu}{\omega^* y^2} = \frac{500\nu}{\frac{\omega^{*+} u_*^2}{\nu} \left(\frac{y^+ \nu}{u_*}\right)^2} = \frac{500\nu}{\frac{\omega^{*+} u_*^2}{\nu} \frac{y^{+2} \nu^2}{u_*^2}} = \frac{500\nu}{\omega^{*+} y^{+2} \nu} = \frac{500}{\omega^{*+} y^{+2}}$$

One last function…

$$\frac{4\sigma_{\omega 2} k}{CD_{k\omega} y^2}$$

Let's deal with the stuff outside…

$$\frac{4\sigma_{\omega 2} k}{y^2} = 4\sigma_{\omega 2} \frac{u_*^2 k^+}{\left(\frac{y^+ \nu}{u_*}\right)^2} = 4\sigma_{\omega 2} \frac{k^+}{y^{+2}} \frac{u_*^4}{\nu^2}$$

$$CD_{k\omega} = \max\left(2\rho\sigma_{\omega 2} \frac{1}{\omega^*} \frac{\partial \omega^*}{\partial x_j} \frac{\partial k}{\partial x_j}, 10^{-10}\right)$$

$$\omega^* = \frac{\omega^{*+} u_*^2}{\nu}$$

$$y = \frac{y^+ \nu}{u_*}$$

We can cheat to find an estimate in the boundary layer…

We can assume in the BL

$$\frac{\partial \omega^*}{\partial x_j}\frac{\partial k}{\partial x_j} = \frac{\partial \omega^*}{\partial y}\frac{\partial k}{\partial y}$$

In BL

$$CD_{k\omega} = \max\left(2\rho\sigma_{\omega 2}\frac{1}{\omega^*}\frac{\partial \omega^*}{\partial y}\frac{\partial k}{\partial y}, 10^{-10}\right)$$

$$\omega^* = \frac{\omega^{*+}u_*^2}{\nu}$$

$$y = \frac{y^+\nu}{u_*}$$

$$k = k^+u_*^2$$

Nondimensionalising

$$CD_{k\omega} = \max\left(2\rho\sigma_{\omega 2}\frac{1}{\frac{\omega^{*+}u_*^2}{\nu}}\frac{\partial\left(\frac{\omega^{*+}u_*^2}{\nu}\right)}{\partial\left(\frac{y^+\nu}{u_*}\right)}\frac{\partial(k^+u_*^2)}{\partial\left(\frac{y^+\nu}{u_*}\right)}, 10^{-10}\right)$$

$$CD_{k\omega} = \max\left(2\rho\sigma_{\omega 2}\frac{1}{\frac{\omega^{*+}u_*^2}{\nu}}\frac{\frac{u_*^2}{\nu}\,\partial\left(\omega^{*+}\right)}{\frac{\nu}{u_*}\,\partial(y^+)}\frac{u_*^2\partial(k^+)}{\frac{\nu}{u_*}\,\partial(y^+)}, 10^{-10}\right)$$

$$CD_{k\omega} = \max\left(2\rho\sigma_{\omega 2}\frac{1}{\omega^{*+}\nu}\frac{\frac{u_*^2}{\nu}\,\partial\left(\omega^{*+}\right)}{\partial(y^+)}\frac{u_*^2\partial(k^+)}{\partial(y^+)}, 10^{-10}\right)$$

$$CD_{k\omega} = \max\left(2\rho\sigma_{\omega 2}\frac{u_*^4}{\nu^2}\frac{1}{\omega^{*+}}\frac{\partial\left(\omega^{*+}\right)}{\partial(y^+)}\frac{\partial(k^+)}{\partial(y^+)}, 10^{-10}\right)$$

So for,

$$\frac{4\sigma_{\omega 2}k}{CD_{k\omega}y^2} = 4\sigma_{\omega 2}\frac{k^+}{y^{+2}}\frac{u_*^4}{\nu^2} * \frac{1}{CD_{k\omega}}$$

$$\frac{4\sigma_{\omega 2}k}{CD_{k\omega}y^2} = 4\sigma_{\omega 2}\frac{k^+}{y^{+2}}\frac{u_*^4}{\nu^2} * \frac{1}{\frac{u_*^4}{\nu^2}\max\left(2\rho\sigma_{\omega 2}\frac{1}{\omega^{*+}}\frac{\partial(\omega^{*+})}{\partial(y^+)}\frac{\partial(k^+)}{\partial(y^+)}, \frac{\nu^2}{u_*^4}10^{-10}\right)}$$

$$\frac{4\sigma_{\omega 2}k}{CD_{k\omega}y^2} = 4\sigma_{\omega 2}\frac{k^+}{y^{+2}} * \frac{1}{\max\left(2\rho\sigma_{\omega 2}\frac{1}{\omega^{*+}}\frac{\partial \omega^{*+}}{\partial y^+}\frac{\partial k^+}{\partial y^+}, \frac{\nu^2}{u_*^4}10^{-10}\right)}$$

Can we nondimensionalise $\frac{\nu^2}{u_*^4}$?

$$u_* = \sqrt{\frac{\tau_{wall}}{\rho}}$$

At the wall

$$\tau = \rho\nu\frac{\partial u}{\partial y}$$

$$u_* = \sqrt{\nu\frac{\partial u}{\partial y}}$$

Hang on, why is there an extra density term in the denominator?

$$F_1 = \tanh(\Gamma^4)$$

$$\Gamma = \min\left(\max\left(\frac{\sqrt{k}}{\beta^*\omega^* d};\frac{500\nu}{\omega^* d^2}\right);\frac{4\rho\sigma_{\omega 2}k}{CD_{k\omega}d^2}\right)$$

Hellsten, A. (1998). Some improvements in Menter's k-omega SST turbulence model. In 29th AIAA, Fluid Dynamics Conference (p. 2554).

http://cfd.mace.manchester.ac.uk/twiki/pub/Main/CDAdapcoMeetingsM4/AIAA_98-2554-CP.PS.pdf

This is from an earlier paper, and there **is** a density term in the numerator…

Okay so let's correct this:

$$\frac{4\rho\sigma_{\omega 2}k}{CD_{k\omega}y^2} = 4\sigma_{\omega 2}\frac{k^+}{{y^+}^2}\frac{\rho u_*^4}{\nu^2} * \frac{1}{CD_{k\omega}}$$

$$CD_{k\omega} = \max\left(2\sigma_{\omega 2}\frac{\rho u_*^4}{\nu^2}\frac{1}{{\omega^*}^+}\frac{\partial {\omega^*}^+}{\partial y^+}\frac{\partial k^+}{\partial y^+}, 10^{-10}\right)$$

Looks like this term is quite important

$$\frac{\rho u_*^4}{\nu^2}$$

$$u_* = \sqrt{\frac{\tau_{wall}}{\rho}}$$

$$\frac{\rho\left(\frac{\tau_{wall}}{\rho}\right)^2}{\nu^2} = \frac{\tau_{wall}^2}{\rho\nu^2}$$

At the wall

$$\tau_{wall} = \rho(\nu + \nu_t)\frac{\partial u}{\partial y}$$

At the wall, $\nu_t = 0$

$$\tau_{wall} = \rho\nu\frac{\partial u}{\partial y}$$

$$y = \frac{y^+\nu}{u_*}$$

$$u = u^+u_*$$

$$\tau_{wall} = \rho\nu\frac{\partial u^+u_*}{\partial\frac{y^+\nu}{u_*}}$$

$$\tau_{wall} = \rho u_*^2\frac{\partial u^+}{\partial y^+}$$

In VSL region

$$\frac{\partial u^+}{\partial y^+} = 1$$

$$\tau_{wall} = \rho u_*^2$$

$$\frac{\rho\left(\frac{\tau_{wall}}{\rho}\right)^2}{\nu^2} = \frac{\tau_{wall}^2}{\rho\nu^2} = \frac{(\rho u_*^2)^2}{\rho\nu^2}$$

Looks like we can't cancel out all the terms here…

$$\frac{4\rho\sigma_{\omega 2}k}{CD_{k\omega}y^2} = 4\sigma_{\omega 2}\frac{k^+}{{y^+}^2}\frac{\rho u_*^4}{\nu^2} * \frac{1}{CD_{k\omega}}$$

$$CD_{k\omega} = \max\left(2\sigma_{\omega 2}\frac{\rho u_*^4}{\nu^2}\frac{1}{{\omega^*}^+}\frac{\partial{\omega^*}^+}{\partial y^+}\frac{\partial k^+}{\partial y^+}, 10^{-10}\right)$$

$$\frac{4\rho\sigma_{\omega 2}k}{CD_{k\omega}y^2} = \frac{4\sigma_{\omega 2}\frac{k^+}{{y^+}^2}}{\max\left(2\sigma_{\omega 2}\frac{1}{{\omega^*}^+}\frac{\partial{\omega^*}^+}{\partial y^+}\frac{\partial k^+}{\partial y^+}, \frac{\nu^2}{\rho u_*^4}10^{-10}\right)}$$

We can do parametric analysis to see the effect of the value of $\frac{\nu^2}{\rho u_*^4(Re, surface\ roughness)}$

So k^+ we already have in excel… (plotted)

$$\frac{\partial{\omega^*}^+}{\partial y^+} = \frac{-12}{\beta_1(y^+)^3}$$

${\omega^*}^+$	${\omega^*}^+ = \frac{6}{\beta_1(y^+)^2}$ $\beta_1 = \frac{3}{40}$	${\omega^*}^+ = \sqrt{{{\omega^*}^+_{VSL}}^2 + {{\omega^*_{TSL}}^+}^2}$	${\omega^*}^+ = \frac{1}{\kappa y^+\sqrt{C_\mu}}$ $\kappa = 0.41$ $C_\mu = 0.09$

$\frac{\partial \omega^{*+}}{\partial y^+}$	$\frac{-12}{\beta_1 (y^+)^3}$	-	$\omega^{*+} = \frac{-1}{\kappa (y^+)^2 \sqrt{C_\mu}}$

$$\frac{\partial k^+}{\partial y^+}$$

For $\frac{\partial k^+}{\partial y^+}$, it is equal to 0 in the BL

k^+	$k^+ = C_k (y^+)^{3.23} = 0$ As $C_k = 0$	$k^+ = \omega^{*+} \nu_t^+$ $1 = \left(1 + \frac{\nu_t}{\nu}\right) \frac{du^+}{dy^+}$ In log law: $\frac{du^+}{dy^+} = \frac{1}{\kappa} \frac{1}{y^+}$ Or in VSL $\frac{du^+}{dy^+} = 1$ Or you can use Van Driest Model	$k^+ = \frac{1}{\sqrt{C_\mu}}$ $C_\mu = 0.09$

In BL

$$\frac{4\rho\sigma_{\omega 2} k}{CD_{k\omega} y^2} = \frac{4\sigma_{\omega 2} \frac{k^+}{y^{+2}}}{\max\left(2\sigma_{\omega 2} \frac{1}{\omega^{*+}} \frac{\partial \omega^{*+}}{\partial y^+} \frac{\partial k^+}{\partial y^+}, \frac{\nu^2}{\rho u_*^4} 10^{-10}\right)} = \frac{4\sigma_{\omega 2} \frac{k^+}{y^{+2}}}{\frac{\nu^2}{\rho u_*^4} 10^{-10}}$$

The first term may be negligible…

$$\frac{4\sigma_{\omega 2} \frac{k^+}{y^{+2}}}{\frac{\nu^2}{\rho u_*^4} 10^{-10}} = 0 \ in\ VSL$$

$$\frac{\frac{4\sigma_{\omega 2}}{\sqrt{C_\mu}}}{\frac{\nu^2}{\rho u_*^4} 10^{-10}} \frac{1}{y^{+2}} = 0 \ in\ VSL$$

$$\sigma_{\omega 2} = 0.856$$

$$\sqrt{C_\mu} = \sqrt{0.09} = 0.3$$

$$\frac{4\sigma_{\omega 2}}{\sqrt{C_\mu} 10^{-10}} = 1.14133E + 11$$

$$Z = \frac{4\rho\sigma_{\omega 2}k}{CD_{k\omega}y^2} \approx \frac{1.14133E + 11}{\frac{\nu^2}{\rho u_*^4}y^{+2}}$$

We show that for this, typical Z values with zeta =0.01 , 1 and 100,

$F_1 = 1$ in the BL → this turns on $k - \omega$ in BL always.

But as $2\sigma_{\omega 2}\frac{1}{\omega^{*+}}\frac{\partial\omega^{*+}}{\partial y^+}\frac{\partial k^+}{\partial y^+}$ increases in turbulent region

Then F_1 decreases to reach 0, and thus turn on the $k - \varepsilon$ model.

$$\zeta = \frac{\nu^2}{\rho u_*^4}$$

Typical value for ζ

https://www.aiche.org/ccps/resources/glossary/process-safety-glossary/friction-velocity-u

for air flow

$$u_* \approx 0.05\frac{m}{s}\ (light\ wind)$$

$$u_* \approx 1\frac{m}{s}\ (strong\ wind)$$

$$\rho = 1\frac{kg}{m^3}$$

$$\nu_{air} = 1.48e - 5\frac{m^2}{s}$$

https://www.engineersedge.com/physics/viscosity_of_air_dynamic_and_kinematic_14483.htm

$$\zeta = \frac{\left(1.48e - 5\frac{m^2}{s}\right)^2}{1\ (1^4)} = 2.1904E - 10$$

From baseline model (with just the blending function only), also called BSL, to $k - \omega$ SST

Problem 1: expt data shows overprediction of reynold's shear stresses in adverse pressure gradient flows…

(Menter, F. R. (1994). Two-equation eddy-viscosity turbulence models for engineering applications. AIAA journal, 32(8), 1598-1605.) → menter's paper for $k - \omega$ SST

What are adverse pressure gradient flows?

https://en.wikipedia.org/wiki/Adverse_pressure_gradient

when $\frac{dp}{dx}$ does not favour flow direction, that's adverse pressure gradient…

- ➔ We haven't quite dealt with that in the previous flows, those wall functions are based on flat plate flow…
- ➔ But for curved surfaces, eg. Airfoil, spheres, this doesn't work as well anymore…

How does one sort this out??

In that same paper,

We find a key: the bradshaw assumption

In the turbulent BL:

$$\tau_{turbulent} = \rho a_1 k$$

Where k is the turbulent kinetic energy

But how do we calculate reynold's shear stress normally?

$$\tau_{turbulent} = \rho \nu_t \frac{\partial u}{\partial y}$$

Or else we know $\frac{\partial U}{\partial y}$ is known as the rate of strain S, sometimes known as Ω

If we were to rewrite this in terms of turbulent KE (not going to show it here)

$$\tau_{turbulent} = \sqrt{\frac{production\ of\ k}{dissipation\ of\ k}}\rho a_1 k$$

For turbulent BL flows, for the Bradshaw assumption ie,

$$\tau_{turbulent} = \rho a_1 k$$

To be true,

$$\frac{production\ of\ k}{dissipation\ of\ k} = 1$$

Unfortunately in k-omega baseline model, this doesn't hold true.

So we need to artificially force:

$$\tau_{turb} = \rho a_1 k$$

In the turbulent BL and

$$\tau_{turb} = \rho \nu_t \left(\frac{\partial u}{\partial y}\right)$$

In the rest of the flow…

In other words, we resort to blending functions again!

Let's call the new blending function Z

$$\tau_{turb} = Z\rho a_1 k + (1 - Z)\rho \nu_t \left(\frac{\partial u}{\partial y}\right)$$

Similar to F_1 we have $Z = 1$ in turbulent BL, $Z = 0$ in the main fluid. (I wont use F2 yet, because it is used in the model differently)

This is the general idea, but instead of writing things like this, Menter uses a different form:

$$\tau_{turb} = \rho \nu_t \left(\frac{\partial u}{\partial y}\right)$$

But we redefine ν_t such that in the turbulent bulk/freestream

$$\nu_t = \frac{k}{\omega^*} = C_\mu \frac{k}{\omega}$$

In BL

ν_t is such that $\tau_{turb} = \rho a_1 k$

So to do that,

We equate

$$\rho a_1 k = \rho \nu_t \left(\frac{\partial u}{\partial y}\right)$$

$$\nu_t = \frac{a_1 k}{\left(\frac{\partial u}{\partial y}\right)} \; in\ the\ BL$$

This forces $\tau_{turb} = \rho a_1 k$ in the BL

$$\tau_{turb} = \rho \nu_t \left(\frac{\partial u}{\partial y}\right) = \frac{a_1 k}{\left(\frac{\partial u}{\partial y}\right)} \rho \left(\frac{\partial u}{\partial y}\right) = \rho a_1 k$$

The way this is done:

$$\nu_t = \frac{a_1 k}{\max\left(a_1 \omega^*, F_2 \left(\frac{\partial u}{\partial y}\right)\right)}$$

$$F_2 = \tanh\left[\left[\max\left(\frac{\sqrt{k}}{\omega y}, \frac{500 \nu C_\mu}{\omega y^2}\right)\right]^2\right]$$

$$F_2 = \tanh\left[\left[\max\left(\frac{\sqrt{k}}{C_\mu \omega^* y}, \frac{500 \nu}{\omega^* y^2}\right)\right]^2\right]$$

Again, $F_2 = 1$ in BL, $F_2 = 0$ in turbulent bulk (freestream or centre of pipeflow) so that

$$F \left(\frac{\partial u}{\partial y}\right) > a_1 \omega \; in\ turbulent\ sublayer$$

As this limits production of turbulent shear stress or turbulent viscosity in the turbulent sublayer, this modification is known as the **production limiter**

$$\nu_t = \frac{a_1 k}{\max\left(a_1\omega^*, F_2\left(\frac{\partial u}{\partial y}\right)\right)}$$

in 3D, we don't just have u velocity and y, hence we use strain rate S instead.

$$S = \left(\frac{\partial u}{\partial y}\right)\ in\ 3D\ (i'm\ oversimplifying\ but\ yeah\ \ldots)$$

$$\nu_t = \frac{a_1 k}{\max(a_1\omega^*, F_2 S)}$$

But producing this change now poses a problem in the sublayer, since we artificially suppress turbulent viscosity, and this kind of upsets the balance of transport equations for k and ω

So we have to modify them…

What are the changes?

1) Some of the constants have to be tweaked from the original $k-\omega$ model
2) The production term of ω must be written a certain way
   a. In original $k-\omega$ / baseline model, $P_\omega = \alpha\frac{1}{\nu_t}\tau_{ij}\frac{\partial u_i}{\partial x_j} = \alpha\frac{\omega^*}{k}\tau_{ij}\frac{\partial u_i}{\partial x_j}$
   b. In SST model, $\nu_t \neq \frac{k}{\omega^*}$, so we can't use this same formulation
      i. To remove ambiguity, $P_\omega = \alpha S^2 = \alpha\frac{\omega^*}{k}\tau_{ij}\frac{\partial u_i}{\partial x_j}$

In most CFD guides eg. Cfd online, we won't see this comparison being made,

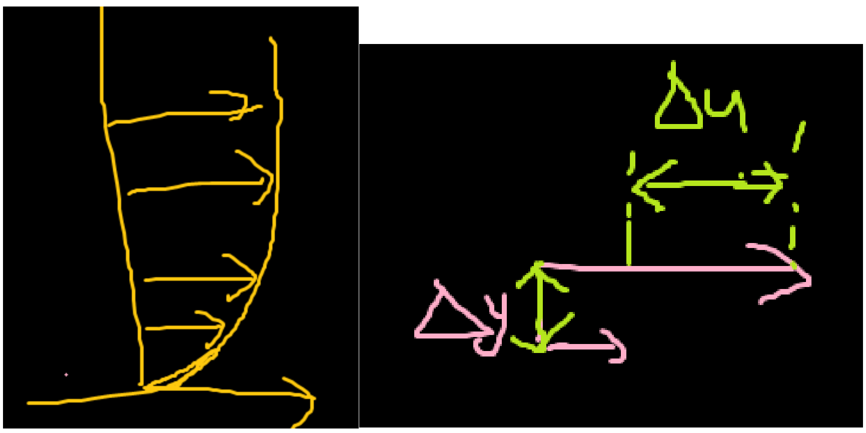
ν_t is removed from the expression even in $k-\omega$ model

This is the 1994 version of the $k-\omega$ SST model.

<u>Problem 2: in stagnation regions</u>

Large normal strain produces excessive turbulent kinetic energy…

Normal strain $= \frac{\partial U}{\partial y}$



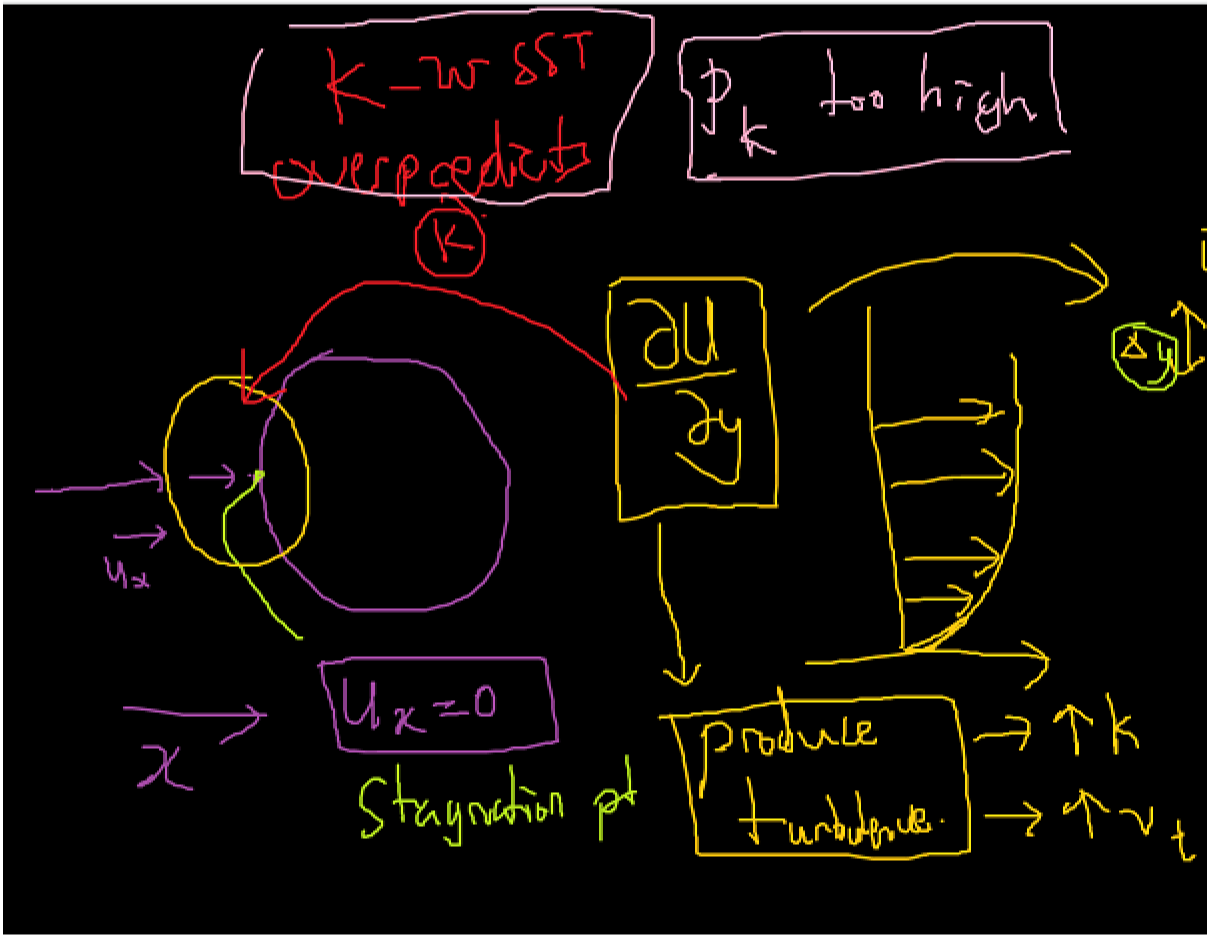
Characteristic length scale $= \Delta y$

Characteristic velocity scale $= \Delta u = \frac{\partial u}{\partial y}(strain\ rate)\Delta y$

$$Re = \frac{Ux}{\nu} = \frac{\Delta u \Delta y}{\nu} = \frac{\Delta y^2 \frac{\partial u}{\partial y}(strain\ rate)}{\nu}$$

Increasing strain rate increases Re

Eg in stagnation regions… take a sphere for example.



To compensate for this, a limiter is used on P_k in the turbulent kinetic energy equation

$$P_k = \min\left(\tau_{ij}\frac{\partial u_i}{\partial x_j}, 10\,\beta^* k\omega^*\right)$$

$$\beta^*\omega^* k = \varepsilon$$

We can P_k at 10ε= 10x dissipation rate.

https://www.openfoam.com/documentation/guides/latest/doc/guide-turbulence-ras-k-omega-sst.html https://turbmodels.larc.nasa.gov/sst.html

http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.460.2814&rep=rep1&type=pdf

Menter, F. R., Kuntz, M., & Langtry, R. (2003). Ten years of industrial experience with the SST turbulence model. Turbulence, heat and mass transfer, 4(1), 625-632.

This is known as the 2003 update, as of 2020, OpenFOAM uses this model…

All right so this is $k-\omega$ SST for you!

Summary of key features

- Note that a lot of it was data fitting, empirical testing…
    - K-omega in BL, k-epsilon in bulk region
    - Shear stress production limiter in adverse pressure gradient BL region (redefines ν_t to reduce it artificially)
    - Turbulent KE production limiter in stagnation regions (limit P_k)

SpalartAllmaras Model

Why study this when we have $k-\omega\ SST$?

1) Sometimes if you like simplicity, this is good, easy to understand/intuitive model
2) Developers of the model also developed Detached Eddy Simulation
    a. Hybrid between RANS and LES

https://www.researchgate.net/publication/236888804_A_One-Equation_Turbulence_Model_for_Aerodynamic_Flows

Spalart, P., & Allmaras, S. (1992, January). A one-equation turbulence model for aerodynamic flows. In 30th aerospace sciences meeting and exhibit (p. 439).

S-A model Fluid Mechanics 101

https://www.youtube.com/watch?v=Xivc0EIGFQw

Follow up paper

http://ae.metu.edu.tr/tuncer/ae546/docs/ICCFD7-1902.pdf

from the paper:

Development of transport equations…

Previously, $k-\omega$ and $k-\varepsilon$ are used

Here

It's just the kinematic viscosity → easy to use and low computation cost (1 less transport equation)

Derviation/Thought Process in development:

Closure model

$$-\overline{u_i'u_j'} = 2\nu_t S_{ij}$$

So we want a transport equation for turbulent kinematic viscosity,

Remember?

$$\nu_t = C_\mu \frac{k^2}{\varepsilon} = C_\mu \frac{k}{\omega} = \frac{k}{\omega^*}$$

$$\omega \equiv \frac{k}{\varepsilon}, \omega = \omega^* C_\mu = \omega^* \beta^*$$

Yes this is our friend, turbulent kinematic viscosity or eddy diffusivity

So we'll have our standard terms:

$$\frac{\partial \nu_t}{\partial t} + u_j \frac{\partial \nu_t}{\partial x_j} = RHS$$

What's in the right hand side?

We'll have a source of turbulent viscosity

Confused viscosity $\nu_{LOL}^{?}$

Remember our turbulence viscosity has its source in the strain rate $\frac{\partial U}{\partial y}$, remember the dimensional argument?

- Just a note though, I said S and Ω were similar in meaning in the last video, but they're NOT
  - $S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$, this is strain rate tensor, rate of deformation of fluid
    - $S = \sqrt{2S_{ij}S_{ij}}$ (strain rate), velocity gradients…
  - $\Omega_{ij} = \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$, this is not the same, and it's more closely related to vorticity (curl of fluid)
    - $\vec{\omega} = \vec{\nabla} \times \vec{u} \equiv \sqrt{\Omega_{ij}\Omega_{ij}}$ (vorticity), tendency of fluid to twist/curl

Source term:

$$source = c_{b1}\, \nu_t\, S$$

Diffusion terms: (see k Omega SST/ or other RANS models)

$$diffusion = \frac{1}{\sigma}\{\nabla \cdot (\nu_t \nabla \nu_t)\}$$

But there are extra terms that pop up… why?

Recall our k and omega equations…

$$\frac{\partial k}{\partial t}+u_j\frac{\partial k}{\partial x_j}=\frac{\partial}{\partial x_j}\left(\left(\nu+\sigma_{K(k\omega SST)}\nu_t\right)\frac{\partial k}{\partial x_j}\right)+\frac{P_{k(dynamic)}}{\rho}-\beta^*\omega^*k$$

$$\frac{\partial \omega^*}{\partial t}+u_i\frac{\partial \omega^*}{\partial x_i}=\frac{\partial}{\partial x_j}\left(\nu+\frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial \omega^*}{\partial x_j}+(1-F_1)2\left(\frac{1}{\omega^*}\right)\frac{\partial \omega^*}{\partial x_j}\frac{\partial k}{\partial x_j}+\alpha S^2-\beta{\omega^*}^2$$

Remember $\omega=\beta^*\omega^*$

Try combining both into one long equation for ν_t

$$\nu_t=\frac{k}{\omega^*}$$

$$k=\omega^*\nu_t$$

Substitute into the k equation

$$\frac{\partial \omega^*\nu_t}{\partial t}+u_j\frac{\partial \omega^*\nu_t}{\partial x_j}=\frac{\partial}{\partial x_j}\left(\left(\nu+\sigma_{K(k\omega SST)}\nu_t\right)\frac{\partial \omega^*\nu_t}{\partial x_j}\right)+\frac{P_{k(dynamic)}}{\rho}-\beta^*\omega^*\omega^*\nu_t$$

Try differentiating it all the way and subtract the ω^* equation from it…

What happens to the viscosity term?

$$\frac{\partial}{\partial x_j}\left(\left(\nu+\sigma_{K(k\omega SST)}\nu_t\right)\frac{\partial \omega^*\nu_t}{\partial x_j}\right)$$

noting

$$\frac{\partial \omega^*\nu_t}{\partial x_j}=\nu_t\frac{\partial \omega^*}{\partial x_j}+\omega^*\frac{\partial \nu_t}{\partial x_j}$$

$$\frac{\partial}{\partial x_j}\left(\left(\nu+\sigma_{K(k\omega SST)}\nu_t\right)\left\{\nu_t\frac{\partial \omega^*}{\partial x_j}+\omega^*\frac{\partial \nu_t}{\partial x_j}\right\}\right)$$

$$=\frac{\partial}{\partial x_j}\left(\left(\nu+\sigma_{K(k\omega SST)}\nu_t\right)\omega^*\frac{\partial \nu_t}{\partial x_j}\right)+\frac{\partial}{\partial x_j}\left(\left(\nu+\sigma_{K(k\omega SST)}\nu_t\right)\nu_t\frac{\partial \omega^*}{\partial x_j}\right)$$

$$\frac{\partial}{\partial x_j}\left(\left(\nu+\sigma_{K(k\omega SST)}\nu_t\right)\omega^*\frac{\partial \nu_t}{\partial x_j}\right)$$
$$=\omega^*\frac{\partial}{\partial x_j}\left(\left(\nu+\sigma_{K(k\omega SST)}\nu_t\right)\frac{\partial \nu_t}{\partial x_j}\right)+\frac{\partial \nu_t}{\partial x_j}\frac{\partial}{\partial x_j}\left(\left(\nu+\sigma_{K(k\omega SST)}\nu_t\right)\frac{\partial \nu_t}{\partial x_j}\right)$$

$$\frac{\partial}{\partial x_j}\left(\left(\nu+\sigma_{K(k\omega SST)}\nu_t\right)\nu_t\frac{\partial \omega^*}{\partial x_j}\right)$$
$$=\nu_t\frac{\partial}{\partial x_j}\left(\left(\nu+\sigma_{K(k\omega SST)}\nu_t\right)\frac{\partial \omega^*}{\partial x_j}\right)+\frac{\partial \omega^*}{\partial x_j}\frac{\partial}{\partial x_j}\left(\left(\nu+\sigma_{K(k\omega SST)}\nu_t\right)\nu_t\right)$$

$$\frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right)\frac{\partial \omega^* \nu_t}{\partial x_j}\right)$$

$$= \omega^* \frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right)\frac{\partial \nu_t}{\partial x_j}\right) + \frac{\partial \nu_t}{\partial x_j}\frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right)\frac{\partial \nu_t}{\partial x_j}\right) + \nu_t \frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right)\frac{\partial \omega^*}{\partial x_j}\right) + \frac{\partial \omega^*}{\partial x_j}\frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right)\nu_t\right)$$

$$\frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right)\nu_t\right) = constant * \frac{\partial}{\partial x_j}\left(\nu_t^2\right) = constant * \nabla\left(\nu_t^2\right)$$

Near wall treatment

<u>Log law layer</u>

Destruction of turbulence near wall

After some dimensional analysis…

$$-c_{w1}\left(\frac{\nu_t}{d}\right)^2$$

$$w = wall$$

In free flows, $d \to \infty$ so doesn't really matter

After some testing: show a great prediction for velocity profile, log law

$$\frac{\partial \nu_t}{\partial t} + u_j \frac{\partial \nu_t}{\partial x_j} = -c_{w1}\left(\frac{\nu_t}{d}\right)^2 + c_{b1}\,\nu_t\,S + \frac{1}{\sigma}\{\nabla \cdot (\nu_t \nabla \nu_t) + c_{b2}(\nabla \nu_t^2)\}$$

Produces a good $u^+ = f(y^+)$ in log law region

Provided have some suitable wall function for VSL region

Problem: skin friction coefficient too low (underpredict wall shear stress)

- ➔ Destruction term "decays too slowly" in outer BL, - in the paper
- ➔ Need to decrease destruction of turbulent viscosity in this area

Introduce a nondimensional coefficient to compensate for this…

- ➔ In log law region, $f_w = 1$
- ➔ It should help dampen the destruction term in the outer BL

Idea:

- Take inspiration from mixing length model
- Decay it with distance from the wall relation

Nondimnesional group

$$r = \left(\frac{mixing\ length\ scale}{characterisc\ distance\ from\ the\ wall}\right)^2 = \left(\frac{\sqrt{\frac{\nu_t}{S}}}{\kappa d}\right)^2 = \frac{\nu_t}{S\kappa^2 d^2}$$

$$f_w = g = \left(r + c_{w2}(r^6 - r)\right)$$

<u>Problem: f_w gets too high in bulk fluid! (upsets numerical stability)</u>

Dampen it again…

$$g = \left(r + c_{w2}(r^6 - r)\right)$$

Damping function for numerical stability= $\left[\frac{1+c_{w3}^6}{g^6+c_{w3}^6}\right]^{\frac{1}{6}}$

$$f_w = g\left[\frac{1 + c_{w3}^6}{g^6 + c_{w3}^6}\right]^{\frac{1}{6}}$$

$\left[\frac{1+c_{w3}^6}{g^6+c_{w3}^6}\right]^{\frac{1}{6}}$ prevents f_w from getting too big and thus upsetting numerical stability

<u>Viscous sublayer</u>

<u>Problem: How can we force</u>

$$\underline{u^+ = y^+}$$

<u>Here? And also correct predictions in the buffer layer?</u>

In log law region

$$\nu_t = \kappa y u_*$$

But in the VSL, this doesn't hold true!

How can we get it to work…

Can we take the turbulent kinematic viscosity above (log law region correct turbulent kinematic viscosity) and dampen it?

In the VSL, buffer reigon, we have this:

$$\nu_t = \kappa y u_* * damping\ function$$

Damping function for VSL/buffer layer

$$f_{v1} = \frac{\chi^3}{\chi^3 + c_{v1}^3}$$

$$v = viscous$$

$$\chi = \kappa y^+$$

$$\frac{\nu_t\ (disclaimer)}{\nu} = \frac{\kappa y u_*}{\nu} = \kappa y^+ = \chi$$

Disclaimer:

The turbulence viscosity, ν_t in the log law region and the bulk region of the fluid is called:

$$\nu_t = \tilde{\nu}\ (in\ bulk\ region\ and\ log - law\ region)$$

$$\chi = \frac{\tilde{\nu}}{\nu} = \kappa y^+$$

(definition starts in the log law region)

$$\frac{\partial \tilde{\nu}}{\partial t} + u_j \frac{\partial \tilde{\nu}}{\partial x_j} = -c_{w1}\left(\frac{\tilde{\nu}}{d}\right)^2 + c_{b1}\,\tilde{\nu}\,S + \frac{1}{\sigma}\{\nabla \cdot (\tilde{\nu}\nabla\tilde{\nu}) + c_{b2}(\nabla\tilde{\nu}^2)\}$$

So that’s how we introduce the <u>spalart allmaras variable ($\tilde{\nu}$)</u>

$$\nu_t = \tilde{\nu}\frac{\chi^3}{\chi^3 + c_{v1}^3}$$

$$f_{v1} = \frac{\chi^3}{\chi^3 + c_{v1}^3}$$

<u>Problem: too much production of turbulence in VSL</u>

Dampen it again!

We replace S with $\tilde{S}$

$$\tilde{S} = f(S)$$

$$\tilde{S} = S + \frac{\tilde{\nu}}{\kappa^2 d^2} f_{v2}$$

$$f_{v2} = 1 - \frac{\chi}{1 + \chi f_{v1}}$$

$$f_{v1} = \frac{\chi^3}{\chi^3 + c_{v1}^3}$$

Now our equation looks like this:

$$\frac{\partial\tilde{\nu}}{\partial t} + u_j\frac{\partial\tilde{\nu}}{\partial x_j} = -c_{w1}f_w\left(\frac{\tilde{\nu}}{d}\right)^2 + c_{b1}\,\tilde{\nu}\,\tilde{S} + \frac{1}{\sigma}\{\nabla\cdot(\tilde{\nu}\nabla\tilde{\nu}) + c_{b2}(\nabla\tilde{\nu}^2)\}$$

So far, we have dealt with:

- Log law region (with f_w)
- VSL ($\tilde{\nu}, \tilde{S}$)
- Bulk region

<u>Problem: numerical instabilities present in this formulation</u>

Long story short,

$\tilde{\nu} = 0$ causes the solvers to blow up… (skipping explanation)

Laminar to turbulent transition for laminar region shear layers

What is a shear layer?

http://thermopedia.com/content/1118/

we need to set a "**trip**" in order to prevent instability, allow $\tilde{\nu} = 0$ in VSL to be a stable solution and not cause blow-ups.

To do so, artificially dampen production term to zero in the BL

Now that we artificially reduce production of $\tilde{\nu}$ in BL, we also need to artificially reduce the destruction of $\tilde{\nu}$ in BL and transition region.

To artificially reduce production term:

$$\frac{\partial\tilde{\nu}}{\partial t} + u_j\frac{\partial\tilde{\nu}}{\partial x_j} = -c_{w1}f_w\left(\frac{\tilde{\nu}}{d}\right)^2 + c_{b1}[1-f_{t2}]\,\tilde{\nu}\,\tilde{S} + \frac{1}{\sigma}\{\nabla\cdot(\tilde{\nu}\nabla\tilde{\nu}) + c_{b2}(\nabla\tilde{\nu}^2)\}$$

To artificially reduce the destruction term:

$$\frac{\partial\tilde{\nu}}{\partial t} + u_j\frac{\partial\tilde{\nu}}{\partial x_j} = -\left[c_{w1}f_w - \frac{c_{b1}}{\kappa^2}f_{t2}\right]\left(\frac{\tilde{\nu}}{d}\right)^2 + c_{b1}[1-f_{t2}]\,\tilde{\nu}\,\tilde{S} + \frac{1}{\sigma}\{\nabla\cdot(\tilde{\nu}\nabla\tilde{\nu}) + c_{b2}(\nabla\tilde{\nu}^2)\}$$

See f_{t2}

$$f_{t2} = c_{t3} * \exp(-c_{t4}\chi^2)$$

Remember

$$\chi = \frac{\tilde{\nu}}{\nu} = \kappa y^+$$

T stands for trip

Problem: transition from laminar to turbulent layer not smooth

We sort of kaboomed our BL $\tilde{\nu}$ source and destruction terms by doing the above (trip terms, f_{t2})…

How can we generate turbulence in the BL?

See f_{t1} and ΔU

$$\frac{\partial \tilde{\nu}}{\partial t} + u_j \frac{\partial \tilde{\nu}}{\partial x_j} = -\left[c_{w1} f_w - \frac{c_{b1}}{\kappa^2} f_{t2}\right]\left(\frac{\tilde{\nu}}{d}\right)^2 + c_{b1}[1 - f_{t2}]\,\tilde{\nu}\,\tilde{S} + \frac{1}{\sigma}\{\nabla \cdot (\tilde{\nu}\nabla\tilde{\nu}) + c_{b2}(\nabla\tilde{\nu}^2)\} + f_{t1}\Delta U^2$$

$$\Delta U = norm\ of\ difference\ between\ trip\ point, and\ point\ where\ we\ are\ calculating$$

Trip point→ remember we wanted to trip our production and destruction terms in the turbulent/laminar transition part of BL…

http://brennen.caltech.edu/fluidbook/basicfluiddynamics/turbulence/lawofthewall.pdf

now what does f_{t1} look like?

$$f_{t1} = c_{t1} g_t \exp\left(-c_{t2}\frac{\omega_t^2}{\Delta U^2}[d^2 + g_t^2 d_t^2]\right)$$

$$g_t \equiv \min\left(0.1, \frac{\Delta U}{\omega_t \Delta x_t}\right)$$

$$\Delta x_t = grid\ spacing\ along\ wall\ at\ trip$$

$$\vec{\omega}_t = vorticity\ of\ wall\ at\ trip\ point \equiv \nabla \times \vec{u}_{trip}$$

log law region → turbulent BL

VSL → laminar BL

Buffer zone → transition region…

https://mathworld.wolfram.com/Norm.html

what is a norm?

well in this context, this is…

if $\vec{x}$ is a vector

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$norm\ (\vec{x}) = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

So it means like for a position, it is the length between two points

<u>Technically correct way: L2 norm</u>

In this case, it's the magnitude difference in velocity,

So it's somewhat like ΔU we are used to

So this serves as the term to build up turbulence somewhat in the transition zone….

That's it!

The end product:

$$\frac{\partial \tilde{\nu}}{\partial t} + u_j \frac{\partial \tilde{\nu}}{\partial x_j} = -\left[c_{w1} f_w - \frac{c_{b1}}{\kappa^2} f_{t2}\right] \left(\frac{\tilde{\nu}}{d}\right)^2 + c_{b1}[1 - f_{t2}]\, \tilde{\nu}\, \tilde{S} + \frac{1}{\sigma}\{\nabla \cdot (\tilde{\nu} \nabla \tilde{\nu}) + c_{b2}(\nabla \tilde{\nu}^2)\} + f_{t1} \Delta U^2$$

$$\nu_t = \tilde{\nu} \frac{\chi^3}{\chi^3 + c_{v1}^3}$$

https://www.cfd-online.com/Wiki/Spalart-Allmaras_model

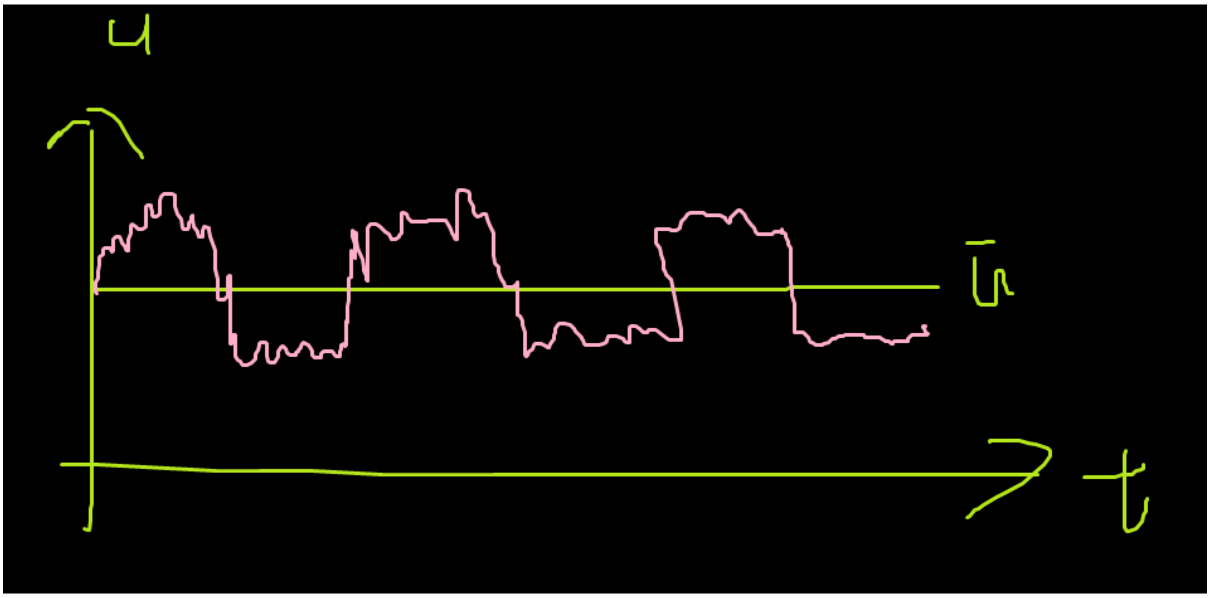
Large eddy simulation LES

https://www.amazon.com/Turbulent-Flows-Stephen-B-Pope/dp/0521598869

Step 1:

So we don't average ALL the turbulence…

But only those of the smallest scales.

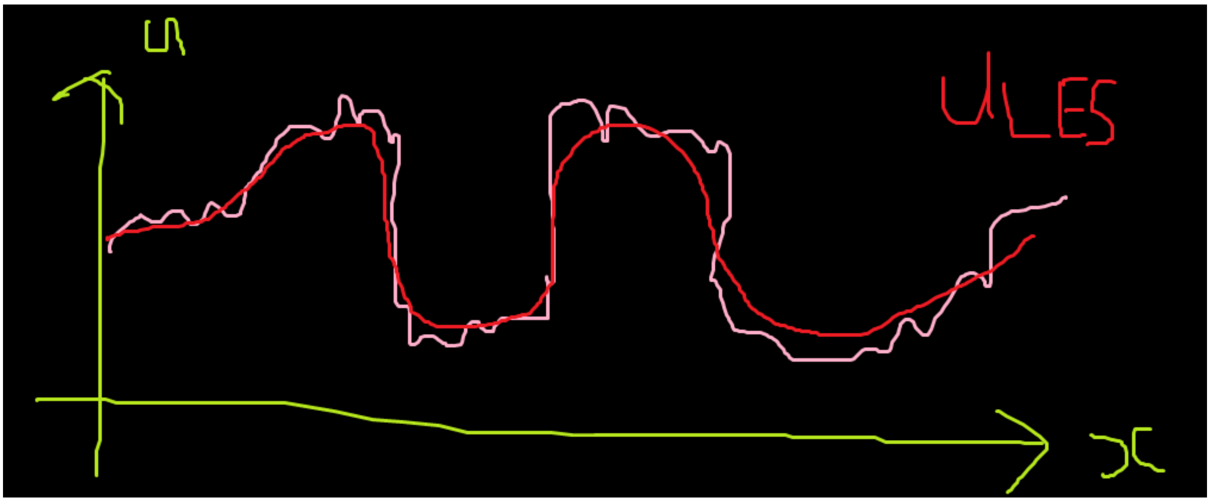


In RANS we average by <u>time</u>

In unsteady RANS,

We may average by a shorter period of <u>time</u>

But LES often averages (or rather filters) by x (<u>distance or length</u>)



$$u = \bar{u} + u'$$

$\bar{u}$ represents the filtered velocity field.

The above method I mentioned is one of the ways of filtering, known as a box filtering method…

In the more strictly correct way of filtering

$$\bar{u}(x,t) = \int G(r,x)u(x-r,t)\,dr$$

We can see from above, r is some sort of length scale since r is subtracted from x.

So this is taken for a length scale about x.

Compare this to time averaging

$$\bar{u}(x,t) = \int_{period} u(x,t)dt$$

What shapes can $G(r)$ take?

Name	Filter Function
General	G(r)
Box	$\frac{1}{\Delta} H\left(\frac{1}{2}\Delta - \lvert r \rvert\right)$
Gaussian	$\sqrt{\frac{6}{\pi\Delta^2}} \exp\left(-\frac{6r^2}{\Delta^2}\right)$
Sharp spectral	$\frac{\sin\left(\frac{\pi r}{\Delta}\right)}{\pi r}$
Cauchy	$\frac{a}{\pi\Delta\left[\left(\frac{r}{\Delta}\right)^2 + a^2\right]}, a = \frac{\pi}{24}$

https://www.amazon.com/Turbulent-Flows-Stephen-B-Pope/dp/0521598869

How to get Δ, the characteristic length scale? Or filter length scale?

One simple way:

$$\Delta = (cell\ volume)^{\frac{1}{3}}$$

You can specify it manually as well…

After filtering

$$u = \bar{u} + u'$$

Step 2:

We substitute these into our equation, and we get pretty much the similar thing:

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z} = 0$$

Momentum conservation (x direction)

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} + \bar{w}\frac{\partial \bar{u}}{\partial z} = -\frac{1}{\rho}\frac{\partial \bar{P}}{\partial x} + \nu\,\nabla^2\bar{u} - \frac{\partial}{\partial x}\left(\overline{u'^2}\right) - \frac{\partial}{\partial y}\overline{u'v'} - \frac{\partial}{\partial z}\overline{u'w'}$$

But for LES filtering, this is slightly different

- In RANS, $u^2 = (\bar{u} + u')^2 = \bar{u}^2 + \bar{u}u' + u'^2$
    - Note $\bar{u}$ is assumed constant with time…
- When we time average, $\overline{u^2} = \overline{(\bar{u} + u')^2} = \overline{\bar{u}^2} + \overline{(\bar{u}u')} + \overline{(u'^2)} = \bar{u}^2 + \overline{(u'^2)}$

What about LES filtering?

Mass conservation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

(a little introduction to Einstein notation)

$$\frac{\partial u_i}{\partial x_i} = \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$i = 1,2 \; and \; 3$$

$$\frac{\partial u_i}{\partial x_i} = 0$$

For x y and z direction,

$$u_i = \overline{u_\iota} + u_i'$$

Substitute in:

$$\frac{\partial(\overline{u_\iota} + u_i')}{\partial x_i} = 0$$

$$\frac{\partial(\overline{u_\iota})}{\partial x_i} + \frac{\partial(u_i')}{\partial x_i} = 0$$

One assumption is this:

$$G(r, x) = G(r)$$

If filter doesn't change with distance, then

$$\frac{\partial(u_i')}{\partial x_i} = 0$$

$$\frac{\partial(u_1')}{\partial x_1} + \frac{\partial(u_2')}{\partial x_2} + \frac{\partial(u_3')}{\partial x_3} = 0$$

Therefore, we are left with:

$$\frac{\partial(\overline{u_\iota})}{\partial x_i} = 0$$

$$\frac{\partial \bar{u}_j}{\partial t} + \frac{\partial \overline{(u_i u_j)}}{\partial x_i} = \nu \frac{\partial^2 \bar{u}_j}{\partial x_i \partial x_i} - \frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_j}$$

We can take a look at our RANS equation for comparison

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} + \bar{w}\frac{\partial \bar{u}}{\partial z} = -\frac{1}{\rho}\frac{\partial \bar{P}}{\partial x} + \nu\,\nabla^2\bar{u} - \frac{\partial}{\partial x}\left(\overline{u'^2}\right) - \frac{\partial}{\partial y}\overline{u'v'} - \frac{\partial}{\partial z}\overline{u'w'}$$

Let's expand the I's out…

X dir: (subs j=1)

$$\frac{\partial \bar{u}_j}{\partial t} + \frac{\partial \overline{(u_1 u_j)}}{\partial x_1} + \frac{\partial \overline{(u_2 u_j)}}{\partial x_2} + \frac{\partial \overline{(u_3 u_j)}}{\partial x_3} = \nu\left(\frac{\partial^2 \bar{u}_j}{\partial x_1 \partial x_1} + \frac{\partial^2 \bar{u}_j}{\partial x_2 \partial x_2} + \frac{\partial^2 \bar{u}_j}{\partial x_3 \partial x_3}\right) - \frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_j}$$

J represents 3 directions (3 equations for velocity vector in x,y,z direction)

Y dir: (subs j=2)

$$\frac{\partial \bar{u}_j}{\partial t} + \frac{\partial \overline{(u_1 u_j)}}{\partial x_1} + \frac{\partial \overline{(u_2 u_j)}}{\partial x_2} + \frac{\partial \overline{(u_3 u_j)}}{\partial x_3} = \nu\left(\frac{\partial^2 \bar{u}_j}{\partial x_1 \partial x_1} + \frac{\partial^2 \bar{u}_j}{\partial x_2 \partial x_2} + \frac{\partial^2 \bar{u}_j}{\partial x_3 \partial x_3}\right) - \frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_j}$$

Z dir: (subs j=3)

$$\frac{\partial \bar{u}_j}{\partial t} + \frac{\partial \overline{(u_1 u_j)}}{\partial x_1} + \frac{\partial \overline{(u_2 u_j)}}{\partial x_2} + \frac{\partial \overline{(u_3 u_j)}}{\partial x_3} = \nu\left(\frac{\partial^2 \bar{u}_j}{\partial x_1 \partial x_1} + \frac{\partial^2 \bar{u}_j}{\partial x_2 \partial x_2} + \frac{\partial^2 \bar{u}_j}{\partial x_3 \partial x_3}\right) - \frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_j}$$

The shorthand (einstien notation) of writing is:

$$\frac{\partial \bar{u}_j}{\partial t} + \frac{\partial \overline{(u_i u_j)}}{\partial x_i} = \nu \frac{\partial^2 \bar{u}_j}{\partial x_i \partial x_i} - \frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_j}$$

Closure terms come from this term:

Momentum convection term:

$$\frac{\partial \overline{(u_i u_j)}}{\partial x_i}$$

We can of course expand this out using filtered quantities

$$u_i = \bar{u}_i + u_i'$$

$$u_j = \bar{u}_j + u_j'$$

$$u_i u_j = (\bar{u}_i + u_i')(\bar{u}_j + u_j') = \bar{u}_i\bar{u}_j + \bar{u}_j u_i' + \bar{u}_i u_j' + u_i' u_j'$$

To find $\overline{(u_i u_j)}$, we need to filter again…

$$\overline{(u_i u_j)}(x,t) = \int G(r,x) u_i u_j (x-r,t)\, dr$$

$$\overline{(u_i u_j)} = \overline{\bar{u}_i\bar{u}_j} + \overline{\bar{u}_j u_i'} + \overline{\bar{u}_i u_j'} + \overline{u_i' u_j'}$$

Note that that certain quantities do not disappear because they are not time averages…

$$\frac{\partial \bar{u}_j}{\partial t} + \frac{\partial \overline{(u_i u_j)}}{\partial x_i} = \nu \frac{\partial^2 \bar{u}_j}{\partial x_i \partial x_i} - \frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_j}$$

What we want at the end of the day…

$$\frac{\partial \bar{u}_j}{\partial t} + \frac{\partial \bar{u}_i \bar{u}_j}{\partial x_i} = \nu \frac{\partial^2 \bar{u}_j}{\partial x_i \partial x_i} - \frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_j} + extra\ term\ \ldots$$

Use product rule on:

$$\frac{\partial \bar{u}_i \bar{u}_j}{\partial x_i} = \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_i} + \bar{u}_i \frac{\partial \bar{u}_j}{\partial x_i}$$

From continuity equation

$$\frac{\partial (\bar{u}_i)}{\partial x_i} = 0$$

Then we have:

$$\frac{\partial \bar{u}_i \bar{u}_j}{\partial x_i} = \bar{u}_j (0) + \bar{u}_i \frac{\partial \bar{u}_j}{\partial x_i} = \bar{u}_i \frac{\partial \bar{u}_j}{\partial x_i}$$

How do we get from:

$$\frac{\partial \bar{u}_j}{\partial t} + \frac{\partial \overline{(u_i u_j)}}{\partial x_i} = \nu \frac{\partial^2 \bar{u}_j}{\partial x_i \partial x_i} - \frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_j}$$

To here:

$$\frac{\partial \bar{u}_j}{\partial t} + \bar{u}_i \frac{\partial \bar{u}_j}{\partial x_i} = \nu \frac{\partial^2 \bar{u}_j}{\partial x_i \partial x_i} - \frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_j} + extra\ term\ \ldots$$

What about?

$\frac{\partial \overline{(u_i u_j)}}{\partial x_i}$ and $\frac{\partial \overline{u_i u_j}}{\partial x_i}$

So what is the difference between:

$$\frac{\partial \overline{(u_i u_j)}}{\partial x_i}\ and\ \frac{\partial \bar{u}_i \bar{u}_j}{\partial x_i}$$

$$\tau_{ij}^R \equiv \overline{U_i U_j} - \bar{U}_i \bar{U}_j$$

Residual reynold's stress

$$\overline{(u_i u_j)} = \bar{u}_i \bar{u}_j + \tau_{ij}^R$$

$$\frac{\partial}{\partial x_i}\overline{(u_i u_j)} = \frac{\partial}{\partial x_i}\bar{u}_i \bar{u}_j + \frac{\partial}{\partial x_i}\tau_{ij}^R$$

Earlier we found that:

$$\frac{\partial \bar{u}_i \bar{u}_j}{\partial x_i} = \bar{u}_j(0) + \bar{u}_i \frac{\partial \bar{u}_j}{\partial x_i} = \bar{u}_i \frac{\partial \bar{u}_j}{\partial x_i}$$

So,

$$\frac{\partial}{\partial x_i} \overline{(u_i u_j)} = \bar{u}_i \frac{\partial \bar{u}_j}{\partial x_i} + \frac{\partial}{\partial x_i} \tau_{ij}^R$$

We substitute this back:

$$\frac{\partial \bar{u}_j}{\partial t} + \bar{u}_i \frac{\partial \bar{u}_j}{\partial x_i} + \frac{\partial}{\partial x_i} \tau_{ij}^R = \nu \frac{\partial^2 \bar{u}_j}{\partial x_i \partial x_i} - \frac{1}{\rho} \frac{\partial \bar{p}}{\partial x_j}$$

$$\frac{\partial \bar{u}_j}{\partial t} + \bar{u}_i \frac{\partial \bar{u}_j}{\partial x_i} = \nu \frac{\partial^2 \bar{u}_j}{\partial x_i \partial x_i} - \frac{\partial}{\partial x_i} \tau_{ij}^R - \frac{1}{\rho} \frac{\partial \bar{p}}{\partial x_j}$$

We can define

$$\frac{\bar{D}}{\bar{D}t} = \frac{\partial}{\partial t} + \bar{u}_i \frac{\partial}{\partial x_i}$$

$$\frac{\bar{D}\bar{u}_j}{\bar{D}t} = \nu \frac{\partial^2 \bar{u}_j}{\partial x_i \partial x_i} - \frac{\partial}{\partial x_i} \tau_{ij}^R - \frac{1}{\rho} \frac{\partial \bar{p}}{\partial x_j}$$

$$\frac{\bar{D}\bar{u}_j}{\bar{D}t} = \nu \frac{\partial \bar{U}_j}{\partial x_i \partial x_i} - \frac{\partial \tau_{ij}^R}{\partial x_i} - \frac{1}{\rho} \frac{\partial \bar{p}}{\partial x_j}$$

Smagorinsky LES model

Step 3:

The next trick is: how do we find $\frac{\partial}{\partial x_i} \tau_{ij}^R$

This differs for each LES model

We can model $\frac{\partial}{\partial x_i} \tau_{ij}^R$ as a turbulent viscosity (momentum diffusivity in subgrid scale)

$$\tau_{ij}^R = -2\nu_t \overline{S_{ij}}$$

$$\overline{S_{ij}} = \frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)$$

$$\tau = \mu \frac{\partial u}{\partial y}$$

$$\tau_{ij}^R = -\nu_t \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)$$

$$\frac{\partial}{\partial x_i}\tau_{ij}^R = -\frac{\partial}{\partial x_i}\nu_t\left(\frac{\partial \bar{u}_i}{\partial x_j}+\frac{\partial \bar{u}_j}{\partial x_i}\right)$$

$$\frac{\bar{D}\bar{u}_j}{\bar{D}t} = \nu\frac{\partial^2 \bar{u}_j}{\partial x_i \partial x_i}+\frac{\partial}{\partial x_i}\nu_t\left(\frac{\partial \bar{u}_i}{\partial x_j}+\frac{\partial \bar{u}_j}{\partial x_i}\right)-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_j}$$

These residuals are modelled as extra shear stresses (more viscosity)

Overall:

$$\frac{\partial \bar{u}_i}{\partial x_i} = 0$$

$$\frac{\partial \bar{u}_i}{\partial t}+\frac{\partial \bar{u}_j \bar{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \bar{P}}{\partial x_i}+\frac{\partial}{\partial x_j}\left\{(\nu+\nu_s)\left[\frac{\partial \bar{u}_i}{\partial x_j}+\frac{\partial \bar{u}_j}{\partial x_i}\right]\right\}+F_i$$

Extra viscosity is added similar to RANS, but derivation is different because the filtering process is not the same as time averaging process

How then do we get ν_t

Smagorinsky Model → 1963 (LES modelling of atmospheric turbulence)

Subgrid scale meaning smaller than Δ

$$\nu_t\left[\frac{m^2}{s}\right] = l_{characteristic}^2 * \left(\frac{1}{t}\right)$$

What is this 1/t?

Characteristic filtered rate of strain

$$\bar{S} \equiv \sqrt{2\overline{S_{ij}}\,\overline{S_{ij}}}$$

Recall:

$$\overline{S_{ij}} = \frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_j}+\frac{\partial \bar{u}_j}{\partial x_i}\right)$$

$$\bar{S}\left[\frac{1}{s}\right] \equiv \sqrt{\frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_j}+\frac{\partial \bar{u}_j}{\partial x_i}\right)\left[\frac{1}{s}\right]\left(\frac{\partial \bar{u}_i}{\partial x_j}+\frac{\partial \bar{u}_j}{\partial x_i}\right)\left[\frac{1}{s}\right]}$$

$$\nu_t\left[\frac{m^2}{s}\right] = l_{characteristic}^2 * \bar{S}$$

What is the characteristic length scale?

$$l_{characteristic} = C_{smagornisky}\Delta$$

$$\nu_t\left[\frac{m^2}{s}\right] = \left(C_{smagornisky}\Delta\right)_{characteristic}^2 * \bar{S}$$

$$\nu_t \left[\frac{m^2}{s}\right] = C_{smagorinsky}^2 \Delta^2 * \bar{S}$$

$$C_{smagorinsky} \approx 0.17, 0.15, 0.2, 0.25 \rightarrow 0.15 - 0.25$$

- 0.1 for channel flow (Shur, Spalart, Strelets, & Travin, 2008)
- 0.2 for free shear flow

The question is then, how to specify $C_{smagorinsky}$?

- Literature online
- Expt data
- Maybe use other more complicated model…?

Step 4: solve numerically

Wall Treatment for Smagorinsky Model

How can smagorinsky model (LES) be done for the wall?

$$Reynolds\ stress\ = \overline{(u'v')}$$

So we expect normally, that turbulent viscosity near the wall ≈ 0.

$$\nu_t \left[\frac{m^2}{s}\right] = \left(C_{smagornisky}\Delta\right)_{characteristic}^2 * \bar{S}$$

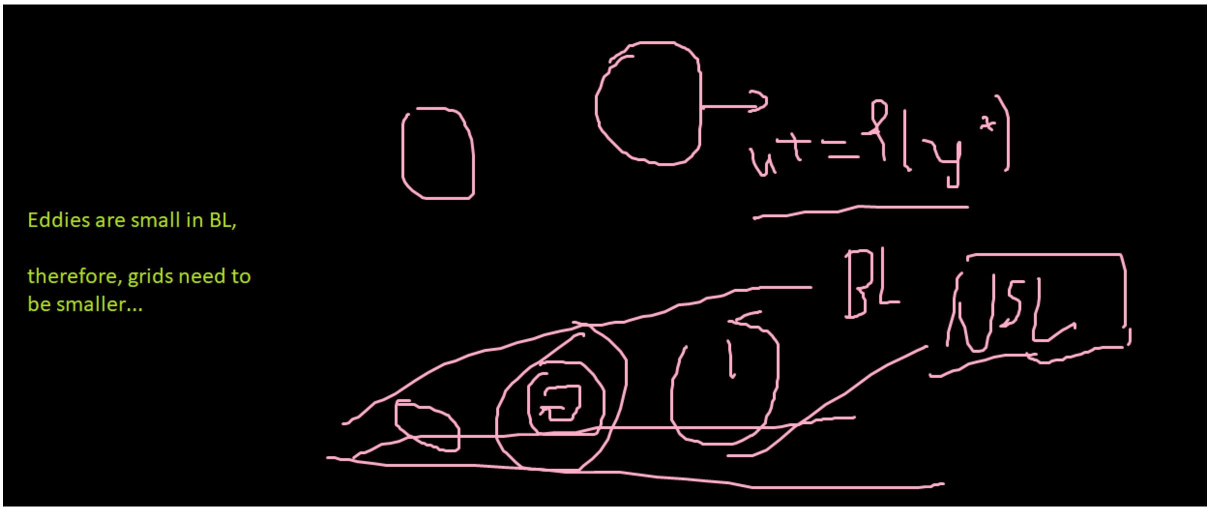
$$l_{characteristic} = C_{smagornisky}\Delta$$

We need the grid size to be VERY small near the wall…

This is LES with resolving of the near wall region (NWR).

$$\bar{S} \equiv \sqrt{2\overline{S_{ij}S_{ij}}}$$

$$\overline{S_{ij}} = \frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)$$



Only problem is this:

The number of grid nodes increases with $Re^{1.76}$ (Chapman, 1979)

Near wall resolution, may not be feasible for high Re flow.

How can we lower computational cost?

LES-Near wall modelling

Adjust turbulent viscosity near the wall…

$$\nu_t\left[\frac{m^2}{s}\right] = \left(C_{smagornisky}\Delta\right)^2_{characteristic} * \bar{S}$$

$$\nu_t = l^2_{characteristic} * \bar{S}$$

We can take inspiration from Van Driest…

$$l_{characteristic} = C_{smagornisky}\Delta\left[1 - \exp\left(-\frac{y^+}{A^+}\right)\right]$$

$$A^+ \approx 26$$

At the wall, $y^+ = 0$, exp(0)=1

$$l_{characteristic} = 0 \; at \; the \; wall!$$

$$l_{characteristic} = \kappa y\left[1 - \exp\left(-\frac{y^+}{A^+}\right)\right]$$

$$\kappa \approx 0.4$$

https://www.openfoam.com/documentation/guides/latest/doc/guide-turbulence-les-delta-vandriest.html

$$l_{characteristic} = \min\left(\kappa y\left[1 - \exp\left(-\frac{y^+}{A^+}\right)\right], C_{smagornisky}\Delta\right)$$

The Van Driest damping function is given by:

$$D = 1 - \exp^{\frac{-y^+}{A^+}}$$

The final length scale is given by:

$$\Delta = \min\left(\frac{\kappa y}{C_s} D, \Delta_g\right)$$

where Δ_g is a geometric-based delta function such as the **cube-root volume** delta.

What OpenFOAM does for LES,

$$l_{characteristic} = C_{smagornisky}\Delta$$

Far away from the wall, $\Delta = V^{\frac{1}{3}} = \Delta_g$

Close to the wall, we want $l_{characteristic} = \kappa y \left[1 - \exp\left(-\frac{y^+}{A^+}\right)\right]$

Near the wall, we define Δ as follows:

$$C_{smagornisky}\Delta = \kappa y \left[1 - \exp\left(-\frac{y^+}{A^+}\right)\right]$$

$$\Delta = \frac{\kappa y}{C_{smagornisky}} \left[1 - \exp\left(-\frac{y^+}{A^+}\right)\right]$$

More LES models!

(smargorinsky and dynamic smargorinksy and more…)

So we've discussed a few major RANS models…

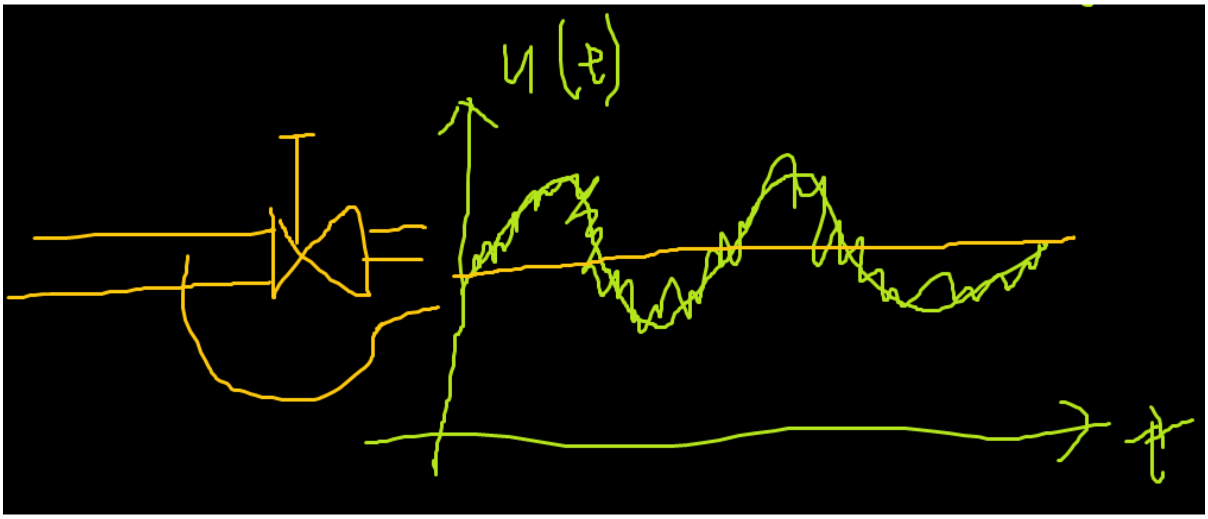
- $k - \varepsilon$
- $k - \omega$
- $k - \omega\ SST$
- Spalart Allmaras model

These are unsteady RANS…

$$\bar{u}(t) = \frac{1}{\Delta t}\int_0^{\Delta t} u(t)\, dt$$

For RANS

$$\bar{u}(t) = \lim_{\Delta t \to \infty} \frac{1}{\Delta t}\int_0^{\Delta t} u(t)\, dt$$



But unsteady RANS

$$\bar{u}(t) = \frac{1}{\Delta t}\int_0^{\Delta t} u(t)\, dt$$

$$\Delta t \neq \infty$$

Very large eddy simulation (VLES)

Weakness: flow results are only as good as the model of turbulence.

Weaknesses of some RANS/URANS(VLES) models…

- $k - \varepsilon$
  - Doesn't do adverse pressure gradients well
  - Wall boundary conditions not as good as $k - \omega$
  - Complex flows not good
- $k - \omega$
  - Freestream ω BC must be specified
  - Flow is sensitive to this BC
- $k - \omega\ SST$
  - Blends $k - \omega$ and $k - \varepsilon$
  - Cannot model some complex flows properly, eg. Turbomachinery (turbines/jets)
- Spalart Allmaras model
  - Cannot model some complex flows properly, eg. Turbomachinery (turbines/jets)

All of these are based on empirical models of wall functions, and these are **not** universal.

Let's look further into the world of LES modelling

Why LES?

- RANS has a lot of empirical constants, equation constant used for one flow may not always be suitable that of another flow

- Unsteady RANS (aka Very large eddy simulations VLES) ≈ Navier stokes (unsteady) + turbulence modelling
  - (Spalart, Jou, Strelets, & Allmaras, 1997)
  - Spalart, P., Jou, W.-H., Strelets, M., & Allmaras, S. (1997). Comments on the feasibility of LES for winds, and on a hybrid RANS/LES approach. 1, 4–8.
- DNS while accurate is too impractical

LES wants more accuracy than RANS but more speed than DNS

10. Numerical methods in LES

- It is important for LES calculations to predict accurately the quantities that led to choosing LES in the first place (e.g., turbulent fluctuations, acoustic sources, mixing,…)

- Numerical dissipation present in most RANS codes is inadequate for LES (c.f. flow over cylinder)

- Ideally in LES nondissipative discretizations (central differencing as opposed to low order upwinding or any extra added numerical dissipation) must be used.

(note the fvSchemes!)

Dynamic Smagorinsky Model

We studied the Smagorinsky Model before:

$$\nu_t \left[\frac{m^2}{s}\right] = C_{smagorinsky}^2 \Delta^2 * \bar{S}$$

$$C_{smagorinsky} \approx 0.17, 0.15, 0.2, 0.25 \rightarrow 0.15 - 0.25$$

Pipe or duct flow, $C_{smagorinsky} \approx 0.11$

➔ Used originally for atmospheric boundary layers (meterology)

How can we deal with this constant? It doesn't seem very constant across all cases!

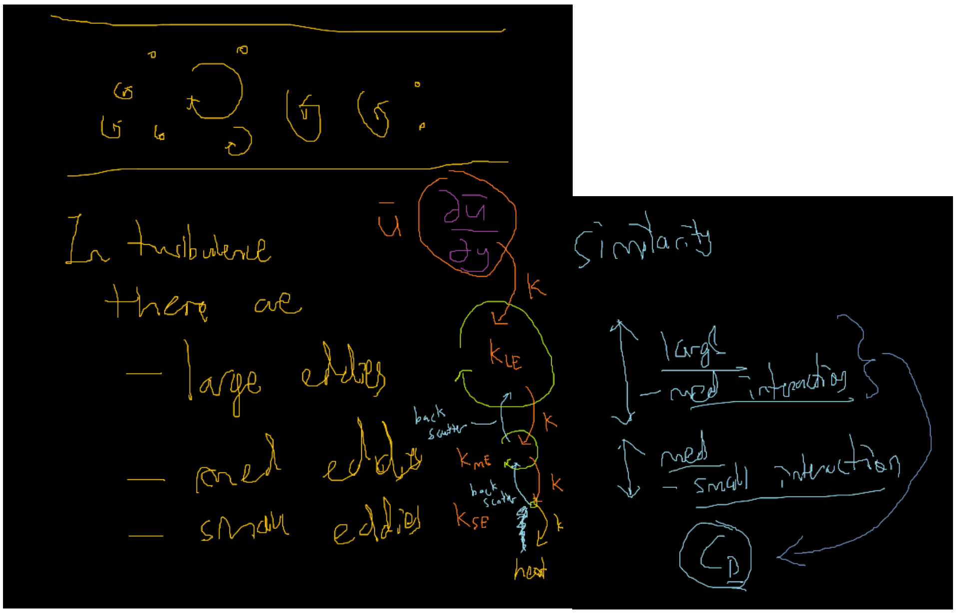
- Good for atmospheric turbulence modelling in meteorology, not so good otherwise

How can we deal with this?

- We can try having a dynamic coefficient based on a similarity approach

What is this similarity approach?

Let’s first paint a picture of turbulence…



LES Dynamic Smagorinsky model:

https://www.cfd-online.com/Wiki/Dynamic_subgrid-scale_model

What is backscatter?

https://en.wikipedia.org/wiki/Large_eddy_simulation

Sagaut, P. (2006). Large eddy simulation for incompressible flows: an introduction. Springer Science & Business Media.

Pope, S. B. (2001). Turbulent flows.

Classic Smagorinsky (Handout: Large Eddy Simulation I G. Constantinescu)

https://www.iihr.uiowa.edu/gconstantinescu/files/2012/10/LES_models_1.pdf

Dynamic Models (Handout: Large Eddy Simulation II G. Constantinescu)

https://www.iihr.uiowa.edu/gconstantinescu/files/2012/10/LES_models_2.pdf

KEquation Models (Handout: Large Eddy Simulation III G. Constantinescu)

https://www.iihr.uiowa.edu/gconstantinescu/files/2012/10/LES_models_3.pdf

Hybrid Models (Handout: Large Eddy Simulation IV G. Constantinescu)

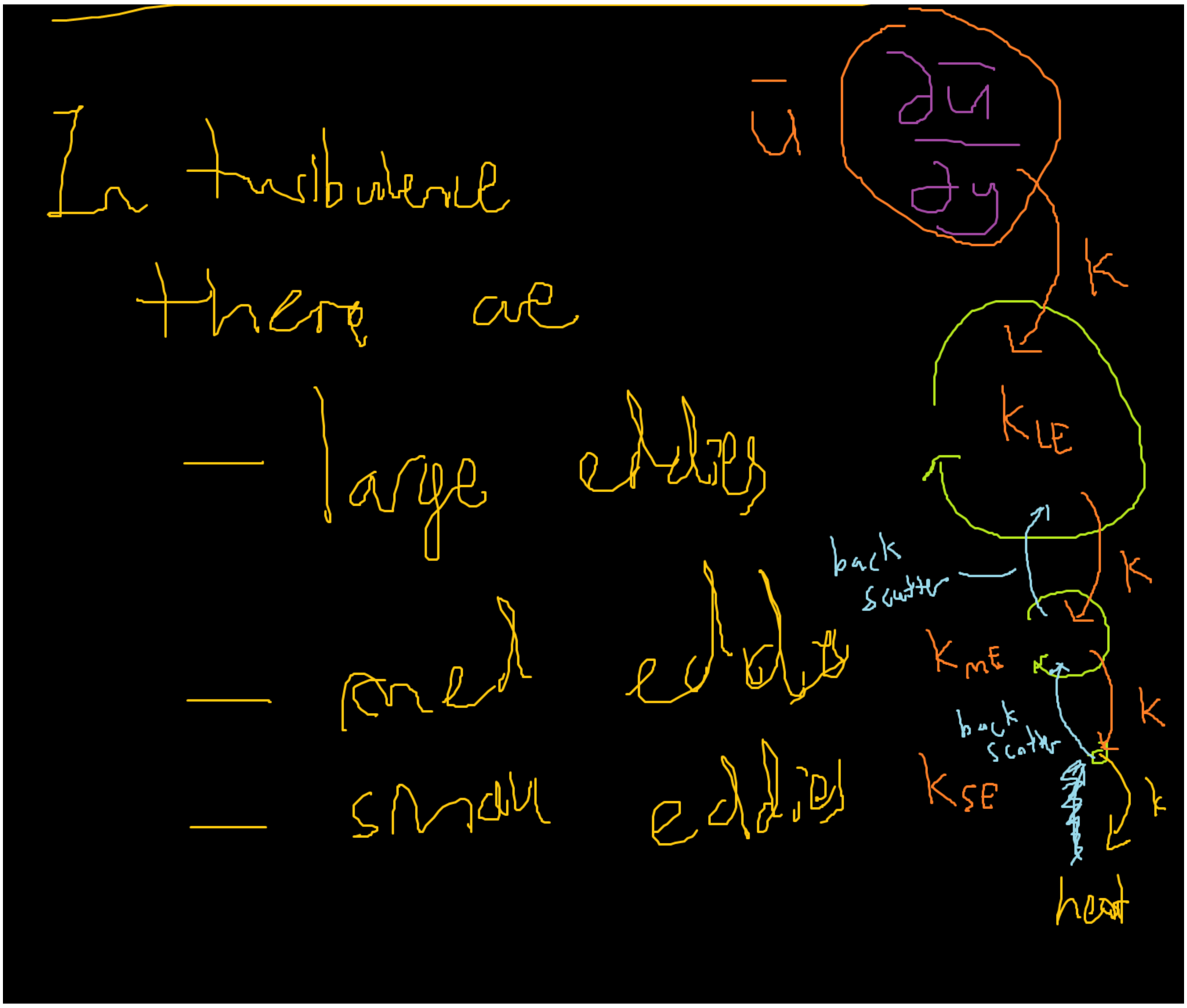
https://www.iihr.uiowa.edu/gconstantinescu/files/2012/10/LES_models_4.pdf

Spalart, P., Jou, W.-H., Strelets, M., & Allmaras, S. (1997). Comments on the feasibility of LES for winds, and on a hybrid RANS/LES approach. 1, 4–8.

How do we get the large eddy-medium eddy interactions?

We need to filter the equations again…

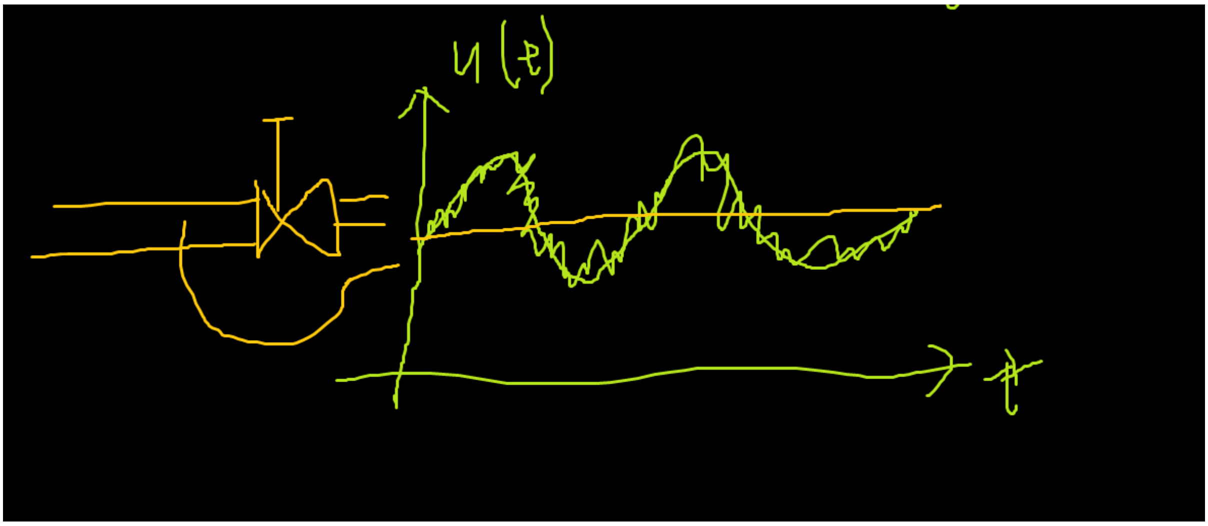
So let's go to our diagram…



Instead of saying large medium and small eddies, let's use a graph instead.

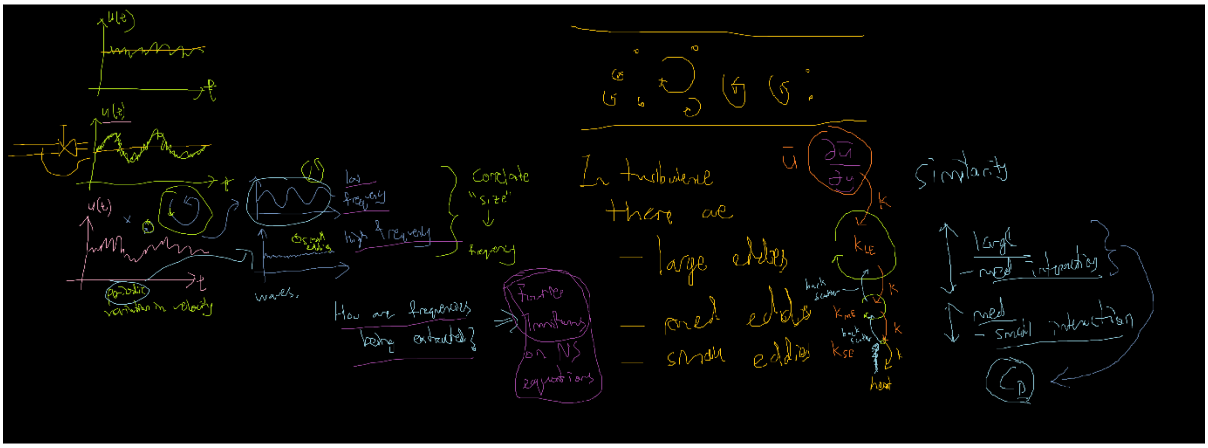
What's commonly used in LES modelling to describe the size of the eddies is often describing the turbulence <u>spectrum</u>

Why?



Recall this?

We can see distinctly here the velocity is a superposition of a big sine wave and a several smaller sine waves



Big sine waves → low frequency

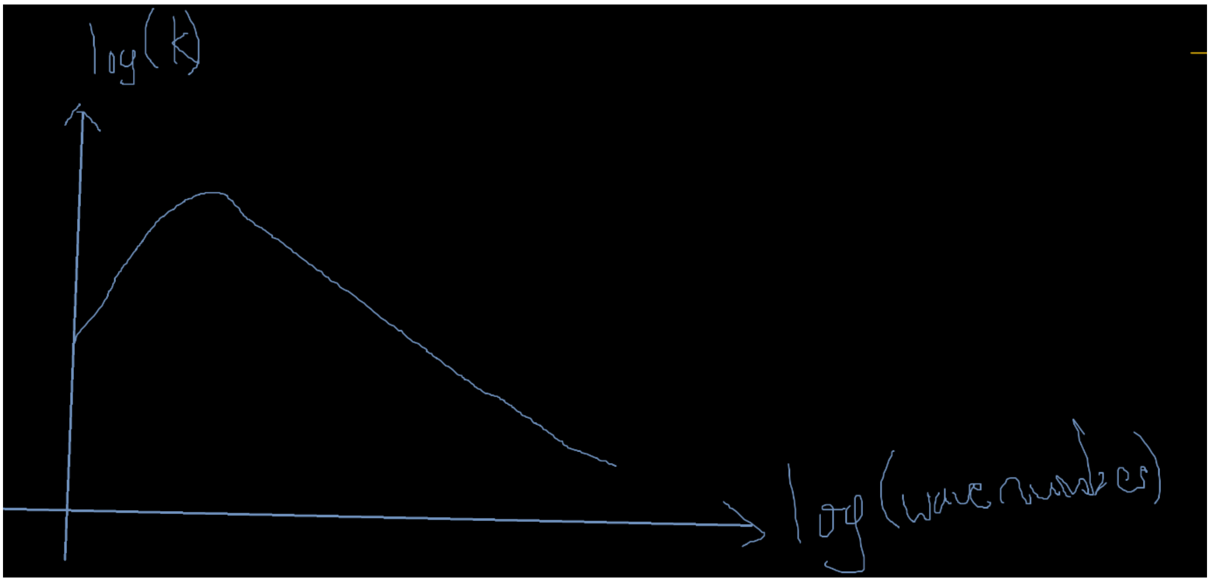
Smaller sine waves → high frequency

Do large eddies correspond to big or small sine waves?

Do small eddies correspond to big or small sine waves?

So a typical frequency spectrum is like this:

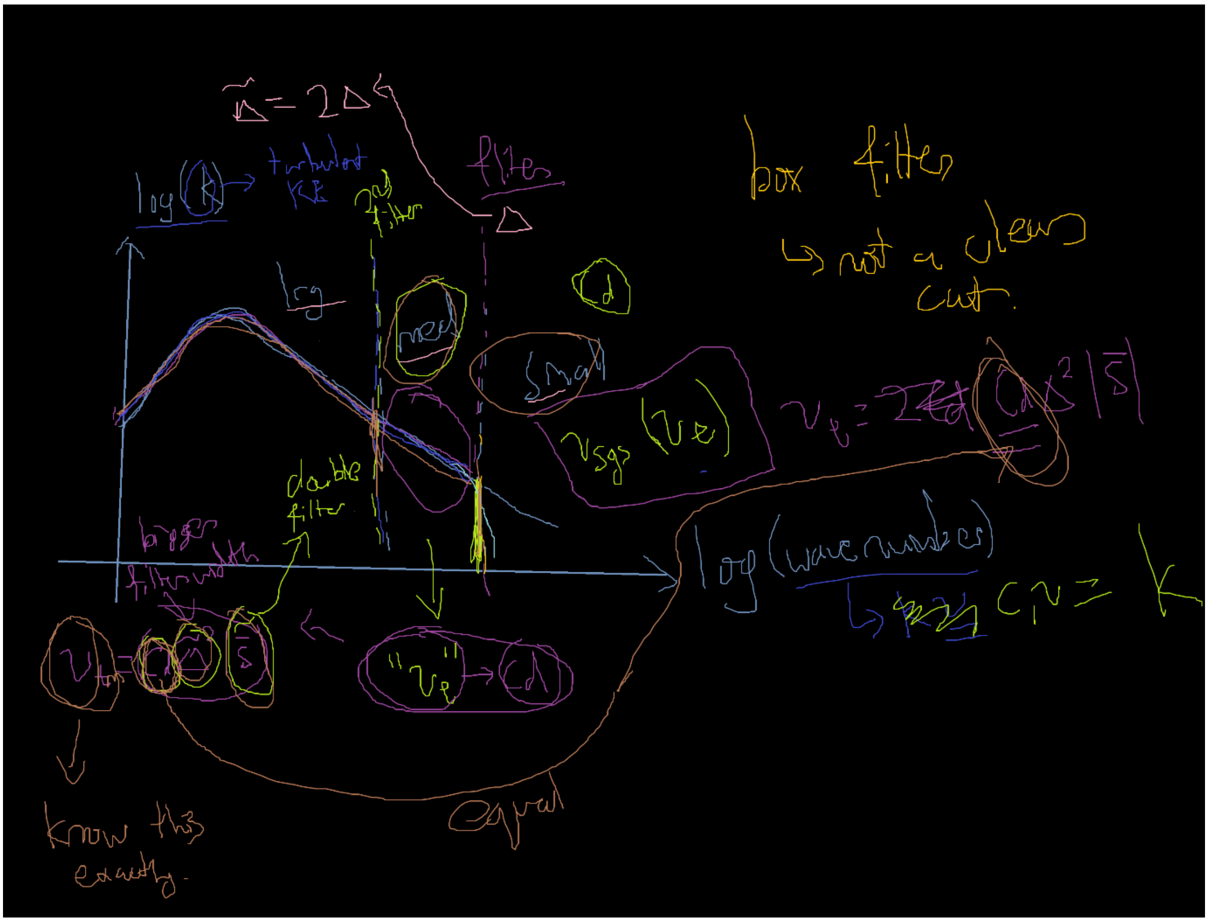
On a log-log plot of kinetic energy vs wavenumber



The small, medium and large eddies can be segregated in the following manner:

In general smaller filters widths correspond to higher frequency.

Let’s see what filtering does to this spectrum…



The process of filtering a second time to get the C_d is known as test filtering

So now that we talked about it briefly, we need to talk about filtering more mathematically.

If you recall our previous videos, we had a very **basic** introduction to filtering, enough to get conceptual understanding, but not quite enough to go through the hard math…

Now we need to go through the hard math part because we need to **double filter/test filtering**

So usually we need to filter our N-S equations, let's recap:

$$[\mu\nabla^2 u] - \frac{\partial P}{\partial x} + \rho\, g_x \; = \rho\left(\frac{D}{Dt}u\right)$$

Or in incompressible form and Einstein combined with tensor notation:

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j}(u_i u_j) = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + g_i$$

Conservation of mass (continuity)

$$\frac{\partial u_i}{\partial x_i} = 0$$

So for our analysis, we need to be more familiar with two terms and how to filter them properly:

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j}(u_i u_j) = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + g_i$$

Just to rule out some confusion in notation:

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Some people use the notation:

$$S_{ij} \approx \frac{1}{2}(\nabla\vec{u} + (\nabla\vec{u})^T)$$

$$(\nabla\vec{u})^T \rightarrow transpose$$

Gradient of velocity is a vector

$$\nabla\vec{u} = \begin{bmatrix} \frac{\partial}{\partial x} v_x & \frac{\partial}{\partial y} v_x & \frac{\partial}{\partial z} v_x \\ \frac{\partial}{\partial x} v_y & \frac{\partial}{\partial y} v_y & \frac{\partial}{\partial z} v_y \\ \frac{\partial}{\partial x} v_z & \frac{\partial}{\partial y} v_z & \frac{\partial}{\partial z} v_z \end{bmatrix}$$

Recall:

$$\nabla u_x = \begin{bmatrix} \frac{\partial}{\partial x} u_x \\ \frac{\partial}{\partial y} u_x \\ \frac{\partial}{\partial z} u_x \end{bmatrix};\ \nabla u_y = \begin{bmatrix} \frac{\partial}{\partial x} u_y \\ \frac{\partial}{\partial y} u_y \\ \frac{\partial}{\partial z} u_y \end{bmatrix};\ \nabla u_z = \begin{bmatrix} \frac{\partial}{\partial x} u_z \\ \frac{\partial}{\partial y} u_z \\ \frac{\partial}{\partial z} u_z \end{bmatrix}$$

The transpose is then:

$$(\nabla\vec{u})^T = \begin{bmatrix} \frac{\partial}{\partial x} v_x & \frac{\partial}{\partial x} v_y & \frac{\partial}{\partial x} v_z \\ \frac{\partial}{\partial y} v_x & \frac{\partial}{\partial y} v_y & \frac{\partial}{\partial y} v_z \\ \frac{\partial}{\partial z} v_x & \frac{\partial}{\partial z} v_y & \frac{\partial}{\partial z} v_z \end{bmatrix}$$

Adding them together:

$$S_{ij} \approx \frac{1}{2}(\nabla\vec{u} + (\nabla\vec{u})^T)$$

$$S_{ij} \approx \frac{1}{2}\left(\begin{bmatrix} \frac{\partial}{\partial x} v_x & \frac{\partial}{\partial y} v_x & \frac{\partial}{\partial z} v_x \\ \frac{\partial}{\partial x} v_y & \frac{\partial}{\partial y} v_y & \frac{\partial}{\partial z} v_y \\ \frac{\partial}{\partial x} v_z & \frac{\partial}{\partial y} v_z & \frac{\partial}{\partial z} v_z \end{bmatrix} + \begin{bmatrix} \frac{\partial}{\partial x} v_x & \frac{\partial}{\partial x} v_y & \frac{\partial}{\partial x} v_z \\ \frac{\partial}{\partial y} v_x & \frac{\partial}{\partial y} v_y & \frac{\partial}{\partial y} v_z \\ \frac{\partial}{\partial z} v_x & \frac{\partial}{\partial z} v_y & \frac{\partial}{\partial z} v_z \end{bmatrix}\right)$$

https://en.wikipedia.org/wiki/Strain-rate_tensor

obviously we don't want to write this all out, so we use tensor notation/Einstein notation to compress each term

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$\frac{\partial u_i}{\partial x_j} = \nabla\vec{u}$$

Element in row I, column j is $\frac{\partial u_i}{\partial x_j}$

so this is short hand for a 3x3 matrix of row i column j, let's say i=1, j=1

$$S_{11} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_1}\right) = \frac{\partial u_1}{\partial x_1} = \frac{\partial u}{\partial x}$$

so for example, the element in row 3 column 1 is

$$S_{31} = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right)$$

https://en.wikipedia.org/wiki/Infinitesimal_strain_theory#Infinitesimal_strain_tensor

no matter how you write it, they take on pretty much the same physical meaning.

If you're interested in the math, there's more nuance to it, but I'm not going into that here.

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j}(u_i u_j) = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + g_i$$

How do we make sense of:

$$\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)?$$

It's supposedly equal to:

$$\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \nabla^2 \vec{u}$$

Well here's a clue, something more familar:

$$\nabla^2 \vec{u} = \begin{bmatrix} \nabla^2 u_x \\ \nabla^2 u_y \\ \nabla^2 u_z \end{bmatrix}$$

So we know this is a 3 row 1 column vector…

How do we know?

In this case

$$\frac{\partial}{\partial x_j} = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$$

$$\frac{\partial}{\partial x_j} = \nabla \cdot$$

Divergence, remember?

$$\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)=\frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial x_j}+\frac{\partial}{\partial x_j}\frac{\partial u_j}{\partial x_i}$$

First term is relatively simple:

We do this first for i=1

$$\frac{\partial}{\partial x_j}\frac{\partial u_1}{\partial x_j}=\frac{\partial^2}{\partial x^2}u_1+\frac{\partial^2}{\partial y^2}u_1+\frac{\partial^2}{\partial z^2}u_1=\nabla^2 u_1$$

If repeated over 3 directions

$$\frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial x_j}=\frac{\partial^2}{\partial x^2}u_i+\frac{\partial^2}{\partial y^2}u_i+\frac{\partial^2}{\partial z^2}u_i=\nabla^2\vec{u}$$

How about the second term?

Well we can apply some partial derivative rules to switch it around…

$$\frac{\partial}{\partial x_j}\frac{\partial u_j}{\partial x_i}=\frac{\partial}{\partial x_i}\frac{\partial u_j}{\partial x_j}=\frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x}u_1+\frac{\partial}{\partial y}u_2+\frac{\partial}{\partial z}u_3\right)=\nabla(\nabla\cdot\vec{u})$$

I didn't prove $\frac{\partial}{\partial x_j}\frac{\partial u_j}{\partial x_i}=\frac{\partial}{\partial x_i}\frac{\partial u_j}{\partial x_j}$

$$\frac{\partial}{\partial x_j}\frac{\partial u_j}{\partial x_i}$$

Let's consider i=1

$$\frac{\partial}{\partial x_j}\frac{\partial u_j}{\partial x_1}=\frac{\partial}{\partial x}\frac{\partial u_x}{\partial x}+\frac{\partial}{\partial y}\frac{\partial u_y}{\partial x}+\frac{\partial}{\partial z}\frac{\partial u_z}{\partial x}=\frac{\partial}{\partial x_1}(\nabla\cdot\vec{u})=\frac{\partial}{\partial x_1}\left(\frac{\partial u_j}{\partial x_j}\right)$$

$$\nabla\cdot\vec{u}=0$$

Remember in incompressible flows, $\nabla(\nabla\cdot\vec{u})=0$

Normally we can just cancel the terms and transform

$$\frac{\partial u_i}{\partial t}+\frac{\partial}{\partial x_j}(u_i u_j)=-\frac{1}{\rho}\frac{\partial P}{\partial x_i}+\nu\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)+g_i$$

into

$$\frac{\partial u_i}{\partial t}+\frac{\partial}{\partial x_j}(u_i u_j)=-\frac{1}{\rho}\frac{\partial P}{\partial x_i}+\nu\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j}\right)+g_i$$

$$[\mu\nabla^2 u]-\frac{\partial P}{\partial x}+\rho\, g_x\ =\rho\left(\frac{D}{Dt}u\right)$$

But now we have filtering, doing so would eliminate some of the reynold's stresses or more correctly, subgrid stresses we need to consider in LES… remember we need to expand everything out and filter them!

So remember how we defined a general filter in 1D?

$$\bar{u}(x) = \int_{-\infty}^{+\infty} u(x')G(x-x')\,dx'$$

If we generalize to 3 coordinates

$$\bar{u}(\vec{x}) = \int_{-\infty}^{+\infty} u(\xi)G(\vec{x}-\xi)\,d^3\xi$$

This is just short hand…

A generic filter can also include a time function, remember RANS?

$$\bar{u}(x,t) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} u(\xi,t')G(x-\xi,t-t')\,d^3\xi\,dt'$$

So we extend what we do with space also in the dimension of time…

G is our filter function which gives sort of a "weighted average", a more formal term is called **convolution kernel**

https://en.wikipedia.org/wiki/Integral_transform

the above has the details…

Properties of filters (important because we going to start filtering NS equations again)

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j}(u_i u_j) = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + g_i$$

What are some properties? (they come from the integral)

1) Conservations of constants: you can take the constants outside the integral
   a. Filter $au(x)$, a is a constant
   b. $\overline{au(x)} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} a * u(\xi,t')G(x-\xi,t-t')\,d^3\xi\,dt'$
   c. $\overline{au(x)} = a\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} u(\xi,t')G(x-\xi,t-t')\,d^3\xi\,dt' = a\bar{u}$
   d. From this we can say: $\overline{au(x)} = a\overline{u(x)}$
2) Linearity: $\overline{\phi+\psi} = \bar{\phi} + \bar{\psi}$
   a. $\overline{\phi+\psi} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(\phi+\psi)(\xi,t')G(x-\xi,t-t')\,d^3\xi\,dt'$

b. $\overline{\phi+\psi} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(\phi)(\xi,t')G(x-\xi,t-t')\,d^3\xi\,dt' + \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(\psi)(\xi,t')G(x-\xi,t-t')\,d^3\xi\,dt'$

c. $\bar{\phi} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(\phi)(\xi,t')G(x-\xi,t-t')\,d^3\xi\,dt'$

d. $\bar{\psi} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(\psi)(\xi,t')G(x-\xi,t-t')\,d^3\xi\,dt'$

e. Hence you can prove: $\overline{\phi+\psi} = \bar{\phi} + \bar{\psi}$

3) Commutation with derivation: $\overline{\frac{\partial\phi}{\partial s}} = \frac{\partial\bar{\phi}}{\partial s}$

a. $\overline{\frac{\partial\phi}{\partial s}} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{\partial\phi}{\partial s}(\xi,t')G(x-\xi,t-t')\,d^3\xi\,dt'$

b. $\overline{\frac{\partial\phi}{\partial s}} = \frac{\partial}{\partial s}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\phi(\xi,t')G(x-\xi,t-t')\,d^3\xi\,dt'$

c. $\overline{\frac{\partial\phi}{\partial s}} = \frac{\partial}{\partial s}\bar{\phi}$

Filtering the Navier Stokes Equation (first time we do it in more detail)

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j}(u_i u_j) = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + g_i$$

Ignore gravity and filter, (remember the above properties, plus it's incompressible)

$$\overline{\left\{\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j}(u_i u_j)\right\}} = \overline{\left\{-\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right\}}$$

$$\overline{\frac{\partial u_i}{\partial t}} + \overline{\left\{\frac{\partial}{\partial x_j}(u_i u_j)\right\}} = \overline{\left\{-\frac{1}{\rho}\frac{\partial P}{\partial x_i}\right\}} + \overline{\left\{\nu\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right\}}$$

$$\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial}{\partial x_j}\overline{(u_i u_j)} = -\frac{1}{\rho}\frac{\partial \bar{P}}{\partial x_i} + \nu\frac{\partial}{\partial x_j}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)$$

$$\frac{\partial \bar{u}_i}{\partial x_i} = 0$$

How do we do filtering of $\overline{(u_i u_j)}$

We'll need to expand out u_i and u_j

And basically

$$u_i = \bar{u}_i + u_i'$$

$$u_j = \bar{u}_j + u_j'$$

The total velocity is the filtered part and non filtered part

The interesting bit (where the reynold's stresses, or rather unfiltered stresses come about) is the

$$\overline{(u_i u_j)} = \overline{(\bar{u}_i + u_i')(\bar{u}_j + u_j')}$$

Expand this out…

$$\overline{(\bar{u}_i + u_i')(\bar{u}_j + u_j')} = \overline{(\bar{u}_i\bar{u}_j + \bar{u}_i u_j' + u_i'\bar{u}_j + u_i'u_j')}$$

$$= \overline{\bar{u}_i\bar{u}_j} + \overline{\bar{u}_i u_j'} + \overline{u_i'\bar{u}_j} + \overline{u_i'u_j'}$$

We give special names to each term:

$$\tau_{ij} = \rho\left(\overline{\bar{u}_i u_j'} + \overline{u_i'\bar{u}_j} + \overline{u_i'u_j'}\right)$$

This is known as the subgrid tensor τ_{ij}…

<u>Subgrid velocity = unfiltered part of the velocity</u>

$$\overline{u_i'u_j'} = R_{ij}\ (Reynold's\ subgrid\ tensor)$$

$$\overline{\bar{u}_i u_j'} + \overline{u_i'\bar{u}_j} = C_{ij}\ (cross-stress\ tensor)$$

Let's go back and make one more adjustment…

$$\frac{\partial\bar{u}_i}{\partial t} + \frac{\partial}{\partial x_j}\left(\overline{\bar{u}_i\bar{u}_j} + \overline{\bar{u}_i u_j'} + \overline{u_i'\bar{u}_j} + \overline{u_i'u_j'}\right) = -\frac{1}{\rho}\frac{\partial\bar{P}}{\partial x_i} + \nu\frac{\partial}{\partial x_j}\left(\frac{\partial\bar{u}_i}{\partial x_j} + \frac{\partial\bar{u}_j}{\partial x_i}\right)$$

$$\frac{\partial\bar{u}_i}{\partial t} + \frac{\partial}{\partial x_j}\overline{\bar{u}_i\bar{u}_j} + \frac{\partial}{\partial x_j}\left(\overline{\bar{u}_i u_j'} + \overline{u_i'\bar{u}_j} + \overline{u_i'u_j'}\right) = -\frac{1}{\rho}\frac{\partial\bar{P}}{\partial x_i} + \nu\frac{\partial}{\partial x_j}\left(\frac{\partial\bar{u}_i}{\partial x_j} + \frac{\partial\bar{u}_j}{\partial x_i}\right)$$

What we actually want…

$$\frac{\partial\bar{u}_i}{\partial t} + \frac{\partial}{\partial x_j}\bar{u}_i\bar{u}_j + \frac{\partial}{\partial x_j}\left(\overline{\bar{u}_i u_j'} + \overline{u_i'\bar{u}_j} + \overline{u_i'u_j'}\right) + \cdots = -\frac{1}{\rho}\frac{\partial\bar{P}}{\partial x_i} + \nu\frac{\partial}{\partial x_j}\left(\frac{\partial\bar{u}_i}{\partial x_j} + \frac{\partial\bar{u}_j}{\partial x_i}\right)$$

https://en.wikipedia.org/wiki/Large_eddy_simulation

Leonard's decomposition

$$\overline{\bar{u}_i\bar{u}_j} = \bar{u}_i\bar{u}_j + L_{ij}$$

$$L_{ij} = Leonard's tensor$$

$$L_{ij} = \overline{\bar{u}_i\bar{u}_j} - \bar{u}_i\bar{u}_j$$

Represents interaction between large (filtered) scales

$$\frac{\partial\bar{u}_i}{\partial t} + \frac{\partial}{\partial x_j}\left(\bar{u}_i\bar{u}_j + L_{ij}\right) + \frac{\partial}{\partial x_j}\left(\overline{\bar{u}_i u_j'} + \overline{u_i'\bar{u}_j} + \overline{u_i'u_j'}\right) = -\frac{1}{\rho}\frac{\partial\bar{P}}{\partial x_i} + \nu\frac{\partial}{\partial x_j}\left(\frac{\partial\bar{u}_i}{\partial x_j} + \frac{\partial\bar{u}_j}{\partial x_i}\right)$$

$$\frac{\partial\bar{u}_i}{\partial t} + \frac{\partial}{\partial x_j}\left(\bar{u}_i\bar{u}_j\right) + \frac{\partial}{\partial x_j}\left(\overline{\bar{u}_i u_j'} + \overline{u_i'\bar{u}_j} + \overline{u_i'u_j'} + L_{ij}\right) = -\frac{1}{\rho}\frac{\partial\bar{P}}{\partial x_i} + \nu\frac{\partial}{\partial x_j}\left(\frac{\partial\bar{u}_i}{\partial x_j} + \frac{\partial\bar{u}_j}{\partial x_i}\right)$$

$$\frac{\partial\bar{u}_i}{\partial t} + \frac{\partial}{\partial x_j}\left(\bar{u}_i\bar{u}_j\right) + \frac{\partial}{\partial x_j}\left(C_{ij} + R_{ij} + L_{ij}\right) = -\frac{1}{\rho}\frac{\partial\bar{P}}{\partial x_i} + \nu\frac{\partial}{\partial x_j}\left(\frac{\partial\bar{u}_i}{\partial x_j} + \frac{\partial\bar{u}_j}{\partial x_i}\right)$$

We need to model $C_{ij} + R_{ij} + L_{ij}$

We define

$$\tau_{ij}^r = \rho\left(C_{ij} + R_{ij} + L_{ij}\right) = \tau_{ij} + \rho\left(L_{ij}\right)$$

The following is known as triple decomposition or Leonard's decomposition

$$\overline{(\overline{u_i} + u_i')(\overline{u_j} + u_j')} = \overline{(\overline{u_i}\,\overline{u_j} + \overline{u_i}u_j' + u_i'\overline{u_j} + u_i'u_j')}$$

$$= \overline{\overline{u_i}\,\overline{u_j}} + \overline{\overline{u_i}u_j'} + \overline{u_i'\overline{u_j}} + \overline{u_i'u_j'}$$

$$= \overline{u_i}\,\overline{u_j} + L_{ij} + C_{ij} + R_{ij}$$

Recall:

We model the subgrid tensor in smagorinsky model as:

https://www.cfd-online.com/Wiki/Smagorinsky-Lilly_model

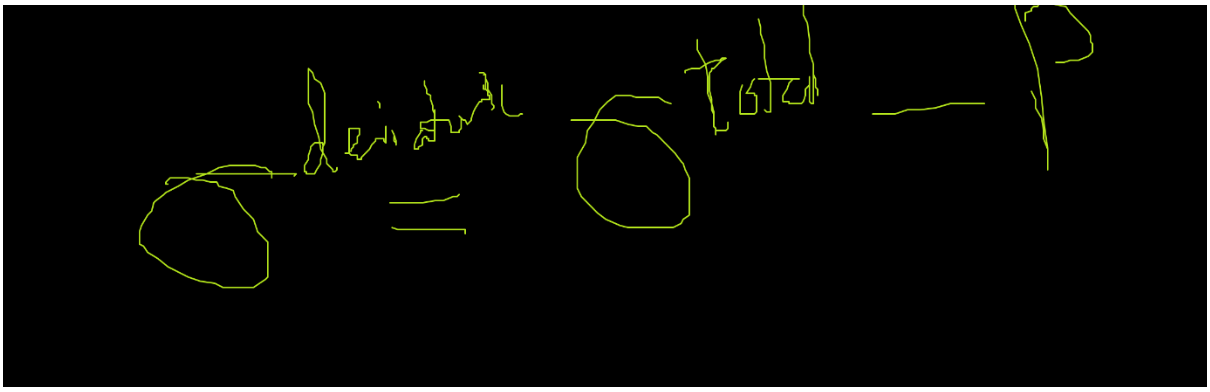
$$\tau_{ij}^r - \frac{1}{3}\tau_{kk}^r\delta_{ij} = -2\rho\nu_{sgs}S_{ij}$$

$$\nu_{sgs} = \left(C_{smagorinsky}\Delta\right)^2|\bar{S}|$$

Where

$$|\bar{S}| = \sqrt{2S_{ij}S_{ij}}$$

$$S_{ij} = \frac{1}{2}\left(\frac{\partial\overline{u_i}}{\partial x_j} + \frac{\partial\overline{u_j}}{\partial x_i}\right)$$



So we model the deviatoric subgrid stresses or residual stresses with the smagorinsky subgrid viscosity model

$$\tau_{ij}^r - \frac{1}{3}\tau_{kk}^r\delta_{ij} = -2\rho\nu_{sgs}S_{ij}$$

$$\tau_{ij}^r - \frac{1}{3}\tau_{kk}^r\delta_{ij} = \begin{bmatrix} \tau_{11}^r & \tau_{12}^r & \tau_{13}^r \\ \tau_{21}^r & \tau_{22}^r & \tau_{23}^r \\ \tau_{31}^r & \tau_{32}^r & \tau_{33}^r \end{bmatrix} - \frac{1}{3}\begin{bmatrix} \tau_{11}^r + \tau_{22}^r + \tau_{33}^r & 0 & 0 \\ 0 & \tau_{11}^r + \tau_{22}^r + \tau_{33}^r & 0 \\ 0 & 0 & \tau_{11}^r + \tau_{22}^r + \tau_{33}^r \end{bmatrix}$$

$$\nu_{sgs} = \left(C_{smagorinsky}\Delta\right)^2 |\bar{S}|$$

Where

$$|\bar{S}| = \sqrt{2S_{ij}S_{ij}}$$

$$S_{ij} = \frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)$$

Result:

$$\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial}{\partial x_j}\left(\bar{u}_i\bar{u}_j\right) = -\frac{1}{\rho}\frac{\partial \bar{P}}{\partial x_i} + \nu\frac{\partial}{\partial x_j}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right) - \frac{\partial}{\partial x_j}\left(C_{ij} + R_{ij} + L_{ij}\right)$$

$$\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial}{\partial x_j}\left(\bar{u}_i\bar{u}_j\right) = -\frac{1}{\rho}\frac{\partial \bar{P}}{\partial x_i} + \nu\frac{\partial}{\partial x_j}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right) + \frac{\partial}{\partial x_j}\left(\nu_{sgs}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)\right)$$

$$\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial}{\partial x_j}\left(\bar{u}_i\bar{u}_j\right) = -\frac{1}{\rho}\frac{\partial \bar{P}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\{\nu + \nu_{sgs}\}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)\right)$$

Cross stress tensor: large and small scale interaction (C_{ij})

Subgrid Tensor: small scales only

Double filtering (not double decomposition)

- So remember how we do our closure relations with the subgrid stresses…

We need to do a second round of filtering to determine $C_d = C_{smagorinsky}^2$, this is known as test filtering…

Filter width of this test filter $\widetilde{\Delta} = 2\Delta$

$$\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial}{\partial x_j}\left(\bar{u}_i\bar{u}_j\right) = -\frac{1}{\rho}\frac{\partial \bar{P}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\{\nu + \nu_{sgs}\}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)\right)$$

$$\left\{\frac{\partial \bar{u}_i}{\partial t} + \widetilde{\frac{\partial}{\partial x_j}\left(\bar{u}_i\bar{u}_j\right)}\right\} = \left\{-\frac{1}{\rho}\frac{\partial \bar{P}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\widetilde{\{\nu + \nu_{sgs}\}}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)\right)\right\}$$

$$\frac{\partial \tilde{\bar{u}}_i}{\partial t} + \frac{\partial}{\partial x_j}\left(\widetilde{\bar{u}_i\bar{u}_j}\right) = -\frac{1}{\rho}\frac{\partial \tilde{\bar{P}}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\widetilde{\{\nu + \nu_{sgs}\}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)}\right)$$

$$\left(\widetilde{\bar{u}_i\bar{u}_j}\right) = \cdots$$

Reference the 1st round of filtering…

$$\overline{(\bar{u}_i + u_i')(\bar{u}_j + u_j')} = \overline{(\bar{u}_i\bar{u}_j + \bar{u}_i u_j' + u_i'\bar{u}_j + u_i' u_j')}$$

$$= \overline{\bar{u}_i\bar{u}_j} + \overline{\bar{u}_i u_j'} + \overline{u_i'\bar{u}_j} + \overline{u_i' u_j'}$$

$$= \bar{u}_i\bar{u}_j + L_{ij} + C_{ij} + R_{ij}$$

So for our second round of filtering, by analogy

$$\bar{u} = \tilde{\bar{u}} + \bar{u}'$$

$$\left(\widetilde{\bar{u}_i\bar{u}_j}\right) = \widetilde{\bar{u}_i\bar{u}_j} + \widetilde{\bar{u}_i\bar{u}_j'} + \widetilde{\bar{u}_i'\bar{u}_j} + \widetilde{\bar{u}_i'\bar{u}_j'}$$

$$\left(\widetilde{\bar{u}_i\bar{u}_j}\right) = \tilde{\bar{u}}_i\tilde{\bar{u}}_j + \tilde{L}_{ij} + \widetilde{\bar{u}_i\bar{u}_j'} + \widetilde{\bar{u}_i'\bar{u}_j} + \widetilde{\bar{u}_i'\bar{u}_j'}$$

$$\left(\widetilde{\bar{u}_i\bar{u}_j}\right) = \tilde{\bar{u}}_i\tilde{\bar{u}}_j + \tilde{L}_{ij} + \tilde{C}_{ij} + \tilde{R}_{ij}$$

Looks like I made another mistake in the previous video…

https://www.cfd-online.com/Wiki/Dynamic_subgrid-scale_model

don't combine the τ into the model yet….

$$\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial}{\partial x_j}(\bar{u}_i\bar{u}_j) = -\frac{1}{\rho}\frac{\partial \bar{P}}{\partial x_i} + \nu\frac{\partial}{\partial x_j}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right) - \frac{\partial}{\partial x_j}(\tau_{ij}^r)$$

$$\frac{\partial \tilde{\bar{u}}_i}{\partial t} + \frac{\partial}{\partial x_j}\left(\widetilde{\bar{u}_i\bar{u}_j}\right) = -\frac{1}{\rho}\frac{\partial \tilde{\bar{P}}}{\partial x_i} + \nu\frac{\partial}{\partial x_j}\left(\frac{\partial \tilde{\bar{u}}_i}{\partial x_j} + \frac{\partial \tilde{\bar{u}}_j}{\partial x_i}\right) - \frac{\partial}{\partial x_j}(\widetilde{\tau_{ij}^r})$$

We decompose this term $\left(\widetilde{\bar{u}_i\bar{u}_j}\right)$ as such

$$\left(\widetilde{\bar{u}_i\bar{u}_j}\right) = \tilde{\bar{u}}_i\tilde{\bar{u}}_j + \tilde{L}_{ij} + \tilde{C}_{ij} + \tilde{R}_{ij}$$

$$\left(\widetilde{\bar{u}_i\bar{u}_j}\right) = \tilde{\bar{u}}_i\tilde{\bar{u}}_j + \mathcal{L}_{ij}^r$$

$$\frac{\partial \tilde{\bar{u}}_i}{\partial t} + \frac{\partial}{\partial x_j}(\tilde{\bar{u}}_i\tilde{\bar{u}}_j + \mathcal{L}_{ij}^r) = -\frac{1}{\rho}\frac{\partial \tilde{\bar{P}}}{\partial x_i} + \nu\frac{\partial}{\partial x_j}\left(\frac{\partial \tilde{\bar{u}}_i}{\partial x_j} + \frac{\partial \tilde{\bar{u}}_j}{\partial x_i}\right) - \frac{\partial}{\partial x_j}(\widetilde{\tau_{ij}^r})$$

$$\frac{\partial \tilde{\bar{u}}_i}{\partial t} + \frac{\partial}{\partial x_j}(\tilde{\bar{u}}_i\tilde{\bar{u}}_j) = -\frac{1}{\rho}\frac{\partial \tilde{\bar{P}}}{\partial x_i} + \nu\frac{\partial}{\partial x_j}\left(\frac{\partial \tilde{\bar{u}}_i}{\partial x_j} + \frac{\partial \tilde{\bar{u}}_j}{\partial x_i}\right) - \frac{\partial}{\partial x_j}(\widetilde{\tau_{ij}^r} + \mathcal{L}_{ij}^r)$$

$$\frac{\partial \tilde{\bar{u}}_i}{\partial t} + \frac{\partial}{\partial x_j}(\tilde{\bar{u}}_i\tilde{\bar{u}}_j + \widetilde{\tau_{ij}^r} + \mathcal{L}_{ij}^r) = -\frac{1}{\rho}\frac{\partial \tilde{\bar{P}}}{\partial x_i} + \nu\frac{\partial}{\partial x_j}\left(\frac{\partial \tilde{\bar{u}}_i}{\partial x_j} + \frac{\partial \tilde{\bar{u}}_j}{\partial x_i}\right)$$

$$\left(\widetilde{\bar{u}_i\bar{u}_j}\right) = \tilde{\bar{u}}_i\tilde{\bar{u}}_j + \mathcal{L}_{ij}^r$$

$$\frac{\partial \tilde{\bar{u}}_i}{\partial t} + \frac{\partial}{\partial x_j}\left(\left(\widetilde{\bar{u}_i\bar{u}_j}\right) + \widetilde{\tau_{ij}^r}\right) = -\frac{1}{\rho}\frac{\partial \tilde{\bar{P}}}{\partial x_i} + \nu\frac{\partial}{\partial x_j}\left(\frac{\partial \tilde{\bar{u}}_i}{\partial x_j} + \frac{\partial \tilde{\bar{u}}_j}{\partial x_i}\right)$$

We use this identity:

$$\overline{u_i u_j} = \bar{u}_i\bar{u}_j + L_{ij} + C_{ij} + R_{ij}$$

$$\widetilde{\overline{u_i u_j}} = \widetilde{\overline{u_i}\,\overline{u_j}} + \widetilde{L_{ij}} + \widetilde{C_{ij}} + \widetilde{R_{ij}}$$

$$\left(\widetilde{\overline{u_i u_j}}\right) = \left(\widetilde{\overline{u_i}\,\overline{u_j}}\right) + \widetilde{\tau_{ij}^r}$$

$$\left(\widetilde{\overline{u_i u_j}}\right) = \tilde{\bar{u}}_i \tilde{\bar{u}}_j + \mathcal{L}_{ij}^r + \widetilde{\tau_{ij}^r}$$

$$\left(\widetilde{\overline{u_i u_j}}\right) = \tilde{\bar{u}}_i \tilde{\bar{u}}_j + T_{ij}$$

$$T_{ij} = \mathcal{L}_{ij}^r + \widetilde{\tau_{ij}^r}$$

This is known as Germano's identity…

$$\mathcal{L}_{ij}^r = T_{ij} - \widetilde{\tau_{ij}^r}$$

$$\frac{\partial \tilde{\bar{u}}_i}{\partial t} + \frac{\partial}{\partial x_j}\left(\tilde{\bar{u}}_i \tilde{\bar{u}}_j + \widetilde{\tau_{ij}^r} + \mathcal{L}_{ij}^r\right) = -\frac{1}{\rho}\frac{\partial \tilde{\bar{P}}}{\partial x_i} + \nu \frac{\partial}{\partial x_j}\left(\frac{\partial \tilde{\bar{u}}_i}{\partial x_j} + \frac{\partial \tilde{\bar{u}}_j}{\partial x_i}\right)$$

$$\frac{\partial \tilde{\bar{u}}_i}{\partial t} + \frac{\partial}{\partial x_j}\left(\tilde{\bar{u}}_i \tilde{\bar{u}}_j\right) = -\frac{1}{\rho}\frac{\partial \tilde{\bar{P}}}{\partial x_i} + \nu \frac{\partial}{\partial x_j}\left(\frac{\partial \tilde{\bar{u}}_i}{\partial x_j} + \frac{\partial \tilde{\bar{u}}_j}{\partial x_i}\right) - \frac{\partial}{\partial x_j}\left(\widetilde{\tau_{ij}^r} + \mathcal{L}_{ij}^r\right)$$

Remember how we dealt with:

We dealt with τ_{ij}^r (the deviatoric part) in the following manner…

$$\left(\tau_{ij}^r - \frac{1}{3}\tau_{kk}^r \delta_{ij}\right)_{dynamic} = -2\rho \nu_{sgs} S_{ij}$$

$$\tau_{ij}^r - \frac{1}{3}\tau_{kk}^r \delta_{ij} = \begin{bmatrix} \tau_{11}^r & \tau_{12}^r & \tau_{13}^r \\ \tau_{21}^r & \tau_{22}^r & \tau_{23}^r \\ \tau_{31}^r & \tau_{32}^r & \tau_{33}^r \end{bmatrix} - \frac{1}{3}\begin{bmatrix} \tau_{11}^r + \tau_{22}^r + \tau_{33}^r & 0 & 0 \\ 0 & \tau_{11}^r + \tau_{22}^r + \tau_{33}^r & 0 \\ 0 & 0 & \tau_{11}^r + \tau_{22}^r + \tau_{33}^r \end{bmatrix}$$

$$\nu_{sgs} = \left(C_{smagorinsky}\Delta\right)^2 |\bar{S}|$$

$$|\bar{S}| = \sqrt{2 S_{ij} S_{ij}}$$

$$S_{ij} = \frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)$$

$$\left(\tau_{ij}^r - \frac{1}{3}\tau_{kk}^r \delta_{ij}\right)_{dynamic} = -2\rho\left(C_{smagorinsky}\Delta\right)^2 |\bar{S}| S_{ij}$$

$$\left(\tau_{ij}^r - \frac{1}{3}\tau_{kk}^r \delta_{ij}\right)_{dynamic} = -2\rho C_{smagorinsky}^2 (\Delta)^2 |\bar{S}| S_{ij}$$

$$\left(\tau_{ij}^r - \frac{1}{3}\tau_{kk}^r \delta_{ij}\right)_{dynamic} = -2\rho C_d (\Delta)^2 |\bar{S}| S_{ij}$$

$$\tau_{ij}^r - \frac{1}{3}\tau_{kk}^r \delta_{ij} = -2 C_d (\Delta)^2 |\bar{S}| S_{ij}$$

We can deal with this $\mathcal{L}_{ij}^{r}$ in a similar manner…

$$\mathcal{L}_{ij}^{r} = T_{ij} - \widetilde{\tau_{ij}^{r}}$$

Doing this means applying the above relation

$$\left(\tau_{ij}^{r} - \frac{1}{3}\tau_{kk}^{r}\delta_{ij}\right)_{dynamic} = -2\rho\nu_{sgs}S_{ij}$$

$$\tau_{ij}^{r} - \frac{1}{3}\tau_{kk}^{r}\delta_{ij} = -2C_d(\Delta)^2|\bar{S}|S_{ij}$$

To T_{ij} as well

$$T_{ij} - \frac{1}{3}T_{kk}\delta_{ij} = -2C_d(\widetilde{\Delta})^2|\tilde{\bar{S}}|\tilde{S}_{ij}$$

Where

$$\tilde{S}_{ij} = \frac{1}{2}\left(\frac{\partial\tilde{\bar{u}}_i}{\partial x_j} + \frac{\partial\tilde{\bar{u}}_j}{\partial x_i}\right)$$

$$|\tilde{\bar{S}}| = \sqrt{2\tilde{S}_{ij}\tilde{S}_{ij}}$$

Substitute back into our $\mathcal{L}_{ij}$ equation

$$\mathcal{L}_{ij}^{r} = T_{ij} - \widetilde{\tau_{ij}^{r}}$$

$$\mathcal{L}_{ij}^{r} - \frac{1}{3}\mathcal{L}_{kk}\delta_{ij} = \left(T_{ij} - \frac{1}{3}T_{kk}\delta_{ij}\right) - \left(\widetilde{\tau_{ij}^{r} - \frac{1}{3}\tau_{kk}^{r}\delta_{ij}}\right)$$

$$\mathcal{L}_{ij}^{r} - \frac{1}{3}\mathcal{L}_{kk}\delta_{ij} = \left(-2C_d(\widetilde{\Delta})^2|\tilde{\bar{S}}|\tilde{S}_{ij}\right) - \left(-2C_d\widetilde{(\Delta)^2|\bar{S}|S_{ij}}\right)$$

$$\mathcal{L}_{ij}^{r} - \frac{1}{3}\mathcal{L}_{kk}\delta_{ij} = \left(-2C_d(\widetilde{\Delta})^2|\tilde{\bar{S}}|\tilde{S}_{ij}\right) - \left(-2C_d\widetilde{(\Delta)^2|\bar{S}|S_{ij}}\right)$$

Let's define for convenience:

$$\alpha_{ij}\ \&\ \beta_{ij}$$

$$\alpha_{ij} = -2(\widetilde{\Delta})^2|\tilde{\bar{S}}|\tilde{S}_{ij}$$

$$\beta_{ij} = -2(\Delta)^2|\bar{S}|S_{ij}$$

$$\mathcal{L}_{ij}^{r} - \frac{1}{3}\mathcal{L}_{kk}\delta_{ij} = (\alpha_{ij}C_d) - (\widetilde{\beta_{ij}C_d})$$

$$\mathcal{L}_{ij}^{r} - \frac{1}{3}\mathcal{L}_{kk}\delta_{ij} = \left(-2C_d(\widetilde{\Delta})^2|\tilde{\bar{S}}|\tilde{S}_{ij}\right) - \left(-2C_d\widetilde{(\Delta)^2|\bar{S}|S_{ij}}\right)$$

We need to assume, C_d is constant w.r.t the filter so we can carry on the calculation…

Filter is usually spatial, we need to assume C_d is locally constant

- ➔ C_d actually changes with space…

➔ $\left(\widetilde{\beta_{ij}C_d}\right) \approx C_d\widetilde{\beta_{ij}}$

$$\left(-2C_d\widetilde{(\Delta)^2|\bar{S}|S_{ij}}\right) \approx C_d\left(-2\widetilde{(\Delta)^2|\bar{S}|S_{ij}}\right)$$

$$\mathcal{L}_{ij}^r - \frac{1}{3}\mathcal{L}_{kk}\delta_{ij} \approx C_d\left\{\left(-2\left(\tilde{\Delta}\right)^2|\tilde{\bar{S}}|\tilde{S}_{ij}\right) - \left(-2\widetilde{(\Delta)^2|\bar{S}|S_{ij}}\right)\right\}$$

Since C_d is not truly constant with space, we need to find an **average** C_d which bets fits the equation locally (ie within the filter)

How do we find C_d now,

We have Germano's identity:

$$\mathcal{L}_{ij}^r = T_{ij} - \widetilde{\tau_{ij}^r}$$

Recall

$$\left(\widetilde{\overline{u_i}\overline{u_j}}\right) = \tilde{\bar{u}}_i\tilde{\bar{u}}_j + \tilde{L}_{ij} + \tilde{C}_{ij} + \tilde{R}_{ij}$$

$$\left(\widetilde{\overline{u_i}\overline{u_j}}\right) = \tilde{\bar{u}}_i\tilde{\bar{u}}_j + \mathcal{L}_{ij}^r$$

$$\mathcal{L}_{ij}^r = \tilde{L}_{ij} + \tilde{C}_{ij} + \tilde{R}_{ij}$$

$$\left(\widetilde{\overline{u_i}\overline{u_j}}\right) - \tilde{\bar{u}}_i\tilde{\bar{u}}_j = \tilde{L}_{ij} + \tilde{C}_{ij} + \tilde{R}_{ij}$$

$$\mathcal{L}_{ij}^r = \left(\widetilde{\overline{u_i}\overline{u_j}}\right) - \tilde{\bar{u}}_i\tilde{\bar{u}}_j$$

Using $\mathcal{L}_{ij}^r$, we can easily compute $\mathcal{L}_{ij}^r - \frac{1}{3}\mathcal{L}_{kk}\delta_{ij}$

$$\mathcal{L}_{ij}^r - \frac{1}{3}\mathcal{L}_{kk}\delta_{ij} \approx C_{d-avg}\left\{\left(-2\left(\tilde{\Delta}\right)^2|\tilde{\bar{S}}|\tilde{S}_{ij}\right) - \left(-2\widetilde{(\Delta)^2|\bar{S}|S_{ij}}\right)\right\}$$

You have roughly one equation for C_d

$$C_d \approx \frac{\mathcal{L}_{ij}^r - \frac{1}{3}\mathcal{L}_{kk}\delta_{ij}}{\left\{\left(-2\left(\tilde{\Delta}\right)^2|\tilde{\bar{S}}|\tilde{S}_{ij}\right) - \left(-2\widetilde{(\Delta)^2|\bar{S}|S_{ij}}\right)\right\}}$$

You're almost done with this dynamic model…

Problem is this:

We have multiple values of C_d

We need to take an average.

$$E_{ij} = \left[\mathcal{L}_{ij}^r - \frac{1}{3}\mathcal{L}_{kk}\delta_{ij}\right] - \left[C_d\left\{\left(-2\left(\tilde{\Delta}\right)^2|\tilde{\bar{S}}|\tilde{S}_{ij}\right) - \left(-2\widetilde{(\Delta)^2|\bar{S}|S_{ij}}\right)\right\}\right]$$

So we have a scalar average error so to speak…

$$E_{ij}S_{ij} = \left[\mathcal{L}_{ij}^r - \frac{1}{3}\mathcal{L}_{kk}\delta_{ij}\right]S_{ij} - \left[C_d\left\{\left(-2\left(\tilde{\Delta}\right)^2|\tilde{\bar{S}}|\tilde{S}_{ij}S_{ij}\right) - \left(-2\widetilde{(\Delta)^2|\bar{S}|S_{ij}}\right)S_{ij}\right\}\right]$$

So to find the minimum error:

$$\frac{\partial E_{ij}S_{ij}}{\partial C_d} = 0$$

$$E_{ij}\, S_{ij} = \left[\mathcal{L}_{ij}^{r} - \frac{1}{3}\mathcal{L}_{kk}\delta_{ij}\right]S_{ij} - \left[C_d\left\{\left(-2\left(\widetilde{\Delta}\right)^2\left|\tilde{\bar{S}}\right|\tilde{S}_{ij}S_{ij}\right) - \left(-2\widetilde{(\Delta)^2|\bar{S}|S_{ij}})S_{ij}\right\}\right]$$

$$E_{ij}S_{ij} = \left[\mathcal{L}_{ij}^{r} - \frac{1}{3}\mathcal{L}_{kk}\delta_{ij}\right]S_{ij} - C_d\left[\left\{\left(-2\left(\widetilde{\Delta}\right)^2\left|\tilde{\bar{S}}\right|\tilde{S}_{ij}S_{ij}\right) - \left(-2\widetilde{(\Delta)^2|\bar{S}|S_{ij}})S_{ij}\right\}\right]$$

$$E_{ij}S_{ij} - \left[\mathcal{L}_{ij}^{r} - \frac{1}{3}\mathcal{L}_{kk}\delta_{ij}\right]S_{ij} = -C_d\left[\left\{\left(-2\left(\widetilde{\Delta}\right)^2\left|\tilde{\bar{S}}\right|\tilde{S}_{ij}S_{ij}\right) - \left(-2\widetilde{(\Delta)^2|\bar{S}|S_{ij}})S_{ij}\right\}\right]$$

$$C_d = \frac{E_{min}S_{ij} - \left[\mathcal{L}_{ij}^{r} - \frac{1}{3}\mathcal{L}_{kk}\delta_{ij}\right]S_{ij}}{\left[\left\{\left(2\left(\widetilde{\Delta}\right)^2\left|\tilde{\bar{S}}\right|\tilde{S}_{ij}S_{ij}\right) - \left(2\widetilde{(\Delta)^2|\bar{S}|S_{ij}})S_{ij}\right\}\right]}$$

$$C_d = \frac{\left\{E_{min} - \left[\mathcal{L}_{ij}^{r} - \frac{1}{3}\mathcal{L}_{kk}\delta_{ij}\right]\right\}S_{ij}}{\left[\left\{\left(2\left(\widetilde{\Delta}\right)^2\left|\tilde{\bar{S}}\right|\tilde{S}_{ij}\right) - \left(2\widetilde{(\Delta)^2|\bar{S}|S_{ij}})\right\}\right]S_{ij}}$$

Let’s find $E_{\min}\ S_{ij}$

$$\frac{\partial}{\partial C_d}\ E_{ij}\, S_{ij} = -\left[\left\{\left(-2\left(\widetilde{\Delta}\right)^2\left|\tilde{\bar{S}}\right|\tilde{S}_{ij}S_{ij}\right) - \left(-2\widetilde{(\Delta)^2|\bar{S}|S_{ij}})S_{ij}\right\}\right]$$

$$\frac{\partial}{\partial C_d}\ E_{ij}\, S_{ij} = -\left[\left\{\left(-2\left(\widetilde{\Delta}\right)^2\left|\tilde{\bar{S}}\right|\tilde{S}_{ij}S_{ij}\right) - \left(-2\widetilde{(\Delta)^2|\bar{S}|S_{ij}})S_{ij}\right\}\right] = 0$$

$$-\left[\left\{\left(-2\left(\widetilde{\Delta}\right)^2\left|\tilde{\bar{S}}\right|\tilde{S}_{ij}S_{ij}\right) - \left(-2\widetilde{(\Delta)^2|\bar{S}|S_{ij}})S_{ij}\right\}\right] = 0$$

$$\left(-2\left(\widetilde{\Delta}\right)^2\left|\tilde{\bar{S}}\right|\tilde{S}_{ij}S_{ij}\right) = \left(-2\widetilde{(\Delta)^2|\bar{S}|S_{ij}})S_{ij}$$

And then what happens to C_d?

We can try a least squares method to solve this problem…

$$E_{ij} = \left[\mathcal{L}_{ij}^{r} - \frac{1}{3}\mathcal{L}_{kk}\delta_{ij}\right] - \left[C_d\left\{\left(-2\left(\widetilde{\Delta}\right)^2\left|\tilde{\bar{S}}\right|\tilde{S}_{ij}\right) - \left(-2\widetilde{(\Delta)^2|\bar{S}|S_{ij}})\right\}\right]$$

$$\frac{\partial}{\partial C_d}\ E_{ij}\, E_{ij} = 0$$

$$\mathcal{L}_{ij}^{d} = \left[\mathcal{L}_{ij}^{r} - \frac{1}{3}\mathcal{L}_{kk}\delta_{ij}\right]$$

$$\left(-2\left(\widetilde{\Delta}\right)^2\left|\tilde{\bar{S}}\right|\tilde{S}_{ij}\right) = \alpha_{ij}$$

$$\left(-2(\Delta)^2|\bar{S}|S_{ij}\right) = \beta_{ij}$$

$$E_{ij} = \mathcal{L}_{ij}^{d} - C_d\left\{\alpha_{ij} - \widetilde{\beta_{ij}}\right\}$$

$$E_{ij}\, E_{ij} = \left(\mathcal{L}_{ij}^{d} - C_d\left\{\alpha_{ij} - \widetilde{\beta_{ij}}\right\}\right)\left(\mathcal{L}_{ij}^{d} - C_d\left\{\alpha_{ij} - \widetilde{\beta_{ij}}\right\}\right)$$

$$E_{ij}\, E_{ij} = \left(\mathcal{L}_{ij}^{d}\right)^2 - 2\mathcal{L}_{ij}^{d}C_d\left\{\alpha_{ij} - \widetilde{\beta_{ij}}\right\} + C_d^2\left\{\alpha_{ij} - \widetilde{\beta_{ij}}\right\}^2$$

$$\frac{\partial}{\partial C_d}\,E_{ij}\,E_{ij}=0$$

$$\frac{\partial}{\partial C_d}\left\{\left(\mathcal{L}_{ij}^{d}\right)^2-2\mathcal{L}_{ij}^{d}C_d\left\{\alpha_{ij}-\widetilde{\beta_{ij}}\right\}+C_d^2\left\{\alpha_{ij}-\widetilde{\beta_{ij}}\right\}^2\right\}=0$$

$$-2\mathcal{L}_{ij}^{d}\left\{\alpha_{ij}-\widetilde{\beta_{ij}}\right\}+2C_d\left\{\alpha_{ij}-\widetilde{\beta_{ij}}\right\}^2=0$$

$$2C_d\left\{\alpha_{ij}-\widetilde{\beta_{ij}}\right\}^2=2\mathcal{L}_{ij}^{d}\left\{\alpha_{ij}-\widetilde{\beta_{ij}}\right\}$$

$$C_d=\frac{2\mathcal{L}_{ij}^{d}\left\{\alpha_{ij}-\widetilde{\beta_{ij}}\right\}}{2\left\{\alpha_{ij}-\widetilde{\beta_{ij}}\right\}^2}$$

$$C_d=\frac{\mathcal{L}_{ij}^{d}\left\{\alpha_{ij}-\widetilde{\beta_{ij}}\right\}}{\left\{\alpha_{ij}-\widetilde{\beta_{ij}}\right\}^2}$$

$$m_{ij}=\alpha_{ij}-\widetilde{\beta_{ij}}$$

$$C_d=\frac{\mathcal{L}_{ij}^{d}\left\{m_{ij}\right\}}{\left\{m_{ij}\right\}^2}$$

$$C_d=\frac{\mathcal{L}_{ij}^{d}\left\{m_{ij}\right\}}{\{m_{kl}\}^2}$$

$$C_d=\frac{\mathcal{L}_{ij}^{d}\left\{\alpha_{ij}-\widetilde{\beta_{ij}}\right\}}{\left\{\alpha_{ij}-\widetilde{\beta_{ij}}\right\}^2}$$

Let’s substitute:

$$\left(-2\left(\widetilde{\Delta}\right)^2|\tilde{\bar{S}}|\tilde{S}_{ij}\right)=\alpha_{ij}$$

$$\left(-2(\Delta)^2|\bar{S}|S_{ij}\right)=\beta_{ij}$$

$$C_d=\frac{-2\mathcal{L}_{ij}^{d}\left\{\left(\left(\widetilde{\Delta}\right)^2|\tilde{\bar{S}}|\tilde{S}_{ij}\right)-\left(\widetilde{(\Delta)^2|\bar{S}|S_{ij}}\right)\right\}}{4\left\{\left(\left(\widetilde{\Delta}\right)^2|\tilde{\bar{S}}|\tilde{S}_{ij}\right)-\left(\widetilde{(\Delta)^2|\bar{S}|S_{ij}}\right)\right\}^2}$$

$$C_d=-\frac{1}{2}\frac{\mathcal{L}_{ij}^{d}\left\{\left(\left(\widetilde{\Delta}\right)^2|\tilde{\bar{S}}|\tilde{S}_{ij}\right)-(\Delta)^2\left(\widetilde{|\bar{S}|S_{ij}}\right)\right\}}{\left\{\left(\left(\widetilde{\Delta}\right)^2|\tilde{\bar{S}}|\tilde{S}_{ij}\right)-(\Delta)^2\left(\widetilde{|\bar{S}|S_{ij}}\right)\right\}^2}$$

According to cfd online:

$$M_{ij}=\left(\widetilde{\Delta}\right)^2|\tilde{\bar{S}}|\tilde{S}_{ij}-(\Delta)^2\left(\widetilde{|\bar{S}|S_{ij}}\right)$$

When you bring this out and replace:

$$C_d=-\frac{1}{2}\frac{\mathcal{L}_{ij}^{d}\left\{M_{ij}\right\}}{M_{ij}M_{ij}}$$

Least squares method by Lilly…

A few notes or disadvantages:

- C_d can be less than 0 in some cases, this may reflect the backward cascade
- Numerical stability problems are also common

In cfd codes, usually they'll be coupled with some other models…

KEqn Model

$k - \varepsilon$ model has an equivalent in LES

Subgrid viscosity ν_{sgs} based on turbulent kinetic energy

$$\nu_{sgs} = \sqrt{\frac{2}{3}\frac{A}{\pi K_0^{1.5}}}\,\overline{\Delta}\,\sqrt{q_{sgs}^2} = C_k\overline{\Delta}\sqrt{k_{sgs}}$$

$$A = 0.438\ (test\ field\ model, TFM\ )$$

$$A = 0.441\ (EDQNM\ theory)$$

$$K_0 = Kolmogorov\ constant$$

$$\nu_{sgs} = \sqrt{\frac{2}{3}\frac{A}{\pi K_0^{1.5}}}\,\overline{\Delta}\,\sqrt{q_{sgs}^2} = C_d\overline{\Delta}\sqrt{k_{sgs}}$$

$$\nu_{sgs} = C_d\overline{\Delta}\sqrt{k_{sgs}}$$

$$C_d = 0.07\ (Yoshizawa\ model)$$

$$k_{sgs} = q_{sgs}^2$$

$$k_{sgs} = \frac{1}{2}\ \overline{(u_i - \overline{u_i})^2}$$

$$k_{sgs} = \frac{1}{2}(\overline{u_i^2} - \overline{u_i}^2)$$

$$u_i = \overline{u_i} + u_i'$$

We can expand the second equation

$$k_{sgs} = \frac{1}{2}(\overline{(\overline{u_\iota} + u_\iota')^2} - \overline{u_\iota}^2)$$

$$k_{sgs} = \frac{1}{2}\left(\overline{\left(\overline{u_\iota}^2 + u_\iota'^2 + 2\overline{u_\iota}u_\iota'\right)} - \overline{u_\iota}^2\right)$$

$$k_{sgs} = \frac{1}{2}\left(\left(\overline{\overline{u_\iota}^2} + \overline{u_\iota'^2} + 2\overline{\overline{u_\iota}u_\iota'}\right) - \overline{u_\iota}^2\right)$$

$$\overline{u_\iota}^2 = (u_i - u_i')^2$$

We can also expand this:

$$k_{sgs} = \frac{1}{2}\ (\overline{u_\iota - \overline{u_\iota}})^2$$

$$k_{sgs} = \frac{1}{2}\ \left(\overline{\overline{u_\iota} + u_\iota' - \overline{u_\iota}}\right)^2$$

$$k_{sgs} = \frac{1}{2}\ \left(\overline{u_\iota'}\right)^2$$

But for simplicity's sake, we can approximate these two to be equal using similarity (Bardina's hypothesis of scale similarity)

$$k_{sgs} = \frac{1}{2}\ \left(\overline{u_\iota'}\right)^2 \approx \frac{1}{2}(\overline{u_i^2} - \overline{u_\iota}^2)$$

Now to derive the subgrid stress equation…

1) Remember our momentum equations?
    a. Get the unfiltered equation
    b. Get the filtered equation
    c. Subtract filtered from unfiltered
2) Multiply by u_i'
    a. Filter
    b. Why filter? Remember CFD only resolves things on scale larger than filter… everything below is modeled.
3) Do double decomposition

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j}\left(u_i u_j\right) = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$\frac{\partial \overline{u_\iota}}{\partial t} + \frac{\partial}{\partial x_j}\left(\overline{u_\iota}\overline{u_J}\right) = -\frac{1}{\rho}\frac{\partial \bar{P}}{\partial x_i} + \nu\frac{\partial}{\partial x_j}\left(\frac{\partial \overline{u_\iota}}{\partial x_j} + \frac{\partial \overline{u_J}}{\partial x_i}\right) - \frac{\partial}{\partial x_j}\left(\tau_{ij}^r\right)$$

We do subtraction:

$$\frac{\partial u_i}{\partial t}+\frac{\partial}{\partial x_j}(u_i u_j)-\left(\frac{\partial \overline{u}_\iota}{\partial t}+\frac{\partial}{\partial x_j}(\overline{u}_\iota \overline{u}_J)\right)$$
$$=-\frac{1}{\rho}\frac{\partial P}{\partial x_i}+\nu\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)-\left(-\frac{1}{\rho}\frac{\partial \overline{P}}{\partial x_i}+\nu\frac{\partial}{\partial x_j}\left(\frac{\partial \overline{u}_\iota}{\partial x_j}+\frac{\partial \overline{u}_J}{\partial x_i}\right)-\frac{\partial}{\partial x_j}(\tau_{ij}^r)\right)$$

$$\frac{\partial u_i}{\partial t}-\frac{\partial \overline{u}_\iota}{\partial t}=\frac{\partial}{\partial t}(u_i-\overline{u}_\iota)=\frac{\partial u_i'}{\partial t}$$

$$\frac{\partial}{\partial x_j}(u_i u_j)-\frac{\partial}{\partial x_j}(\overline{u}_\iota \overline{u}_J)=\frac{\partial}{\partial x_j}(u_i u_j-\overline{u}_\iota \overline{u}_J)$$

$$\frac{\partial}{\partial x_j}\left((\overline{u}_\iota+u_i')(\overline{u}_J+u_j')-\overline{u}_\iota \overline{u}_J\right)=\frac{\partial}{\partial x_j}(\overline{u}_\iota \overline{u}_J+\overline{u}_\iota u_j'+\overline{u}_J u_i'+u_i' u_j'-\overline{u}_\iota \overline{u}_J)$$

$$\frac{\partial}{\partial x_j}(\overline{u}_\iota u_j'+\overline{u}_J u_i'+u_i' u_j')$$

$$\frac{\partial u_i'}{\partial t}+\frac{\partial}{\partial x_j}(\overline{u}_\iota u_j'+\overline{u}_J u_i'+u_i' u_j')$$

$$=-\frac{1}{\rho}\frac{\partial P}{\partial x_i}+\nu\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)-\left(-\frac{1}{\rho}\frac{\partial \overline{P}}{\partial x_i}+\nu\frac{\partial}{\partial x_j}\left(\frac{\partial \overline{u}_\iota}{\partial x_j}+\frac{\partial \overline{u}_J}{\partial x_i}\right)-\frac{\partial}{\partial x_j}(\tau_{ij}^r)\right)$$

$$-\frac{1}{\rho}\frac{\partial P}{\partial x_i}+\frac{1}{\rho}\frac{\partial \overline{P}}{\partial x_i}=-\frac{1}{\rho}\frac{\partial P'}{\partial x_i}$$

$$--\frac{\partial}{\partial x_j}(\tau_{ij}^r)=\frac{\partial}{\partial x_j}(\tau_{ij}^r)$$

$$\nu\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)-\nu\frac{\partial}{\partial x_j}\left(\frac{\partial \overline{u}_\iota}{\partial x_j}+\frac{\partial \overline{u}_J}{\partial x_i}\right)=\nu\frac{\partial}{\partial x_j}\left(\frac{\partial u_i'}{\partial x_j}+\frac{\partial u_j'}{\partial x_i}\right)$$

$$\frac{\partial u_i'}{\partial t}+\frac{\partial}{\partial x_j}(\overline{u}_\iota u_j'+\overline{u}_J u_i'+u_i' u_j')=-\frac{1}{\rho}\frac{\partial P'}{\partial x_i}+\nu\frac{\partial}{\partial x_j}\left(\frac{\partial u_i'}{\partial x_j}+\frac{\partial u_j'}{\partial x_i}\right)+\frac{\partial}{\partial x_j}(\tau_{ij}^r)$$

We have the unfiltered part of the NS equations…

Multiply by u_i', remember $k_{sgs}=\frac{1}{2}\left(\overline{u_\iota'}\right)^2$

$$u_i'\frac{\partial u_i'}{\partial t}+u_i'\frac{\partial}{\partial x_j}(\overline{u}_\iota u_j'+\overline{u}_J u_i'+u_i' u_j')=-u_i'\frac{1}{\rho}\frac{\partial P'}{\partial x_i}+\nu u_i'\frac{\partial}{\partial x_j}\left(\frac{\partial u_i'}{\partial x_j}+\frac{\partial u_j'}{\partial x_i}\right)+u_i'\frac{\partial}{\partial x_j}(\tau_{ij}^r)$$

Time dependent term:

$$u_i'\frac{\partial u_i'}{\partial t}$$

Let's consider

$$\frac{\partial}{\partial t} u_i'^2 = 2u_i' \frac{\partial u_i'}{\partial t}$$

$$u_i' \frac{\partial u_i'}{\partial t} = \frac{1}{2} \frac{\partial}{\partial t} u_i'^2 = \frac{\partial}{\partial t}\left(\frac{1}{2} u_i'^2\right)$$

$$u_i' \frac{\partial u_i'}{\partial t} \rightarrow \frac{\partial}{\partial t} k_{sgs}$$

After you filter:

$$k_{sgs} = \frac{1}{2}\left(\overline{u_i'}\right)^2$$

Let's consider this term:

$$u_i' \frac{\partial}{\partial x_j}\left(\overline{u}_i u_j' + \overline{u}_j u_i' + u_i' u_j'\right)$$

$$u_i' \frac{\partial}{\partial x_j}\left(\overline{u}_i u_j' + \overline{u}_j u_i' + u_i' u_j'\right) = u_i'\left(\frac{\partial}{\partial x_j} \overline{u}_i u_j' + \frac{\partial}{\partial x_j} \overline{u}_j u_i' + \frac{\partial}{\partial x_j} u_i' u_j'\right)$$

$$u_i' \frac{\partial}{\partial x_j} \overline{u}_j u_i'$$

$$\frac{\partial}{\partial x_j} \overline{u}_j u_i' u_i' = u_i' \frac{\partial}{\partial x_j} u_i' \overline{u}_j + \overline{u}_j u_i' \frac{\partial}{\partial x_j} u_i'$$

$$\frac{\partial}{\partial x_j} \overline{u}_j u_i' + \frac{\partial}{\partial x_j} u_i' u_j' = \frac{\partial}{\partial x_j} u_i' u_j$$

$$u_i'\left(\frac{\partial}{\partial x_j} \overline{u}_j u_i' + \frac{\partial}{\partial x_j} u_i' u_j'\right) = u_i' \frac{\partial}{\partial x_j} u_i' u_j$$

So the following also applies:

$$u_i'\left(\frac{\partial}{\partial x_j} \overline{u}_j u_i' + \frac{\partial}{\partial x_j} u_i' u_j'\right) = u_i' \frac{\partial}{\partial x_j} u_i' u_j$$

$$\frac{\partial}{\partial x_j} u_j u_i' u_i' = u_i' \frac{\partial}{\partial x_j} u_i' u_j + u_j u_i' \frac{\partial}{\partial x_j} u_i'$$

$$\frac{\partial}{\partial x_j} u_j \left(\frac{1}{2} u_i' u_i'\right) = \frac{1}{2} u_i' \frac{\partial}{\partial x_j} u_i' u_j + \frac{1}{2} u_j u_i' \frac{\partial}{\partial x_j} u_i'$$

$$u_i' \frac{\partial}{\partial x_j} u_i' u_j = 2 \frac{\partial}{\partial x_j} u_j \left(\frac{1}{2} u_i' u_i'\right) - u_j u_i' \frac{\partial}{\partial x_j} u_i'$$

$$u_i' \frac{\partial}{\partial x_j}\left(\overline{u}_i u_j' + \overline{u}_j u_i' + u_i' u_j'\right) = u_i'\left(\frac{\partial}{\partial x_j} \overline{u}_i u_j'\right) + 2 \frac{\partial}{\partial x_j} u_j \left(\frac{1}{2} u_i' u_i'\right) - u_j u_i' \frac{\partial}{\partial x_j} u_i'$$

$$u_i'\left(\frac{\partial}{\partial x_j}\bar{u}_\imath u_j'\right)+2\frac{\partial}{\partial x_j}u_j\left(\frac{1}{2}\,u_i'u_i'\right)-u_ju_i'\frac{\partial}{\partial x_j}u_i'=2\frac{\partial}{\partial x_j}u_j\left(\frac{1}{2}\,u_i'u_i'\right)+u_i'\left(\frac{\partial}{\partial x_j}\bar{u}_\imath u_j'\right)-u_ju_i'\frac{\partial}{\partial x_j}u_i'$$

Now for pressure effects:

$$-u_i'\frac{1}{\rho}\frac{\partial P'}{\partial x_i}=-\frac{1}{\rho}u_i'\frac{\partial(P-\bar{P})}{\partial x_i}$$

$$\frac{\partial}{\partial x_i}u_i'P'=u_i'\frac{\partial}{\partial x_i}P'+P'\frac{\partial}{\partial x_i}u_i'$$

Now for diffusion terms:

$$\nu u_i'\frac{\partial}{\partial x_j}\left(\frac{\partial u_i'}{\partial x_j}+\frac{\partial u_j'}{\partial x_i}\right)$$

$$u_i'\frac{\partial}{\partial x_j}\frac{\partial u_i'}{\partial x_j}=u_i'\frac{\partial^2}{\partial x_j^2}u_i'$$

$$\frac{\partial^2}{\partial x_j^2}(u_i')^2=\frac{\partial}{\partial x_j}\left\{\frac{\partial}{\partial x_j}(u_i')^2\right\}=2\frac{\partial}{\partial x_j}\left\{\frac{\partial}{\partial x_j}\frac{1}{2}(u_i')^2\right\}$$

$$\frac{\partial}{\partial x_j}\left\{\frac{\partial}{\partial x_j}(u_i')^2\right\}=\frac{\partial}{\partial x_j}2u_i'\left\{\frac{\partial}{\partial x_j}u_i'\right\}=2\left[\left\{\frac{\partial}{\partial x_j}u_i'\right\}^2+u_i'\frac{\partial^2}{\partial x_j^2}u_i'\right]$$

$$\left[\left\{\frac{\partial}{\partial x_j}u_i'\right\}^2+u_i'\frac{\partial^2}{\partial x_j^2}u_i'\right]=\frac{\partial}{\partial x_j}\left\{\frac{\partial}{\partial x_j}\frac{1}{2}(u_i')^2\right\}$$

$$u_i'\frac{\partial^2}{\partial x_j^2}u_i'=-\left\{\frac{\partial}{\partial x_j}u_i'\right\}^2+\frac{\partial}{\partial x_j}\left\{\frac{\partial}{\partial x_j}\left(\frac{1}{2}(u_i')^2\right)\right\}=-\left\{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}\right\}+\frac{\partial}{\partial x_j}\left\{\frac{\partial}{\partial x_j}\left(\frac{1}{2}(u_i')^2\right)\right\}$$

So let's deal with the other term…

$$u_i'\frac{\partial}{\partial x_j}\frac{\partial u_j'}{\partial x_i}$$

$$\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}(u_i'u_j')=\frac{\partial}{\partial x_j}\left\{u_i'\,\frac{\partial}{\partial x_i}u_j'+u_j'\frac{\partial}{\partial x_i}u_i'\right\}$$

Replace terms with KE terms:

$$u_i'\frac{\partial u_i'}{\partial t}+u_i'\frac{\partial}{\partial x_j}(\bar{u}_\imath u_j'+\bar{u}_\jmath u_i'+u_i'u_j')=-u_i'\frac{1}{\rho}\frac{\partial P'}{\partial x_i}+\nu u_i'\frac{\partial}{\partial x_j}\left(\frac{\partial u_i'}{\partial x_j}+\frac{\partial u_j'}{\partial x_i}\right)+u_i'\frac{\partial}{\partial x_j}(\tau_{ij}^r)$$

$$\frac{\partial}{\partial t}k_{sgs}+\frac{\partial}{\partial x_j}u_jk_{sgs}+\frac{1}{2}u_i'\frac{\partial}{\partial x_j}u_i'u_j+\frac{1}{2}u_ju_i'\frac{\partial}{\partial x_j}u_i'+u_i'\left(\frac{\partial}{\partial x_j}\bar{u}_\imath u_j'\right)-u_ju_i'\frac{\partial}{\partial x_j}u_i'$$

$$=-u_i'\frac{1}{\rho}\frac{\partial P'}{\partial x_i}+\nu\left\{u_i'\frac{\partial}{\partial x_j}\left(\frac{\partial u_j'}{\partial x_i}\right)-\left\{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}\right\}+\frac{\partial}{\partial x_j}\left\{\frac{\partial}{\partial x_j}k_{sgs}\right\}\right\}+u_i'\frac{\partial}{\partial x_j}(\tau_{ij}^r)$$

Some important closure relations for the terms:

Diffusion term:

$$\frac{\partial}{\partial x_j}\left(\frac{1}{2}\overline{u_i'u_i'u_j'}+\overline{u'p}\right)=C_2\frac{\partial}{\partial x_j}\left(\bar{\Delta}\sqrt{k_{sgs}}\frac{\partial}{\partial x_j}k_{sgs}\right)$$

Usually we can neglect in turbulent zone:

$$\nu_{sgs}=\nu_t\gg\nu$$

Dissipation term:

$$\varepsilon=\overline{\frac{\nu}{2}\left(\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}\right)}=C_1\frac{k_{sgs}^{1.5}}{\bar{\Delta}}$$

Dynamic KEqn

https://www.openfoam.com/documentation/guides/latest/doc/guide-turbulence-les-dynamic-k-eqn.html

What is backscatter?

https://en.wikipedia.org/wiki/Large_eddy_simulation

WALE Model

Paper:

F. Nicoud and F. Ducros. Subgrid-Scale Stress Modelling Based on the Square of the Velocity Gradient Tensor. Flow Turbulence and Combustion, 62(3):183–200, 1999.

http://doi.org/10.1023/A:1009995426001

https://www.cfd-online.com/Wiki/Wall-adapting_local_eddy-viscosity_(WALE)_model

recall our standard defintions:

$$\bar{S}_{ij}=\frac{1}{2}\left(\frac{\partial\bar{u}_i}{\partial x_j}+\frac{\partial\bar{u}_j}{\partial x_i}\right)\rightarrow strain\ rate$$

$$\bar{\Omega}_{ij}=\frac{1}{2}\left(\frac{\partial\bar{u}_i}{\partial x_j}-\frac{\partial\bar{u}_j}{\partial x_i}\right)\rightarrow rotation$$

Recall smagorinsky:

$$\nu_t=\left(C_{Smagorinsky}^2\right)\Delta^2|\bar{S}|=\left(C_{Smagorinsky}^2\right)\Delta^2\sqrt{2S_{ij}S_{ij}}$$

Problem:

$$\underline{\nu_t \neq 0 \; at \; wall}$$

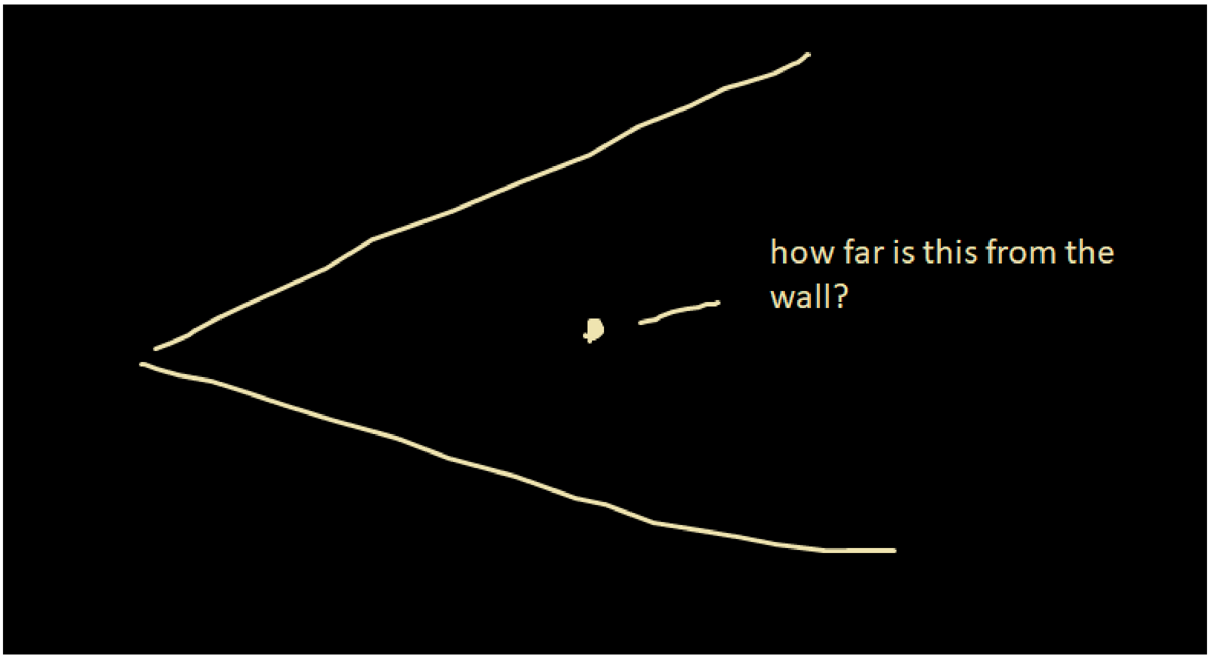
Use van driest damping function such that

$$\nu_t = (C_{Smagorinsky}^2 D^2)\Delta^2|\bar{S}| = (C_{Smagorinsky}^2)\Delta^2\sqrt{2S_{ij}S_{ij}}$$

$$D = 1 - \exp\left(\frac{y^+}{A^+}\right), A^+ = 25.0$$

According to Nicoud, this is rather ad-hoc, a scrappy way of forcing $\nu_t = 0$ at wall,

May not work for complex geometries…



Sometimes CFD code has a way of calculating that…

But Nicoud goes further…

- No explicit dependence on y
- But shows correct behavior from wall surface y behavior

He also tries to incorporate some flow physics

- "wall function" or damping function should capture both <u>strain</u> and <u>rotation</u> rate…

So the equation is in the form:

$$\nu_t = C_m\Delta^2\overline{(OP(x,t))}$$

In smagorinsky,

$$\overline{(OP(x,t))} = \sqrt{2S_{ij}S_{ij}}$$

$$C_m = C_{smagorinsky}^2$$

Well, he wanted $\overline{(OP(x,t))}$ defined so that it has correct behavior from wall, ie $\nu_t = 0$ at wall,

So he uses this function:

$$\mathcal{S}_{ij}^d\mathcal{S}_{ij}^d = \frac{1}{6}\left(\bar{S}_{ij}\bar{S}_{ij}\bar{S}_{ij}\bar{S}_{ij} + \bar{\Omega}_{ij}\bar{\Omega}_{ij}\bar{\Omega}_{ij}\bar{\Omega}_{ij}\right) + \frac{2}{3}\left(\bar{\Omega}_{ij}\bar{\Omega}_{ij}\bar{S}_{ij}\bar{S}_{ij}\right) + 2\left(\bar{S}_{ik}\bar{S}_{kj}\bar{\Omega}_{jl}\bar{\Omega}_{li}\right)$$

Structure this way:

- Large strain rate produces turbulence
- Large rotation rate produces turbulence
- Rotation is more important than shear in turbulence generation (physically true)
- For pure shear flows → no rotation

For pure shear, $\mathcal{S}_{ij}^d\mathcal{S}_{ij}^d = 0$

- No turbulence generated in laminar zone, ie wall, laminar is pure shear only
- Smagorinsky predicts turbulence in such a zone

<u>Problem: incorrect wall behavior</u>

- $\mathcal{S}_{ij}^d\mathcal{S}_{ij}^d(y^2\ behaviour\ from\ wall) \rightarrow \left(\mathcal{S}_{ij}^d\mathcal{S}_{ij}^d\right)^{\frac{3}{2}} \rightarrow y^3\ behaviour$

<u>Problem: wrong dimension (ie wrong units)</u>

Correct dimensions without upsetting y^3 behavior:

$$\left(\mathcal{S}_{ij}^d\mathcal{S}_{ij}^d\right)^{\frac{3}{2}} \rightarrow frequency^6 \rightarrow (s^{-1})^6$$

Remember $\mathcal{S}_{ij}^d \rightarrow units\ of\ g^2 \rightarrow frequency^2$

You want some quantity with magnitude $\mathcal{O}(1)$, but with correct units.

$$\left(\bar{S}_{ij}\bar{S}_{ij}\right)^{\frac{5}{2}} \rightarrow frequency^5 \rightarrow (s^{-1})^5$$

- $\mathcal{S}_{ij}^d\mathcal{S}_{ij}^d \rightarrow \left(\mathcal{S}_{ij}^d\mathcal{S}_{ij}^d\right)^{\frac{3}{2}} \rightarrow \frac{\left(\mathcal{S}_{ij}^d\mathcal{S}_{ij}^d\right)^{\frac{3}{2}}}{\left(\bar{S}_{ij}\bar{S}_{ij}\right)^{\frac{5}{2}}}(y^3\ behaviour + correct\ units)$

<u>Problem: numerical instability at the wall</u>

$$\left(\bar{S}_{ij}\bar{S}_{ij}\right)^{\frac{5}{2}} \rightarrow 0\ at\ the\ wall$$

You want a denominator that doesn't go to 0 at the wall, preserves the units, but also has $\mathcal{O}(1)$ at the wall.

Solution, add

$$\left(\mathcal{S}_{ij}^d\mathcal{S}_{ij}^d\right)^{\frac{5}{4}}$$

To denominator, term is negligible in BL, but nonzero at wall.

$$\left(\mathcal{S}_{ij}^{d}\mathcal{S}_{ij}^{d}\right)^{\frac{5}{4}} \rightarrow (frequency^2 * frequency^2)^{\frac{5}{4}} \rightarrow frequency^5$$

Correct units!

$$\nu_t = C_m \Delta^2 \overline{(OP(x,t))}$$

$$\overline{(OP(x,t))} = \frac{\left(\mathcal{S}_{ij}^{d}\mathcal{S}_{ij}^{d}\right)^{\frac{3}{2}}}{\left(\bar{S}_{ij}\bar{S}_{ij}\right)^{\frac{5}{2}} + \left(\mathcal{S}_{ij}^{d}\mathcal{S}_{ij}^{d}\right)^{\frac{5}{4}}}$$

Choice of C_w

Best fitting of experimental data, original paper says 0.55-0.60 for Smagorinsky coefficient of 0.18.

But smagorinsky coefficient $C_s \neq 0.18$ for all flows, it depends on flow type and grid size.

Good compromise over several flow types: $C_w = 0.325$ → corresponds to $C_s \approx 0.1 \rightarrow$

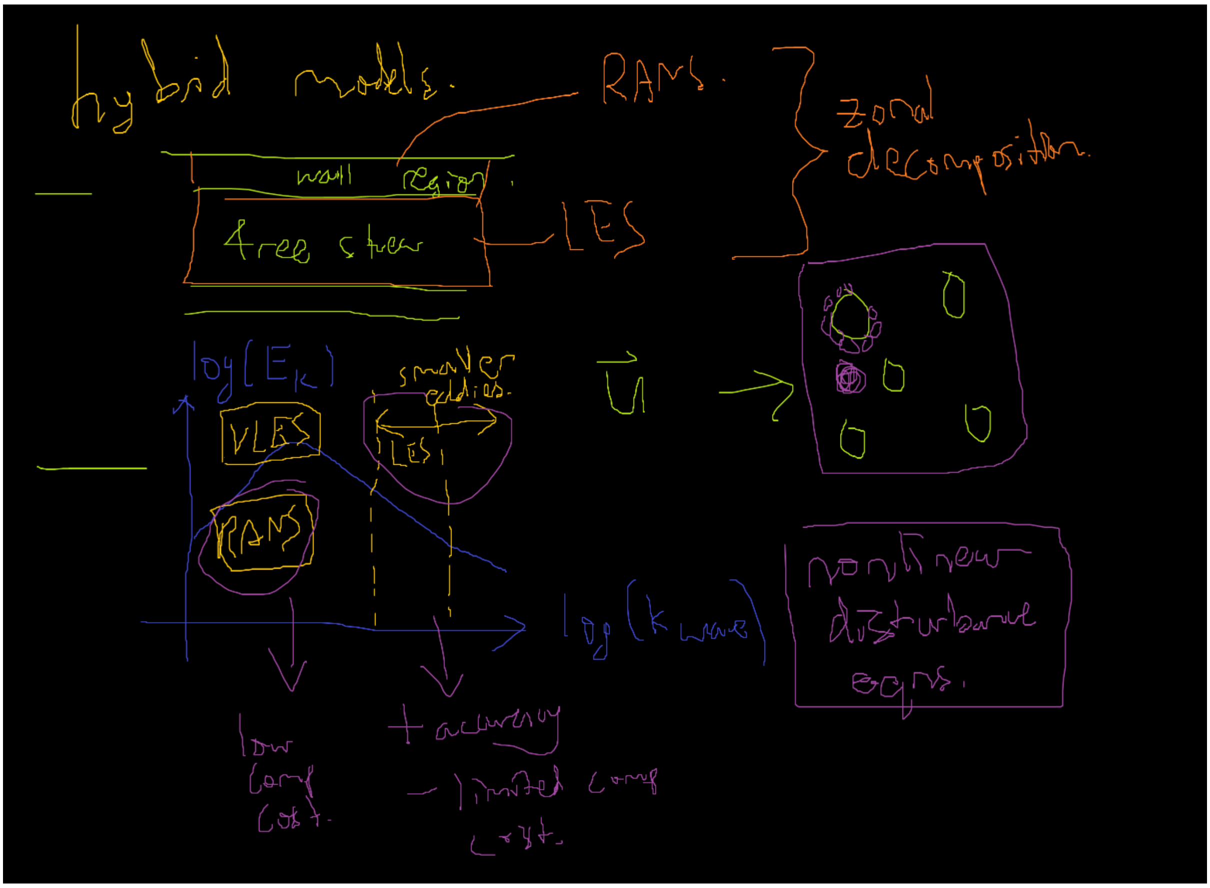
Good for bounded flows

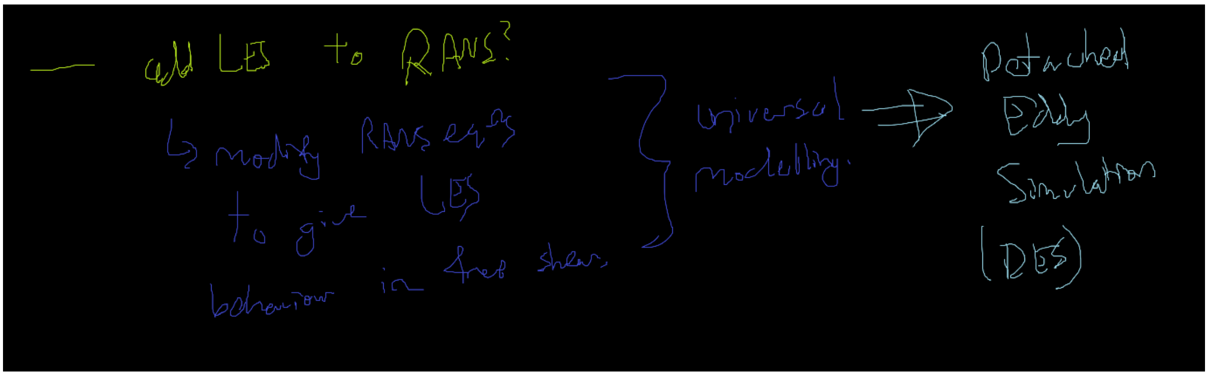
https://www.slideshare.net/mmer547/les-customization-en

Pipe flow is being used here, $C_s \approx 0.1$

http://users.monash.edu.au/~bburn/pdf/csiro99_pipe.pdf

Hybrid RANS-LES





DES

P.R. Spalart, W-H Jou, M. Strelets, and S.R. Allmaras. Comments on the Feasibility of LES for Wings, and on a Hybrid RANS/LES. In Advances in DNS/LES, pages 137–147, Columbus, OH, USA, 1997. Greyden Press.

Original: Spalart-Almaras RANS-LES (DES)

By Fluid Mechanics 101 Spalart-Allmaras

https://www.youtube.com/watch?v=Xivc0EIGFQw

Spalart Allmaras Model – DES

https://www.researchgate.net/profile/Michael_Strelets/publication/307943024_Comments_on_the_feasibility_of_LES_for_winds_and_on_a_hybrid_RANSLES_approach/links/57d7ceae08ae0c0081ecd0d5/Comments-on-the-feasibility-of-LES-for-winds-and-on-a-hybrid-RANS-LES-approach.pdf

Good Reference here:

https://www.iihr.uiowa.edu/gconstantinescu/files/2012/10/LES_models_4.pdf

Let's recall the Spalart Allmaras RANS model:

$$\frac{\partial \tilde{\nu}}{\partial t} + u_j \frac{\partial \tilde{\nu}}{\partial x_j} = -\left[c_{w1} f_w - \frac{c_{b1}}{\kappa^2} f_{t2}\right] \left(\frac{\tilde{\nu}}{d}\right)^2 + c_{b1}[1 - f_{t2}]\, \tilde{\nu}\, \tilde{S} + \frac{1}{\sigma}\{\nabla \cdot (\tilde{\nu}\nabla\tilde{\nu}) + c_{b2}(\nabla\tilde{\nu}^2)\} + f_{t1}\Delta U^2$$

$$\nu_t = \tilde{\nu} \frac{\chi^3}{\chi^3 + c_{v1}^3}$$

Key change for DES version

- Replace $d \; with \; \tilde{d}$
- $\tilde{d} \equiv \min(d, C_{DES}\Delta)$
- $C_{DES} = \mathcal{O}(1), ie \approx \; 0.5 - 2.0$
    - $C_{DES} = 0.65$ → calibrated for isotropic turbulence at equilibrium

$$\frac{\partial \tilde{\nu}}{\partial t} + u_j \frac{\partial \tilde{\nu}}{\partial x_j} = -\left[c_{w1} f_w - \frac{c_{b1}}{\kappa^2} f_{t2}\right] \left(\frac{\tilde{\nu}}{\tilde{d}}\right)^2 + c_{b1}[1 - f_{t2}]\, \tilde{\nu}\, \tilde{S} + \frac{1}{\sigma}\{\nabla \cdot (\tilde{\nu}\nabla\tilde{\nu}) + c_{b2}(\nabla\tilde{\nu}^2)\} + f_{t1}\Delta U^2$$

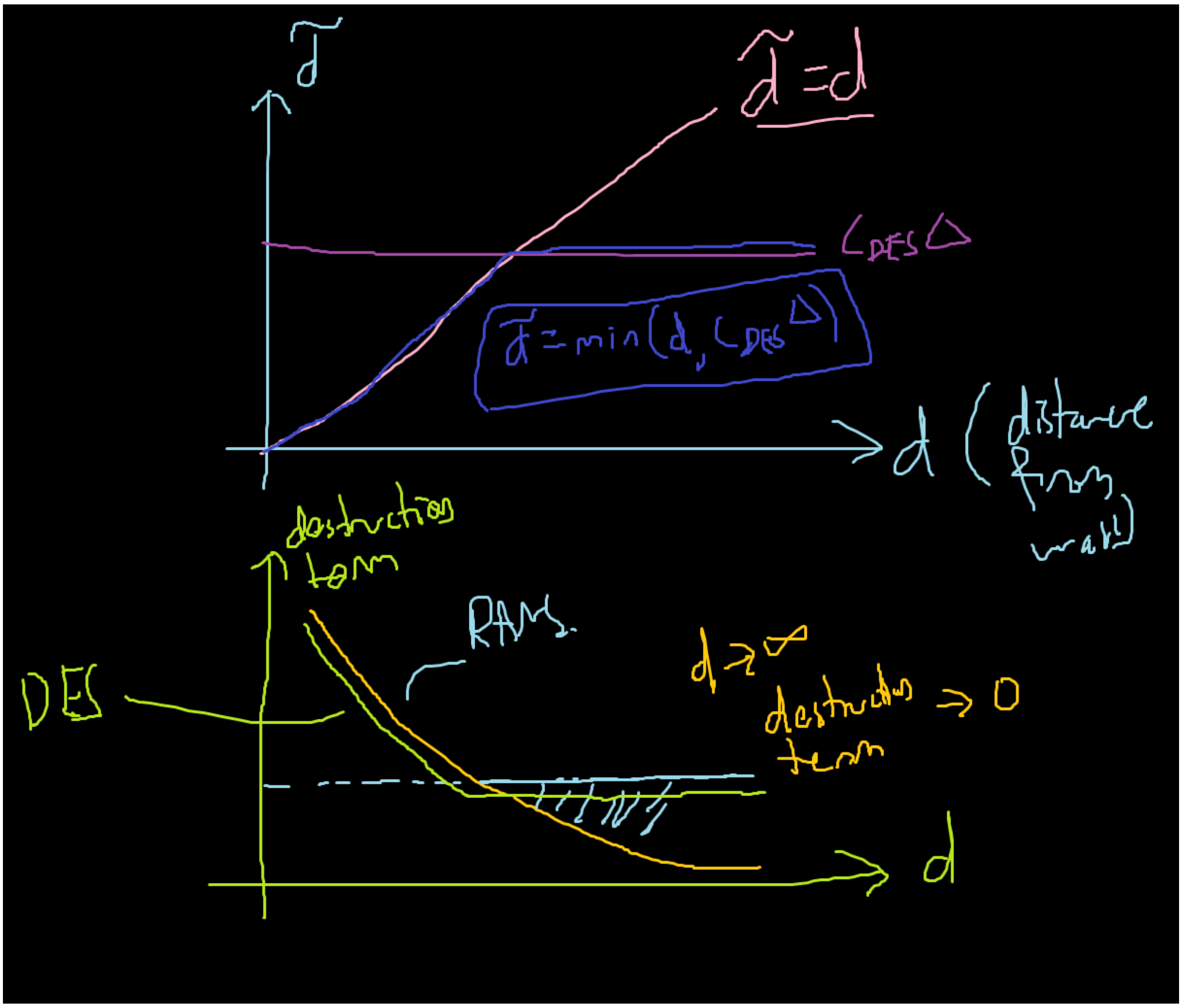
$$\nu_t = \tilde{\nu} \frac{\chi^3}{\chi^3 + c_{v1}^3}$$

$$\tilde{d} \equiv \min(d, C_{DES}\Delta)$$

$$\Delta = \mathrm{V}^{\frac{1}{3}}$$

Far from the wall:

$$-\left[c_{w1} f_w - \frac{c_{b1}}{\kappa^2} f_{t2}\right] \left(\frac{\tilde{\nu}}{C_{DES}\Delta}\right)^2$$



Effect: destruction term

It is finite in the free shear region, thus ν_t will decrease in free shear region…

This is how we get LES behaviour, as ν_t drops, the apparent Re increases, and we can form eddies!

One note:

$$\tilde{d} \equiv \min(d, C_{DES}\Delta)$$

If you decrease Δ by having ultra fine mesh… more of the simulation behaves like LES,

If you decrease Δ even smaller, such that $\nu \gg \nu_{sgs}\ or\ \nu_t$ then you have something almost like DNS

- This is called quasi-DNS

This relationship of more mesh refinement → more accurate, only true for LES/DES models or at least SA-DES…

For URANS → eg. SA model

- Refining Δ doesn't result in DNS

- Doesn't mean more accurate simulation
- Turbulence viscosity is still being modelled, rather than destroyed.
    - Accuracy is only as good as your turbulence model.

This is the key difference between URANS and LES/DES

Key difference between URANS and LES:

Take a look at the viscosity term for RANS/URANS ($k - \omega\ SST\ or\ k - \varepsilon$):

$$\nu_t = C_\mu \frac{k^2}{\varepsilon} = \frac{k}{\omega^*}$$

Where $\omega^* = \frac{\omega}{C_\mu} = \frac{\varepsilon}{C_\mu k}$

And for spalart allmaras

$$\nu_t = \frac{\chi^3}{C_{v1}^3 + \chi^3} \tilde{\nu}$$

Neither of them are grid dependent…

But for smagorinsky

$$\nu_t = \nu_{sgs} = C_{smagorinsky}^2 \Delta^2 * \bar{S}$$

ν_t being modelled depends on the Δ, and reducing $\Delta \to 0$ results in $\nu_t \to 0$ (DNS)

But reducing grid size for URANS does not mean more accuracy or DNS like calculations.

Note: according to paper, SA-DES behaves like smagorinsky model in free shear zone not kEqn

M. Strelets. Detached eddy simulation of massively separated flows. In *39th Aerospace Sciences Meeting and Exhibit*, Reno, NV, USA, 2001.

We also have: kOmega-SST (DES)

Recall our k and omega equations…

$$\frac{\partial k}{\partial t} + u_j \frac{\partial k}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)} \nu_t\right) \frac{\partial k}{\partial x_j}\right) + \frac{P_{k(dynamic)}}{\rho} - \beta^* \omega^* k$$

$$\frac{\partial \omega^*}{\partial t} + u_i \frac{\partial \omega^*}{\partial x_i} = \frac{\partial}{\partial x_j}\left(\nu + \frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial \omega^*}{\partial x_j} + (1 - F_1)2\left(\frac{1}{\omega^*}\right)\frac{\partial \omega^*}{\partial x_j}\frac{\partial k}{\partial x_j} + \alpha S^2 - \beta \omega^{*2}$$

Remember $\omega = \beta^* \omega^*$, $\omega \equiv \frac{\varepsilon}{k}$

$$\beta^* = C_\mu = 0.09$$

The idea is to follow the SA-DES model and replace some length scale to get LES effect in free shear zone.

From mixing length model:

$$\nu_t = l^2[m^2]\left(\frac{\partial \bar{u}}{\partial y}\right)[s^{-1}]$$

From $k - \omega$

$$\nu_t = C_\mu \frac{k}{\omega} = \frac{k\ [m^2 s^{-2}]}{\omega^*\ [s^{-1}]}$$

We can also change the length scale here:

$$l_{k-\omega} = \frac{k^{0.5}\ [m^2 s^{-2}]^{0.5}}{\omega\ [s^{-1}]}$$

$$\tilde{l} = \min\left(\frac{k^{0.5}}{\omega}, C_{DES}\Delta\right)$$

Change the dissipative term:

$$\beta^* \omega^* k$$

$$\beta^* \omega^* k = \beta^* \omega^* k * \frac{l_{k-\omega}}{l_{k-\omega}}$$

$$\beta^* \omega^* k * \frac{l_{k-\omega}}{l_{k-\omega}} = \frac{\beta^* \omega^* k l_{k-\omega}}{l_{k-\omega}} = \frac{\omega k \frac{k^{0.5}}{\omega}}{l_{k-\omega}} = \frac{k^{1.5}}{l_{k-\omega}}$$

$$\frac{\partial k}{\partial t} + u_j \frac{\partial k}{\partial x_j} = \frac{\partial}{\partial x_j}\left((\nu + \sigma_{K(k\omega SST)}\nu_t)\frac{\partial k}{\partial x_j}\right) + \frac{P_{k(dynamic)}}{\rho} - \frac{k^{1.5}}{l_{k-\omega}}$$

Given that

$$l_{k-\omega} = \frac{k^{0.5}\ [m^2 s^{-2}]^{0.5}}{\omega\ [s^{-1}]}$$

Now we replace the length scale to make the model DES

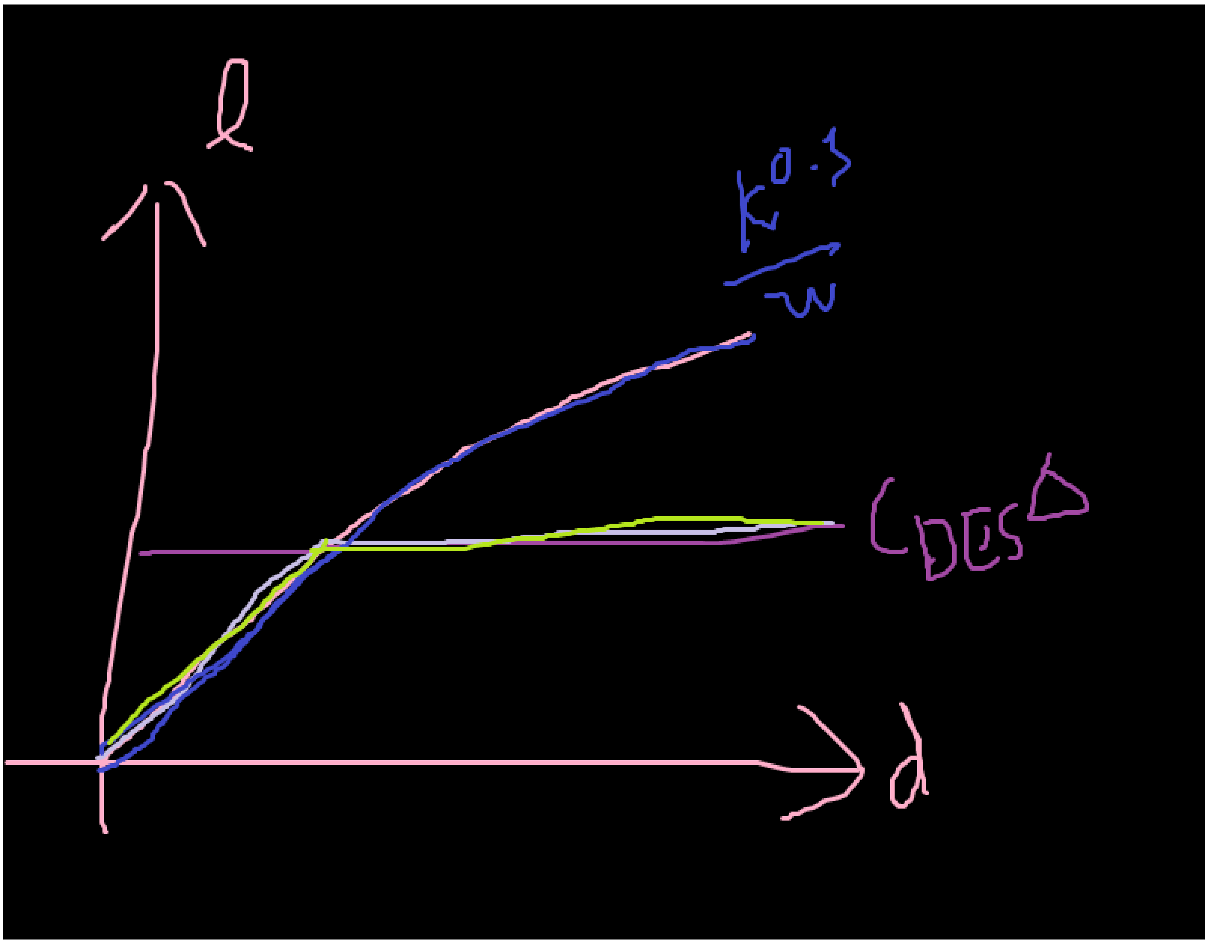
$$\frac{\partial k}{\partial t} + u_j \frac{\partial k}{\partial x_j} = \frac{\partial}{\partial x_j}\left((\nu + \sigma_{K(k\omega SST)}\nu_t)\frac{\partial k}{\partial x_j}\right) + \frac{P_{k(dynamic)}}{\rho} - \frac{k^{1.5}}{\tilde{l}}$$

$$\tilde{l} = \min\left(\frac{k^{0.5}}{\omega}, C_{DES}\Delta\right)$$

$$\frac{\partial k}{\partial t}+u_j\frac{\partial k}{\partial x_j}=\frac{\partial}{\partial x_j}\left(\left(\nu+\sigma_{K(k\omega SST)}\nu_t\right)\frac{\partial k}{\partial x_j}\right)+\frac{P_{k(dynamic)}}{\rho}-\frac{k^{1.5}}{\min\left(\frac{k^{0.5}}{\omega},C_{DES}\Delta\right)}$$

$$\frac{\partial \omega^*}{\partial t}+u_i\frac{\partial \omega^*}{\partial x_i}=\frac{\partial}{\partial x_j}\left(\nu+\frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial \omega^*}{\partial x_j}+(1-F_1)2\left(\frac{1}{\omega^*}\right)\frac{\partial \omega^*}{\partial x_j}\frac{\partial k}{\partial x_j}+\alpha S^2-\beta\omega^{*2}$$

The omega equation remains untouched.



Making length scale smaller in free shear zone, makes dissipation bigger…

You destroy more **modelled** turbulence KE in free shear zone as a result.

What is C_{DES}?

One is calibrated for each RANS model for $k-\omega\ SST$

rmb $k-\omega\ SST$ is a hybrid model between $k-\varepsilon$ and $k-\omega$

$$C_{DES\,(k-\varepsilon)}\approx 0.60$$

$$C_{DES\,(k-\omega)}\approx 0.82$$

$$C_{DES}=(1-F_1)C_{DES\,(k-\varepsilon)}+F_1C_{DES\,(k-\omega)}$$

A note on numerical schemes

M. Strelets. Detached eddy simulation of massively separated flows. In *39th Aerospace Sciences Meeting and Exhibit*, Reno, NV, USA, 2001.

RANS → stability, upwind schemes.

LES → accuracy, linear schemes.

DES → hybrid? Which do you use?

$$F = F_{upwind}(\sigma) + F_{linear}(1-\sigma)$$

Disadvantage of upwind: diffusive (ie less accurate)

Fine mesh with upwind scheme, can have similar accuracy to coarse mesh with linear scheme

Fluid Mechanics 101 (Upwind schemes vs linear vs central difference)

https://www.youtube.com/watch?v=JVE0fNkc540

$k-\omega\ SST\ DES$ and $SA-DES$ are especially well calibrated for aircraft.

Tweaks to SA-DES (DDES)

P.R. Spalart, S. Deck, M.L. Shur, K.D. Squires, M.K. Strelets, and A. Travin. A New Version of Detached-eddy Simulation, Resistant to Ambiguous Grid Densities. *Theoretical and Computational Fluid Dynamics*, 20(3):181–195, 2006.

https://www.researchgate.net/publication/225137433_A_New_Version_of_Detached-eddy_Simulation_Resistant_to_Ambiguous_Grid_Densities

(pdf available on researchgate)

Problem: LES Switches on too soon in some grids

$$\tilde{d} = \min(d, C_{DES}\Delta)$$

$$\Delta = V^{\frac{1}{3}}$$

$$\Delta = \max(\Delta x, \Delta y, \Delta z)$$

Latter only applies well for cuboid shapes

Normally for RANS to switch on in BL

$$RANS\ switch\ on\ \ (d\ (distance\ from\ wall) < C_{DES}\Delta)$$

For BL, max d is δ

Hence

$$\Delta > \frac{\delta}{C_{DES}}$$

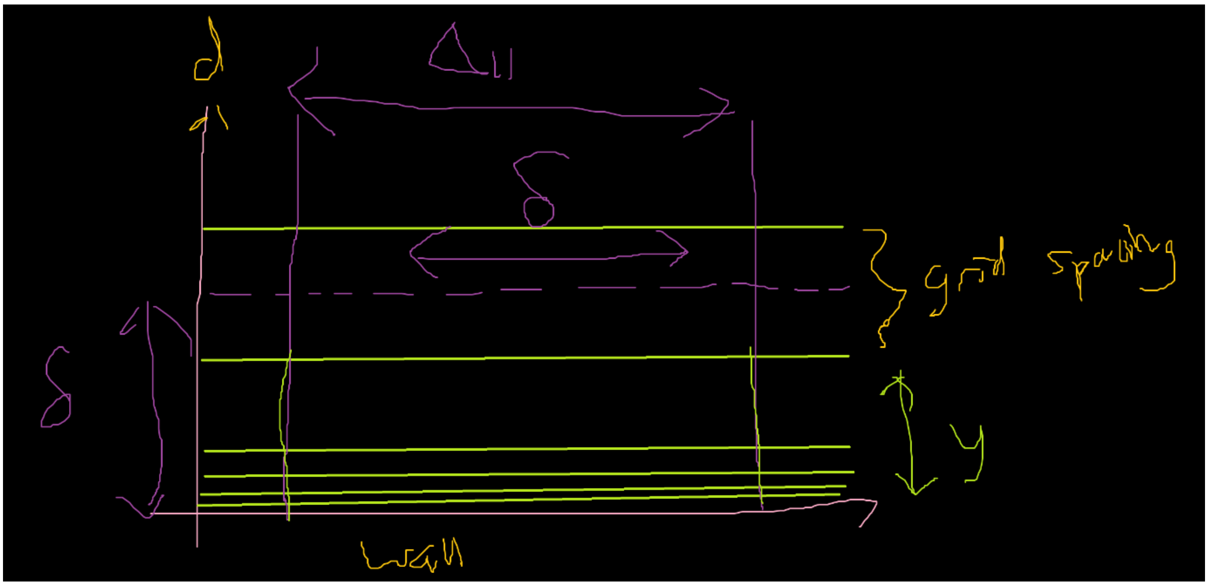
We need to adjust Δ

$$V^{\frac{1}{3}} > \frac{\delta}{C_{DES}}$$

Or

$$\max(\Delta x, \Delta y, \Delta z) > \frac{\delta}{C_{DES}}$$

Forcing RANS to switch on:



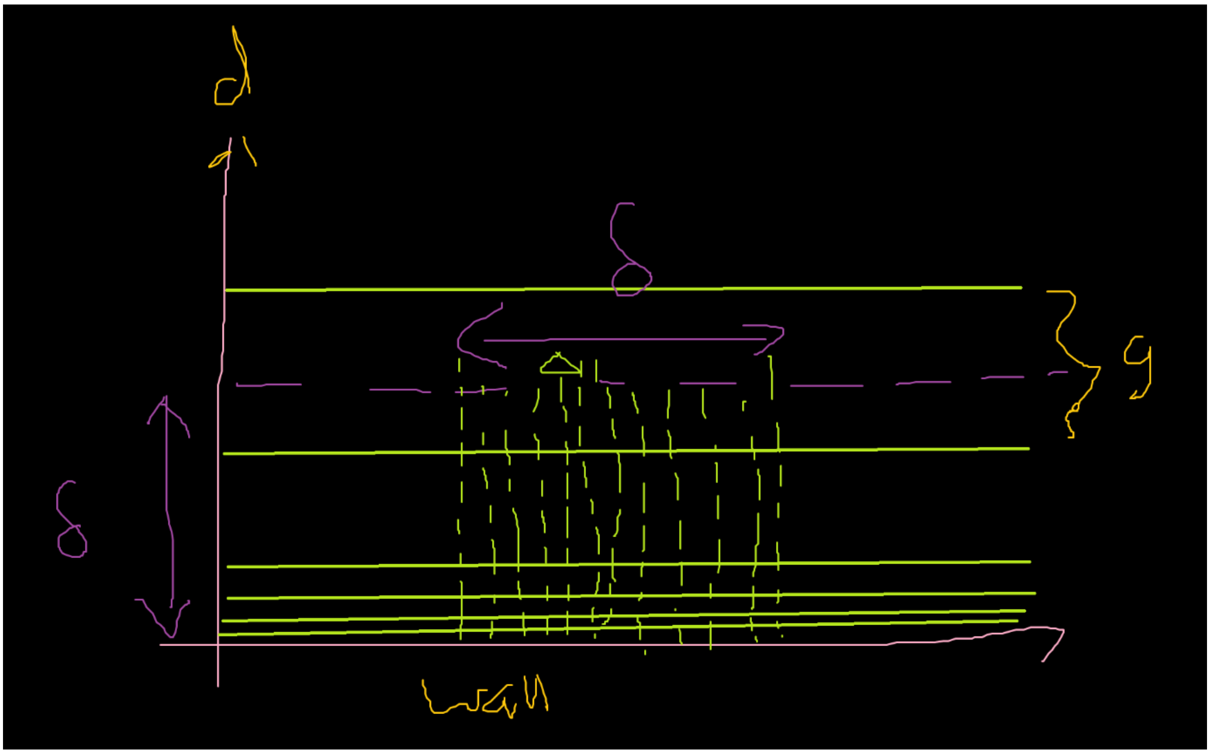
If you want to switch on LES, you need to make sure the converse is true,

So that in BL,

$$\max(\Delta x, \Delta y, \Delta z) \ll \frac{\delta}{C_{DES}}$$

This means typical values of

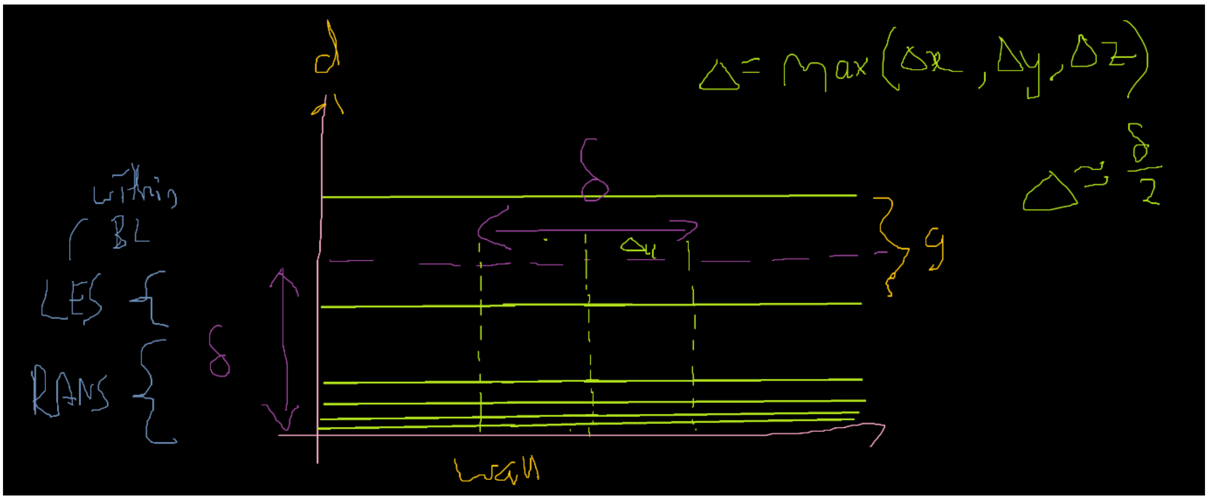
$$\Delta \approx \frac{\delta}{20}$$



Makes it so that you have LES on in BL…

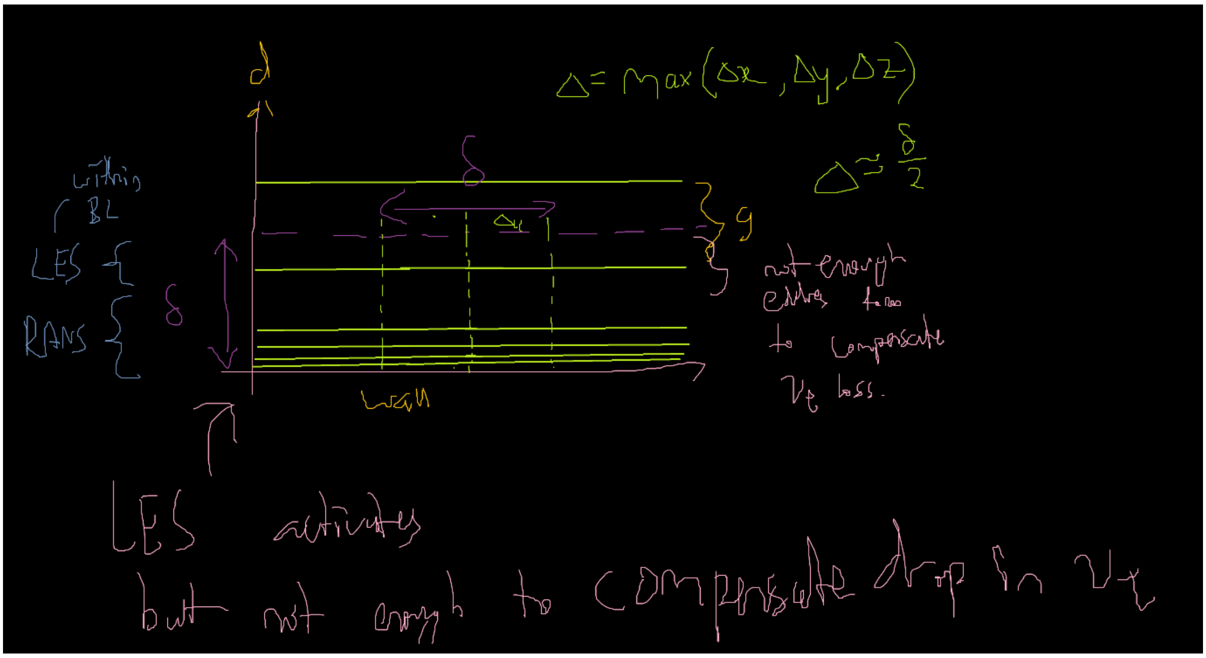
The gray area or ambiguous region

$$\Delta \approx \frac{\delta}{2}$$



So for DES we have to be careful about how we space our grids… otherwise LES kicks in too early…

This early onset of LES causes problems…



Technical Term: Model Stress Depletion (MSD)

DES destroys modelled reynold's stress in the RANS (ie ν_t)→ LES transitions without producing enough eddies in LES region.

- Why is this important?
- Degree of MSD is dependent on how you structure your grid
- Different grids in BL → different results

We need to stop (or delay) onset of LES within the BL.

Solution: Take inspiration from k Omega SST's Blending Functions

Preserve RANS in Boundary Layer → Delay onset of LES, hence the name Delayed…

If you remember,

$k - \omega\ SST$→ blending functions F_1 and F_2 using tanh

Spalart's idea was to take blending function to switch from RANS in BL to LES in Free shear. This blending function is the key change from DES to DDES.

$$f_d \equiv 1 - \tanh([8r_d]^3)$$

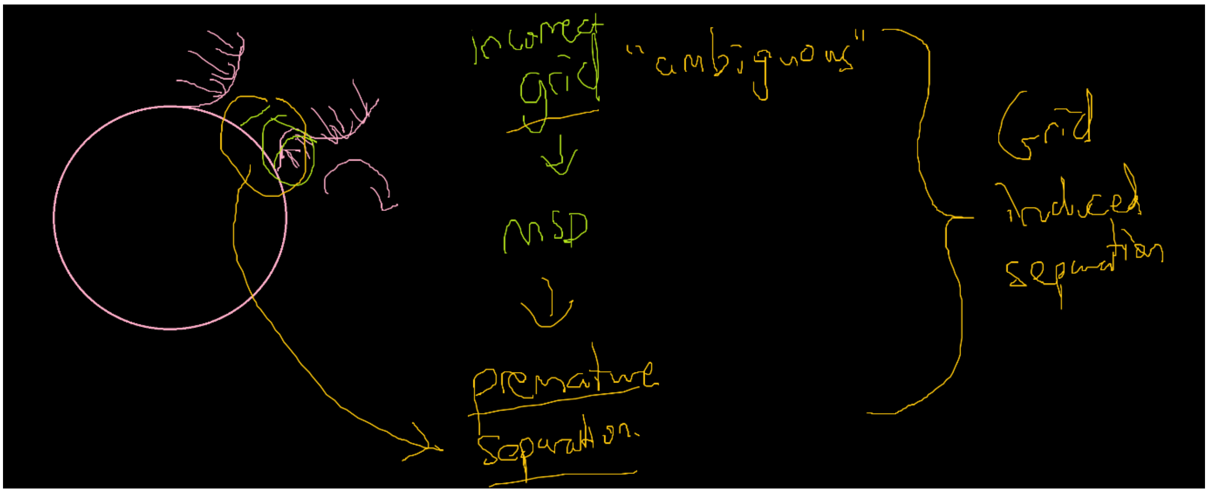
$$r_d \equiv \frac{\nu_t + \nu}{|\nabla U|\kappa^2 d^2}$$

$$|\nabla U| = \sqrt{\nabla \mathrm{U} \cdot \nabla \mathrm{U}}$$

$$\tilde{d} \equiv d - f_d \max(0, d - C_{DES}\Delta)$$

Blending functions get rid of Model stress depletion which is based on grid structure in BL. Different grids in BL should give more or less same results.

➔ The different result is: premature flow separation



Blending functions prevent onset of LES within BL when it's unwanted.

Let's investigate the behaviour of the blending function

$$f_d \equiv 1 - \tanh([8r_d]^3)$$

$$r_d \equiv \frac{\nu_t + \nu}{|\nabla U|\kappa^2 d^2}$$

$$|\nabla U| = \sqrt{\nabla \mathrm{U} \cdot \nabla \mathrm{U}}$$

$$\tilde{d} \equiv d - f_d \max(0, d - C_{DES}\Delta)$$

At low r_D, $f_d \to 1$ → classic DES behaviour

$$\tilde{d} \equiv d - f_d \max(0, d - C_{DES}\Delta)$$

$$f_d \to 1$$

$$\tilde{d} \equiv d - \max(0, d - C_{DES}\Delta)$$

If $d - C_{DES}\Delta < 0$

$$\tilde{d} = d$$

(RANS mode)

Otherwise if $d - C_{DES}\Delta > 0$

$$\tilde{d} \equiv d - (d - C_{DES}\Delta) = C_{DES}\Delta$$

(LES mode)

If $f_d \to 0$, we get RANS

$$\tilde{d} = d \; (RANS \; mode)$$

What triggers $f_d \to 0$?

$$f_d \equiv 1 - \tanh([8r_d]^3)$$

Large r_d triggers $f_d \to 0$

Large r_d triggers RANS mode.

$$r_d \equiv \frac{\nu_t + \nu}{|\nabla U|\kappa^2 d^2}$$

Large r_d means

- ➔ Large turbulent viscosity in comparison to $|\nabla U|$
- ➔ Large turbulent viscosity means large r_d

Effect of large turbulent viscosity:

- ➔ Large turbulent viscosity triggers RANS mode, or rather prevents LES from switching on.

Large D makes r_d small

Large $|\nabla U|$ makes r_d small → large velocity gradient → naturally generates eddies or flow instabilities. This makes it ok for DES or LES to kick in.

Tweaks to SA-DES (Low Re Correction)

P.R. Spalart, S. Deck, M.L. Shur, K.D. Squires, M.K. Strelets, and A. Travin. A New Version of Detached-eddy Simulation, Resistant to Ambiguous Grid Densities. *Theoretical and Computational Fluid Dynamics*, 20(3):181–195, 2006.

https://www.researchgate.net/publication/225137433_A_New_Version_of_Detached-eddy_Simulation_Resistant_to_Ambiguous_Grid_Densities

(pdf available on researchgate)

Refining grid in regions of free shear switches LES on

Recall:

$$l_{des} = \min(l_{RANS}, C_{DES}\Delta)$$

For SA model:

$$\tilde{d} = \min(d, C_{DES}\Delta)$$

Refine grid: $\Delta \to too\ small$, eddy viscosity (SA variable $\tilde{\nu}$) destruction term → big → "DNS mode"

No turbulent viscosity.

We destroy eddy viscosity when $\tilde{d}$ is small as if it's near a wall.

$$\tilde{d} \to 0, \nu_t \to 0$$

$\mathrm{C_{DES}}\Delta \to 0$ means behavior as if it's near a wall.

$$\tilde{d} = \min(d, C_{DES}\Delta)$$

How can we prevent $C_{DES}\Delta$ from approaching 0 in free shear regions where there is excessive refinement ie $\Delta \to 0\ (small\ and\ finite)$?

Use a booster function Ψ

$$\tilde{d} = \min(d, \Psi C_{DES}\Delta)$$

Ψ is the booster function where you increase the value of $\Psi C_{DES}\Delta$ such that it is not approaching 0 and destroys eddy viscosity as if it were near a wall.

$$\Psi \geq 1$$

$\Psi > 1$ should activate in low Re Regions (not enough flow disturbances generated to compensate for loss in ν_t) and far from wall (ie free shear)

$$\mathrm{l_{DES}} = \Psi C_{DES}\Delta$$

This is known as the low Re correction.

How should we design Ψ?

Basic principle

- ➔ Calibrate Ψ such that subgrid stress model (ν_t) should behave like smagorinsky model in the free shear zone even in low Re
  - $\nu_t = \left(C_{smagorinsky}\Delta\right)^2 |S|$

Lots of math later… and calibration

$$\Psi^2 = \frac{\frac{f_w}{f_w^*} - \frac{c_{b1}}{c_{w1}\kappa^2 f_w^*}[f_{t1} + (1 - f_{t2})f_{v2}]}{f_{v1}(1 - f_{t2})}$$

$$f_w^* = 0.424$$

For rest of the constants, please refer to spalart allmaras RANS model

https://www.cfd-online.com/Wiki/Spalart-Allmaras_model

now we make approximations to tweak Ψ numerically

$$\Psi^2 = \frac{\frac{f_w}{f_w^*} - \frac{c_{b1}}{c_{w1}\kappa^2 f_w^*}[f_{t1} + (1-f_{t2})f_{v2}]}{f_{v1}(1-f_{t2})}$$

Ψ kicks in when there should be some meaningful amount of ν_t ($\nu_t > \nu/10$)

But in this range, we find that $\frac{f_w}{f_w^*} \approx 1 \pm 2\%$

$$\Psi^2 = \frac{1 - \frac{c_{b1}}{c_{w1}\kappa^2 f_w^*}[f_{t1} + (1-f_{t2})f_{v2}]}{f_{v1}(1-f_{t2})}$$

We want to make sure Ψ^2 doesn't blow up because denominator reaches 0

$\nu_t \to 0\ means\ \chi = \frac{\nu_t}{\nu} \to 0$ means $f_{t2} \to 1$

$$f_{t2} = C_{t3}\exp(-C_{t4}\chi^2)$$

$$\Psi^2 = \frac{1 - \frac{c_{b1}}{c_{w1}\kappa^2 f_w^*}[f_{t1} + (1-f_{t2})f_{v2}]}{f_{v1}\max[10^{-10}, (1-f_{t2})]}$$

Another way to cap Ψ

$$\Psi^2 = \min\left\{10^2, \frac{1 - \frac{c_{b1}}{c_{w1}\kappa^2 f_w^*}[f_{t1} + (1-f_{t2})f_{v2}]}{f_{v1}\max[10^{-10}, (1-f_{t2})]}\right\}$$

This limits:

$$1 \le \Psi \le 10$$

You can also apply this principle to $k - \omega\ SST\ DES$

<u>You also can apply this low Re correction to SA-DDES</u>

- We have an extra variable f_d to correct the modelled stress depletion phenomenon
- This can also appear in Ψ
- $\Psi^2 = \min\left\{10^2, \frac{1 - \frac{c_{b1}}{c_{w1}\kappa^2 f_w^*}[f_{t1} + (1-f_{t2})f_{v2}]}{f_{v1}\max[10^{-10}, (1-f_{t2})]}\right\}$
    - These are functions of χ
    - Everywhere you see χ, replace it with $\max(\chi, 20f_d)$

<u>Problem:</u>

<u>Refining too much in low Re causes grid induced modelled stress depletion, thus lowering eddy viscosity too much.</u>

- Usually fine grid regions are near wall
- However, fine grid can also cause the DES model to treat the fine grid as if it were a wall…

SA-IDDES (A combined model of Wall Modelled LES and DDES)

https://scholar.google.com.sg/scholar?q=a+hybrid+rans+les+approach+with+delayed+des&hl=en&as_sdt=0&as_vis=1&oi=scholart

Shur, M. L., Spalart, P. R., Strelets, M. K., & Travin, A. K. (2008). A hybrid RANS-LES approach with delayed-DES and wall-modelled LES capabilities. *International Journal of Heat and Fluid Flow*, *29*(6), 1638-1649.

https://www.researchgate.net/publication/223933800_A_hybrid_RANS-LES_approach_with_delayed-DES_and_wall-modelled_LES_capabilities

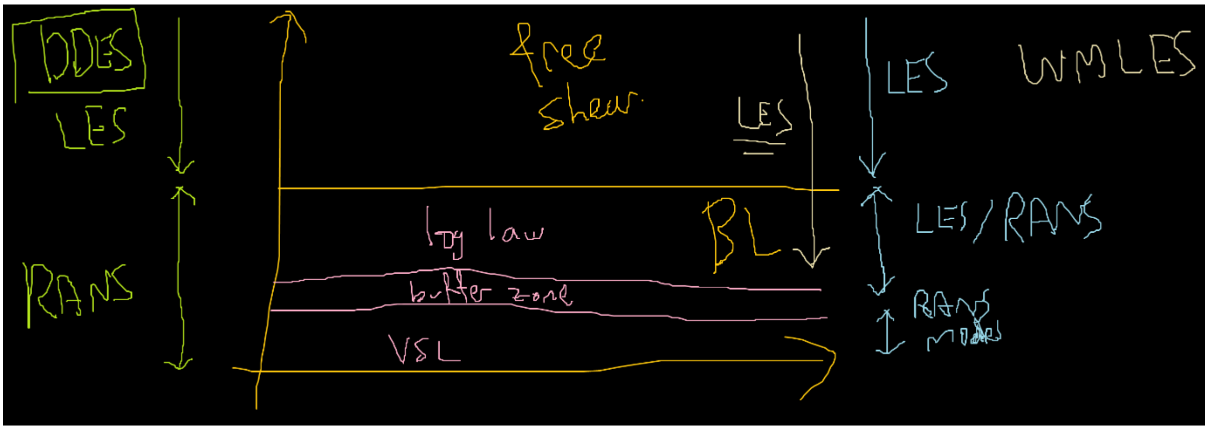
(pdf available on researchgate)

Source Code in OpenFoam

https://develop.openfoam.com/Development/openfoam/tree/master/src/TurbulenceModels/turbulenceModels/DES/SpalartAllmarasIDDES

Tutorial case in OpenFoam

https://www.openfoam.com/documentation/guides/latest/doc/verification-validation-turbulent-surface-mounted-cube.html



Unlike WALE, WMLES uses a RANS model for viscous sublayer within the BL.

Can DES be used in LES mode for BL?

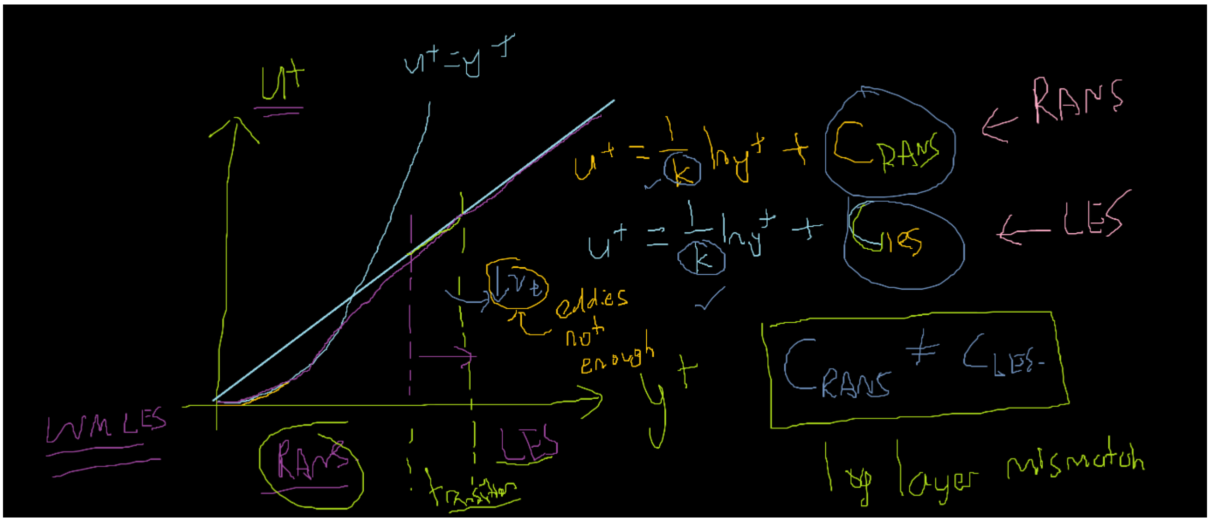
Ie RANS for viscous sublayer but LES in other parts of BL (this is called Wall Modelled LES, WMLES)

- Note this is different from WALE model discussed, WALE uses an empirical function in viscous sublayer but **not RANS**

Sort of… but there are a few problems…

Problem: log layer mismatch

Recall log law of the wall: $u^+ = \frac{1}{\kappa} \ln y^+ + C$



Result: 15-20% error in skin friction (too much error for aerospace application) → from paper (Shur et al., 2008)

Solution: adopt correct blending function for WMLES… (again more empirical complicated looking functions)

So what does WMLES look like for SA model?

Similar to DES, we just change the length scale and blend it with complicated empirical functions…

$$l_{DDES} = l_{RANS} - f_d \max\{0, (l_{RANS} - l_{LES})\}$$

$$l_{WMLES} = f_B(1 + f_e) l_{RANS} + (1 - f_B) l_{LES}$$

F_B is quite straightforward

- Empirical blending function to switch between RANS/LES length scale
- $f_B = 1$ is RANS mode, $f_B = 0$ is LES mode

$$f_B = \min\{2\exp(-9\alpha^2), 1.0\}$$

The 1.0 is there to cap f_B at 1

$$\alpha = 0.25 - \frac{d_w}{h_{\max}}$$

$$h_{\max} = \max(h_x, h_y, h_z)$$

What's f_e for? It's to help with modelled stress depletion. (this one isn't grid dependent, it's just that reduction in RANS eddy viscosity is not compensated with enough resolved turbulence)

So we want f_e to be zero at all times unless it is here to slow down the eddy viscosity reduction from RANS model. (similar to DES)

$$f_e = \max\{(f_{e1} - 1), 0\}\,\Psi f_{e2}$$

We note that Ψ makes a reappearance here.

$$\Psi^2 = \min\left\{10^2, \frac{1 - \frac{c_{b1}}{c_{w1}\kappa^2 f_w^*}[f_{t1} + (1 - f_{t2}) f_{v2}]}{f_{v1}\max[10^{-10}, (1 - f_{t2})]}\right\}$$

What we are doing here has **nothing** to do with low Re correction in DDES/DES conceptually speaking. It's just a convenient function to use and exhibits correct numerical behavior.

$$f_{e1} = \begin{cases} 2\exp(-11.09\alpha^2) & if\ \alpha \geq 0 \\ 2\exp(-9.0\alpha^2) & if\ \alpha < 0 \end{cases}$$

f_{e1} preserves RANS eddy viscosity <u>based on length scale</u>

$$f_{e2} = 1.0 - \max\{f_t, f_l\}$$

$$f_t = \tanh[(c_t^2 r_{dt})^3]$$

$$f_l = \tanh\left[\left(c_l^2 r_{dl}\right)^{10}\right]$$

What are typical values,

Refer to OpenFOAM SA-IDDES source code

https://develop.openfoam.com/Development/openfoam/blob/master/src/TurbulenceModels/turbulenceModels/DES/SpalartAllmarasIDDES/SpalartAllmarasIDDES.C

line 66 to 78 provides the formula…

line 208 to 225 provides the values…

$$c_l = 3.35, c_t = 1.63$$

for r_{dt} and r_{dl}

recall this from SA-RANS

$$r_d \equiv \frac{\nu_t + \nu}{|\nabla U| \kappa^2 d_w^2}$$

To make it more numerically stable, we make sure denominator doesn't go to zero

$$r_d \equiv \frac{\nu_t + \nu}{\max\{10^{-10}, |\nabla U|\} \kappa^2 d_w^2}$$

$$r_{dl} = \frac{\nu}{\max\{10^{-10}, |\nabla U|\} \kappa^2 d_w^2}$$

$$r_{dt} = \frac{\nu_t}{\max\{10^{-10}, |\nabla U|\} \kappa^2 d_w^2}$$

f_{e2} controls drop in eddy viscosity based on ν_t

Blending DDES with WMLES

This is key change for IDDES

Blending 2 length scales:

$$l_{WMLES} = f_B(1 + f_e) l_{RANS} + (1 - f_B) l_{LES}$$

$$l_{DDES} = l_{RANS} - f_d \max\{0, (l_{RANS} - l_{LES})\}$$

Modify original DDES length scale:

$$\tilde{l}_{DDES} = \tilde{f}_d l_{RANS} + \left(1 - \tilde{f}_d\right) l_{LES}$$

$$\tilde{f}_d = \max\{(1 - f_{dt}), f_B\}$$

$$f_{dt} = 1 - \tanh[(8 r_{dt})^3]$$

Combine with WMLES properties:

$$\tilde{l}_{IDDES} = \tilde{f}_d (1 + f_e) l_{RANS} + \left(1 - \tilde{f}_d\right) l_{LES}$$

Viola…

Behaviour discussion:

Large resolved turbulence in inflow → $r_{dt} \ll 1$

$$r_{dt} = \frac{\nu_t}{\max\{10^{-10}, |\nabla U|\} \kappa^2 d_w^2}$$

Low modelled eddy viscosity compared to strain rates (velocity gradient).

Strain rates are huge in resolved turbulent flow…

Reminder: resolved means that you solve for the velocity field in CFD code.

With $r_{dt} \ll 1$

$$f_{dt} = 1 - \tanh[(8r_{dt})^3] \rightarrow 1$$

$$1 - f_{dt} \rightarrow 0$$

This triggers

$$\tilde{f}_d = \max\{(1 - f_{dt}), f_B\} \rightarrow \max\{0, f_B\}$$

$$\tilde{f}_d = f_B \rightarrow WMLES$$

$$\tilde{l}_{WMLES} = f_B(1 + f_e)l_{RANS} + (1 - f_B)l_{LES}$$

Otherwise (not enough resolved turbulence in flow), $f_e \rightarrow 0$ triggers DDES.

$$\tilde{l}_{DDES} = \tilde{f}_d l_{RANS} + (1 - \tilde{f}_d)l_{LES}$$

K omega SST DDES and IDDES

Gritskevich, M. S., Garbaruk, A. V., Schütze, J., & Menter, F. R. (2012). Development of DDES and IDDES formulations for the k-ω shear stress transport model. *Flow, turbulence and combustion*, *88*(3), 431-449.

K omega SST DDES/IDDES ResearchGate pdf

https://www.researchgate.net/publication/257518951_Development_of_DDES_and_IDDES_Formulations_for_the_k-o_Shear_Stress_Transport_Model

K omega SST DDES/IDDES Springer Link

https://link.springer.com/article/10.1007/s10494-011-9378-4

K omega SST DDES/IDDES OpenFoam Source Code

https://develop.openfoam.com/Development/openfoam/tree/master/src/TurbulenceModels/turbulenceModels/DES/kOmegaSSTDDES

https://develop.openfoam.com/Development/openfoam/tree/master/src/TurbulenceModels/turbulenceModels/DES/kOmegaSSTIDDES

Recall k Omega SST DES

$$\frac{\partial k}{\partial t} + u_j \frac{\partial k}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right) \frac{\partial k}{\partial x_j} \right) + \frac{P_{k(dynamic)}}{\rho} - \frac{k^{1.5}}{\tilde{l}}$$

$$\tilde{l} = \min\left(\frac{k^{0.5}}{\omega}, C_{DES}\Delta \right)$$

$$\frac{\partial k}{\partial t}+u_j\frac{\partial k}{\partial x_j}=\frac{\partial}{\partial x_j}\left(\left(\nu+\sigma_{K(k\omega SST)}\nu_t\right)\frac{\partial k}{\partial x_j}\right)+\frac{P_{k(dynamic)}}{\rho}-\frac{k^{1.5}}{\min\left(\frac{k^{0.5}}{\omega},C_{DES}\Delta\right)}$$

$$\frac{\partial \omega^*}{\partial t}+u_i\,\frac{\partial \omega^*}{\partial x_i}=\frac{\partial}{\partial x_j}\left(\nu+\frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial \omega^*}{\partial x_j}+(1-F_1)2\left(\frac{1}{\omega^*}\right)\frac{\partial \omega^*}{\partial x_j}\frac{\partial k}{\partial x_j}+\alpha S^2-\beta\omega^{*2}$$

The omega equation remains untouched.

For DDES

$$l_{DDES}=l_{RANS}-f_d\max(0,l_{RANS}-l_{LES})$$

$$l_{LES}=C_{DES}h_{max}=C_{DES}\Delta$$

$$l_{RANS}=\frac{k^{0.5}}{\omega}$$

$$\omega\equiv\frac{\varepsilon}{k}$$

$$l_{RANS}=\frac{k^{0.5}}{C_\mu\omega^*}$$

$$\omega=C_\mu\omega^*$$

$$l_{DDES}=\frac{k^{0.5}}{C_\mu\omega^*}-f_d\max\left(0,\frac{k^{0.5}}{C_\mu\omega^*}-C_{DES}\Delta\right)$$

$$C_{DES}=C_{DES1}F_1+C_{DES2}(1-F_1)$$

DDES part:

$$f_d=1-\tanh[(C_{d1}r_d)^{C_{d2}}]$$

In SA DDES model

$$f_d=1-\tanh[(8r_d)^3]$$

But in this paper, to calibrate f_d to prevent onset of LES in BL

$$f_d=1-\tanh[(20r_d)^3]$$

$$C_{d1}=20$$

In SA DDES model

$$r_d\equiv\frac{\nu_t+\nu}{\max\{10^{-10},|\nabla U|\}\,\kappa^2 d_w^2}$$

Over here:

$$r_d\equiv\frac{\nu_t+\nu}{\sqrt{0.5\,(S^2+\Omega^2)}\kappa^2 d_w^2}$$

$$C_\mu=0.09,\kappa=0.41$$

$$C_{DES1} = 0.78, C_{DES2} = 0.61, C_{d1} = 20, C_{d2} = 3$$

For IDDES

$$l_{IDDES} = \widetilde{f_d}(1 + f_e)l_{RANS} + \left(1 - \widetilde{f_d}\right)l_{LES}$$

$$l_{LES} = C_{DES}\Delta$$

$$\Delta = \min\{C_w \max[d_w, h_{\max}\ ], h_{\max}\ \}$$

$$l_{RANS} = \frac{k^{0.5}}{C_\mu \omega^*}$$

$$C_{DES} = C_{DES1}F_1 + C_{DES2}(1 - F_1)$$

$$\widetilde{f_d} = \max\{(1 - f_{dt}), f_b\}$$

$$f_{dt} = 1 - \tanh[(C_{dt1}r_{dt})^{C_{dt2}}]$$

$$f_b = \min\{2\exp(-9\alpha^2)\,, 1.0\}$$

$$\alpha = 0.25 - \frac{d_w}{h_{\max}}$$

Now for f_e

$$f_e = \max\{(f_{e1} - 1), 0\}\,f_{e2}$$

$$f_{e1} = \begin{cases} 2\exp(-11.09\alpha^2) & if\ \alpha \geq 0 \\ 2\exp(-9.0\alpha^2) & if\ \alpha < 0 \end{cases}$$

$$f_{e2} = 1.0 - \max\{f_t, f_l\}$$

$$f_t = \tanh[(c_t^2 r_{dt})^3]$$

$$f_l = \tanh\left[\left(c_l^2 r_{dl}\right)^{10}\right]$$

$$r_d \equiv \frac{\nu_t + \nu}{\sqrt{0.5\ (S^2 + \Omega^2)}\kappa^2 d_w^2}$$

$$r_{dl} = \frac{\nu}{\sqrt{0.5\ (S^2 + \Omega^2)}\kappa^2 d_w^2}$$

$$r_{dt} = \frac{\nu_t}{\sqrt{0.5\ (S^2 + \Omega^2)}d_w^2}$$

This sums up the IDDES model for k omega SST.

C_w	C_{dt1}	C_{dt2}	C_l	C_t
0.15	20	3	5.0	1.87

Comments from the Paper

For DDES, calibrating C_{dt1} from 8 to 20 is VERY important

For IDDES C_{dt1} doesn't change results much from 8 to 20, but 25 is too high, so C_{dt1} was left at 20, but for IDDES changing it to 8 changes the result marginally.

Simplified version of IDDES for k Omega SST

It can also be simplified by making $f_e = 0$

So in simplified k omega SST IDDES

$$l_{IDDES} = \widetilde{f_d} l_{RANS} + (1 - \widetilde{f_d}) l_{LES}$$

- Omitting f_e causes stronger log layer mismatch in developed channel flow, but effect is marginal according to authors (download the paper and refer to figure 6 for reference)
- They also set $C_{dt} = 20$ for IDDES model and change some other constants:

So technically according to this paper you can omit f_e, this is called the simplified IDDES model.

Implementation in OpenFOAM

See constants for each one…

and also

In OpenFoam, comments for Simplified Model are found in lines 147 to 154

It's not actually implemented!

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