K-Omega SST Model

To have the best of both worlds, we have a blend of k-Omega and k-Epsilon in the k-Omega SST model.

So the models switches to k-Omega near the wall and k-Epsilon near the free stream.

Dr Aiden (Fluid Mechanics 101)

https://www.youtube.com/watch?v=myv-ityFnS4

$$\frac{\partial k}{\partial t} + u_j \frac{\partial k}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right)\frac{\partial k}{\partial x_j}\right) + \frac{P_{k(dynamic)}}{\rho} - \varepsilon$$

$$\varepsilon = \beta^* \omega^* k;\ \omega = \omega^* \beta^* = \omega^* C_\mu;\ \beta^* = 0.09$$

$$\frac{\partial k}{\partial t} + u_j \frac{\partial k}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right)\frac{\partial k}{\partial x_j}\right) + \frac{P_{k(dynamic)}}{\rho} - \beta^* \omega^* k$$

$$\frac{\partial k}{\partial t} + u_j \frac{\partial k}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right)\frac{\partial k}{\partial x_j}\right) + \frac{P_{k(dynamic)}}{\rho} - \omega k$$

(note: $k-\varepsilon, k-\omega$ models, k equation remains the same-ish, so $k-\omega\ SST$ is also the same)

From $k-\varepsilon$

$$\frac{\partial \omega}{\partial t} + u_i \frac{\partial \omega}{\partial x_i} = \frac{\partial}{\partial x_j}\left(\nu + \frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial \omega}{\partial x_j} + \frac{2}{k}\left(\nu + \frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial \omega}{\partial x_j}\frac{\partial k}{\partial x_j} + P[C_{\varepsilon 1} - 1]\frac{\omega}{k} - \omega^2[C_{\varepsilon 2} - 1]$$

From $k-\omega$

$$\frac{\partial \omega}{\partial t} + u_i \frac{\partial \omega}{\partial x_i} = \frac{\partial}{\partial x_j}\left(\nu + \frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial \omega}{\partial x_j} + P\alpha\frac{\omega}{k} - \omega^2\frac{\beta}{\beta^*}$$

$$\omega \equiv \frac{\varepsilon}{k}$$

Let's say we want to blend…

In deriving the $k-\omega\ SST$ formulation,

$$\frac{\partial \omega}{\partial t} + u_i \frac{\partial \omega}{\partial x_i} = \frac{\partial}{\partial x_j}\left(\nu + \frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial \omega}{\partial x_j} + (f_{blending})\frac{2}{k}\left(\nu + \frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial \omega}{\partial x_j}\frac{\partial k}{\partial x_j} + P[C_{\varepsilon 1} - 1]\frac{\omega}{k} - \omega^2[C_{\varepsilon 2} - 1]$$

$$f_{blending} = 1 - F_1, 0 < F_1 < 1$$

At the wall, $F_1 \to 1$, in bulk region $F_1 \to 0$

$$\frac{\partial \omega}{\partial t} + u_i \frac{\partial \omega}{\partial x_i} = \frac{\partial}{\partial x_j}\left(\nu + \frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial \omega}{\partial x_j} + (1 - F_1)\frac{2}{k}\left(\nu + \frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial \omega}{\partial x_j}\frac{\partial k}{\partial x_j} + P[C_{\varepsilon 1} - 1]\frac{\omega}{k} - \omega^2[C_{\varepsilon 2} - 1]$$

$$\left(\nu + \frac{\nu_T}{\sigma_\varepsilon}\right) \approx \frac{\nu_T}{\sigma_\varepsilon}\ in\ turbulent\ bulk\ fluid\ region$$

$$\nu_T \approx C_\mu \frac{k^2}{\varepsilon} = \beta^* \frac{k^2}{\varepsilon}$$

$$\left(\nu + \frac{\nu_T}{\sigma_\varepsilon}\right) * \frac{1}{k} \approx \frac{\beta^*}{\sigma_\varepsilon}\frac{k^2}{\varepsilon}\frac{1}{k} = \frac{\beta^*}{\sigma_\varepsilon}\frac{k}{\varepsilon} = \frac{\beta^*}{\sigma_\varepsilon}\frac{1}{\omega} = \frac{1}{\sigma_\varepsilon}\frac{1}{\omega}$$

$$\omega = \omega^* \beta^* \to \omega^* = \frac{\omega}{\beta^*}$$

To make it look like the

$$\frac{2}{k}\left(\nu + \frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial \omega}{\partial x_j}\frac{\partial k}{\partial x_j} \approx 2\frac{1}{\sigma_\varepsilon}\frac{1}{\omega}\frac{\partial \omega}{\partial x_j}\frac{\partial k}{\partial x_j}\ (only\ applies\ when \frac{\nu_t}{\sigma_\varepsilon} \gg \nu, ie\ far\ from\ wall)$$

With blending function

$$2(1 - F_1)\frac{1}{\sigma_\varepsilon}\frac{1}{\omega}\frac{\partial \omega}{\partial x_j}\frac{\partial k}{\partial x_j} \to 0\ at\ the\ wall, otherwise\ finite\ far\ from\ wall$$

The blending function F_1. F_1 is the degree to which $k - \omega$ model activates

$$F_1 = \tanh\left\{\left\{\min\left[\max\left(\frac{\sqrt{k}}{\omega y}, \frac{500\nu C_\mu}{\omega y^2}\right), \frac{4\sigma_{\omega 2} k}{CD_{k\omega} y^2}\right]\right\}^4\right\}$$

So what do $\frac{\sqrt{k}}{\omega y}$, $\frac{500\nu C_\mu}{\omega y^2}$ and $\frac{4\sigma_{\omega 2} k}{CD_{k\omega} y^2}$

We want to discuss near the wall. Why?

Because we want $k - \omega$ to activate near the wall. $F_1 \to 1$

How does F_1 change in Viscous Sublayer (VSL) and then the log law region (turbulent sublayer), see how this velocity profile goes for large y^+ values.

Of course, this is for flat plate, BL flow, which we can assume applies to most flow types, **except convection and flow with strong pressure gradient→ this negates the constant shear stress hypothesis**

$$\frac{\tau_{app}}{\rho} = (\nu + \nu_t)\frac{\partial \bar{u}}{\partial y}$$

Eventually we want to see how F_1 varies with y or y^+…

For that, we need wall fns for k and ω

$\frac{\sqrt{k}}{\omega y}$, $\frac{500\nu C_\mu}{\omega y^2}$ and $\frac{4\sigma_{\omega 2} k}{CD_{k\omega} y^2}$

ω should vary with y^+ and k also…

How did this k equation come about

$$\frac{\partial k}{\partial t}+u_j\frac{\partial k}{\partial x_j}=\frac{\partial}{\partial x_j}\left(\left(\nu+\sigma_{K(k\omega SST)}\nu_t\right)\frac{\partial k}{\partial x_j}\right)+\frac{P_{k(dynamic)}}{\rho}-\omega k$$

Nondimensionalising it…

$$k^+=\frac{k}{u_*^2},\omega^+=\frac{\omega\nu}{u_*^2}$$

$$u_*=\sqrt{\frac{\tau_{wall}}{\rho}}$$

We can substitute:

$$k=k^+u_*^2\,;\omega=\frac{\omega^+u_*^2}{\nu}$$

$$\frac{\partial k}{\partial t}+u_j\frac{\partial k}{\partial x_j}=\frac{\partial}{\partial x_j}\left(\left(\nu+\sigma_{K(k\omega SST)}\nu_t\right)\frac{\partial k}{\partial x_j}\right)+\frac{P_{k(dynamic)}}{\rho}-\omega k$$

$$u_*^2\left(\frac{\partial k^+}{\partial t}+u_j\frac{\partial k^+}{\partial x_j}\right)=u_*^2\frac{\partial}{\partial x_j}\left(\left(\nu+\sigma_{K(k\omega SST)}\nu_t\right)\frac{\partial k^+}{\partial x_j}\right)+\frac{P_{k(dynamic)}}{\rho}-\frac{\omega^+u_*^2}{\nu}k^+u_*^2$$

$$u_*^2\left(\frac{\partial k^+}{\partial t}+u_j\frac{\partial k^+}{\partial x_j}\right)=u_*^2\frac{\partial}{\partial x_j}\left(\left(\nu+\sigma_{K(k\omega SST)}\nu_t\right)\frac{\partial k^+}{\partial x_j}\right)+\frac{P_{k(dynamic)}}{\rho}-\frac{\omega^+k^+u_*^4}{\nu}$$

$$\left(\frac{\partial k^+}{\partial t}+u_j\frac{\partial k^+}{\partial x_j}\right)=\frac{\partial}{\partial x_j}\left(\left(\nu+\sigma_{K(k\omega SST)}\nu_t\right)\frac{\partial k^+}{\partial x_j}\right)+\frac{P_{k(dynamic)}}{\rho}-\frac{\omega^+k^+u_*^2}{\nu}$$

$$u_j=u_j^+u_*$$

$$\left(\frac{\partial k^+}{\partial t}+u_j^+u_*\frac{\partial k^+}{\partial x_j}\right)=\frac{\partial}{\partial x_j}\left(\left(\nu+\sigma_{K(k\omega SST)}\nu_t\right)\frac{\partial k^+}{\partial x_j}\right)+\frac{P_{k(dynamic)}}{\rho}-\frac{\omega^+k^+u_*^2}{\nu}$$

So in BL, we assume

$$\frac{\partial k^+}{\partial t}+u_j^+u_*\frac{\partial k^+}{\partial x_j}\approx 0$$

In BL, the production of turbulent KE is negligible (more so in VSL)

$$\frac{P_{k(dynamic)}}{\rho}\approx 0$$

So we get:

$$0=\frac{\partial}{\partial x_j}\left(\left(\nu+\sigma_{K(k\omega SST)}\nu_t\right)\frac{\partial k^+}{\partial x_j}\right)-\frac{\omega^+k^+u_*^2}{\nu}$$

$$0=\frac{\partial}{\partial x_j}\left(\left(1+\sigma_{K(k\omega SST)}\frac{\nu_t}{\nu}\right)\frac{\partial k^+}{\partial x_j}\right)-\frac{\omega^+k^+u_*^2}{\nu^2}$$

In BL, we only consider y direction, so we simplify

$$0 = \frac{\partial}{\partial y}\left(\left(1 + \sigma_{K(k\omega SST)}\frac{\nu_t}{\nu}\right)\frac{\partial k^+}{\partial y}\right) - \frac{\omega^+ k^+ u_*^2}{\nu^2}$$

$$y = \frac{y^+ \nu}{u_*}$$

$$0 = \frac{\partial}{\partial \frac{y^+ \nu}{u_*}}\left(\left(1 + \sigma_{K(k\omega SST)}\frac{\nu_t}{\nu}\right)\frac{\partial k^+}{\partial \frac{y^+ \nu}{u_*}}\right) - \frac{\omega^+ k^+ u_*^2}{\nu^2}$$

$$0 = \frac{u_*^2}{\nu^2}\frac{\partial}{\partial y^+}\left(\left(1 + \sigma_{K(k\omega SST)}\frac{\nu_t}{\nu}\right)\frac{\partial k^+}{\partial y^+}\right) - \frac{\omega^+ k^+ u_*^2}{\nu^2}$$

$$0 = \frac{u_*^2}{\nu^2}\frac{\partial}{\partial y^+}\left(\left(1 + \sigma_{K(k\omega SST)}\frac{\nu_t}{\nu}\right)\frac{\partial k^+}{\partial y^+}\right) - \frac{\omega^+ k^+ u_*^2}{\nu^2}$$

$$0 = \frac{\partial}{\partial y^+}\left(\left(1 + \sigma_{K(k\omega SST)}\frac{\nu_t}{\nu}\right)\frac{\partial k^+}{\partial y^+}\right) - \omega^+ k^+$$

In VSL, we assume $\nu_t \ll \nu$

$$0 = \frac{\partial}{\partial y^+}\left(\frac{\partial k^+}{\partial y^+}\right) - \omega^+ k^+$$

I think im missing a C_μ

$$\omega = \omega^* \beta^* \rightarrow \omega^* = \frac{\omega}{\beta^*}, \beta^* = C_\mu$$

If we use ω_*

$$0 = \frac{\partial}{\partial y^+}\left(\frac{\partial k^+}{\partial y^+}\right) - C_\mu \omega^{*+} k^+$$

I'll skip the ω^{*+} derivation…

$$0 = \frac{\partial}{\partial y^+}\left(\frac{\partial \omega^{*+}}{\partial y^+}\right) - \beta_1 \left(\omega^{*+}\right)^2$$

We solve the ODE's in the VSL,

$$\omega^{*+} = \frac{6}{\beta_1 (y^+)^2}$$

$$\beta_1 = \frac{3}{40}\ (k - \omega\ equation\ constant)$$

Subs this into:

$$0 = \frac{\partial}{\partial y^+}\left(\frac{\partial k^+}{\partial y^+}\right) - C_\mu \frac{6}{\beta_1 (y^+)^2} k^+$$

And solve the ODE here, we get:

$$k^+ = C_k (y^+)^{3.23}$$

By intuition, there is no turbulence in the VSL,

So $C_k = 0$

We also have a log law region:

$$\omega^{*+} = \frac{1}{\kappa \sqrt{C_\mu} y^+}$$

$$k^{*+} = \frac{1}{\sqrt{C_\mu}}$$

To derive the above,

recall

$$\left(\frac{\partial k^+}{\partial t} + u_j^+ u_* \frac{\partial k^+}{\partial x_j}\right) = \frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)} \nu_t\right) \frac{\partial k^+}{\partial x_j}\right) + \frac{P_{k(dynamic)}}{\rho} - \frac{\omega^+ k^+ u_*^2}{\nu}$$

$$\left(\frac{\partial k^+}{\partial t} + u_j^+ u_* \frac{\partial k^+}{\partial x_j}\right) = 0$$

But for turbulent sublayer

$$\frac{P_{k(dynamic)}}{\rho} \neq 0$$

$$0 = \frac{\partial}{\partial y^+}\left(\left(1 + \sigma_{K(k\omega SST)} \frac{\nu_t}{\nu}\right) \frac{\partial k^+}{\partial y^+}\right) + \nu_t^+ \left|\frac{dU^+}{dy^+}\right|^2 - C_\mu \omega^{*+} k^+$$

$$\frac{P_{k(dynamic)}}{\rho} = \overline{u_i' u_j'} \frac{\partial U}{\partial y}$$

$$\frac{P_{k(dynamic)}}{\rho} = \nu_t \frac{\partial U}{\partial y} \frac{\partial U}{\partial y}$$

$$0 = \frac{\partial}{\partial y^+}\left(\left(1 + \sigma_{K(k\omega SST)} \frac{\nu_t}{\nu}\right) \frac{\partial k^+}{\partial y^+}\right) + \nu_t^+ \left|\frac{dU^+}{dy^+}\right|^2 - C_\mu \omega^{*+} k^+$$

In VSL, we assume $\nu_t \ll \nu$

But in turbulent sublayer, we assume $\nu_t \gg \nu$

$$0 = \frac{\partial}{\partial y^+}\left(\left(\sigma_{K(k\omega SST)} \frac{\nu_t}{\nu}\right) \frac{\partial k^+}{\partial y^+}\right) + \nu_t^+ \left|\frac{dU^+}{dy^+}\right|^2 - C_\mu \omega^{*+} k^+$$

We assume here in turbulent sublayer:

$$\frac{\partial}{\partial y^+}\left(\left(\sigma_{K(k\omega SST)}\frac{\nu_t}{\nu}\right)\frac{\partial k^+}{\partial y^+}\right) \ll \nu_t^+\left|\frac{dU^+}{dy^+}\right|^2$$

$$\frac{\partial}{\partial y^+}\left(\left(\sigma_{K(k\omega SST)}\frac{\nu_t}{\nu}\right)\frac{\partial k^+}{\partial y^+}\right) \approx 0$$

$$\left(\sigma_{K(k\omega SST)}\frac{\nu_t}{\nu}\right)\frac{\partial k^+}{\partial y^+} = C_k$$

$$\nu_t^+\left|\frac{dU^+}{dy^+}\right|^2 - C_\mu \omega^{*+}k^+ = 0$$

$$\nu_t^+ = \frac{k^+}{\omega^{*+}}$$

$$\omega^{*+} = \frac{k^+}{\nu_t^+}$$

$$\nu_t^+\left|\frac{dU^+}{dy^+}\right|^2 - C_\mu \frac{k^+}{\nu_t^+}k^+ = 0$$

$$\nu_t^{+2}\left|\frac{dU^+}{dy^+}\right|^2 - C_\mu k^{+2} = 0$$

Or to eliminate k^+

$$k^+ = \omega^{*+}\nu_t^+$$

$$\nu_t^+\left|\frac{dU^+}{dy^+}\right|^2 - C_\mu \omega^{*+}\omega^{*+}\nu_t^+ = 0$$

$$\left|\frac{dU^+}{dy^+}\right|^2 - C_\mu \omega^{*+2} = 0$$

$$\omega^{*+} = \frac{\left|\frac{dU^+}{dy^+}\right|}{\sqrt{C_\mu}}$$

What is $\frac{dU^+}{dy^+}$ in log law region

$$u^+ = 2.5\ln y^+ + 5.5\ (\log\ \ law\ of\ the\ wall)$$

$$u^+ = \frac{1}{\kappa}\ln y^+ + 5.5, \kappa \approx 0.4$$

$$\frac{du^+}{dy^+} = \frac{1}{\kappa}\frac{1}{y^+}$$

Subs into the ω^{*+} and we get

$$\omega^{*+} = \frac{1}{\kappa y^+ \sqrt{C_\mu}}$$

Now we want to get back our k equation…

Recall:

$$k^+ = \omega^{*+} \nu_t^+$$

$$k^+ = \frac{1}{\kappa y^+ \sqrt{C_\mu}} \nu_t^+$$

What is ν_t^+

$$\tau_{wall} = \rho(\nu + \nu_t)\frac{\partial u}{\partial y}$$

Nondimensionalise,

$$1 = \left(1 + \frac{\nu_t}{\nu}\right)\frac{du^+}{dy^+}$$

In log law region:

$$\nu_t \gg \nu$$

So we get

$$\frac{\nu_t}{\nu}\frac{du^+}{dy^+} = 1$$

$$\nu_t^+ = \frac{1}{\frac{du^+}{dy^+}}$$

Substitute this back:

$$k^+ = \frac{1}{\kappa y^+ \sqrt{C_\mu}} \frac{1}{\frac{du^+}{dy^+}}$$

$$k^+ = \frac{1}{\kappa y^+ \sqrt{C_\mu}} \kappa y^+$$

$$k^+ = \frac{1}{\sqrt{C_\mu}}$$

This is k^+ for log law region…

How about k^+ in VSL

Recall:

$$k^+ = C_k (y^+)^{3.23}$$

In TSL,

$$\left(\sigma_{K(k\omega SST)}\frac{\nu_t}{\nu}\right)\frac{\partial k^+}{\partial y^+} = C_k$$

$$k^+ = \frac{1}{\sqrt{C_\mu}}, so\ \frac{\partial k^+}{\partial y^+} = 0$$

Therefore

$$C_k = 0$$

In intermediate region, we take root mean sqr of VSL and TSL ω^*

	VSL	Intermediate (buffer) layer	Turbulent Sublayer
k^+	$k^+ = C_k(y^+)^{3.23} = 0$ As $C_k = 0$	$k^+ = \omega^{*+}\nu_t^+$ $1 = \left(1+\frac{\nu_t}{\nu}\right)\frac{du^+}{dy^+}$ In log law: $\frac{du^+}{dy^+} = \frac{1}{\kappa}\frac{1}{y^+}$ Or in VSL $\frac{du^+}{dy^+} = 1$ Or you can use Van Driest Model	$k^+ = \frac{1}{\sqrt{C_\mu}}$ $C_\mu = 0.09$
ω^{*+}	$\omega^{*+} = \frac{6}{\beta_1(y^+)^2}$ $\beta_1 = \frac{3}{40}$	ω^{*+} $= \sqrt{{\omega^{*+}}_{VSL}^{2} + {\omega_{TSL}^{*}}^{+2}}$	$\omega^{*+} = \frac{1}{\kappa y^+\sqrt{C_\mu}}$ $\kappa = 0.41$ $C_\mu = 0.09$

$\frac{\sqrt{k}}{C_\mu\omega^* y}$, $\frac{500\nu}{\omega^* y^2}$ and $\frac{4\sigma_{\omega 2}k}{CD_{k\omega}y^2}$

So first we want to nondimensionalise them…. So we can get these functions as a function of y^+

$$k^+ = \frac{k}{u_*^2}$$

$$\omega^{*+} = \frac{\omega^*\nu}{u_*^2}$$

$$y^+ = \frac{yu_*}{\nu}$$

We can nondimensionalise

$$\frac{\sqrt{k}}{C_\mu \omega^* y} = \frac{\sqrt{u_*^2 k^+}}{C_\mu \frac{\omega^{*+} u_*^2}{\nu} \frac{y^+ \nu}{u_*}} = \frac{u_* \sqrt{k^+}}{C_\mu \omega^{*+} u_* y^+} = \frac{\sqrt{k^+}}{C_\mu \omega^{*+} y^+}$$

Now we can substitute in…

$$y^+ < 11.6$$

$$u^+ = y^+ \ (viscous\ sublyaer\ equation)$$

$$y^+ > 11.6$$

$$u^+ = 2.5 \ln y^+ + 5.5 \ (\log\ \ law\ of\ the\ wall)$$

The next function is this:

$$\frac{500\nu}{\omega^* y^2}$$

$$y = \frac{y^+ \nu}{u_*}$$

$$\omega^* = \frac{\omega^{*+} u_*^2}{\nu}$$

Substitute in…

$$\frac{500\nu}{\omega^* y^2} = \frac{500\nu}{\frac{\omega^{*+} u_*^2}{\nu} \left(\frac{y^+ \nu}{u_*}\right)^2} = \frac{500\nu}{\frac{\omega^{*+} u_*^2}{\nu} \frac{y^{+2} \nu^2}{u_*^2}} = \frac{500\nu}{\omega^{*+} y^{+2} \nu} = \frac{500}{\omega^{*+} y^{+2}}$$

One last function…

$$\frac{4\sigma_{\omega 2} k}{CD_{k\omega} y^2}$$

Let's deal with the stuff outside…

$$\frac{4\sigma_{\omega 2} k}{y^2} = 4\sigma_{\omega 2} \frac{u_*^2 k^+}{\left(\frac{y^+ \nu}{u_*}\right)^2} = 4\sigma_{\omega 2} \frac{k^+}{y^{+2}} \frac{u_*^4}{\nu^2}$$

$$CD_{k\omega} = \max\left(2\rho\sigma_{\omega 2} \frac{1}{\omega^*} \frac{\partial \omega^*}{\partial x_j} \frac{\partial k}{\partial x_j}, 10^{-10}\right)$$

$$\omega^* = \frac{\omega^{*+} u_*^2}{\nu}$$

$$y = \frac{y^+ \nu}{u_*}$$

We can cheat to find an estimate in the boundary layer…

We can assume in the BL

$$\frac{\partial \omega^*}{\partial x_j}\frac{\partial k}{\partial x_j} = \frac{\partial \omega^*}{\partial y}\frac{\partial k}{\partial y}$$

In BL

$$CD_{k\omega} = \max\left(2\rho\sigma_{\omega 2}\frac{1}{\omega^*}\frac{\partial \omega^*}{\partial y}\frac{\partial k}{\partial y}, 10^{-10}\right)$$

$$\omega^* = \frac{\omega^{*+}u_*^2}{\nu}$$

$$y = \frac{y^+\nu}{u_*}$$

$$k = k^+u_*^2$$

Nondimensionalising

$$CD_{k\omega} = \max\left(2\rho\sigma_{\omega 2}\frac{1}{\frac{\omega^{*+}u_*^2}{\nu}}\frac{\partial\left(\frac{\omega^{*+}u_*^2}{\nu}\right)}{\partial\left(\frac{y^+\nu}{u_*}\right)}\frac{\partial(k^+u_*^2)}{\partial\left(\frac{y^+\nu}{u_*}\right)}, 10^{-10}\right)$$

$$CD_{k\omega} = \max\left(2\rho\sigma_{\omega 2}\frac{1}{\frac{\omega^{*+}u_*^2}{\nu}}\frac{\frac{u_*^2}{\nu}\,\partial\left(\omega^{*+}\right)}{\frac{\nu}{u_*}\,\partial(y^+)}\frac{u_*^2\partial(k^+)}{\frac{\nu}{u_*}\,\partial(y^+)}, 10^{-10}\right)$$

$$CD_{k\omega} = \max\left(2\rho\sigma_{\omega 2}\frac{1}{\omega^{*+}\nu}\frac{\frac{u_*^2}{\nu}\,\partial\left(\omega^{*+}\right)}{\partial(y^+)}\frac{u_*^2\partial(k^+)}{\partial(y^+)}, 10^{-10}\right)$$

$$CD_{k\omega} = \max\left(2\rho\sigma_{\omega 2}\frac{u_*^4}{\nu^2}\frac{1}{\omega^{*+}}\frac{\partial\left(\omega^{*+}\right)}{\partial(y^+)}\frac{\partial(k^+)}{\partial(y^+)}, 10^{-10}\right)$$

So for,

$$\frac{4\sigma_{\omega 2}k}{CD_{k\omega}y^2} = 4\sigma_{\omega 2}\frac{k^+}{y^{+2}}\frac{u_*^4}{\nu^2} * \frac{1}{CD_{k\omega}}$$

$$\frac{4\sigma_{\omega 2}k}{CD_{k\omega}y^2} = 4\sigma_{\omega 2}\frac{k^+}{y^{+2}}\frac{u_*^4}{\nu^2} * \frac{1}{\frac{u_*^4}{\nu^2}\max\left(2\rho\sigma_{\omega 2}\frac{1}{\omega^{*+}}\frac{\partial(\omega^{*+})}{\partial(y^+)}\frac{\partial(k^+)}{\partial(y^+)}, \frac{\nu^2}{u_*^4}10^{-10}\right)}$$

$$\frac{4\sigma_{\omega 2}k}{CD_{k\omega}y^2} = 4\sigma_{\omega 2}\frac{k^+}{y^{+2}} * \frac{1}{\max\left(2\rho\sigma_{\omega 2}\frac{1}{\omega^{*+}}\frac{\partial\omega^{*+}}{\partial y^+}\frac{\partial k^+}{\partial y^+}, \frac{\nu^2}{u_*^4}10^{-10}\right)}$$

Can we nondimensionalise $\frac{\nu^2}{u_*^4}$?

$$u_* = \sqrt{\frac{\tau_{wall}}{\rho}}$$

At the wall

$$\tau = \rho \nu \frac{\partial u}{\partial y}$$

$$u_* = \sqrt{\nu \frac{\partial u}{\partial y}}$$

Hang on, why is there an extra density term in the denominator?

stream ω. The blending function is given by

$$F_1 = \tanh\left(\Gamma^4\right) \tag{8}$$

where

$$\Gamma = \min\left(\max\left(\frac{\sqrt{k}}{\beta^* \omega d}; \frac{500\nu}{\omega d^2}\right); \frac{4\rho\sigma_{\omega 2} k}{CD_{k\omega} d^2}\right) \tag{9}$$

Hellsten, A. (1998). Some improvements in Menter's k-omega SST turbulence model. In 29th AIAA, Fluid Dynamics Conference (p. 2554).

http://cfd.mace.manchester.ac.uk/twiki/pub/Main/CDAdapcoMeetingsM4/AIAA_98-2554-CP.PS.pdf

This is from an earlier paper, and there **is** a density term in the numerator…

Okay so let's correct this:

$$\frac{4\rho\sigma_{\omega 2} k}{CD_{k\omega} y^2} = 4\sigma_{\omega 2} \frac{k^+}{y^{+2}} \frac{\rho u_*^4}{\nu^2} * \frac{1}{CD_{k\omega}}$$

$$CD_{k\omega} = \max\left(2\sigma_{\omega 2} \frac{\rho u_*^4}{\nu^2} \frac{1}{\omega^{*+}} \frac{\partial \omega^{*+}}{\partial y^+} \frac{\partial k^+}{\partial y^+}, 10^{-10}\right)$$

Looks like this term is quite important

$$\frac{\rho u_*^4}{\nu^2}$$

$$u_* = \sqrt{\frac{\tau_{wall}}{\rho}}$$

$$\frac{\rho \left(\frac{\tau_{wall}}{\rho}\right)^2}{\nu^2} = \frac{\tau_{wall}^2}{\rho \nu^2}$$

At the wall

$$\tau_{wall} = \rho(\nu + \nu_t)\frac{\partial u}{\partial y}$$

At the wall, $\nu_t = 0$

$$\tau_{wall} = \rho\nu\frac{\partial u}{\partial y}$$

$$y = \frac{y^+\nu}{u_*}$$

$$u = u^+u_*$$

$$\tau_{wall} = \rho\nu\frac{\partial u^+u_*}{\partial \frac{y^+\nu}{u_*}}$$

$$\tau_{wall} = \rho u_*^2\frac{\partial u^+}{\partial y^+}$$

In VSL region

$$\frac{\partial u^+}{\partial y^+} = 1$$

$$\tau_{wall} = \rho u_*^2$$

$$\frac{\rho\left(\frac{\tau_{wall}}{\rho}\right)^2}{\nu^2} = \frac{\tau_{wall}^2}{\rho\nu^2} = \frac{(\rho u_*^2)^2}{\rho\nu^2}$$

Looks like we can't cancel out all the terms here…

$$\frac{4\rho\sigma_{\omega 2}k}{CD_{k\omega}y^2} = 4\sigma_{\omega 2}\frac{k^+}{{y^+}^2}\frac{\rho u_*^4}{\nu^2} * \frac{1}{CD_{k\omega}}$$

$$CD_{k\omega} = \max\left(2\sigma_{\omega 2}\frac{\rho u_*^4}{\nu^2}\frac{1}{{\omega^*}^+}\frac{\partial {\omega^*}^+}{\partial y^+}\frac{\partial k^+}{\partial y^+}, 10^{-10}\right)$$

$$\frac{4\rho\sigma_{\omega 2}k}{CD_{k\omega}y^2} = \frac{4\sigma_{\omega 2}\frac{k^+}{{y^+}^2}}{\max\left(2\sigma_{\omega 2}\frac{1}{{\omega^*}^+}\frac{\partial {\omega^*}^+}{\partial y^+}\frac{\partial k^+}{\partial y^+}, \frac{\nu^2}{\rho u_*^4}10^{-10}\right)}$$

We can do parametric analysis to see the effect of the value of $\frac{\nu^2}{\rho u_*^4(Re, surface\ roughness)}$

So k^+ we already have in excel… (plotted)

$$\frac{\partial {\omega^*}^+}{\partial y^+} = \frac{-12}{\beta_1(y^+)^3}$$

ω^{*+}	$\omega^{*+} = \frac{6}{\beta_1 (y^+)^2}$ $\beta_1 = \frac{3}{40}$	ω^{*+} $= \sqrt{\omega^{*+2}_{VSL} + \omega^{*\,+2}_{TSL}}$	$\omega^{*+} = \frac{1}{\kappa y^+ \sqrt{C_\mu}}$ $\kappa = 0.41$ $C_\mu = 0.09$
$\frac{\partial \omega^{*+}}{\partial y^+}$	$\frac{-12}{\beta_1 (y^+)^3}$	-	$\omega^{*+} = \frac{-1}{\kappa (y^+)^2 \sqrt{C_\mu}}$

$$\frac{\partial k^+}{\partial y^+}$$

For $\frac{\partial k^+}{\partial y^+}$, it is equal to 0 in the BL

k^+	$k^+ = C_k (y^+)^{3.23} = 0$ As $C_k = 0$	$k^+ = \omega^{*+} \nu_t^+$ $1 = \left(1 + \frac{\nu_t}{\nu}\right)\frac{du^+}{dy^+}$ In log law: $\frac{du^+}{dy^+} = \frac{1}{\kappa}\frac{1}{y^+}$ Or in VSL $\frac{du^+}{dy^+} = 1$ Or you can use Van Driest Model	$k^+ = \frac{1}{\sqrt{C_\mu}}$ $C_\mu = 0.09$

In BL

$$\frac{4\rho\sigma_{\omega 2} k}{CD_{k\omega} y^2} = \frac{4\sigma_{\omega 2}\frac{k^+}{y^{+2}}}{\max\left(2\sigma_{\omega 2}\frac{1}{\omega^{*+}}\frac{\partial \omega^{*+}}{\partial y^+}\frac{\partial k^+}{\partial y^+}, \frac{\nu^2}{\rho u_*^4}10^{-10}\right)} = \frac{4\sigma_{\omega 2}\frac{k^+}{y^{+2}}}{\frac{\nu^2}{\rho u_*^4}10^{-10}}$$

The first term may be negligible…

$$\frac{4\sigma_{\omega 2}\frac{k^+}{y^{+2}}}{\frac{\nu^2}{\rho u_*^4}10^{-10}} = 0 \; in \; VSL$$

$$\frac{\frac{4\sigma_{\omega 2}}{\sqrt{C_\mu}}}{\frac{\nu^2}{\rho u_*^4}10^{-10}}\frac{1}{y^{+2}} = 0 \; in \; VSL$$

$$\sigma_{\omega 2} = 0.856$$

$$\sqrt{C_\mu} = \sqrt{0.09} = 0.3$$

$$\frac{4\sigma_{\omega 2}}{\sqrt{C_\mu}10^{-10}} = 1.14133E + 11$$

$$Z = \frac{4\rho\sigma_{\omega 2}k}{CD_{k\omega}y^2} \approx \frac{1.14133E + 11}{\frac{\nu^2}{\rho u_*^4}y^{+2}}$$

We show that for this, typical Z values with zeta =0.01 , 1 and 100,

$F_1 = 1$ in the BL → this turns on $k - \omega$ in BL always.

But as $2\sigma_{\omega 2}\frac{1}{\omega^{*+}}\frac{\partial\omega^{*+}}{\partial y^+}\frac{\partial k^+}{\partial y^+}$ increases in turbulent region

Then F_1 decreases to reach 0, and thus turn on the $k - \varepsilon$ model.

$$\zeta = \frac{\nu^2}{\rho u_*^4}$$

Typical value for ζ

https://www.aiche.org/ccps/resources/glossary/process-safety-glossary/friction-velocity-u

for air flow

$$u_* \approx 0.05\frac{m}{s}\ (light\ wind)$$

$$u_* \approx 1\frac{m}{s}\ (strong\ wind)$$

$$\rho = 1\frac{kg}{m^3}$$

$$\nu_{air} = 1.48e - 5\frac{m^2}{s}$$

https://www.engineersedge.com/physics/viscosity_of_air_dynamic_and_kinematic_14483.htm

$$\zeta = \frac{\left(1.48e - 5\frac{m^2}{s}\right)^2}{1\ (1^4)} = 2.1904E - 10$$

From baseline model (with just the blending function only), also called BSL, to $k - \omega$ SST

Problem 1: expt data shows overprediction of reynold's shear stresses in adverse pressure gradient flows…

(Menter, F. R. (1994). Two-equation eddy-viscosity turbulence models for engineering applications. AIAA journal, 32(8), 1598-1605.) → menter's paper for $k - \omega$ SST

What are adverse pressure gradient flows?

https://en.wikipedia.org/wiki/Adverse_pressure_gradient

when $\frac{dp}{dx}$ does not favour flow direction, that's adverse pressure gradient…

- ➔ We haven't quite dealt with that in the previous flows, those wall functions are based on flat plate flow…
- ➔ But for curved surfaces, eg. Airfoil, spheres, this doesn't work as well anymore…

How does one sort this out??

In that same paper,

We find a key: the bradshaw assumption

In the turbulent BL:

$$\tau_{turbulent} = \rho a_1 k$$

Where k is the turbulent kinetic energy

But how do we calculate reynold's shear stress normally?

$$\tau_{turbulent} = \rho \nu_t \frac{\partial u}{\partial y}$$

Or else we know $\frac{\partial U}{\partial y}$ is known as the rate of strain S, sometimes known as Ω

If we were to rewrite this in terms of turbulent KE (not going to show it here)

$$\tau_{turbulent} = \sqrt{\frac{production\ of\ k}{dissipation\ of\ k}} \rho a_1 k$$

For turbulent BL flows, for the Bradshaw assumption ie,

$$\tau_{turbulent} = \rho a_1 k$$

To be true,

$$\frac{production\ of\ k}{dissipation\ of\ k} = 1$$

Unfortunately in k-omega baseline model, this doesn't hold true.

So we need to artificially force:

$$\tau_{turb} = \rho a_1 k$$

In the turbulent BL and

$$\tau_{turb} = \rho\nu_t\left(\frac{\partial u}{\partial y}\right)$$

In the rest of the flow…

In other words, we resort to blending functions again!

Let's call the new blending function Z

$$\tau_{turb} = Z\rho a_1 k + (1-Z)\rho\nu_t\left(\frac{\partial u}{\partial y}\right)$$

Similar to F_1 we have $Z = 1$ in turbulent BL, $Z = 0$ in the main fluid. (I wont use F2 yet, because it is used in the model differently)

This is the general idea, but instead of writing things like this, Menter uses a different form:

$$\tau_{turb} = \rho\nu_t\left(\frac{\partial u}{\partial y}\right)$$

But we redefine ν_t such that in the turbulent bulk/freestream

$$\nu_t = \frac{k}{\omega^*} = C_\mu\frac{k}{\omega}$$

In BL

ν_t is such that $\tau_{turb} = \rho a_1 k$

So to do that,

We equate

$$\rho a_1 k = \rho\nu_t\left(\frac{\partial u}{\partial y}\right)$$

$$\nu_t = \frac{a_1 k}{\left(\frac{\partial u}{\partial y}\right)}\ in\ the\ BL$$

This forces $\tau_{turb} = \rho a_1 k$ in the BL

$$\tau_{turb} = \rho\nu_t\left(\frac{\partial u}{\partial y}\right) = \frac{a_1 k}{\left(\frac{\partial u}{\partial y}\right)}\rho\left(\frac{\partial u}{\partial y}\right) = \rho a_1 k$$

The way this is done:

$$\nu_t = \frac{a_1 k}{\max\left(a_1\omega^*, F_2\left(\frac{\partial u}{\partial y}\right)\right)}$$

$$F_2 = \tanh\left[\left[\max\left(\frac{\sqrt{k}}{\omega y}, \frac{500\nu C_\mu}{\omega y^2}\right)\right]^2\right]$$

$$F_2 = \tanh\left[\left[\max\left(\frac{\sqrt{k}}{C_\mu \omega^* y}, \frac{500\nu}{\omega^* y^2}\right)\right]^2\right]$$

Again, $F_2 = 1$ in BL, $F_2 = 0$ in turbulent bulk (freestream or centre of pipeflow) so that

$$F\left(\frac{\partial u}{\partial y}\right) > a_1 \omega \text{ in turbulent sublayer}$$

As this limits production of turbulent shear stress or turbulent viscosity in the turbulent sublayer, this modification is known as the **production limiter**

$$\nu_t = \frac{a_1 k}{\max\left(a_1 \omega^*, F_2\left(\frac{\partial u}{\partial y}\right)\right)}$$

in 3D, we don't just have u velocity and y, hence we use strain rate S instead.

$$S = \left(\frac{\partial u}{\partial y}\right) \text{ in } 3D \text{ (i'm oversimplifying but yeah ...)}$$

$$\nu_t = \frac{a_1 k}{\max(a_1 \omega^*, F_2 S)}$$

But producing this change now poses a problem in the sublayer, since we artificially suppress turbulent viscosity, and this kind of upsets the balance of transport equations for k and ω

So we have to modify them…

What are the changes?

1) Some of the constants have to be tweaked from the original $k - \omega$ model
2) The production term of ω must be written a certain way
    a. In original $k - \omega$ / baseline model, $P_\omega = \alpha \frac{1}{\nu_t} \tau_{ij} \frac{\partial u_i}{\partial x_j} = \alpha \frac{\omega^*}{k} \tau_{ij} \frac{\partial u_i}{\partial x_j}$
    b. In SST model, $\nu_t \neq \frac{k}{\omega^*}$, so we can't use this same formulation
        i. To remove ambiguity, $P_\omega = \alpha S^2 = \alpha \frac{\omega^*}{k} \tau_{ij} \frac{\partial u_i}{\partial x_j}$

In most CFD guides eg. Cfd online, we won't see this comparison being made,

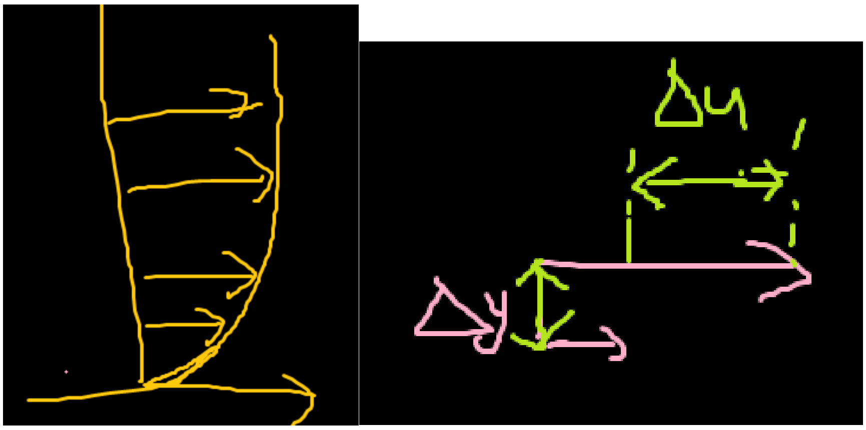
ν_t is removed from the expression even in $k - \omega$ model

This is the 1994 version of the $k - \omega$ SST model.

<u>Problem 2: in stagnation regions</u>

Large normal strain produces excessive turbulent kinetic energy…

Normal strain $= \frac{\partial U}{\partial y}$



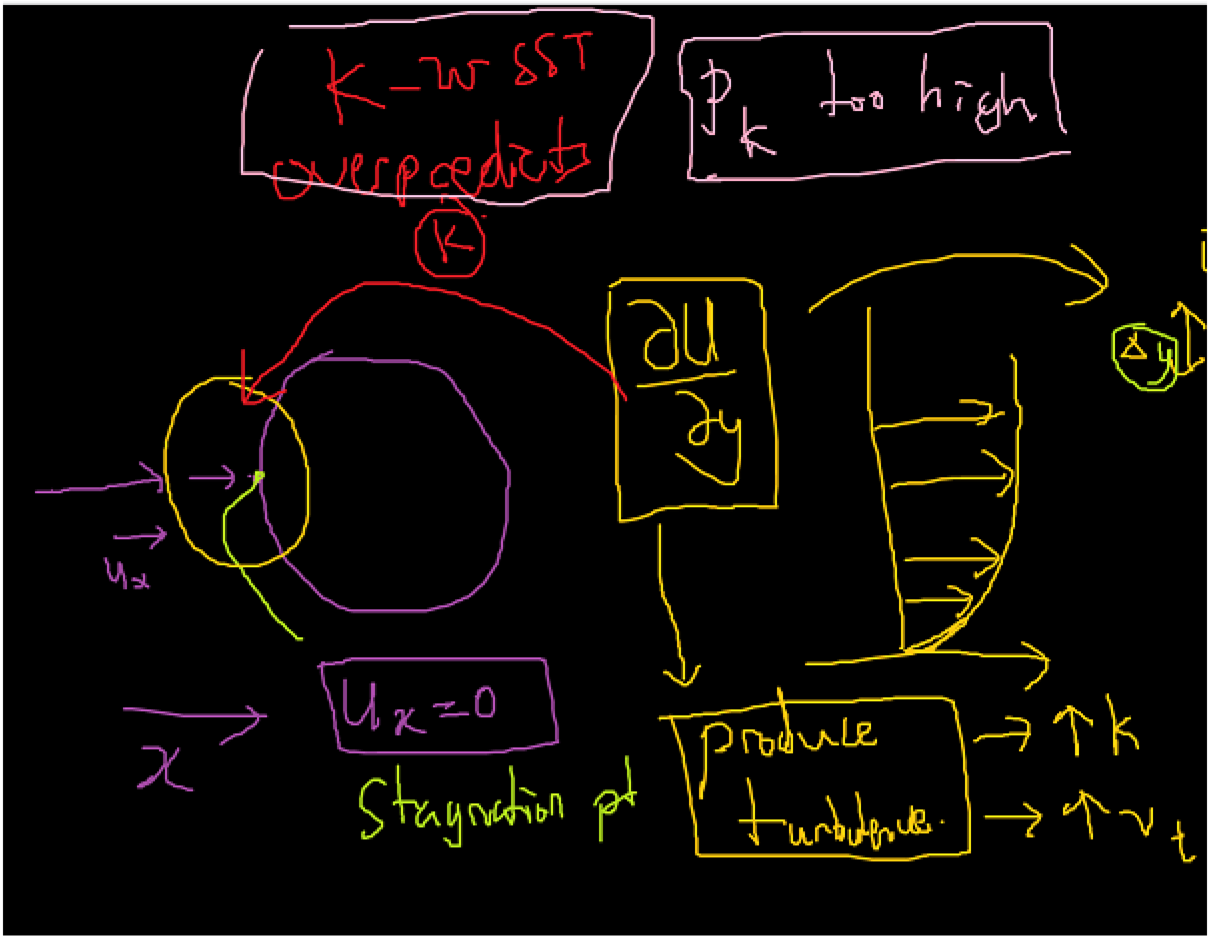
Characteristic length scale $= \Delta y$

Characteristic velocity scale $= \Delta u = \frac{\partial u}{\partial y}(strain\ rate)\Delta y$

$$Re = \frac{Ux}{\nu} = \frac{\Delta u \Delta y}{\nu} = \frac{\Delta y^2 \frac{\partial u}{\partial y}(strain\ rate)}{\nu}$$

Increasing strain rate increases Re

Eg in stagnation regions… take a sphere for example.



To compensate for this, a limiter is used on P_k in the turbulent kinetic energy equation

$$P_k = \min\left(\tau_{ij}\frac{\partial u_i}{\partial x_j}, 10\,\beta^* k\omega^*\right)$$

$$\beta^*\omega^* k = \varepsilon$$

We can P_k at 10ε= 10x dissipation rate.

https://www.openfoam.com/documentation/guides/latest/doc/guide-turbulence-ras-k-omega-sst.html

https://turbmodels.larc.nasa.gov/sst.html

http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.460.2814&rep=rep1&type=pdf

Menter, F. R., Kuntz, M., & Langtry, R. (2003). Ten years of industrial experience with the SST turbulence model. Turbulence, heat and mass transfer, 4(1), 625-632.

This is known as the 2003 update, as of 2020, OpenFOAM uses this model…

All right so this is $k-\omega$ SST for you!

Summary of key features

- Note that a lot of it was data fitting, empirical testing…
  - K-omega in BL, k-epsilon in bulk region
  - Shear stress production limiter in adverse pressure gradient BL region (redefines ν_t to reduce it artificially)
  - Turbulent KE production limiter in stagnation regions (limit P_k)

SpalartAllmaras Model

Why study this when we have $k-\omega\ SST$?

1) Sometimes if you like simplicity, this is good, easy to understand/intuitive model
2) Developers of the model also developed Detached Eddy Simulation
   a. Hybrid between RANS and LES

https://www.researchgate.net/publication/236888804_A_One-Equation_Turbulence_Model_for_Aerodynamic_Flows

Spalart, P., & Allmaras, S. (1992, January). A one-equation turbulence model for aerodynamic flows. In 30th aerospace sciences meeting and exhibit (p. 439).

S-A model Fluid Mechanics 101

https://www.youtube.com/watch?v=Xivc0EIGFQw

Follow up paper

http://ae.metu.edu.tr/tuncer/ae546/docs/ICCFD7-1902.pdf

from the paper:

Development of transport equations…

Previously, $k-\omega$ and $k-\varepsilon$ are used

Here

It's just the kinematic viscosity → easy to use and low computation cost (1 less transport equation)

Derviation/Thought Process in development:

Closure model

$$-\overline{u_i' u_j'} = 2\nu_t S_{ij}$$

So we want a transport equation for turbulent kinematic viscosity,

Remember?

$$\nu_t = C_\mu \frac{k^2}{\varepsilon} = C_\mu \frac{k}{\omega} = \frac{k}{\omega^*}$$

$$\omega \equiv \frac{k}{\varepsilon}, \omega = \omega^* C_\mu = \omega^* \beta^*$$

Yes this is our friend, turbulent kinematic viscosity or eddy diffusivity

So we'll have our standard terms:

$$\frac{\partial \nu_t}{\partial t} + u_j \frac{\partial \nu_t}{\partial x_j} = RHS$$

What's in the right hand side?

We'll have a source of turbulent viscosity

Confused viscosity $\nu_{LOL}^{?}$

Remember our turbulence viscosity has its source in the strain rate $\frac{\partial U}{\partial y}$, remember the dimensional argument?

- Just a note though, I said S and Ω were similar in meaning in the last video, but they're NOT
  - $S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$, this is strain rate tensor, rate of deformation of fluid
    - $S = \sqrt{2 S_{ij} S_{ij}}$ (strain rate), velocity gradients…

- $\Omega_{ij} = \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$, this is not the same, and it's more closely related to vorticity (curl of fluid)
  - $\vec{\omega} = \vec{\nabla} \times \vec{u} \equiv \sqrt{\Omega_{ij}\Omega_{ij}}$ (vorticity), tendency of fluid to twist/curl

Source term:

$$source = c_{b1}\, \nu_t\, S$$

Diffusion terms: (see k Omega SST/ or other RANS models)

$$diffusion = \frac{1}{\sigma}\{\nabla \cdot (\nu_t \nabla \nu_t)\}$$

But there are extra terms that pop up… why?

Recall our k and omega equations…

$$\frac{\partial k}{\partial t} + u_j \frac{\partial k}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right)\frac{\partial k}{\partial x_j}\right) + \frac{P_{k(dynamic)}}{\rho} - \beta^* \omega^* k$$

$$\frac{\partial \omega^*}{\partial t} + u_i \frac{\partial \omega^*}{\partial x_i} = \frac{\partial}{\partial x_j}\left(\nu + \frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial \omega^*}{\partial x_j} + (1 - F_1) 2\left(\frac{1}{\omega^*}\right)\frac{\partial \omega^*}{\partial x_j}\frac{\partial k}{\partial x_j} + \alpha S^2 - \beta \omega^{*2}$$

Remember $\omega = \beta^* \omega^*$

Try combining both into one long equation for ν_t

$$\nu_t = \frac{k}{\omega^*}$$

$$k = \omega^* \nu_t$$

Substitute into the k equation

$$\frac{\partial \omega^* \nu_t}{\partial t} + u_j \frac{\partial \omega^* \nu_t}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right)\frac{\partial \omega^* \nu_t}{\partial x_j}\right) + \frac{P_{k(dynamic)}}{\rho} - \beta^* \omega^* \omega^* \nu_t$$

Try differentiating it all the way and subtract the ω^* equation from it…

What happens to the viscosity term?

$$\frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right)\frac{\partial \omega^* \nu_t}{\partial x_j}\right)$$

noting

$$\frac{\partial \omega^* \nu_t}{\partial x_j} = \nu_t \frac{\partial \omega^*}{\partial x_j} + \omega^* \frac{\partial \nu_t}{\partial x_j}$$

$$\frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right)\left\{\nu_t \frac{\partial \omega^*}{\partial x_j} + \omega^* \frac{\partial \nu_t}{\partial x_j}\right\}\right)$$

$$= \frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right)\omega^* \frac{\partial \nu_t}{\partial x_j}\right) + \frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right)\nu_t \frac{\partial \omega^*}{\partial x_j}\right)$$

$$\frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right)\omega^* \frac{\partial \nu_t}{\partial x_j}\right)$$

$$= \omega^* \frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right)\frac{\partial \nu_t}{\partial x_j}\right) + \frac{\partial \nu_t}{\partial x_j}\frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right)\frac{\partial \nu_t}{\partial x_j}\right)$$

$$\frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right)\nu_t \frac{\partial \omega^*}{\partial x_j}\right)$$

$$= \nu_t \frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right)\frac{\partial \omega^*}{\partial x_j}\right) + \frac{\partial \omega^*}{\partial x_j}\frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right)\nu_t\right)$$

$$\frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right)\frac{\partial \omega^* \nu_t}{\partial x_j}\right)$$

$$= \omega^* \frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right)\frac{\partial \nu_t}{\partial x_j}\right) + \frac{\partial \nu_t}{\partial x_j}\frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right)\frac{\partial \nu_t}{\partial x_j}\right) + \nu_t \frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right)\frac{\partial \omega^*}{\partial x_j}\right) + \frac{\partial \omega^*}{\partial x_j}\frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right)\nu_t\right)$$

$$\frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right)\nu_t\right) = constant * \frac{\partial}{\partial x_j}(\nu_t^2) = constant * \nabla(\nu_t^2)$$

Near wall treatment

Log law layer

Destruction of turbulence near wall

After some dimensional analysis…

$$-c_{w1}\left(\frac{\nu_t}{d}\right)^2$$

$$w = wall$$

In free flows, $d \to \infty$ so doesn't really matter

After some testing: show a great prediction for velocity profile, log law

$$\frac{\partial \nu_t}{\partial t} + u_j \frac{\partial \nu_t}{\partial x_j} = -c_{w1}\left(\frac{\nu_t}{d}\right)^2 + c_{b1}\,\nu_t\,S + \frac{1}{\sigma}\{\nabla \cdot (\nu_t \nabla \nu_t) + c_{b2}(\nabla \nu_t^2)\}$$

Produces a good $u^+ = f(y^+)$ in log law region

Provided have some suitable wall function for VSL region

Problem: skin friction coefficient too low (underpredict wall shear stress)

- ➔ Destruction term “decays too slowly” in outer BL, - in the paper
- ➔ Need to decrease destruction of turbulent viscosity in this area

Introduce a nondimensional coefficient to compensate for this…

- ➔ In log law region, $f_w = 1$
- ➔ It should help dampen the destruction term in the outer BL

Idea:

- Take inspiration from mixing length model
- Decay it with distance from the wall relation

Nondimnesional group

$$r = \left(\frac{mixing\ length\ scale}{characterisc\ distance\ from\ the\ wall}\right)^2 = \left(\frac{\sqrt{\frac{\nu_t}{S}}}{\kappa d}\right)^2 = \frac{\nu_t}{S\kappa^2 d^2}$$

$$f_w = g = \left(r + c_{w2}(r^6 - r)\right)$$

<u>Problem: f_w gets too high in bulk fluid! (upsets numerical stability)</u>

Dampen it again…

$$g = \left(r + c_{w2}(r^6 - r)\right)$$

Damping function for numerical stability= $\left[\frac{1+c_{w3}^6}{g^6+c_{w3}^6}\right]^{\frac{1}{6}}$

$$f_w = g\left[\frac{1 + c_{w3}^6}{g^6 + c_{w3}^6}\right]^{\frac{1}{6}}$$

$\left[\frac{1+c_{w3}^6}{g^6+c_{w3}^6}\right]^{\frac{1}{6}}$ prevents f_w from getting too big and thus upsetting numerical stability

<u>Viscous sublayer</u>

<u>Problem: How can we force</u>

$$\underline{u^+ = y^+}$$

Here? And also correct predictions in the buffer layer?

In log law region

$$\nu_t = \kappa y u_*$$

But in the VSL, this doesn't hold true!

How can we get it to work…

Can we take the turbulent kinematic viscosity above (log law region correct turbulent kinematic viscosity) and dampen it?

In the VSL, buffer reigon, we have this:

$$\nu_t = \kappa y u_* * damping\ function$$

Damping function for VSL/buffer layer

$$f_{v1} = \frac{\chi^3}{\chi^3 + c_{v1}^3}$$

$$v = viscous$$

$$\chi = \kappa y^+$$

$$\frac{\nu_t\ (disclaimer)}{\nu} = \frac{\kappa y u_*}{\nu} = \kappa y^+ = \chi$$

Disclaimer:

The turbulence viscosity, ν_t in the log law region and the bulk region of the fluid is called:

$$\nu_t = \tilde{\nu}\ (in\ bulk\ region\ and\ log - law\ region)$$

$$\chi = \frac{\tilde{\nu}}{\nu} = \kappa y^+$$

(definition starts in the log law region)

$$\frac{\partial \tilde{\nu}}{\partial t} + u_j \frac{\partial \tilde{\nu}}{\partial x_j} = -c_{w1}\left(\frac{\tilde{\nu}}{d}\right)^2 + c_{b1}\,\tilde{\nu}\,S + \frac{1}{\sigma}\{\nabla \cdot (\tilde{\nu}\nabla\tilde{\nu}) + c_{b2}(\nabla\tilde{\nu}^2)\}$$

So that's how we introduce the spalart allmaras variable ($\tilde{\nu}$)

$$\nu_t = \tilde{\nu}\frac{\chi^3}{\chi^3 + c_{v1}^3}$$

$$f_{v1} = \frac{\chi^3}{\chi^3 + c_{v1}^3}$$

Problem: too much production of turbulence in VSL

Dampen it again!

We replace S with $\tilde{S}$

$$\tilde{S} = f(S)$$

$$\tilde{S} = S + \frac{\tilde{\nu}}{\kappa^2 d^2} f_{v2}$$

$$f_{v2} = 1 - \frac{\chi}{1 + \chi f_{v1}}$$

$$f_{v1} = \frac{\chi^3}{\chi^3 + c_{v1}^3}$$

Now our equation looks like this:

$$\frac{\partial \tilde{\nu}}{\partial t} + u_j \frac{\partial \tilde{\nu}}{\partial x_j} = -c_{w1} f_w \left(\frac{\tilde{\nu}}{d}\right)^2 + c_{b1}\, \tilde{\nu}\, \tilde{S} + \frac{1}{\sigma}\{\nabla \cdot (\tilde{\nu} \nabla \tilde{\nu}) + c_{b2}(\nabla \tilde{\nu}^2)\}$$

So far, we have dealt with:

- Log law region (with f_w)
- VSL $(\tilde{\nu}, \tilde{S})$
- Bulk region

<u>Problem: numerical instabilities present in this formulation</u>

Long story short,

$\tilde{\nu} = 0$ causes the solvers to blow up… (skipping explanation)

Laminar to turbulent transition for laminar region shear layers

What is a shear layer?

http://thermopedia.com/content/1118/

we need to set a “**trip**” in order to prevent instability, allow $\tilde{\nu} = 0$ in VSL to be a stable solution and not cause blow-ups.

To do so, artificially dampen production term to zero in the BL

Now that we artificially reduce production of $\tilde{\nu}$ in BL, we also need to artificially reduce the destruction of $\tilde{\nu}$ in BL and transition region.

To artificially reduce production term:

$$\frac{\partial \tilde{\nu}}{\partial t} + u_j \frac{\partial \tilde{\nu}}{\partial x_j} = -c_{w1} f_w \left(\frac{\tilde{\nu}}{d}\right)^2 + c_{b1}[1 - f_{t2}]\, \tilde{\nu}\, \tilde{S} + \frac{1}{\sigma}\{\nabla \cdot (\tilde{\nu} \nabla \tilde{\nu}) + c_{b2}(\nabla \tilde{\nu}^2)\}$$

To artificially reduce the destruction term:

$$\frac{\partial \tilde{\nu}}{\partial t} + u_j \frac{\partial \tilde{\nu}}{\partial x_j} = -[c_{w1} f_w - \frac{c_{b1}}{\kappa^2} f_{t2}] \left(\frac{\tilde{\nu}}{d}\right)^2 + c_{b1}[1 - f_{t2}]\, \tilde{\nu}\, \tilde{S} + \frac{1}{\sigma}\{\nabla \cdot (\tilde{\nu}\nabla\tilde{\nu}) + c_{b2}(\nabla\tilde{\nu}^2)\}$$

See f_{t2}

$$f_{t2} = c_{t3} * \exp(-c_{t4}\chi^2)$$

Remember

$$\chi = \frac{\tilde{\nu}}{\nu} = \kappa y^+$$

T stands for trip

<u>Problem: transition from laminar to turbulent layer not smooth</u>

We sort of kaboomed our BL $\tilde{\nu}$ source and destruction terms by doing the above (trip terms, f_{t2})…

How can we generate turbulence in the BL?

See f_{t1} and ΔU

$$\frac{\partial \tilde{\nu}}{\partial t} + u_j \frac{\partial \tilde{\nu}}{\partial x_j} = -\left[c_{w1} f_w - \frac{c_{b1}}{\kappa^2} f_{t2}\right] \left(\frac{\tilde{\nu}}{d}\right)^2 + c_{b1}[1 - f_{t2}]\, \tilde{\nu}\, \tilde{S} + \frac{1}{\sigma}\{\nabla \cdot (\tilde{\nu}\nabla\tilde{\nu}) + c_{b2}(\nabla\tilde{\nu}^2)\} + f_{t1}\Delta U^2$$

$$\Delta U = norm\ of\ difference\ between\ trip\ point, and\ point\ where\ we\ are\ calculating$$

Trip point→ remember we wanted to trip our production and destruction terms in the turbulent/laminar transition part of BL…

http://brennen.caltech.edu/fluidbook/basicfluiddynamics/turbulence/lawofthewall.pdf

now what does f_{t1} look like?

$$f_{t1} = c_{t1} g_t \exp\left(-c_{t2} \frac{\omega_t^2}{\Delta U^2}[d^2 + g_t^2 d_t^2]\right)$$

$$g_t \equiv \min\left(0.1, \frac{\Delta U}{\omega_t \Delta x_t}\right)$$

$$\Delta x_t = grid\ spacing\ along\ wall\ at\ trip$$

$$\vec{\omega}_t = vorticity\ of\ wall\ at\ trip\ point \equiv \nabla \times \vec{u}_{trip}$$

log law region → turbulent BL

VSL → laminar BL

Buffer zone → transition region…

https://mathworld.wolfram.com/Norm.html

what is a norm?

well in this context, this is…

if $\vec{x}$ is a vector

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$norm\ (\vec{x}) = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

So it means like for a position, it is the length between two points

<u>Technically correct way: L2 norm</u>

In this case, it's the magnitude difference in velocity,

So it's somewhat like ΔU we are used to

So this serves as the term to build up turbulence somewhat in the transition zone….

That's it!

The end product:

$$\frac{\partial \tilde{\nu}}{\partial t} + u_j \frac{\partial \tilde{\nu}}{\partial x_j} = -\left[c_{w1} f_w - \frac{c_{b1}}{\kappa^2} f_{t2}\right]\left(\frac{\tilde{\nu}}{d}\right)^2 + c_{b1}[1 - f_{t2}]\,\tilde{\nu}\,\tilde{S} + \frac{1}{\sigma}\{\nabla \cdot (\tilde{\nu}\nabla\tilde{\nu}) + c_{b2}(\nabla\tilde{\nu}^2)\} + f_{t1}\Delta U^2$$

$$\nu_t = \tilde{\nu}\frac{\chi^3}{\chi^3 + c_{v1}^3}$$

https://www.cfd-online.com/Wiki/Spalart-Allmaras_model

More LES models!

(smargorinsky and dynamic smargorinksy and more…)

So we've discussed a few major RANS models…

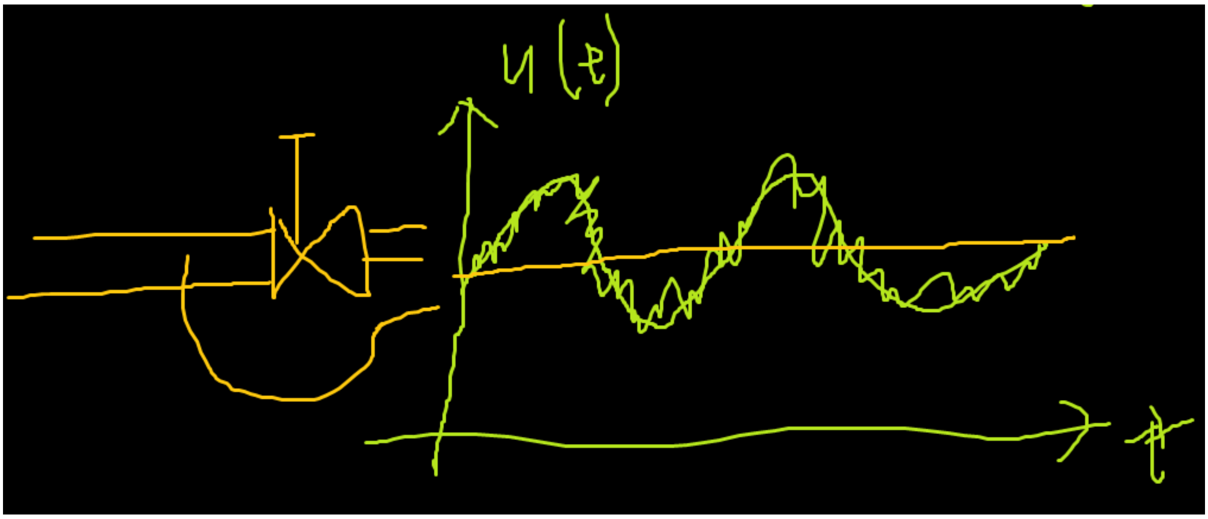
- $k - \varepsilon$
- $k - \omega$
- $k - \omega\ SST$
- Spalart Allmaras model

These are unsteady RANS…

$$\bar{u}(t) = \frac{1}{\Delta t}\int_0^{\Delta t} u(t)\, dt$$

For RANS

$$\bar{u}(t) = \lim_{\Delta t \to \infty} \frac{1}{\Delta t}\int_0^{\Delta t} u(t)\, dt$$



But unsteady RANS

$$\bar{u}(t) = \frac{1}{\Delta t}\int_0^{\Delta t} u(t)\, dt$$

$$\Delta t \neq \infty$$

Very large eddy simulation (VLES)

Weakness: flow results are only as good as the model of turbulence.

Weaknesses of some RANS/URANS(VLES) models…

- $k - \varepsilon$
  - Doesn't do adverse pressure gradients well
  - Wall boundary conditions not as good as $k - \omega$

    - Complex flows not good
- $k - \omega$
    - Freestream ω BC must be specified
    - Flow is sensitive to this BC
- $k - \omega\ SST$
    - Blends $k - \omega$ and $k - \varepsilon$
    - Cannot model some complex flows properly, eg. Turbomachinery (turbines/jets)
- Spalart Allmaras model
    - Cannot model some complex flows properly, eg. Turbomachinery (turbines/jets)

All of these are based on empirical models of wall functions, and these are **not** universal.

Let's look further into the world of LES modelling

Why LES?

- RANS has a lot of empirical constants, equation constant used for one flow may not always be suitable that of another flow
    - Unsteady RANS (aka Very large eddy simulations VLES) ≈ Navier stokes (unsteady) + turbulence modelling
        - (Spalart, Jou, Strelets, & Allmaras, 1997)
        - Spalart, P., Jou, W.-H., Strelets, M., & Allmaras, S. (1997). Comments on the feasibility of LES for winds, and on a hybrid RANS/LES approach. 1, 4–8.
- DNS while accurate is too impractical

LES wants more accuracy than RANS but more speed than DNS

10. Numerical methods in LES

- It is important for LES calculations to predict accurately the quantities that led to choosing LES in the first place (e.g., turbulent fluctuations, acoustic sources, mixing,…)
- Numerical dissipation present in most RANS codes is inadequate for LES (c.f. flow over cylinder)
- Ideally in LES nondissipative discretizations (central differencing as opposed to low order upwinding or any extra added numerical dissipation) must be used.

(note the fvSchemes!)

WALE Model

Paper:

F. Nicoud and F. Ducros. Subgrid-Scale Stress Modelling Based on the Square of the Velocity Gradient Tensor. Flow Turbulence and Combustion, 62(3):183–200, 1999.

http://doi.org/10.1023/A:1009995426001

https://www.cfd-online.com/Wiki/Wall-adapting_local_eddy-viscosity_(WALE)_model

recall our standard defintions:

$$\bar{S}_{ij} = \frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right) \rightarrow strain\ rate$$

$$\bar{\Omega}_{ij} = \frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_j} - \frac{\partial \bar{u}_j}{\partial x_i}\right) \rightarrow rotation$$

Recall smagorinsky:

$$\nu_t = (C_{Smagorinsky}^2)\Delta^2|\bar{S}| = (C_{Smagorinsky}^2)\Delta^2\sqrt{2S_{ij}S_{ij}}$$

<u>Problem:</u>

$$\underline{\nu_t \neq 0\ at\ wall}$$

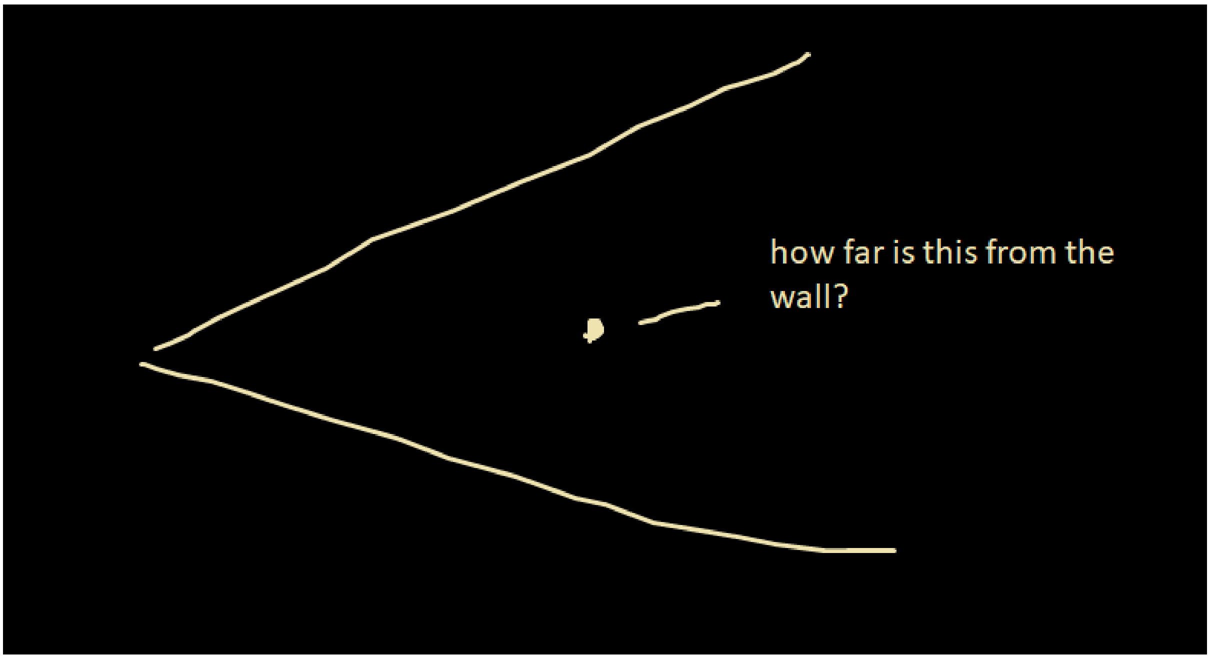
Use van driest damping function such that

$$\nu_t = (C_{Smagorinsky}^2 D^2)\Delta^2|\bar{S}| = (C_{Smagorinsky}^2)\Delta^2\sqrt{2S_{ij}S_{ij}}$$

$$D = 1 - \exp\left(\frac{y^+}{A^+}\right), A^+ = 25.0$$

According to Nicoud, this is rather ad-hoc, a scrappy way of forcing $\nu_t = 0$ at wall,

May not work for complex geometries…



Sometimes CFD code has a way of calculating that…

But Nicoud goes further…

- No explicit dependence on y
- But shows correct behavior from wall surface y behavior

He also tries to incorporate some flow physics

- "wall function" or damping function should capture both strain and rotation rate…

So the equation is in the form:

$$\nu_t = C_m \Delta^2 \overline{(OP(x,t))}$$

In smagorinsky,

$$\overline{(OP(x,t))} = \sqrt{2S_{ij}S_{ij}}$$

$$C_m = C_{smagorinsky}^2$$

Well, he wanted $\overline{(OP(x,t))}$ defined so that it has correct behavior from wall, ie $\nu_t = 0$ at wall,

So he uses this function:

$$S_{ij}^d S_{ij}^d = \frac{1}{6}\left(\bar{S}_{ij}\bar{S}_{ij}\bar{S}_{ij}\bar{S}_{ij} + \bar{\Omega}_{ij}\bar{\Omega}_{ij}\bar{\Omega}_{ij}\bar{\Omega}_{ij}\right) + \frac{2}{3}\left(\bar{\Omega}_{ij}\bar{\Omega}_{ij}\bar{S}_{ij}\bar{S}_{ij}\right) + 2\left(\bar{S}_{ik}\bar{S}_{kj}\bar{\Omega}_{jl}\bar{\Omega}_{li}\right)$$

Structure this way:

- Large strain rate produces turbulence

- Large rotation rate produces turbulence
- Rotation is more important than shear in turbulence generation (physically true)
- For pure shear flows → no rotation

For pure shear, $\mathcal{S}_{ij}^d \mathcal{S}_{ij}^d = 0$

- No turbulence generated in laminar zone, ie wall, laminar is pure shear only
- Smagorinsky predicts turbulence in such a zone

<u>Problem: incorrect wall behavior</u>

- $\mathcal{S}_{ij}^d \mathcal{S}_{ij}^d (y^2 \; behaviour \; from \; wall) \rightarrow \left(\mathcal{S}_{ij}^d \mathcal{S}_{ij}^d\right)^{\frac{3}{2}} \rightarrow y^3 \; behaviour$

<u>Problem: wrong dimension (ie wrong units)</u>

Correct dimensions without upsetting y^3 behavior:

$$\left(\mathcal{S}_{ij}^d \mathcal{S}_{ij}^d\right)^{\frac{3}{2}} \rightarrow frequency^6 \rightarrow (s^{-1})^6$$

Remember $\mathcal{S}_{ij}^d \rightarrow units \; of \; g^2 \rightarrow frequency^2$

You want some quantity with magnitude $\mathcal{O}(1)$, but with correct units.

$$\left(\bar{S}_{ij} \bar{S}_{ij}\right)^{\frac{5}{2}} \rightarrow frequency^5 \rightarrow (s^{-1})^5$$

- $\mathcal{S}_{ij}^d \mathcal{S}_{ij}^d \rightarrow \left(\mathcal{S}_{ij}^d \mathcal{S}_{ij}^d\right)^{\frac{3}{2}} \rightarrow \frac{\left(\mathcal{S}_{ij}^d \mathcal{S}_{ij}^d\right)^{\frac{3}{2}}}{\left(\bar{S}_{ij} \bar{S}_{ij}\right)^{\frac{5}{2}}} (y^3 \; behaviour + correct \; units)$

<u>Problem: numerical instability at the wall</u>

$$\left(\bar{S}_{ij} \bar{S}_{ij}\right)^{\frac{5}{2}} \rightarrow 0 \; at \; the \; wall$$

You want a denominator that doesn't go to 0 at the wall, preserves the units, but also has $\mathcal{O}(1)$ at the wall.

Solution, add

$$\left(\mathcal{S}_{ij}^d \mathcal{S}_{ij}^d\right)^{\frac{5}{4}}$$

To denominator, term is negligible in BL, but nonzero at wall.

$$\left(\mathcal{S}_{ij}^d \mathcal{S}_{ij}^d\right)^{\frac{5}{4}} \rightarrow (frequency^2 * frequency^2)^{\frac{5}{4}} \rightarrow frequency^5$$

Correct units!

$$\nu_t = C_m \Delta^2 \overline{(OP(x,t))}$$

$$\overline{(OP(x,t))} = \frac{\left(\mathcal{S}_{ij}^d \mathcal{S}_{ij}^d\right)^{\frac{3}{2}}}{\left(\bar{S}_{ij} \bar{S}_{ij}\right)^{\frac{5}{2}} + \left(\mathcal{S}_{ij}^d \mathcal{S}_{ij}^d\right)^{\frac{5}{4}}}$$

Choice of C_w

Best fitting of experimental data, original paper says 0.55-0.60 for Smagorinsky coefficient of 0.18.

But smagorinsky coefficient $C_s \neq 0.18$ for all flows, it depends on flow type and grid size.

Good compromise over several flow types: $C_w = 0.325$ → corresponds to $C_s \approx 0.1 \rightarrow$

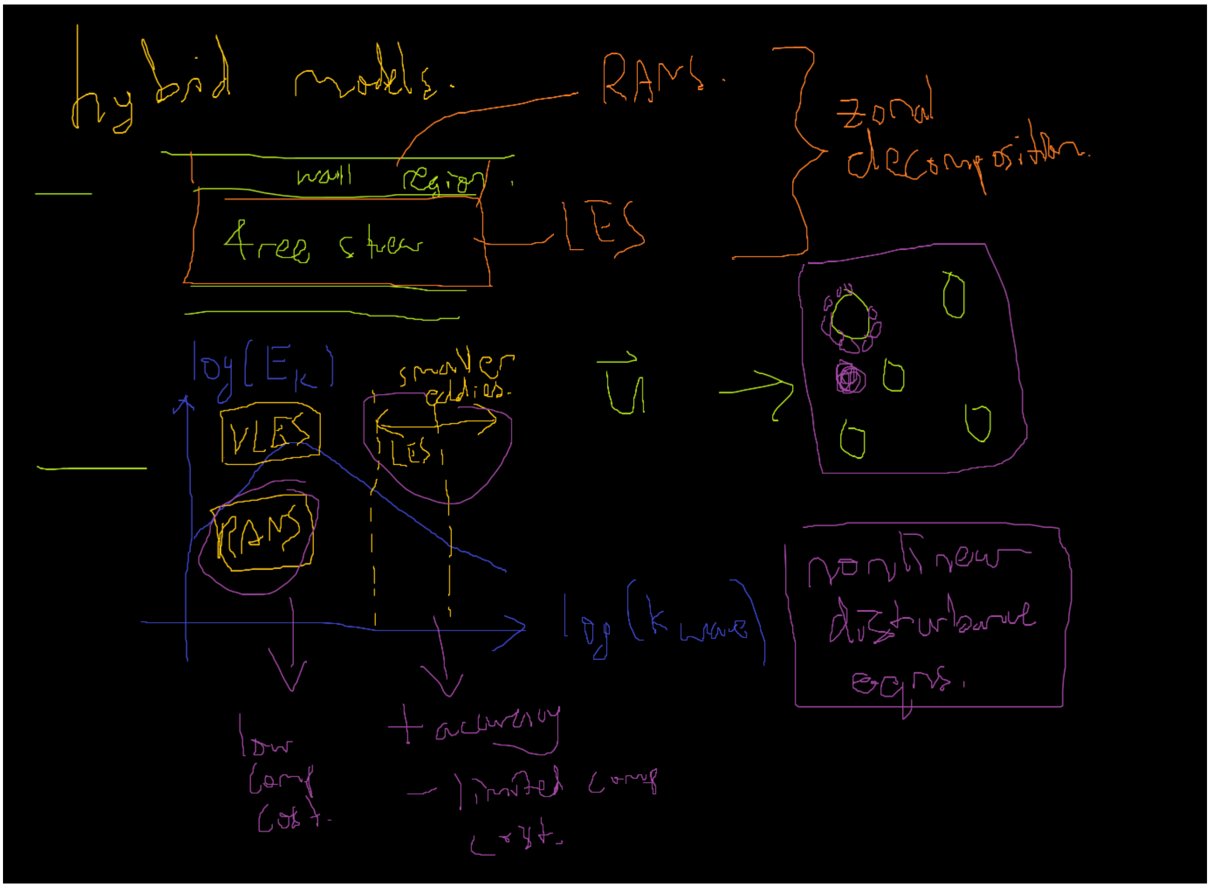
Good for bounded flows

https://www.slideshare.net/mmer547/les-customization-en

Pipe flow is being used here, $C_s \approx 0.1$

http://users.monash.edu.au/~bburn/pdf/csiro99_pipe.pdf

Hybrid RANS-LES



— add LES to RANS?

↳ modify RANS eqns to give LES behaviour in free shear } universal modelling. ⇒ Detached Eddy Simulation (DES)

DES

P.R. Spalart, W-H Jou, M. Strelets, and S.R. Allmaras. Comments on the Feasibility of LES for Wings, and on a Hybrid RANS/LES. In Advances in DNS/LES, pages 137–147, Columbus, OH, USA, 1997. Greyden Press.

Original: Spalart-Almaras RANS-LES (DES)

By Fluid Mechanics 101 Spalart-Allmaras

https://www.youtube.com/watch?v=Xivc0EIGFQw

Spalart Allmaras Model – DES

https://www.researchgate.net/profile/Michael_Strelets/publication/307943024_Comments_on_the_feasibility_of_LES_for_winds_and_on_a_hybrid_RANSLES_approach/links/57d7ceae08ae0c0081ecd0d5/Comments-on-the-feasibility-of-LES-for-winds-and-on-a-hybrid-RANS-LES-approach.pdf

Good Reference here:

https://www.iihr.uiowa.edu/gconstantinescu/files/2012/10/LES_models_4.pdf

Let's recall the Spalart Allmaras RANS model:

$$\frac{\partial \tilde{\nu}}{\partial t} + u_j \frac{\partial \tilde{\nu}}{\partial x_j} = -\left[c_{w1} f_w - \frac{c_{b1}}{\kappa^2} f_{t2}\right]\left(\frac{\tilde{\nu}}{d}\right)^2 + c_{b1}[1 - f_{t2}]\,\tilde{\nu}\,\tilde{S} + \frac{1}{\sigma}\{\nabla\cdot(\tilde{\nu}\nabla\tilde{\nu}) + c_{b2}(\nabla\tilde{\nu}^2)\} + f_{t1}\Delta U^2$$

$$\nu_t = \tilde{\nu}\frac{\chi^3}{\chi^3 + c_{v1}^3}$$

Key change for DES version

- Replace d $with$ $\tilde{d}$
- $\tilde{d} \equiv \min(d, C_{DES}\Delta)$
- $C_{DES} = \mathcal{O}(1), ie \approx\ 0.5 - 2.0$
  - $C_{DES} = 0.65$ → calibrated for isotropic turbulence at equilibrium

$$\frac{\partial \tilde{\nu}}{\partial t} + u_j \frac{\partial \tilde{\nu}}{\partial x_j} = -\left[c_{w1} f_w - \frac{c_{b1}}{\kappa^2} f_{t2}\right] \left(\frac{\tilde{\nu}}{\tilde{d}}\right)^2 + c_{b1}[1 - f_{t2}]\, \tilde{\nu}\, \tilde{S} + \frac{1}{\sigma}\{\nabla \cdot (\tilde{\nu} \nabla \tilde{\nu}) + c_{b2}(\nabla \tilde{\nu}^2)\} + f_{t1} \Delta U^2$$

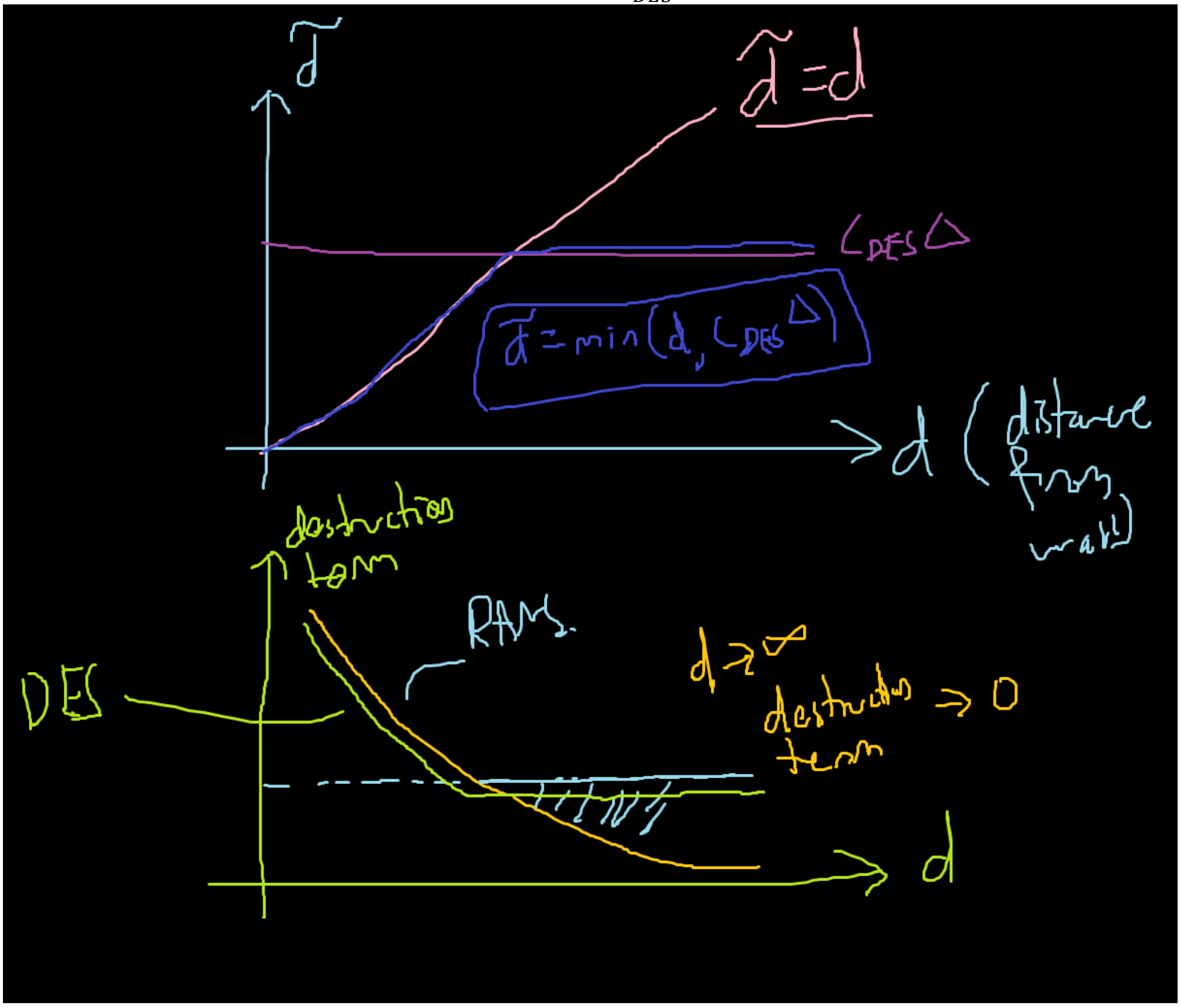
$$\nu_t = \tilde{\nu} \frac{\chi^3}{\chi^3 + c_{v1}^3}$$

$$\tilde{d} \equiv \min(d, C_{DES}\Delta)$$

$$\Delta = V^{\frac{1}{3}}$$

Far from the wall:

$$-\left[c_{w1} f_w - \frac{c_{b1}}{\kappa^2} f_{t2}\right] \left(\frac{\tilde{\nu}}{C_{DES}\Delta}\right)^2$$



Effect: destruction term

It is finite in the free shear region, thus ν_t will decrease in free shear region…

This is how we get LES behaviour, as ν_t drops, the apparent Re increases, and we can form eddies!

One note:

$$\tilde{d} \equiv \min(d, C_{DES}\Delta)$$

If you decrease Δ by having ultra fine mesh… more of the simulation behaves like LES, If you decrease Δ even smaller, such that $\nu \gg \nu_{sgs}\ or\ \nu_t$ then you have something almost like DNS

- This is called quasi-DNS

This relationship of more mesh refinement → more accurate, only true for LES/DES models or at least SA-DES…

For URANS → eg. SA model

- Refining Δ doesn't result in DNS
- Doesn't mean more accurate simulation
- Turbulence viscosity is still being modelled, rather than destroyed.
    - Accuracy is only as good as your turbulence model.

This is the key difference between URANS and LES/DES

Key difference between URANS and LES:

Take a look at the viscosity term for RANS/URANS ($k - \omega\ SST\ or\ k - \varepsilon$):

$$\nu_t = C_\mu \frac{k^2}{\varepsilon} = \frac{k}{\omega^*}$$

Where $\omega^* = \frac{\omega}{C_\mu} = \frac{\varepsilon}{C_\mu k}$

And for spalart allmaras

$$\nu_t = \frac{\chi^3}{C_{v1}^3 + \chi^3}\tilde{\nu}$$

Neither of them are grid dependent…

But for smagorinsky

$$\nu_t = \nu_{sgs} = C_{smagorinsky}^2 \Delta^2 * \bar{S}$$

ν_t being modelled depends on the Δ, and reducing $\Delta \to 0$ results in $\nu_t \to 0$ (DNS)

But reducing grid size for URANS does not mean more accuracy or DNS like calculations.

Note: according to paper, SA-DES behaves like smagorinsky model in free shear zone not kEqn

M. Strelets. Detached eddy simulation of massively separated flows. In *39th Aerospace Sciences Meeting and Exhibit*, Reno, NV, USA, 2001.

We also have: kOmega-SST (DES)

Recall our k and omega equations…

$$\frac{\partial k}{\partial t}+u_j\frac{\partial k}{\partial x_j}=\frac{\partial}{\partial x_j}\left(\left(\nu+\sigma_{K(k\omega SST)}\nu_t\right)\frac{\partial k}{\partial x_j}\right)+\frac{P_{k(dynamic)}}{\rho}-\beta^*\omega^*k$$

$$\frac{\partial \omega^*}{\partial t}+u_i\frac{\partial \omega^*}{\partial x_i}=\frac{\partial}{\partial x_j}\left(\nu+\frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial \omega^*}{\partial x_j}+(1-F_1)2\left(\frac{1}{\omega^*}\right)\frac{\partial \omega^*}{\partial x_j}\frac{\partial k}{\partial x_j}+\alpha S^2-\beta{\omega^*}^2$$

Remember $\omega=\beta^*\omega^*$, $\omega\equiv\frac{\varepsilon}{k}$

$$\beta^*=C_\mu=0.09$$

The idea is to follow the SA-DES model and replace some length scale to get LES effect in free shear zone.

From mixing length model:

$$\nu_t=l^2[m^2]\left(\frac{\partial\bar{u}}{\partial y}\right)[s^{-1}]$$

From $k-\omega$

$$\nu_t=C_\mu\frac{k}{\omega}=\frac{k\ [m^2s^{-2}]}{\omega^*\ [s^{-1}]}$$

We can also change the length scale here:

$$l_{k-\omega}=\frac{k^{0.5}\ [m^2s^{-2}]^{0.5}}{\omega\ [s^{-1}]}$$

$$\tilde{l}=\min\left(\frac{k^{0.5}}{\omega},C_{DES}\Delta\right)$$

Change the dissipative term:

$$\beta^*\omega^*k$$

$$\beta^*\omega^*k=\beta^*\omega^*k*\frac{l_{k-\omega}}{l_{k-\omega}}$$

$$\beta^*\omega^*k*\frac{l_{k-\omega}}{l_{k-\omega}}=\frac{\beta^*\omega^*kl_{k-\omega}}{l_{k-\omega}}=\frac{\omega k\frac{k^{0.5}}{\omega}}{l_{k-\omega}}=\frac{k^{1.5}}{l_{k-\omega}}$$

$$\frac{\partial k}{\partial t}+u_j\frac{\partial k}{\partial x_j}=\frac{\partial}{\partial x_j}\left(\left(\nu+\sigma_{K(k\omega SST)}\nu_t\right)\frac{\partial k}{\partial x_j}\right)+\frac{P_{k(dynamic)}}{\rho}-\frac{k^{1.5}}{l_{k-\omega}}$$

Given that

$$l_{k-\omega} = \frac{k^{0.5}\ [m^2 s^{-2}]^{0.5}}{\omega\ [s^{-1}]}$$

Now we replace the length scale to make the model DES

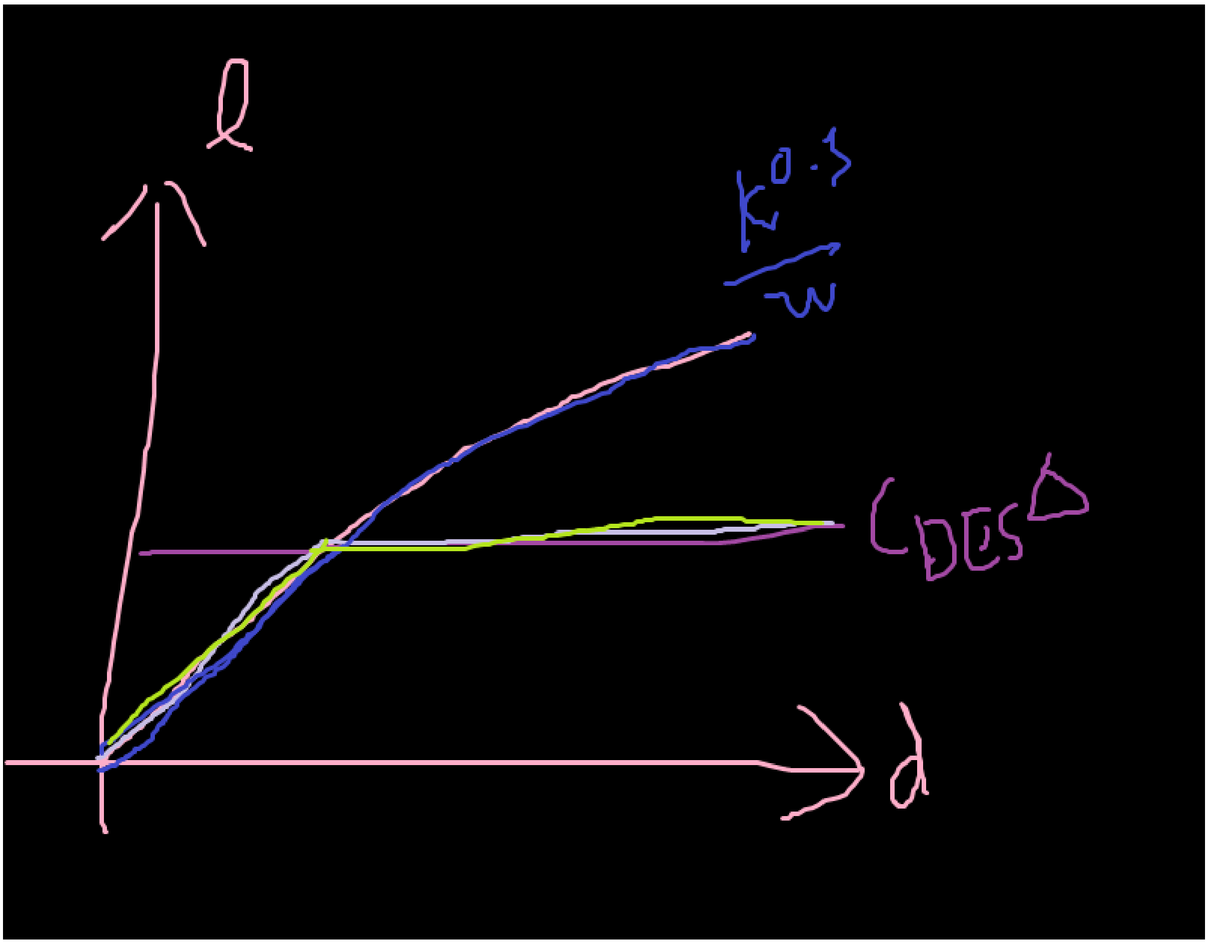
$$\frac{\partial k}{\partial t} + u_j \frac{\partial k}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right)\frac{\partial k}{\partial x_j}\right) + \frac{P_{k(dynamic)}}{\rho} - \frac{k^{1.5}}{\tilde{l}}$$

$$\tilde{l} = \min\left(\frac{k^{0.5}}{\omega}, C_{DES}\Delta\right)$$

$$\frac{\partial k}{\partial t} + u_j \frac{\partial k}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right)\frac{\partial k}{\partial x_j}\right) + \frac{P_{k(dynamic)}}{\rho} - \frac{k^{1.5}}{\min\left(\frac{k^{0.5}}{\omega}, C_{DES}\Delta\right)}$$

$$\frac{\partial \omega^*}{\partial t} + u_i \frac{\partial \omega^*}{\partial x_i} = \frac{\partial}{\partial x_j}\left(\nu + \frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial \omega^*}{\partial x_j} + (1 - F_1)2\left(\frac{1}{\omega^*}\right)\frac{\partial \omega^*}{\partial x_j}\frac{\partial k}{\partial x_j} + \alpha S^2 - \beta {\omega^*}^2$$

The omega equation remains untouched.



Making length scale smaller in free shear zone, makes dissipation bigger…

You destroy more **modelled** turbulence KE in free shear zone as a result.

What is C_{DES}?

One is calibrated for each RANS model for $k - \omega\ SST$

rmb $k - \omega\ SST$ is a hybrid model between $k - \varepsilon$ and $k - \omega$

$$C_{DES\,(k-\varepsilon)} \approx 0.60$$

$$C_{DES\,(k-\omega)} \approx 0.82$$

$$C_{DES} = (1 - F_1)C_{DES\,(k-\varepsilon)} + F_1 C_{DES\,(k-\omega)}$$

<u>A note on numerical schemes</u>

M. Strelets. Detached eddy simulation of massively separated flows. In *39th Aerospace Sciences Meeting and Exhibit*, Reno, NV, USA, 2001.

RANS → stability, upwind schemes.

LES → accuracy, linear schemes.

DES → hybrid? Which do you use?

$$F = F_{upwind}(\sigma) + F_{linear}(1 - \sigma)$$

Disadvantage of upwind: diffusive (ie less accurate)

Fine mesh with upwind scheme, can have similar accuracy to coarse mesh with linear scheme

Fluid Mechanics 101 (Upwind schemes vs linear vs central difference)

https://www.youtube.com/watch?v=JVE0fNkc540

$k - \omega\ SST\ DES$ and $SA - DES$ are especially well calibrated for aircraft.

Tweaks to SA-DES (DDES)

P.R. Spalart, S. Deck, M.L. Shur, K.D. Squires, M.K. Strelets, and A. Travin. A New Version of Detached-eddy Simulation, Resistant to Ambiguous Grid Densities. *Theoretical and Computational Fluid Dynamics*, 20(3):181–195, 2006.

https://www.researchgate.net/publication/225137433_A_New_Version_of_Detached-eddy_Simulation_Resistant_to_Ambiguous_Grid_Densities

(pdf available on researchgate)

Problem: LES Switches on too soon in some grids

$$\tilde{d} = \min(d, C_{DES}\Delta)$$

$$\Delta = V^{\frac{1}{3}}$$

$$\Delta = \max(\Delta x, \Delta y, \Delta z)$$

Latter only applies well for cuboid shapes

Normally for RANS to switch on in BL

$$RANS\ switch\ on\ \ (d\ (distance\ from\ wall) < C_{DES}\Delta)$$

For BL, max d is δ

Hence

$$\Delta > \frac{\delta}{C_{DES}}$$

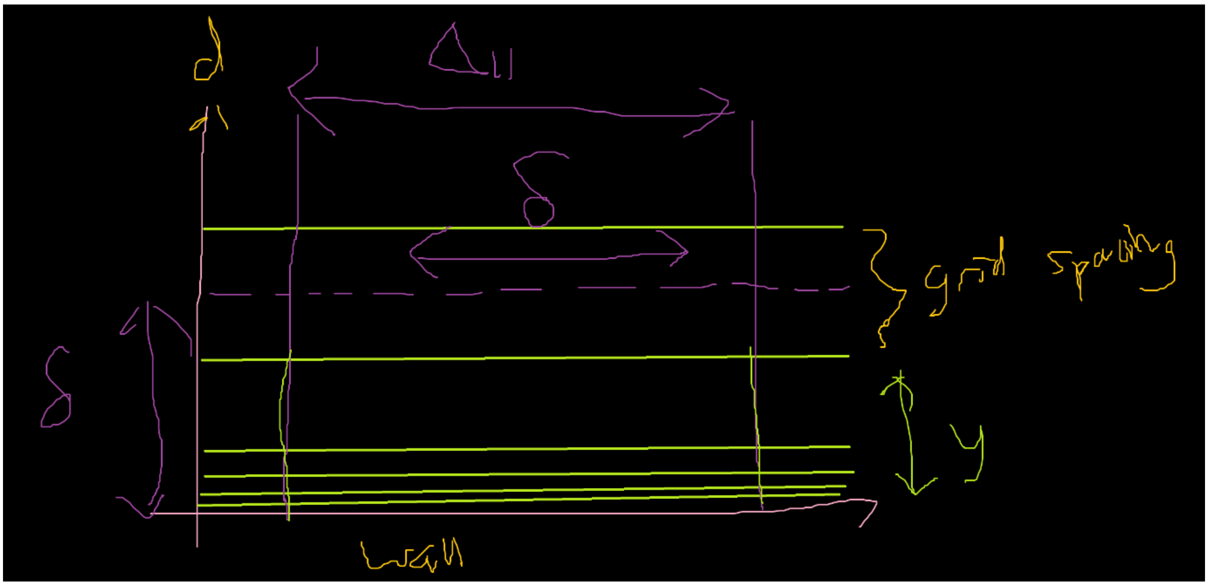
We need to adjust Δ

$$V^{\frac{1}{3}} > \frac{\delta}{C_{DES}}$$

Or

$$\max(\Delta x, \Delta y, \Delta z) > \frac{\delta}{C_{DES}}$$

Forcing RANS to switch on:



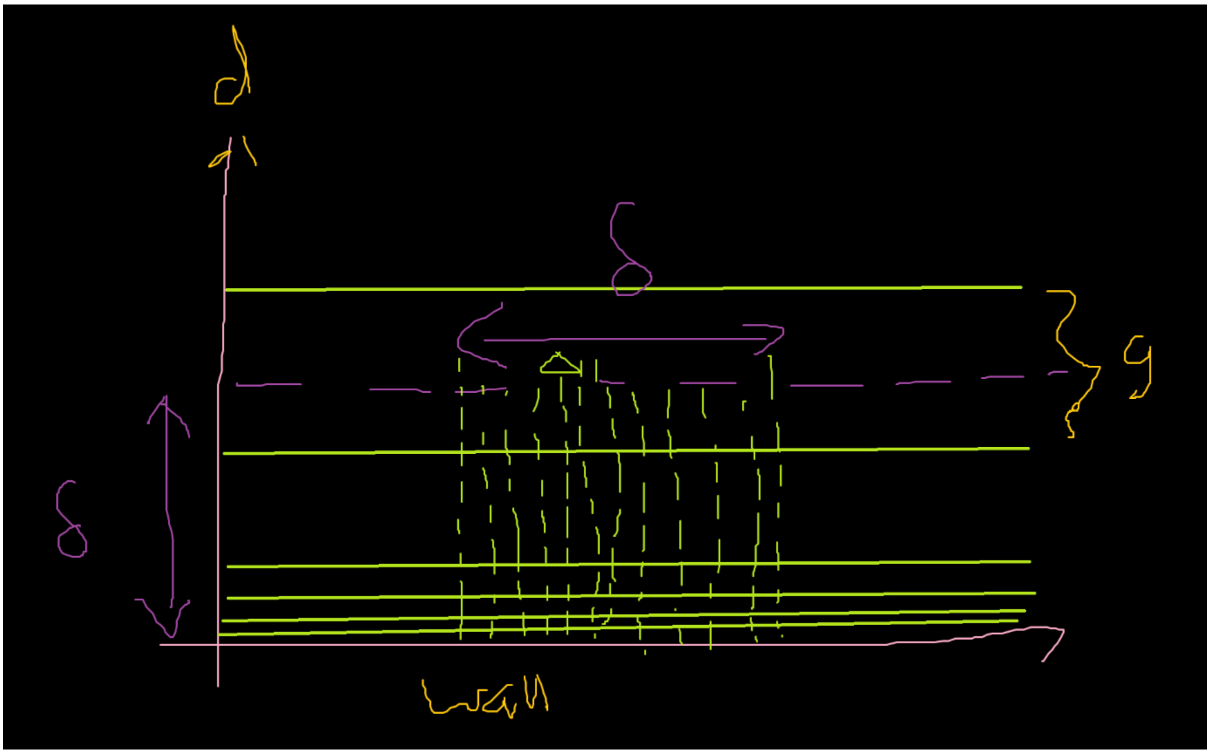
If you want to switch on LES, you need to make sure the converse is true,

So that in BL,

$$\max(\Delta x, \Delta y, \Delta z) \ll \frac{\delta}{C_{DES}}$$

This means typical values of

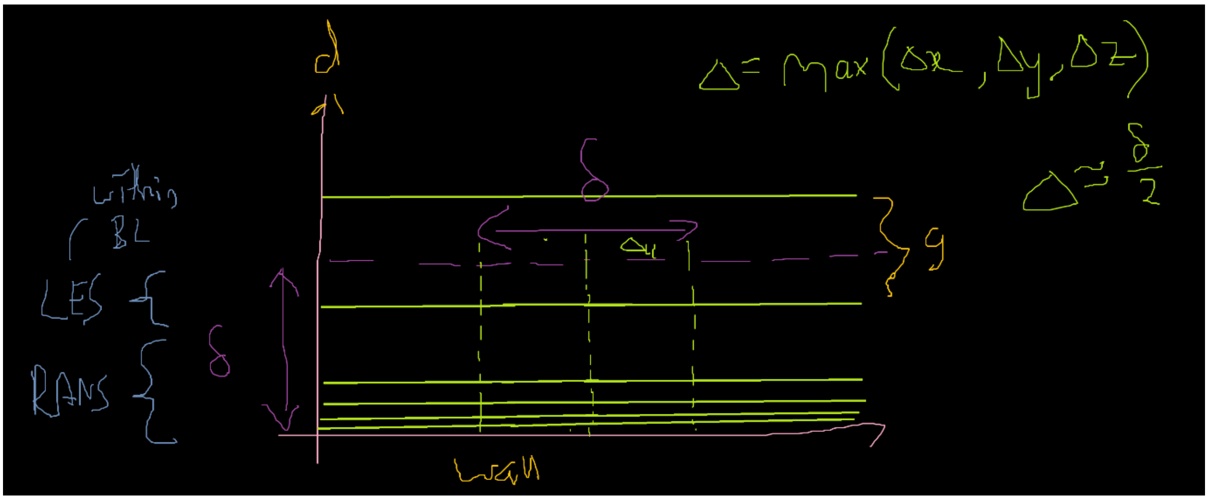
$$\Delta \approx \frac{\delta}{20}$$



Makes it so that you have LES on in BL...

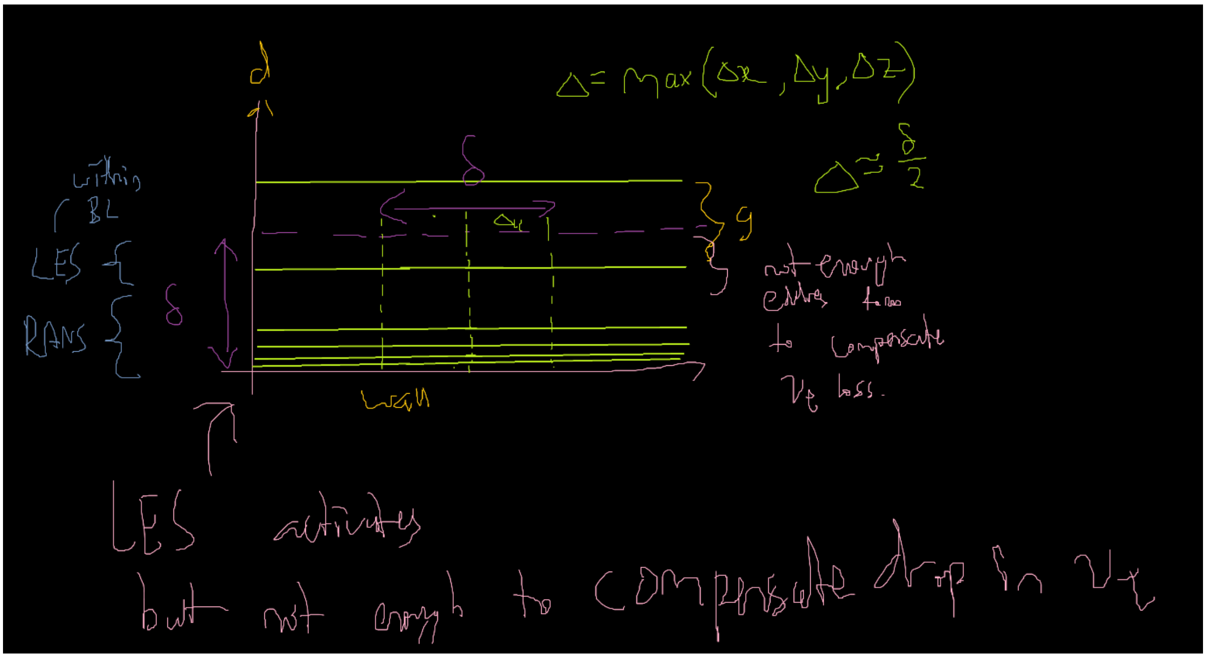
The gray area or ambiguous region

$$\Delta \approx \frac{\delta}{2}$$



So for DES we have to be careful about how we space our grids… otherwise LES kicks in too early…

This early onset of LES causes problems…



Technical Term: Model Stress Depletion (MSD)

DES destroys modelled reynold's stress in the RANS (ie ν_t)→ LES transitions without producing enough eddies in LES region.

- Why is this important?
- Degree of MSD is dependent on how you structure your grid
- Different grids in BL → different results

We need to stop (or delay) onset of LES within the BL.

<u>Solution: Take inspiration from k Omega SST's Blending Functions</u>

Preserve RANS in Boundary Layer → Delay onset of LES, hence the name Delayed…

If you remember,

$k-\omega\ SST$→ blending functions F_1 and F_2 using tanh

Spalart's idea was to take blending function to switch from RANS in BL to LES in Free shear. This blending function is the key change from DES to DDES.

$$f_d \equiv 1 - \tanh([8r_d]^3)$$

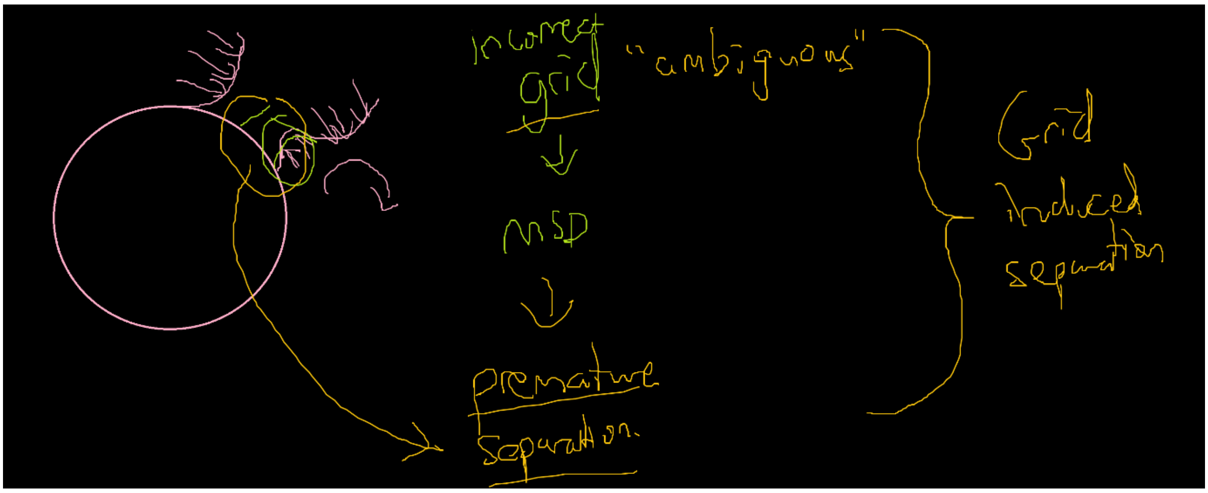
$$r_d \equiv \frac{\nu_t + \nu}{|\nabla U|\kappa^2 d^2}$$

$$|\nabla U| = \sqrt{\nabla \mathrm{U} \cdot \nabla \mathrm{U}}$$

$$\tilde{d} \equiv d - f_d \max(0, d - C_{DES}\Delta)$$

Blending functions get rid of Model stress depletion which is based on grid structure in BL. Different grids in BL should give more or less same results.

➔ The different result is: premature flow separation



Blending functions prevent onset of LES within BL when it's unwanted.

Let's investigate the behaviour of the blending function

$$f_d \equiv 1 - \tanh([8r_d]^3)$$

$$r_d \equiv \frac{\nu_t + \nu}{|\nabla U|\kappa^2 d^2}$$

$$|\nabla U| = \sqrt{\nabla \mathrm{U} \cdot \nabla \mathrm{U}}$$

$$\tilde{d} \equiv d - f_d \max(0, d - C_{DES}\Delta)$$

At low r_D, $f_d \to 1$ → classic DES behaviour

$$\tilde{d} \equiv d - f_d \max(0, d - C_{DES}\Delta)$$

$$f_d \to 1$$

$$\tilde{d} \equiv d - \max(0, d - C_{DES}\Delta)$$

If $d - C_{DES}\Delta < 0$

$$\tilde{d} = d$$

(RANS mode)

Otherwise if $d - C_{DES}\Delta > 0$

$$\tilde{d} \equiv d - (\, d - C_{DES}\Delta) = C_{DES}\Delta$$

(LES mode)

If $f_d \to 0$, we get RANS

$$\tilde{d} = d\ (RANS\ mode)$$

What triggers $f_d \to 0$?

$$f_d \equiv 1 - \tanh([8r_d]^3)$$

Large r_d triggers $f_d \to 0$

Large r_d triggers RANS mode.

$$r_d \equiv \frac{\nu_t + \nu}{|\nabla U|\kappa^2 d^2}$$

Large r_d means

- ➔ Large turbulent viscosity in comparison to $|\nabla U|$
- ➔ Large turbulent viscosity means large r_d

Effect of large turbulent viscosity:

- ➔ Large turbulent viscosity triggers RANS mode, or rather prevents LES from switching on.

Large D makes r_d small

Large $|\nabla U|$ makes r_d small → large velocity gradient → naturally generates eddies or flow instabilities. This makes it ok for DES or LES to kick in.

Tweaks to SA-DES (Low Re Correction)

P.R. Spalart, S. Deck, M.L. Shur, K.D. Squires, M.K. Strelets, and A. Travin. A New Version of Detached-eddy Simulation, Resistant to Ambiguous Grid Densities. *Theoretical and Computational Fluid Dynamics*, 20(3):181–195, 2006.

https://www.researchgate.net/publication/225137433_A_New_Version_of_Detached-eddy_Simulation_Resistant_to_Ambiguous_Grid_Densities

(pdf available on researchgate)

Refining grid in regions of free shear switches LES on

Recall:

$$l_{des} = \min(l_{RANS}, C_{DES}\Delta)$$

For SA model:

$$\tilde{d} = \min(d, C_{DES}\Delta)$$

Refine grid: $\Delta \to too\ small$, eddy viscosity (SA variable $\tilde{\nu}$) destruction term → big → "DNS mode"

No turbulent viscosity.

We destroy eddy viscosity when $\tilde{d}$ is small as if it's near a wall.

$$\tilde{d} \to 0, \nu_t \to 0$$

$C_{DES}\Delta \to 0$ means behavior as if it's near a wall.

$$\tilde{d} = \min(d, C_{DES}\Delta)$$

How can we prevent $C_{DES}\Delta$ from approaching 0 in free shear regions where there is excessive refinement ie $\Delta \to 0\ (small\ and\ finite)$?

Use a booster function Ψ

$$\tilde{d} = \min(d, \Psi C_{DES}\Delta)$$

Ψ is the booster function where you increase the value of $\Psi C_{DES}\Delta$ such that it is not approaching 0 and destroys eddy viscosity as if it were near a wall.

$$\Psi \geq 1$$

$\Psi > 1$ should activate in <u>low Re Regions</u> (not enough flow disturbances generated to compensate for loss in ν_t) and <u>far from wall (ie free shear)</u>

$$\mathrm{l_{DES}} = \Psi C_{DES}\Delta$$

This is known as the low Re correction.

How should we design Ψ?

Basic principle

- ➔ Calibrate Ψ such that subgrid stress model (ν_t) should behave like smagorinsky model in the free shear zone even in low Re
  - $\nu_t = \left(C_{smagorinsky}\Delta\right)^2|S|$

Lots of math later… and calibration

$$\Psi^2 = \frac{\frac{f_w}{f_w^*} - \frac{c_{b1}}{c_{w1}\kappa^2 f_w^*}[f_{t1} + (1 - f_{t2})f_{v2}]}{f_{v1}(1 - f_{t2})}$$

$$f_w^* = 0.424$$

For rest of the constants, please refer to spalart allmaras RANS model

https://www.cfd-online.com/Wiki/Spalart-Allmaras_model

now we make approximations to tweak Ψ numerically

$$\Psi^2 = \frac{\frac{f_w}{f_w^*} - \frac{c_{b1}}{c_{w1}\kappa^2 f_w^*}[f_{t1} + (1 - f_{t2})f_{v2}]}{f_{v1}(1 - f_{t2})}$$

Ψ kicks in when there should be some meaningful amount of ν_t ($\nu_t > \nu/10$)

But in this range, we find that $\frac{f_w}{f_w^*} \approx 1 \pm 2\%$

$$\Psi^2 = \frac{1 - \frac{c_{b1}}{c_{w1}\kappa^2 f_w^*}[f_{t1} + (1 - f_{t2})f_{v2}]}{f_{v1}(1 - f_{t2})}$$

We want to make sure Ψ^2 doesn't blow up because denominator reaches 0

$\nu_t \to 0\ means\ \chi = \frac{\nu_t}{\nu} \to 0$ means $f_{t2} \to 1$

$$f_{t2} = C_{t3} \exp(-C_{t4}\chi^2)$$

$$\Psi^2 = \frac{1 - \frac{c_{b1}}{c_{w1}\kappa^2 f_w^*}[f_{t1} + (1 - f_{t2})f_{v2}]}{f_{v1} \max[10^{-10}, (1 - f_{t2})]}$$

Another way to cap Ψ

$$\Psi^2 = \min\left\{10^2, \frac{1 - \frac{c_{b1}}{c_{w1}\kappa^2 f_w^*}[f_{t1} + (1 - f_{t2})f_{v2}]}{f_{v1} \max[10^{-10}, (1 - f_{t2})]}\right\}$$

This limits:

$$1 \le \Psi \le 10$$

You can also apply this principle to $k - \omega\ SST\ DES$

You also can apply this low Re correction to SA-DDES

- We have an extra variable f_d to correct the modelled stress depletion phenomenon
- This can also appear in Ψ
- $\Psi^2 = \min\left\{10^2, \frac{1 - \frac{c_{b1}}{c_{w1}\kappa^2 f_w^*}[f_{t1} + (1 - f_{t2})f_{v2}]}{f_{v1} \max[10^{-10}, (1 - f_{t2})]}\right\}$
    - These are functions of χ
    - Everywhere you see χ, replace it with $\max(\chi, 20f_d)$

Problem:

Refining too much in low Re causes grid induced modelled stress depletion, thus lowering eddy viscosity too much.

- Usually fine grid regions are near wall
- However, fine grid can also cause the DES model to treat the fine grid as if it were a wall…

SA-IDDES (A combined model of Wall Modelled LES and DDES)

https://scholar.google.com.sg/scholar?q=a+hybrid+rans+les+approach+with+delayed+des&hl=en&as_sdt=0&as_vis=1&oi=scholart

Shur, M. L., Spalart, P. R., Strelets, M. K., & Travin, A. K. (2008). A hybrid RANS-LES approach with delayed-DES and wall-modelled LES capabilities. *International Journal of Heat and Fluid Flow*, *29*(6), 1638-1649.

https://www.researchgate.net/publication/223933800_A_hybrid_RANS-LES_approach_with_delayed-DES_and_wall-modelled_LES_capabilities

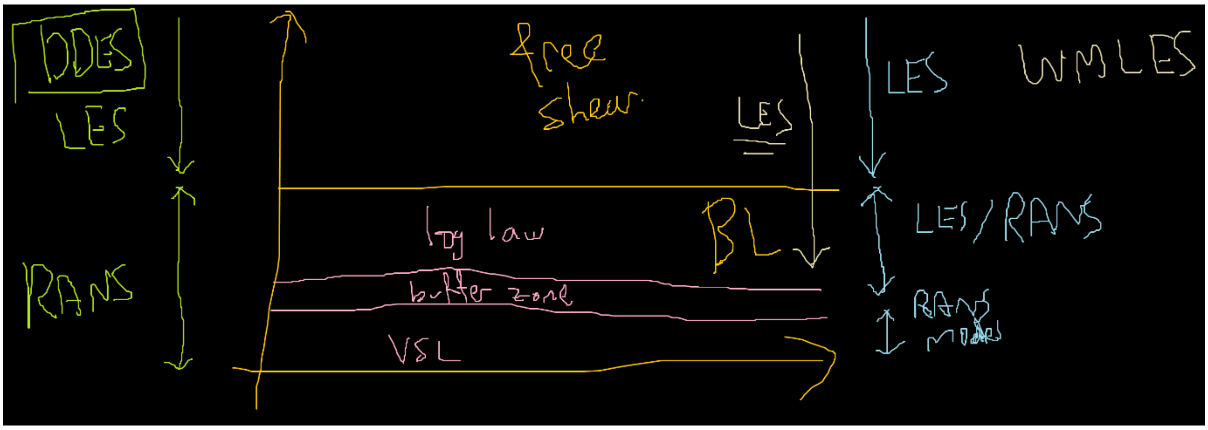
(pdf available on researchgate)

Source Code in OpenFoam

https://develop.openfoam.com/Development/openfoam/tree/master/src/TurbulenceModels/turbulenceModels/DES/SpalartAllmarasIDDES

Tutorial case in OpenFoam

https://www.openfoam.com/documentation/guides/latest/doc/verification-validation-turbulent-surface-mounted-cube.html



Unlike WALE, WMLES uses a RANS model for viscous sublayer within the BL.

Can DES be used in LES mode for BL?

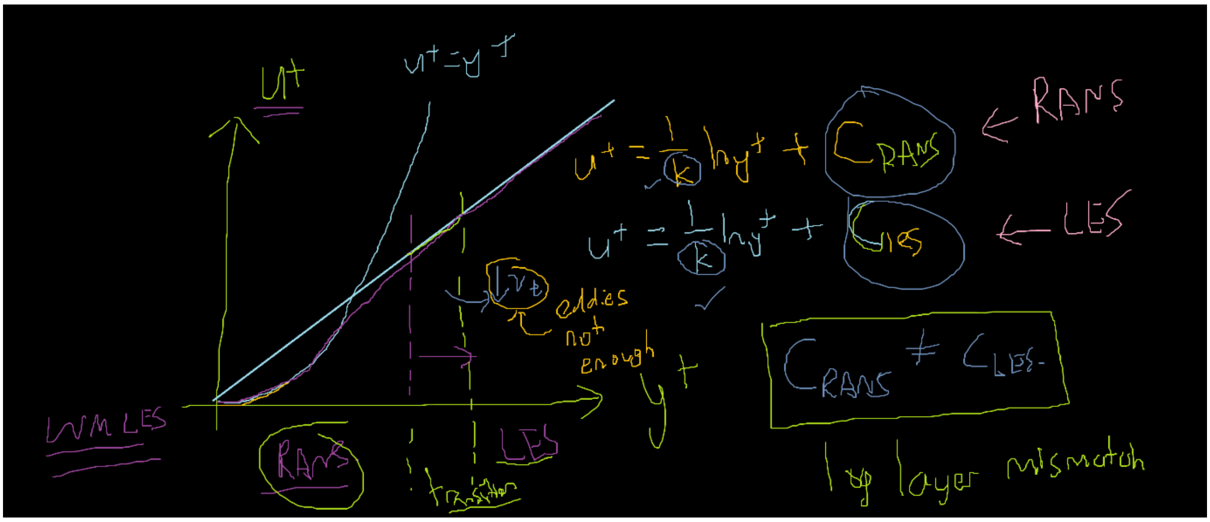
Ie RANS for viscous sublayer but LES in other parts of BL (this is called Wall Modelled LES, WMLES)

- Note this is different from WALE model discussed, WALE uses an empirical function in viscous sublayer but **not RANS**

Sort of… but there are a few problems…

<u>Problem: log layer mismatch</u>

Recall log law of the wall: $u^+ = \frac{1}{\kappa}\ln y^+ + C$



Result: 15-20% error in skin friction (too much error for aerospace application) → from paper (Shur, Spalart, Strelets, & Travin, 2008)

Solution: adopt correct blending function for WMLES… (again more empirical complicated looking functions)

<u>So what does WMLES look like for SA model?</u>

Similar to DES, we just change the length scale and blend it with complicated empirical functions…

$$l_{DDES} = l_{RANS} - f_d \max\{0, (l_{RANS} - l_{LES})\}$$

$$l_{WMLES} = f_B(1 + f_e)l_{RANS} + (1 - f_B)l_{LES}$$

F_B is quite straightforward

- Empirical blending function to switch between RANS/LES length scale
- $f_B = 1$ is RANS mode, $f_B = 0$ is LES mode

$$f_B = \min\{2\exp(-9\alpha^2), 1.0\}$$

The 1.0 is there to cap f_B at 1

$$\alpha = 0.25 - \frac{d_w}{h_{\max}}$$

$$h_{\max} = \max(h_x, h_y, h_z)$$

What's f_e for? It's to help with modelled stress depletion. (this one isn't grid dependent, it's just that reduction in RANS eddy viscosity is not compensated with enough resolved turbulence)

So we want f_e to be zero at all times unless it is here to slow down the eddy viscosity reduction from RANS model. (similar to DES)

$$f_e = \max\{(f_{e1} - 1), 0\}\,\Psi f_{e2}$$

We note that Ψ makes a reappearance here.

$$\Psi^2 = \min\left\{10^2, \frac{1 - \frac{c_{b1}}{c_{w1}\kappa^2 f_w^*}[f_{t1} + (1 - f_{t2})f_{v2}]}{f_{v1}\max[10^{-10}, (1 - f_{t2})]}\right\}$$

What we are doing here has **nothing** to do with low Re correction in DDES/DES conceptually speaking. It's just a convenient function to use and exhibits correct numerical behavior.

$$f_{e1} = \begin{cases} 2\exp(-11.09\alpha^2) & if\ \alpha \geq 0 \\ 2\exp(-9.0\alpha^2) & if\ \alpha < 0 \end{cases}$$

f_{e1} preserves RANS eddy viscosity <u>based on length scale</u>

$$f_{e2} = 1.0 - \max\{f_t, f_l\}$$

$$f_t = \tanh[(c_t^2 r_{dt})^3]$$

$$f_l = \tanh\left[\left(c_l^2 r_{dl}\right)^{10}\right]$$

What are typical values,

Refer to OpenFOAM SA-IDDES source code

https://develop.openfoam.com/Development/openfoam/blob/master/src/TurbulenceModels/turbulenceModels/DES/SpalartAllmarasIDDES/SpalartAllmarasIDDES.C

line 66 to 78 provides the formula…

line 208 to 225 provides the values…

$$c_l = 3.35, c_t = 1.63$$

for r_{dt} and r_{dl}

recall this from SA-RANS

$$r_d \equiv \frac{\nu_t + \nu}{|\nabla U| \kappa^2 d_w^2}$$

To make it more numerically stable, we make sure denominator doesn't go to zero

$$r_d \equiv \frac{\nu_t + \nu}{\max\{10^{-10}, |\nabla U|\} \, \kappa^2 d_w^2}$$

$$r_{dl} = \frac{\nu}{\max\{10^{-10}, |\nabla U|\} \, \kappa^2 d_w^2}$$

$$r_{dt} = \frac{\nu_t}{\max\{10^{-10}, |\nabla U|\} \, \kappa^2 d_w^2}$$

f_{e2} controls drop in eddy viscosity based on ν_t

Blending DDES with WMLES

This is key change for IDDES

Blending 2 length scales:

$$l_{WMLES} = f_B(1 + f_e) l_{RANS} + (1 - f_B) l_{LES}$$

$$l_{DDES} = l_{RANS} - f_d \max\{0, (l_{RANS} - l_{LES})\}$$

Modify original DDES length scale:

$$\tilde{l}_{DDES} = \tilde{f}_d l_{RANS} + \left(1 - \tilde{f}_d\right) l_{LES}$$

$$\tilde{f}_d = \max\{(1 - f_{dt}), f_B\}$$

$$f_{dt} = 1 - \tanh[(8 r_{dt})^3]$$

Combine with WMLES properties:

$$\tilde{l}_{IDDES} = \tilde{f}_d (1 + f_e) l_{RANS} + \left(1 - \tilde{f}_d\right) l_{LES}$$

Viola…

Behaviour discussion:

Large resolved turbulence in inflow → $r_{dt} \ll 1$

$$r_{dt} = \frac{\nu_t}{\max\{10^{-10}, |\nabla U|\}\,\kappa^2 d_w^2}$$

Low modelled eddy viscosity compared to strain rates (velocity gradient).

Strain rates are huge in resolved turbulent flow…

Reminder: resolved means that you solve for the velocity field in CFD code.

With $r_{dt} \ll 1$

$$f_{dt} = 1 - \tanh[(8r_{dt})^3] \to 1$$

$$1 - f_{dt} \to 0$$

This triggers

$$\tilde{f_d} = \max\{(1 - f_{dt}), f_B\} \to \max\{0, f_B\}$$

$$\tilde{f_d} = f_B \to WMLES$$

$$\tilde{l}_{WMLES} = f_B(1 + f_e)l_{RANS} + (1 - f_B)l_{LES}$$

Otherwise (not enough resolved turbulence in flow), $f_e \to 0$ triggers DDES.

$$\tilde{l}_{DDES} = \tilde{f_d} l_{RANS} + (1 - \tilde{f_d})l_{LES}$$

K omega SST DDES and IDDES

Gritskevich, M. S., Garbaruk, A. V., Schütze, J., & Menter, F. R. (2012). Development of DDES and IDDES formulations for the k-ω shear stress transport model. *Flow, turbulence and combustion*, *88*(3), 431-449.

K omega SST DDES/IDDES ResearchGate pdf

https://www.researchgate.net/publication/257518951_Development_of_DDES_and_IDDES_Formulations_for_the_k-o_Shear_Stress_Transport_Model

K omega SST DDES/IDDES Springer Link

https://link.springer.com/article/10.1007/s10494-011-9378-4

K omega SST DDES/IDDES OpenFoam Source Code

https://develop.openfoam.com/Development/openfoam/tree/master/src/TurbulenceModels/turbulenceModels/DES/kOmegaSSTDDES

https://develop.openfoam.com/Development/openfoam/tree/master/src/TurbulenceModels/turbulenceModels/DES/kOmegaSSTIDDES

Recall k Omega SST DES

$$\frac{\partial k}{\partial t} + u_j \frac{\partial k}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right)\frac{\partial k}{\partial x_j}\right) + \frac{P_{k(dynamic)}}{\rho} - \frac{k^{1.5}}{\tilde{l}}$$

$$\tilde{l} = \min\left(\frac{k^{0.5}}{\omega}, C_{DES}\Delta\right)$$

$$\frac{\partial k}{\partial t} + u_j \frac{\partial k}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\left(\nu + \sigma_{K(k\omega SST)}\nu_t\right)\frac{\partial k}{\partial x_j}\right) + \frac{P_{k(dynamic)}}{\rho} - \frac{k^{1.5}}{\min\left(\frac{k^{0.5}}{\omega}, C_{DES}\Delta\right)}$$

$$\frac{\partial \omega^*}{\partial t} + u_i \frac{\partial \omega^*}{\partial x_i} = \frac{\partial}{\partial x_j}\left(\nu + \frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial \omega^*}{\partial x_j} + (1 - F_1)2\left(\frac{1}{\omega^*}\right)\frac{\partial \omega^*}{\partial x_j}\frac{\partial k}{\partial x_j} + \alpha S^2 - \beta {\omega^*}^2$$

The omega equation remains untouched.

For DDES

$$l_{DDES} = l_{RANS} - f_d \max(0, l_{RANS} - l_{LES})$$

$$l_{LES} = C_{DES} h_{max} = C_{DES}\Delta$$

$$l_{RANS} = \frac{k^{0.5}}{\omega}$$

$$\omega \equiv \frac{\varepsilon}{k}$$

$$l_{RANS} = \frac{k^{0.5}}{C_\mu \omega^*}$$

$$\omega = C_\mu \omega^*$$

$$l_{DDES} = \frac{k^{0.5}}{C_\mu \omega^*} - f_d \max\left(0, \frac{k^{0.5}}{C_\mu \omega^*} - C_{DES}\Delta\right)$$

$$C_{DES} = C_{DES1}F_1 + C_{DES2}(1 - F_1)$$

DDES part:

$$f_d = 1 - \tanh[(C_{d1} r_d)^{C_{d2}}]$$

In SA DDES model

$$f_d = 1 - \tanh[(8 r_d)^3]$$

But in this paper, to calibrate f_d to prevent onset of LES in BL

$$f_d = 1 - \tanh[(20 r_d)^3]$$

$$C_{d1} = 20$$

In SA DDES model

$$r_d \equiv \frac{\nu_t + \nu}{\max\{10^{-10}, |\nabla U|\}\,\kappa^2 d_w^2}$$

Over here:

$$r_d \equiv \frac{\nu_t + \nu}{\sqrt{0.5\,(S^2 + \Omega^2)}\kappa^2 d_w^2}$$

$$C_\mu = 0.09, \kappa = 0.41$$

$$C_{DES1} = 0.78, C_{DES2} = 0.61, C_{d1} = 20, C_{d2} = 3$$

For IDDES

$$l_{IDDES} = \widetilde{f_d}(1 + f_e)l_{RANS} + \left(1 - \widetilde{f_d}\right)l_{LES}$$

$$l_{LES} = C_{DES}\Delta$$

$$\Delta = \min\{C_w \max[d_w, h_{\max}\ ], h_{\max}\ \}$$

$$l_{RANS} = \frac{k^{0.5}}{C_\mu \omega^*}$$

$$C_{DES} = C_{DES1}F_1 + C_{DES2}(1 - F_1)$$

$$\widetilde{f_d} = \max\{(1 - f_{dt}), f_b\}$$

$$f_{dt} = 1 - \tanh[(C_{dt1}r_{dt})^{C_{dt2}}]$$

$$f_b = \min\{2\exp(-9\alpha^2)\,, 1.0\}$$

$$\alpha = 0.25 - \frac{d_w}{h_{\max}}$$

Now for f_e

$$f_e = \max\{(f_{e1} - 1), 0\}\,f_{e2}$$

$$f_{e1} = \begin{cases} 2\exp(-11.09\alpha^2) & if\ \alpha \geq 0 \\ 2\exp(-9.0\alpha^2) & if\ \alpha < 0 \end{cases}$$

$$f_{e2} = 1.0 - \max\{f_t, f_l\}$$

$$f_t = \tanh[(c_t^2 r_{dt})^3]$$

$$f_l = \tanh\left[\left(c_l^2 r_{dl}\right)^{10}\right]$$

$$r_d \equiv \frac{\nu_t + \nu}{\sqrt{0.5\,(S^2 + \Omega^2)}\kappa^2 d_w^2}$$

$$r_{dl} = \frac{\nu}{\sqrt{0.5\,(S^2 + \Omega^2)}\kappa^2 d_w^2}$$

$$r_{dt} = \frac{\nu_t}{\sqrt{0.5\,(S^2 + \Omega^2)} d_w^2}$$

This sums up the IDDES model for k omega SST.

C_w	C_{dt1}	C_{dt2}	C_l	C_t
0.15	20	3	5.0	1.87

Comments from the Paper

For DDES, calibrating C_{dt1} from 8 to 20 is VERY important

For IDDES C_{dt1} doesn't change results much from 8 to 20, but 25 is too high, so C_{dt1} was left at 20, but for IDDES changing it to 8 changes the result marginally.

Simplified version of IDDES for k Omega SST

It can also be simplified by making $f_e = 0$

So in simplified k omega SST IDDES

$$l_{IDDES} = \widetilde{f_d} l_{RANS} + (1 - \widetilde{f_d}) l_{LES}$$

- Omitting f_e causes stronger log layer mismatch in developed channel flow, but effect is marginal according to authors (download the paper and refer to figure 6 for reference)
- They also set $C_{dt} = 20$ for IDDES model and change some other constants:

So technically according to this paper you can omit f_e, this is called the simplified IDDES model.

Implementation in OpenFOAM

See constants for each one…

and also

In OpenFoam, comments for Simplified Model are found in lines 147 to 154

It's not actually implemented!

Transition flow (laminar – turbulence)

- K-Omega SST LM
- https://www.openfoam.com/documentation/guides/latest/doc/guide-turbulence-ras-k-omega-sst-lm.html
- https://www.openfoam.com/documentation/guides/latest/doc/verification-validation-turbulent-t3a.html
- https://develop.openfoam.com/Development/openfoam/tree/master/tutorials/incompressible/simpleFoam/T3A

Heat Transfer

Some Reference Material

Bejan, A. (2013). Convection heat transfer. John wiley & sons.

OpenFOAM EEqn.H (from buoyantPimpleFoam folder)

https://develop.openfoam.com/Development/openfoam/blob/master/applications/solvers/heatTransfer/buoyantPimpleFoam/EEqn.H

OpenFOAM source code

https://develop.openfoam.com/Development/openfoam

Let's start with equations of heat transfer…(Bejan, 2013)

$$\rho\frac{De}{Dt} + e\left(\frac{D\rho}{Dt} + \rho\nabla\cdot\vec{v}\right) = -\nabla\cdot\overrightarrow{q''} + q''' - P\nabla\cdot\vec{v} + \mu\Phi$$

$$\mu\Phi = \text{dissipation term}$$

$$P\nabla\cdot\vec{v} = PV\ work\ or\ flow\ work\ (or\ part\ of\ it)$$

$$P\nabla\cdot\vec{v} \rightarrow work\ done\ term$$

Mass conservation:

$$\left(\frac{D\rho}{Dt} + \rho\nabla\cdot\vec{v}\right) = 0$$

$$\rho\frac{De}{Dt} = -\nabla\cdot\overrightarrow{q''} + q''' - P\nabla\cdot\vec{v} + \mu\Phi$$

In terms of enthalpy

$$\left(\frac{\partial H}{\partial T}\right)_p = c_p$$

$$h \equiv U + PV$$

But u is used as velocity, so we write it as e here to avoid confusion

$$h \equiv e + PV$$

$$V = \frac{1}{\rho}\ (specific\ volume)$$

$$h \equiv e + \frac{P}{\rho}$$

so we differentiate…

$$\frac{Dh}{Dt} = \frac{D}{Dt}\left(e + \frac{P}{\rho}\right)$$

$$\frac{Dh}{Dt} = \frac{De}{Dt} + \frac{D}{Dt}\left(\frac{P}{\rho}\right) = \frac{De}{Dt} + \left(\frac{\rho \frac{D}{Dt}P - P\frac{D}{Dt}\rho}{\rho^2}\right)$$

$$\frac{Dh}{Dt} = \frac{De}{Dt} + \frac{1}{\rho}\frac{D}{Dt}P - \frac{P}{\rho^2}\frac{D}{Dt}\rho$$

$$\frac{De}{Dt} = \frac{Dh}{Dt} - \left(\frac{1}{\rho}\frac{D}{Dt}P - \frac{P}{\rho^2}\frac{D}{Dt}\rho\right)$$

Substitute:

$$\rho\left\{\frac{Dh}{Dt} - \left(\frac{1}{\rho}\frac{D}{Dt}P - \frac{P}{\rho^2}\frac{D}{Dt}\rho\right)\right\} = -\nabla \cdot \overrightarrow{q''} + q''' - P\nabla \cdot \vec{v} + \mu\Phi$$

$$\left\{\rho\frac{Dh}{Dt} - \left(\frac{D}{Dt}P - \frac{P}{\rho}\frac{D}{Dt}\rho\right)\right\} = -\nabla \cdot \overrightarrow{q''} + q''' - P\nabla \cdot \vec{v} + \mu\Phi$$

$$\rho\frac{Dh}{Dt} = -\nabla \cdot \overrightarrow{q''} + q''' - P\nabla \cdot \vec{v} + \mu\Phi + \left(\frac{D}{Dt}P - \frac{P}{\rho}\frac{D}{Dt}\rho\right)$$

$$\rho\frac{Dh}{Dt} = -\nabla \cdot \overrightarrow{q''} + q''' - \frac{P}{\rho}\rho\nabla \cdot \vec{v} + \mu\Phi + \frac{DP}{Dt} + \left(-\frac{P}{\rho}\frac{D}{Dt}\rho\right)$$

$$\rho\frac{Dh}{Dt} = -\nabla \cdot \overrightarrow{q''} + q''' + \mu\Phi + \frac{DP}{Dt} - \frac{P}{\rho}\left(\frac{D\rho}{Dt} + \rho\nabla \cdot \vec{v}\right)$$

$$\rho\frac{Dh}{Dt} = -\nabla \cdot \overrightarrow{q''} + q''' + \mu\Phi + \frac{DP}{Dt}$$

$$de = Tds - PdV$$

$$dh = Tds + VdP$$

From thermodynamic calculus

$$\beta = -\frac{1}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_P$$

$$\left(\frac{\partial s}{\partial T}\right)_p = \frac{c_p}{T}$$

$$dh = c_p dT + \frac{1}{\rho}(1 - \beta T)dP$$

$$\rho\, dh = \rho c_p dT + (1 - \beta T)dP$$

And we replace dh with Dh/Dt

$$\rho \frac{Dh}{Dt} = \rho c_p \frac{DT}{Dt} + (1-\beta T)\frac{DP}{Dt}$$

$$\rho c_p \frac{DT}{Dt} + (1-\beta T)\frac{DP}{Dt} = -\nabla \cdot \overrightarrow{q''} + q''' + \mu\Phi + \frac{DP}{Dt}$$

$$\rho c_p \frac{DT}{Dt} = -\nabla \cdot \overrightarrow{q''} + q''' + \mu\Phi + \beta T\frac{DP}{Dt}$$

If incompressible, $\beta = 0$

$$\rho c_p \frac{DT}{Dt} = -\nabla \cdot \overrightarrow{q''} + q''' + \mu\Phi$$

For ideal gas $\beta = \frac{1}{T}$

$$\rho c_p \frac{DT}{Dt} = -\nabla \cdot \overrightarrow{q''} + q''' + \mu\Phi + \frac{DP}{Dt}$$

Also we can use fourier's law here

$$\overrightarrow{q''} = -k\nabla T$$

Final equation for ideal gas:

$$\rho c_p \frac{DT}{Dt} = \nabla \cdot (k\nabla T) + q''' + \mu\Phi + \frac{DP}{Dt}$$

Final equation (general):

$$\rho c_p \frac{DT}{Dt} = \nabla \cdot (k\nabla T) + q''' + \mu\Phi + \beta T\frac{DP}{Dt}$$

For incompressible fluid:

$$\rho c_p \frac{DT}{Dt} = \nabla \cdot (k\nabla T) + q''' + \mu\Phi$$

The dissipation term is usually negligible

$$\Phi = 2\left[\sum\left(\frac{\partial u_i}{\partial x_i}\right)^2\right] + \left[\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)^2\right] - \frac{2}{3}\left[\sum\left(\frac{\partial u_i}{\partial x_i}\right)\right]^2$$

Except in flows, eg. Lubrication and flow of viscous fluid, eg. Crude oil (Bejan, 2013)

Averaging/Filtering

So then we do averaging again for RANS, and filtering for LES type simulations…

We do incompressible assumption for simplicity's sake,

We can also assume k and c_p don't change much.

Recall: $\alpha = \frac{k}{\rho c_P}$

$$\rho c_p \frac{DT}{Dt} = \nabla \cdot (k \nabla T) + q''' + \mu \Phi$$

$$\frac{DT}{Dt} = \nabla \cdot (\alpha\ \nabla T) + \frac{q'''}{\rho c_p} + \frac{\mu \Phi}{\rho c_p}$$

Again for simplicity, I will ignore $\frac{q'''}{\rho c_p} + \frac{\mu \Phi}{\rho c_p}$

$$\frac{DT}{Dt} = \nabla \cdot (\alpha\ \nabla T)$$

$$\frac{DT}{Dt} = \alpha \nabla^2 T$$

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} = \alpha \nabla^2 T$$

We can time average for steady state solution (RANS)

$$\overline{\frac{\partial T}{\partial t}} = 0\ (for\ RANS)$$

$$\overline{\frac{\partial T}{\partial t}} = \frac{\partial \bar{T}}{\partial t}\ (for\ LES, filter\ process)$$

$$\overline{\left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z}\right)} = \overline{\alpha \nabla^2 T}$$

$$\overline{u \frac{\partial T}{\partial x}} + \overline{v \frac{\partial T}{\partial y}} + \overline{w \frac{\partial T}{\partial z}} = \alpha \nabla^2 \bar{T}$$

$$\overline{u \frac{\partial T}{\partial x}} = \frac{1}{\Delta t} \int_0^{\Delta t} u \frac{\partial T}{\partial x}\ dt\, , usually\ \Delta t \to \infty$$

$$\overline{u \frac{\partial T}{\partial x}} = \frac{1}{\Delta t} \int_0^{\Delta t} u \frac{\partial T}{\partial x}\ dt$$

$$u = \bar{u} + u'$$

$$T = \bar{T} + T'$$

$$\overline{u \frac{\partial T}{\partial x}} = \frac{1}{\Delta t} \int_0^{\Delta t} (\bar{u} + u') \frac{\partial (\bar{T} + T')}{\partial x}\ dt$$

$$\overline{u \frac{\partial T}{\partial x}} = \frac{1}{\Delta t} \int_0^{\Delta t} \bar{u} \frac{\partial \bar{T}}{\partial x} + \bar{u} \frac{\partial T'}{\partial x} + u' \frac{\partial \bar{T}}{\partial x} + u' \frac{\partial T'}{\partial x}\ dt$$

$$\left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z}\right)$$

$$\frac{\partial uT}{\partial x} + \frac{\partial vT}{\partial y} + \frac{\partial wT}{\partial z} = \left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z}\right) + T\ \left(\frac{\partial u_i}{\partial x_i}\right)$$

$\left(\frac{\partial \mathrm{u_i}}{\partial x_i}\right) = 0$ from continuty

$$\frac{\partial uT}{\partial x} + \frac{\partial vT}{\partial y} + \frac{\partial wT}{\partial z} = \left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}\right)$$

$$\overline{\left(\frac{\partial uT}{\partial x} + \frac{\partial vT}{\partial y} + \frac{\partial wT}{\partial z}\right)}$$

$$\frac{\overline{\partial uT}}{\partial x} = \frac{\partial}{\partial x}\frac{1}{\Delta \mathrm{t}}\int_0^{\Delta t} (uT)dt$$

$$\frac{\overline{\partial uT}}{\partial x} = \frac{\partial}{\partial x}\frac{1}{\Delta \mathrm{t}}\int_0^{\Delta t} (\bar{u} + u')(\bar{T} + T')dt$$

$$\frac{\overline{\partial uT}}{\partial x} = \frac{\partial}{\partial x}\frac{1}{\Delta \mathrm{t}}\int_0^{\Delta t} \bar{u}\bar{T} + \bar{u}T' + u'\bar{T} + u'T'dt$$

$$\frac{\overline{\partial uT}}{\partial x} = \frac{\partial}{\partial x}(\bar{u}\bar{T} + \overline{u'T'})$$

$$\frac{\partial}{\partial x}(\bar{u}\bar{T} + \overline{u'T'}) + \frac{\partial}{\partial y}(\bar{v}\bar{T} + \overline{v'T'}) + \frac{\partial}{\partial z}(\bar{w}\bar{T} + \overline{w'T'}) = \alpha\nabla^2\bar{T}$$

$$\frac{\partial}{\partial x}(\bar{u}\bar{T}) + \frac{\partial}{\partial y}(\bar{v}\bar{T}) + \frac{\partial}{\partial z}(\bar{w}\bar{T}) = \alpha\nabla^2\bar{T} - \frac{\partial}{\partial x}\overline{u'T'} - \frac{\partial}{\partial y}\overline{v'T'} - \frac{\partial}{\partial z}\overline{w'T'}$$

$$\frac{\partial}{\partial x}(\bar{u}\bar{T}) + \frac{\partial}{\partial y}(\bar{v}\bar{T}) + \frac{\partial}{\partial z}(\bar{w}\bar{T}) = \alpha\nabla^2\bar{T} + \bar{T}\left(\frac{\partial \bar{u}_i}{\partial x_i}\right) + \bar{u}\frac{\partial \bar{T}}{\partial x} + \bar{v}\frac{\partial \bar{T}}{\partial y} + \bar{w}\frac{\partial \bar{T}}{\partial z}$$

$$\left(\frac{\partial \bar{u}_i}{\partial x_i}\right) = 0 \;(mass\ conservation\ in\ time\ averaged\ sense)$$

$$\frac{\partial}{\partial x}(\bar{u}\bar{T}) + \frac{\partial}{\partial y}(\bar{v}\bar{T}) + \frac{\partial}{\partial z}(\bar{w}\bar{T}) = \bar{u}\frac{\partial \bar{T}}{\partial x} + \bar{v}\frac{\partial \bar{T}}{\partial y} + \bar{w}\frac{\partial \bar{T}}{\partial z}$$

$$\bar{u}\frac{\partial \bar{T}}{\partial x} + \bar{v}\frac{\partial \bar{T}}{\partial y} + \bar{w}\frac{\partial \bar{T}}{\partial z} = \alpha\nabla^2\bar{T} - \frac{\partial}{\partial x}\overline{u'T'} - \frac{\partial}{\partial y}\overline{v'T'} - \frac{\partial}{\partial z}\overline{w'T'}$$

Just like before, we'll need to define:

$$-\frac{\partial}{\partial x}\overline{u'T'}$$

$$-\frac{\partial}{\partial y}\overline{v'T'}$$

$$-\frac{\partial}{\partial z}\overline{w'T'}$$

LES version of clousure…

We do <u>filtering</u> instead of time averaging.

$$\overline{\left(\frac{\partial uT}{\partial x} + \frac{\partial vT}{\partial y} + \frac{\partial wT}{\partial z}\right)}$$

$$\frac{\overline{\partial uT}}{\partial x} = \frac{\partial}{\partial x}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(uT)\,G(x-r,t-t')dt'\,dr$$

Expand out…

$$uT = \bar{u}\bar{T} + \bar{u}T' + u'\bar{T} + u'T'$$

$$\frac{\overline{\partial uT}}{\partial x} = \frac{\partial}{\partial x}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\bar{u}\bar{T} + \bar{u}T' + u'\bar{T} + u'T')\,G(x-r,t-t')dt'\,dr$$

$$\frac{\overline{\partial uT}}{\partial x} = \frac{\partial}{\partial x}\left(\overline{\bar{u}\bar{T}} + \overline{\bar{u}T'} + \overline{u'\bar{T}} \;+ \overline{(u'T')}\right)$$

$$\frac{\overline{\partial uT}}{\partial x} = \frac{\partial}{\partial x}\overline{\bar{u}\bar{T}} + \frac{\partial}{\partial x}\overline{\bar{u}T'} + \frac{\partial}{\partial x}\overline{u'\bar{T}} \;+ \frac{\partial}{\partial x}\overline{(u'T')}$$

$$\frac{\overline{\partial uT}}{\partial x} = \frac{\partial}{\partial x}\bar{u}\bar{T} + \left(\frac{\partial}{\partial x}\overline{\bar{u}\bar{T}} - \frac{\partial}{\partial x}\bar{u}\bar{T}\right) + \frac{\partial}{\partial x}\overline{\bar{u}T'} + \frac{\partial}{\partial x}\overline{u'\bar{T}} \;+ \frac{\partial}{\partial x}\overline{(u'T')}$$

In x, y and z

$$\frac{\overline{\partial u_\iota T}}{\partial x_\iota} = \frac{\partial}{\partial x_i}\bar{u}_\iota\bar{T} + \left(\frac{\partial}{\partial x_i}\overline{\bar{u}_\iota\bar{T}} - \frac{\partial}{\partial x_i}\bar{u}_\iota\bar{T}\right) + \frac{\partial}{\partial x_i}\overline{\bar{u}_\iota T'} + \frac{\partial}{\partial x_i}\overline{u_\iota'\bar{T}} \;+ \frac{\partial}{\partial x_i}\overline{(u_\iota'T')}$$

$$\frac{\overline{\partial u_\iota T}}{\partial x_\iota} = \frac{\partial}{\partial x_i}\bar{u}_\iota\bar{T} + \frac{\partial}{\partial x_i}\frac{q''_{sgs}}{\rho c_p}$$

$$\frac{\partial\bar{T}}{\partial t} + \overline{\left(\frac{\partial uT}{\partial x} + \frac{\partial vT}{\partial y} + \frac{\partial wT}{\partial z}\right)} = \alpha\nabla^2\bar{T}$$

$$\frac{\partial\bar{T}}{\partial t} + \frac{\partial}{\partial x_i}\bar{u}_\iota\bar{T} + \frac{\partial}{\partial x_i}\frac{q''_{sgs}}{\rho c_p} = \alpha\nabla^2\bar{T}$$

$$\frac{\partial\bar{T}}{\partial t} + \frac{\partial}{\partial x_i}\bar{u}_\iota\bar{T} = \alpha\nabla^2\bar{T} - \frac{\partial}{\partial x_i}\frac{q''_{sgs}}{\rho c_p}$$

$$\frac{\partial\bar{T}}{\partial t} + \bar{u}_\iota\frac{\partial}{\partial x_i}\bar{T} + \bar{T}\frac{\partial}{\partial x_i}\bar{u}_\iota = \alpha\nabla^2\bar{T} - \frac{\partial}{\partial x_i}\frac{q''_{sgs}}{\rho c_p}$$

$$\frac{\partial}{\partial x_i}\bar{u}_\iota = 0$$

LES version:

$$\frac{\partial\bar{T}}{\partial t} + \bar{u}_\iota\frac{\partial}{\partial x_i}\bar{T} = \alpha\nabla^2\bar{T} - \frac{\partial}{\partial x_i}\frac{q''_{sgs}}{(\rho c_p)}$$

RANS version:

$$\bar{u}\frac{\partial\bar{T}}{\partial x} + \bar{v}\frac{\partial\bar{T}}{\partial y} + \bar{w}\frac{\partial\bar{T}}{\partial z} = \alpha\nabla^2\bar{T} - \frac{\partial}{\partial x}\overline{u'T'} - \frac{\partial}{\partial y}\overline{v'T'} - \frac{\partial}{\partial z}\overline{w'T'}$$

URANS version:

$$\frac{\partial \bar{T}}{\partial t} + \bar{u}\frac{\partial \bar{T}}{\partial x} + \bar{v}\frac{\partial \bar{T}}{\partial y} + \bar{w}\frac{\partial \bar{T}}{\partial z} = \alpha\nabla^2\bar{T} - \frac{\partial}{\partial x}\overline{u'T'} - \frac{\partial}{\partial y}\overline{v'T'} - \frac{\partial}{\partial z}\overline{w'T'}$$

Closure is done this way:

$$\frac{\partial \bar{T}}{\partial t} + \bar{u}_i\frac{\partial}{\partial x_i}\bar{T} = \alpha\nabla^2\bar{T} - \frac{\partial}{\partial x_i}\frac{q''_{sgs}}{\rho c_p}$$

→$q''_{sgs} = -k_{turb}\frac{\partial \bar{T}}{\partial x_i}$

$$\frac{\partial \bar{T}}{\partial t} + \bar{u}_i\frac{\partial}{\partial x_i}\bar{T} = \alpha\nabla^2\bar{T} - \frac{\partial}{\partial x_i}\frac{-k_{turb}\frac{\partial \bar{T}}{\partial x_i}}{\rho c_p}$$

$$\frac{\partial \bar{T}}{\partial t} + \bar{u}_i\frac{\partial}{\partial x_i}\bar{T} = \alpha\nabla^2\bar{T} + \frac{\partial}{\partial x_i}\alpha_t\frac{\partial \bar{T}}{\partial x_i}$$

$$\frac{\partial \bar{T}}{\partial t} + \bar{u}_i\frac{\partial}{\partial x_i}\bar{T} = \alpha\nabla^2\bar{T} + \alpha_t\nabla^2\bar{T}$$

$$\frac{\partial \bar{T}}{\partial t} + \bar{u}_i\frac{\partial}{\partial x_i}\bar{T} = (\alpha + \alpha_t)\nabla^2\bar{T}$$

$$(\alpha + \alpha_t) = \alpha_{eff}$$

$$\frac{\partial \bar{T}}{\partial t} + \bar{u}\frac{\partial \bar{T}}{\partial x} + \bar{v}\frac{\partial \bar{T}}{\partial y} + \bar{w}\frac{\partial \bar{T}}{\partial z} = \alpha\nabla^2\bar{T} - \frac{\partial}{\partial x}\overline{u'T'} - \frac{\partial}{\partial y}\overline{v'T'} - \frac{\partial}{\partial z}\overline{w'T'}$$

$$-\frac{\partial}{\partial x}\overline{u'T'} - \frac{\partial}{\partial y}\overline{v'T'} - \frac{\partial}{\partial z}\overline{w'T'} = \frac{\partial}{\partial x_i}\alpha_t\frac{\partial \bar{T}}{\partial x_i}$$

$$\frac{\partial \bar{T}}{\partial t} + \bar{u}_i\frac{\partial}{\partial x_i}\bar{T} = (\alpha + \alpha_t)\nabla^2\bar{T}$$

So regardless whether we use RANS or LES, or URANS

$$\frac{\partial \bar{T}}{\partial t} + \bar{u}_i\frac{\partial}{\partial x_i}\bar{T} = (\alpha + \alpha_t)\nabla^2\bar{T}$$

The equation form looks the same, but α_t is somewhat different between RANS/URANS and LES.

Closure Model and Prandtl Number (RANS first, then LES later)

Review Paper

Kirillov, P. L. (2017). Heat exchange in turbulent flow. Part 1. Turbulent Prandtl number. Atomic Energy, 122(3), 156-171.

Bejan, A. (2013). Convection heat transfer. John wiley & sons.

Liquid Metal turbulent Pr model

Weigand, B., Ferguson, J. R., & Crawford, M. E. (1997). An extended Kays and Crawford turbulent Prandtl number model. International journal of heat and mass transfer, 40(17), 4191-4196.

LES subgrid models for turbulent Pr

Li, D. (2016). Revisiting the subgrid-scale Prandtl number for large-eddy simulation. Journal of Fluid Mechanics, 802.

Moin, P., Squires, K., Cabot, W., & Lee, S. (1991). A dynamic subgrid‐scale model for compressible turbulence and scalar transport. Physics of Fluids A: Fluid Dynamics, 3(11), 2746-2757.

Peng, S. H., & Davidson, L. (2002). On a subgrid-scale heat flux model for large eddy simulation of turbulent thermal flow. International Journal of Heat and Mass Transfer, 45(7), 1393-1405.

DES Models

Viswanathan, A. K., & Tafti, D. K. (2006). Detached eddy simulation of flow and heat transfer in fully developed rotating internal cooling channel with normal ribs. International journal of heat and fluid flow, 27(3), 351-370.

Turnow, J., & Kornev, N. EVALUATION OF FLOW STRUCTURES AND HEAT TRANSFER OVER DIMPLED SURFACES IN A NARROW CHANNEL FOR HIGH REYNOLDS AND PRANDTL NUMBER USING HYBRID RANS-LES METHODS.

WALE for Heat Transfer

Ben-Nasr, O., Hadjadj, A., Chaudhuri, A., & Shadloo, M. S. (2017). Assessment of subgrid-scale modeling for large-eddy simulation of a spatially-evolving compressible turbulent boundary layer. *Computers & Fluids*, *151*, 144–158. https://doi.org/https://doi.org/10.1016/j.compfluid.2016.07.004

How do we determine α_t(a.k.a thermal eddy diffusivity)?

1. New transport equation?
    a. Remember with ν_t we had developed transport equations in RANS models
    b. For SGS models in LES we also had some equations
2. Piggyback on ν_t (momentum eddy diffusivity)
    a. The same turbulence which help momentum diffuse also help heat energy diffuse
        i. Eg. Eddies

Side note: if you are designing heat exchanger→ almost always better to have some degree of turbulence → additional heat transfer

Idea:

$$\mathcal{P}r = \frac{\nu}{\alpha}$$

$$\frac{\nu_t}{\alpha_t} = constant = \mathcal{P}r_t$$

How is Pr_t (turbulent Prandtl Number) calculated?

$$\mathcal{P}r_t = \frac{\nu_t}{\alpha_t}$$

Reynold's stresses (near wall region → flat plate BL flow)

$$-\overline{u'v'} = \nu_t \frac{\partial \bar{u}}{\partial y}$$

$$-\frac{q''_{turb}}{\rho c_p} = -\overline{v'T'} = \alpha_t \frac{\partial \bar{T}}{\partial y}$$

$$\nu_t = \frac{-\overline{u'v'}}{\frac{\partial \bar{u}}{\partial y}}$$

$$\alpha_t = \frac{-\overline{v'T'}}{\frac{\partial \bar{T}}{\partial y}}$$

Substituting accordingly, (for flat plate BL flow)

$$\mathcal{P}r_t = \frac{\frac{-\overline{u'v'}}{\frac{\partial \bar{u}}{\partial y}}}{\frac{-\overline{v'T'}}{\frac{\partial \bar{T}}{\partial y}}} = \frac{\overline{u'v'}}{\overline{v'T'}} \frac{\frac{\partial \bar{T}}{\partial y}}{\frac{\partial \bar{u}}{\partial y}}$$

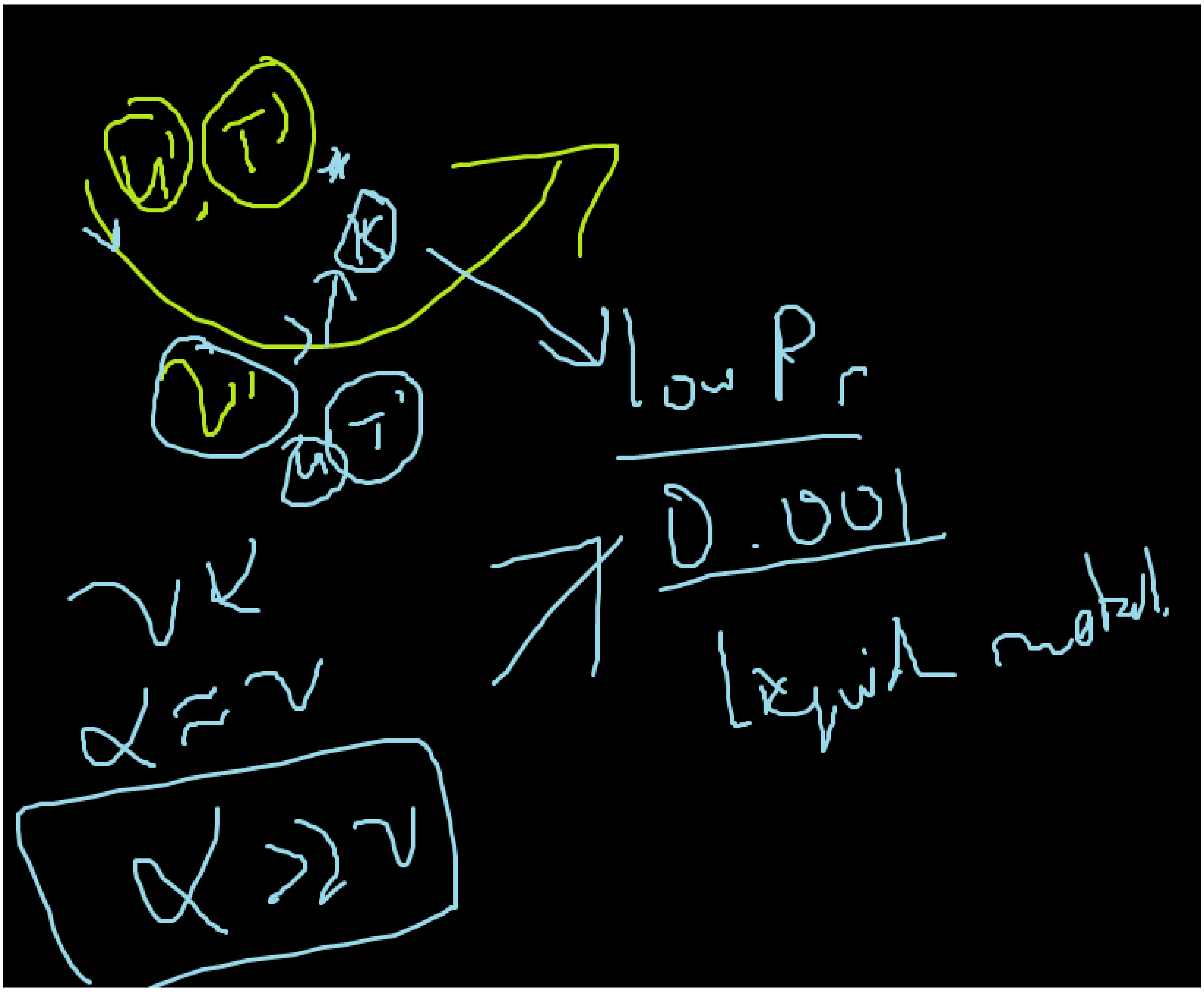
We can first take a look at near wall region, because BL flow is the most basic of concerns.

$$\mathcal{P}r_t = \frac{\overline{u'v'}}{\overline{v'T'}} \frac{\frac{\partial \bar{T}}{\partial y}}{\frac{\partial \bar{u}}{\partial y}}$$

First we make an assumption:

$$\frac{\overline{u'v'}}{\overline{v'T'}} \approx 1$$

The above is true given that temperature has no additional transport mechanism as compared to u momentum. Ie, if the fluid isn't too conductive.



Ie $\mathcal{P}r \approx 1$

$$\mathcal{P}r_t = \frac{\frac{\partial \bar{T}}{\partial y}}{\frac{\partial \bar{u}}{\partial y}}$$

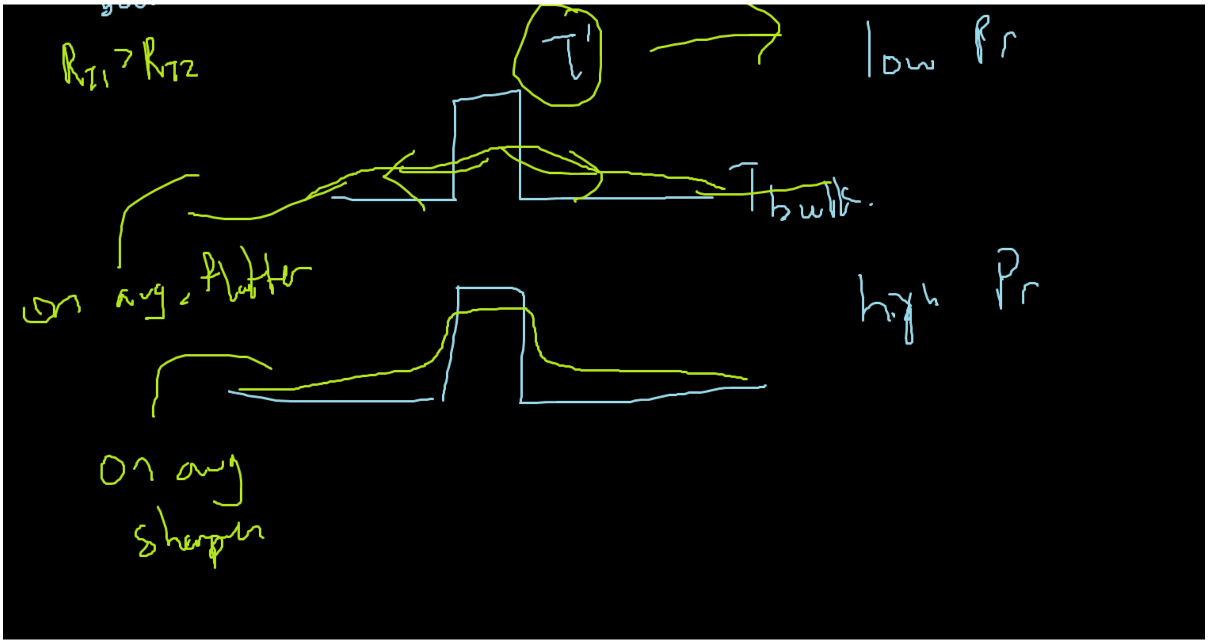
What happens if Pr changes? What is its effect on $\mathcal{P}r_t$

If Pr is low, means heat flows through the fluid very easily compared to momentum, ie. Liquid metals

$$\frac{\overline{u'v'}}{\overline{v'T'}} \approx 1$$

What will happen to T'?

$$T' \rightarrow small\ in\ high\ thermal\ conductivity\ environments$$



$$\frac{\overline{u'v'}}{\overline{v'T'}} \approx 1 \; if \; \mathcal{P}r \approx 1$$

If $\mathcal{P}r$ is high, then for constant $\overline{u'v'}$, $\overline{v'T'}$ is higher

If $\mathcal{P}r$ is low, then for constant $\overline{u'v'}$, $\overline{v'T'}$ is lower…

$$\frac{\overline{u'v'}}{\overline{v'T'}} = f(\mathcal{P}r) \; in \; reality$$

$$\frac{\overline{u'v'}}{\overline{v'T'}} \rightarrow inversely \; proportional \; to \; \mathcal{P}r$$

How then do we find $\mathcal{P}r_t$ in simplified form?

$$\mathcal{P}r_t = \frac{\frac{\partial \bar{T}}{\partial y}}{\frac{\partial \bar{u}}{\partial y}}$$

We can find the resolved temperature gradients near the wall!

Experimental data(Kirillov, 2017) provides a law of the wall:

Recall the log law region

$$u^+ = 2.44 \ln y^+ + 5$$

We also have this for temperature:

This is with experimental data from 0.7-100 (Kirillov, 2017)

$$T^+ = 2.075 \ln y^+ + 3.9$$

$$\mathcal{P}r_t = \frac{\frac{\partial \bar{T}}{\partial y}}{\frac{\partial \bar{u}}{\partial y}}$$

Where

$$T^+ = \frac{T - T_0}{T_*}$$

$$T_* = \frac{q_{wall}}{\rho c_p u_*}$$

$$u_* = \sqrt{\frac{\tau_{wall}}{\rho}}$$

$$y^+ = \frac{y u_*}{\nu}$$

$$\mathcal{P}r_t = \frac{\frac{\partial \bar{T}}{\partial y^+}}{\frac{\partial \bar{u}}{\partial y^+}}$$

We start with dT

$$d\bar{T} = d(\bar{T} - T_{wall})$$

apply

$$T^+ T_* = T - T_0$$

$$d\bar{T} = d(T^+ T_*) = T_* dT^+$$

$$d\bar{u} = d(u_* u^+) = u_* du^+$$

$$\mathcal{P}r_t = \frac{\frac{\partial \bar{T}}{\partial y^+}}{\frac{\partial \bar{u}}{\partial y^+}} = \frac{T_* \frac{\partial \bar{T}^+}{\partial y^+}}{u_* \frac{\partial \bar{u}^+}{\partial y^+}}$$

$$\frac{T_*}{u_*} = \frac{q_{wall}}{\rho c_p u_*} \frac{1}{u_*} = \frac{q_{wall}}{\rho c_p u_*^2}$$

$$\mathcal{P}r_t = \frac{\overline{u'v'}}{\overline{v'T'}} \frac{\frac{\partial \bar{T}}{\partial y}}{\frac{\partial \bar{u}}{\partial y}} = \frac{\overline{u'v'}}{\overline{v'T'}} \frac{T_* \frac{\partial \bar{T}^+}{\partial y^+}}{u_* \frac{\partial \bar{u}^+}{\partial y^+}}$$

$$\mathcal{P}r_t = \frac{q_{wall}}{\rho c_p u_*^2}\frac{\overline{u'v'}}{\overline{v'T'}}\frac{\frac{\partial \bar{T}^+}{\partial y^+}}{\frac{\partial \bar{u}^+}{\partial y^+}} = \frac{\left\{\frac{\overline{u'v'}}{u_*^2}\right\}}{\left\{\frac{\overline{v'T'}}{\left(\frac{q_{wall}}{\rho c_p}\right)}\right\}}\frac{\frac{\partial \bar{T}^+}{\partial y^+}}{\frac{\partial \bar{u}^+}{\partial y^+}}$$

$$\frac{\left\{\frac{\overline{u'v'}}{u_*^2}\right\}}{\left\{\frac{\overline{v'T'}}{\left(\frac{q_{wall}}{\rho c_p}\right)}\right\}} = f(Re, \mathcal{P}\mathrm{r}) \rightarrow general$$

Simple case:

$$\mathcal{P}r \rightarrow 1, \frac{\left\{\frac{\overline{u'v'}}{u_*^2}\right\}}{\left\{\frac{\overline{v'T'}}{\left(\frac{q_{wall}}{\rho c_p}\right)}\right\}} \rightarrow 1$$

For this case

$$\mathcal{P}r_t = \frac{\left\{\frac{\overline{u'v'}}{u_*^2}\right\}}{\left\{\frac{\overline{v'T'}}{\left(\frac{q_{wall}}{\rho c_p}\right)}\right\}}\frac{\frac{\partial \bar{T}^+}{\partial y^+}}{\frac{\partial \bar{u}^+}{\partial y^+}} = \frac{\frac{\partial \bar{T}^+}{\partial y^+}}{\frac{\partial \bar{u}^+}{\partial y^+}}$$

In log law region:

$$T^+ = 2.075 \ln y^+ + 3.9$$

$$\frac{\partial \bar{T}^+}{\partial y^+} = 2.075\frac{1}{y^+}$$

$$u^+ = 2.44 \ln y^+ + 5$$

$$\frac{\partial \bar{u}^+}{\partial y^+} = 2.44\frac{1}{y^+}$$

$$\mathcal{P}r_t = \frac{2.075\frac{1}{y^+}}{2.44\frac{1}{y^+}} = 0.85 \ (\mathcal{P}r\ 0.7\ to\ 100)$$

Expt data shows that using $\mathcal{P}r_t = 0.85$ for this range in most **<u>RANS</u>** equations is quite acceptable.

For LES it's different, but $\mathcal{O}(\mathcal{P}r_t) = 1$

There may be some different correlations:

$$T^+ = 2.195 \ln y^+ + 13.2\mathcal{P}r - 5.66$$

This is for $\mathcal{Pr}$ from 0.5 to 5 (Bejan, 2013)

If u use this correlation $\mathcal{Pr}_t \to 0.9$

Some basic derivation (Bejan, 2013)

$$(\alpha + \alpha_t)\frac{\partial \bar{T}}{\partial y} = \left[(\alpha + \alpha_t)\frac{\partial \bar{T}}{\partial y}\right]_{y=0}$$

This is constant heat flux assumption

$$\left[(\alpha + \alpha_t)\frac{\partial \bar{T}}{\partial y}\right]_{y=0} = -\frac{q''_{wall}}{\rho c_p}$$

Bringing the (roughly) constant terms to one side for simplicity, we then integrate

Noting that

$$d(\bar{T}) = d(\bar{T} - T_0)$$

For simplicity

$$(\alpha + \alpha_t)\frac{\partial(\bar{T} - T_0)}{\partial y} = -\frac{q''_{wall}}{\rho c_p}$$

Nondimensionalisation of Pr_t equation yields…

$$(\alpha + \alpha_t)\frac{T_* \partial T^+}{\frac{\nu}{u_*}\,\partial y^+} = -\frac{q''_{wall}}{\rho c_p}$$

$$(\alpha + \alpha_t)\frac{T_* \partial T^+}{\frac{\nu}{u_*}\,\partial y^+} = -\frac{q''_{wall}}{\rho c_p}$$

$$T_* = \frac{q_{wall}}{\rho c_p u_*}$$

$$(\alpha + \alpha_t)\frac{\frac{q_{wall}}{\rho c_p u_*}\partial T^+}{\frac{\nu}{u_*}\,\partial y^+} = -\frac{q''_{wall}}{\rho c_p}$$

$$(\alpha + \alpha_t)\frac{\partial T^+}{\frac{\nu}{1}\,\partial y^+} = -1$$

If we want to get rid of negative sign, we define and do the substitution differenyl

$$\frac{T^+}{T_*} = (T_{wall} - \bar{T})$$

$$(\alpha + \alpha_t)\frac{\partial T^+}{\frac{\nu}{1}\,\partial y^+} = 1$$

$$\left(\frac{\alpha}{\nu}+\frac{\alpha_t}{\nu}\right)\frac{\partial T^+}{\partial y^+}=1$$

$$\left(\frac{1}{\mathcal{P}r}+\frac{\alpha_t \nu_t}{\nu_t \nu}\right)\frac{\partial T^+}{\partial y^+}=1$$

$$\left(\frac{1}{\mathcal{P}r}+\left(\frac{1}{\mathcal{P}r_t}\right)\frac{\nu_t}{\nu}\right)\frac{\partial T^+}{\partial y^+}=1$$

$$\frac{\partial T^+}{\partial y^+}=\frac{1}{\left(\frac{1}{\mathcal{P}r}+\left(\frac{1}{\mathcal{P}r_t}\right)\frac{\nu_t}{\nu}\right)}$$

$$T^+=\int_0^{y^+}\frac{1}{\left(\frac{1}{\mathcal{P}r}+\left(\frac{1}{\mathcal{P}r_t}\right)\frac{\nu_t}{\nu}\right)}d\left(y^{+\prime}\right)$$

In VSL $\nu_t\to 0$, $\left(\frac{1}{\mathcal{P}r_t}\right)\frac{\nu_t}{\nu}$ disappears compared to $\frac{1}{\mathcal{P}r}$, this we call it the conduction sublayer

The other regime is where: $\left(\frac{1}{\mathcal{P}r_t}\right)\frac{\nu_t}{\nu}\gg\frac{1}{\mathcal{P}r}$, that usually occurs when $\nu_t\gg\nu$, we have a log layer regime…

$$T^+=\int_0^{y^+}\frac{1}{\left(\frac{1}{\mathcal{P}r}+\left(\frac{1}{\mathcal{P}r_t}\right)\frac{\nu_t}{\nu}\right)}d\left(y^{+\prime}\right)$$

$$T^+=\int_0^{y^+_{CSL}}\frac{1}{\left(\frac{1}{\mathcal{P}r}\right)}d\left(y^{+\prime}\right)+\int_{y^+_{CSL}}^{y^+}\frac{1}{\left(\left(\frac{1}{\mathcal{P}r_t}\right)\frac{\nu_t}{\nu}\right)}d\left(y^{+\prime}\right)$$

$$T^+=\mathcal{P}r\int_0^{y^+_{CSL}}d\left(y^{+\prime}\right)+\int_{y^+_{CSL}}^{y^+}\frac{1}{\left(\left(\frac{1}{\mathcal{P}r_t}\right)\frac{\nu_t}{\nu}\right)}d\left(y^{+\prime}\right)$$

In conduction sublayer:

$$T^+=\mathcal{P}r\,y^+\ (from\ y^+=0\ to\ y^+=y^+_{CSL})$$

In log law region:

$$T^+=\int_{y^+_{CSL}}^{y^+}\frac{1}{\left(\left(\frac{1}{\mathcal{P}r_t}\right)\frac{\nu_t}{\nu}\right)}d\left(y^{+\prime}\right)$$

Recall when we derived our log law region for velocity:

$$\left(1+\frac{\nu_t}{\nu}\right)\frac{\partial u^+}{\partial y^+}=1$$

In log law region

$$\frac{\partial u^+}{\partial y^+}=\frac{\nu}{\nu_t}$$

$$T^+ = \int_{y^+_{CSL}}^{y^+} \frac{1}{\left(\frac{\left(\frac{1}{\mathcal{P}r_t}\right)1}{\frac{\partial u^+}{\partial y^+}}\right)} d(y^{+\prime})$$

What is $\frac{\partial u^+}{\partial y^+}$?

In log law region:

$$\frac{\partial u^+}{\partial y^+} = \frac{1}{0.41}\frac{1}{y^+} = \frac{1}{\kappa y^+}$$

$$T^+ = \int_{y^+_{CSL}}^{y^+} \frac{1}{\left(\frac{1}{\mathcal{P}r_t}\right)\kappa y^+} d(y^{+\prime})$$

$$T^+ = \frac{\mathcal{P}r_t}{\kappa}\int_{y^+_{CSL}}^{y^+} \frac{1}{y^+} d(y^{+\prime})$$

$$T^+ = \frac{\mathcal{P}r_t}{\kappa}\ln\frac{y^+}{y^+_{CSL}}$$

So we can get $\mathcal{P}r_t$ from these equations and compare to experimental data.

Conduction sublayer

$$T^+ = \mathcal{P}r\, y^+\ (from\ y^+ = 0\ to\ y^+ = y^+_{CSL})$$

Log law region

$$T^+ = \frac{\mathcal{P}r_t}{\kappa}\ln\frac{y^+}{y^+_{CSL}}$$

Compare with expt data, substitute $\mathcal{P}r_t = 0.9$, $y^+_{CSL} = 13.2$, $\kappa = 0.41$

That results in this equation for Pr=0.5-5

$$T^+ = 2.195\ln y^+ + 13.2\mathcal{P}r - 5.66$$

Outside the Boundary Layer (RANS Treatment)

McEligot, D. M., & Taylor, M. F. (1996). The turbulent Prandtl number in the near-wall region for low-Prandtl-number gas mixtures. International Journal of Heat and Mass Transfer, 39(6), 1287-1295.

Kays, W. M. (May 1, 1994). "Turbulent Prandtl Number—Where Are We?." ASME. J. Heat Transfer. May 1994; 116(2): 284–295. https://doi.org/10.1115/1.2911398

$$\mathcal{P}r_t = \frac{q_{wall}}{\rho c_p u_*^2} \frac{\overline{u'v'}}{\overline{v'T'}} \frac{\frac{\partial \bar{T}^+}{\partial y^+}}{\frac{\partial \bar{u}^+}{\partial y^+}} = \frac{\left\{\frac{\overline{u'v'}}{u_*^2}\right\}}{\left\{\frac{\overline{v'T'}}{\left(\frac{q_{wall}}{\rho c_p}\right)}\right\}} \frac{\frac{\partial \bar{T}^+}{\partial y^+}}{\frac{\partial \bar{u}^+}{\partial y^+}}$$

$$\frac{\left\{\frac{\overline{u'v'}}{u_*^2}\right\}}{\left\{\frac{\overline{v'T'}}{\left(\frac{q_{wall}}{\rho c_p}\right)}\right\}} = f(Re, \mathcal{P}r)$$

Experiments show Pr is roughly constant in free shear zone, similar to log law BL…

But there is still some actual variation below log law layer (McEligot & Taylor, 1996).

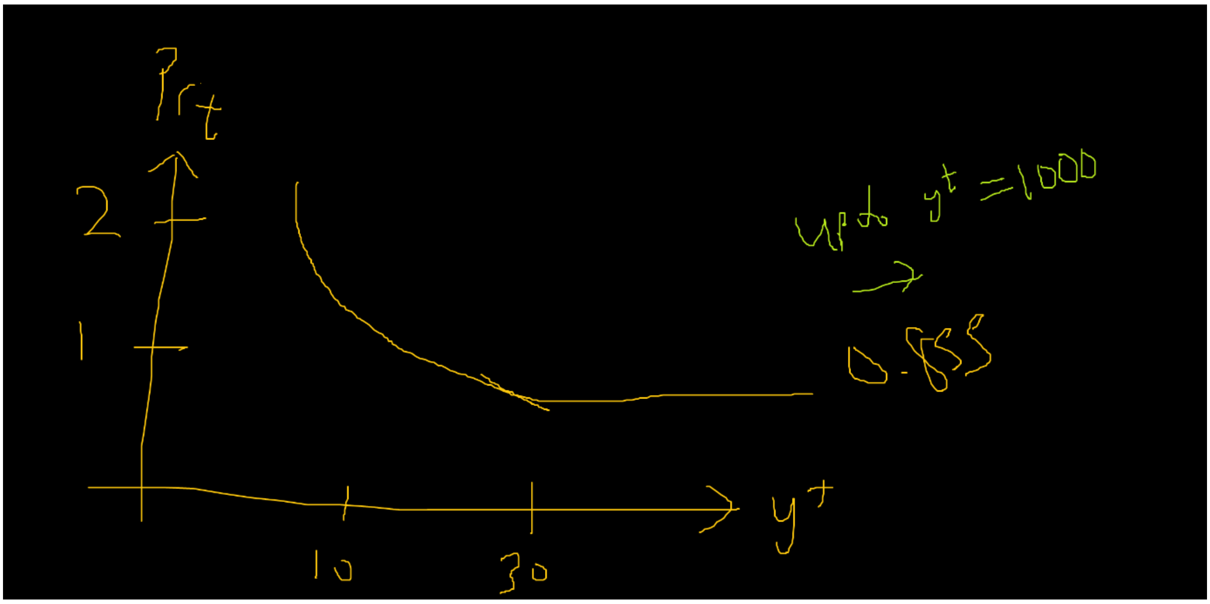
One simple model is:

For air and water at $\mathcal{P}r \approx 0.7$, we get (Kays, 1994):

$$\mathcal{P}r_t = 0.855 + \left(1 - \tanh\left(0.2(y^+ - 7.5)\right)\right)$$

Also known as Hollingsworth Equation.

This is important for $y^+ < 30$



Variation of Pr_t (RANS treatment)

Two important cases of note:

- Liquid metals (Low Pr)(Weigand, Ferguson, & Crawford, 1997)
    - Varies with Re and Pr and y^+
- Oil and Glycerin (Ultra high Pr)

<u>Liquid metal case</u>(Weigand et al., 1997)

$$\mathcal{P}r = 0.008\ Liquid\ Na$$

Pipe and channel flow, average Pr_t (Weigand et al., 1997):

$$\mathcal{P}r_{t(average)} = 0.9 + \frac{182.4}{\mathcal{P}r\ (Re^{0.888})}$$

(Jischa and Rieke)

$$\mathcal{P}r_{t(average)} = 0.85\left(\frac{0.9}{0.85} + \frac{\frac{182.4}{0.85}}{\mathcal{P}r\ (Re^{0.888})}\right)$$

$$\left(\frac{0.9}{0.85} + \frac{\frac{182.4}{0.85}}{\mathcal{P}r\ (Re^{0.888})}\right) = f(Re, \mathcal{P}r)$$

For liquid metals, the log region actually has funny shapes, I mean they do not share the same shapes as for $\mathcal{P}r = 1$

So

$$\left(\frac{0.9}{0.85} + \frac{\frac{182.4}{0.85}}{\mathcal{P}r\ (Re^{0.888})}\right) = \frac{\left\{\frac{\overline{u'v'}}{u_*^2}\right\}}{\left\{\frac{\overline{v'T'}}{\left(\frac{q_{wall}}{\rho c_p}\right)}\right\}} = f(Re, \mathcal{P}r) \rightarrow \boldsymbol{caution\ not\ an\ offical\ correlation}$$

Doesn't quite hold, but it's interesting to see the Re and Pr dependence.

Variation with wall distance:

$$\mathcal{P}r_t = \frac{1}{\frac{1}{2\mathcal{P}r_{t(free\ shear)}} + C\ Pe_t\sqrt{\frac{1}{\mathcal{P}r_{t(free\ shear)}}} - (C\ Pe_t)^2\left[1 - \exp\left(-\frac{1}{C\ Pe_t\sqrt{\mathcal{P}r_{t(free\ shear)}}}\right)\right]}$$

$$C = 0.3$$

$$\mathcal{P}r_{t(free\ shear)} = 0.85 + \frac{182.4}{\mathcal{P}r\ (Re^{0.888})}$$

$$Pe_t = \mathcal{P}r \frac{\nu_t}{\nu}$$

$$\nu_t = f(y^+)$$

$$\mathcal{P}e = Re * \mathcal{P}r$$

Pr>1 case(Kirillov, 2017)

For flow in tube ($1 < \mathcal{P}r < 45$):

Water (5-180°C → pressurised water), CO_2 25°C, Ethylene glycol at 70-100°C

$$\mathcal{P}r_t = 1.01 - 0.09\,\mathcal{P}r^{0.36}$$

For Flow in Tube (145<Pr<1800):

Ethylene glycol (0-24°C), Glycerin 50°C

$$\mathcal{P}r_t = 1.01 - 0.25 \ln \mathcal{P}r$$

For Flow in Tube (1800<Pr<12500):

Glycerin $0 - 45$°C

$$\mathcal{P}r_t = 0.99 - 0.44\,(\ln \mathcal{P}r)^{0.5}$$

Near Wall Regions (FD- pipe flow)

Prandtl Number Models for oils and other high Pr fluids (Pr=7-1000), (Re~80,000-380,000)

$$y^+ < 5$$

$$\mathcal{P}r_t = 1.07$$

$$\nu_t = 0.001\, {y^+}^3$$

$$y^+ > 5$$

$$\mathcal{P}r_t = \frac{1}{\left\{0.5882 + 0.288\left(\frac{\nu_t}{\nu}\right) - 0.0441\left(\frac{\nu_t}{\nu}\right)^2 \left[1 - \exp\left(-\frac{5.165}{\left(\frac{\nu_t}{\nu}\right)}\right)\right]\right\}}$$

In mixing length model near wall:

$$\nu_t = l^2 \left|\left(\frac{\partial u}{\partial y}\right)\right|$$

Van Driest Relation

$$l = \kappa y \left[1 - \exp\left(-\frac{y^+}{25}\right)\right]$$

$$\kappa = von\ karman\ coefficient\ 0.40 - 0.41$$

The above correlation fits well

However, we often like to avoid using complicated models… as far as possible

Pr ~48.6 - 64.3 (Zukausukas and Slanciaukas) $Re_m = \frac{\delta_2 u_\infty}{\nu}$ $(momentum\ thickness\ Re) = 1810$

Pr ~5.63 - 6.19 Water (Hollingsworth) $Re_m = 1552$

Similar to above, except we use Hollingsworth equation instead of Kay's and Crawford model.

$$y^+ < 5$$

$$\mathcal{P}r_t = 1.07$$

$$\nu_t = 0.001\, y^{+3}$$

$$y^+ > 5$$

$$\mathcal{P}r_t = 0.855 + \left(1 - \tanh\left(0.2(y^+ - 7.5)\right)\right)$$

- ➔ This means $\mathcal{P}r_t \rightarrow 2.76\ near\ wall, but\ 0.855\ far\ from\ wall\ (y^+ \sim 24)$

In mixing length model near wall:

$$\nu_t = l^2 \left|\left(\frac{\partial u}{\partial y}\right)\right|$$

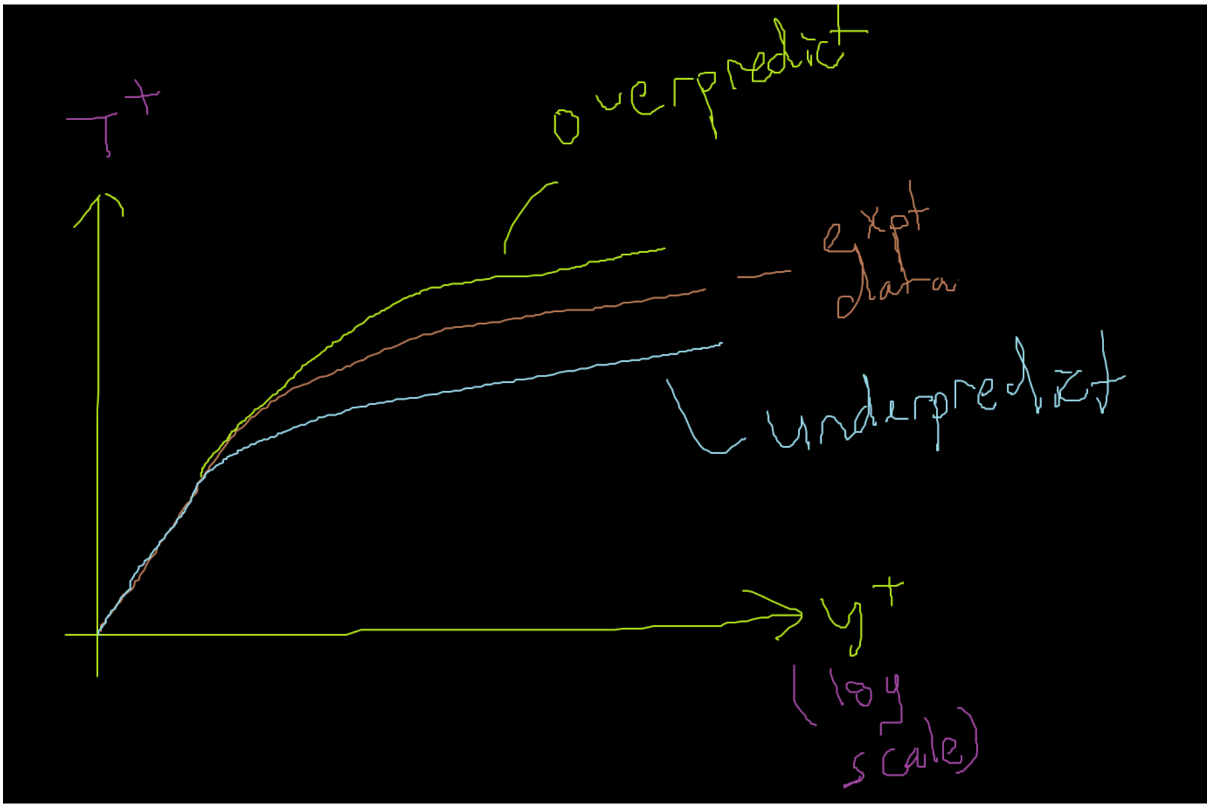
Van Driest Relation

$$l = \kappa y \left[1 - \exp\left(-\frac{y^+}{25}\right)\right]$$

$$\kappa = von\ karman\ coefficient\ 0.40 - 0.41$$

Pr~4.53 – 5.38 $Re_m = 3140$

Hollingsworth equation doesn't fit perfectly, Hollingsworth overpredicts T^+ in log layer by ~12%



What if we used $\mathcal{P}r_t = 0.85$ throughout?

Zaitsev, D. K., & Smirnov, E. M. (2019). Method of calculation of turbulent Prandtl number for the SST turbulence model, St. Petersburg Polytechnical State University Journal. *Physics and Mathematics*, *12*(1), 35-44.

https://physmath.spbstu.ru/userfiles/files/articles/2019/1/3Zaytsev-Smirnov-eng.pdf

Oil Flows (Pr=48.6-64.3) (Kays, 1994)

15% underprediction for oil at $Re_m = 1810$

Water Flows (Pr=5.63-6.19)(Kays, 1994)

15-20% underprediction for water at $Re_m = 1552$

Water Flows (Pr=4.53-5.38)(Kays, 1994)

5% underprediction $Re_m = \frac{\delta_2 u_\infty}{\nu}$ $(momentum\ thickness\ Re) = 3140$

Underprediction means for us: overpredicting heat flux at fixed temperature difference, flow condition and same fluid properties

$$T^+ = \frac{T - T_0}{T_*}$$

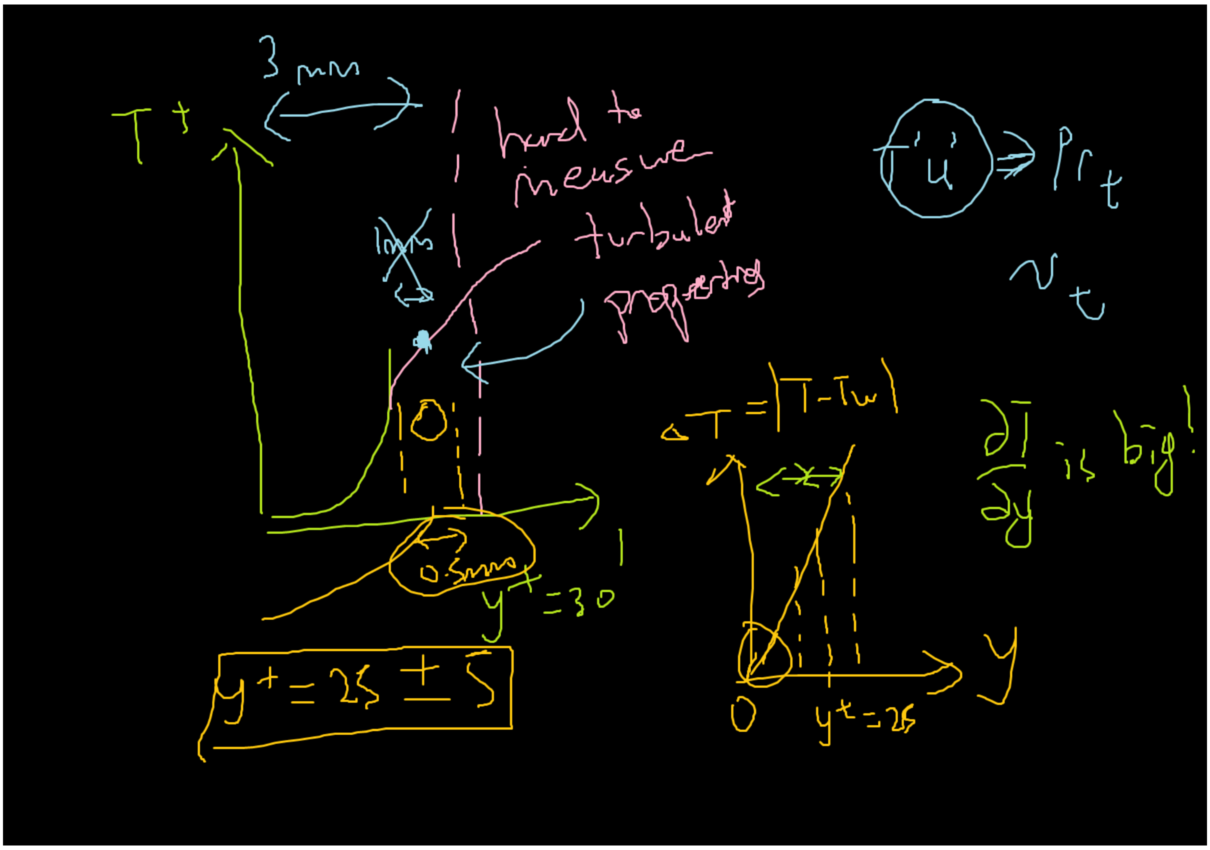
$$T_* = \frac{q_{wall}}{\rho c_p u_*}$$

<u>Effect of Pressure Gradient</u>

Adverse Pressure gradient makes $\mathcal{P}r_t$ change a lot (there was some discrepancy as to whether it increased or decreased(Kays, 1994) → airflow data

Current Difficulties in $\mathrm{Pr_t}$

- Measuring near wall region → most critical!!
    - Critical for natural convection
    - Critical for mixed flows (ie both natural and forced convection occur)



Being off in your measurement by just a bit throws off your temperature reading.

There are also turbulent fluctuations in the BL (buffer layer and log law).

- ➔ Since there's error, there will be some uncertainty in temperature measurement
- ➔ Your temperature measurement error will be bigger sometimes than T'

Technique is to work backwards from heat transfer rate → $\mathcal{P}r_t$

How accurate is it? Not too sure.

$\mathcal{P}r_t$ becomes one value for the wall region.

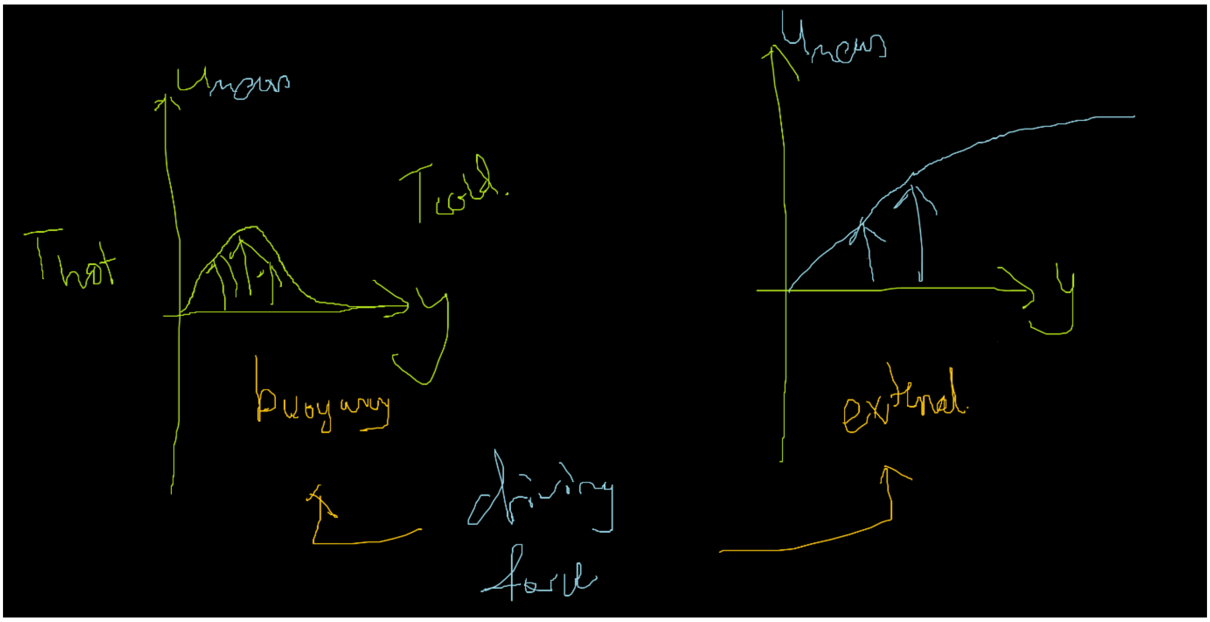
Next best thing is DNS data near the wall and compare with RANS.

Natural Convection

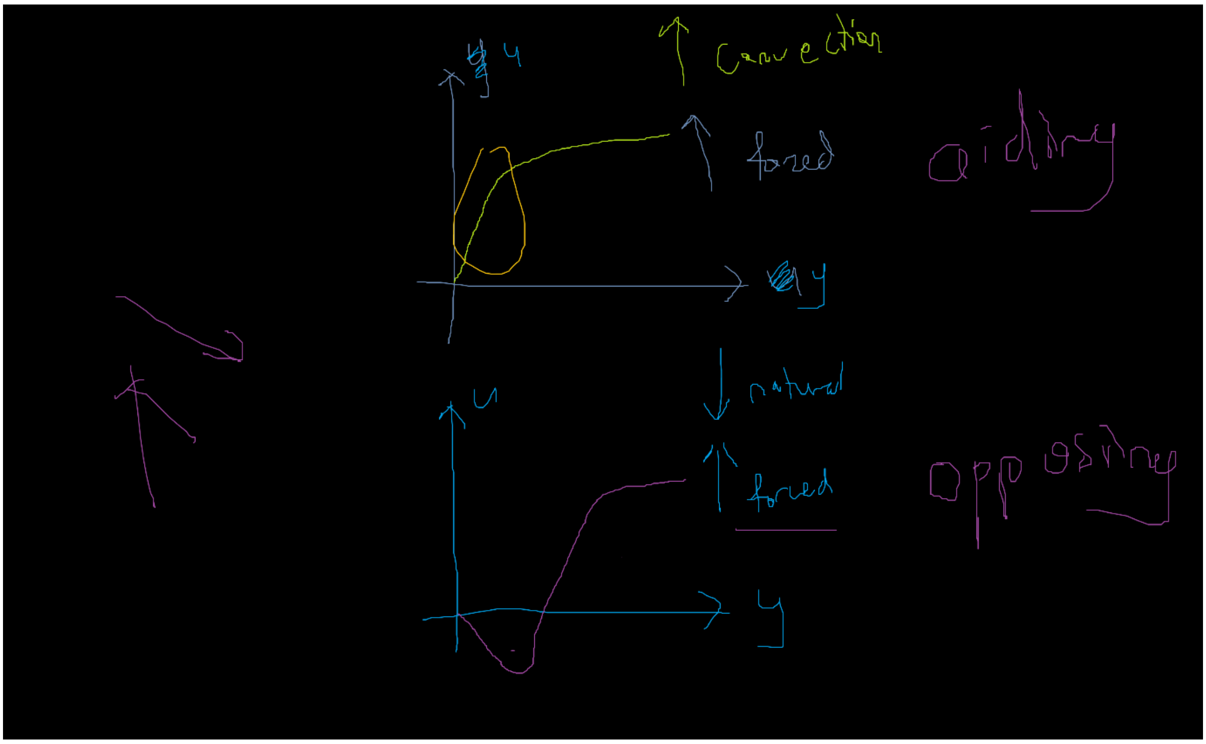
Henkes, R. A. W. M., Van Der Vlugt, F. F., & Hoogendoorn, C. J. (1991). Natural-convection flow in a square cavity calculated with low-Reynolds-number turbulence models. *International Journal of Heat and Mass Transfer*, *34*(2), 377-388.

Yuan, X., Moser, A., & Suter, P. (1993). Wall functions for numerical simulation of turbulent natural convection along vertical plates. *International journal of heat and mass transfer*, *36*(18), 4477-4485.

Standard k epsilon model with normal log-law functions overpredicts natural convection heat transfer by ~30%! (Henkes, Van Der Vlugt, & Hoogendoorn, 1991)



Hence new wall functions were developed for **natural convection** (Yuan, Moser, & Suter, 1993).



But it's difficult to have empirical wall functions for every single kind of flow, eg. Mixed convection.

First let's introduce our new dimensionless parameters (Yuan et al., 1993):

	Forced Convection	Natural Convection
Distance from wall	$y^+ \equiv \frac{yu_\tau}{\nu}$	$y^* \equiv \frac{yu_q}{\alpha}$
Velocity	$u^+ \equiv \frac{u}{u_\tau}$	$u^{**} \equiv \frac{u_q^3 u}{u_\tau^4}$
Temperature	$T^+ \equiv \frac{(T_{wall} - T)u_\tau}{\left(\frac{q_{wall}}{\rho c_p}\right)}$	$T^*_{natl} \equiv \frac{(T_{wall} - T)}{T_q}$
Reference dimensionless quantities	$u_\tau \equiv \sqrt{\frac{\tau_{wall}}{\rho}}$	$u_q \equiv \left(\frac{g\beta\alpha q_{wall}}{\rho c_P}\right)^{\frac{1}{4}}$ $T_q \equiv \left(\frac{q_{wall}^3}{g\beta\alpha\left(\rho c_p\right)^3}\right)^{\frac{1}{4}}$

➔ Which Pr and densities do we use since temperature varies?

Note: we often use <u>film thickness fluid properties</u> to calculate the density, heat capacity for the above

$$T_{film} = \frac{T_{wall} + T_\infty}{2}$$

The temperature wall function

$$T^*_{natl} = y^* \; (y^* \leq 1)$$

$$T^*_{natl} = 1 + 1.36 \ln y^* - 0.135(\ln y^*)^2 \; (1 < y^* \leq 100)$$

$$T^*_{natl} = 4.4 \; (y^* > 100)$$

The velocity wall function:

$$u^{**} = \min\{f_1; f_0\}$$

What is f_1?

$$f_1 = 1.41 y_1^{**} - 3.11 y_1^{**2} + 2.38 y_1^{**3} \; ; \; y_1^{**} \leq 0.53$$

$$y_1^{**} = \frac{y u_q^3}{\alpha u_\tau^2}$$

$$f_1 = 0.228 \; ; \; y_1^{**} > 0.53$$

f_1 deals with velocity in the inner sublayer

What is f_0?

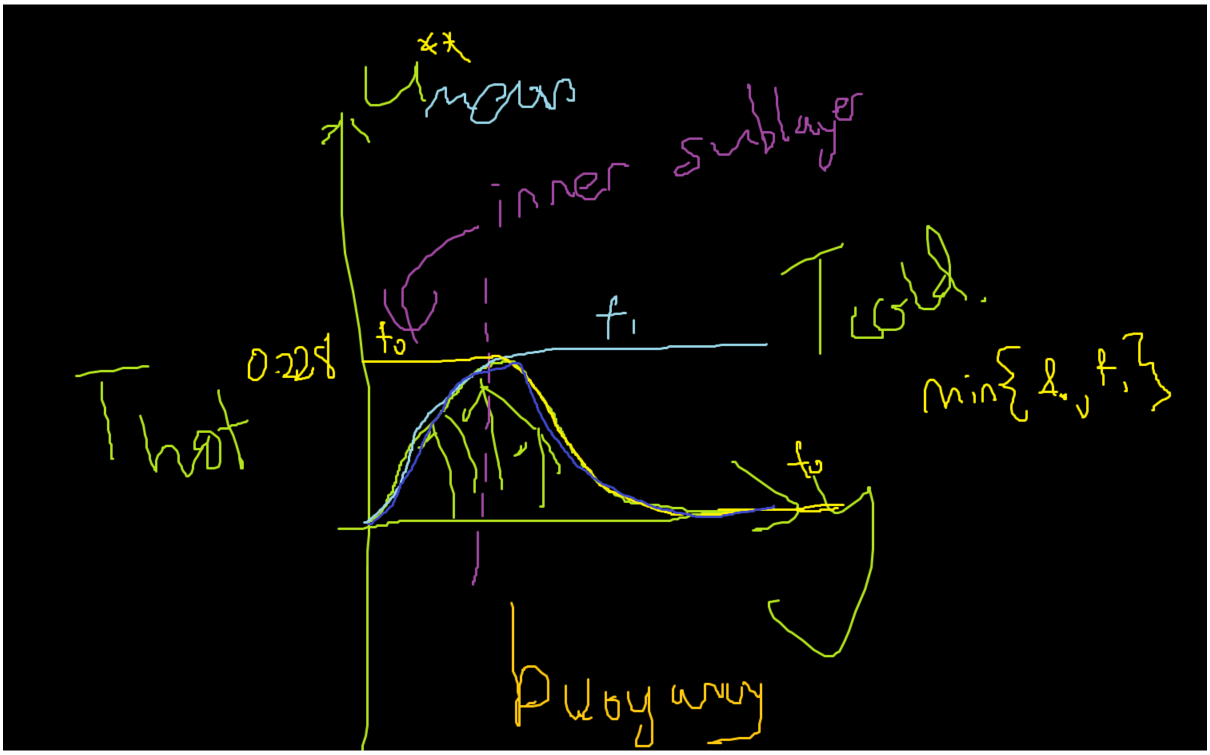
$$f_0 = 0.228 \; ; \; y_0^{**} < 0.005$$

$$y_0^{**} = \frac{y u_q^7}{\alpha u_\tau^6}$$

$$f_0 = -0.458 - 0.258 \ln y_0^{**} - 0.02425(\ln y_0^{**})^2 \; ; 0.005 \leq \; y_0^{**} \leq 0.1$$

$$f_0 = 0 \; ; y_0^{**} > 0.1$$

f_0 deals with velocity in the outer sublayer



Validated for air natural convection data.

While not very useful outside natural convection, we can still compare this to using a log layer profile…

- ➔ would still take a lot of effort and we'd have to assume some heat flux value or something…
- ➔ Alternatively lots and lots of manipulation to get T^+ out from T^*_{natl} equations

	Forced Convection	Natural Convection
Distance from wall	$y^+ \equiv \frac{yu_\tau}{\nu}$	$y^* \equiv \frac{yu_q}{\alpha}$
Velocity	$u^+ \equiv \frac{u}{u_\tau}$	$u^{**} \equiv \frac{u_q^3 u}{u_\tau^4}$
Temperature	$T^+ \equiv \frac{(T_{wall} - T)u_\tau}{\left(\frac{q_{wall}}{\rho c_p}\right)}$	$T^*_{natl} \equiv \frac{(T_{wall} - T)}{T_q}$
Reference dimensionless quantities	$u_\tau \equiv \sqrt{\frac{\tau_{wall}}{\rho}}$	$u_q \equiv \left(\frac{g\beta\alpha q_{wall}}{\rho c_P}\right)^{\frac{1}{4}}$ $T_q \equiv \left(\frac{q_{wall}^3}{g\beta\alpha(\rho c_p)^3}\right)^{\frac{1}{4}}$

$$u_q \equiv \left(\frac{g\beta\alpha q_{wall}}{\rho c_P}\right)^{\frac{1}{4}}$$

We note:

$$\left(\frac{q_{wall}}{\rho c_p}\right) = \frac{(T_{wall} - T)u_\tau}{T^+} = u_\tau T_*$$

Noting:

$$T_* = \frac{q_{wall}}{\rho c_p u_*}$$

We can use this to substitute:

$$u_q \equiv (g\beta\alpha u_\tau T_*)^{\frac{1}{4}}$$

$$T_q \equiv \left(\frac{(u_\tau T_*)^3}{g\beta\alpha}\right)^{\frac{1}{4}}$$

Now showing the following in terms of these coordinates…

$$y^* \equiv \frac{yu_q}{\alpha} = \frac{y(g\beta\alpha u_\tau T_*)^{\frac{1}{4}}}{\alpha}$$

Noting:

$$y^+ \equiv \frac{yu_\tau}{\nu}$$

$$y = \frac{y^+ \nu}{u_\tau}$$

$$y^* = \frac{\frac{y^+ \nu}{u_\tau}(g\beta\alpha u_\tau T_*)^{\frac{1}{4}}}{\alpha}$$

$$y^* = y^+ \frac{\nu (g\beta\alpha u_\tau T_*)^{\frac{1}{4}}}{\alpha u_\tau}$$

Note how these are scaled according to the buoyancy terms…

So let's get a temperature profile out first:

$$T^*_{natl} \equiv \frac{(T_{wall} - T)}{T_q}$$

Let's substitute T^+ in first,

$$T^+ \equiv \frac{(T_{wall} - T)u_\tau}{\left(\frac{q_{wall}}{\rho c_p}\right)}$$

$$(T_{wall} - T) = \frac{T^+ \left(\frac{q_{wall}}{\rho c_p}\right)}{u_\tau}$$

$$T^*_{natl} \equiv \frac{\frac{T^+ \left(\frac{q_{wall}}{\rho c_p}\right)}{u_\tau}}{T_q}$$

$$T^*_{natl} \equiv T^+ \frac{\left(\frac{q_{wall}}{\rho c_p}\right)}{u_\tau T_q}$$

Let's substitute in T_q

$$T_q \equiv \left(\frac{(u_\tau T_*)^3}{g\beta\alpha}\right)^{\frac{1}{4}}$$

$$T^*_{natl} \equiv T^+ \frac{\left(\frac{q_{wall}}{\rho c_p}\right)}{u_\tau \left(\frac{(u_\tau T_*)^3}{g\beta\alpha}\right)^{\frac{1}{4}}}$$

$$T^*_{natl} \equiv T^+ \frac{\left(\frac{q_{wall}}{\rho c_p u_\tau}\right)}{\left(\frac{(u_\tau T_*)^3}{g\beta\alpha}\right)^{\frac{1}{4}}}$$

$$T_* = \frac{q_{wall}}{\rho c_p u_\tau}$$

$$T^*_{natl} \equiv T^+ \frac{T_*}{\left(\frac{(u_\tau T_*)^3}{g\beta\alpha}\right)^{\frac{1}{4}}}$$

$$T^*_{natl} \equiv T^+ \frac{1}{\left(\frac{(u_\tau T_*)^3}{T_*^4 g\beta\alpha}\right)^{\frac{1}{4}}}$$

$$T^*_{natl} \equiv T^+ \frac{1}{\left(\frac{u_\tau^3}{T_* g\beta\alpha}\right)^{\frac{1}{4}}}$$

$$T^*_{natl} \equiv T^+ \frac{1}{\left(\frac{u_\tau^4}{u_\tau T_* g\beta\alpha}\right)^{\frac{1}{4}}}$$

$$T^*_{natl} \equiv T^+ \frac{(u_\tau T_* g\beta\alpha)^{\frac{1}{4}}}{u_\tau}$$

All right we can now see the log layer profile

$$T^*_{natl} = 1 + 1.36 \ln y^* - 0.135(\ln y^*)^2 \ (1 < y^* \leq 100)$$

$$y^* = y^+ \frac{\nu(g\beta\alpha u_\tau T_*)^{\frac{1}{4}}}{\alpha u_\tau}$$

In the log layer profile…

$$\ln y^* = \ln y^+ + \ln \frac{\nu(g\beta\alpha u_\tau T_*)^{\frac{1}{4}}}{\alpha u_\tau}$$

$$T^*_{natl} = 1 + 1.36 \ln y^* - 0.135(\ln y^*)^2$$

Substitute $\ln y^*$,

$$T^*_{natl} = 1 + 1.36\left(\ln y^+ + \ln \frac{\nu(g\beta\alpha u_\tau T_*)^{\frac{1}{4}}}{\alpha u_\tau}\right) - 0.135\left(\ln y^+ + \ln \frac{\nu(g\beta\alpha u_\tau T_*)^{\frac{1}{4}}}{\alpha u_\tau}\right)^2$$

We can now substitute

$$T^*_{natl} \equiv T^+ \frac{(u_\tau T_* g\beta\alpha)^{\frac{1}{4}}}{u_\tau}$$

$$T^+ \frac{(u_\tau T_* g\beta\alpha)^{\frac{1}{4}}}{u_\tau} = 1 + 1.36\left(\ln y^+ + \ln \frac{\nu(g\beta\alpha u_\tau T_*)^{\frac{1}{4}}}{\alpha u_\tau}\right) - 0.135\left(\ln y^+ + \ln \frac{\nu(g\beta\alpha u_\tau T_*)^{\frac{1}{4}}}{\alpha u_\tau}\right)^2$$

We can now see a term $\frac{(u_\tau T_* g\beta\alpha)^{\frac{1}{4}}}{u_\tau}$ pops out pretty frequently here, let's name it C_{convBL}

And if you notice,

$$Gr = \frac{g\beta\Delta T\, L^3}{\nu^2}$$

You'd notice some terms in common yeah?

$$T^+ C_{convBL} = 1 + 1.36\left(\ln y^+ + \ln \frac{\nu C_{convBL}}{\alpha}\right) - 0.135\left(\ln y^+ + \ln \frac{\nu C_{convBL}}{\alpha}\right)^2$$

Ahh, and look, $\frac{\nu}{\alpha} = \mathcal{P}r$

$$T^+ C_{convBL} = 1 + 1.36(\ln y^+ + \ln \mathcal{P}r C_{convBL}) - 0.135(\ln y^+ + \ln \mathcal{P}r C_{convBL})^2$$

Pretty evident here, that Prandtl number and C_{conv} are quite important…

$$T^+ C_{convBL} = 1 + 1.36(\ln y^+ + \ln \mathcal{P}r C_{convBL}) - 0.135(\ln y^+ + \ln \mathcal{P}r C_{convBL})^2$$

$$T^+ = \frac{1}{C_{convBL}} + \frac{1.36}{C_{convBL}}(\ln y^+ + \ln \mathcal{P}r C_{convBL}) - \frac{0.135}{C_{convBL}}(\ln y^+ + \ln \mathcal{P}r C_{convBL})^2$$

We see things are difficult to match due to the $(\ln y^+)^2$ term

For reference:

$$T^+ = A \ln y^+ + B$$

The above equation is in forced convection flow, and shows the temperature BL profile in log law region

$$T^+ = \frac{1}{C_{convBL}} + \frac{1.36}{C_{convBL}}(\ln y^+ + \ln \mathcal{P}r + \ln C_{convBL}) - \frac{0.135}{C_{convBL}}(\ln y^+ + \ln \mathcal{P}r C_{convBL})^2$$

Let's expand:

$$(\ln y^+ + \ln \mathcal{P}r C_{convBL})^2 = (\ln y^+)^2 + 2(\ln y^+)(\ln \mathcal{P}r C_{convBL}) + (\ln \mathcal{P}r C_{convBL})^2$$

$$\begin{aligned} T^+ = {} & \frac{1}{C_{convBL}} + \frac{1.36}{C_{convBL}}(\ln y^+ + \ln \mathcal{P}r + \ln C_{convBL}) \\ & - \frac{0.135}{C_{convBL}}\{(\ln y^+)^2 + 2(\ln y^+)(\ln \mathcal{P}r C_{convBL}) + (\ln \mathcal{P}r C_{convBL})^2\} \end{aligned}$$

Now we can combine the terms:

$$T^+ = \left[\frac{1}{C_{convBL}} + \frac{1.36}{C_{convBL}} \ln \mathcal{P}r C_{convBL} - \frac{0.135}{C_{convBL}} (\ln \mathcal{P}r C_{convBL})^2\right] + \left\{\frac{1.36}{C_{convBL}} - \frac{2(0.135)}{C_{convBL}} (\ln \mathcal{P}r C_{convBL})\right\} (\ln y^+) - \frac{0.135}{C_{convBL}} \{(\ln y^+)^2\}$$

Now in theory if we neglected the $\{(\ln y^+)^2\}$ term,

we could match $\left\{\frac{1.36}{C_{convBL}} - \frac{2(0.135)}{C_{convBL}} (\ln \mathcal{P}r C_{convBL})\right\}$ to the coeffcient in front of $\ln y^+$

And $\left[\frac{1}{C_{convBL}} + \frac{1.36}{C_{convBL}} \ln \mathcal{P}r C_{convBL} - \frac{0.135}{C_{convBL}} (\ln \mathcal{P}r C_{convBL})^2\right]$ to the constant

For in the log layer,

$$T^+ = A \ln y^+ + B$$

But do notice:

$$T^*_{natl} = y^* \; (y^* \leq 1)$$

After this

$$(1 < y^* \leq 100)$$

$$T^+ = \left[\frac{1}{C_{convBL}} + \frac{1.36}{C_{convBL}} \ln \mathcal{P}r C_{convBL} - \frac{0.135}{C_{convBL}} (\ln \mathcal{P}r C_{convBL})^2\right] + \left\{\frac{1.36}{C_{convBL}} - \frac{2(0.135)}{C_{convBL}} (\ln \mathcal{P}r C_{convBL})\right\} (\ln y^+) - \frac{0.135}{C_{convBL}} \{(\ln y^+)^2\}$$

The transition range changes with y^* and that also is convection dependent…

Thus we can see it's pretty difficult to match and compare forced convection and natural convection BLs as we have additional variables to consider → ie q_w, β and α all will determine the temperature profile.

Even if we could ignore the $(\ln y^+)^2$ term, we'd have to know the wall heat flux, which could change at different points for an isothermal wall!

Probably not a good idea to do this in general… so we shall not do it for u^+ and u^{**}

It's hard to compare <u>natural convection</u> and <u>forced convection</u> BL profiles.

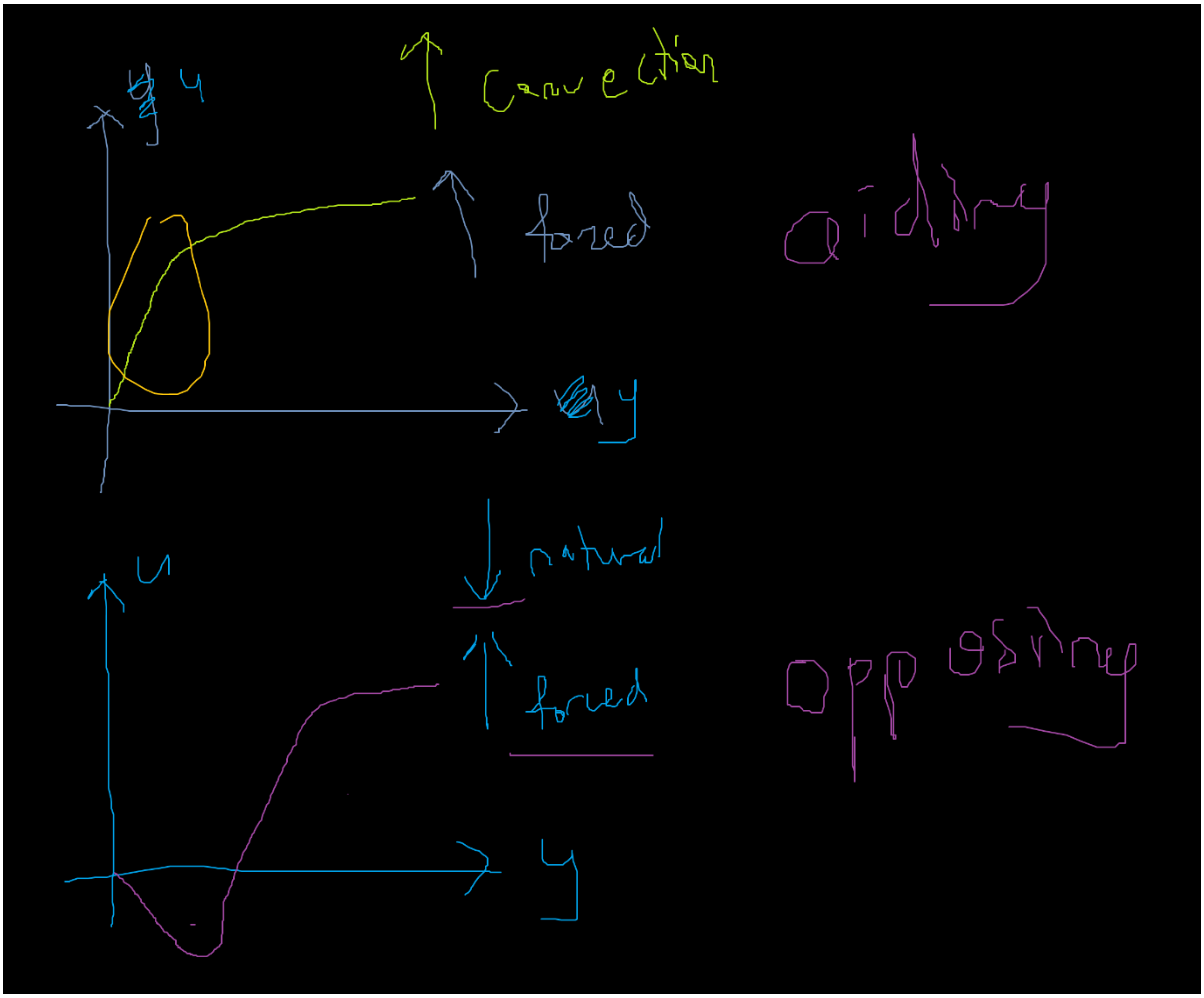
So herein lies the difficulty behind empirical correlations:

<u>Do we develop a wall function for every kind of flow</u>?

How about <u>mixed convection</u> flows and flows of complex geometry?

- Natural convection and forced convection occur at the same time

    - Aiding mixed convection
    - Opposing Mixed convection
- Non flat plate or simple pipe flows



For complex flows, may want to turn to LES → more “brute force” method

LES treatment of Pr_t and Turbulent Heat Transfer in General

<u>LES Fire Simulation (natural convection)</u>

Wang, Y., Chatterjee, P., & de Ris, J. L. (2011). Large eddy simulation of fire plumes. Proceedings of the Combustion Institute, 33(2), 2473-2480.

<u>LES Simulation between Heated Surfaces (natural convection)</u>

Lau, G. E., Yeoh, G. H., Timchenko, V., & Reizes, J. A. (2012). Large-eddy simulation of natural convection in an asymmetrically-heated vertical parallel-plate channel: Assessment of subgrid-scale models. Computers & fluids, 59, 101-116.

Peng, S. H., & Davidson, L. (2001). Large eddy simulation for turbulent buoyant flow in a confined cavity. International Journal of Heat and Fluid Flow, 22(3), 323-331.

Barhaghi, D. G., Davidson, L., & Karlsson, R. (2006). Large-eddy simulation of natural convection boundary layer on a vertical cylinder. International journal of heat and fluid flow, 27(5), 811-820.

Wang, W. P., & Pletcher, R. H. (1996). On the large eddy simulation of a turbulent channel flow with significant heat transfer. Physics of Fluids, 8(12), 3354-3366.

LES treatment of Pr_t is slightly more complex due to filter scale (Li, 2016).

As we let $\Delta \to 0$, $\mathcal{P}r_t$ change along with since it approaches DNS… (that's one way to look at things)

- ν_t or ν_{sgs} → 0

$\mathcal{P}r_t$ derived through spectral considerations of filtered NS-equations (Li, 2016), assume scale similarity between resolved scales of motion and Subgrid scales of motion. (skipping this derivation)

$$\mathcal{P}r_t \sim \mathcal{O}(1)$$

$$\mathcal{P}r_t = 0.47$$

This was from theoretical considerations.

Most common value is $\mathcal{P}r_t = 0.4$ (Peng & Davidson, 2002)

From turbulent cavity experiments, using a Smagorinsky LES model with $\mathcal{P}r_t = 0.4, C = 0.0441$ seems to fit turbulent natural convection well (Peng & Davidson, 2001). Van Driest model (Peng & Davidson, 2001) used.

$$l = \kappa y \left(1 - \exp\left(-\frac{y^+}{25}\right)\right)$$

Denominator can be 26 also sometimes.

$$\nu_t = l^2 |S|$$

Van driest function can look like this:

$$\nu_t = \left(C_{Smagorinsky} l \left(1 - \exp\left(-\frac{y^+}{25}\right)\right)\right)^2 |S|$$

A dynamic subgrid model treatment was presented by Moin, helps us see some typical values of Pr_t (Moin, Squires, Cabot, & Lee, 1991)

- ➔ 0.4-0.6 is good

From dynamic model (a priori → this is from theory rather than expt)

$$\mathcal{P}r_t \to 0.4 - 0.6\ (fine\ grid)$$

$$\mathcal{P}r_t \to 0.9\ (coarse\ grid)$$

With coarse grid, Pr_t from LES approaches Pr_t of RANS because more turbulence is modelled as a turbulent viscosity and turbulent conduction.

Remember LES resolves turbulence → ideally 80% and models the rest

RANS model → model all turbulence

Forced convection Flat Plate Channel Flow

Note: times u get forced convection heat transfer, there is always some natural convection like it or not, it's just whether we ignore it. Ie by assuming incompressibility

And in channel/pipe flow, RANS is usually better than LES due to wall functions etc etc.

Flat plate BL Flow

A good model for use is the WALE model (Ding, Wang, & Chen, 2020)

Turbulent Prandtl number of 0.89 (model value fixed at 0.9) agrees well with expt data in forced convection **supersonic** flow (Ben-Nasr, Hadjadj, Chaudhuri, & Shadloo, 2017).

If you wanted to do a dynamic $\mathcal{P}r_t$, you could do it in a similar way to the dynamic smagorinsky model (I skip details)(Lee, Xu, & Pletcher, 2004):

$$\mathcal{P}r_t = -C_d\overline{\Delta}^2 \frac{\langle F_K F_K \rangle}{\langle E_K F_K \rangle}$$

$$E_K = \frac{1}{\hat{\bar{\rho}}}\widehat{\bar{\rho}\tilde{u}_k}\widehat{\bar{\rho}\tilde{T}} - \widehat{\bar{\rho}\tilde{u}_k\tilde{T}}$$

$$F_K = \widehat{\overline{\Delta}}^2\hat{\bar{\rho}}\left|\tilde{\hat{S}}\right|\frac{\partial \hat{\tilde{T}}}{\partial x_k} - \overline{\Delta}^2\bar{\rho}\,\widehat{\left|\hat{S}\right|\frac{\partial \tilde{T}}{\partial x_k}}$$

Remember we do double filtering here, even density can change.

The above applies to compressible equations. Of course, if ρ is constant, you could easily take it out of the filtering integral…

The tilde line means favre filtering:

We haven't introduced this before due to us dealing mostly with incompressible flow

Let's say you have a quantity you want to filter

$$\overline{\rho u} = \bar{\rho}\tilde{u}$$

$$\tilde{u} = \frac{\overline{\rho u}}{\bar{\rho}}$$

Note that brackets mean spatial averaging…

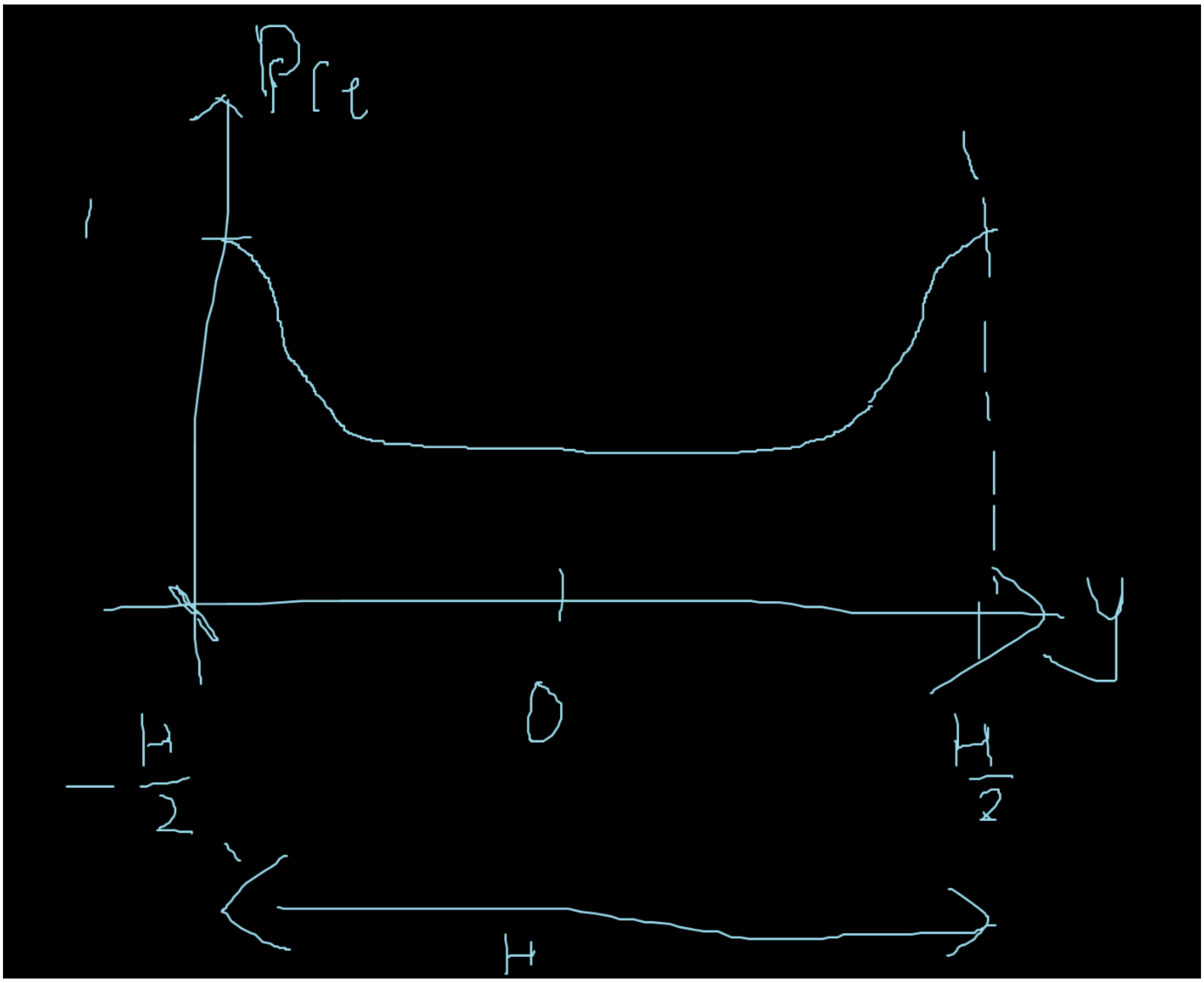
Okay so we have LES simulation of such flows (W. Wang & Pletcher, 1996)

- ➔ Low mach numbers: mach 0.01 → incompressible

<u>From dynamic procedure</u>

Interesting to note: $\mathcal{P}r_t \to 1$ near wall $\mathcal{P}r_t \to 0.5$ near centreline

Near centreline $\mathcal{P}r_t \sim 0.5$



In BL ultra near wall eg. Conduction sublayer $\mathcal{P}r_t \sim 0.9$

In log law layer $\frac{y}{\delta} > 0.5$, $\mathcal{P}r_t \to 0.5$

Compare this to the RANS $\mathcal{P}r_t \to 0.85\ or\ 0.75$ depending on $\mathcal{P}r$ for centreline calcs.

Though remember, it's not compared to experimental data. Very difficult to measure quantities in turbulent flux…

Timestep: 3×10^{-2} dimensionless timestep

- ➔ LES experiment trying to go for steady state flow

65*65*65 grid

$$2\pi\delta * 2\delta * 2\pi\delta$$

$$\delta = \frac{channel\ height}{2}$$

$$\pi H * H * \pi H$$

$$H = Channel\ Height$$

Channel Dimension…

<u>Drawback (error) of LES in convection</u>

LES tends to underpredict skin friction and Nu

C_f underpredicted by ~12.5% low Mach numbers (incompressible) (W. Wang & Pletcher, 1996)

Most of these come from LES having difficulty predicting properties near wall.

Profiles are mostly well matched,

Except for some cases we get a 10% underprediction of near wall velocity compared to DNS data.

Goes to show LES has some difficulties near walls unless you go into the DNS regime.

<u>Lack of turbulent channel LES data online for incompressible flows</u>

- ➔ Quite trivial TBH, you can DIY

But for supersonic flow, some papers were found, and temperature profile near wall is within 2% of DNS data(Ben-Nasr et al., 2017). WALE, dynamic smagorinsky

This is $\frac{T}{T_\infty}$

Now skin friction is overestimated, WALE overestimates by 20% compared to DNS(Ben-Nasr et al., 2017).

Still better to use RANS in these cases, or LES-RANS hybrid models for BL.

Error: $\pm 20\%$ compared to DNS of C_f and Nu

Compared to RANS simulation overpredicting Nu by 30%, quite an improvement!

- ➔ This is for <u>forced convection</u>

<u>**Natural Convection LES**</u>

Natural convection is considered to by asymmetric in nature when it comes to turbulence (Lau, Yeoh, Timchenko, & Reizes, 2012).

<u>LES performs better here</u>:

- Smagorinsky model was designed for atmosphere (meterology): → driven by natural convection and far from the ground!

Smagorinsky model with $\mathcal{P}r_t = 0.4$ has been used successfully in isothermal vertical cylinder verified by experiment (Barhaghi, Davidson, & Karlsson, 2006).

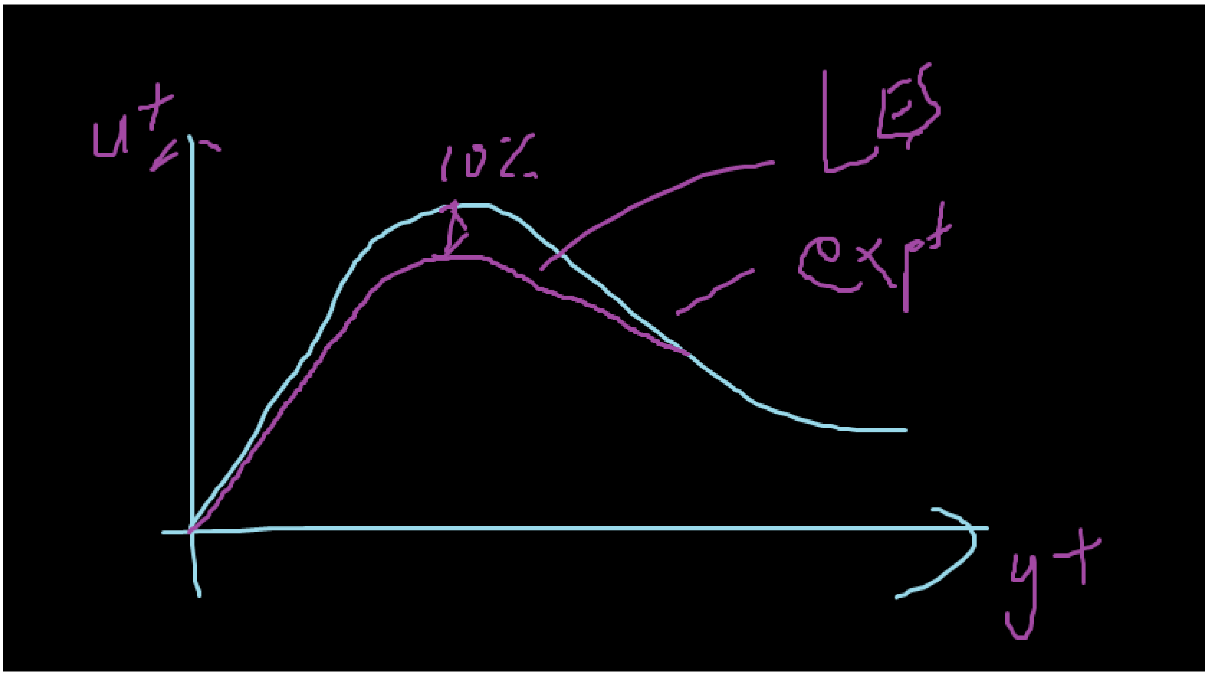
Also $u^+ = y^+$ valid in $y^+ < 5$ (Barhaghi et al., 2006).

Forced convection VSL formula works → even if you put your normal forced convection VSL wall function

- It works for <u>natural convection!!!</u>

It works, T^+ agrees to within ~7% (biggest discrepancy) just by looking at the graph. Eg if experiment is $T^+ = 15$ @ $y^+ = 10^3$, the simulation would say $T^+ = 16$

Velocity profile follows the same shape: biggest discrepancy is about ~10%.



Other things compared as well

Eg.

$$\frac{u'v'}{T'v'}$$

Turbulence statistics

Matching turbulence statistics is more difficult in general.

Above results were using **smagorinsky model (without Van Driest Damping I think, I couldn't find it mentioned in the paper).**

<u>kEqn model with constant $\mathcal{P}r_t$</u>

$\mathcal{P}r_t = 0.4$ is acceptable (Lau et al., 2012)

General trends from literature:

- ➔ Even constant $\mathcal{P}r_t$ performs well with kEqn
- ➔ Dynamic and kEqn better than Smagorinsky in terms of predictive performance
    - $\nu_t = C_d \Delta^2 \, |S|$
- ➔ Having dynamic $\mathcal{P}r_t$ is even better

Usually dynamic models outperform smagorinsky type models (Lau et al., 2012) these dynamic models perform with constant Prt rather than dynamically calculated Prt. Dynamic Prt is common in literature for LES(Peng & Davidson, 2001).

Tests were done with LES for fire simulations, for those requirements, (may not be <1% accurate) it was sufficient (Y. Wang, Chatterjee, & de Ris, 2011)

- kEqn model with constant $\mathcal{P}r_t$ performed satisfactorily
- do note to use the van driest damping function when using smagorinsky type models.

For fires, we compare flame height in (m) vs entrainment (mass flow rate)..

- compared to expt
    - LES simulation at most underpredicts flame height by max 15-20%
        - Eg. 5m flame, max error is 4m height predicition, most of the time, it's 4.5m-4.8m ish (4% off, pretty good!) (Y. Wang et al., 2011)

<u>**Mixed Convection**</u>

Lee, J. S., Xu, X., & Pletcher, R. H. (2004). Large eddy simulation of heated vertical annular pipe flow in fully developed turbulent mixed convection. International Journal of Heat and Mass Transfer, 47(3), 437-446.

Wei Zhang, Qingyan Chen (2000) Large Eddy Simulation of Natural and Mixed Convection Airflow Indoors with Two Simple Filtered Dynamic Subgrid Scale Models, Numerical

Heat Transfer: Part A: Applications, 37:5, 447-463, DOI: 10.1080/104077800274154

For case with Favre Filtering and dynamic smagorinsky model (by Germano).

$$\mathcal{P}r_t = -C_d \overline{\Delta}^2 \frac{\langle F_K F_K \rangle}{\langle E_K F_K \rangle}$$

$$E_K = \frac{1}{\hat{\bar{\rho}}} \widehat{\bar{\rho}\tilde{u}_k} \widehat{\bar{\rho}\tilde{T}} - \widehat{\hat{\rho}\tilde{u}_k \tilde{T}}$$

$$F_K = \widehat{\bar{\Delta}}^2 \hat{\bar{\rho}} |\hat{\tilde{S}}| \frac{\partial \hat{\tilde{T}}}{\partial x_k} - \bar{\Delta}^2 \bar{\rho} \widehat{|\hat{S}| \frac{\partial \tilde{T}}{\partial x_k}}$$

Mixed convection of air was simulated with this method.

Re from 6200-9700 was tested in pipe (concentric cylinders).

Velocity profiles agreed well with DNS data.

- Most of the time, error was about 3-5% at most
    - 7-10% for outliers

Crosses → DNS

What they did in the test:

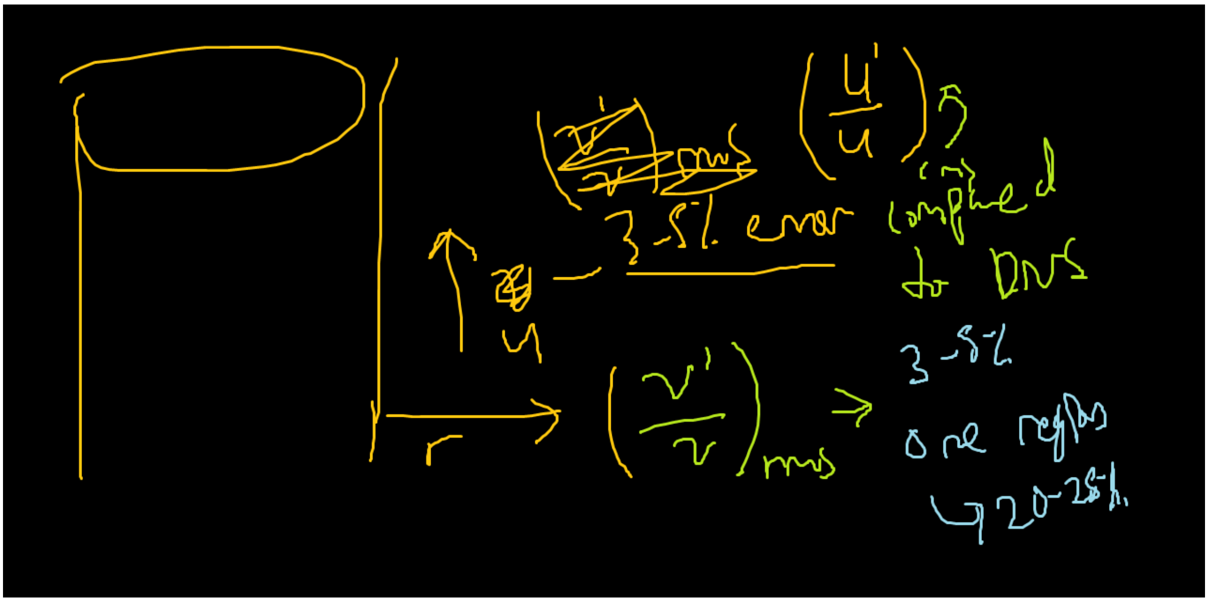
- Fixed q_w
- Tested the mixed convection that came about.

Grid: (48*48*80) was sufficient to produce this.

➔ Note that this is for aiding flows.

For further comparison they measured turbulence statistics

$$I = \frac{u'}{\bar{u}}$$



<u>For experimental data, for HVAC (heating ventilation air conditioning) (Chen Qingyan, 2000)</u>

Dynamic smagorinsky model was compared against experimental data in

1.04*1.04*0.7 m^3 room,

Side wall temp = 15℃

Floor temp = 35℃

Mesh: 62 * 62 * 12

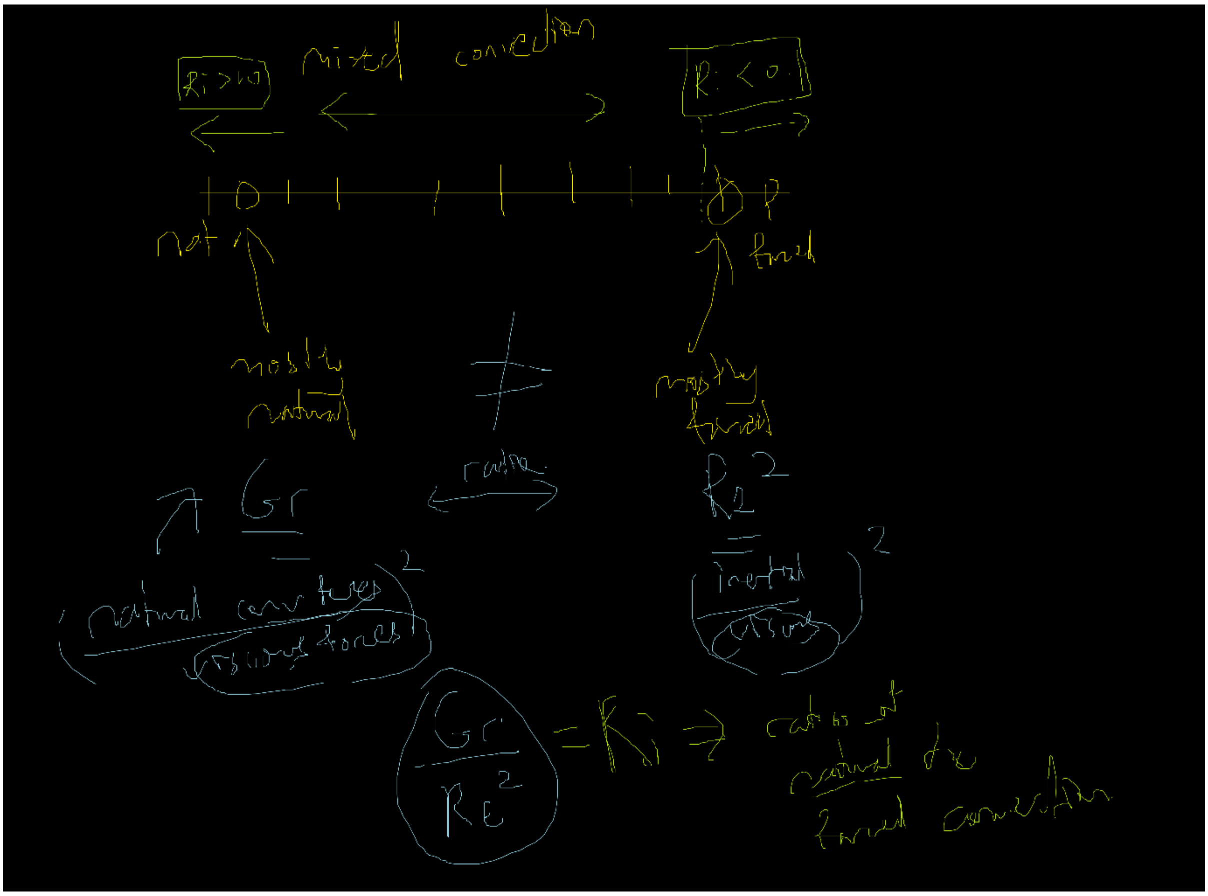
- Eg. Expt data shows 20℃ air temp, smagorinsky predicts 21℃ air temp. → ~5% temperature profile agreement? Taking 20℃ as reference
  - If you were to use kelvin, then the temperature is super close …. So which temperature scale is a good reference?
    - $\frac{1K}{273+20} = 0.3\%$
    - Which is correct? Do we use 5% or 0.3%?
  - Or you could say $(T_{max} - T_{min}) = 20℃$
    - 5% would be closer to correct.
- Eg2. Expt data shows $0.25\frac{m}{s}$, LES shows $0.20\frac{m}{s}$
  - $U_{max} - U_{min} = 0.5\frac{m}{s}$
  - Error % → 10% $= \frac{0.25-0.20}{0.5}$

Agreement is decent!

Turbulent statistics harder to match… → 20-40% off!

(natural convection is more dominant)

We can move on to opposing mixed convection:



WALE model tested for mixed convection (both aiding and opposing)(Ding et al., 2020).

$$Ri \approx 0.048$$

Compared friction Re_τ and Nu

$$Re_\tau = \frac{u_\tau \delta}{\nu}$$

For channel width, $\delta = \frac{1}{2}(channel\ width)$

You will see Re_τ pop up a lot here

How does Re_τ compare to our usual Re?

$$Re_m = \frac{u_m l}{\nu}$$

Our usual Re is Re_m, reynold's number based on mean velocity

In channel flow, $l = 2\delta$, $\delta = \frac{1}{2}(channel\ width)$

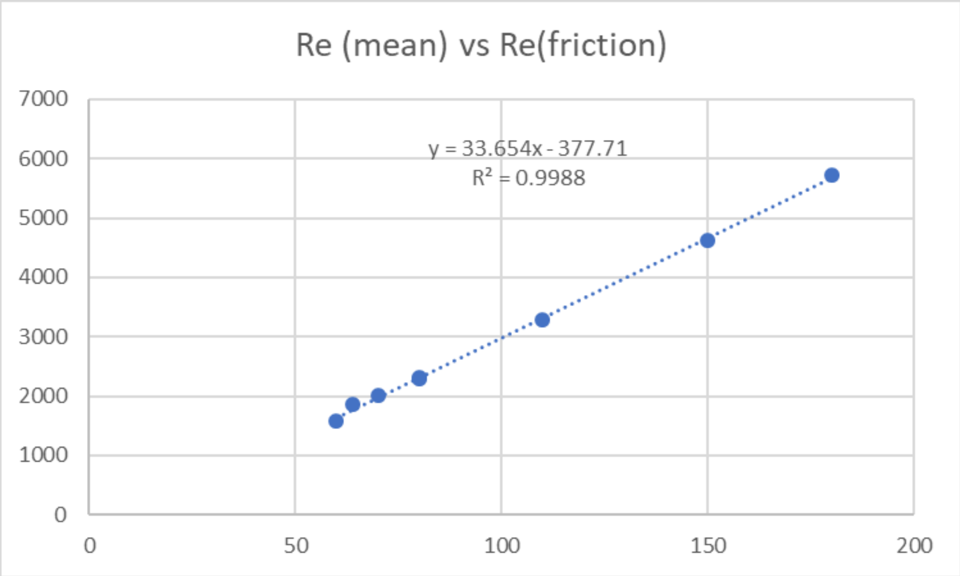
$$l = full\ channel\ width$$

There was data to compare Re_τ vs Re_m (Tsukahara, Seki, Kawamura, & Tochio, 2005)

Tsukahara, T., Seki, Y., Kawamura, H., & Tochio, D. (2005). DNS of turbulent channel flow at very low Reynolds numbers. In *TSFP Digital Library Online*. Begel House Inc..

Both of these are DNS data

Re_τ	60	64	70	80	80	110	150	180
Re_m	1580	1860	2010	2320	2290	3290	4620	5730



Here it shows:

$$Re_m = 33.654 * Re_\tau - 377.71$$

Or if you force intercept to zero, you get

$$Re_m = 30.409\, Re_\tau$$

$$R^2 = 0.9879$$

This formula gives you a rough idea how to estimate Re_τ from Re_m for channel flow

Different flow type, eg. Pipe flows, have different configurations.

Good thing about Re_τ is that you can compare this across all kinds of flow geometry… (almost though…)

- No argument as to what the characteristic velocity is, it is $u_\tau = \sqrt{\frac{\tau_{wall}}{\rho}}$
- Only need to define what the length scale is

$$Nu = \frac{hD}{k}$$

Compared to DNS

From Ding, (Ding et al., 2020)

Method	Skin friction C_f aiding mixed convection		*Skin friction C_f Opposing mixed convection*		Nu aiding mixed convection		Nu Opposing Mixed Convection	
DNS	0.00990	–	*0.00790*	–	7.42	–	20.94	–
WALE (coarse mesh → mesh is coarser than it needs to be)	0.00941	− 4.9%	*0.00690*	*− 12.6%*	6.60	− 10.8%	18.93	− 10.6%
IDDES	0.00973	− 1.7%	0.00694	− 12.1%	7.55	1.8%	18.58	− 11.3%
PL-DDES	0.01021	3.1%	0.00764	− 3.3%	7.62	2.7%	19.85	− 5.2%

$$\mathcal{P}r_t = 0.9$$

Note: RANS wall functions seem to work okay for aiding mixed convection, not so good for opposing mixed convection.

Conclusion

LES can in fact simulate natural convection and mixed convection flows. Preferably, we would like to have dynamically calculated coefficients and/or a transport equation eg. kEquation. It's best if $\mathcal{P}r_t$ is calculated dynamically but you can still get decent results if it were constant.

Detached Eddy Simulation (various types)

Based on the above, we can "guess" appropriate DES/DDES/IDDES values for Pr_t

Note that DES/DDES both use RANS in full BL.

Pr_t should be~ 0.85 for most flows there.

In free shear zone, depending on the grid size we adjust

- ~0.5 for LES style grid (fine grid)
- ~0.85 for RANS style grid (coarse grid)

Take note of Molecular Prandtl Number Dependence!

There are papers using DES models for rotating cooling channels (Viswanathan & Tafti, 2006) and also for dimpled surfaces (Turnow & Kornev, n.d.). In latter paper, $k - \omega\ SST\ IDDES$ is tested.

For IDDES, of dimpled flow, $\mathcal{P}r_t = 0.9$ (Turnow & Kornev, n.d.)(Turnow, Kasper, & Kornev, 2018) for Pr = 0.71, 3. Tested against empirical Petukhov Gnielinski correlation, found to be somewhat reasonable.

However, $\mathcal{P}r_t = 0.9$ causes temperature log layer mismatch in BL → .

Some ways to solve this: either

- use an intermediate value from 0.5-0.9, throughout
- or use adjust Pr in wallfunction and free shear separately. OpenFoam seems to be able to do this somehow.

Notes on above experiment (Ding et al., 2020):

$$Re_m = 4494$$

$$\mathcal{P}r = 0.71$$

$$Gr = 9.6 * 10^5 \ (Re\ equivalent\ for\ Natural\ convection)$$

Richardson number (https://en.wikipedia.org/wiki/Richardson_number wikipedia)

or better yet: https://www.sciencedirect.com/science/article/abs/pii/S0017931003005866

Oztop, H. F., & Dagtekin, I. (2004). Mixed convection in two-sided lid-driven differentially heated square cavity. International Journal of Heat and mass transfer, 47(8-9), 1761-1769.

$$Ri = \frac{Gr}{Re^2} = \frac{9.6e5}{4494^2} = 0.04753 \ (paper\ states\ 0.048)$$

Shows relative importance of natural convection in determining Nu

$$Re_\tau = 300 \ (DNS)$$

Why the discrepancy??

- In aiding mixed convection, turbulence is less than in opposing mixed convection
- You need a finer mesh to resolve turbulence in opposing mixed convection (Ding et al., 2020)

PL-DDES (production limited DDES) (Ding, Wang, & Chen, 2019) was built to counter this.

PL-DDES model for Temperature Log layer Mismatch

Tweaks IDDES for turbulent heat transfer especially for turbulent BL. (Ding et al., 2019).

Built on baseline k-omega SST model.

This is $k-\omega$ SST without production limiter

Recall that the difference between the baseline and SST version of the k-omega model is:

This limiter is used on P_k in the turbulent kinetic energy equation

$$P_k = \min\left(\tau_{ij}\frac{\partial u_i}{\partial x_j}, 10\,\beta^* k\omega^*\right)$$

$$\beta^*\omega^* k = \varepsilon$$

We can cap P_k at 10ε= 10x dissipation rate.

Instead of limiting production like so, an artificial function is used instead to clip turbulent viscosity.

$$\nu_t = Fr\left(\frac{k}{\omega^*}\right)$$

$$\omega = \omega^*\beta^*$$

$$\omega \equiv \frac{\varepsilon}{k}$$

Impose an artificial maximum on ν_t

For cell size dependent modelled stress depletion (MSD) a.k.a grid induced separation (GIS), shielding functions similar to DDES are used.

$$\nu_t = (1-f_d)Fr\left(\frac{k}{\omega^*}\right) + f_d\frac{k}{\omega^*}$$

$$0 \le Fr \le 1$$

$$Fr = \min\left(\frac{L_c}{L_t}, 1.0\right)$$

$$L_c = 0.2(\Delta x\Delta y\Delta z)^{\frac{1}{3}}$$

$$L_t = \frac{\sqrt{k}}{\beta^*\omega^*}$$

$$f_d = \max(f_{d1}, f_{d2})$$

$$f_{d1} = \min\{2\exp(-9r_1^2), 1.0\}$$

$$r_1 = 0.25 - \frac{d_w}{h_{max}}$$

$$f_{d2} = \tanh\{(14r_2)^3\}$$

Of course in f_{d2} you can calibrate the constants to something other than 14 and 3.

These help solve the thermal log layer mismatch. It’s a relatively new model, so not many CFD codes have it at this point (may 2020).

But good potential for natural convection turbulent heat trf with $\mathcal{P}r_t = 0.9$.

Near Wall Behaviour

For above experiment

T^+ profiles in $y^+ < 10$ are very well matched to DNS by IDDES model, WALE and PL DDES for both aiding and opposing flow.

T^+ profiles at $10 \leq y^+ \leq 60$ very well matched for all 3 models for aiding mixed convection

T^+ profiles for $10 \leq y^+ \leq 100$ overpredicted for all 3 models (max error 20-25%).

u^+ profiles well matched in shape for all models, max error 12.5% from DNS, for PL DDES, max error ~3%.

CFD implementation

In CFD, the h field is usually solved instead, so to leave the equation above is completely okay… esp since ρ and c_p are not always constant.

$$\rho c_p \frac{DT}{Dt} = -\nabla \cdot \overrightarrow{q''} + q''' + \mu\Phi + \beta T \frac{DP}{Dt}$$

$$\frac{\partial \bar{T}}{\partial t} + \bar{u}_i \frac{\partial}{\partial x_i} \bar{T} = (\alpha + \alpha_t)\nabla^2 \bar{T}$$

https://develop.openfoam.com/Development/openfoam

https://develop.openfoam.com/Development/openfoam/blob/master/applications/solvers/heatTransfer/buoyantPimpleFoam/EEqn.H

Note that here, this energy equation is the same regardless of turbulence model.

https://develop.openfoam.com/Development/openfoam/tree/master/src/TurbulenceModels/compressible/EddyDiffusivity

So in EddyDiffusivity.C line 30-43, the idea is that we have a constant Prt to calculate α_t at all places except for near boundary where in line 42, α_t is corrected in the boundary conditions.

In OpenFOAM, turbulent Prandtl number is fixed…

```
theodore_ong@LAPTOP-FK3MEHTI:/opt/OpenFOAM/OpenFOAM-v1912/src$ grep -r Prt
TurbulenceModels/compressible/EddyDiffusivity/EddyDiffusivity.C:    // Read Prt if provided
TurbulenceModels/compressible/EddyDiffusivity/EddyDiffusivity.C:    Prt_ = dimensionedScalar("Prt", dimless, 1.0, this->
coeffDict());
TurbulenceModels/compressible/EddyDiffusivity/EddyDiffusivity.C:    alphat_ = this->rho_*this->nut()/Prt_;
TurbulenceModels/compressible/EddyDiffusivity/EddyDiffusivity.C:    // Cannot read Prt yet
TurbulenceModels/compressible/EddyDiffusivity/EddyDiffusivity.C:    Prt_("Prt", dimless, 1.0),
TurbulenceModels/compressible/EddyDiffusivity/EddyDiffusivity.C:        Prt_.readIfPresent(this->coeffDict());
TurbulenceModels/compressible/EddyDiffusivity/EddyDiffusivity.H:            dimensionedScalar Prt_;
TurbulenceModels/compressible/RAS/buoyantKEpsilon/buoyantKEpsilon.H:    1/Prt coefficient is replaced by Cg to provide m
ore general control of
```

https://github.com/OpenFOAM/OpenFOAM-7/tree/master/src/TurbulenceModels/compressible/EddyDiffusivity

```
theodore_ong@LAPTOP-FK3MEHTI:/opt/openfoam7/src$ grep -r Prt
TurbulenceModels/compressible/EddyDiffusivity/EddyDiffusivity.C:    // Read Prt if provided
TurbulenceModels/compressible/EddyDiffusivity/EddyDiffusivity.C:    Prt_ = dimensioned<scalar>::lookupOrDefault
TurbulenceModels/compressible/EddyDiffusivity/EddyDiffusivity.C:        "Prt",
TurbulenceModels/compressible/EddyDiffusivity/EddyDiffusivity.C:    alphat_ = this->rho_*this->nut()/Prt_;
TurbulenceModels/compressible/EddyDiffusivity/EddyDiffusivity.C:    // Cannot read Prt yet
TurbulenceModels/compressible/EddyDiffusivity/EddyDiffusivity.C:    Prt_("Prt", dimless, 1.0),
TurbulenceModels/compressible/EddyDiffusivity/EddyDiffusivity.C:        Prt_.readIfPresent(this->coeffDict());
TurbulenceModels/compressible/EddyDiffusivity/EddyDiffusivity.H:            dimensionedScalar Prt_;
TurbulenceModels/compressible/RAS/buoyantKEpsilon/buoyantKEpsilon.H:    1/Prt coefficient is replaced by Cg to provide m
ore general control of
TurbulenceModels/compressible/turbulentFluidThermoModels/derivedFvPatchFields/wallFunctions/alphatWallFunctions/alphatJa
yatillekeWallFunction/alphatJayatillekeWallFunctionFvPatchScalarField.C:    Prt_(0.85)
TurbulenceModels/compressible/turbulentFluidThermoModels/derivedFvPatchFields/wallFunctions/alphatWallFunctions/alphatJa
yatillekeWallFunction/alphatJayatillekeWallFunctionFvPatchScalarField.C:    Prt_(ptf.Prt_)
TurbulenceModels/compressible/turbulentFluidThermoModels/derivedFvPatchFields/wallFunctions/alphatWallFunctions/alphatJa
```

We also have Jayatilleke's wall function here:

https://www.openfoam.com/documentation/guides/latest/api/compressible_2turbulentFluidThermoModels_2derivedFvPatchFields_2wallFunctions_2alphatWallFunctiobd40b4b476d66110b6c94657dff90953_source.html

Jayatilleke, C. L. V. (1966). The influence of Prandtl number and surface roughness on the resistance of the laminar sub-layer to momentum and heat transfer.

https://spiral.imperial.ac.uk/bitstream/10044/1/17357/2/Jayatilleke-CLV-1966-PhD-Thesis.pdf

On the calculation of heat transfer rates in fully turbulent wall flows

https://www.sciencedirect.com/science/article/pii/0307904X87901430

Malin, M. R. (1987). On the calculation of heat transfer rates in fully turbulent wall flows. *Applied mathematical modelling*, *11*(4), 281-284.

So you can access the above paper to see the Jayatilleke wall function (Malin, 1987). Another also shows that UMIST scheme is good for complex turbulent flows (Craft, Gant, Gerasimov, Iacovides, & Launder, 2006). → Jayatilleke wall function developed for forced convection, also used for jet flow.

The wall function is as follows:

$$u^+ = \frac{1}{\kappa} \ln y^+ + A$$

$$T^+ = \mathcal{P}r_{t(average)} \left[U^+ + 9.24 * \left(\left(\frac{\mathcal{P}r}{\mathcal{P}r_{t(average)}} \right)^{0.75} - 1 \right) \left(1 + 0.28 * \exp\left\{ -0.007 * \frac{\mathcal{P}r}{\mathcal{P}r_{t(average)}} \right\} \right) \right]$$

Takes into account variation with Viscous Sublayer

Used in OpenFOAM

alphatJayatillekeWallFunction

Is this suitable for natural convection and mixed convection? Nope! It was made for pipe flow:

LES Models in OpenFOAM for natural convection

Wang, Y., Chatterjee, P., & de Ris, J. L. (2011). Large eddy simulation of fire plumes. Proceedings of the Combustion Institute, 33(2), 2473-2480.

https://www.openfoam.com/documentation/guides/latest/api/dynamicKEqn_8C_source.html

http://glossary.ametsoc.org/wiki/Gradient_transport_theory

used in fire simulation(Y. Wang et al., 2011)

➔ Combustion/fireFoam/LES/compartmentFire

In the paper: fireFoam is used

- snappyHexMesh

Mesh setup

- smallest cells: 1.25*1.25*1.25 cm
- placed inside 60cm * 100cm * 60 cm box
- one equation kEqn SGS equation used
    - SGS scalar fluxes, eg temperature and concentration use gradient transport model and constant turbulent Prandtl number $\mathcal{P}r_t$
    - Gradient transport model: essentially $\overline{w'c'} = -K\frac{\partial \bar{c}}{\partial z}$
    - C is some concentration
    - http://glossary.ametsoc.org/wiki/Gradient_transport_theory

Just for DES

Btw DES blended divergence scheme here:

https://www.openfoam.com/documentation/guides/latest/doc/guide-schemes-divergence-deshybrid.html

➔ Doesn't quite work for heat transfer though!

Checks with Experiments: Forced Convection

For channel flow with dimples (Turnow & Kornev, n.d.), $k - \omega\ SST\ IDDES$ is tested against empirical Petukhov Gnielinski correlation. $\mathcal{P}r_t = 0.9$

Checks with Experiments: Natural Convection

Aiding and mixed convection flow… (Ding et al., 2020) $\mathcal{P}r_t = 0.9$ compared against DNS.

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