Real Analysis

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1 Cauchy's limit theorem

Let $\varepsilon > 0$ and $N \in \mathbb{N}$ such that

$$|a_k - a| \leq \frac{\varepsilon}{2}$$

for all $k \geq N$. Due to

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{N} (a_k - a) = 0,$$

there exists an $M \in \mathbb{N}$ such that

$$\left| \frac{1}{n} \sum_{k=1}^{N} (a_k - a) \right| \leq \frac{\varepsilon}{2}$$

for all $n \geq M$.
Now for all $n \geq \max(N, M)$, we have:

$$\begin{aligned} \left| \frac{1}{n} \left(\sum_{k=1}^{n} a_k \right) - a \right| = \left| \frac{1}{n} \sum_{k=1}^{n} (a_k - a) \right| &= \left| \frac{1}{n} \sum_{k=1}^{N} (a_k - a) + \frac{1}{n} \sum_{k=N+1}^{n} (a_k - a) \right| \\ &\leq \left| \frac{1}{n} \sum_{k=1}^{N} (a_k - a) \right| + \frac{1}{n} \sum_{k=N+1}^{n} |a_k - a| \\ &\leq \frac{\varepsilon}{2} + \frac{(n-N)\varepsilon}{2n} \leq \varepsilon. \end{aligned}$$

References

[1] Tyler Perlman, The Development of the Hardy Inequality `https://www.sas.rochester.edu/mth/undergraduate/honorspaperspdfs/tylerperlman2021.pdf`