Claim.

Write $P(p) \equiv \bar{p}(p)$ so I can type fast.

Suppose $P(p)$ is increasing from $P(0) = k \in (0,1)$ to $P(1) = 1$. Then on $(0,1)$:

$$LHS(p) \equiv \frac{P(p) - p}{1 - p} \text{ is decreasing in } p \text{ if } P(p) \text{ is strictly convex.}$$

Proof.

$P(p)$ is convex on $(0,1)$ if and only if for all $0 < p_1 < p_2 < 1$ and $a \in (0,1)$,

$$P(ap_1 + (1-a)p_2) < aP(p_1) + (1-a)P(p_2).$$

For arbitrary p_1 and $a \in (0,1)$, take $p_2 = 1$ and write $p_3 = ap_1 + (1-a)$.

Now write $q = aP(p_1) + (1-a)$,i.e., q is the point on the line segment between $P(p_1)$ and 1. Then

$$\begin{aligned}
&\frac{q - p_3}{1 - p_3} \\
=&\frac{aP(p_1) + (1-a) - [ap_1 + (1-a)]}{1 - [ap_1 + (1-a)]} \\
=&\frac{aP(p_1) - ap_1}{a(1 - p_1)} \\
=&\frac{P(p_1) - p_1}{1 - p_1}.
\end{aligned}$$

Last, observe that by convexity of P,

$$\begin{aligned}
P(p_3) &< aP(p_1) + (1-a)P(p_2) \\
&= aP(p_1) + (1-a)P(1) \\
&= aP(p_1) + (1-a) \\
&= q.
\end{aligned}$$

Putting these together

$$\frac{P(p_3) - p_3}{1 - p_3} < \frac{P(p_1) - p_1}{1 - p_1}$$

and because p_1 and $p_3 > p_1$ were arbitrary we have proved that $LHS(p)$ is decreasing.