THE CONFIGURATION SPACES OF FINITE REPRESENTATION TYPE ALGEBRAS

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1. Introduction

Let Λ be a finite-dimensional algebra over $\mathbb{C}$, of finite representation type. We introduce and study two affine varieties associated to Λ, $\mathcal{U}_\Lambda$ and $\widetilde{\mathcal{M}}_\Lambda$. These are defined by polynomial equations in variables, u_X, labelled by the indecomposable objects, X, in a certain category closely related to the module category. We also study $\mathcal{U}_\Lambda^{\geq 0}$, the real, non-negative points of $\mathcal{U}_\Lambda$. This has a boundary stratification, with strata that are themselves the non-negative parts of the $\mathcal{U}$-varieties for Jasso reductions of Λ. for the real, non-negative solutions to the u-equations.

The u-equations associated to a Dynkin type A_n quiver were first written down by physicists Koba and Nielsen in 1969 [**?**, **?**, **?**]. They played a key role in the first definition of the tree-level string amplitude, which is an integral over $\mathcal{U}_{A_n}^{\geq 0}$. Koba and Nielsen showed that the boundary structure of $\mathcal{U}_{A_n}^{\geq 0}$ agrees with that of the n-dimensional associahedron. This property captures the important physical property of 'factorisation'.

Recently, Arkani-Hamed, He, and Lam [4] showed that similar equations can be defined for any Dynkin quiver, Q, with n vertices. For simplicity in this introduction, take Q to be simply-laced. These authors introduced systems of $\widehat{F}$-equations, and showed that the variety they define, $\widetilde{\mathcal{M}}_{\mathbb{C}Q}$, is irreducible. They also defined certain rational functions $v_X \in \mathbb{C}(y_1, \dots, y_n)$, and showed that they provide a rational parametrisation of $\widetilde{\mathcal{M}}_{\mathbb{C}Q}$. They further showed that

$$\mathcal{U}_{\mathbb{C}Q} = \widetilde{\mathcal{M}}_{\mathbb{C}Q} \supset \mathcal{M}_{\mathbb{C}Q}^{\geq 0},$$

and that the boundary stratification of $\mathcal{M}_{\mathbb{C}Q}^{\geq 0}$ is that of the generalized associahedron of type Q.

In this paper, we give a generalisation of these results, beyond the Dynkin quivers, to all algebras of finite representation type, including non-hereditary algebras. For any such algebra, Λ, we define and study the u-equation varieties. We find that they continue to enjoy the properties considered by Koba and Nielsen, and Arkani-Hamed, He, and Lam. In particular, $\widetilde{\mathcal{M}}_\Lambda$ is irreducible, and

$$\mathcal{U}_\Lambda \supseteq \widetilde{\mathcal{M}}_\Lambda \supset \mathcal{M}_\Lambda^{\geq 0}.$$

The boundary strata of the positive part, $M_\Lambda^{\geq 0}$, are themselves the u-equation varieties for the Jasso reductions of Λ. And we also show that the varieties behave well under taking algebra quotients. These results show that the u-equation varieties are not special features of hereditary algebras. These varieties, and their positive parts, are a fascinating meeting point between algebraic geometry, tropical geometry, and representation theory. This paper advances our understanding of these intimate connections for the finite representation type case. These results may well extend naturally to all finite-dimensional algebras. To this end, results for locally gentle algebras will be presented elsewhere.

1.1. **Summary of main results.** We restrict our algebra Λ to be representation finite. As we shall see, in this setting, we can make a number of surprising statements.

(1) The boundary structure of $\mathcal{M}_\Lambda^{\geq 0}$ is dual to the g-vector fan of Λ.

(2) For M a rigid indecomposable in K_Λ, the boundary of $\mathcal{M}_\Lambda^{\geq 0}$ where $u_M = 0$ (dual to the ray in the g-vector fan corresponding to M), equals $\mathcal{M}_{B_M}^{\geq 0}$, where B_M is the Jasso reduction of Λ at M.
(3) Functions $v_M \in \mathbb{C}(y_1, \ldots, y_n)$ can be defined in this generality, as a ratio of F-polynomials, and provide a rational parametrization of $\widetilde{\mathcal{M}}_\Lambda$, while restricting to $y_i \geq 0$ provides a rational parametrization of $\mathcal{M}_\Lambda^{\geq 0}$.
(4) $\widetilde{\mathcal{M}}_\Lambda$ can be identified as an affine open set inside the toric variety associated to the g-vector fan of Λ.
(5) Dilogarithm identities: the sum of the Rogers dilogarithms of the v_M is constant (i.e., independent of $y_1, \ldots, y_n$).
(6) Functoriality under quotients: if A is a quotient of Λ, then there is a natural surjective map from $\widetilde{\mathcal{M}}_\Lambda$ to $\widetilde{\mathcal{M}}_A$, and this map agrees with the blowdown map of the toric varieties associated to the g-vector fans of Λ and A.

Properties (1)–(4) are analogues of results established in the Dynkin setting in [4], while property (5) was established, again in the Dynkin setting, in [12]. Property (6) is new.

The rational functions v_M have an interest in their own right. For M a rigid indecomposable in K_Λ, we have that the tropicalization of v_M, evaluated at $g \in \mathbb{Z}^n$, tells us the multiplicity of M as a summand of a generic presentation of weight g. This extends to arbitrary finite-dimensional algebras the recent result of Kus–Reineke [24] for Dynkin quivers. (For this one result, we drop our running assumption that our algebras are of finite representation type.)

At the moment, there is one obvious question which we are not able to resolve:

Question 1.1. *Is* $\mathcal{U}_\Lambda = \widetilde{\mathcal{M}}_\Lambda$*?*

Our view is that, since $\widetilde{\mathcal{M}}_\Lambda$ is irreducible and contains the region $\mathcal{M}_\Lambda^{\geq 0}$ (whose boundary structure is analogous to that relevant for Koba and Nielsen), $\widetilde{\mathcal{M}}_\Lambda$ is to be preferred to $\mathcal{U}_\Lambda$, and we make it the focus of our analysis. Nonetheless, it would be satisfying to have a resolution, either positive or negative, to Question 1.1.

1.2. **Introduction from a cluster algebras point of view.** The results of [4] are couched in the language of cluster algebras. Here we give a summary of their results, in order to clarify how our work is related to theirs, and to explain how our results generalise their results to our larger setting. However, a reader for whom cluster algebras are unfamiliar (or unappealing) can safely skip this discussion. We do not otherwise make use of cluster algebras in this paper.

Fix a Dynkin type Φ (not necessarily simply-laced) of rank n, and choose an orientation of the edges of the Dynkin diagram. This determines a skew-symmetrizable $n \times n$ exchange matrix B. Further rows can be added to B if desired, obtaining a matrix $\widetilde{B}$ from which a cluster algebra $\mathcal{A}(\widetilde{B})$ is then defined. One way to add further rows provides the cluster algebra of type Φ with *universal coefficients*; this will be particularly relevant for us, but much of the structure of $\mathcal{A}(\widetilde{B})$ depends only on B (or Φ), not $\widetilde{B}$.

The cluster variables of $\mathcal{A}(\widetilde{B})$ are indexed by $\Phi_{\geq -1}$, the *almost positive roots*. For $\gamma, \beta \in \Phi_{\geq -1}$, there is a *compatibility degree*, denoted $(\gamma \parallel \beta) \in \mathbb{Z}_{\geq 0}$.

The system of u-equations, as presented in [4], can then be described as follows: we introduce a variable u_γ for each $\gamma \in \Phi_{\geq -1}$. For each $\gamma \in \Phi_{\geq -1}$, we also have an equation

$$u_\gamma + \prod_{\beta \in \Phi_{\geq -1}} u_\beta^{(\beta \parallel \gamma)}.$$

In type A_n, the almost positive roots can be identified with the diagonals of an $(n+3)$-gon, and the compatibility degree $(\beta \parallel \gamma)$ is 1 if β and γ cross in their interiors, and zero otherwise. In this way, the u-equations of Koba–Nielsen can be recovered.

From the same starting point (an oriented Dynkin diagram), [4] also consider another system of equations, which we call the $\widehat{F}$ equations. For $\beta \in \Phi_{\geq -1}$, let x_β be the corresponding variable in the cluster algebra with universal coefficients, where the coefficient variables, also indexed by $\Phi_{\geq -1}$

are denoted u_β. For a negative simple root $-\alpha_i$, the corresponding $x_{-\alpha_i}$ is an initial variable. The polynomial $\widehat{F}_\beta$ is then given by

$$\widehat{F}_\beta = x_\beta|_{x_{-\alpha_i}\leftarrow 1}.$$

That is to say, $\widehat{F}_\beta$ is obtained from x_β by setting the initial cluster variables equal to 1. This is analogous to the procedure for defining the F-polynomials from [?]; there, however, one starts with the cluster algebra with principal coefficients, rather than the cluster algebra with universal coefficients, before setting the initial cluster variables to one. Because the cluster algebra with principal coefficients in some sense controls all cluster algebras, it is possible to calculate $\widehat{F}_\beta$ from the corresponding F-polynomial F_β. The $\widehat{F}$-equations are then that $\widehat{F}_\beta = 1$ for $\beta \in \Phi_{>0}$. (The $\widehat{F}$ polynomials corresponding to negative simple roots are identically equal to 1, so they do not yield any corresponding equation.)

It is shown in [4] that in this setting the u-equations and the $\widehat{F}$-equations define the same ideal. Also, it is shown how to give a rational parametrization of the variety cut out by this ideal. Essentially, u_γ can be described as (a slight transformation of) a certain $\mathcal{X}$-cluster variable, in the sense of Fock and Goncharov.

More precisely, suppose that x_γ and x_δ are two cluster variables of $\mathcal{A}(\widetilde{B})$ related by mutation. We then have an "exchange relation" $x_\gamma x_\delta = M_1 + M_2$, where M_1 and M_2 are two monomials in the remaining variables in the cluster, plus coefficients. Then M_2/M_1 is an $\mathcal{X}$-cluster variable.

Only a subset of the $\mathcal{X}$-cluster variables will be relevant for our purposes. Specifically, these the cluster variables which correspond to a mutation at a vertex i such that all the non-zero entries B_{ij} have the same sign (or, in the quiver language, a mutation at a source or sink). For any $\gamma \in \Phi_{\geq 0}$, it turns out that there is a unique $\delta \in \Phi_{\geq 0}$ such that, in any cluster containing x_γ in which it mutates to x_δ, the entries of the B-matrix in the row corresponding to γ are all positive. For such a mutation, one of the monomials in the exchange relation, say M_2, will have no mutable cluster variables appearing in it. Then, provided $\widetilde{B}$ was full rank (which can always be accomplished by adding sufficiently many rows), we can set

$$u_\gamma = \frac{M_1}{x_\gamma x_\delta} = \frac{1}{1 + M_2/M_1},$$

and this provides a rational parametrization of the variety cut out by the u-equations or the $\widehat{F}$-equations.

In the present paper, we greatly generalize these results. Starting from an arbitrary algebra of finite representation type Λ, there is no corresponding cluster algebra. There are F-polynomials, as introduced by Derksen–Weyman–Zelevinsky. We introduce a suitable analogue of F-polynomials with universal coefficients, which we denote $\widehat{F}$, allowing us to write down $\widehat{F}$-equations in our setting.

Further, we can give a rational parametrization of the variety cut out by the $\widehat{F}$-equations as a ratio, not of cluster variables, as in [4], but of F-polynomials.

2. Algebra background

Let Λ be an associative $\mathbb{C}$-algebra, finite-dimensional over $\mathbb{C}$. Let $e_1, \ldots, e_n$ be a complete set of pairwise orthogonal primitive idempotents of Λ. Then, as a right Λ-module, Λ is a direct sum of the indecomposable projective modules: $P_1 = e_1\Lambda, \ldots, P_n = e_n\Lambda$. We assume that Λ is *basic*; i.e. the P_i are pairwise non-isomorphic. We write $\operatorname{mod}\Lambda$ for the abelian category of finite-dimensional right Λ-modules, and $\operatorname{proj}\Lambda$ and $\operatorname{inj}\Lambda$ for the full additive subcategories of projective and injective modules, respectively. We write $S_i = P_i/\operatorname{rad} P_i$ for the *simple* right Λ modules. The indecomposable injective modules, I_i, are the injective envelopes of S_i. In particular, note that the socle of I_i is $\operatorname{soc} I_i = S_i$.

We write $\hom_\Lambda(\ ,\ )$ for the dimension of $\operatorname{Hom}_\Lambda(\ ,\ )$ over $\mathbb{C}$, and similarly $\hom_{\mathcal{C}}(\ ,\ )$ for the dimension of $\operatorname{Hom}_{\mathcal{C}}(\ ,\ )$ in a category $\mathcal{C}$. Similarly, we write $\operatorname{ext}^1_\Lambda(\ ,\ )$ for the dimension of $\operatorname{Ext}^1_\Lambda(\ ,\ )$. We write DV for the $\mathbb{C}$-linear dual vector space of a $\mathbb{C}$-vector space V.

2.1. **Almost split morphisms.** We briefly recall some key elements of Auslander–Reiten theory. The reader interested in further details is invited to consult [6, Chapter IV].

Consider a short exact sequence, $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$, in $\operatorname{mod}\Lambda$. It is *split* if it is isomorphic to $0 \to L \to L \oplus N \to N \to 0$. In a split short exact sequence, f is called a *section* and g is called a *retraction*. Obviously, in a split exact sequence, any map $h : Z \to N$ can be lifted to a map $h' : Z \to M$ such that $h = gh'$. Under the same hypothesis, any map $j : L \to Z$ can be extended to a map $j' : M \to Z$ with $j = j'f$.

An important class of short exact sequences are the *almost split* or *Auslander–Reiten* sequences. They are very close to sharing the above lifting/extension properties. For N indecomposable, a map $g : M \to N$ is *right almost split* if it is not a retraction, and for any indecomposable module Z and non-isomorphism $h : Z \to N$, there exists a map $h' : Z \to M$ such that $h = gh'$. Similarly, for L indecomposable, a map $f : L \to M$ is *left almost split* if it is not a section, and for any indecomposable module Z and non-isomorphism $j : Z \to L$, there exists a map $j' : M \to Z$ with $j = j'f$.

A short exact sequence $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ is *almost split* if and only if N is indecomposable and f is left almost split, or equivalently if and only if L is indecomposable and g is right almost split. [6, Theorem IV.1.13]. Any non-projective indecomposable module N is the right-most term of an almost split sequence, $0 \to \tau N \to M \to N \to 0$, unique up to isomorphism [6, Theorem IV.3.1]. Similarly, any non-injective indecomposable L is the left-most term of an almost split sequence, $0 \to L \to M \to \tau^{-1}L \to 0$. This defines the *Auslander-Reiten translation*, τ, of $\operatorname{mod}\Lambda$.

Note that the almost split sequences provide a supply of left and right almost split maps (a right almost split map to every non-projective indecomposable, and a left almost split map from every non-injective indecomposable). We will also make use of right almost split maps into indecomposable projective modules and left almost split maps out of indecomposable injective modules. If P is indecomposable projective, the inclusion $\operatorname{rad} P \to P$ is right almost split. Moreover, if I is indecomposable injective, then the quotient $I \to I/\operatorname{soc} I$ is left almost split [6, Proposition IV.3.5].

The following proposition follows directly from the almost splitness of the maps mentioned above. We use this, in particular, in Section 4, where we show how these identities define a useful set of dual bases of the split Grothendieck group.

Proposition 2.1. *Suppose that $0 \to \tau N \to M \to N \to 0$ is an almost split sequence, and X is an indecomposable module. Then:*

(1) $\hom_\Lambda(X, \tau N) - \hom_\Lambda(X, M) + \hom_\Lambda(X, N) = 0$ *unless $X \simeq N$, in which case it equals* 1.
(2) $-\hom_\Lambda(X, \operatorname{rad} P_i) + \hom_\Lambda(X, P_i) = 0$ *unless $X \simeq P_i$, in which case it equals* 1.
(3) $\hom_\Lambda(N, X) - \hom_\Lambda(M, X) + \hom_\Lambda(\tau N, X) = 0$ *unless $X \simeq \tau N$, in which case it equals* 1.
(4) $\hom_\Lambda(I_i, M) - \hom_\Lambda(I_i/S_i, M) = 0$*, unless $M \simeq I_i$, in which case it is* 1.

The *injectively stable morphisms* from X to Y in $\operatorname{mod}\Lambda$, denoted $\overline{\operatorname{Hom}}_\Lambda(X, Y)$ are by definition $\operatorname{Hom}_\Lambda(X, Y)$ quotiented by the subspace of those morphisms factoring through an injective Λ-module. One of the Auslander–Reiten formulas [6, Theorem IV.2.13] says the following:

Theorem 2.2. $\operatorname{Ext}^1_\Lambda(X, Y) \simeq D\overline{\operatorname{Hom}}_\Lambda(Y, \tau X)$.

We will need the following lemma:

Lemma 2.3. *Let M be an indecomposable Λ-module. Then* $\operatorname{Ext}^1_\Lambda(M, S_i) \simeq D\operatorname{Hom}_\Lambda(S_i, \tau M)$.

Proof. By the Auslander–Reiten formula of Theorem 2.2, $\operatorname{Ext}^1_\Lambda(M, S_i) \simeq D\overline{\operatorname{Hom}}_\Lambda(S_i, \tau M)$. The only indecomposable injective which admits a non-zero morphism from S_i is I_i. An injection from I_i to τM splits, so in that case τM, being indecomposable, would have to be isomorphic to I_i, which is impossible. Thus, a map from S_i to τM would necessarily factor through a proper quotient of I_i, and thus a quotient of $I_i/\operatorname{soc} I_i$. The only possible non-zero image of S_i in I_i being $\operatorname{soc} I_i$, no non-zero map from S_i to τM factors through I_i, and therefore $D\overline{\operatorname{Hom}}_\Lambda(S_i, \tau M) \simeq D\operatorname{Hom}_\Lambda(S_i, \tau M)$. □

2.2. **The category K_Λ.** Let $K^b := K^b(\operatorname{proj}\Lambda)$ be the homotopy category of bounded complexes of projective modules. Write Σ for the shift functor of K^b, that shifts all degrees of an object in K^b by -1.

Definition 2.4. The category, $K_\Lambda := K^{[-1,0]}(\operatorname{proj}\Lambda)$, of 2-*term complexes of projectives*, is the full subcategory of K^b whose objects are complexes supported only in degrees -1 and 0. K_Λ is an extension-closed subcategory of $K^b(\operatorname{proj}\Lambda)$, and so is an extriangulated category in the sense of [25], by a result of [21]. As an extriangulated category, K_Λ has enough projective objects and enough injective objects.

For an object, X, in K_Λ, we write $X = (X^{-1} \to X^0)$, for projective modules X^{-1}, X^0 in $\operatorname{mod}\Lambda$. The indecomposable projective objects of K_Λ are $P_i = (0 \to P_i)$. (Note that we write P_i both for the projective indecomposables of $\operatorname{mod}\Lambda$ and of K_Λ.) The indecomposable injective objects are the shifted projectives, $\Sigma P_i = (P_i \to 0)$.

Definition 2.5. A *conflation* in K_Λ is a distinguished triangle of $K^b(\operatorname{proj}\Lambda)$ whose objects and morphisms are in K_Λ. Given such a distinguished triangle $X \to E \to Y \to \Sigma X$, we write $X \rightarrowtail E \twoheadrightarrow Y$ for the corresponding conflation. A conflation in K_Λ is split (resp. almost split) if it is split (resp. almost split) as a distinguished triangle in $K^b(\operatorname{proj}\Lambda)$.

We write $\mathcal{I}$ for the set of indecomposable objects of K_Λ. The category K_Λ has *Auslander–Reiten–Serre duality* [22]. In particular, for any non-projective indecomposable object $X \in \mathcal{I}$, there is an almost split conflation of the form $\tau X \rightarrowtail E \twoheadrightarrow X$. For any non-injective indecomposable object $Y \in \mathcal{I}$, there is an almost split conflation $Y \rightarrowtail F \twoheadrightarrow \tau^{-1}Y$. (Note that we write τ both for the Auslander-Reiten translation in $\operatorname{mod}\Lambda$ and in K_Λ.)

2.3. **The H^0 functor.** The category K_Λ is closely related to $\operatorname{mod}\Lambda$. Indeed, there is a functor

$$H^0 : K_\Lambda \to \operatorname{mod}\Lambda,$$

which sends $X = (X^{-1} \xrightarrow{f} X^0)$ to its cohomology in degree zero, $H^0(X) = X^0/\operatorname{Im}(f)$. In other words, we can identify objects $\operatorname{mod}\Lambda$ with their (minimal) projective presentations, which are objects of K_Λ. However, note that the injectives of K_Λ, ΣP_i, are not projective presentations of a non-zero module in $\operatorname{mod}\Lambda$, and $H^0(\Sigma P_i) = 0$. This leads to:

Lemma 2.6. *H^0 induces an equivalence of k-linear categories*

$$K_\Lambda/\langle\Sigma\Lambda\rangle \xrightarrow{\sim} \operatorname{mod}\Lambda.$$

Here, $\langle\Sigma\Lambda\rangle$ is the ideal of all morphisms that factor through an object of $\operatorname{add}(\Sigma\Lambda)$.

Lemma 2.7. *A conflation, $X \rightarrowtail E \twoheadrightarrow Y$, in K_Λ is split if and only if the corresponding exact sequence*

$$H^0X \to H^0E \to H^0Y$$

is a split short exact sequence in $\operatorname{mod}\Lambda$.

Moreover, H^0 relates the Auslander–Reiten translations of K_Λ and $\operatorname{mod}\Lambda$.

Lemma 2.8. *For a non-projective indecomposable, $X \in \mathcal{I}$, consider its Auslander-Reiten conflation in K_Λ, $\tau X \rightarrowtail E \twoheadrightarrow X$. Then*

1. *If X is non-injective, then this conflation is sent by H^0 to the Auslander-Reiten sequence in* $\operatorname{mod}\Lambda$,
$$0 \to H^0(\tau X) \to H^0E \to H^0X \to 0.$$
In particular, $H^0(\tau X) = \tau H^0X$.
2. *If X is injective in K_Λ, i.e. $X = \Sigma P_i$, then the above conflation is sent by H^0 to the exact sequence*
$$I_i \to I_i/S_i \to 0,$$
where I_i is the injective module in $\operatorname{mod}\Lambda$. *In particular, $H^0(\tau\Sigma P_i) = I_i$.*

(3) *Finally, applying Σ^{-1} to the Auslander-Reiten conflation for $X = \Sigma P_i$ gives a triangle in $K^b(\operatorname{proj}\Lambda)$:*

$$\Sigma^{-1}(\tau P_i) \rightarrowtail \Sigma^{-1} E \twoheadrightarrow P_i.$$

This is sent by H^0 to the exact sequence

$$0 \to \operatorname{rad} P_i \to P_i.$$

The following form of Auslander-Reiten duality relates morphisms in the module category and the homotopy category.

Lemma 2.9. *For indecomposables $X, Y \in \mathcal{I}$,*

$$\operatorname{Hom}_{K^b}(X, \Sigma Y) \cong D \operatorname{Hom}_\Lambda(H^0 Y, H^0(\tau X)),$$

In particular,

$$\hom_{K^b}(X, \Sigma Y) = \hom_\Lambda(H^0 Y,\, H^0(\tau X)).$$

Proof. For $X \neq \Sigma P_i$, we have $H^0(\tau X) = \tau H^0 X$ (Lemma 2.8) and the result can be proved using Serre duality and the Nakayama functor ν; a proof can be found in [27, Lemma 2.6]. For $X = \Sigma P_i$, we have $H^0(\tau X) = I_i$. Indeed, in this case we have $\operatorname{Hom}_{K^b}(X, \Sigma Y) \cong \operatorname{Hom}_\Lambda(P_i, H^0 Y)$. And $\operatorname{Hom}_\Lambda(P_i, H^0 Y) \simeq D \operatorname{Hom}_\Lambda(H^0 Y, I_i)$. □

2.4. **The g-vector fan.** A general reference for the material in this subsection is [1]. For each object X in K_Λ, we define a g-vector, $g(X)$, in $\mathbb{Z}^n$. If $X = (X^{-1} \to X^0)$, then the entry $g(X)_i$ of $g(X)$ is the multiplicity of P_i in X^0 minus the multiplicity of P_i in X^{-1}.

An object X in K_Λ is called *rigid* if $\operatorname{Hom}(X, \Sigma X) = 0$. If X is rigid, then P_i cannot appear in both X^{-1} and X^0 [1, Proposition 2.5]. So, in particular, if X is rigid and non-zero then $g(X) \neq 0$. There is at most one rigid object with any given g-vector [1, Theorem 5.5]. We say that two rigid objects X, Y in K_Λ are *compatible* if $\operatorname{Hom}_{K^b}(X, \Sigma Y) = 0$ and $\operatorname{Hom}_{K^b}(Y, \Sigma X) = 0$.

The g-vector fan in $\mathbb{Z}^n$ has rays generated by the g-vectors of indecomposable rigid objects. The cones of the g-vector fan are generated by the g-vectors of sets of pairwise compatible indecomposable rigid objects. As the name suggests, the g-vector fan is indeed a fan [14, Corollary 6.7]. In other words, two cones in the fan intersect either in another cone of the fan. Moreover, if Λ has only finitely many indecomposable rigid objects, then the g-vector fan covers $\mathbb{R}^n$ [5, Proposition 4.8].

2.5. **Dimension vectors.** For each object M in $\operatorname{mod}\Lambda$, define the *dimension vector*, $d(M) \in \mathbb{Z}^n$, with $d(M)_i = \dim(M e_i)$. For $X \in K_\Lambda$, we define its dimension vector as the dimension vector of $H^0(X)$, and write $d(X) = d(H^0(X))$. Write $\langle -, - \rangle$ for the standard inner product on $\mathbb{Z}^n$. Then, for $X \in K_\Lambda$ and $M \in \operatorname{mod}\Lambda$, we have the pairing

$$\langle g(X),\, d(M) \rangle = \sum_{i=1}^{n} g(X)_i\, d(M)_i.$$

If $X = (X^{-1} \to X^0)$, then

$$\langle g(X),\, d(M) \rangle = \hom_\Lambda(X^0, M) - \hom_\Lambda(X^{-1}, M). \tag{2.1}$$

The following is a variation on [8, Theorem 1.4a] (See also [1, Proposition 2.4(a)].) We include a short proof for the convenience of the reader.

Lemma 2.10. *For an indecomposable $X \in \mathcal{I}$, and $M \in \operatorname{mod}\Lambda$,*

$$\langle g(X), d(M) \rangle = \hom_\Lambda(H^0 X, M) - \hom_\Lambda(M,\, H^0(\tau X)).$$

Proof. If $X = \Sigma P_i$, $H^0 X = 0$ and $H^0(\tau X) = I_i$. But $\hom_\Lambda(P_i, M) = \hom_\Lambda(M, I_i)$. So the statement follows directly from (2.1). Now take X non-injective. We have $X^{-1} \xrightarrow{f} X^0 \xrightarrow{q} H^0 X \to 0$. Dually, there is a short exact sequence $0 \xrightarrow{\iota} \tau H^0 X \to \nu X^{-1} \xrightarrow{\nu f} \nu X^0$, where ν denotes the

Nakayama functor. Then, applying $\mathrm{Hom}_\Lambda(M,-)$ and $\mathrm{Hom}_\Lambda(-,M)$, respectively, we get the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}(M,\tau H^0X) \arrow[r, "\iota_*"] & \mathrm{Hom}(M,\nu X^{-1}) \arrow[r, "(\nu f)_*"] \arrow[d, leftrightarrow, "\simeq"] & \mathrm{Hom}(M,\nu X^0) \arrow[d, leftrightarrow, "\simeq"] & & \\ & & D\,\mathrm{Hom}(X^{-1},M) \arrow[r, "Df^*"] & D\,\mathrm{Hom}(X^0,M) \arrow[r, "Dq^*"] & D\,\mathrm{Hom}(H^0X,M) \arrow[r] & 0, \end{tikzcd}$$

which defines a four-term exact sequence. Exactness of this implies

$$\hom_\Lambda(M,\tau H^0X) - \hom_\Lambda(X^{-1},M) + \hom_\Lambda(X^0,M) - \hom_\Lambda(H^0X,M) = 0.$$

Then (2.1) gives the result. □

2.6. F**-polynomials.** Given a vector $\mathbf{d} \in \mathbb{Z}_{\geq 0}$, let $\mathrm{Gr}_{\mathbf{d}}(M)$ be the *submodule Grassmannian* of M. As a set, the elements of $\mathrm{Gr}_{\mathbf{d}}(M)$ are in bijection with submodules of M of dimension $\mathbf{d}$. It can be viewed as a Zariski-closed subset of the product $\prod_{i=1}^n \mathrm{Gr}_{d_i}(Me_i)$ of Grassmannians [11], and in this setting it is a projective variety.

Definition 2.11. Let M be a module in $\mathrm{mod}\,\Lambda$. Its F*-polynomial* is

$$F_M = \sum_{\mathbf{d}\in\mathbb{Z}_{\geq 0}} \chi\left(\mathrm{Gr}_{\mathbf{d}}(M)\right) y^{\mathbf{d}} \in \mathbb{Z}[y_1,\ldots,y_n],$$

where χ is the Euler characteristic of topological spaces and $y^{\mathbf{d}} = \prod_{i=1}^n y_i^{d_i}$. Moreover, for an object X of K_Λ, we define its F-polynomial as

$$F_X := F_{H^0X}.$$

2.7. $\widehat{F}$**-polynomials.** Let $\mathcal{I}$ be a set of representatives of isomorphism classes of indecomposable objects of K_Λ. We now define a second set of polynomials, $\widehat{F}_M$, in the ring $\mathbb{Z}[u_X \mid X \in \mathcal{I}]$. For this, it is important to assume the following

Assumption 2.12 (Finiteness assumption)**.** The set $\mathcal{I}$ of isomorphism classes of indecomposable objects of K_Λ is finite. Equivalently, Λ is of finite representation type.

We make this assumption for the rest of the paper.

Definition 2.13. Given a module M in $\mathrm{mod}\,\Lambda$, its $\widehat{F}$*-polynomial* is

$$\widehat{F}_M = \sum_{\mathbf{d}\in\mathbb{Z}_{\geq 0}} \chi(\mathrm{Gr}_{\mathbf{d}}(M)) \prod_{V\in\mathcal{I}} u_V^{\hom_\Lambda(H^0V,M)-\langle g(V),\mathbf{d}\rangle}.$$

And, for $X \in K_\Lambda$, we define

$$\widehat{F}_X := \widehat{F}_{H^0X}.$$

The $\widehat{F}$-polynomials are closely related to the F-polynomials by the following map from the y-variables to the u-variables.

Definition 2.14. Define a homomorphism $\Psi : \mathbb{C}[y_1,\ldots,y_n] \to \mathbb{C}[u_V^{\pm 1} \mid V \in \mathcal{I}]$ that maps

$$\Psi :\; y_i \longmapsto \prod_{V\in\mathcal{I}} u_V^{-g(V)_i}.$$

In particular, Ψ maps the monomial $y^{\mathbf{d}}$ to $\prod_V u_V^{-\langle g(V),\,\mathbf{d}\rangle}$. So

Lemma 2.15. *$\widehat{F}_M$ is given by*

$$\widehat{F}_M = \left(\prod_{V\in\mathcal{I}} u_V^{\hom_\Lambda(H^0V,\,M)}\right) \Psi(F_M).$$

We verify that

Lemma 2.16. *For any $M \in \mathrm{mod}\,\Lambda$, $\widehat{F}_M$ is indeed a polynomial. That is, $\widehat{F}_M \in \mathbb{Z}[u_V \mid V \in \mathcal{I}]$.*

Proof. Take any $\mathbf{d}$ with $\chi(\mathrm{Gr}_{\mathbf{d}}(M))$ non-zero. Take some submodule of M, L, of dimension $d(L) = \mathbf{d}$. Then we claim $\hom_\Lambda(H^0V, M) - \langle g(V), \mathbf{d}\rangle$ is non-negative for all $V \in \mathcal{I}$. Let $V = (V^{-1} \to V^0)$. Then we get an exact sequence

$$V^{-1} \to V^0 \to H^0V \to 0$$

in $\operatorname{mod}\Lambda$. Moreover, applying the functor $\mathrm{Hom}_\Lambda(-, L)$, gives the exact sequence of vector spaces,

$$0 \to \mathrm{Hom}_\Lambda(H^0V, L) \to \mathrm{Hom}_\Lambda(V^0, L) \to \mathrm{Hom}_\Lambda(V^{-1}, L).$$

But exactness implies that

$$\hom_\Lambda(H^0V, L) - \hom_\Lambda(V^0, L) + \hom_\Lambda(V^{-1}, L) \geq 0.$$

Since L is a submodule of M, $\hom_\Lambda(H^0V, M) \geq \hom_\Lambda(H^0V, L)$. And so, recalling also (2.1), $\hom_\Lambda(H^0V, M) - \langle g(V), \mathbf{d}\rangle \geq 0$. □

3. Varieties defined by $\widehat{F}$-equations and u-equations

3.1. **The varieties $\mathcal{M}_\Lambda$ and $\widetilde{\mathcal{M}}_\Lambda$.** The equations

$$\widehat{F}_X = 1, \tag{3.1}$$

for all $X \in \mathcal{I}$, define the affine varieties that are the main subject of this paper.

Definition 3.1. In the ring $\mathbb{C}[u_V^{\pm 1} \mid V \in \mathcal{I}]$, consider the ideal $I_\Lambda = \langle \widehat{F}_W - 1 \mid W \in \mathcal{I}\rangle$. Similarly, write $\widetilde{I}_\Lambda$ for the ideal $\widetilde{I}_\Lambda = \langle \widehat{F}_W - 1 \mid W \in \mathcal{I}\rangle$ in the polynomial ring $\mathbb{C}[u_V \mid V \in \mathcal{I}]$, generated by the same polynomials. This defines two affine varieties:

$$\begin{aligned} R_\Lambda &:= \mathbb{C}[u_X^{\pm 1} \mid X \in \mathcal{I}]/I_\Lambda & \qquad \widetilde{R}_\Lambda &:= \mathbb{C}[u_X \mid X \in \mathcal{I}]/\widetilde{I}_\Lambda \\ \mathcal{M}_\Lambda &:= \mathrm{Spec}(R_\Lambda) & \widetilde{\mathcal{M}}_\Lambda &:= \mathrm{Spec}(\widetilde{R}_\Lambda). \end{aligned}$$

Thus $\mathcal{M}_\Lambda$ is the localisation of $\widetilde{\mathcal{M}}_\Lambda$ along the divisors $u_V = 0$. In other words, $\mathcal{M}_\Lambda$ is obtained by removing the points where some $u_V = 0$. We emphasize that, in the case that Λ is the path algebra of the Dynkin quiver A_{n-3}, $\widetilde{\mathcal{M}}_\Lambda$ is *not* the Deligne-Mumford compactification of the moduli space of points on $\mathbb{P}^1$, $\overline{\mathcal{M}}_{0,n}$. The latter is a projective variety, whereas $\widetilde{M}_\Lambda$ is an affine open in $\overline{\mathcal{M}}_{0,n}$. To avoid confusion, we therefore use a tilde, and not a bar, to denote these affine varieties. Note that these affine varieties are studied by Brown in [9].

3.2. **The u-equations and configuration spaces.** The varieties in Definition 3.1 generalize the cluster configuration spaces of finite type of [4], along with other varieties previously considered in the physics and mathematics literature [**?**, **?**, **?**, **?**]. We now show how the varieties defined by the equations (3.1) are related to another set of equations called the u-equations.

Definition 3.2. Under the finiteness assumption 2.12, the *u-equation* of an object $X \in \mathcal{I}$ is the following equation in the polynomial ring $\mathbb{Z}[u_V \mid V \in \mathcal{I}]$:

$$u_X + \prod_{Y \in \mathcal{I}} u_Y^{c(X,Y)} = 1,$$

where the *compatibility degree* of $X, Y \in \mathcal{I}$ is

$$c(X, Y) = \hom_{K_\Lambda}(X, \Sigma Y) + \hom_{K_\Lambda}(Y, \Sigma X).$$

The u-equations define an affine variety

$$\widetilde{\mathcal{U}}_\Lambda = \mathrm{Spec}(\widetilde{S}_\Lambda),$$

where $\widetilde{S}_\Lambda := \mathbb{C}[u_X \mid X \in \mathcal{I}]/\langle u\text{-equations}\rangle$.

Definition 3.3. The configuration space, $\widetilde{\mathcal{U}}_\Lambda^{\geq 0}$, is the semialgebraic set of real, non-negative points of $\widetilde{\mathcal{U}}_\Lambda$.

Finally, the exponents $c(X, Y)$ appearing in the u-equations can be written in terms of the module category $\operatorname{mod}\Lambda$ as follows.

Lemma 3.4. *For two $X, Y \in \mathcal{I}$, the compatibility degree $c(X,Y)$ is also given by*

$$c(X,Y) = \hom_\Lambda(H^0X, H^0(\tau Y)) + \hom_\Lambda(H^0Y,\, H^0(\tau X)),$$

or, equivalently, by

$$c(X,Y) = -\langle g(X), d(H^0Y)\rangle + \hom_\Lambda(H^0X, H^0Y) + \hom_\Lambda(H^0X, H^0(\tau Y)).$$

Proof. Recall from Lemma 2.9 that $\hom_{K_\Lambda}(Y, \Sigma X) = \hom_\Lambda(H^0X, H^0(\tau Y))$. The second formula follows from Lemma 2.10. □

3.3. **Examples.**

3.3.1. *Dynkin types A_n, D_n, E_6, E_7 and E_8.* If the algebra Λ is the path algebra $\mathbb{C}Q$ of a quiver Q of Dynkin type A, D or E, then the u-equations defining $\widetilde{\mathcal{U}}_\Lambda$ are the same as the ones defining the variety $\widetilde{\mathcal{M}}$ of Arkani-Hamed, He and Lam [4]. This is a consequence of the additive categorification of cluster algebras, which was worked out for Dynkin quivers in [11]. In particular, for type A_{n-3}, we recover the variety $\widetilde{\mathcal{M}}_{0,n}$ of Brown [9]. We work out types A_3 and D_4 in detail in Section 13.

3.4. **The varieties $\mathcal{M}$ and $\mathcal{U}$.**

The main result of this section is that the $\widehat{F}$-equations (3.1) imply the u-equations. In other words, $\widetilde{\mathcal{U}}_\Lambda \supseteq \widetilde{\mathcal{M}}_\Lambda$. The main tool in order to prove this is a kind of "exchange relation" for the $\widehat{F}$-polynomials.

Proposition 3.5. *Let $X \in \mathcal{I}$.*

1. *If X is neither projective nor injective, with almost split conflation $\tau X \rightarrowtail E_X \twoheadrightarrow X$, then $\widehat{F}_X \widehat{F}_{\tau X} = u_X \widehat{F}_{E_X} + \prod_{W\in\mathcal{I}} u_W^{c(W,X)}$.*
2. *If $X = P_i$, then $\widehat{F}_X = u_X \widehat{F}_{\operatorname{rad} P_i} + \prod_{W\in\mathcal{I}} u_W^{c(W,X)}$.*
3. *If $X = \Sigma P_i$, then $\widehat{F}_{\tau X} = u_X \widehat{F}_{I_i/S_i} + \prod_{W\in\mathcal{I}} u_W^{c(W,X)}$.*

From Proposition 3.5, it follows immediately that

Corollary 3.6. *If $\widehat{F}_X = 1$ for all $X \in \mathcal{I}$, then $u_X + \prod_{W\in\mathcal{I}} u_W^{c(W,X)} = 1$ for all $X \in \mathcal{I}$.*

To prove Proposition 3.5, we need some relations between different F-polynomials.

Lemma 3.7. *For an indecomposable projective $P_i \in \operatorname{mod}\Lambda$, $F_{P_i} = F_{\operatorname{rad} P_i} + y^{d(P_i)}$. For an indecomposable injective $I_i \in \operatorname{mod}\Lambda$, $F_{I_i} = y_i F_{I_i/S_i} + 1$.*

Proof. If a submodule of P is not P itself, then it is a submodule of the radical, $\operatorname{rad} P$. I_i contains the zero submodule, and all other submodules contain $\operatorname{soc} I_i = S_i$. □

Lemma 3.8. *For $V \rightarrowtail X \twoheadrightarrow W$ a conflation in K_Λ:*

1. *if the conflation is split, then $F_V F_W = F_X$,*
2. *if the conflation is almost split, and W is not injective, then $F_V F_W = F_X + y^{d(W)}$.*

Proof. Recall that for $X \in K_\Lambda$, $F_X := F_{H^0X}$. The identities then follow from the corresponding identities in $\operatorname{mod}\Lambda$, together with Lemma 2.8, which relates short exact sequences in $\operatorname{mod}\Lambda$ and conflations in K_Λ. Indeed, (1) holds for split short exact sequences in $\operatorname{mod}\Lambda$ by [28, Prop. 2.13] (see also [11, 16]). And, by Lemma 2.7, H^0 sends a split conflation to a split short exact sequence in $\operatorname{mod}\Lambda$. Whereas (2) holds for an almost split short exact sequence in $\operatorname{mod}\Lambda$, which is the main result of [16]. And, since W is non-injective, H^0 sends an almost split conflation to an almost split short exact sequence (Lemma 2.8). □

Proof of Proposition 3.5. (1) Take $X \in \mathcal{I}$ neither projective nor injective. By Lemma 3.8(2), we have $F_X F_{\tau X} = F_{E_X} + y^{d(X)}$. We can turn this into a relation among $\widehat{F}$-polynomials. Using the homomorphism Ψ and Lemma 2.15,

$$\widehat{F}_X \widehat{F}_{\tau X} = \Psi(F_{E_X}) \prod_{W\in\mathcal{I}} u_W^{\hom_\Lambda(H^0W,H^0X)+\hom_\Lambda(H^0W,\tau H^0X)} + \prod_{W\in\mathcal{I}} u_W^{-\langle g(W),d(X)\rangle+\hom_\Lambda(H^0W,H^0X)+\hom_\Lambda(H^0W,\tau H^0X)}$$

The first term on the right hand side is equal to $u_X\widehat{F}_{E_X}$, by Proposition 2.1, which implies that

$$\hom_\Lambda(H^0W,H^0X)+\hom_\Lambda(H^0W,\tau H^0X) = \begin{cases} \hom_\Lambda(H^0W,H^0E_X) & \text{if } X\neq W \\ \hom_\Lambda(H^0W,H^0E_X)+1 & \text{if } X=W. \end{cases}$$

And the second term is equal to $\prod_{W\in\mathcal{I}} u_W^{c(W,X)}$, by Lemma 3.4.

(2) Take $X = P_i$. By Lemma 3.7, $F_X = F_{\operatorname{rad} H^0X} + y^{d(X)}$ and, using Ψ and Lemma 2.15,

$$\widehat{F}_X = \Psi(F_{\operatorname{rad} H^0X}) \prod_{W\in\mathcal{I}} u_W^{\hom_\Lambda(H^0W,H^0X)} + \prod_{W\in\mathcal{I}} u_W^{-\langle g(W),d(X)\rangle+\hom_\Lambda(H^0W,H^0X)}.$$

The first term is equal to $u_X\widehat{F}_{\operatorname{rad} P_i}$, by Proposition 2.1. The second term is equal to $\prod_{W\in\mathcal{I}} u_W^{c(W,X)}$, by Lemma 3.4, noting that $\tau H^0X = 0$.

(3) Take $X = \Sigma P_i$ injective. By Lemma 3.7, $F_{\tau X} = y_iF_E + 1$, where $H^0(\tau X) = I_i$ and $H^0E = I_i/S_i$. Using Ψ and Lemma 2.15, this becomes

$$\widehat{F}_{\tau X} = \Psi(F_E) \prod_{W\in\mathcal{I}} u_W^{\hom_\Lambda(H^0W,H^0(\tau X))-g(W)_i} + \prod_{W\in\mathcal{I}} u_W^{\hom_\Lambda(H^0W,I_i)}.$$

The second term is equal to $\prod_{W\in\mathcal{I}} u_W^{c(W,X)}$, by Lemma 3.4, noting that $H^0X = 0$. The first term is equal to $u_X\widehat{F}_E$, because

$$\hom_\Lambda(H^0W,H^0(\tau X)) - g(W)_i = \begin{cases} \hom_\Lambda(H^0W,H^0E) & \text{if } X\neq W \\ \hom_\Lambda(H^0W,H^0E)+1 & \text{if } X=W. \end{cases} \tag{3.2}$$

This follows from the short exact sequence $0\to S_i\to I_i\to I_i/S_i\to 0$, which gives

$$0\to \operatorname{Hom}_\Lambda(H^0W,S_i)\to \operatorname{Hom}_\Lambda(H^0W,I_i)\to \operatorname{Hom}_\Lambda(H^0W,I_i/S_i)\to \operatorname{Ext}^1(H^0W,S_i)\to 0.$$

But, by Lemma 2.3, $\operatorname{ext}^1(M,S_i) = \hom(S_i,\tau M)$, so that

$$\hom_\Lambda(H^0W,I_i/S_i) - \hom_\Lambda(H^0W,I_i) = \hom_\Lambda(H^0W,S_i) - \hom_\Lambda(S_i,\tau H^0W),$$

which implies (3.2), by Lemma 2.10) and observing that $H^0(\Sigma P_i) = 0$, but $g(\Sigma P_i)_i = -1$. □

We can prove a converse to Corollary 3.6 in case all indecomposable objects of K_Λ are in the τ-orbits of the shifted projectives.

Corollary 3.9. *Assume that any $X\in\mathcal{I}$ is isomorphic to $\tau^m(\Sigma P_i)$ for some m and some i. Then the u-equations imply that $\widehat{F}_X = 1$ for all $X\in\mathcal{I}$. In particular, $\widetilde{M}_\Lambda = \widetilde{U}_\Lambda$.*

Proof. If follows from observing that we always have $\widehat{F}_{\Sigma P_i} = 1$, and by knitting. □

Example 3.10. Here are some families of examples to which Corollary 3.9:

(1) if Q is any orientation of a Dynkin quiver of type A, D or E and $\Lambda = \mathbb{C}Q$ is its path algebra.
(2) if Λ is the path algebra of $1\to 2\to\cdots\to n$ modulo any ideal generated by paths of length at least 2;
(3) by a theorem of Skowroński [**?**] and Liu [**?**], any tilted algebra of finite representation type (since the Auslander–Reiten quiver of such an algebra contains a complete slice, and thus satisfies the hypotheses of Corollary 3.9).

4. Split Grothendieck groups

In this section, we give bases for the split Grothendieck groups of $\operatorname{mod}\Lambda$ and also study the split Grothendieck group of K_Λ.

Definition 4.1. For any extriangulated category $\mathcal{C}$, its *split Grothendieck group* is

$$K_0^{\mathrm{sp}}(\mathcal{C}) := \bigoplus_{A\in \operatorname{ind}\mathcal{C}} \mathbb{Z}\cdot [A]/\langle [X]-[E]+[Y] \mid X \rightarrowtail E \twoheadrightarrow Y \text{ split conflation}\rangle.$$

In the case of the module category, $\mathcal{C} = \operatorname{mod}\Lambda$, split conflation means split short exact sequence.

Consider first the split Grothendieck group of $\operatorname{mod}\Lambda$. The set of classes $[M]$ for all indecomposable modules is a basis of $K_0^{\mathrm{sp}}(\operatorname{mod}\Lambda)$, which we call the standard basis. We will give two other bases of $K_0^{\mathrm{sp}}(\operatorname{mod}\Lambda)$, each of which is dual in a suitable sense to the standard basis.

It is natural to consider classes associated to almost split short exact sequences. For any non-projective indecomposable object M of $\operatorname{mod}\Lambda$, we write

$$r_M = [\tau M] - [E] + [M]$$

for the class associated to its Auslander-Reiten sequence, $0 \to \tau M \to E \to M \to 0$. In the case that $M = P_i$, we write

$$r_{P_i} = -[\operatorname{rad} P_i] + [P_i],$$

for the class associated to the right almost split inclusion. Similarly, for any non-injective indecomposable M of $\operatorname{mod}\Lambda$, write

$$\ell_M = [M] - [E] + [\tau^{-1}M],$$

for the class associated to $0 \to M \to E \to \tau^{-1}M \to 0$. In the case $M = I_i$, write

$$\ell_{I_i} = [I_i] - [I_i/S_i],$$

for the class associated to the left almost split quotient. Note that for non-projective M, $\ell_{\tau M} = r_M$.

There is a natural pairing on $K_0^{\mathrm{sp}}(\operatorname{mod}\Lambda)$ given by

$$([M],[N]) = \hom_\Lambda(M,N).$$

With respect to this pairing we have

Lemma 4.2. *The set of classes r_M are a basis for $K_0^{\mathrm{sp}}(\operatorname{mod}\Lambda)$, and this is right-dual to the standard basis, in the sense that*

$$([M], r_N) = \begin{cases} 0 & M \neq N, \\ 1 & M = N. \end{cases}$$

Similarly, the set of classes ℓ_M are also a basis, and this basis is left-dual to the standard basis:

$$(\ell_M, [N]) = \begin{cases} 0 & M \neq N, \\ 1 & M = N. \end{cases}$$

In particular, the pairing $(\ ,\ )$ is non-degenerate.

Proof. As already mentioned, the set of classes $[M]$ for all indecomposables $M \in \operatorname{mod}\Lambda$ are a basis of $K_0^{\mathrm{sp}}(\operatorname{mod}\Lambda)$. On the other hand, $([M], r_N)$ and $(\ell_M, [N])$ satisfy the stated properties by Proposition 2.1. But this shows that the r_M (resp. ℓ_M) also define bases of $K_0^{\mathrm{sp}}(\operatorname{mod}\Lambda)$. □

The following lemma is immediate (see for instance [7]):

Lemma 4.3. *Any class $x \in K_0^{\mathrm{sp}}$ admits basis expansions*

$$x = \sum_{M\in \operatorname{ind}\Lambda} (M,x) r_M, \qquad x = \sum_{M\in\operatorname{ind}\Lambda} (x,M)\ell_M$$

into the basis of r_M and the basis of ℓ_M classes, respectively.

We now consider the split Grothendieck group $K_0^{\mathrm{sp}}(K_\Lambda)$. Again, the standard basis for K_0^{sp} consists of the isomorphism classes of indecomposable objects, i.e., the objects from $\mathcal{I}$.

It will be useful to extend H^0 to a map from $K_0^{\mathrm{sp}}(K_\Lambda)$ to $K_0^{\mathrm{sp}}(\operatorname{mod}\Lambda)$, sending $[X]$ to $[H^0(X)]$.

For non-projective $X \in \mathcal{I}$, let $\tau X \rightarrowtail E \twoheadrightarrow X$ be its almost split conflation, and write

$$b_X = [X] - [E] + [\tau X] \in K_0^{\mathrm{sp}}(K_\Lambda).$$

For X indecomposable projective, define $b_X = [X] - [\operatorname{rad} X]$ (where we use the slight abuse of notation that $\operatorname{rad} X$ denotes the minimal projective resolution of the radical of the projective module X).

The assignment of a g-vector $g(X) \in \mathbb{Z}^n$ to each $X \in \mathcal{I}$ (see Section 2.4) can be extended to a linear map $g : K_0^{\mathrm{sp}}\big(K_\Lambda\big) \to \mathbb{Z}^n$. The surjection g can be regarded as the natural surjection of $K_0^{\mathrm{sp}}(K_\Lambda)$ onto the Grothendieck group $K_0(K_\Lambda) \simeq \mathbb{Z}^n$, which has a basis given by the classes $[P_i]$ of the projectives. The following characterises the kernel of g.

Theorem 4.4 (Theorem 4.13 of [26])**.** *The map $g : K_0^{\mathrm{sp}}\big(K_\Lambda\big) \to \mathbb{Z}^n$ is a split epimorphism. The set*

$$\{b_X \mid X \in \mathcal{I} \text{ and } X \neq P_i\}$$

is a basis of $\ker g$ as a free abelian group. This basis can be completed to a basis of $K_0^{\mathrm{sp}}\big(K_\Lambda\big)$ by adding the classes $[P_i]$ of the indecomposable projectives.

5. Geometry of $\mathcal{M}_\Lambda$

In this section, we show that $\mathcal{M}_\Lambda$ is irreducible by realizing it as a torus quotient of an open set in affine space, and we give several different descriptions of its coordinate ring.

Define a grading on $\mathbb{C}[u_M^{\pm} \mid M \in \mathcal{I}]$ with grading group $K_0^{\mathrm{sp}}(\Lambda)$ by $d(u_M) = H^0(b_M)$.

Lemma 5.1. *$d(\Psi(y_i)) = 0 \in K_0^{\mathrm{sp}}(\operatorname{mod}\Lambda)$.*

Proof. The multiplicity of $[M]$ in $d(\Psi(y_i))$ is the i-th coefficient of $g(b_M)$. Theorem 4.4 says that b_M is in the kernel of g, and the result follows. $\square$

Now we can establish the following lemma:

Lemma 5.2. *For $M \in \operatorname{mod}\Lambda$, we have that $\widehat{F}_M$ is homogeneous with respect to d, and its degree is $[M]$.*

Proof. First, by the previous lemma, $d(\Psi(F_M)) = 0$. It follows that $\widehat{F}_M$ is homogeneous with degree $\sum_{L \in \operatorname{ind}\Lambda} \hom(L, M) H^0(b_L)$. To determine the coefficient of $[N]$ in this sum, let $N \rightarrowtail E \twoheadrightarrow \tau^{-1}N$ be the almost split conflation in K_Λ beginning at N. Then the coefficient of $[N]$ in the degree of $\widehat{F}_M$ is (ℓ_N, M). This is 1 if $N = M$ and zero otherwise, which proves the lemma. $\square$

This grading on $\mathbb{C}[u_M^{\pm} \mid M \in \mathcal{I}]$ defines an action of the torus $T = \operatorname{Spec}\mathbb{C}[K_0^{\mathrm{sp}}(\operatorname{mod}\Lambda)]$ on $\operatorname{Spec}\mathbb{C}[u_M^{\pm} \mid M \in \mathcal{I}]$. Because the $\widehat{F}_M$ are homogeneous, the T-action restricts to an action on $X^\circ = \operatorname{Spec}\mathbb{C}[u_M^{\pm} \mid M \in \mathcal{I}][\widehat{F}_M^{-1} \mid M \in \operatorname{ind}\Lambda]$, which is the locus in $\operatorname{Spec}\mathbb{C}[u_M^{\pm} \mid M \in \mathcal{I}]$ where the $\widehat{F}_M$ are non-zero.

We now have the following theorem:

Theorem 5.3. $\mathbb{C}[u_M^{\pm}, \widehat{F}_M^{-1}] = \mathbb{C}[\widehat{F}_M^{\pm} \mid M \in \operatorname{ind}\Lambda] \otimes_{\mathbb{C}} R_\Lambda$

Proof. $\mathbb{C}[\widehat{F}_M^{\pm}]$ is a subring of $\mathbb{C}[u_M^{\pm}, \widehat{F}_M^{-1}]$, and contains exactly one Laurent monomial (in the $\widehat{F}_M$'s) in each degree in $K_0^{\mathrm{sp}}(\operatorname{mod}\Lambda)$. Thus, any homogeneous element of $\mathbb{C}[u_M^{\pm}, \widehat{F}_M^{-1}]$ can be uniquely written as a Laurent monomial in $\mathbb{C}[\widehat{F}_M^{\pm}]$ times an element of $\mathbb{C}[u_M^{\pm}, \widehat{F}_M^{-1}]_0$, where the subscript zero indicates the subring of degree 0. Thus, we have an isomorphism

$$\mathbb{C}[u_M^{\pm}, \widehat{F}_M^{-1}] \xrightarrow{\sim} \mathbb{C}[\widehat{F}_M^{\pm}] \otimes_{\mathbb{C}} \mathbb{C}[u_M^{\pm}, \widehat{F}_M^{-1}]_0$$

which sends $\widehat{F}_M^{\pm 1}$ to $\widehat{F}_M^{\pm 1} \otimes 1$. Thus the ideal I_Λ is sent on the right-hand side to the ideal generated by the $(\widehat{F}_M - 1) \otimes 1$. Taking the quotient, we get that $R_\Lambda \cong \mathbb{C}[u_M^{\pm}, \widehat{F}_M^{-1}]_0$, proving the theorem. $\square$

Corollary 5.4. *$\mathcal{M}_\Lambda$ is isomorphic to X°/T, and is thus an irreducible variety.*

Proof. $\mathcal{M}_\Lambda = \mathrm{Spec}(R_\Lambda)$. We have showed that $R_\Lambda = \mathbb{C}[u_M^\pm, \widehat{F}_M^{-1}]_0$. The coordinate ring of X° is $\mathbb{C}[u_M^\pm, \widehat{F}_M^{-1}]$, so $\mathbb{C}[u_M^\pm, \widehat{F}_M^{-1}]_0$ is the coordinate ring of X°/T. □

For $M \in \mathcal{I}$, let

$$\widehat{v}_M = \frac{u_M \widehat{F}_E}{\widehat{F}_{\tau M}\widehat{F}_M},$$

where $\tau M \to E \to M$ is the Auslander-Reiten conflation ending at M, unless M is projective, in which case set $E = \mathrm{rad}\, M$ and $\tau M = 0$.

Lemma 5.5. *$\widehat{v}_M$ is homogeneous of degree zero (and thus T-invariant).*

Proof. This is immediate from the fact that $d(u_M) = H^0(b_M)$, together with Lemma 5.2. □

Theorem 5.6. *The ring $\mathbb{C}[u_M^\pm, \widehat{F}_M^{-1}]_0$ is generated by the $\widehat{v}_M^\pm$.*

Proof. The isomorphism $\mathbb{C}[u_M^\pm, \widehat{F}_M^{-1}] \xrightarrow{\sim} \mathbb{C}[\widehat{F}_M^\pm] \otimes_{\mathbb{C}} \mathbb{C}[u_M^\pm, \widehat{F}_M^{-1}]_0$ of Theorem 5.3 sends each u_M to $\frac{\widehat{F}_{\tau M}\widehat{F}_M}{\widehat{F}_E} \otimes \widehat{v}_M$. Taking the quotient by I_Λ yields the result. □

We will now obtain another expression for R_Λ as a subring of the rational function field in n variables, $\mathbb{C}(y_1, \ldots, y_n)$.

Proposition 5.7. *The map ψ from $\mathbb{C}[u_M^\pm, \widehat{F}_M^{-1}]$ to $\mathbb{C}[y_1^\pm, \ldots, y_n^\pm, F_M^{-1}]$, sending:*

$$u_M \to 1 \text{ for } M \in \mathrm{ind}\,\Lambda$$
$$u_{(P_i \to 0)} \to y_i$$

sends $\widehat{F}_M$ to F_M, and is an isomorphism from $\mathbb{C}[u_M^\pm, \widehat{F}_M^{-1}]_0$ to $\mathbb{C}[y_1^\pm, \ldots, y_n^\pm, F_M^{-1}]$.

Proof. It is easy to check from the definition of $\widehat{F}_M$ that ψ does indeed send $\widehat{F}_M$ to F_M, and thus is a well-defined map of rings from $\mathbb{C}[u_M^\pm, \widehat{F}_M^{-1}]$ to $\mathbb{C}[y_1^\pm, \ldots, y_n^\pm, F_M^{-1}]$.

The argument now is similar to the argument for Theorem 5.3. This time, we use the fact that the degrees of $u_M \mid M \in \mathrm{ind}\,\Lambda$ form a basis for $K_0^{\mathrm{sp}}(\mathrm{mod}\,\Lambda)$ to write

$$\mathbb{C}[u_M^\pm, \widehat{F}_M^{-1}] \xrightarrow{\sim} \mathbb{C}[u_M^\pm \mid M \in \mathrm{ind}\,\Lambda] \otimes_{\mathbb{C}} \mathbb{C}[u_M^\pm, \widehat{F}_M^{-1}]_0.$$

The ideal on the left-hand side generated by the $u_M - 1$ for $M \in \mathrm{ind}\,\Lambda$ is sent to the ideal generated by the $(u_M - 1) \otimes 1$, so taking the quotient gives

$$\mathbb{C}[u_M^\pm, \widehat{F}_M^{-1}]/\langle u_M - 1 \mid M \in \mathrm{ind}\,\Lambda\rangle \simeq \mathbb{C}[u_M^\pm, \widehat{F}_M^{-1}]_0.$$

But the left hand-side is isomorphic to $\mathbb{C}[y_1, \ldots, y_n, F_M^{-1}]$ under the map ψ, which yields the result. □

Define $v_M = \psi(\widehat{v}_M)$. Explicitly, the v_M are given in terms of F-polynomials as

$$v_M = \frac{F_E}{F_M\, F_{\tau M}}, \qquad v_{P_i} = \frac{F_{\mathrm{rad}\ P_i}}{F_{P_i}}, \qquad v_{\Sigma P_i} = y_i\, \frac{F_{I_i/S_i}}{F_{I_i}},$$

for $M \neq P_i, \Sigma P_i$, and where we write $0 \to \tau M \to E \to M \to 0$ for the AR-sequence ending with M.

We can now prove an analogue of Theorem 5.6, regarding $\mathbb{C}[y_1^\pm, \ldots, y_n^\pm, F_M^{-1}]$ (which is isomorphic to R_Λ by Proposition 5.7).

Proposition 5.8. *$\mathbb{C}[y_1^\pm, \ldots, y_n^\pm, F_M^{-1}]$ is generated by the $v_M^\pm$, with $M \in \mathcal{I}$.*

Proof. By Theorem 5.6, $\mathbb{C}[u_M^\pm, \widehat{F}_M^{-1}]_0$ is generated by $\widehat{v}_M^\pm$. Applying the map ψ from Proposition 5.7, its image under ψ is generated by the images under ψ of the $\widehat{v}_M^\pm$, which are the $v_M^\pm$. □

Corollary 5.9. *The rational functions v_M satisfy the $\widehat{F} = 1$ equations and the u-equations.*

Proof. The isomorphism from $\mathbb{C}[v_M^{\pm}]$ to $R_\Lambda = \mathbb{C}[u_M \mid M \in \mathcal{I}]/\langle \widehat{F}_M = 1 \mid M \in \operatorname{ind}\Lambda\rangle$, sending v_M to u_M, implies that the v_M satisfy the $\widehat{F} = 1$ equations. Corollary 3.6 shows that they therefore also satisfy the u-equations. □

Corollary 5.10. *There is an isomorphism from an open set in $\mathbb{C}^n$ to $\mathcal{M}_\Lambda$, sending $(y_1, \dots, y_n)$ to $(v_M(y_1, \dots, y_n))_{M\in\mathcal{I}}$.*

Proof. Theorem 5.3 and Proposition 5.7 give an isomorphism from R_Λ to $\mathbb{C}[y_1^{\pm}, \dots, y_n^{\pm}]$ localized at the product of all the F_M for $M \in \operatorname{ind}\Lambda$. The desired map is the corresponding map of varieties. □

6. Explicit proof that the v_X satisfy the $\widehat{F}_M = 1$ equations

In this section, we give a more direct proof of Corollary 5.9, by showing explcitly that when we substitute v_X for u_X (for all $X \in \mathcal{I}$) into $\widehat{F}_M$, for $M \in \operatorname{ind}\Lambda$, the result equals 1. The identities that we prove here will also be useful in Section 9.

We write F for the homomorphism $F : K_0^{\mathrm{sp}}(\operatorname{mod}\Lambda) \to \mathbb{Q}(y_1, \dots, y_n)$ that maps classes in the split Grothendieck group to their F-polynomials:

$$F : \sum_i \lambda_i [A_i] \longmapsto \prod_i F_{A_i}^{\lambda_i},$$

for integers λ_i. By Lemma 3.8(1), this is indeed well defined. Moreover, note that

$$F(r_X) = v_X^{-1}, \qquad y_i F(\ell_{I_i}) = v_{\Sigma P_i}^{-1},$$

for $X \neq \Sigma P_i$. Then the basis expansions in K_0^{sp} given by Lemma 4.3 imply the following identities:

Lemma 6.1. *For any module M in $\operatorname{mod}\Lambda$,*

$$\frac{1}{F_M} = \prod_{N\in\operatorname{ind}\Lambda} v_N^{\hom_\Lambda(N,M)}$$

$$\frac{y^{d(M)}}{F_M} = \prod_{X\in\mathcal{I}} v_X^{\hom_\Lambda(M,H^0(\tau X))}$$

Proof. The first equation follows by applying Lemma 4.3 to expand $x = [M]$ in the r_N basis, and then applying F to the resulting identity in $K_0^{\mathrm{sp}}(\operatorname{mod}\Lambda)$. Similarly expanding $x = [M]$ in the ℓ_N basis and applying F, we get

$$\frac{1}{F_M} = \prod_{N\in\operatorname{ind}\Lambda} F(\ell_N)^{-\hom_\Lambda(M,N)} = \prod_{X\in\mathcal{I}} F(\ell_{H^0(\tau X)})^{-\hom_\Lambda(M,H^0(\tau X))}$$

Now, $F(\ell_{H^0(\tau X)}) = v_X^{-1}$ unless $X = \Sigma P_i$, in which case $y_i F(\ell_{H^0(\tau\Sigma P_i)}) = v_{\Sigma P_i}^{-1}$. Thus, to be left with only powers of the v_X on the righthand side, we need a compensating monomial on the lefthand side, which is just $y^{d(M)}$ because $\hom(M, H^0(\tau\Sigma P_i)) = d(M)_i$. This proves the second equation. □

Lemma 6.2. *For any module M in $\operatorname{mod}\Lambda$, with dimension vector $d(M)$,*

$$y^{d(M)} = \prod_{Y\in\mathcal{I}} v_Y^{-\langle g(Y),d(M)\rangle}.$$

Proof. By the identities for $1/F_M$ and $y^{d(M)}/F_M$ (Lemma 6.1), we find

$$y^{d(M)} = \prod_{Y\in\mathcal{I}} v_Y^{\hom_\Lambda(M,H^0\tau Y)-\hom_\Lambda(H^0 Y,M)}.$$

Then the result follows from Lemma 2.10. □

We can now strengthen Proposition 5.8. We write Φ for the map from $\mathbb{C}[u_M \mid M \in \mathcal{I}]$ to $\mathbb{C}(y_1, \dots, y_n)$ sending u_M to v_M.

Theorem 6.3. *The map Φ induces an isomorphism $\bar{\Phi} : R_\Lambda \to \mathbb{C}[y_1^{\pm 1}, \dots, y_n^{\pm 1}; F_M^{-1}, M \in \mathcal{I}]$.*

Proof. Recall, Definition 2.13, that for $X \in \mathcal{I}$,

$$\widehat{F}_X = \sum_{\mathbf{d} \in \mathbb{Z}_{\geq 0}} \chi\left(\mathrm{Gr}_{\mathbf{d}}(X)\right) \prod_{W \in \mathcal{I}} u_W^{\hom(H^0W, H^0X) - \langle g_W, \mathbf{d} \rangle}. \tag{6.1}$$

By Lemmas 6.1 and 6.2,

$$\Phi\left(\prod_{W \in \mathcal{I}} u_W^{\hom(H^0W, H^0X)}\right) = \frac{1}{F_X} \quad \text{and} \quad \Phi\left(\prod_{W \in \mathcal{I}} u_W^{-\langle g_W, \mathbf{d}\rangle}\right) = y^{\mathbf{d}}.$$

So it follows that Φ is surjective onto $\mathbb{C}[y_1^{\pm 1}, \ldots, y_n^{\pm 1}; F_M^{-1}, M \in \mathcal{I}]$. Moreover,

$$\Phi\left(\widehat{F}_X\right) = \frac{1}{F_X} \sum_{\mathbf{d} \in \mathbb{Z}_{\geq 0}} \chi\left(\mathrm{Gr}_{\mathbf{d}}(X)\right) y^{\mathbf{d}} = \frac{F_X}{F_X} = 1, \tag{6.2}$$

so the kernel of Φ contains all $\widehat{F} = 1$ equations and Φ induces a surjective map $\bar{\Phi}$ with domain R_Λ.

We will define an inverse to $\bar{\Phi}$ by extending the map Ψ of Definition 2.14 to a morphism

$$\Psi : \mathbb{C}[y_1^{\pm 1}, \ldots, y_n^{\pm 1}; F_M^{-1}, M \in \mathcal{I}] \to R_\Lambda.$$

For this morphism to be well-defined, $\Psi(F_M^{-1})$ has to be the inverse of $\Psi(F_M)$. With reference to Lemma 6.1, we define

$$\Psi(F_M^{-1}) = \prod_{W \in \mathcal{I}} u_W^{\hom(H^0W, H^0X)}.$$

So $\Psi(F_M^{-1})\Psi(F_M) = \widehat{F}_M$, but $\widehat{F}_M$ is equal to 1 in R_Λ. So Ψ is indeed well-defined.

We claim that the composition

$$R_\Lambda \xrightarrow{\bar{\Phi}} \mathbb{C}[y_1^{\pm 1}, \ldots, y_n^{\pm 1}; F_M^{-1}, M \in \mathcal{I}] \xrightarrow{\Psi} R_\Lambda$$

is the identity map. It suffices to prove that $\Psi\bar{\Phi}(u_X) = u_X$ for all $X \in \mathcal{I}$. If X is not projective nor a shifted projective, then

$$\begin{aligned} \Psi\bar{\Phi}(u_X) &= \Psi\left(\frac{F_{E_X}}{F_X F_{\tau X}}\right) \\ &= \prod_{W \in \mathcal{I}} u_W^{\hom(H^0W, H^0X) + \hom(H^0W, \tau H^0X) - \hom(H^0W, H^0E_X)} = u_X, \end{aligned}$$

where we use Proposition 2.1 and Lemma 2.8. Next, if $X = P_i$ is projective, then

$$\begin{aligned} \Psi\bar{\Phi}(u_{P_i}) &= \Psi\left(\frac{F_{\mathrm{rad}\,P_i}}{F_{P_i}}\right) \\ &= \prod_{W \in \mathcal{I}} u_W^{\hom(H^0W, P_i) - \hom(H^0W, \mathrm{rad}\,P_i)} = u_{P_i}, \end{aligned}$$

again using Proposition 2.1 and Lemma 2.8.

Finally, if $X = \Sigma P_i$ is a shifted projective, then

$$\begin{aligned} \Psi\bar{\Phi}(u_{\Sigma P_i}) &= \Psi\left(y_i \frac{F_{I_i/\operatorname{soc} I_i}}{F_{I_i}}\right) = \Psi\left(\frac{F_{I_i} - 1}{F_{I_i}}\right) \\ &= 1 - \prod_{W \in \mathcal{I}} u_W^{\hom(H^0W, I_i)} = 1 - \prod_{W \in \mathcal{I}} u_W^{\hom(P_i, W)} \\ &= 1 - \prod_{W \in \mathcal{I}} u_W^{\hom(\Sigma P_i, \Sigma W) + \hom(W, \Sigma\Sigma P_i)} = u_X, \end{aligned}$$

where the last equality is a consequence of the u-equations (Corollary 3.6).

So $\Psi\bar{\Phi}$ is the identity, and so $\bar{\Phi}$ is injective. This proves that $\bar{\Phi}$ is an isomorphism. $\square$

7. Tropical u-variables

In this section, we prove some results on the tropicalization of the rational functions v_M for $M \in \mathcal{I}$, as defined in Section 6. These are of independent interest, generalizing results of [24] for Dynkin quivers; we will also use them in Section 9

Because of the independent interest of these results, in this section we drop the assumption that our finite-dimensional algebra Λ is of finite representation type. We continue to write $\mathcal{I}$ for the set of indecomposable objects of K_Λ, though this set may no longer be finite.

For $g \in \mathbb{Z}^n$, we can consider two-term complexes of projectives $P^{-1} \to P^0$, with $P^0 = \bigoplus_{g_i>0} P_i^{g_i}$, $P^{-1} = \bigoplus_{g_i<0} P_i^{-g_i}$. We call such complexes *presentations of weight* g. The space of such complexes is just an affine space, so in particular it is irreducible, and we can look for generic behaviours. In particular, we will be interested in the multiplicity of a given $X \in \mathcal{I}$ as an indecomposable summand of a generic presentation of weight g. Also, for M a Λ-module, define $\hom(g, M)$ to be the dimension of Hom from the cokernel of a generic presentation of weight g to M.

Any $g \in \mathbb{Z}^n$ admits a canonical decomposition, similar to Kac's canonical decomposition for representations [15, Section 4]: the canonical decomposition of $g = g_1 + \cdots + g_s$ if a generic presentation of weight g decomposes as a direct sum of presentations of weights $g_1, \ldots, g_s$.

Tropicalization of a polynomial drops coefficients, replaces product by sum and sum by maximum. The tropicalization of a monomial is therefore a linear function, and the tropicalization of a polynomial is the maximum of the linear functions corresponding to the terms of the polynomial. Generally tropicalization is only considered for polynomials with non-negative coefficients, but it will be convenient for us to neglect that condition.

Lemma 7.1. $\operatorname{trop} F_M(x) = \max_{L \subset M} \underline{\dim} L \cdot x$, *where the maximum is taken over subrepresentations of M.*

Proof. This is not immediate from our definition, because of the Euler characteristics in F_M. Suppose that L is a submodule of M of dimension $\mathbf{e}$ but $\chi(\mathrm{Gr}_{\mathbf{e}}(M)) = 0$. Then the term $\mathbf{e} \cdot y$ appears in the maximum on the righthand side but not on the lefthand side. However, let P be the convex hull of the dimension vectors of the subrepresentations of M. Clearly, the maximum on the righthand side of the equation in the statement in the lemma can be restricted to the vertices of P, and [**?**] shows that for $\mathbf{e}$ a vertex of P, we have $\mathrm{Gr}_{\mathbf{e}}(M) = \{\mathrm{pt}\}$, so the Euler characteristic is 1. Thus, the terms appearing on the righthand side but but missing from the lefthand side (if any) correspond to points contained within the convex hull of the vertices of P, and these points will never provide the maximum. $\square$

Fei shows [18, Theorem 3.6] the following theorem:

Theorem 7.2. *For any representation M of Λ, and any $g \in \mathbb{Z}^n$, there exists a positive integer m such that* $\hom(mg, M) = m(\operatorname{trop} F_M)(g)$. *Further, if the statement holds for m, it also holds for any multiple of m.*

If a vector $g \in \mathbb{Z}^n$ lies in the support of the g-vector fan, then the generic presentation of weight mg is just m copies of the generic presentation of weight g, so we can take $m = 1$ in Fei's theorem.

Theorem 7.3. *For $g \in \mathbb{Z}^n$ and $X \in \mathcal{I}$, we have that $-\operatorname{trop} v_X(g)$ equals the multiplicity of X in a generic presentation of weight g.*

Proof. By replacing g successively by each of the summands in its canonical decomposition, we can suppose that the generic presentation of weight g is indecomposable. We split into three cases, based on whether a generic presentation of weight g is

(1) the minimal presentation of an indecomposable τ-rigid module,
(2) the minimal presentation of an indecomposable non τ-rigid module, or
(3) a complex of the form $P_i \to 0$ (i.e., a shifted projective).

We further divide into cases based on whether X is

(a) a minimal presentation of a non-projective module M, with Auslander–Reiten sequence

$$0 \to \tau M \to E \to M \to 0,$$

(b) a complex of the form $0 \to P_j$ (i.e., a minimal presentation of a projective module P_j), or
(c) a complex of the form $P_j \to 0$.

We first consider case 1. Let N be the cokernel of a generic representation of weight g, which is well-defined up to isomorphism since we are in case 1.

We consider case 1(a). In this case,

$$\operatorname{trop}(v_X)(g) = -\operatorname{trop} F_{\tau M}(g) + \operatorname{trop} F_E(g) - \operatorname{trop} F_M(g).$$

Applying Theorem 7.2 (and choosing $m = 1$, as we may since g is in the support of the g-vector fan), we obtain

$$\begin{aligned} -\operatorname{trop} v_X(g) &= \hom(N, \tau M) - \hom(N, E) + \hom(N, M) \\ &= (N, r_M) \end{aligned}$$

By Lemma 4.3, (N, r_M) equals the multiplicity of M as a summand of N. (Note that nothing up to Lemma 4.3 in Section 4 relied on the finite representation type assumption.) This proves the desired statement in this case.

Case 1(b), where X is an indecomposable projective P_j is very similar, except that the Auslander–Reiten sequence ending at M is replaced by $\operatorname{rad} P_j \to P_j$. Similarly, $-\operatorname{trop} v_X(g) = (N, r_{P_j})$, and we draw the same conclusion.

Now consider case 1(c), where $X = \Sigma P_j$. Then

$$\begin{aligned} -\operatorname{trop}(v_X)(g) &= \operatorname{trop} F_{I_j}(g) - \operatorname{trop} y_j F_{I_j/S_j}(g) \\ &= \hom(N, I_j) - \hom(N, I_j/S_j) - g_j \end{aligned}$$

We have a long exact sequence

$$0 \to \operatorname{Hom}(N, S_j) \to \operatorname{Hom}(N, I_j) \to \operatorname{Hom}(N, I_j/S_j) \to \operatorname{Ext}^1(N, S_j) \to \operatorname{Ext}^1(N, I_j) = 0$$

So

$$\hom(N, I_j) - \hom(N, I_j/S_j) = \hom(N, S_j) - \operatorname{ext}^1(N, S_j)$$

.

Now $\hom(N, S_j)$ is just $\max(g_j, 0)$, while $\operatorname{ext}(N, S_j) = -\max(-g_j, 0)$, so $\hom(N, S_j) - \operatorname{ext}^1(N, S_j) = g_j$, so $-\operatorname{trop}(v_X)(g) = 0$ in this case, as desired.

Now consider case 2. In this case, the multiplicity of any particular X in a generic presentation of weight g is zero, because, while X might possibly appear as some presentation of weight g, it will not be the generic presentation, since g is not τ-rigid. We must therefore show that, in this case, $\operatorname{trop} v_X(g) = 0$.

In case 2(a),

$$\operatorname{trop}(v_X)(g) = -\operatorname{trop} F_{\tau M}(g) + \operatorname{trop} F_E(g) - \operatorname{trop} F_M(g).$$

Applying Theorem 7.2, there is a positive integer m such that

$$-m \operatorname{trop} v_X(g) = \hom(mg, \tau M) - \hom(mg, E) + \hom(mg, M).$$

Let N be the cokernel of some presentation of weight mg. Then $\hom(N, \tau M) - \hom(N, E) + \hom(N, M)$ is (N, r_M), so by Lemma 4.3, it equals the multiplicity of M as a summand of N. Since mg is not τ-rigid, for a generic choice of N, this multiplicity will be zero, so $-m \operatorname{trop} v_X(g) = 0$, as desired.

Case 2(b) is again very similar to case 2(a), in exactly the same way as cases 1(b) and 1(a). The same thing holds for case 2(c).

Finally, consider case 3, where $g = -e_i$. In case 3(a),

$$\operatorname{trop}(v_X)(g) = -\operatorname{trop} F_{\tau M}(-e_i) + \operatorname{trop} F_E(-e_i) - \operatorname{trop} F_M(-e_i)$$

The tropical evaluation of any F-polynomial at $-e_i$ is zero, so $\operatorname{trop}(v_X)(g) = 0$. Case 3(b) is disposed of in the same way. For case 3(c),

$$\begin{aligned} -\operatorname{trop}(v_X)(g) &= \operatorname{trop} F_{I_j}(-e_i) - \operatorname{trop} y_j F_{I_j/S_j}(-e_i) && (7.1) \\ &= \delta_{ij} && (7.2) \end{aligned}$$

as desired. □

The previous result extends to arbitrary finite-dimensional algebras a recent result of Kus–Reineke for Dynkin quivers [24].

Returning to the case where Λ is representation-finite, everything there is to know about generic presentations is encoded in the g-vector fan. Specifically, for any $g \in \mathbb{Z}^n$, we express it as a sum of the generators of the mininal cone containing g in the g-fan:

$$g = c_1 g_{X_1} + \cdots + c_r g_{X_r}.$$

Then the multiplicity of X_i in a generic representation of weight g is c_i, and the multiplicity of any other complex from $\mathcal{I}$ is zero. This implies the following corollary, which completely characterizes $\operatorname{trop} v_M$.

Corollary 7.4. *For Λ of finite representation type,* $\operatorname{trop} v_M$ *is linear on each cone of the g-vector fan. If $N \in \mathcal{I}$ is rigid, then* $\operatorname{trop} v_M(g_N) = -\delta_{MN}$*, where $\delta_{MN} = 1$ is M and N are isomorphic, and zero otherwise.*

In particular, note that if M is not rigid, then $\operatorname{trop} v_M$ is identically zero.

8. Divisors and Jasso Reduction

In order to understand the locus in $\widetilde{\mathcal{M}}_\Lambda$, where all the entries are non-negative, it is important to consider the stratification of $\widetilde{\mathcal{M}}_\Lambda$ according to which of the u_M are zero. In order to understand the strata, we need to understand what happens when we add u_M into the ideal $\widetilde{I}_\Lambda$. It turns out that $\mathbb{C}[u_\alpha]/\widetilde{I}_\Lambda + \langle u_M \rangle$ is isomorphic to the ring defined by the $\widehat{F}$ equations for another algebra of lower rank, called the Jasso reduction of Λ at M, and denoted B_M.

There is a surjective map from $K_0(\Lambda)$ to $K_0(B_M)$, which quotients the class in K_0 of M. There is a bijection which we denote r_M from the objects of $\mathcal{I}_\Lambda$ which are compatible with M (other than M itself), to the objects of $\mathcal{I}_{B_M}$. The map r_M also restricts to a bijection from the rays of the g-vector fan of Λ which are compatible with M other than M, to the rays of the g-vector fan of B_M. This bijection induces an identification of the link of the cone generated by the summands of M in the g-vector fan of Λ with the g-vector fan of B_M, which is also induced by the quotient map from $K_0(\operatorname{proj}\Lambda)$ to $K_0(\operatorname{proj}\Lambda)/[M] = K_0(\operatorname{proj} B_M)$.

Theorem 8.1. *Let $M \in \mathcal{I}$ be rigid. Then there is an isomorphism from $\widetilde{R}_\Lambda/\langle u_M \rangle$ to $\widetilde{R}_{B_M}$ which sends u_N to 1 if N is incompatible with M, and sends u_N to $\breve{u}_{r_M(N)}$ if N is compatible with M and not equal to M.*

Proof. It suffices to show that for every indecomposable $N \in \operatorname{mod}\Lambda,$, $\pi\widehat{F}_M \in \mathcal{I}_{B_M}$, and for every indecomposable $Y \in \operatorname{mod} B_M$, there exists a $\hat{Y} \in \operatorname{mod}\Lambda$ such that $\pi\widehat{F}_{\hat{Y}} = \widehat{F}_Y$.

Recall that there is a map J_M from $\operatorname{mod}\Lambda$ to $\operatorname{mod} B_M$, sending X to $T_M(X)/t_M(X)$, where T_M and t_M are two torsion classes associated to M. The map J_M is an equivalence of additive categories from the Λ modules with $T_M(X) = X$ and $t_M(X) = 0$, to the B_M modules.

Let us denote by π the map sending u_M to zero, u_N to 1 if N is not compatible with M, and otherwise sending it to $\breve{u}_{r(N)}$.

We know that the $\widehat{F} = 1$ equations imply the u-equations, and once we have $u_M = 0$, the u-equations imply that $u_N = 1$ for every N incompatible with M. Thus, for N incompatible with M, we have $u_N - 1 \in \widetilde{I} + \langle u_M \rangle$.

Now let N be an indecomposable Λ-module. Consider $\widehat{F}_N|_{u_M \leftarrow 0}$. By Theorem 7.2, there exists an m such that $(\operatorname{trop} F_N)(me_M) = \hom(mg_M, N)$. Since M is rigid, a generic presentation of g-vector mg_M is isomorphic to m copies of M, so we conclude that $(\operatorname{trop} F_N)(e_M) = \hom(M, N)$. This shows that the monomial factor in the definition of $\widehat{F}_N$ is exactly such as to ensure that the minimum exponent of u_M appearing in $\widehat{F}_N$ is zero. This means that the lattice points of the Newton polytopes of N which maximize $\langle g_M, \cdot \rangle$ correspond to terms in $\widehat{F}_N$ with degree of u_M of zero, while all the other lattice points have u_M appearing with strictly positive degree. Thus, $\widehat{F}_N \mid_{u_M \leftarrow 0}$ is the polynomial obtained by restricting $\widehat{F}_N$ to the face where $\langle g_M, \cdot \rangle$ is maximized.

Let us denote $P_M(N)$ the face of the Harder–Narasimhan polytope of N on which $\langle g_M, \cdot\rangle$ is maximized. As [?] explain, the Harder–Narasimhan polytope of $J_M(N)$ can be identified with $P_M(N)$. Further, [17, Section 6] explains that, after that identification, the F-polynomial of $J_M(N)$ equals the restriction of F_N to $P_M(N)$. Write $p_M(N)$ for the bottommost point of $P_M(N)$.

Now:

$$\begin{aligned}\widehat{F}_{J_M(N)} &= \sum_{\mathbf{d}\in P_M(N)\cap\mathbb{Z}^n} \chi(\mathrm{Gr}_{\mathbf{d}-p_M(N)}(J_M(N)) \prod_{V\in\mathcal{I}_M} \breve{u}_V^{\hom(H^0V, J_M(N))-\langle g_V, \mathbf{d}-p_M(N)\rangle} \\ &= \sum_{\mathbf{d}\in P_M(N)\cap\mathbb{Z}^n} \chi(\mathrm{Gr}_{\mathbf{d}-p_M(N)}(J_M(N)) \prod_{Z\in\mathcal{I}, c(Z,M)=0} \breve{u}_{r_M(Z)}^{\hom(H^0 r_M(Z), J_M(N))-\langle g_{r(Z)}, \mathbf{d}-p_M(N)\rangle}\end{aligned}$$

Now, $g_{r(Z)}$ can be understood as g_Z in the quotient by g_M. Since $\mathbf{d} - p_M(N)$ is perpendicular to g_M, we can replace $g_{r(Z)}$ by g_Z. Further, we can replace $\mathrm{Hom}(H^0 r_M(Z), J_M(N))$ by $\mathrm{Hom}(H^0 Z, J_M(N))$. And by [17, Section 6], $\mathrm{Gr}_{\mathbf{d}-p_M(N)}(J_M(N))$ is isomorphic to $\mathrm{Gr}_{\mathbf{d}}(N)$. We obtain:

$$\widehat{F}_{J_M(N)} = \sum_{\mathbf{d}\in P_M(N)\cap\mathbb{Z}^n} \chi(\mathrm{Gr}_{\mathbf{d}}(N) \prod_{Z\in\mathcal{I}, c(Z,M)=0} \breve{u}_{r_M(Z)}^{\hom(H^0(Z), J_M(N))-\langle g_Z, \mathbf{d}-p_M(N)\rangle}$$

On the other hand,

$$\pi\widehat{F}_N = \sum_{\mathbf{d}\in P_M(N)\cap\mathbb{Z}^n} \chi(\mathrm{Gr}_{\mathbf{d}}(N) \prod_{Z\in\mathcal{I}, c(Z,M)=0} \breve{u}_{r_M(Z)}^{\hom(H^0(Z), N)-\langle g_Z, \mathbf{d}\rangle}$$

To get these two expressions to agree, we need to show that

$$\hom(H^0(Z), N) - \hom(H^0(Z), J_M(N)) - \langle g_Z, p_M(N)\rangle = 0$$

Consider the exact sequence

$$0 \to t_M(N) \to N \to N/t_M(N) \to 0$$

Apply $\mathrm{Hom}(Z, -)$ to it. Since $t_M(N)$ is a quotient of M, and Z is compatible with M, $\mathrm{Ext}^1(Z, t_M(N)) = 0$. Thus, we still have a short exact sequence:

$$0 \to \mathrm{Hom}(Z, t_M(N)) \to \mathrm{Hom}(Z, N) \to \mathrm{Hom}(Z, N/t_M(N)) \to 0$$

Since $Z \in {}^\perp(\tau M)$, the image of any map from Z to N is contained in $T_M(N)$, and thus the image of any map from Z to $N/t_M(N)$ is contained in $J_M(N) = T_M(N)/t_M(N)$. So we have:

$$0 \to \mathrm{Hom}(Z, t_M(N)) \to \mathrm{Hom}(Z, N) \to \mathrm{Hom}(Z, J_M(N)) \to 0$$

So the quantity that we want to show equals zero is $\hom(H^0(Z)), t_M(N)) - \langle g_Z, t_M(N)\rangle = \hom(t_M(N), \tau H^0(Z))$, and since $\mathrm{Hom}(M, \tau H^0(Z)) = 0$, this is zero.

This shows that, for any $N \in \mathrm{mod}\,\Lambda$, $\pi\widehat{F}_N \in \mathcal{I}_{B_M}$. We now want to show that the $\pi\widehat{F}_N$ generate $\mathcal{I}_{B_M}$.

Let Y be an indecomposable B_M module. There is an equivalence of additive categories from $M^\perp \cap {}^\perp(\tau M)$ to $\mathrm{mod}\,B_M$, with the morphism given by J_M. Thus, there is an indecomposable module $\hat{Y} \in M^\perp \cap {}^\perp(\tau M)$ with $J_M(\hat{Y}) = Y$. Consequently, $\pi\widehat{F}_{\hat{Y}} = \widehat{F}_Y$, and we are done. □

The statement corresponding to Theorem 8.1 for the variety $\mathcal{U}_\Lambda$ (defined by the vanishing of the u-equations) also holds.

Theorem 8.2. *Let $M \in \mathcal{I}$ be rigid. Then there is an isomorphism from $\widetilde{S}_\Lambda/\langle u_M\rangle$ to $\widetilde{S}_{B_M}$ which sends u_N to 1 if N is incompatible with M, and sends u_N to $\breve{u}_{J_M(N)}$ if N is compatible with M.*

Proof. Let $\mathcal{Z}_M$ be the full subcategory of K_Λ consisting of objects N which satisfy that $\mathrm{Ext}^1(N, M) = 0 = \mathrm{Ext}^1(M, N)$. By [20, Lemma 3.13], the additive quotient category $\mathcal{Z}_M/[M]$ is equivalent to $K^{[-1,0]}(\mathrm{proj}\,B_M)$ (and the equivalence is induced by the silting reduction functor $J_M : K^b(\mathrm{proj}\,\Lambda)/\,\mathrm{thick}(M) \to K^b(\mathrm{proj}\,B_M)$). Moreover, by [23, Lemma 3.4], if $X, Y \in \mathcal{Z}_M$, then $\mathrm{Ext}^1(X, Y) \cong \mathrm{Ext}^1(J_M(X), J_M(Y))$. Thus the morphism of rings

$$\phi_M : \mathbb{C}[u_X \mid X \in \mathcal{I}] \longrightarrow \mathbb{C}[\breve{u}_Y \mid Y \in \mathcal{I}_M]$$

sending u_M to 0, u_N to 1 if N is incompatible with M and u_N to $u_{J_M(N)}$ if N is compatible to M is surjective and sends u-equations to u-equations. Thus it induces a surjective morphism

$$\phi_M : \widetilde{S}_\Lambda/\langle u_M\rangle \longrightarrow \widetilde{S}_{B_M}.$$

Moreover, the morphism

$$\psi_M : \mathbb{C}[\breve{u}_Y \mid Y \in \mathcal{I}_M] \longrightarrow \mathbb{C}[u_X \mid X \in \mathcal{I}]$$

sending $\breve{u}_{J_M(X)}$ to u_X sends the $\breve{u}$-equations to u-equations after sending u_M to 0 and u_N to 1 for all N incompatible with M. Thus it induces a surjective morphism

$$\psi_M : \widetilde{S}_{B_M} \longrightarrow \widetilde{S}_\Lambda/\langle u_M\rangle.$$

Since ψ_M is clearly a right-inverse to ϕ_M, it is injective. Thus it is an isomorphism, and so is ϕ_M. □

9. Geometry of $\widetilde{\mathcal{M}}_\Lambda$

In this section, we put together the results from Section 5 on the geometry of $\mathcal{M}_\Lambda$ together with results from Sections 7 and 8 to provide a description of the geometry of $\widetilde{\mathcal{M}}_\Lambda$.

Recall that the v_M for $M \in \mathcal{I}$ were defined in section 5, and that $R_\Lambda = \mathbb{C}[u_M^{\pm 1} \mid M \in \mathcal{I}]/I_\Lambda$, the coordinate ring of $\mathcal{M}_\Lambda$, was shown in Proposition 5.8 to be isomorphic to the subring of $\mathbb{C}(y_1,\dots,y_n)$ generated by $v_M^{\pm 1}$.

We will now prove a similar statement for $\widetilde{R}_\Lambda = \mathbb{C}[u_M \mid M \in \mathcal{I}]/\widetilde{I}_\Lambda$, the coordinate ring of $\widetilde{\mathcal{M}}_\Lambda$. To pass from the statement for R_Λ to the statement for $\widetilde{R}_\Lambda$, we will need the following lemma:

Lemma 9.1 ([4, Lemma A.1])**.** *Let $h : A \to B$ be a surjective homomorphism of Noetherian commutative rings with identity. Let $S \subset A$ be the muliplicative set generated by elements $x_1,\dots,x_p$ such that $h(x_1),\dots,h(x_p)$ are not zero divisors in B. Suppose that:*

1. *the localized homomorphism $S^{-1}h : S^{-1}A \to S^{-1}B$ is an isomorphism.*
2. *for $1 \le i \le p$, the induced homomorphism $h_i : A/\langle x_i\rangle \to B/\langle h(x_i)\rangle$ is an isomorphism.*

Then h is an isomorphism.

Let us write $\widetilde{V}_\Lambda$ for the subring of $\mathbb{C}(y_1,\dots,y_n)$ generated by the v_M.

Theorem 9.2. *$\widetilde{R}_\Lambda$ is isomorphic to $\widetilde{V}_\Lambda$.*

Proof. Recall that $\widetilde{I}_\Lambda$ is the ideal in $\mathbb{C}[u_M]$ generated by $\widehat{F}_M - 1$ for $M \in \mathcal{I}$.

Consider the map from the polynomial ring $\mathbb{C}[u_M]$ to $\widetilde{V}_\Lambda$ sending u_M to v_M. By Corollary 5.9, we know that $\widetilde{I}_\Lambda$ is contained in its kernel. We want to show that the induced map, which we shall call $f : \mathbb{C}[u_M]/\widetilde{I}_\Lambda \to \widetilde{V}_\Lambda$ is an isomorphism.

Our goal is to apply Lemma 9.1 to the map f and the elements $\{x_i\} = \{u_M\}$ to conclude that f is an isomorphism. Since the elements v_M are non-zero elements of a field, they are certainly non zero divisors. By Proposition 5.8, f is an isomorphism after inverting the u_M (in the source) and the v_M (in the target); this establishes the condition (1) of the lemma.

By definition, $\widetilde{M}_\Lambda(N)$ is cut out of $\widetilde{M}_\Lambda$ by the ideal u_N. Thus, $\mathbb{C}[u_M]/(\widetilde{I}_\Lambda + \langle u_N\rangle)$ is isomorphic to $\mathbb{C}[\widetilde{M}_\Lambda(N)]$. By results from Section 8, there is an algebra B_N, of rank one less than that of Λ, such that $\widetilde{M}_\Lambda(N) = \widetilde{M}_{B_N}$. By induction on the rank of Λ, the map $f_N : \mathbb{C}[u_M]/(\widetilde{I}_\Lambda + \langle u_N\rangle) \to \widetilde{V}/\langle v_N\rangle$ is an isomorphism. This establishes condition (2) for the lemma, and the desired result now follows. □

We now characterize $\widetilde{R}_\Lambda$ as a subring of $\mathbb{C}(y_1,\dots,y_n)$ in terms of tropical properties.

Proposition 9.3. *The subring of $\mathbb{C}(y_1,\dots,y_n)$ generated by the v_M for $M \in \mathcal{I}$ (which we denote $\mathbb{C}[v_M]$) is exactly the subring generated by the rational functions w such that:*

1. *w is a Laurent monomial in the y_i and F_M, and*
2. *$\mathrm{trop}(w)$ is non-positive.*

Proof. That v_M for $M \in \mathcal{I}$ satisfy (1) and (2) is immediate from Theorem 7.3. By Lemmas 6.1 and 6.2, we can re-express a Laurent monomial w as in (1) as a Laurent monomial in the v_M. By Theorem 7.3, the power of each v_M must be positive, except perhaps for any v_M with M non-rigid. But observe that if M is non-rigid, then the u-equation for M has a factor of u_M on the lefthand side, so dividing both sides by it, we get an expression for u_M^{-1} with both terms having all u_N appearing only with positive powers. □

Write F for $\prod_M F_M$. Let X_Λ be the toric variety defined by the g-vector fan, with y_i the coordinate corresponding to the ray $e_i = g(P_i)$.

A subsemigroup S of $\mathbb{Z}^n$ is called saturated if for any $v \in \mathbb{Z}^n$ such that for some positive integer k we have $kv \in S$, it follows that $v \in S$ [13, Definition 1.3.4]. A lattice polytope P in $\mathbb{Z}^n$ is called very ample if for every vertex v of P, the semigroup generated by $(P \cap \mathbb{Z}^n) - v$ is saturated [13, Definition 2.2.17]. It is also true that P is very ample if and only if for k sufficiently large, any lattice point in kP can be written as a sum of k lattice points of P [10, Exercise 2.23].

If P is a full-dimensional very ample polytope in $\mathbb{Z}^n$, we can take $P \cap \mathbb{Z}^n = \{p_1, \dots, p_s\}$. Then the toric variety associated to the outer normal fan to P can be defined as the Zariski closure of the image of the map sending $T = (\mathbb{C}^*)^n$ to $\mathbb{P}^{s-1}$, sending t to $[t^{p_1} : \cdots : t^{p_s}]$. (See [13, Section 2.3].)

Theorem 9.4. *The affine scheme $\widetilde{\mathcal{M}}_\Lambda$ is isomorphic to the affine open subset of X_Λ where F is non-vanishing. The subvariety $\mathcal{M}_\Lambda$ is the intersection of $F \neq 0$ with the open torus orbit in X_Λ.*

Proof. The outer normal fan of P, the Newton polytope of F, coincides with the g-vector fan of Λ by [2, Corollary 5.25(b)]. Since the g-vector fan of Λ is unimodular, P is automatically very ample [13, Proposition 2.4.4]. (This resolves in the affirmative the first part of [4, Question 5.4].)

Let $P \cap \mathbb{Z}^n = \{p_1, \dots, p_s\}$, and consider the embedding of X_Λ into $\mathbb{P}s - 1$ as above. The polynomial F, the exponents of whose monomials are a subset of $P \cap \mathbb{Z}^n$, defines a hyperplane in $\mathbb{P}^{s-1}$. The complement of this hyperplane is an affine space. The coordinate functions from $\mathbb{P}^{s-1}$ pull back to y^{p_i}/F on X_Λ. Since $\mathrm{trop}(y^{p_i}/F)$ is non-positive, by Proposition 9.3, $\mathbb{C}[y^{p_i}/F]$ is contained in $\mathbb{C}[u_M]$.

For the opposite inclusion, we want to show that v_M is contained in $\mathbb{C}[y^{p_i}/F]$ for any $M \in \mathcal{I}$. Pick a particular M. Its denominator divides F. Since P is very ample, we can choose k sufficiently large so that any lattice point in kP can be written as a sum of k lattice points of P. We can then write $v_M = \overline{A}(y)/F^k(y)$ for some polynomial $\overline{A}(y)$. Because $\mathrm{trop}(v_M) \leq 0$, the exponents of monomials appearing in $A(y)$ are all contained in kP. Let y^u be a monomial appearing in $\overline{A}(y)$. By the assumption on k, we can write $u = u_1 + u_2 + \cdots + u_k$ with each $u_i \in P$. Then $y^u/F^k(y) = (y^{u_1}/F) \dots (y^{u_k}/F)$ belongs to $\mathbb{C}[y^{p_i}/F]$, and summing over all the monomials of $\overline{A}$ we have the desired result. □

10. Positive Real Part of $\widetilde{\mathcal{M}}_\Lambda$

In this section, we investigate the geometry of $\widetilde{\mathcal{M}}_\Lambda^{>0}$ and $\widetilde{\mathcal{M}}_\Lambda^{\geq 0}$, the loci inside $\widetilde{\mathcal{M}}_\Lambda$ where the coordinates u_M are real and positive (respectively, non-negative). We refer to these as the totally positive (respectively, totally non-negative) parts of $\widetilde{\mathcal{M}}_\Lambda$.

We will study them by showing that they coincide with the corresponding parts of the toric variety X_Λ. Recall that X_Λ, being a non-singular toric variety, admits a covering by a finite number of patches isomorphic to $\mathbb{C}^n$. The totally non-negative part of X_Λ is glued together from the copy of $\mathbb{R}^n_{\geq 0}$ contained in each copy of $\mathbb{C}^n$. (See [**?**, Section 4.2], [13, Section 12.2] for more details.) For P a lattice polytope having Σ_Λ as its outer normal fan, the algebraic moment map μ_P provides a homeomorphism from $X_\Lambda^{\geq 0}$ to P. $X_\Lambda^{\geq 0}$ inherits a stratification from X_Λ. The moment map sends the part associated to the cone $\sigma \in \Sigma_\Lambda$ to the face of P with outer normal cone σ. The totally positive part, $X_\Lambda^{>0}$ corresponds under μ_P to the interior of P.

As before, let F be the product of all the F_M for $M \in \operatorname{ind}\Lambda$, and let P be the Newton polytope of F. This choice of P satisfies the hypotheses from the previous paragraph. Let $\{p_1, \dots, p_s\}$ be the lattice points of P. Then, as already mentioned, X_Λ can be constructed inside $\mathbb{P}^{s-1}$ as the Zariski closure of the image of $T = (\mathbb{C}^*)^n$ under the map sending $t \in T$ to $[t^{p_1} : \dots, t^{p_s}]$. Constructing X_Λ in this way, the totally positive part of X_Λ is the image under the same map of the positive real torus $T_{\mathbb{R}}^{>0} = (\mathbb{R}_{>0})^n$. The totally non-negative part $X_\Lambda^{\geq 0}$ is the closure of $X_\Lambda^{>0}$ in the analytic topology.

Lemma 10.1. *Thought of as a (rational) function of $y_1, \dots, y_n$, evaluated at points with all y_i positive, v_M is real and positive.*

Proof. Clearly, it is sufficient to show that $F_M(y) > 0$ for all positive y. If all the y_i are close to zero, then this is certainly true, because the terms in F_M are dominated by the constant term. Now consider a path in $\mathbb{R}^n_{>0}$ from a point close to the origin to some other point in y-space. At the beginning of the path, all the F_M are positive. Suppose that at some point, some F_M becomes zero. As we approach that point, v_M tends towards plus or minus infinity. But we know that the v_N are all real and, up to this point, positive, so v_M would have to between zero and 1, which gives us a contradiction. Thus, all the F_M stay positive the whole length of the path. $\square$

Lemma 10.2. **??** *Conversely, if all the v_M are real and positive, the y_i are real and positive.*

Proof. This is immediate from Lemma 6.2. $\square$

Corollary 10.3. *The totally positive parts of $\widetilde{\mathcal{M}}_\Lambda$ and X_Λ coincide.*

Proof. By definition, the totally positive part of $\widetilde{\mathcal{M}}_\Lambda$ is where the v_M are real and positive. By Lemmas 10.1 and **??**, this is equivalent to the y_i being real and positive. The totally positive part of M_Λ therefore coincides with the intersection of M_Λ with the totally positive part of X_Λ. However, since the locus of X_Λ which is missing from $\widetilde{\mathcal{M}}_\Lambda$ is given by zeros of the F_M, it follows that the totally positive parts of X_Λ and $\widetilde{\mathcal{M}}_\Lambda$ agree. $\square$

$\widetilde{\mathcal{M}}_\Lambda$ has a natural top form on it, namely the top form of the big torus contained in the toric variety X_Λ, which is $dy_1/y_1 \wedge \cdots \wedge dy_n/y_n$. It is natural to ask whether $\widetilde{\mathcal{M}}_\Lambda$ equipped with this differential form is a positive geometry in the sense of [3]. For a rather trivial reason, it isn't: namely, it is not projective. However, except for that issue, it is indeed a positive geometry, because its totally positive part agrees with the totally positive part of its ambient toric variety, and toric varieties are shown in [3] to be totally positive. (We are also implicitly using the fact that the boundaries of the totally positive part are again varieties of the same form.)

Stratification of boundary: Jasso / g vector fan.

Moment map to polytope, whose normal fan is the g vector fan.

11. Functoriality under algebra quotients

Let J be a two-sided ideal of Λ, and consider $A = \Lambda/J$. Since indecomposable A-modules are also indecomposable Λ-modules, A also satisfies our finiteness assumption.

Our goal in the section is to show:

Theorem 11.1. *Let $A = \Lambda/J$. There is a surjective map from $\widetilde{M}_\Lambda$ to $\widetilde{M}_A$.*

This map also restricts to a map from M_Λ to M_A, and is natural in the sense that if we consider some further quotient B of A, the map from $\widetilde{M}_\Lambda$ to $\widetilde{M}_B$ factors as a composition of the maps from $\widetilde{M}_\Lambda$ to $\widetilde{M}_A$ and that from $\widetilde{M}_A$ to $\widetilde{M}_B$.

For $\overline{M}$ an indecomposable A-module, and $\overline{N}$ an A-module, define $[\overline{N} : \overline{M}]$ to be the multiplicity of M as an indecomposable summand of N.

Let π be the functor from K_Λ to K_A defined by $- \otimes_\Lambda A$.

Write $\overline{\mathcal{I}}$ for the indecomposable objects of K_A, and consider a new set of variables $\overline{u}_N$ for $N \in \overline{\mathcal{I}}$.

We prove Theorem 11.1 by establishing the following proposition:

Proposition 11.2. *There is a ring map ϕ from $\widetilde{R}_A$ to $\widetilde{R}_\Lambda$ defined by:*

$$\phi(\overline{u}_{\overline{M}}) = z_{\overline{M}} := \prod_{N \in \mathcal{I}} u_N^{[\pi N : \overline{M}]}$$

For $\overline{L} \in \overline{\mathcal{I}}$, note that we can consider its F-polynomial either as an A-module or as a Λ-module. We distinguish them by writing $F_{\overline{L}_A}$ and $F_{\overline{L}_\Lambda}$ (and similarly for $\widehat{F}$-polynomials).

Proof. To establish the proposition, we need to check that for each $\overline{L} \in \overline{\mathcal{I}}$,

$$\widehat{F}_{\overline{L}_A}|_{\overline{u}_{\overline{M}} \leftarrow z_{\overline{M}}} = 1,$$

that is to say, the result of substituting $z_{\overline{M}}$ for $\overline{u}_{\overline{M}}$ (for all $\overline{M} \in \overline{\mathcal{I}}$) gives 1. As a shorthand, we write $\widehat{F}_{\overline{L}_A}(z)$ for the result of this substitution.

In fact, we will show that $\widehat{F}_{\overline{L}_A}(z) = \widehat{F}_{\overline{L}_\Lambda}$. This establishes what we need, since $\widehat{F}_{\overline{L}_\Lambda} = 1$ in $\widetilde{R}_\Lambda$.

We now prove this claim. By definition,

$$\widehat{F}_{\overline{L}_A}(z) = \sum_{\mathbf{d} \in \mathbb{Z}_{\geq 0}} \chi\left(\mathrm{Gr}_{\mathbf{d}}(H^0\overline{L})\right) \prod_{\overline{N} \in \overline{\mathcal{I}}} \prod_{M \in \mathcal{I}} u_M^{[\pi M:\overline{N}](\hom(H^0\overline{N}, H^0\overline{L}) - \langle g(\overline{N}), \mathbf{d}\rangle)}$$

Collecting all the u_M terms together, we see that they come from the indecomposable summands of πM, so we can rewrite the previous equation as:

$$\widetilde{F}_{\overline{L}_A}(z) = \sum_{\mathbf{d} \in \mathbb{Z}_{\geq 0}} \chi\left(\mathrm{Gr}_{\mathbf{d}}(H^0\overline{L})\right) \prod_{M \in \mathcal{I}} u_M^{\hom(H^0\pi M, H^0\overline{L}) - \langle g(\pi M), \mathbf{d}\rangle)}$$

The g-vector of πM coincides with the g-vector of M by definition, unless P_i is contained in J for some i. But for such i, $\mathbf{d}_i$ is necessarily 0. Thus, we can replace $g(\pi M)$ by $g(M)$. Further, any morphism from H^0M into an A-module necessarily factors through $M \otimes_\Lambda A = H^0(\pi M)$. So we have:

$$\begin{aligned} \widetilde{F}_{\overline{L}_A}(z) &= \sum_{\mathbf{d} \in \mathbb{Z}_{\geq 0}} \chi\left(\mathrm{Gr}_{\mathbf{d}}(H^0\overline{L})\right) \prod_{M \in \mathcal{I}} u_M^{\hom(H^0M, H^0\overline{L}) - \langle g(M), \mathbf{d}\rangle)} \\ &= \widetilde{F}_{\overline{L}_\Lambda}, \end{aligned}$$

as desired. □

We note that $[\pi N : \overline{M}]$ can be calculated solely on the basis of g_N if N is rigid.

Proposition 11.3. *Let $N \in \mathcal{I}$ be rigid. The indecomposable summands of πM can be determined by looking at the minimal cone of the g-vector fan for A that g_N appears in. The expression for g_N as a positive combination of the generators of that cone gives the expression of πN as a sum of indecomposable objects from $\overline{\mathcal{I}}$.*

Proof. The key point to establish is that πN is again rigid. We know the projective presentation of πN (it is that of N, unless the ideal J includes idempotents), so once πN is rigid, it must be a sum of compatible g-vectors from $\overline{\mathcal{I}}$, and there is only one way to combine them to get g_N.

In the case that J contains an idempotent, we don't have an isomorphism of the two spaces of g-vectors, but rather a projection; however, everything works the same way.

We now establish the key point, that πN is again rigid. Suppose that there is a non-zero map from $N \otimes_\Lambda A$ to $(N \otimes_\Lambda A)[1]$. This defines a map from N to $(N \otimes_\Lambda A)[1]$, and this map lifts to a map from N to $N[1]$. Suppose it is null-homotopic. The homotopy descends to a homotopy from N to $(N \otimes_\Lambda A)[1]$, and this factors through a homotopy from $N \otimes_\Lambda A$ to $(N \otimes_\Lambda A)[1]$. Thus, there are no non null-homotopic maps from πN to $\pi N[1]$. □

The map from $\widetilde{\mathcal{M}}_\Lambda$ to $\widetilde{\mathcal{M}}_A$ admits a completely different description in terms of the toric geometry of Section 9.

Theorem 11.4. *Let $A = \Lambda/J$. It is known that the g-vector fan of A coarsens that of Λ, and therefore there is an induced map $X_\Sigma \to X_A$. This map agrees with the map from $\widetilde{\mathcal{M}}_\Lambda$ to $\widetilde{\mathcal{M}}_A$ from Theorem 11.1.*

Proof. The blowdown map from X_Λ to X_A identifies the function fields $k(y_1, \ldots, y_n)$ of the two toric varieties. Since the map from Theorem 11.1 does this too, the two maps coincide. □

12. Dilogarithm identities

Write $L(x)$ for the Rogers dilogarithm of x. In this section, we show how to adapt the argument from [12] to prove the following theorem:

Theorem 12.1. *Let Λ be a finite-dimensional algebra of finite representation type, with n simple modules. Then $\sum_N L(1-v_N) = n\pi^2/6$, where the sum is taken over all indecomposable N in K_Λ.*

The main ingredient in the proof goes back to Frenkel–Szenes [19], see also [**?**]. Let G be the continuous functions from $\mathbb{R}^n_{>0}$ to $\mathbb{R}_{>0}$, which we think of as an abelian group written multiplicatively. Consider $G \otimes_{\mathbb{Z}} G$. Write S^2G for the subgroup of $G \otimes_{\mathbb{Z}} G$ generated by elements of the form $a \otimes b + b \otimes a$.

The following is essentially [19, Proposition 3.1], but slightly reformulated for our convenience. In particular, their functions f_i only take one argument, but the same approach proves the version where the functions take multiple arguments.

Proposition 12.2 ([19, Proposition 3.1])**.** *Suppose that $f_1, \dots, f_r$ are differentiable functions in G, whose range lies in $(0,1)$. Suppose further that $\sum_{i=1}^r f_i \otimes (1-f_i) \in S^2G$. Then $\sum_{i=1} rL(f_i(y_1,\dots,y_n))$ does not depend on the values of $y_1,\dots,y_r$.*

In order to apply the proposition, let us consider $\sum_N v_N \otimes (1-v_N)$. We have

$$\sum_N v_N \otimes (1-v_N) = \sum_{N,M} v_N \otimes v_M^{c(N,M)} = \sum_{N,M} c(N,M) v_N \otimes v_M.$$

Since $c(N,M) = c(M,N)$, this expression falls in S^2G.

13. Small Examples

13.1. A_2 **path algebra.** Let $\Lambda = \mathbb{C}Q$ be the path algebra of the quiver $1 \xrightarrow{a} 2$. Then the category K_Λ has 5 indecomposable objects which we designate by the following shorthand.

$$\begin{array}{lll} P_1 = (0 \to P_1) & P_2 = (0 \to P_2) & S_2 = (P_1 \xrightarrow{a} P_2) \\ \Sigma P_1 = (P_1 \to 0) & \Sigma P_2 = (P_2 \to 0) & \end{array}$$

Then the u-equations are as follows.

$$\begin{array}{lll} u_{P_1} + u_{S_2} u_{\Sigma P_1} = 1 & u_{P_2} + u_{\Sigma P_1} u_{\Sigma P_2} = 1 & u_{S_2} + u_{P_1} u_{\Sigma P_2} = 1 \\ u_{\Sigma P_1} + u_{P_1} u_{P_2} = 1 & u_{\Sigma P_2} + u_{P_2} u_{S_2} = 1 & \end{array}$$

The $\widehat{F}$-polynomials are as follows.

$$\widehat{F}_{P_1} = u_{P_1} + u_{S_2} u_{\Sigma P_1} \qquad \widehat{F}_{P_2} = u_{P_1} u_{P_2} + u_{P_2} u_{S_2} u_{\Sigma P_1} + u_{\Sigma P_1} u_{\Sigma P_2} \qquad \widehat{F}_{S_2} = u_{P_2} u_{S_2} + u_{\Sigma P_2}$$

13.2. A_2 **preprojective algebra** Π_{A_2}**.** The algebra Π_{A_2} is the path algebra of the quiver

$$\begin{tikzcd} 1 \arrow[r, bend left, "a"] & 2 \arrow[l, bend left, "b"] \end{tikzcd}$$

modulo the relations $ab = 0$ and $ba = 0$. The category $K_{\Pi_{A_2}}$ has 6 indecomposable objects. Its Auslander–Reiten quiver is periodic and is depicted below, with the dotted arrows representation the action of τ.

$$\begin{tikzcd}
\cdots \arrow[dr] & & 0 \to P_1 \arrow[dr] & & P_2 \to 0 \arrow[ll, dotted] \arrow[dr] & & \cdots \\
& P_2 \xrightarrow{b} P_1 \arrow[ur] \arrow[dr] & & P_1 \xrightarrow{a} P_2 \arrow[ll, dotted] \arrow[ur] \arrow[dr] & & P_2 \xrightarrow{b} P_1 \arrow[ll, dotted] \arrow[ur] \arrow[dr] & \\
\cdots \arrow[ur] & & P_1 \to 0 \arrow[ll, dotted] \arrow[ur] & & 0 \to P_2 \arrow[ur] & & \cdots \arrow[ll, dotted]
\end{tikzcd}$$

We compute the u-equations and $\widehat{F}$-polynomials. We use the following shorthand for the objects of $K_{\Pi_{A_2}}$:

$$\begin{array}{ccc} P_1 = (0 \to P_1) & P_2 = (0 \to P_2) & S_1 = (P_2 \xrightarrow{b} P_1) \\ S_2 = (P_1 \xrightarrow{a} P_2) & \Sigma P_1 = (P_1 \to 0) & \Sigma P_2 = (P_2 \to 0) \end{array}$$

Then the u-equations are as follows.

$$\begin{array}{ll} u_{P_1} + u_{S_2} u_{\Sigma P_1} u_{\Sigma P_2} = 1 & u_{P_2} + u_{S_1} u_{\Sigma P_1} u_{\Sigma P_2} = 1 \\ u_{S_1} + u_{P_2} u_{S_2}^2 u_{\Sigma P_1} = 1 & u_{S_2} + u_{P_1} u_{S_1}^2 u_{\Sigma P_2} = 1 \\ u_{\Sigma P_1} + u_{P_1} u_{P_2} u_{S_1} = 1 & u_{\Sigma P_2} + u_{P_1} u_{P_2} u_{S_2} = 1 \end{array}$$

The $\widehat{F}$-polynomials are as follows.

$$\begin{array}{ll} \widehat{F}_{P_1} = u_{P_1} u_{P_2} u_{S_2} + u_{P_1} u_{S_1} u_{\Sigma P_2} + u_{S_2} u_{\Sigma P_1} u_{\Sigma P_2} & \widehat{F}_{P_2} = u_{P_1} u_{P_2} u_{S_1} + u_{P_2} u_{S_2} u_{\Sigma P_1} + u_{S_1} u_{\Sigma P_1} u_{\Sigma P_2} \\ \widehat{F}_{S_1} = u_{P_1} u_{S_1} + u_{S_2} u_{\Sigma P_1} & \widehat{F}_{S_2} = u_{P_2} u_{S_2} + u_{S_1} u_{\Sigma P_2} \end{array}$$

We can check using computer algebra software that the ideal generated by the $\widehat{F} = 1$ equations is equal to the ideal generated by the u-equations. In other words, the varieties $\widetilde{\mathcal{M}}_{\Pi_{A_2}}$ and $\widetilde{\mathcal{U}}_{\Pi_{A_2}}$ are equal.

13.3. A_3 **path algebra.** Let $\Lambda = \mathbb{C}Q$ be the path algebra of the quiver $1 \xrightarrow{a} 2 \xrightarrow{b} 3$. Then the category K_Λ has 9 indecomposable objects and its Auslander–Reiten quiver looks like this (with the dotted arrows representing the action of τ).

$$\begin{tikzcd}[column sep=small]
 & & 0 \to P_3 \arrow[dr] & & P_1 \to 0 \arrow[ll, dotted] \arrow[dr] & & \\
 & 0 \to P_2 \arrow[ur] \arrow[dr] & & P_1 \xrightarrow{ba} P_3 \arrow[ll, dotted] \arrow[ur] \arrow[dr] & & P_2 \to 0 \arrow[ll, dotted] \arrow[dr] & \\
0 \to P_1 \arrow[ur] & & P_1 \xrightarrow{a} P_2 \arrow[ll, dotted] \arrow[ur] & & P_2 \xrightarrow{b} P_3 \arrow[ll, dotted] \arrow[ur] & & P_3 \to 0 \arrow[ll, dotted]
\end{tikzcd}$$

We use the following shorthand for the objects of $K_{\Pi_{A_2}}$:

$$\begin{array}{ccc} P_1 = (0 \to P_1) & P_2 = (0 \to P_2) & P_3 = (0 \to P_3) \\ S_2 = (P_1 \xrightarrow{a} P_2) & I_2 = (P_1 \xrightarrow{ba} P_3) & S_3 = (P_2 \xrightarrow{b} P_3 \\ \Sigma P_1 = (P_1 \to 0) & \Sigma P_2 = (P_2 \to 0) & \Sigma P_3 = (P_3 \to 0) \end{array}$$

Then the u-equations are as follows.

$$\begin{array}{lll} u_{P_1} + u_{S_2} u_{I_2} u_{\Sigma P_1} = 1 & u_{P_2} + u_{I_2} u_{S_3} u_{\Sigma P_1} u_{\Sigma P_2} = 1 & u_{P_3} + u_{\Sigma P_1} u_{\Sigma P_2} u_{\Sigma P_3} = 1 \\ u_{S_2} + u_{P_1} u_{S_3} u_{\Sigma P_2} = 1 & u_{I_2} + u_{P_1} u_{P_2} u_{\Sigma P_2} u_{\Sigma P_3} = 1 & u_{S_3} + u_{P_2} u_{S_2} u_{\Sigma P_3} = 1 \\ u_{\Sigma P_1} + u_{P_1} u_{P_2} u_{P_3} = 1 & u_{\Sigma P_2} + u_{P_2} u_{P_3} u_{S_2} u_{I_2} = 1 & u_{\Sigma P_3} + u_{P_3} u_{I_2} u_{S_3} = 1 \end{array}$$

The $\widehat{F}$-polynomials are as follows.

$$\widehat{F}_{P_1} = u_{P_1} + u_{S_2} u_{I_2} u_{\Sigma P_1}$$
$$\widehat{F}_{P_2} = u_{P_1} u_{P_2} + u_{P_2} u_{S_2} u_{I_2} u_{\Sigma P_1} + u_{I_2} u_{S_3} u_{\Sigma P_1} u_{\Sigma P_2}$$
$$\widehat{F}_{P_3} = u_{P_1} u_{P_2} u_{P_3} + u_{P_2} u_{P_3} u_{S_2} u_{I_2} u_{\Sigma P_1} + u_{P_3} u_{I_2} u_{S_3} u_{\Sigma P_1} u_{\Sigma P_2} + u_{\Sigma P_1} u_{\Sigma P_2} u_{\Sigma P_3}$$
‘ $$\widehat{F}_{S_2} = u_{P_2} u_{S_2} + u_{S_3} u_{\Sigma P_2}$$
$$\widehat{F}_{I_2} = u_{P_2} u_{P_3} u_{S_2} u_{I_2} + u_{P_3} u_{I_2} u_{S_3} u_{\Sigma P_2} + u_{\Sigma P_2} u_{\Sigma P_3}$$
$$\widehat{F}_{S_3} = u_{P_3} u_{I_2} u_{S_3} + u_{\Sigma P_3}$$

13.4. A_3 **with a relation.** Let Λ be $\mathbb{C}\langle 1 \leftarrow 2 \leftarrow 3 \rangle$ with the composition of the two arrows being zero. We refer to the indecomposables as $1, 2, 3, 21, 32$.

1. $g_1 = (1, 0, 0)$. $F_1 = 1 + y_1$. $\widehat{F}_1 = u_1 + u_2 u_{P_1[1]}$. $v_1 = 1/F_1$.
2. $g_2 = (-1, 1, 0)$. $F_2 = 1 + y_2$. $\widehat{F}_2 = u_2 u_{21} + u_3 u_{P_2[1]}$. $v_2 = F_{21}/F_2 F_1$
3. $g_3 = (0, -1, 1)$. $F_3 = 1 + y_3$. $\widehat{F}_3 = u_3 u_{32} + u_{P_3[1]}$. $v_3 = F_{32}/F_3 F_2$.

21. $g_{21} = (0, 1, 0)$. $F_{21} = 1 + y_1 + y_1 y_2$. $\widehat{F}_{21} = u_1 u_{21} + u_2 u_{21} u_{P_1[1]} + u_3 u_{P_1[1]} u_{P_2[1]}$. $v_{21} = F_1/F_{21}$.

32. $g_{32} = (0, 0, 1)$. $F_{32} = 1 + y_2 + y_2 y_3$. $u_{32} = F_2/F_{32}$. $\widehat{F}_{32} = u_2 u_{21} u_{32} + u_{32} u_3 u_{P_2[1]} + u_{P_2[1]} u_{P_3[1]}$. $v_{32} = F_2/F_{32}$.

$P_1[1]$. $v_{P_1[1]} = y_1 F_2 / F_{21}$.

$P_2[1]$. $v_{P_2[1]} = y_2F_3/F_{32}$
$P_3[1]$ $v_{P_3[1]} = y_3/F_3$.
The u-equations are:

$$\begin{aligned} u_1 + u_2u_{P_1[1]} &= 1 \\ u_2 + u_1u_3u_{P_2[1]} &= 1 \\ u_3 + u_{21}u_2u_{P_3[1]} &= 1 \\ u_{21} + u_3u_{P_1[1]}u_{P_2[1]} &= 1 \\ u_{32} + u_{P_2[1]}u_{P_3[1]} &= 1 \\ u_{P_1[1]} + u_1u_{21} &= 1 \\ u_{P_2[1]} + u_{21}u_2u_{32} &= 1 \\ u_{P_3[1]} + u_{32}u_3 &= 1 \end{aligned}$$

13.5. **Pellytopes.** Let A be the path algebra of the quiver $1 \to 2 \to \cdots \to n$, modulo the ideal generated by the relations that the composition of any two consecutive arrows is zero.

There are $3n-1$ elements of $\mathcal{I}$: $P_1, \dots, P_n = S_n; \Sigma P_1, \dots \Sigma P_n; S_1, \dots, S_{n-1}$. It will turn out to be more convenient to view P_n as part of the family of simples, rather than the family of projectives, so we will consistently write S_n instead of P_n (though we will still write ΣP_n). The u-equations are as follows:

$$\begin{aligned} u_{P_i} + u_{S_{i-1}}u_{\Sigma P_i}u_{\Sigma P_{i+1}} &= 1 \\ u_{\Sigma P_i} + u_{P_{i-1}}u_{P_i}u_{S_i} &= 1 \\ u_{S_i} + u_{P_{i+1}}u_{S_{i+1}}u_{\Sigma P_i}u_{S_{i-1}} &= 1 \end{aligned}$$

where i runs from 1 to $n-1$ in the first equation, and from 1 to n in the second and third, and we drop the variables $u_{S_0}, u_{S_{n+1}}, u_{P_0}, u_{P_n}, u_{P_{n+1}}$ whenever they arise. These are exactly the u-equations of [**?**].

We have numbered the vertices of our quiver so that the g-vectors of the objects of $\mathcal{I}$ agree with [**?**]. The g-vector of S_i is $\mathbf{e}_i - \mathbf{e}_{i+1}$ for $1 \le i \le n-1$, while the g-vector of $S_n = P_n$ is $\mathbf{e}_n$. The g-vector of P_i is of course $\mathbf{e}_i$, while the g-vector of ΣP_i is $-\mathbf{e}_i$.

The F-polynomials are given by $F_{P_i} = 1 + y_{i+1} + y_iy_{i+1}$ for $1 \le i \le n-1$, and $F_{S_i} = 1 + y_i$ for $1 \le i \le n$.

The $\widehat{F}$-polynomials are

$$\begin{aligned} \widehat{F}_{P_i} &= u_{P_i}u_{P_{i+1}}u_{S_{i+1}} + u_{\Sigma P_{i+1}}u_{S_i}u_{P_i} + u_{\Sigma P_{i+1}}u_{\Sigma P_i}u_{S_{i-1}} \\ \widehat{F}_{S_i} &= u_{P_i}u_{S_i} + u_{\Sigma P_i}u_{S_{i-1}}, \end{aligned}$$

where i runs from 1 to $n-1$ in the first equation and from 1 to n in the second. Again, we drop u_{S_0}, u_{P_n}.

The solutions v_X are given by:

$$\begin{aligned} v_{P_i} &= \frac{F_{S_{i+1}}}{F_{P_i}} \\ v_{\Sigma P_i} &= \frac{y_iF_{S_{i-1}}}{F_{P_{i-1}}} \text{ for } 2 \le i \le n \\ v_{\Sigma P_1} &= \frac{y_1}{1+y_1} \\ v_{S_i} &= \frac{F_{P_i}}{F_{S_{i+1}}F_{S_i}} \end{aligned}$$

We drop the terms $F_{P_n}, F_{S_{n+1}}$ when they appear. (We could also use the formulas given above for them, and then decree that $y_{n+1} = 0$.)

Now consider the $\hat{F}$-equations (or equivalently u-equations) for Λ, the path algebra of the same A_n quiver, with no relations. We set u_{ij} to correspond to M_{ij}, the indecomposable representation supported at vertices k with $i < k < j$. This accounts for all u_{ij} with $0 \le i$, $i \le j-2$, $j \le n+1$.

We let $u_{-1,i}$ correspond to ΣP_i. The index set $\mathcal{I}$ is thereby identified with the diagonals of an $(n+3)$-gon, with vertices numbered -1 to $n+1$.

To find solutions to the Pellytope u-equations in terms of those from Λ (which we denote v_{ij}), as in Section 11, we observe that for any representation of A_n which is neither simple nor projective, its projective resolution, when tensored by the radical square zero quotient of the path algebra, splits into a direct sum of an indecomposable projective and a shifted indecomposable projective. Concretely, for $i \leq j-2$, we have the presentation

$$\widehat{P}_j \to \widehat{P}_{i+1} \to M_{ij} \to 0.$$

(Here we write $\widehat{P}_i$ for the projective Λ-module with simple top S_i, keeping the notation P_i for the corresponding projective A-module.)

If $i = j-2$, when we tensor the presentation of $M_{j-2,j}$ by A, it becomes a minimal projective presentation of S_{j-1}. However, if $i \leq j-3$, the result of tensoring the presentation of M_{ij} by A is a complex of the form $P_j \to P_{i+1}$. Since $\mathrm{Hom}(P_j, P_{i+1})$ is 0, the map must be zero, so the complex splits as $(0 \to P_{i+1}) \oplus (P_j \to 0)$. The result is that each $u_{i,j}$ with $i \neq -1$, $j \neq n+1$, and $j-i > 2$, appears in the product for P_{i+1} and that for ΣP_j.

The solution to the Pellytope u-equations in terms of the solutions for Λ is therefore given by:

$$\begin{aligned} v_{S_i} &= v_{i-1,i+1} \\ v_{P_i} &= \prod_{j=i+2}^{n+1} v_{i-1,j} \\ v_{\Sigma P_i} &= \prod_{\ell=-1}^{i-3} v_{\ell,i} \end{aligned}$$

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