

THE CONFIGURATION SPACES OF FINITE REPRESENTATION TYPE ALGEBRAS

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1. INTRODUCTION

To every finite-dimensional algebra over \mathbb{C} , Λ , of finite representation type, we define and study an affine variety $\widetilde{\mathcal{M}}_\Lambda$. We show that this assignment of varieties to algebras behaves functorially under algebra quotients. We also study the real, non-negative points of these varieties, $\widetilde{\mathcal{M}}_\Lambda^{\geq 0}$. The boundaries of these regions decompose into strata that are themselves the non-negative parts of $\widetilde{\mathcal{M}}$ -varieties for other algebras, obtained as Jasso reductions of Λ . Our results illustrate a fascinating meeting point between algebraic geometry, tropical geometry, and representation theory.

This paper is a large generalization of earlier work, by both physicists and mathematicians, in the setting of Dynkin type algebras. The “ u -equations” associated to Dynkin type A quivers were first introduced by physicists Koba and Nielsen in 1969 [40, 39, 38]. These equations define an affine variety that we denote $\widetilde{\mathcal{U}}_{A_n}$.¹ For example, in the case of A_2 , the u -equations are five equations in five variables, and define a 2-dimensional variety:

$$\begin{aligned} u_{13} + u_{24}u_{25} &= 1 & u_{24} + u_{35}u_{13} &= 1 & u_{35} + u_{14}u_{24} &= 1 \\ u_{14} + u_{25}u_{35} &= 1 & u_{25} + u_{13}u_{14} &= 1 \end{aligned}$$

Here, the variables can be identified with the chords of a pentagon and the monomials appearing in the equations are determined by how the chords cross. The u -equations led Koba and Nielsen to the first definition of tree-level string amplitudes, which are integrals over the non-negative points, $\mathcal{U}_{A_n}^{\geq 0}$. Koba and Nielsen showed that the boundary strata of $\mathcal{U}_{A_n}^{\geq 0}$ intersect in the same way as the faces of the n -dimensional associahedron. This property captures the important physical property of ‘factorisation’, and played a key role in the early development of string theory.

Arkani-Hamed, He, and Lam [6] showed that similar u -equations can be defined for any Dynkin quiver, Q . These define affine varieties, $\widetilde{\mathcal{U}}_{CQ}$, that we view as associated to the path algebra, CQ , of the quiver. Motivated by cluster algebras, the same authors also introduced a second system of \widehat{F} -equations, and showed that they define an n -dimensional irreducible affine variety, $\widetilde{\mathcal{M}}_{CQ}$. For example, in the case of A_2 , the \widehat{F} -equations are three equations in the same five variables introduced above:

$$u_{13} + u_{24}u_{25} = 1 \quad u_{13}u_{14} + u_{14}u_{24}u_{25} + u_{25}u_{35} = 1 \quad u_{14}u_{24} + u_{35} = 1$$

These equations again define a 2-dimensional variety. In fact, [6] proved that these two families of varieties are identical, $\widetilde{\mathcal{U}}_{CQ} = \widetilde{\mathcal{M}}_{CQ}$. They found a rational parametrisation of $\widetilde{\mathcal{M}}_{CQ}$. And they showed that the boundary strata of the non-negative part, $\widetilde{\mathcal{M}}_{CQ}^{\geq 0}$, intersect in the same way as the face lattice of the generalized associahedron of type Q .

In this paper, we define affine varieties $\widetilde{\mathcal{U}}_\Lambda$ and $\widetilde{\mathcal{M}}_\Lambda$ for every finite-dimensional algebra, Λ , of finite representation type. Note that this includes non-hereditary algebras. These varieties agree with those defined by [6] when we take Λ to be the path algebra of a Dynkin quiver. We show that our assignment of varieties to algebras is functorial under algebra quotients, which correspond to toric blowdowns of the $\widetilde{\mathcal{M}}_\Lambda$ varieties. Moreover, we show that the divisors $u_X = 0$ of $\widetilde{\mathcal{M}}_\Lambda$, where one variable is set to zero, correspond to so-called *Jasso reductions* of Λ . See below (Section 1.1)

¹This variety can be viewed as an affine open chart on $\mathcal{M}_{0,n}$, the moduli space of n points in \mathbb{P}^1 .

for an overview of our results. Taken together, our results generalize results of Arkani-Hamed, He, and Lam [6], Chapoton [19], and Kus-Reineke [42] for the Dynkin case, and illustrate how the geometric properties of the varieties $\widetilde{\mathcal{M}}_\Lambda$ can be intimately identified with representation theory data.

1.1. Summary of main results. Fix a finite representation type \mathbb{C} -algebra, Λ , with n isomorphism classes of simple modules. We consider the homotopy category K_Λ of 2-term complexes of projective modules. We write the objects of K_Λ as $X = (X^{-1} \rightarrow X^0)$, for X^{-1}, X^0 in $\text{proj } \Lambda$. The indecomposable objects $X^{-1} \rightarrow X^0$ of K_Λ are of two forms: minimal projective presentations of the module $H^0(X)$, or $P_i \rightarrow 0$, where P_i is an indecomposable projective module. Both categories, K_Λ and $\text{mod } \Lambda$, play a key role in our definitions. We use the indecomposable objects of K_Λ , $X \in \text{ind } K_\Lambda$, to index the set of u -variables, u_X . Then we introduce *compatibility degree*, $c(X, Y)$, of $X, Y \in \text{ind } K_\Lambda$. This can be written in terms of the dimensions of Hom groups in $\text{mod } \Lambda$ (Lemma 3.4):

$$c(X, Y) = \text{hom}_\Lambda(H^0(X), H^0(\tau Y)) + \text{hom}_\Lambda(H^0(Y), H^0(\tau X)).$$

Here, τ is the *Auslander-Reiten translation* on K_Λ . Note that this definition of $c(X, Y)$ involves both categories. Given this, the u -equations are

$$u_X + \prod_{Y \in \text{ind } K_\Lambda} u_Y^{c(X, Y)} = 1,$$

for each $X \in \text{ind } K_\Lambda$. This system of equations defines an affine variety $\widetilde{\mathcal{U}}_\Lambda$. We can also define a polynomial in the same variables, $\widehat{F}_M \in \mathbb{Z}[u_X \mid X \in \text{ind } K_\Lambda]$, for each indecomposable module $M \in \text{mod } \Lambda$ (Definition 2.13). Then the \widehat{F} -equations are,

$$\widehat{F}_M = 1,$$

and these define an affine variety $\widetilde{\mathcal{M}}_\Lambda$ (Definition 3.1).

In this paper we make $\widetilde{\mathcal{M}}_\Lambda$ the focus of our study. These varieties are a natural generalization of the affine varieties introduced by [6] for the Dynkin cases. However, it is interesting to comment on how the two families of affine varieties, $\widetilde{\mathcal{M}}_\Lambda$ and $\widetilde{\mathcal{U}}_\Lambda$, are related. We prove (Corollary 3.6) that $\widetilde{\mathcal{M}}_\Lambda \subseteq \widetilde{\mathcal{U}}_\Lambda$. Moreover, we prove (Proposition 3.9) that $\widetilde{\mathcal{M}}_\Lambda = \widetilde{\mathcal{U}}_\Lambda$ in the case that the *Auslander-Reiten quiver* of K_Λ has no oriented cycles. However, we have also verified in examples (Example 3.11), that this equality holds more generally, leading to the following conjecture.

Conjecture 1.1. *For any finite representation type algebra Λ , $\widetilde{\mathcal{U}}_\Lambda = \widetilde{\mathcal{M}}_\Lambda$.*

We establish the following properties of the $\widetilde{\mathcal{M}}_\Lambda$ varieties.

- (1) $\widetilde{\mathcal{M}}_\Lambda$ is an irreducible variety of dimension n . Moreover, there is an open embedding of $\widetilde{\mathcal{M}}_\Lambda$ into a toric variety (Theorem 9.4).

The fan of this toric variety is the g -vector fan of Λ (see Section 2.4). The rays of the g -vector fan correspond to indecomposable objects of K_Λ which are *rigid* (i.e. satisfy $\text{Hom}(X, \Sigma X) = 0$). These rigid modules correspond to interesting subvarieties of $\widetilde{\mathcal{M}}_\Lambda$:

- (2) For any rigid indecomposable $X \in \text{ind } K_\Lambda$, the divisor $u_X = 0$ in $\widetilde{\mathcal{M}}_\Lambda$ is isomorphic to $\widetilde{\mathcal{M}}_B$, for some algebra B which is called the *Jasso reduction* of Λ at $H^0(X)$ (this is Theorem 8.1). A similar statement (Theorem 8.2) holds for the $\widetilde{\mathcal{U}}_\Lambda$ varieties.

For every $X \in \text{ind } K_\Lambda$, we define (Section 5) a rational function v_X of n variables y_1, \dots, y_n . Then

- (3) The functions $v_X \in \mathbb{C}(y_1, \dots, y_n)$ provide a rational parametrization of $\widetilde{\mathcal{M}}_\Lambda$, i.e. the coordinate ring of $\widetilde{\mathcal{M}}_\Lambda$ is isomorphic to the subring of $\mathbb{C}(y_1, \dots, y_n)$ generated by the v_X (Theorem 9.2). In other words, setting $u_X = v_X(y)$ solves the $\widehat{F}_M = 1$ equations and all solutions are obtained in this way (Corollary 5.11).

It is natural to consider the real, non-negative points of these varieties, $\widetilde{\mathcal{M}}_\Lambda^{\geq 0}$, which is closely related to the toric variety and corresponding g -vector fan mentioned above.

- (4) Restricting the variables y_i to be real, non-negative, $y_i \geq 0$, the rational parametrization above restricts to the non-negative part of the variety, $\widetilde{\mathcal{M}}_{\Lambda}^{\geq 0}$.
- (5) The strata of $\widetilde{\mathcal{M}}_{\Lambda}^{\geq 0}$ are in bijection with the cones of the g -vector fan of Λ . Namely, the codimension-1 boundary strata correspond to the rays of the fan, and the higher codimension strata correspond to its other cones.

The five properties above generalize properties established in [6] for the Dynkin case to all finite representation type algebras. Our more general setting allows us to further establish that our assignment of varieties to algebras is functorial in the sense that:

- (6) If A is a quotient of Λ , then there is a surjective map from $\widetilde{\mathcal{M}}_{\Lambda}$ to $\widetilde{\mathcal{M}}_A$ (Theorem 11.1). As a homomorphism of coordinate rings, this map is a monomial map. However, this map can also be viewed as the blowdown map of the toric varieties associated to the g -vector fans of Λ and A (Theorem 11.4).

We also establish the following property of the *Rogers dilogarithm*,

$$L(x) = - \int_0^x \frac{\log(1-t)}{t} dt + \frac{1}{2} \log(x) \log(1-x),$$

which generalizes the dilogarithm identities proved by Chapoton [19] in the Dynkin setting.

- (7) The sum of the Rogers dilogarithms $L(1-v_X)$ over all $X \in \text{ind } K_{\Lambda}$ is a constant, i.e., independent of y_1, \dots, y_n (Theorem 12.1).

Finally, the rational functions v_X , which play a key role in establishing the above properties, also have a further interest in their own right. For X a rigid indecomposable in K_{Λ} , the *tropicalization* of v_X , $\text{trop } v_X$, is a piecewise linear function on \mathbb{R}^n . Evaluated on some vector $g \in \mathbb{Z}^n$, $\text{trop } v_X$ evaluates to the multiplicity of X as a summand of a generic 2-term projective complex of weight g (see Section 2.4 for the definitions). This is Theorem 7.3, and we prove this for arbitrary finite-dimensional algebras; the assumption that Λ is of finite representation type is not used here. This theorem generalizes the recent result of Kus–Reineke for Dynkin quivers [42].

1.2. Introduction from a cluster algebras point of view. The results of [6] are couched in the language of cluster algebras. We summarize their results here, in order to clarify how our work is related to theirs, and to explain how this paper generalizes their results. A reader for whom cluster algebras are unfamiliar (or unappealing) can safely skip this discussion. We do not otherwise make use of cluster algebras in this paper.

Given a Dynkin type Φ (not necessarily simply-laced) of rank n , with orientation, there is an associated cluster algebra with *universal coefficients*, \mathcal{A} . The cluster variables of \mathcal{A} , x_{γ} , are indexed by the *almost positive roots* $\gamma \in \Phi_{\geq -1}$. Moreover, there is a pairing on the almost positive roots, $(\gamma \parallel \beta) \in \mathbb{Z}_{\geq 0}$, for $\gamma, \beta \in \Phi_{\geq -1}$ (this is the *Fomin–Zelevinsky compatibility degree*).

The affine varieties introduced in [6] can then be described as follows. Introduce a variable u_{γ} for each $\gamma \in \Phi_{\geq -1}$. Then we can consider the variety cut out by the system of equations,

$$u_{\gamma} + \prod_{\beta \in \Phi_{\geq -1}} u_{\beta}^{(\beta \parallel \gamma)} = 1,$$

where we have one equation for each $\gamma \in \Phi_{\geq -1}$. In particular, in type A_n , the almost positive roots can be identified with the diagonals of an $(n+3)$ -gon, and the compatibility degree $(\beta \parallel \gamma)$ is 1 if β and γ cross in their interiors, and zero otherwise. This recovers the u -equations of Koba–Nielsen [40, 39, 38].

A second family of affine varieties, also introduced in [6], can be defined using the same data. These are cut out by equations we call the \widehat{F} -equations. In the Dynkin setting, the \widehat{F} - and u -equations turn out to define the same varieties. However, the equations look very different, and are defined as follows. For each $\beta \in \Phi_{\geq -1}$, let x_{β} be the corresponding variable in the cluster algebra with universal coefficients, where the coefficient variables, also indexed by $\Phi_{\geq -1}$, are denoted u_{β} . For a negative simple root, $-\alpha_i$, the corresponding $x_{-\alpha_i}$ is an initial variable. The

polynomial \widehat{F}_β is then given by

$$\widehat{F}_\beta = x_\beta|_{x_{-\alpha_i}=1}.$$

That is to say, \widehat{F}_β is obtained from x_β by setting all initial cluster variables equal to 1, leaving a polynomial in the coefficient variables $\{u_\gamma\}_{\gamma \in \Phi_{\geq -1}}$. (The procedure to produce \widehat{F}_β is analogous to the procedure for defining the F -polynomials given in [28]; there, however, one starts with the cluster algebra with principal coefficients.) The system of \widehat{F} -equations are simply

$$\widehat{F}_\beta = 1,$$

for each $\beta \in \Phi_{>0}$.

The positive real points of these varieties have interesting boundary stratifications. In fact, the divisors where a single variable vanishes, $u_\beta = 0$, are themselves affine varieties of the same kind.

Moreover, [6] show in the Dynkin setting how to give a rational parametrization of these varieties. Suppose that x_γ and x_δ are two cluster variables related by mutation. Then we have an “exchange relation”, $x_\gamma x_\delta = M_1 + M_2$, where M_1 and M_2 are two monomials in the remaining variables in the cluster and also in the coefficient variables.² To each x_γ (with $\gamma \in \Phi_{\geq 0}$), [6] define a specific cluster containing x_γ . Mutating x_γ in this cluster gives some exchange relation, $x_\gamma x_\delta = M_1 + M_2$, and [6] show that setting

$$u_\gamma = \frac{M_1}{x_\gamma x_\delta} = \frac{1}{1 + M_2/M_1}$$

solves the u -equations (equivalently, the \widehat{F} -equations). In other words, these rational functions of the cluster variables parametrize the varieties $\widetilde{\mathcal{M}}_{CQ}$.

The present paper greatly generalizes these results. We define u -equations and \widehat{F} equations for any algebra of finite representation type, Λ . Note that most such algebras do not correspond to a cluster algebra. However, there are suitable generalizations of all the key ingredients in the cluster algebra definitions. In place of cluster compatibility degree, we introduce a compatibility degree expressed using the dimensions of the Hom groups in $\text{mod } \Lambda$. In place of cluster variables, we use the F -polynomials associated to Λ , as introduced by Derksen–Weyman–Zelevinsky. Finally, we define a suitable generalization of “ F -polynomials with universal coefficients” to our setting, which we continue to denote as \widehat{F} . The results we obtain in our setting are summarized above, Section 1.1, and in particular generalize the results for the Dynkin cases.

1.3. Motivation from physics and number theory. Moduli spaces and their geometry play a central role in both mathematics and theoretical physics. In physics, moduli spaces are important for string theory, matrix models [41], and scattering amplitudes. Recent work has shown that so-called u -equations furnish physicists with new tools for the study of scattering amplitudes and related quantities [6]. In particular, the tropical geometry of the varieties defined by u -equations has applications to amplitude calculations [4, 5].

Families of u -equations have also appeared outside of physics. The u -equations in [12] arise in the study of a special class of polytopes, called pelytopes. From the perspective of this paper, these u -equations arise from quotients of type A_n algebras (see Section 13).

In number theory, u -equations have also played a prominent role. Brown showed that u -equations define affine open charts of $\mathcal{M}_{0,n}$ and used these to prove that all periods of $\mathcal{M}_{0,n}$ can be expressed as multiple zeta values (MZVs) [14]. A key feature of this proof is that u -equations behave well under the natural forgetful morphisms $\mathcal{M}_{0,n+1} \rightarrow \mathcal{M}_{0,n}$. The resulting iterated integrals bear a striking resemblance to string amplitude integrals and provide a geometric framework for understanding the appearance of MZVs in string theory amplitudes [15, 13]. In addition, Brown’s work suggests that the geometry of u -equation varieties may be useful for studying relations among MZVs—such as Zagier’s height-two identity [48].

Finally, there is a close connection to special functions. The Rogers dilogarithm $L(x)$ satisfies an identity that may be viewed as a u -equation for the A_2 root system [37]. This identity played a key role in the formulation of Nahm’s conjecture about the modular properties of hypergeometric

²The ratio M_2/M_1 can be identified as an \mathcal{X} -cluster variable, in the sense of Fock and Goncharov.

q -series. Generalisations of Rogers dilogarithm identities have been established for algebras of Dynkin type in [19]. In this paper, we use our approach to extend these results. We present a natural family of Rogers dilogarithm identities, one for every finite representation type algebra (Section 12).

2. ALGEBRA BACKGROUND

Let Λ be an associative \mathbb{C} -algebra, finite-dimensional over \mathbb{C} . Let e_1, \dots, e_n be a complete set of pairwise orthogonal primitive idempotents of Λ . Then, as a right Λ -module, Λ is a direct sum of the indecomposable projective modules: $P_1 = e_1\Lambda, \dots, P_n = e_n\Lambda$. We assume that Λ is *basic*; i.e. the P_i are pairwise non-isomorphic. We write $\text{mod } \Lambda$ for the abelian category of finite-dimensional right Λ -modules, and $\text{proj } \Lambda$ and $\text{inj } \Lambda$ for the full additive subcategories of projective and injective modules, respectively. We write $S_i = P_i / \text{rad } P_i$ for the *simple* right Λ modules. The indecomposable injective modules, I_i , are the injective envelopes of S_i . In particular, note that the socle of I_i is $\text{soc } I_i = S_i$.

We write $\text{hom}_\Lambda(,)$ for the dimension of $\text{Hom}_\Lambda(,)$ over \mathbb{C} , and similarly $\text{hom}_\mathcal{C}(,)$ for the dimension of $\text{Hom}_\mathcal{C}(,)$ in a category \mathcal{C} . Similarly, we write $\text{ext}_\Lambda^1(,)$ for the dimension of $\text{Ext}_\Lambda^1(,)$. We write DV for the \mathbb{C} -linear dual vector space of a \mathbb{C} -vector space V .

2.1. Almost split morphisms. We briefly recall some key elements of Auslander–Reiten theory. The reader interested in further details is invited to consult [8, Chapter IV].

Consider a short exact sequence, $0 \rightarrow L \xrightarrow{f} M \xrightarrow{g} N \rightarrow 0$, in $\text{mod } \Lambda$. It is *split* if it is isomorphic to $0 \rightarrow L \rightarrow L \oplus N \rightarrow N \rightarrow 0$. In a split short exact sequence, f is called a *section* and g is called a *retraction*. Obviously, in a split exact sequence, any map $h : Z \rightarrow N$ can be lifted to a map $h' : Z \rightarrow M$ such that $h = gh'$. Under the same hypothesis, any map $j : L \rightarrow Z$ can be extended to a map $j' : M \rightarrow Z$ with $j = j'f$.

An important class of short exact sequences are the *almost split* or *Auslander–Reiten* sequences. They are very close to sharing the above lifting/extension properties. For N indecomposable, a map $g : M \rightarrow N$ is *right almost split* if it is not a retraction, and for any indecomposable module Z and non-isomorphism $h : Z \rightarrow N$, there exists a map $h' : Z \rightarrow M$ such that $h = gh'$. Similarly, for L indecomposable, a map $f : L \rightarrow M$ is *left almost split* if it is not a section, and for any indecomposable module Z and non-isomorphism $j : Z \rightarrow L$, there exists a map $j' : M \rightarrow Z$ with $j = j'f$.

A short exact sequence $0 \rightarrow L \xrightarrow{f} M \xrightarrow{g} N \rightarrow 0$ is *almost split* if and only if N is indecomposable and f is left almost split, or equivalently if and only if L is indecomposable and g is right almost split. [8, Theorem IV.1.13]. Any non-projective indecomposable module N is the right-most term of an almost split sequence, $0 \rightarrow \tau N \rightarrow M \rightarrow N \rightarrow 0$, unique up to isomorphism [8, Theorem IV.3.1]. Similarly, any non-injective indecomposable L is the left-most term of an almost split sequence, $0 \rightarrow L \rightarrow M \rightarrow \tau^{-1}L \rightarrow 0$. This defines the *Auslander–Reiten translation*, τ , of $\text{mod } \Lambda$.

Note that the almost split sequences provide a supply of left and right almost split maps (a right almost split map to every non-projective indecomposable, and a left almost split map from every non-injective indecomposable). We will also make use of right almost split maps into indecomposable projective modules and left almost split maps out of indecomposable injective modules. If P is indecomposable projective, the inclusion $\text{rad } P \rightarrow P$ is right almost split. Moreover, if I is indecomposable injective, then the quotient $I \rightarrow I / \text{soc } I$ is left almost split [8, Proposition IV.3.5].

The following proposition follows directly from the almost splitness of the maps mentioned above. We use this, in particular, in Section 4, where we show how these identities define a useful set of dual bases of the split Grothendieck group.

Proposition 2.1. *Suppose that $0 \rightarrow \tau N \rightarrow M \rightarrow N \rightarrow 0$ is an almost split sequence, and X is an indecomposable module. Then:*

- (1) $\text{hom}_\Lambda(X, \tau N) - \text{hom}_\Lambda(X, M) + \text{hom}_\Lambda(X, N) = 0$ unless $X \simeq N$, in which case it equals 1.
- (2) $-\text{hom}_\Lambda(X, \text{rad } P_i) + \text{hom}_\Lambda(X, P_i) = 0$ unless $X \simeq P_i$, in which case it equals 1.

- (3) $\text{hom}_\Lambda(N, X) - \text{hom}_\Lambda(M, X) + \text{hom}_\Lambda(\tau N, X) = 0$ unless $X \simeq \tau N$, in which case it equals 1.
- (4) $\text{hom}_\Lambda(I_i, M) - \text{hom}_\Lambda(I_i/S_i, M) = 0$, unless $M \simeq I_i$, in which case it is 1.

The *injectively stable morphisms* from X to Y in $\text{mod } \Lambda$, denoted $\overline{\text{Hom}}_\Lambda(X, Y)$ are by definition $\text{Hom}_\Lambda(X, Y)$ quotiented by the subspace of those morphisms factoring through an injective Λ -module. One of the Auslander–Reiten formulas [8, Theorem IV.2.13] says the following:

Theorem 2.2. $\text{Ext}_\Lambda^1(X, Y) \simeq D\overline{\text{Hom}}_\Lambda(Y, \tau X)$.

We will need the following lemma:

Lemma 2.3. *Let M be an indecomposable Λ -module. Then $\text{Ext}_\Lambda^1(M, S_i) \simeq D\text{Hom}_\Lambda(S_i, \tau M)$.*

Proof. By the Auslander–Reiten formula of Theorem 2.2, $\text{Ext}_\Lambda^1(M, S_i) \simeq D\overline{\text{Hom}}_\Lambda(S_i, \tau M)$. The only indecomposable injective which admits a non-zero morphism from S_i is I_i . An injection from I_i to τM splits, so in that case τM , being indecomposable, would have to be isomorphic to I_i , which is impossible. Thus, a map from S_i to τM would necessarily factor through a proper quotient of I_i , and thus a quotient of $I_i/\text{soc } I_i$. The only possible non-zero image of S_i in I_i being $\text{soc } I_i$, no non-zero map from S_i to τM factors through I_i , and therefore $D\overline{\text{Hom}}_\Lambda(S_i, \tau M) \simeq D\text{Hom}_\Lambda(S_i, \tau M)$. \square

2.2. The category K_Λ . Let $K^b := K^b(\text{proj } \Lambda)$ be the homotopy category of bounded complexes of projective modules. Write Σ for the shift functor of K^b , that shifts all degrees of an object in K^b by -1 .

Definition 2.4. The category, $K_\Lambda := K^{[-1,0]}(\text{proj } \Lambda)$, of *2-term complexes of projectives*, is the full subcategory of K^b whose objects are complexes supported only in degrees -1 and 0 . K_Λ is an extension-closed subcategory of $K^b(\text{proj } \Lambda)$, and so is an extriangulated category in the sense of [44], by a result of [33]. As an extriangulated category, K_Λ has enough projective objects and enough injective objects.

For an object, X , in K_Λ , we write $X = (X^{-1} \rightarrow X^0)$, for projective modules X^{-1}, X^0 in $\text{mod } \Lambda$. The indecomposable projective objects of K_Λ are $P_i = (0 \rightarrow P_i)$. (Note that we write P_i both for the projective indecomposables of $\text{mod } \Lambda$ and of K_Λ .) The indecomposable injective objects are the shifted projectives, $\Sigma P_i = (P_i \rightarrow 0)$.

Definition 2.5. A *conflation* in K_Λ is a distinguished triangle of $K^b(\text{proj } \Lambda)$ whose objects and morphisms are in K_Λ . Given such a distinguished triangle $X \rightarrow E \rightarrow Y \rightarrow \Sigma X$, we write $X \rightarrowtail E \twoheadrightarrow Y$ for the corresponding conflation. A conflation in K_Λ is split (resp. almost split) if it is split (resp. almost split) as a distinguished triangle in $K^b(\text{proj } \Lambda)$.

We write $\text{ind } K_\Lambda$ for the set of indecomposable objects of K_Λ . The category K_Λ has *Auslander–Reiten–Serre duality* [34]. In particular, for any non-projective indecomposable object $X \in \text{ind } K_\Lambda$, there is an almost split conflation of the form $\tau X \rightarrowtail E \twoheadrightarrow X$. For any non-injective indecomposable object $Y \in \text{ind } K_\Lambda$, there is an almost split conflation $Y \rightarrowtail F \twoheadrightarrow \tau^{-1}Y$. (Note that we write τ both for the Auslander–Reiten translation in $\text{mod } \Lambda$ and in K_Λ .)

2.3. The H^0 functor. The category K_Λ is closely related to $\text{mod } \Lambda$. Indeed, there is a functor

$$H^0 : K_\Lambda \rightarrow \text{mod } \Lambda,$$

which sends $X = (X^{-1} \xrightarrow{f} X^0)$ to its cohomology in degree zero, $H^0(X) = X^0/\text{Im}(f)$. In other words, we can identify objects $\text{mod } \Lambda$ with their (minimal) projective presentations, which are objects of K_Λ . However, note that the injectives of K_Λ , ΣP_i , are not projective presentations of a non-zero module in $\text{mod } \Lambda$, and $H^0(\Sigma P_i) = 0$. This leads to:

Lemma 2.6. H^0 induces an equivalence of k -linear categories

$$K_\Lambda / \langle \Sigma \Lambda \rangle \xrightarrow{\sim} \text{mod } \Lambda.$$

Here, $\langle \Sigma \Lambda \rangle$ is the ideal of all morphisms that factor through an object of $\text{add}(\Sigma \Lambda)$.

Lemma 2.7. *A conflation, $X \rightarrowtail E \twoheadrightarrow Y$, in K_Λ is split if and only if the corresponding exact sequence*

$$H^0 X \rightarrow H^0 E \rightarrow H^0 Y$$

is a split short exact sequence in $\text{mod } \Lambda$.

Moreover, H^0 relates the Auslander–Reiten translations of K_Λ and $\text{mod } \Lambda$.

Lemma 2.8. *For a non-projective indecomposable, $X \in \text{ind } K_\Lambda$, consider its Auslander–Reiten conflation in K_Λ , $\tau X \rightarrowtail E \twoheadrightarrow X$. Then*

- (1) *If X is non-injective, then this conflation is sent by H^0 to the Auslander–Reiten sequence in $\text{mod } \Lambda$,*

$$0 \rightarrow H^0(\tau X) \rightarrow H^0 E \rightarrow H^0 X \rightarrow 0.$$

In particular, $H^0(\tau X) = \tau H^0 X$.

- (2) *If X is injective in K_Λ , i.e. $X = \Sigma P_i$, then the above conflation is sent by H^0 to the exact sequence*

$$I_i \rightarrow I_i/S_i \rightarrow 0,$$

where I_i is the injective module in $\text{mod } \Lambda$. In particular, $H^0(\tau \Sigma P_i) = I_i$.

- (3) *Finally, applying Σ^{-1} to the Auslander–Reiten conflation for $X = \Sigma P_i$ gives a triangle in $K^b(\text{proj } \Lambda)$:*

$$\Sigma^{-1}(\tau P_i) \rightarrow \Sigma^{-1} E \rightarrow P_i \rightarrow .$$

This is sent by H^0 to the exact sequence

$$0 \rightarrow \text{rad } P_i \rightarrow P_i.$$

The following form of Auslander–Reiten duality relates morphisms in the module category and the homotopy category.

Lemma 2.9. *For $X, Y \in \text{ind } K_\Lambda$,*

$$\text{Hom}_{K^b}(X, \Sigma Y) \cong D \text{Hom}_\Lambda(H^0 Y, H^0(\tau X)),$$

In particular,

$$\text{hom}_{K^b}(X, \Sigma Y) = \text{hom}_\Lambda(H^0 Y, H^0(\tau X)).$$

Proof. For $X \neq \Sigma P_i$, we have $H^0(\tau X) = \tau H^0 X$ (Lemma 2.8) and the result can be proved using Serre duality and the Nakayama functor ν ; a proof can be found in [46, Lemma 2.6]. For $X = \Sigma P_i$, we have $H^0(\tau X) = I_i$. Indeed, in this case we have $\text{Hom}_{K^b}(X, \Sigma Y) \cong \text{Hom}_\Lambda(P_i, H^0 Y)$. And $\text{Hom}_\Lambda(P_i, H^0 Y) \simeq D \text{Hom}_\Lambda(H^0 Y, I_i)$. \square

2.4. The g -vector fan. A general reference for the material in this subsection is [1]. For each object X in K_Λ , we define a g -vector, $g(X)$, in \mathbb{Z}^n . If $X = (X^{-1} \rightarrow X^0)$, then the entry $g(X)_i$ of $g(X)$ is the multiplicity of P_i in X^0 minus the multiplicity of P_i in X^{-1} .

An object X in K_Λ is called *rigid* if $\text{Hom}_{K^b(\text{proj } \Lambda)}(X, \Sigma X) = 0$. If X is rigid, then P_i cannot appear in both X^{-1} and X^0 [1, Proposition 2.5]. So, in particular, if X is rigid and non-zero then $g(X) \neq 0$. There is at most one rigid object with any given g -vector [1, Theorem 5.5]. We say that two rigid objects X, Y in K_Λ are *compatible* if $\text{Hom}_{K^b}(X, \Sigma Y) = 0$ and $\text{Hom}_{K^b}(Y, \Sigma X) = 0$.

The g -vector fan in \mathbb{Z}^n has rays generated by the g -vectors of indecomposable rigid objects. The cones of the g -vector fan are generated by the g -vectors of sets of pairwise compatible indecomposable rigid objects. As the name suggests, the g -vector fan is indeed a fan [21, Corollary 6.7]. In other words, two cones in the fan intersect either in another cone of the fan. Moreover, if Λ has only finitely many indecomposable rigid objects, then the g -vector fan covers \mathbb{R}^n [7, Proposition 4.8].

2.5. Dimension vectors. For each object M in $\text{mod } \Lambda$, define the *dimension vector*, $d(M) \in \mathbb{Z}^n$, with $d(M)_i = \dim(Me_i)$. For $X \in K_\Lambda$, we define its dimension vector as the dimension vector of $H^0(X)$, and write $d(X) = d(H^0(X))$. Write $\langle -, - \rangle$ for the standard inner product on \mathbb{Z}^n . Then, for $X \in K_\Lambda$ and $M \in \text{mod } \Lambda$, we have the pairing

$$\langle g(X), d(M) \rangle = \sum_{i=1}^n g(X)_i d(M)_i.$$

If $X = (X^{-1} \rightarrow X^0)$, then

$$\langle g(X), d(M) \rangle = \text{hom}_\Lambda(X^0, M) - \text{hom}_\Lambda(X^{-1}, M). \quad (2.1)$$

The following is a variation on [10, Theorem 1.4a] (See also [1, Proposition 2.4(a)].) We include a short proof for the convenience of the reader.

Lemma 2.10. *For an indecomposable $X \in \text{ind } K_\Lambda$, and $M \in \text{mod } \Lambda$,*

$$\langle g(X), d(M) \rangle = \text{hom}_\Lambda(H^0 X, M) - \text{hom}_\Lambda(M, H^0(\tau X)).$$

In the special case that $X = \Sigma P_i$, this identity becomes

$$\langle g(\Sigma P_i), d(M) \rangle = -\text{hom}_\Lambda(M, I_i),$$

where I_i is the injective module in $\text{mod } \Lambda$.

Proof. If $X = \Sigma P_i$, $H^0 X = 0$ and $H^0(\tau X) = I_i$. But $\text{hom}_\Lambda(P_i, M) = \text{hom}_\Lambda(M, I_i)$. So the statement follows directly from (2.1), after using that $\text{hom}_\Lambda(P_i, M) = \text{hom}_\Lambda(M, I_i)$.

Now take X non-injective. We have $X^{-1} \xrightarrow{f} X^0 \xrightarrow{g} H^0 X \rightarrow 0$. Dually, there is a short exact sequence $0 \xrightarrow{\iota} \tau H^0 X \rightarrow \nu X^{-1} \xrightarrow{\nu f} \nu X^0$, where ν denotes the Nakayama functor. Then, applying $\text{Hom}_\Lambda(M, -)$ and $\text{Hom}_\Lambda(-, M)$, respectively, we get the following commutative diagram

$$\begin{array}{ccccc} 0 \rightarrow \text{Hom}(M, \tau H^0 X) & \xrightarrow{\iota^*} & \text{Hom}(M, \nu X^{-1}) & \xrightarrow{(\nu f)^*} & \text{Hom}(M, \nu X^0) \\ & & \uparrow \simeq & & \uparrow \simeq \\ & & D \text{Hom}(X^{-1}, M) & \xrightarrow{Df^*} & D \text{Hom}(X^0, M) & \xrightarrow{Dg^*} & D \text{Hom}(H^0 X, M) \rightarrow 0, \end{array}$$

which defines a four-term exact sequence. Exactness of this implies

$$\text{hom}_\Lambda(M, \tau H^0 X) - \text{hom}_\Lambda(X^{-1}, M) + \text{hom}_\Lambda(X^0, M) - \text{hom}_\Lambda(H^0 X, M) = 0.$$

Then (2.1) gives the result. \square

2.6. F -polynomials. Given a vector $\mathbf{d} \in \mathbb{Z}_{\geq 0}$, let $\text{Gr}_{\mathbf{d}}(M)$ be the *submodule Grassmannian* of M . As a set, the elements of $\text{Gr}_{\mathbf{d}}(M)$ are in bijection with submodules of M of dimension \mathbf{d} . It can be viewed as a Zariski-closed subset of the product $\prod_{i=1}^n \text{Gr}_{d_i}(Me_i)$ of Grassmannians [18], and in this setting it is a projective variety.

Definition 2.11. Let M be a module in $\text{mod } \Lambda$. Its *F -polynomial* is

$$F_M = \sum_{\mathbf{d} \in \mathbb{Z}_{\geq 0}} \chi(\text{Gr}_{\mathbf{d}}(M)) y^{\mathbf{d}} \in \mathbb{Z}[y_1, \dots, y_n],$$

where χ is the Euler characteristic of topological spaces and $y^{\mathbf{d}} = \prod_{i=1}^n y_i^{d_i}$. Moreover, for an object X of K_Λ , we define its F -polynomial as

$$F_X := F_{H^0 X}.$$

2.7. \widehat{F} -polynomials. We now define a second set of polynomials, \widehat{F}_M , in the ring $\mathbb{Z}[u_X \mid X \in \text{ind } K_\Lambda]$. For this, it is important to assume the following

Assumption 2.12 (Finiteness assumption). The set $\text{ind } K_\Lambda$ of isomorphism classes of indecomposable objects of K_Λ is finite. Equivalently, Λ is of finite representation type.

We make this assumption for the rest of the paper.

Definition 2.13. Given a module M in $\text{mod } \Lambda$, its \widehat{F} -polynomial is

$$\widehat{F}_M = \sum_{\mathbf{d} \in \mathbb{Z}_{\geq 0}} \chi(\text{Gr}_{\mathbf{d}}(M)) \prod_{V \in \text{ind } K_\Lambda} u_V^{\text{hom}_\Lambda(H^0 V, M) - \langle g(V), \mathbf{d} \rangle}.$$

And, for $X \in K_\Lambda$, we define

$$\widehat{F}_X := \widehat{F}_{H^0 X}.$$

The \widehat{F} -polynomials are closely related to the F -polynomials by the following map from the y -variables to the u -variables.

Definition 2.14. Define a homomorphism $\Psi : \mathbb{C}[y_1, \dots, y_n] \rightarrow \mathbb{C}[u_V^{\pm 1} \mid V \in \text{ind } K_\Lambda]$ that maps

$$\Psi : y_i \mapsto \prod_{V \in \text{ind } K_\Lambda} u_V^{-g(V)_i}.$$

In particular, Ψ maps the monomial $y^{\mathbf{d}}$ to $\prod_V u_V^{-\langle g(V), \mathbf{d} \rangle}$. So

Lemma 2.15. \widehat{F}_M is given by

$$\widehat{F}_M = \left(\prod_{V \in \text{ind } K_\Lambda} u_V^{\text{hom}_\Lambda(H^0 V, M)} \right) \Psi(F_M).$$

We verify that

Lemma 2.16. For any $M \in \text{mod } \Lambda$, \widehat{F}_M is indeed a polynomial. That is, $\widehat{F}_M \in \mathbb{Z}[u_V \mid V \in \text{ind } K_\Lambda]$.

Proof. Take any \mathbf{d} with $\chi(\text{Gr}_{\mathbf{d}}(M))$ non-zero. Take some submodule of M , L , of dimension $d(L) = \mathbf{d}$. Then we claim $\text{hom}_\Lambda(H^0 V, M) - \langle g(V), \mathbf{d} \rangle$ is non-negative for all $V \in \text{ind } K_\Lambda$. Let $V = (V^{-1} \rightarrow V^0)$. Then we get an exact sequence

$$V^{-1} \rightarrow V^0 \rightarrow H^0 V \rightarrow 0$$

in $\text{mod } \Lambda$. Moreover, applying the functor $\text{Hom}_\Lambda(-, L)$, gives the exact sequence of vector spaces,

$$0 \rightarrow \text{Hom}_\Lambda(H^0 V, L) \rightarrow \text{Hom}_\Lambda(V^0, L) \rightarrow \text{Hom}_\Lambda(V^{-1}, L).$$

But exactness implies that

$$\text{hom}_\Lambda(H^0 V, L) - \text{hom}_\Lambda(V^0, L) + \text{hom}_\Lambda(V^{-1}, L) \geq 0.$$

Since L is a submodule of M , $\text{hom}_\Lambda(H^0 V, M) \geq \text{hom}_\Lambda(H^0 V, L)$. And so, recalling also (2.1), $\text{hom}_\Lambda(H^0 V, M) - \langle g(V), \mathbf{d} \rangle \geq 0$. \square

3. VARIETIES DEFINED BY \widehat{F} -EQUATIONS AND u -EQUATIONS

3.1. The varieties \mathcal{M}_Λ and $\widetilde{\mathcal{M}}_\Lambda$. The equations

$$\widehat{F}_X = 1, \tag{3.1}$$

for all $X \in \text{ind } K_\Lambda$, define the affine varieties that are the main subject of this paper.

Definition 3.1. In the ring $\mathbb{C}[u_V^{\pm 1} \mid V \in \text{ind } K_\Lambda]$, consider the ideal $I_\Lambda = \langle \widehat{F}_W - 1 \mid W \in \text{ind } K_\Lambda \rangle$. Similarly, write \widetilde{I}_Λ for the ideal $\widetilde{I}_\Lambda = \langle \widehat{F}_W - 1 \mid W \in \text{ind } K_\Lambda \rangle$ in the polynomial ring $\mathbb{C}[u_V \mid V \in \text{ind } K_\Lambda]$, generated by the same polynomials. This defines two affine varieties:

$$\begin{aligned} R_\Lambda &:= \mathbb{C}[u_X^{\pm 1} \mid X \in \text{ind } K_\Lambda] / I_\Lambda & \widetilde{R}_\Lambda &:= \mathbb{C}[u_X \mid X \in \text{ind } K_\Lambda] / \widetilde{I}_\Lambda \\ \mathcal{M}_\Lambda &:= \text{Spec}(R_\Lambda) & \widetilde{\mathcal{M}}_\Lambda &:= \text{Spec}(\widetilde{R}_\Lambda). \end{aligned}$$

Thus \mathcal{M}_Λ is the localisation of $\widetilde{\mathcal{M}}_\Lambda$ along the divisors $u_V = 0$. In other words, \mathcal{M}_Λ is obtained by removing the points where some $u_V = 0$. We emphasize that, in the case that Λ is the path algebra of the Dynkin quiver A_{n-3} , $\widetilde{\mathcal{M}}_\Lambda$ is *not* the Deligne-Mumford compactification of the moduli space of points on \mathbb{P}^1 , $\overline{\mathcal{M}}_{0,n}$. The latter is a projective variety, whereas $\widetilde{\mathcal{M}}_\Lambda$ is an affine open in $\overline{\mathcal{M}}_{0,n}$. To avoid confusion, we therefore use a tilde, and not a bar, to denote these affine varieties. Note that these affine varieties are studied by Brown in [14].

3.2. The u -equations and configuration spaces. The varieties in Definition 3.1 generalize the cluster configuration spaces of finite type of [6], along with other varieties previously considered in the physics and mathematics literature [40, 39, 38, 14]. We now show how the varieties defined by the equations (3.1) are related to another set of equations called the u -equations.

Definition 3.2. Under the finiteness assumption 2.12, the u -equation of an object $X \in \text{ind } K_\Lambda$ is the following equation in the polynomial ring $\mathbb{Z}[u_V \mid V \in \text{ind } K_\Lambda]$:

$$u_X + \prod_{Y \in \text{ind } K_\Lambda} u_Y^{c(X,Y)} = 1,$$

where the *compatibility degree* of $X, Y \in \text{ind } K_\Lambda$ is

$$c(X, Y) = \text{hom}_{K_\Lambda}(X, \Sigma Y) + \text{hom}_{K_\Lambda}(Y, \Sigma X).$$

The u -equations define an affine variety

$$\widetilde{\mathcal{U}}_\Lambda = \text{Spec}(\widetilde{\mathcal{S}}_\Lambda),$$

where $\widetilde{\mathcal{S}}_\Lambda := \mathbb{C}[u_X \mid X \in \text{ind } K_\Lambda] / \langle u\text{-equations} \rangle$.

Definition 3.3. The configuration space, $\widetilde{\mathcal{U}}_\Lambda^{\geq 0}$, is the semialgebraic set of real, non-negative points of $\widetilde{\mathcal{U}}_\Lambda$.

Finally, the exponents $c(X, Y)$ appearing in the u -equations can be written in terms of the module category $\text{mod } \Lambda$ as follows.

Lemma 3.4. For $X, Y \in \text{ind } K_\Lambda$, the compatibility degree $c(X, Y)$ is also given by

$$c(X, Y) = \text{hom}_\Lambda(H^0 X, H^0(\tau Y)) + \text{hom}_\Lambda(H^0 Y, H^0(\tau X)),$$

or, equivalently, by

$$c(X, Y) = -\langle g(X), d(H^0 Y) \rangle + \text{hom}_\Lambda(H^0 X, H^0 Y) + \text{hom}_\Lambda(H^0 X, H^0(\tau Y)).$$

Proof. Recall from Lemma 2.9 that $\text{hom}_{K_\Lambda}(Y, \Sigma X) = \text{hom}_\Lambda(H^0 X, H^0(\tau Y))$. The second formula follows from Lemma 2.10. \square

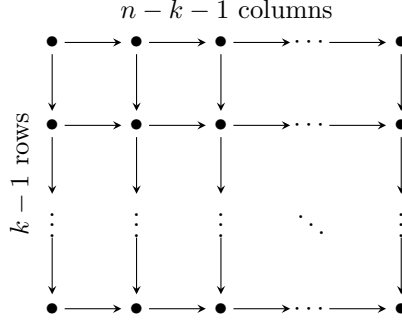
3.3. Examples. The literature includes several families of examples of u -equations, that we summarize below. Note that Definition 3.2, above, generalizes these examples to the case of any finite representation type algebra. For new examples of u -equations, see Section 13.

3.3.1. Dynkin types. If the algebra Λ is the path algebra $\mathbb{C}Q$ of a quiver Q of Dynkin type A, D or E , then the u -equations defining $\widetilde{\mathcal{U}}_\Lambda$ are the same as the ones defining the variety $\widetilde{\mathcal{M}}$ of Arkani-Hamed, He and Lam [6]. This is a consequence of the additive categorification of cluster algebras, which was worked out for Dynkin quivers in [18]. In particular, for type A_{n-3} , $\widetilde{\mathcal{U}}_\Lambda$ is the affine variety $\widetilde{\mathcal{M}}_{0,n}$ of Brown [14]. For example, for A_2 , $\text{ind } K_\Lambda$ has five modules: the projectives P_1, P_2 , the shifted projectives $\Sigma P_1, \Sigma P_2$, and the simple module S_2 . Then the u -equations for this algebra are

$$\begin{aligned} u_{P_1} + u_{S_2} u_{\Sigma P_1} &= 1 & u_{P_2} + u_{\Sigma P_1} u_{\Sigma P_2} &= 1 & u_{S_2} + u_{P_1} u_{\Sigma P_2} &= 1 \\ u_{\Sigma P_1} + u_{P_1} u_{P_2} &= 1 & u_{\Sigma P_2} + u_{P_2} u_{S_2} &= 1 \end{aligned}$$

See Section 13.1 for more details. We also work out type A_3 in detail in Section 13.3.

3.3.2. *The Grassmannian.* The CEGM u -equations are defined in [24]. As will be discussed in more detail elsewhere [25], they are the u -equations for the path algebra, $\Lambda_{k,n}$, of the quiver



with the relations that the composition of any right-arrow with any down-arrow vanishes. For example, for the case of $k = 3$, $n = 6$, the indecomposable modules can be labelled by non cyclically-consecutive 3-tuples of $[6]$, and the u -equations are:

$$\begin{aligned}
 u_{124} + u_{135}u_{136}u_{235}u_{236} &= 1 & u_{125} + u_{136}u_{146}u_{236}u_{246}u_{346} &= 1 \\
 u_{134} + u_{235}u_{236}u_{245}u_{246}u_{256} &= 1 & u_{135} + u_{124}u_{146}u_{236}u_{245}u_{256}u_{346}u_{246}^2 &= 1 \\
 u_{136} + u_{124}u_{125}u_{245}u_{246}u_{256} &= 1 & u_{145} + u_{246}u_{256}u_{346}u_{356} &= 1 \\
 u_{146} + u_{125}u_{135}u_{235}u_{256}u_{356} &= 1 & u_{235} + u_{124}u_{134}u_{146}u_{246}u_{346} &= 1 \\
 u_{236} + u_{124}u_{125}u_{134}u_{135} &= 1 & u_{245} + u_{134}u_{135}u_{136}u_{346}u_{356} &= 1 \\
 u_{246} + u_{125}u_{134}u_{136}u_{145}u_{235}u_{356}u_{135}^2 &= 1 & u_{256} + u_{134}u_{135}u_{136}u_{145}u_{146} &= 1 \\
 u_{346} + u_{125}u_{135}u_{145}u_{235}u_{245} &= 1 & u_{356} + u_{145}u_{146}u_{245}u_{246} &= 1
 \end{aligned}$$

3.4. **The varieties \mathcal{M} and \mathcal{U} .** The main result of this section is that the \widehat{F} -equations (3.1) imply the u -equations. In other words, $\widehat{\mathcal{U}}_\Lambda \supseteq \widehat{\mathcal{M}}_\Lambda$. The main tool in order to prove this is a kind of “exchange relation” for the \widehat{F} -polynomials.

Proposition 3.5. *Let $X \in \text{ind } K_\Lambda$.*

- (1) *If X is neither projective nor injective, with almost split conflation $\tau X \rightarrowtail E_X \twoheadrightarrow X$, then $\widehat{F}_X \widehat{F}_{\tau X} = u_X \widehat{F}_{E_X} + \prod_{W \in \text{ind } K_\Lambda} u_W^{c(W,X)}$.*
- (2) *If $X = P_i$, then $\widehat{F}_X = u_X \widehat{F}_{\text{rad } P_i} + \prod_{W \in \text{ind } K_\Lambda} u_W^{c(W,X)}$.*
- (3) *If $X = \Sigma P_i$, then $\widehat{F}_{\tau X} = u_X \widehat{F}_{I_i/S_i} + \prod_{W \in \text{ind } K_\Lambda} u_W^{c(W,X)}$.*

From Proposition 3.5, it follows immediately that

Corollary 3.6. *If $\widehat{F}_X = 1$ for all $X \in \text{ind } K_\Lambda$, then $u_X + \prod_{W \in \text{ind } K_\Lambda} u_W^{c(W,X)} = 1$ for all $X \in \text{ind } K_\Lambda$.*

To prove Proposition 3.5, we need some relations between different F -polynomials.

Lemma 3.7. *For an indecomposable projective $P_i \in \text{mod } \Lambda$, $F_{P_i} = F_{\text{rad } P_i} + y^{d(P_i)}$. For an indecomposable injective $I_i \in \text{mod } \Lambda$, $F_{I_i} = y_i F_{I_i/S_i} + 1$.*

Proof. If a submodule of P is not P itself, then it is a submodule of the radical, $\text{rad } P$. I_i contains the zero submodule, and all other submodules contain $\text{soc } I_i = S_i$. \square

Lemma 3.8. *For $V \rightarrowtail X \twoheadrightarrow W$ a conflation in K_Λ :*

- (1) *if the conflation is split, then $F_V F_W = F_X$,*
- (2) *if the conflation is almost split, and W is not injective, then $F_V F_W = F_X + y^{d(W)}$.*

Proof. Recall that for $X \in K_\Lambda$, $F_X := F_{H^0 X}$. The identities then follow from the corresponding identities in $\text{mod } \Lambda$, together with Lemma 2.8, which relates short exact sequences in $\text{mod } \Lambda$ and conflations in K_Λ . Indeed, (1) holds for split short exact sequences in $\text{mod } \Lambda$ by [47, Prop. 2.13] (see also [18, 23]). And, by Lemma 2.7, H^0 sends a split conflation to a split short exact sequence in $\text{mod } \Lambda$. Whereas (2) holds for an almost split short exact sequence in $\text{mod } \Lambda$, which is the main result of [23]. And, since W is non-injective, H^0 sends an almost split conflation to an almost split short exact sequence (Lemma 2.8). \square

Proof of Proposition 3.5. (1) Take $X \in \text{ind } K_\Lambda$ neither projective nor injective. By Lemma 3.8(2), we have $F_X F_{\tau X} = F_{E_X} + y^{d(X)}$. We can turn this into a relation among \widehat{F} -polynomials. Using the homomorphism Ψ and Lemma 2.15,

$$\begin{aligned} \widehat{F}_X \widehat{F}_{\tau X} = \Psi(F_{E_X}) \prod_{W \in \text{ind } K_\Lambda} u_W^{\text{hom}_\Lambda(H^0 W, H^0 X) + \text{hom}_\Lambda(H^0 W, \tau H^0 X)} \\ + \prod_{W \in \text{ind } K_\Lambda} u_W^{-\langle g(W), d(X) \rangle + \text{hom}_\Lambda(H^0 W, H^0 X) + \text{hom}_\Lambda(H^0 W, \tau H^0 X)} \end{aligned}$$

The first term on the right hand side is equal to $u_X \widehat{F}_{E_X}$, by Proposition 2.1, which implies that

$$\text{hom}_\Lambda(H^0 W, H^0 X) + \text{hom}_\Lambda(H^0 W, \tau H^0 X) = \begin{cases} \text{hom}_\Lambda(H^0 W, H^0 E_X) & \text{if } X \neq W \\ \text{hom}_\Lambda(H^0 W, H^0 E_X) + 1 & \text{if } X = W. \end{cases}$$

And the second term is equal to $\prod_{W \in \text{ind } K_\Lambda} u_W^{c(W, X)}$, by Lemma 3.4.

(2) Take $X = P_i$. By Lemma 3.7, $F_X = F_{\text{rad } H^0 X} + y^{d(X)}$ and, using Ψ and Lemma 2.15,

$$\widehat{F}_X = \Psi(F_{\text{rad } H^0 X}) \prod_{W \in \text{ind } K_\Lambda} u_W^{\text{hom}_\Lambda(H^0 W, H^0 X)} + \prod_{W \in \text{ind } K_\Lambda} u_W^{-\langle g(W), d(X) \rangle + \text{hom}_\Lambda(H^0 W, H^0 X)}.$$

The first term is equal to $u_X \widehat{F}_{\text{rad } P_i}$, by Proposition 2.1. The second term is equal to $\prod_{W \in \text{ind } K_\Lambda} u_W^{c(W, X)}$, by Lemma 3.4, noting that $\tau H^0 X = 0$.

(3) Take $X = \Sigma P_i$ injective. By Lemma 3.7, $F_{\tau X} = y_i F_E + 1$, where $H^0(\tau X) = I_i$ and $H^0 E = I_i/S_i$. Using Ψ and Lemma 2.15, this becomes

$$\widehat{F}_{\tau X} = \Psi(F_E) \prod_{W \in \text{ind } K_\Lambda} u_W^{\text{hom}_\Lambda(H^0 W, H^0(\tau X)) - g(W)_i} + \prod_{W \in \text{ind } K_\Lambda} u_W^{\text{hom}_\Lambda(H^0 W, I_i)}.$$

The second term is equal to $\prod_{W \in \text{ind } K_\Lambda} u_W^{c(W, X)}$, by Lemma 3.4, noting that $H^0 X = 0$. The first term is equal to $u_X \widehat{F}_E$, because

$$\text{hom}_\Lambda(H^0 W, H^0(\tau X)) - g(W)_i = \begin{cases} \text{hom}_\Lambda(H^0 W, H^0 E) & \text{if } X \neq W \\ \text{hom}_\Lambda(H^0 W, H^0 E) + 1 & \text{if } X = W. \end{cases} \quad (3.2)$$

This follows from the short exact sequence $0 \rightarrow S_i \rightarrow I_i \rightarrow I_i/S_i \rightarrow 0$, which gives

$$0 \rightarrow \text{Hom}_\Lambda(H^0 W, S_i) \rightarrow \text{Hom}_\Lambda(H^0 W, I_i) \rightarrow \text{Hom}_\Lambda(H^0 W, I_i/S_i) \rightarrow \text{Ext}^1(H^0 W, S_i) \rightarrow 0.$$

But, by Lemma 2.3, $\text{ext}^1(M, S_i) = \text{hom}(S_i, \tau M)$, so that

$$\text{hom}_\Lambda(H^0 W, I_i/S_i) - \text{hom}_\Lambda(H^0 W, I_i) = \text{hom}_\Lambda(H^0 W, S_i) - \text{hom}_\Lambda(S_i, \tau H^0 W),$$

which implies (3.2), by Lemma 2.10) and observing that $H^0(\Sigma P_i) = 0$, but $g(\Sigma P_i)_i = -1$. \square

We can prove a converse to Corollary 3.6 in some special cases.

Proposition 3.9. *Assume that the Auslander–Reiten quiver of K_Λ has no oriented cycle. Then the u -equations imply that $\widehat{F}_X = 1$ for all $X \in \text{ind } K_\Lambda$. In particular, $\widetilde{\mathcal{M}}_\Lambda = \widetilde{\mathcal{U}}_\Lambda$.*

Proof. We can assume that the algebra Λ is connected. Unless $\Lambda = \mathbb{C}$, in which case the statement is trivially true, the Auslander–Reiten quiver Γ of K_Λ is connected. Let Γ^+ be the quiver obtained by adding arrows $\tau X \rightarrow X$ for each non-projective X . Then Γ^+ does not have oriented cycles.

We prove the result by an inductive procedure as follows: suppose that we have a non-empty subset \mathcal{I} of $\text{ind } K_\Lambda$ consisting of objects Y for which $\widehat{F}_Y = 1$ under the u -equations. We start the procedure by letting \mathcal{I} be the set of shifted projectives. If $\mathcal{I} \neq \text{ind } K_\Lambda$, then we can find an indecomposable object X not in \mathcal{I} such that $\widehat{F}_X = 1$ as follows. Let $\bar{\Gamma}^+$ be the quiver obtained by removing all vertices of Γ^+ corresponding to objects in \mathcal{I} and all arrows connected to them. Since this quiver has no oriented cycle, it has a sink X . If X is a shifted projective, then it is already in \mathcal{I} , contrary to our hypothesis. Otherwise, it sits in a mesh

$$\begin{array}{ccccc} & & E_1 & & \\ & \nearrow & & \searrow & \\ X & \longrightarrow & E_2 & \longrightarrow & \tau^{-1}X \\ & \searrow & \vdots & \nearrow & \\ & & E_r & & \end{array}$$

and by Proposition 3.5, we have that

$$\widehat{F}_X \widehat{F}_{\tau^{-1}X} = u_{\tau^{-1}X} \prod_{i=1}^r \widehat{F}_{E_i} + \prod_{W \in \text{ind } K_\Lambda} u_W^{c(W, \tau^{-1}X)}.$$

By assumption on X , the objects E_i and $\tau^{-1}X$ are all in \mathcal{I} . Therefore, we get that $\widehat{F}_X = u_{\tau^{-1}X} + \prod_{W \in \text{ind } K_\Lambda} u_W^{c(W, \tau^{-1}X)}$, which is equal to 1 under the u -equations. Thus we can add X to \mathcal{I} and apply induction. The result is proved. \square

Example 3.10. Here are some families of examples for which Proposition 3.9 applies:

- (1) if Q is any orientation of a Dynkin quiver of type A , D or E and $\Lambda = \mathbb{C}Q$ is its path algebra;
- (2) if Λ is the path algebra of $1 \rightarrow 2 \rightarrow \dots \rightarrow n$ modulo any ideal generated by paths of length at least 2;
- (3) if Λ is a tilted algebra of Dynkin type A, D or E . Indeed, in this case, $\text{mod } \Lambda$ is equivalent to the heart of some t -structure in $\mathcal{D}^b(\text{mod } \mathbb{C}Q)$, with Q a quiver of the corresponding Dynkin type, and thus cannot have oriented cycles by Happel's theorem on the structure of $\mathcal{D}^b(\text{mod } \mathbb{C}Q)$ [32].

Example 3.11. Using a computer, we verified the equality $\widetilde{\mathcal{M}}_\Lambda = \widetilde{\mathcal{U}}_\Lambda$ for the following algebras: $\mathbb{C}[x]/(x^d)$ for $d = 1, 2, \dots, 9$; a 2-cycle $1 \xrightarrow{a} 2 \xrightarrow{b} 1$ with relation $ab = 0$; a 4-cycle where all paths of length two vanish; and the quiver

$$1 \begin{array}{c} \xrightarrow{a} \\ \xleftarrow{b} \end{array} 2 \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} x$$

where all path of length two vanish.

4. SPLIT GROTHENDIECK GROUPS

In this section, we give bases for the split Grothendieck groups of $\text{mod } \Lambda$ and also study the split Grothendieck group of K_Λ .

Definition 4.1. For any extriangulated category \mathcal{C} , its *split Grothendieck group* is

$$K_0^{\text{sp}}(\mathcal{C}) := \bigoplus_{A \in \text{ind } \mathcal{C}} \mathbb{Z} \cdot [A] / \langle [X] - [E] + [Y] \mid X \rightarrowtail E \twoheadrightarrow Y \text{ split conflation} \rangle.$$

In the case of the module category, $\mathcal{C} = \text{mod } \Lambda$, split conflation means split short exact sequence.

Consider first the split Grothendieck group of $\text{mod } \Lambda$. The set of classes $[M]$ for all indecomposable modules is a basis of $K_0^{\text{sp}}(\text{mod } \Lambda)$, which we call the standard basis. We will give two other bases of $K_0^{\text{sp}}(\text{mod } \Lambda)$, each of which is dual in a suitable sense to the standard basis.

It is natural to consider classes associated to almost split short exact sequences. For any non-projective indecomposable object M of $\text{mod } \Lambda$, we write

$$r_M = [\tau M] - [E] + [M]$$

for the class associated to its Auslander-Reiten sequence, $0 \rightarrow \tau M \rightarrow E \rightarrow M \rightarrow 0$. In the case that $M = P_i$, we write

$$r_{P_i} = -[\text{rad } P_i] + [P_i],$$

for the class associated to the right almost split inclusion. Similarly, for any non-injective indecomposable M of $\text{mod } \Lambda$, write

$$\ell_M = [M] - [E] + [\tau^{-1} M],$$

for the class associated to $0 \rightarrow M \rightarrow E \rightarrow \tau^{-1} M \rightarrow 0$. In the case $M = I_i$, write

$$\ell_{I_i} = [I_i] - [I_i/S_i],$$

for the class associated to the left almost split quotient. Note that for non-projective M , $\ell_{\tau M} = r_M$.

There is a natural pairing on $K_0^{\text{sp}}(\text{mod } \Lambda)$ given by

$$([M], [N]) = \text{hom}_\Lambda(M, N).$$

With respect to this pairing we have

Lemma 4.2. *The set of classes r_M are a basis for $K_0^{\text{sp}}(\text{mod } \Lambda)$, and this is right-dual to the standard basis, in the sense that*

$$([M], r_N) = \begin{cases} 0 & M \neq N, \\ 1 & M = N. \end{cases}$$

Similarly, the set of classes ℓ_M are also a basis, and this basis is left-dual to the standard basis:

$$(\ell_M, [N]) = \begin{cases} 0 & M \neq N, \\ 1 & M = N. \end{cases}$$

In particular, the pairing $(\ , \)$ is non-degenerate.

Proof. As already mentioned, the set of classes $[M]$ for all indecomposables $M \in \text{mod } \Lambda$ are a basis of $K_0^{\text{sp}}(\text{mod } \Lambda)$. On the other hand, $([M], r_N)$ and $(\ell_M, [N])$ satisfy the stated properties by Proposition 2.1. But this shows that the r_M (resp. ℓ_M) also define bases of $K_0^{\text{sp}}(\text{mod } \Lambda)$. \square

The following lemma is immediate (see for instance [9]):

Lemma 4.3. *Any class $x \in K_0^{\text{sp}}$ admits basis expansions*

$$x = \sum_{M \in \text{ind } \Lambda} (M, x) r_M, \quad x = \sum_{M \in \text{ind } \Lambda} (x, M) \ell_M$$

into the basis of r_M and the basis of ℓ_M classes, respectively.

We now consider the split Grothendieck group $K_0^{\text{sp}}(K_\Lambda)$. Again, the standard basis for K_0^{sp} consists of the isomorphism classes of indecomposable objects, i.e., $\text{ind } K_\Lambda$.

It will be useful to extend H^0 to a map from $K_0^{\text{sp}}(K_\Lambda)$ to $K_0^{\text{sp}}(\text{mod } \Lambda)$, sending $[X]$ to $[H^0(X)]$.

For non-projective $X \in \text{ind } K_\Lambda$, let $\tau X \rightarrow E \rightarrow X$ be its almost split conflation, and write

$$b_X = [X] - [E] + [\tau X] \in K_0^{\text{sp}}(K_\Lambda).$$

For X indecomposable projective, define $b_X = [X] - [\text{rad } X]$ (where we use the slight abuse of notation that $\text{rad } X$ denotes the minimal projective resolution of the radical of the projective module X).

The assignment of a g -vector $g(X) \in \mathbb{Z}^n$ to each $X \in \text{ind } K_\Lambda$ (see Section 2.4) can be extended to a linear map $g : K_0^{\text{sp}}(K_\Lambda) \rightarrow \mathbb{Z}^n$. The surjection g can be regarded as the natural surjection of $K_0^{\text{sp}}(K_\Lambda)$ onto the Grothendieck group $K_0(K_\Lambda) \simeq \mathbb{Z}^n$, which has a basis given by the classes $[P_i]$ of the projectives. The following characterises the kernel of g .

Theorem 4.4 (Theorem 4.13 of [45]). *The map $g : K_0^{\text{sp}}(K_\Lambda) \rightarrow \mathbb{Z}^n$ is a split epimorphism. The set*

$$\{b_X \mid X \in \text{ind } K_\Lambda \text{ and } X \neq P_i\}$$

is a basis of $\ker g$ as a free abelian group.

5. GEOMETRY OF \mathcal{M}_Λ

In this section, we show that \mathcal{M}_Λ is irreducible by realizing it as a torus quotient of an open set in affine space, and we give several different descriptions of its coordinate ring.

Define a grading on $\mathbb{C}[u_M^\pm \mid M \in \text{ind } K_\Lambda]$ with grading group $K_0^{\text{sp}}(\Lambda)$ by $d(u_M) = H^0(b_M)$.

Lemma 5.1. $d(\Psi(y_i)) = 0 \in K_0^{\text{sp}}(\text{mod } \Lambda)$.

Proof. The degree $d(\Psi(y_i))$ is, by definition, $-\sum_{X \in \text{ind } K_\Lambda} g(X)_i H^0(b_X)$. For $M \in \text{ind } \Lambda$, the multiplicity of $[M]$ in $d(\Psi(y_i))$ is the i -th component of $-g(b_{\tau_{K_\Lambda}^{-1}(M)})$. Theorem 4.4 says that $b_{\tau_{K_\Lambda}^{-1}(M)}$ is in the kernel of g , and the result follows. \square

Now we can establish the following lemma:

Lemma 5.2. *For $M \in \text{mod } \Lambda$, we have that \widehat{F}_M is homogeneous with respect to d , and its degree is $[M]$.*

Proof. First, by the previous lemma, $d(\Psi(F_M)) = 0$. It follows that \widehat{F}_M is homogeneous with degree $\sum_{L \in \text{ind } \Lambda} \text{hom}(L, M) H^0(b_L)$. To determine the coefficient of $[N]$ in this sum, let $N \rightarrowtail E \twoheadrightarrow \tau^{-1}N$ be the almost split conflation in K_Λ beginning at N . Then the coefficient of $[N]$ in the degree of \widehat{F}_M is (ℓ_N, M) . This is 1 if $N = M$ and zero otherwise, which proves the lemma. \square

This grading on $\mathbb{C}[u_M^\pm \mid M \in \text{ind } K_\Lambda]$ defines an action of the torus $T = \text{Spec } \mathbb{C}[K_0^{\text{sp}}(\text{mod } \Lambda)]$ on $\text{Spec } \mathbb{C}[u_M^\pm \mid M \in \text{ind } K_\Lambda]$. Because the \widehat{F}_M are homogeneous, the T -action restricts to an action on $X^\circ = \text{Spec } \mathbb{C}[u_M^\pm \mid M \in \text{ind } K_\Lambda][\widehat{F}_M^{-1} \mid M \in \text{ind } \Lambda]$, which is the locus in $\text{Spec } \mathbb{C}[u_M^\pm \mid M \in \text{ind } K_\Lambda]$ where the \widehat{F}_M are non-zero.

We now have the following theorem:

Theorem 5.3. *There is an isomorphism $\mathbb{C}[u_M^\pm, \widehat{F}_M^{-1}] \cong \mathbb{C}[\widehat{F}_M^\pm \mid M \in \text{ind } \Lambda] \otimes_{\mathbb{C}} \mathbb{C}[u_M^\pm, \widehat{F}_M^{-1}]_0$ which induces an isomorphism $R_\Lambda \cong \mathbb{C}[u_M^\pm, \widehat{F}_M^{-1}]_0$.*

Proof. $\mathbb{C}[\widehat{F}_M^\pm]$ is a subring of $\mathbb{C}[u_M^\pm, \widehat{F}_M^{-1}]$, and contains exactly one Laurent monomial (in the \widehat{F}_M 's) in each degree in $K_0^{\text{sp}}(\text{mod } \Lambda)$. Thus, any homogeneous element of $\mathbb{C}[u_M^\pm, \widehat{F}_M^{-1}]$ can be uniquely written as a Laurent monomial in $\mathbb{C}[\widehat{F}_M^\pm]$ times an element of $\mathbb{C}[u_M^\pm, \widehat{F}_M^{-1}]_0$, where the subscript zero indicates the subring of degree 0. We therefore have an isomorphism

$$\mathbb{C}[u_M^\pm, \widehat{F}_M^{-1}] \xrightarrow{\sim} \mathbb{C}[\widehat{F}_M^\pm] \otimes_{\mathbb{C}} \mathbb{C}[u_M^\pm, \widehat{F}_M^{-1}]_0$$

which sends $\widehat{F}_M^{\pm 1}$ to $\widehat{F}_M^{\pm 1} \otimes 1$. The ideal I_Λ is sent on the right-hand side to the ideal generated by the $(\widehat{F}_M - 1) \otimes 1$. Taking the quotient, we get that $R_\Lambda \cong \mathbb{C}[u_M^\pm, \widehat{F}_M^{-1}]_0$, proving the theorem. \square

Corollary 5.4. \mathcal{M}_Λ is isomorphic to X°/T , and is thus an irreducible variety.

Proof. $\mathcal{M}_\Lambda = \text{Spec}(R_\Lambda)$. We have showed that $R_\Lambda = \mathbb{C}[u_M^\pm, \widehat{F}_M^{-1}]_0$. The coordinate ring of X° is $\mathbb{C}[u_M^\pm, \widehat{F}_M^{-1}]$, so $\mathbb{C}[u_M^\pm, \widehat{F}_M^{-1}]_0$ is the coordinate ring of X°/T . \square

For $M \in \text{ind } K_\Lambda$, let

$$\widehat{v}_M = \frac{u_M \widehat{F}_E}{\widehat{F}_{\tau M} \widehat{F}_M},$$

where $\tau M \rightarrow E \rightarrow M$ is the Auslander-Reiten conflation ending at M , unless M is projective, in which case set $E = \text{rad } M$ and $\tau M = 0$.

Lemma 5.5. \widehat{v}_M is homogeneous of degree zero (and thus T -invariant).

Proof. This is immediate from the fact that $d(u_M) = H^0(b_M)$, together with Lemma 5.2. \square

Theorem 5.6. *The ring $\mathbb{C}[u_M^\pm, \widehat{F}_M^{-1}]_0$ is generated by the \widehat{v}_M^\pm .*

Proof. The isomorphism $\mathbb{C}[u_M^\pm, \widehat{F}_M^{-1}] \xrightarrow{\sim} \mathbb{C}[\widehat{F}_M^\pm] \otimes_{\mathbb{C}} \mathbb{C}[u_M^\pm, \widehat{F}_M^{-1}]_0$ of Theorem 5.3 sends each u_M to $\frac{\widehat{F}_{\tau M} \widehat{F}_M}{\widehat{F}_E} \otimes \widehat{v}_M$. Taking the quotient by I_Λ yields the result. \square

We will now obtain another expression for R_Λ as a subring of the rational function field in n variables, $\mathbb{C}(y_1, \dots, y_n)$.

Proposition 5.7. *The map ψ from $\mathbb{C}[u_M^\pm, \widehat{F}_M^{-1}]$ to $\mathbb{C}[y_1^\pm, \dots, y_n^\pm, F_M^{-1}]$, sending:*

$$u_M \rightarrow 1 \text{ for } M \in \text{ind } \Lambda$$

$$u_{\Sigma P_i} \rightarrow y_i$$

sends \widehat{F}_M to F_M , and is an isomorphism from $\mathbb{C}[u_M^\pm, \widehat{F}_M^{-1}]_0$ to $\mathbb{C}[y_1^\pm, \dots, y_n^\pm, F_M^{-1}]$.

Proof. It is easy to check from the definition of \widehat{F}_M that ψ does indeed send \widehat{F}_M to F_M , and thus is a well-defined map of rings from $\mathbb{C}[u_M^\pm, \widehat{F}_M^{-1}]$ to $\mathbb{C}[y_1^\pm, \dots, y_n^\pm, F_M^{-1}]$.

The argument now is similar to the argument for Theorem 5.3. This time, we use the fact that the degrees of u_M for $M \in \text{ind } \Lambda$ form a basis for $K_0^{\text{sp}}(\text{mod } \Lambda)$ to write

$$\mathbb{C}[u_M^\pm, \widehat{F}_M^{-1}] \xrightarrow{\sim} \mathbb{C}[u_M^\pm \mid M \in \text{ind } \Lambda] \otimes_{\mathbb{C}} \mathbb{C}[u_M^\pm, \widehat{F}_M^{-1}]_0.$$

The ideal on the left-hand side generated by the $u_M - 1$ for $M \in \text{ind } \Lambda$ is sent to the ideal generated by the $(u_M - 1) \otimes 1$, so taking the quotient gives

$$\mathbb{C}[u_M^\pm, \widehat{F}_M^{-1}] / \langle u_M - 1 \mid M \in \text{ind } \Lambda \rangle \simeq \mathbb{C}[u_M^\pm, \widehat{F}_M^{-1}]_0.$$

But the left hand-side is isomorphic to $\mathbb{C}[y_1, \dots, y_n, F_M^{-1}]$ under the map ψ , which yields the result. \square

Define $v_M = \psi(\widehat{v}_M)$. Explicitly, the v_M are given in terms of F -polynomials as

$$v_M = \frac{F_E}{F_M F_{\tau M}}, \quad v_{P_i} = \frac{F_{\text{rad } P_i}}{F_{P_i}}, \quad v_{\Sigma P_i} = y_i \frac{F_{I_i/S_i}}{F_{I_i}},$$

for $M \neq P_i, \Sigma P_i$, and where we write $0 \rightarrow \tau M \rightarrow E \rightarrow M \rightarrow 0$ for the AR-sequence ending with M .

Remark 5.8. The v_X can be described without splitting into three cases by observing that

$$v_X = \frac{F_X F_{\tau X} - y^{\dim H^0 X}}{F_X F_{\tau X}}.$$

We can now prove an analogue of Theorem 5.6, regarding $\mathbb{C}[y_1^\pm, \dots, y_n^\pm, F_M^{-1}]$ (which is isomorphic to R_Λ by Proposition 5.7).

Proposition 5.9. *$\mathbb{C}[y_1^\pm, \dots, y_n^\pm, F_M^{-1}]$ is generated by the v_M^\pm , with $M \in \text{ind } K_\Lambda$.*

Proof. By Theorem 5.6, $\mathbb{C}[u_M^\pm, \widehat{F}_M^{-1}]_0$ is generated by \widehat{v}_M^\pm . Applying the map ψ from Proposition 5.7, the image under ψ of $\mathbb{C}[u_M^\pm, \widehat{F}_M^{-1}]_0$ is generated by the images under ψ of the \widehat{v}_M^\pm , which are the v_M^\pm . \square

We write Φ for the map from $\mathbb{C}[u_M \mid M \in \text{ind } K_\Lambda]$ to $\mathbb{C}(y_1, \dots, y_n)$ sending u_M to v_M .

Theorem 5.10. *The map Φ induces an isomorphism $\bar{\Phi} : R_\Lambda \rightarrow \mathbb{C}[y_1^{\pm 1}, \dots, y_n^{\pm 1}; F_M^{-1} \mid M \in \text{ind } K_\Lambda]$.*

Proof. The map $\bar{\Phi}$ is the composition of the map $R_\Lambda \rightarrow \mathbb{C}[u_M^\pm, \widehat{F}_M^{-1}]_0$ from Theorem 5.3 and the map ψ from Proposition 5.7, so the result holds. \square

Corollary 5.11. *The rational functions v_M satisfy the $\widehat{F} = 1$ equations and the u -equations.*

Proof. The isomorphism of Theorem 5.10 from $R_\Lambda = \mathbb{C}[u_M \mid M \in \text{ind } K_\Lambda] / \langle \widehat{F}_M = 1 \mid M \in \text{ind } \Lambda \rangle$ to $\mathbb{C}[v_M]$ sending u_M to v_M , implies that the v_M satisfy the $\widehat{F} = 1$ equations. Corollary 3.6 shows that they therefore also satisfy the u -equations. \square

Corollary 5.12. *There is an isomorphism from an open set in \mathbb{C}^n to \mathcal{M}_Λ , sending (y_1, \dots, y_n) to $(v_M(y_1, \dots, y_n))_{M \in \text{ind } K_\Lambda}$.*

Proof. This is the isomorphism of varieties corresponding to the isomorphism of rings from Theorem 5.10. \square

6. IDENTITIES RELATING v_X AND F_M

In this section, we prove some equations relating the rational functions v_X and the polynomials F_M . These identities will be needed in Section 9. They can further be used to check directly that the v_X satisfy the $\widehat{F}_M = 1$ equations.

We write F for the homomorphism $F : K_0^{\text{sp}}(\text{mod } \Lambda) \rightarrow \mathbb{Q}(y_1, \dots, y_n)$ that maps classes in the split Grothendieck group to their F -polynomials:

$$F : \sum_i \lambda_i [A_i] \mapsto \prod_i F_{A_i}^{\lambda_i},$$

for integers λ_i . By Lemma 3.8(1), this is indeed well defined. Moreover, note that

$$F(r_X) = v_X^{-1}, \quad y_i F(\ell_{I_i}) = v_{\Sigma P_i}^{-1},$$

for $X \neq \Sigma P_i$. Then the basis expansions in K_0^{sp} given by Lemma 4.3 imply the following identities:

Lemma 6.1. *For any module M in $\text{mod } \Lambda$,*

$$\begin{aligned} \frac{1}{F_M} &= \prod_{N \in \text{ind } \Lambda} v_N^{\text{hom}_\Lambda(N, M)} \\ \frac{y^{d(M)}}{F_M} &= \prod_{X \in \text{ind } K_\Lambda} v_X^{\text{hom}_\Lambda(M, H^0(\tau X))} \end{aligned}$$

Proof. The first equation follows by applying Lemma 4.3 to expand $x = [M]$ in the r_N basis, and then applying F to the resulting identity in $K_0^{\text{sp}}(\text{mod } \Lambda)$. Similarly expanding $x = [M]$ in the ℓ_N basis and applying F , we get

$$\frac{1}{F_M} = \prod_{N \in \text{ind } \Lambda} F(\ell_N)^{-\text{hom}_\Lambda(M, N)} = \prod_{X \in \text{ind } K_\Lambda} F(\ell_{H^0(\tau X)})^{-\text{hom}_\Lambda(M, H^0(\tau X))}$$

Now, $F(\ell_{H^0(\tau X)}) = v_X^{-1}$ unless $X = \Sigma P_i$, in which case $y_i F(\ell_{H^0(\tau \Sigma P_i)}) = v_{\Sigma P_i}^{-1}$. Thus, to be left with only powers of the v_X on the righthand side, we need a compensating monomial on the lefthand side, which is just $y^{d(M)}$ because $\text{hom}(M, H^0(\tau \Sigma P_i)) = d(M)_i$. This proves the second equation. \square

Lemma 6.2. *For any module M in $\text{mod } \Lambda$, with dimension vector $d(M)$,*

$$y^{d(M)} = \prod_{Y \in \text{ind } K_\Lambda} v_Y^{-\langle g(Y), d(M) \rangle}.$$

Proof. By the identities for $1/F_M$ and $y^{d(M)}/F_M$ (Lemma 6.1), we find

$$y^{d(M)} = \prod_{Y \in \text{ind } K_\Lambda} v_Y^{\text{hom}_\Lambda(M, H^0 \tau Y) - \text{hom}_\Lambda(H^0 Y, M)}.$$

Then the result follows from Lemma 2.10. \square

The two lemmas above can be used to give a direct proof of Corollary 5.11.

7. TROPICAL u -VARIABLES

In this section, we prove some results on the tropicalization of the rational functions v_M for $M \in \text{ind } K_\Lambda$, as defined just before Proposition 5.9. These are of independent interest, generalizing results of [42] for Dynkin quivers; we will also use them in Section 9. Because of the independent interest of these results, in this section we drop the assumption that our finite-dimensional algebra Λ is of finite representation type.

For $g \in \mathbb{Z}^n$, we can consider two-term complexes of projectives $P^{-1} \rightarrow P^0$, with $P^0 = \bigoplus_{g_i > 0} P_i^{g_i}$, $P^{-1} = \bigoplus_{g_i < 0} P_i^{-g_i}$. We call such complexes *presentations of weight g* . The space of such complexes is just an affine space, so in particular it is irreducible, and we can look for generic behaviours. In particular, we will be interested in the multiplicity of a given $X \in \text{ind } K_\Lambda$ as an indecomposable summand of a generic presentation of weight g . Also, for M a Λ -module, define $\text{hom}(g, M)$ to be the dimension of Hom from the cokernel of a generic presentation of weight g to M .

Any $g \in \mathbb{Z}^n$ admits a canonical decomposition, similar to Kac's canonical decomposition for representations [22, Section 4]: the canonical decomposition of $g = g_1 + \dots + g_s$ if a generic presentation of weight g decomposes as a direct sum of presentations of weights g_1, \dots, g_s .

Tropicalization of a polynomial drops coefficients, replaces product by sum and sum by maximum. The tropicalization of a monomial is therefore a linear function, and the tropicalization of a polynomial is the maximum of the linear functions corresponding to the terms of the polynomial. Generally tropicalization is only considered for polynomials with non-negative coefficients, but it will be convenient for us to neglect that condition.

Lemma 7.1. $\text{trop } F_M(x) = \max_{L \subset M} \underline{\dim} L \cdot x$, where the maximum is taken over subrepresentations of M .

Proof. This is not immediate from our definition, because of the Euler characteristics in F_M . Suppose that L is a submodule of M of dimension \mathbf{e} but $\chi(\text{Gr}_{\mathbf{e}}(M)) = 0$. Then the term $\mathbf{e} \cdot y$ appears in the maximum on the righthand side but not on the lefthand side. However, let P be the convex hull of the dimension vectors of the subrepresentations of M . Clearly, the maximum on the righthand side of the equation in the statement in the lemma can be restricted to the vertices of P , and [11] shows that for \mathbf{e} a vertex of P , we have $\text{Gr}_{\mathbf{e}}(M) = \{\text{pt}\}$, so the Euler characteristic is 1. Thus, the terms appearing on the righthand side but missing from the lefthand side (if any) correspond to points contained within the convex hull of the vertices of P , and these points will never provide the maximum. \square

Fei shows [27, Theorem 3.6] the following theorem:

Theorem 7.2. *For any representation M of Λ , and any $g \in \mathbb{Z}^n$, there exists a positive integer m such that $\text{hom}(mg, M) = m(\text{trop } F_M)(g)$. Further, if the statement holds for m , it also holds for any multiple of m .*

If a vector $g \in \mathbb{Z}^n$ lies in the support of the g -vector fan, then the generic presentation of weight mg is just m copies of the generic presentation of weight g , so we can take $m = 1$ in Fei's theorem.

The following Theorem generalizes a recent result of Kus–Reineke for Dynkin quivers [42] to arbitrary finite-dimensional algebras.

Theorem 7.3. *For $g \in \mathbb{Z}^n$ and $X \in \text{ind } K_\Lambda$, we have that $-\text{trop } v_X(g)$ equals the multiplicity of X in a generic presentation of weight g .*

Proof. By replacing g successively by each of the summands in its canonical decomposition, we can suppose that the generic presentation of weight g is indecomposable. We split into three cases, based on whether a generic presentation of weight g is

- (1) the minimal presentation of an indecomposable τ -rigid module,
- (2) the minimal presentation of an indecomposable non τ -rigid module, or
- (3) a complex of the form $P_i \rightarrow 0$ (i.e., a shifted projective).

We further divide into cases based on whether X is

- (a) a minimal presentation of a non-projective module M , with Auslander–Reiten sequence

$$0 \rightarrow \tau M \rightarrow E \rightarrow M \rightarrow 0,$$

- (b) a complex of the form $0 \rightarrow P_j$ (i.e., a minimal presentation of a projective module P_j), or
 (c) a complex of the form $P_j \rightarrow 0$.

First, consider case 1. Let N be the cokernel of a generic representation of weight g , which is well-defined up to isomorphism since we are in case 1.

In case 1(a),

$$\text{trop}(v_X)(g) = -\text{trop } F_{\tau M}(g) + \text{trop } F_E(g) - \text{trop } F_M(g).$$

Applying Theorem 7.2 (and choosing $m = 1$, as we may since g is in the support of the g -vector fan), we obtain

$$\begin{aligned} -\text{trop } v_X(g) &= \text{hom}(N, \tau M) - \text{hom}(N, E) + \text{hom}(N, M) \\ &= (N, r_M) \end{aligned}$$

By Lemma 4.3, (N, r_M) equals the multiplicity of M as a summand of N . (Note that nothing up to Lemma 4.3 in Section 4 relied on the finite representation type assumption.) This proves the desired statement in this case.

Case 1(b), where X is an indecomposable projective P_j , is very similar. The difference is that the Auslander–Reiten sequence ending at M is replaced by $\text{rad } P_j \rightarrow P_j$. However, $-\text{trop } v_X(g) = (N, r_{P_j})$, and we draw the same conclusion.

In case 1(c), we have $X = \Sigma P_j$. Then

$$\begin{aligned} -\text{trop}(v_X)(g) &= \text{trop } F_{I_j}(g) - \text{trop } y_j F_{I_j/S_j}(g) \\ &= \text{hom}(N, I_j) - \text{hom}(N, I_j/S_j) - g_j \end{aligned}$$

We have a long exact sequence

$$0 \rightarrow \text{Hom}(N, S_j) \rightarrow \text{Hom}(N, I_j) \rightarrow \text{Hom}(N, I_j/S_j) \rightarrow \text{Ext}^1(N, S_j) \rightarrow \text{Ext}^1(N, I_j) = 0$$

So

$$\text{hom}(N, I_j) - \text{hom}(N, I_j/S_j) = \text{hom}(N, S_j) - \text{ext}^1(N, S_j)$$

. Now $\text{hom}(N, S_j)$ is just $\max(g_j, 0)$, while $\text{ext}^1(N, S_j) = -\max(-g_j, 0)$, so $\text{hom}(N, S_j) - \text{ext}^1(N, S_j) = g_j$, so $-\text{trop}(v_X)(g) = 0$ in this case, as desired.

Now consider case 2. In this case, the multiplicity of any particular X in a generic presentation of weight g is zero, because, while X might possibly appear as some presentation of weight g , it will not be the generic presentation, since g is not τ -rigid. We must therefore show that, in this case, $\text{trop } v_X(g) = 0$.

In case 2(a),

$$\text{trop}(v_X)(g) = -\text{trop } F_{\tau M}(g) + \text{trop } F_E(g) - \text{trop } F_M(g).$$

Applying Theorem 7.2, there is a positive integer m such that

$$-m \text{trop } v_X(g) = \text{hom}(mg, \tau M) - \text{hom}(mg, E) + \text{hom}(mg, M).$$

Let N be the cokernel of some presentation of weight mg . Then $\text{hom}(N, \tau M) - \text{hom}(N, E) + \text{hom}(N, M)$ is (N, r_M) , so by Lemma 4.3, it equals the multiplicity of M as a summand of N . Since mg is not τ -rigid, for a generic choice of N , this multiplicity will be zero, so $-m \text{trop } v_X(g) = 0$, as desired.

Cases 2(b) and 2(c) are again very similar to case 2(a).

Finally, consider case 3, where $g = -e_i$. In case 3(a),

$$\text{trop}(v_X)(g) = -\text{trop } F_{\tau M}(-e_i) + \text{trop } F_E(-e_i) - \text{trop } F_M(-e_i)$$

The tropical evaluation of any F -polynomial at $-e_i$ is zero, so $\text{trop}(v_X)(g) = 0$. Case 3(b) is disposed of in the same way. For case 3(c),

$$-\text{trop}(v_X)(g) = \text{trop } F_{I_j}(-e_i) - \text{trop } y_j F_{I_j/S_j}(-e_i) = \delta_{ij},$$

as desired. □

We now return to studying algebras Λ of finite representation type. In this case, generic presentations of modules are completely encoded by the g -vector fan. Specifically, for any $g \in \mathbb{Z}^n$, we express it as a sum of the generators of the minimal cone containing g in the g -fan:

$$g = c_1 g_{X_1} + \cdots + c_r g_{X_r}.$$

Then the multiplicity of X_i in a generic representation of weight g is c_i , and the multiplicity of any other complex from $\text{ind } K_\Lambda$ is zero. This implies the following corollary, which completely characterizes $\text{trop } v_M$.

Corollary 7.4. *For Λ of finite representation type, $\text{trop } v_M$ is linear on each cone of the g -vector fan. If $N \in \text{ind } K_\Lambda$ is rigid, then $\text{trop } v_M(g_N) = -\delta_{MN}$, where $\delta_{MN} = 1$ if M and N are isomorphic, and zero otherwise.*

In particular, note that if M is not rigid, then $\text{trop } v_M$ is identically zero.

8. DIVISORS AND JASSO REDUCTION

In order to understand the locus in $\widetilde{\mathcal{M}}_\Lambda$, where all the entries are non-negative, it is important to consider the stratification of $\widetilde{\mathcal{M}}_\Lambda$ according to which of the u_M are zero. In order to understand the strata, we need to understand what happens when we add u_M into the ideal \widetilde{I}_Λ . It turns out that $\mathbb{C}[u_\alpha]/\langle \widetilde{I}_\Lambda + \langle u_M \rangle \rangle$ is isomorphic to the ring defined by the \widehat{F} equations for another algebra of lower rank, called the Jasso reduction of Λ at M , and denoted B . See [36] for details.

There is a surjective map from $K_0(\Lambda)$ to $K_0(B)$, which sends the class of M to zero. There is a bijection which we denote s_M from the objects of $\text{ind } K_\Lambda$ which are compatible with M (other than M itself), to the objects of $\text{ind } K_B$. The map s_M also restricts to a bijection from the rays of the g -vector fan of Λ which are compatible with M other than M , to the rays of the g -vector fan of B . This bijection induces an identification of the link of the cone generated by the summands of M in the g -vector fan of Λ with the g -vector fan of B , which is also induced by the quotient map from $K_0(\text{proj } \Lambda)$ to $K_0(\text{proj } \Lambda)/[M] = K_0(\text{proj } B)$.

In what follows, we will write u_X for the variables defining \widetilde{R}_Λ and \check{u}_Y for the ones defining \widetilde{R}_B .

Theorem 8.1. *Let $M \in \text{ind } K_\Lambda$ be rigid. The morphism $\pi : \mathbb{C}[u_X \mid X \in \text{ind } K_\Lambda] \rightarrow \mathbb{C}[\check{u}_Y \mid Y \in \text{ind } K_B]$ which sends u_N to 1 if N is incompatible with M , and sends u_N to $\check{u}_{s_M(N)}$ if N is compatible with M and not equal to M , induces an isomorphism from $\widetilde{R}_\Lambda/\langle u_M \rangle$ to \widetilde{R}_B .*

Proof. It suffices to show that for every indecomposable $N \in \text{mod } \Lambda$, $\pi \widehat{F}_M \in \widetilde{I}_B$, and for every indecomposable $Y \in \text{mod } B$, there exists a $\hat{Y} \in \text{mod } \Lambda$ such that $\pi \widehat{F}_{\hat{Y}} = \widehat{F}_Y$.

Recall that there is a map J_M from $\text{mod } \Lambda$ to $\text{mod } B$, sending X to $T_M(X)/t_M(X)$, where T_M and t_M are two torsion classes associated to M . The map J_M is an equivalence of additive categories from the Λ modules with $T_M(X) = X$ and $t_M(X) = 0$, to the B modules.

We know that the $\widehat{F} = 1$ equations imply the u -equations by Corollary 3.6, and once we have $u_M = 0$, the u -equations imply that $u_N = 1$ for every N incompatible with M . Thus, for N incompatible with M , we have $u_N - 1 \in \widetilde{I}_\Lambda + \langle u_M \rangle$.

Now let N be an indecomposable Λ -module. Consider $\widehat{F}_N|_{u_M \leftarrow 0}$. By Theorem 7.2, there exists an m such that $(\text{trop } F_N)(mg_M) = \text{hom}(mg_M, N)$. Since M is rigid, a generic presentation of g -vector mg_M is isomorphic to m copies of M , so we conclude that $(\text{trop } F_N)(g_M) = \text{hom}(M, N)$. This shows that the monomial factor in the definition of \widehat{F}_N is exactly such as to ensure that the minimum exponent of u_M appearing in \widehat{F}_N is zero. This means that the lattice points of the Newton polytopes of N which maximize $\langle g_M, \cdot \rangle$ correspond to terms in \widehat{F}_N with degree of u_M of zero, while all the other lattice points have u_M appearing with strictly positive degree. Thus, $\widehat{F}_N|_{u_M \leftarrow 0}$ is the polynomial obtained by restricting \widehat{F}_N to the face where $\langle g_M, \cdot \rangle$ is maximized.

Let us denote $P_M(N)$ the face of the Harder–Narasimhan polytope of N on which $\langle g_M, \cdot \rangle$ is maximized. As [11] explain, the Harder–Narasimhan polytope of $J_M(N)$ can be identified with $P_M(N)$. Further, [26, Section 6] explains that, after that identification, the F -polynomial of $J_M(N)$ equals the restriction of F_N to $P_M(N)$. Write $p_M(N)$ for the bottommost point of $P_M(N)$.

Now:

$$\begin{aligned}\widehat{F}_{J_M(N)} &= \sum_{\mathbf{d} \in P_M(N) \cap \mathbb{Z}^n} \chi(\mathrm{Gr}_{\mathbf{d}-p_M(N)}(J_M(N))) \prod_{V \in \mathrm{ind} K_B} \check{u}_V^{\mathrm{hom}(H^0 V, J_M(N)) - \langle g_V, \mathbf{d} - p_M(N) \rangle} \\ &= \sum_{\mathbf{d} \in P_M(N) \cap \mathbb{Z}^n} \chi(\mathrm{Gr}_{\mathbf{d}-p_M(N)}(J_M(N))) \prod_{Z \in \mathrm{ind} K_\Lambda, c(Z, M)=0} \check{u}_{s_M(Z)}^{\mathrm{hom}(H^0 s_M(Z), J_M(N)) - \langle g_{r(Z)}, \mathbf{d} - p_M(N) \rangle}\end{aligned}$$

Now, $g_{r(Z)}$ can be understood as g_Z in the quotient by g_M . Since $\mathbf{d} - p_M(N)$ is perpendicular to g_M , we can replace $g_{r(Z)}$ by g_Z . Further, we can replace $\mathrm{Hom}(H^0 s_M(Z), J_M(N))$ by $\mathrm{Hom}(H^0 Z, J_M(N))$. And by [26, Section 6], $\mathrm{Gr}_{\mathbf{d}-p_M(N)}(J_M(N))$ is isomorphic to $\mathrm{Gr}_{\mathbf{d}}(N)$. We obtain:

$$\widehat{F}_{J_M(N)} = \sum_{\mathbf{d} \in P_M(N) \cap \mathbb{Z}^n} \chi(\mathrm{Gr}_{\mathbf{d}}(N)) \prod_{Z \in \mathrm{ind} K_\Lambda, c(Z, M)=0} \check{u}_{s_M(Z)}^{\mathrm{hom}(H^0(Z), J_M(N)) - \langle g_Z, \mathbf{d} - p_M(N) \rangle}$$

On the other hand,

$$\pi \widehat{F}_N = \sum_{\mathbf{d} \in P_M(N) \cap \mathbb{Z}^n} \chi(\mathrm{Gr}_{\mathbf{d}}(N)) \prod_{Z \in \mathrm{ind} K_\Lambda, c(Z, M)=0} \check{u}_{s_M(Z)}^{\mathrm{hom}(H^0(Z), N) - \langle g_Z, \mathbf{d} \rangle}$$

To get these two expressions to agree, we need to show that

$$\mathrm{hom}(H^0(Z), N) - \mathrm{hom}(H^0(Z), J_M(N)) - \langle g_Z, p_M(N) \rangle = 0$$

Consider the exact sequence

$$0 \rightarrow t_M(N) \rightarrow N \rightarrow N/t_M(N) \rightarrow 0$$

Apply $\mathrm{Hom}(Z, -)$ to it. Since $t_M(N)$ is a quotient of M , and Z is compatible with M , $\mathrm{Ext}^1(Z, t_M(N)) = 0$. Thus, we still have a short exact sequence:

$$0 \rightarrow \mathrm{Hom}(Z, t_M(N)) \rightarrow \mathrm{Hom}(Z, N) \rightarrow \mathrm{Hom}(Z, N/t_M(N)) \rightarrow 0$$

Since $Z \in {}^\perp(\tau M)$, the image of any map from Z to N is contained in $T_M(N)$, and thus the image of any map from Z to $N/t_M(N)$ is contained in $J_M(N) = T_M(N)/t_M(N)$. So we have:

$$0 \rightarrow \mathrm{Hom}(Z, t_M(N)) \rightarrow \mathrm{Hom}(Z, N) \rightarrow \mathrm{Hom}(Z, J_M(N)) \rightarrow 0$$

So the quantity that we want to show equals zero is $\mathrm{hom}(H^0(Z), t_M(N)) - \langle g_Z, t_M(N) \rangle = \mathrm{hom}(t_M(N), \tau H^0(Z))$, and since $\mathrm{Hom}(M, \tau H^0(Z)) = 0$, this is zero.

This shows that, for any $N \in \mathrm{mod} \Lambda$, $\pi \widehat{F}_N \in \mathrm{ind} K_B$. We now want to show that the $\pi \widehat{F}_N$ generate $\mathrm{ind} K_B$.

Let Y be an indecomposable B module. There is an equivalence of additive categories from $M^\perp \cap {}^\perp(\tau M)$ to $\mathrm{mod} B$, with the morphism given by J_M . Thus, there is an indecomposable module $\hat{Y} \in M^\perp \cap {}^\perp(\tau M)$ with $J_M(\hat{Y}) = Y$. Consequently, $\pi \widehat{F}_{\hat{Y}} = \widehat{F}_Y$, and we are done. \square

The statement corresponding to Theorem 8.1 for the variety \mathcal{U}_Λ (defined by the vanishing of the u -equations) also holds.

Theorem 8.2. *Let $M \in \mathrm{ind} K_\Lambda$ be rigid. Then there is an isomorphism from $\widetilde{S}_\Lambda / \langle u_M \rangle$ to \widetilde{S}_B which sends u_N to 1 if N is incompatible with M , and sends u_N to $\check{u}_{J_M(N)}$ if N is compatible with M .*

Proof. Let \mathcal{Z}_M be the full subcategory of K_Λ consisting of objects N which satisfy that $\mathrm{Ext}^1(N, M) = 0 = \mathrm{Ext}^1(M, N)$. By [31, Lemma 3.13], the additive quotient category $\mathcal{Z}_M/[M]$ is equivalent to $K^{[-1,0]}(\mathrm{proj} B)$ (and the equivalence is induced by the silting reduction functor $J_M : K^b(\mathrm{proj} \Lambda) / \mathrm{thick}(M) \rightarrow K^b(\mathrm{proj} B)$). Moreover, by [35, Lemma 3.4], if $X, Y \in \mathcal{Z}_M$, then $\mathrm{Ext}^1(X, Y) \cong \mathrm{Ext}^1(J_M(X), J_M(Y))$. Thus the morphism of rings

$$\phi_M : \mathbb{C}[u_X \mid X \in \mathrm{ind} K_\Lambda] \longrightarrow \mathbb{C}[\check{u}_Y \mid Y \in \mathrm{ind} K_B]$$

sending u_M to 0, u_N to 1 if N is incompatible with M and u_N to $u_{J_M(N)}$ if N is compatible to M is surjective and sends u -equations to u -equations. Thus it induces a surjective morphism

$$\phi_M : \tilde{S}_\Lambda / \langle u_M \rangle \longrightarrow \tilde{S}_B.$$

Moreover, the morphism

$$\psi_M : \mathbb{C}[\check{u}_Y \mid Y \in \text{ind } K_B] \longrightarrow \mathbb{C}[u_X \mid X \in \text{ind } K_\Lambda]$$

sending $\check{u}_{J_M(X)}$ to u_X sends the \check{u} -equations to u -equations after sending u_M to 0 and u_N to 1 for all N incompatible with M . Thus it induces a surjective morphism

$$\psi_M : \tilde{S}_B \longrightarrow \tilde{S}_\Lambda / \langle u_M \rangle.$$

Since ψ_M is clearly a right-inverse to ϕ_M , it is injective. Thus it is an isomorphism, and so is ϕ_M . \square

9. GEOMETRY OF $\tilde{\mathcal{M}}_\Lambda$

In this section, we put together the results from Section 5 on the geometry of \mathcal{M}_Λ together with results from Sections 7 and 8 to provide a description of the geometry of $\tilde{\mathcal{M}}_\Lambda$.

Recall that the v_M for $M \in \text{ind } K_\Lambda$ were defined in section 5, and that $R_\Lambda = \mathbb{C}[u_M^{\pm 1} \mid M \in \text{ind } K_\Lambda] / I_\Lambda$, the coordinate ring of \mathcal{M}_Λ , was shown in Proposition 5.9 to be isomorphic to the subring of $\mathbb{C}(y_1, \dots, y_n)$ generated by $v_M^{\pm 1}$.

We will now prove a similar statement for $\tilde{R}_\Lambda = \mathbb{C}[u_M \mid M \in \text{ind } K_\Lambda] / \tilde{I}_\Lambda$, the coordinate ring of $\tilde{\mathcal{M}}_\Lambda$. To pass from the statement for R_Λ to the statement for \tilde{R}_Λ , we will need the following lemma:

Lemma 9.1 ([6, Lemma A.1]). *Let $h : A \rightarrow B$ be a surjective homomorphism of Noetherian commutative rings with identity. Let $S \subset A$ be the multiplicative set generated by elements x_1, \dots, x_p such that $h(x_1), \dots, h(x_p)$ are not zero divisors in B . Suppose that:*

- (1) *the localized homomorphism $S^{-1}h : S^{-1}A \rightarrow S^{-1}B$ is an isomorphism.*
- (2) *for $1 \leq i \leq p$, the induced homomorphism $h_i : A / \langle x_i \rangle \rightarrow B / \langle h(x_i) \rangle$ is an isomorphism.*

Then h is an isomorphism.

Let us write \tilde{V}_Λ for the subring of $\mathbb{C}(y_1, \dots, y_n)$ generated by the v_M .

Theorem 9.2. *\tilde{R}_Λ is isomorphic to \tilde{V}_Λ .*

Proof. Recall that \tilde{I}_Λ is the ideal in $\mathbb{C}[u_M]$ generated by $\hat{F}_M - 1$ for $M \in \text{ind } K_\Lambda$.

Consider the map from the polynomial ring $\mathbb{C}[u_M]$ to \tilde{V}_Λ sending u_M to v_M . By Corollary 5.11, we know that \tilde{I}_Λ is contained in its kernel. We want to show that the induced map, which we shall call $f : \mathbb{C}[u_M] / \tilde{I}_\Lambda \rightarrow \tilde{V}_\Lambda$ is an isomorphism.

Our goal is to apply Lemma 9.1 to the map f and the elements $\{x_i\} = \{u_M\}$ to conclude that f is an isomorphism. Since the elements v_M are non-zero elements of a field, they are certainly non zero divisors. By Proposition 5.9, f is an isomorphism after inverting the u_M (in the source) and the v_M (in the target); this establishes the condition (1) of the lemma.

Two prove condition (2), consider first the case where $N \in \text{ind } K_\Lambda$ is not rigid. Then u_N appears in both terms in the u -equation of N , so u_N is invertible in \tilde{R}_Λ . Thus, $\mathbb{C}[u_M] / (\tilde{I}_\Lambda + \langle u_N \rangle)$ is trivial, and so is $\tilde{V}_\Lambda / \langle f(u_N) \rangle$.

Now, if N is rigid, then by results from Section 8, there is an algebra B_N , of rank one less than that of Λ , such that $\tilde{R}_\Lambda / \langle u_N \rangle \cong \tilde{R}_{B_N}$. By induction on the rank of Λ , the map $f_N : \mathbb{C}[u_M] / (\tilde{I}_\Lambda + \langle u_N \rangle) \rightarrow \tilde{V}_\Lambda / \langle v_N \rangle$ is an isomorphism. This establishes condition (2) for the lemma, and the desired result now follows. \square

We now characterize \tilde{R}_Λ as a subring of $\mathbb{C}(y_1, \dots, y_n)$ in terms of tropical properties.

Proposition 9.3. *The subring of $\mathbb{C}(y_1, \dots, y_n)$ generated by the v_M for $M \in \text{ind } K_\Lambda$ (which we denote $\mathbb{C}[v_M]$) is exactly the subring generated by the rational functions w such that:*

- (1) w is a Laurent monomial in the y_i and F_M , and
- (2) $\text{trop}(w)$ is non-positive.

Proof. That v_M for $M \in \text{ind } K_\Lambda$ satisfy (1) and (2) is immediate from Theorem 7.3. By Lemmas 6.1 and 6.2, we can re-express a Laurent monomial w as in (1) as a Laurent monomial in the v_M . By Theorem 7.3, the power of each v_M must be positive, except perhaps for any v_M with M non-rigid. But observe that if M is non-rigid, then the u -equation for M has a factor of u_M on the lefthand side, so dividing both sides by it, we get an expression for u_M^{-1} with both terms having all u_N appearing only with positive powers. \square

Write F for $\prod_M F_M$. Let X_Λ be the toric variety defined by the g -vector fan, with y_i the coordinate corresponding to the ray $e_i = g(P_i)$.

A subsemigroup S of \mathbb{Z}^n is called saturated if for any $v \in \mathbb{Z}^n$ such that for some positive integer k we have $kv \in S$, it follows that $v \in S$ [20, Definition 1.3.4]. A lattice polytope P in \mathbb{Z}^n is called very ample if for every vertex v of P , the semigroup generated by $(P \cap \mathbb{Z}^n) - v$ is saturated [20, Definition 2.2.17]. It is also true that P is very ample if and only if for k sufficiently large, any lattice point in kP can be written as a sum of k lattice points of P [16, Exercise 2.23].

If P is a full-dimensional very ample polytope in \mathbb{Z}^n , we can take $P \cap \mathbb{Z}^n = \{p_1, \dots, p_s\}$. Then the toric variety associated to the outer normal fan to P can be defined as the Zariski closure of the image of the map sending $T = (\mathbb{C}^*)^n$ to \mathbb{P}^{s-1} , sending t to $[t^{p_1} : \dots : t^{p_s}]$. (See [20, Section 2.3].)

Theorem 9.4. *The affine scheme $\widetilde{\mathcal{M}}_\Lambda$ is isomorphic to the affine open subset of X_Λ where F is non-vanishing. The subvariety \mathcal{M}_Λ is the intersection of $F \neq 0$ with the open torus orbit in X_Λ .*

Proof. The outer normal fan of P , the Newton polytope of F , coincides with the g -vector fan of Λ by [2, Corollary 5.25(b)]. Since the g -vector fan of Λ is unimodular, P is automatically very ample [20, Proposition 2.4.4]. (This resolves in the affirmative the first part of [6, Question 5.4].)

Let $P \cap \mathbb{Z}^n = \{p_1, \dots, p_s\}$, and consider the embedding of X_Λ into \mathbb{P}^{s-1} as above. The polynomial F , the exponents of whose monomials are a subset of $P \cap \mathbb{Z}^n$, defines a hyperplane in \mathbb{P}^{s-1} . The complement of this hyperplane is an affine space. The coordinate functions from \mathbb{P}^{s-1} pull back to y^{p_i}/F on X_Λ . Since $\text{trop}(y^{p_i}/F)$ is non-positive, by Proposition 9.3, $\mathbb{C}[y^{p_i}/F]$ is contained in $\mathbb{C}[u_M]$.

For the opposite inclusion, we want to show that v_M is contained in $\mathbb{C}[y^{p_i}/F]$ for any $M \in \text{ind } K_\Lambda$. Pick a particular M . Its denominator divides F . Since P is very ample, we can choose k sufficiently large so that any lattice point in kP can be written as a sum of k lattice points of P . We can then write $v_M = \overline{A}(y)/F^k(y)$ for some polynomial $\overline{A}(y)$. Because $\text{trop}(v_M) \leq 0$, the exponents of monomials appearing in $\overline{A}(y)$ are all contained in kP . Let y^u be a monomial appearing in $\overline{A}(y)$. By the assumption on k , we can write $u = u_1 + u_2 + \dots + u_k$ with each $u_i \in P$. Then $y^u/F^k(y) = (y^{u_1}/F) \dots (y^{u_k}/F)$ belongs to $\mathbb{C}[y^{p_i}/F]$, and summing over all the monomials of \overline{A} we have the desired result. \square

10. POSITIVE REAL PART OF $\widetilde{\mathcal{M}}_\Lambda$

In this section, we investigate the geometry of $\widetilde{\mathcal{M}}_\Lambda^{>0}$ and $\widetilde{\mathcal{M}}_\Lambda^{\geq 0}$, the loci inside $\widetilde{\mathcal{M}}_\Lambda$ where the coordinates u_M are real and positive (respectively, non-negative). We refer to these as the totally positive (respectively, totally non-negative) parts of $\widetilde{\mathcal{M}}_\Lambda$.

We will study them by showing that they coincide with the corresponding parts of the toric variety X_Λ . Recall that X_Λ , being a non-singular toric variety, admits a covering by a finite number of patches isomorphic to \mathbb{C}^n . The totally non-negative part of X_Λ is glued together from the copy of $\mathbb{R}_{\geq 0}^n$ contained in each copy of \mathbb{C}^n . (See [30, Section 4.2], [20, Section 12.2] for more details.) For P a lattice polytope having Σ_Λ as its outer normal fan, the algebraic moment map μ_P provides a homeomorphism from $X_\Lambda^{\geq 0}$ to P . $X_\Lambda^{\geq 0}$ inherits a stratification from X_Λ . The moment map sends the part associated to the cone $\sigma \in \Sigma_\Lambda$ to the face of P with outer normal cone σ . The totally positive part, $X_\Lambda^{>0}$ corresponds under μ_P to the interior of P .

As before, let F be the product of all the F_M for $M \in \text{ind } \Lambda$, and let P be the Newton polytope of F . This choice of P satisfies the hypotheses from the previous paragraph. Let $\{p_1, \dots, p_s\}$ be the

lattice points of P . Then, as already mentioned, X_Λ can be constructed inside \mathbb{P}^{s-1} as the Zariski closure of the image of $T = (\mathbb{C}^*)^n$ under the map sending $t \in T$ to $[t^{p_1} : \dots, t^{p_s}]$. Constructing X_Λ in this way, the totally positive part of X_Λ is the image under the same map of the positive real torus $T_{\mathbb{R}}^{>0} = (\mathbb{R}_{>0})^n$. The totally non-negative part $X_\Lambda^{\geq 0}$ is the closure of $X_\Lambda^{>0}$ in the analytic topology.

Lemma 10.1. *Thought of as a (rational) function of y_1, \dots, y_n , evaluated at points with all y_i positive, v_M is real and positive.*

Proof. Clearly, it is sufficient to show that $F_M(y) > 0$ for all positive y . If all the y_i are close to zero, then this is certainly true, because the terms in F_M are dominated by the constant term. Now assume there is a path in $\mathbb{R}_{>0}^n$ from a point close to the origin to some point in y -space where v_N is non-positive. Then on that path, some of the F_M have to be non-positive. Consider the first point p on the path where some F_M vanishes. Then, using Remark 5.8, we see that v_M approaches $-\infty$ as we approach that point. But this means that v_M has to vanish on some earlier point on the path, and so some $F_{M'}$ vanishes too, contradicting the minimality of p . \square

Lemma 10.2. *Conversely, if all the v_M are real and positive, the y_i are real and positive.*

Proof. This is immediate from Lemma 6.2. \square

Corollary 10.3. *The totally positive parts of $\widetilde{\mathcal{M}}_\Lambda$ and X_Λ coincide.*

Proof. By definition, the totally positive part of $\widetilde{\mathcal{M}}_\Lambda$ is where the v_M are real and positive. By Lemmas 10.1 and 10.2, this is equivalent to the y_i being real and positive. The totally positive part of \mathcal{M}_Λ therefore coincides with the intersection of \mathcal{M}_Λ with the totally positive part of X_Λ . However, since the locus of X_Λ which is missing from $\widetilde{\mathcal{M}}_\Lambda$ is given by zeros of the F_M , it follows that the totally positive parts of X_Λ and $\widetilde{\mathcal{M}}_\Lambda$ agree. \square

$\widetilde{\mathcal{M}}_\Lambda$ has a natural top form on it, namely the top form of the big torus contained in the toric variety X_Λ , which is $dy_1/y_1 \wedge \dots \wedge dy_n/y_n$. It is natural to ask whether $\widetilde{\mathcal{M}}_\Lambda$ equipped with this differential form is a positive geometry in the sense of [3]. For a rather trivial reason, it isn't: namely, it is not projective. However, except for that issue, it is indeed a positive geometry, because its totally positive part agrees with the totally positive part of its ambient toric variety, and toric varieties are shown in [3] to be totally positive. (We are also implicitly using the fact that the boundaries of the totally positive part are again varieties of the same form.)

11. FUNCTORIALITY UNDER ALGEBRA QUOTIENTS

Let J be a two-sided ideal of Λ , and consider $A = \Lambda/J$. Since indecomposable A -modules are also indecomposable Λ -modules, A also satisfies our finiteness assumption.

Our goal in the section is to show:

Theorem 11.1. *Let $A = \Lambda/J$. There is a surjective map from $\widetilde{\mathcal{M}}_\Lambda$ to $\widetilde{\mathcal{M}}_A$.*

This map also restricts to a map from \mathcal{M}_Λ to \mathcal{M}_A , and is natural in the sense that if we consider some further quotient B of A , the map from $\widetilde{\mathcal{M}}_\Lambda$ to $\widetilde{\mathcal{M}}_B$ factors as a composition of the maps from $\widetilde{\mathcal{M}}_\Lambda$ to $\widetilde{\mathcal{M}}_A$ and that from $\widetilde{\mathcal{M}}_A$ to $\widetilde{\mathcal{M}}_B$.

For \overline{M} an indecomposable A -module, and \overline{N} an A -module, define $[\overline{N} : \overline{M}]$ to be the multiplicity of M as an indecomposable summand of N .

Let π be the functor from K_Λ to K_A defined by $- \otimes_\Lambda A$.

Consider a new set of variables \overline{u}_N for $N \in \text{ind } K_A$.

We prove Theorem 11.1 by establishing the following proposition:

Proposition 11.2. *There is a ring map ϕ from \widetilde{R}_Λ to \widetilde{R}_A defined by:*

$$\phi(\overline{u}_{\overline{M}}) = z_{\overline{M}} := \prod_{N \in \text{ind } K_A} u_N^{[\pi N : \overline{M}]}.$$

For $\bar{L} \in \text{ind } K_A$, note that we can consider its F -polynomial either as an A -module or as a Λ -module. We distinguish them by writing $F_{\bar{L}_A}$ and $F_{\bar{L}_\Lambda}$ (and similarly for \hat{F} -polynomials).

Proof. To establish the proposition, we need to check that for each $\bar{L} \in \text{ind } K_A$,

$$\hat{F}_{\bar{L}_A} |_{\bar{u}_{\bar{M}} \leftarrow z_{\bar{M}}} = 1,$$

that is to say, the result of substituting $z_{\bar{M}}$ for $\bar{u}_{\bar{M}}$ (for all $\bar{M} \in \text{ind } K_A$) gives 1. As a shorthand, we write $\hat{F}_{\bar{L}_A}(z)$ for the result of this substitution.

In fact, we will show that $\hat{F}_{\bar{L}_A}(z) = \hat{F}_{\bar{L}_\Lambda}$. This establishes what we need, since $\hat{F}_{\bar{L}_\Lambda} = 1$ in \tilde{R}_Λ . We now prove this claim. By definition,

$$\hat{F}_{\bar{L}_A}(z) = \sum_{\mathbf{d} \in \mathbb{Z}_{\geq 0}} \chi(\text{Gr}_{\mathbf{d}}(H^0 \bar{L})) \prod_{\bar{N} \in \text{ind } K_A} \prod_{M \in \text{ind } K_\Lambda} u_M^{[\pi M : \bar{N}](\text{hom}(H^0 \bar{N}, H^0 \bar{L}) - \langle g(\bar{N}), \mathbf{d} \rangle)}$$

Collecting all the u_M terms together, we see that they come from the indecomposable summands of πM , so we can rewrite the previous equation as:

$$\tilde{F}_{\bar{L}_A}(z) = \sum_{\mathbf{d} \in \mathbb{Z}_{\geq 0}} \chi(\text{Gr}_{\mathbf{d}}(H^0 \bar{L})) \prod_{M \in \text{ind } K_\Lambda} u_M^{\text{hom}(H^0 \pi M, H^0 \bar{L}) - \langle g(\pi M), \mathbf{d} \rangle)}$$

The g -vector of πM coincides with the g -vector of M by definition, unless P_i is contained in J for some i . But for such i , \mathbf{d}_i is necessarily 0. Thus, we can replace $g(\pi M)$ by $g(M)$. Further, any morphism from $H^0 M$ into an A -module necessarily factors through $M \otimes_\Lambda A = H^0(\pi M)$. So we have:

$$\begin{aligned} \tilde{F}_{\bar{L}_A}(z) &= \sum_{\mathbf{d} \in \mathbb{Z}_{\geq 0}} \chi(\text{Gr}_{\mathbf{d}}(H^0 \bar{L})) \prod_{M \in \text{ind } K_\Lambda} u_M^{\text{hom}(H^0 M, H^0 \bar{L}) - \langle g(M), \mathbf{d} \rangle)} \\ &= \tilde{F}_{\bar{L}_\Lambda}, \end{aligned}$$

as desired. \square

We note that $[\pi N : \bar{M}]$ can be calculated solely on the basis of g_N if N is rigid.

Proposition 11.3. *Let $N \in \text{ind } K_\Lambda$ be rigid. The indecomposable summands of πM can be determined by looking at the minimal cone of the g -vector fan for A that g_N appears in. The expression for g_N as a positive combination of the generators of that cone gives the expression of πN as a sum of indecomposable objects from $\text{ind } K_A$.*

Proof. The key point to establish is that πN is again rigid. We know the projective presentation of πN (it is that of N , unless the ideal J includes idempotents), so once πN is rigid, it must be a sum of compatible g -vectors from $\tilde{\mathcal{T}}$, and there is only one way to combine them to get g_N .

In the case that J contains an idempotent, we don't have an isomorphism of the two spaces of g -vectors, but rather a projection; however, everything works the same way.

We now establish the key point, that πN is again rigid. Suppose that there is a non-zero map from $N \otimes_\Lambda A$ to $\Sigma(N \otimes_\Lambda A)$. This defines a map from N to $\Sigma(N \otimes_\Lambda A)$, and this map lifts to a map from N to ΣN . Suppose it is null-homotopic. The homotopy descends to a homotopy from N to $\Sigma(N \otimes_\Lambda A)$, and this factors through a homotopy from $N \otimes_\Lambda A$ to $\Sigma(N \otimes_\Lambda A)$. Thus, there are no non null-homotopic maps from πN to $\Sigma(\pi N)$. \square

The map from $\widetilde{\mathcal{M}}_\Lambda$ to $\widetilde{\mathcal{M}}_A$ admits a completely different description in terms of the toric geometry of Section 9.

The maximal cones of Σ_Λ are in bijection with the torsion classes of Λ : if θ lies in the interior of a maximal cone σ , then the modules M in the corresponding torsion class can be characterized by the fact that $\langle \theta, \underline{\dim} N \rangle$ is non-negative for every quotient N of M [17, Corollary 4.26]. It immediately follows that if A is a quotient of Λ , then if θ and θ' determine the same torsion class for Λ , they will also do so for A . It follows that the image under the natural map from $K_0(\text{proj } \Lambda)$

to $K_0(\text{proj } A)$ of a maximal cone (and hence of any cone) of Σ_Λ is contained in a cone of Σ_A . (Note that this applies when Λ and A have the same number of simple modules, so the natural map is an identification, but also when A has fewer.) This condition relating the fans Σ_Λ and Σ_A is exactly the condition that the natural map from $K_0(\text{proj } \Lambda)$ to $K_0(\text{proj } A)$ induces a map between the corresponding toric varieties [20, Theorem 3.3.4].

Theorem 11.4. *Let $A = \Lambda/J$. The map of toric varieties from X_Λ to X_A induced by the natural quotient from $K_0(\text{proj } \Lambda)$ to $K_0(\text{proj } A)$ agrees with the map from $\widehat{\mathcal{M}}_\Lambda$ to $\widehat{\mathcal{M}}_A$ from Theorem 11.1.*

Proof. Suppose that $A = \Lambda/J$. Let $J' = J \cap \text{rad } A$, where $\text{rad } A$ is the ideal of A generated by its arrows. We prove the theorem in two steps, first passing from Λ to Λ/J' , and then from Λ/J' to Λ/J . At the first step, J' has no idempotents. At the second step, we are quotienting by idempotents corresponding to isolated vertices in Λ/J' .

It therefore suffices to prove the theorem in two special cases: when J has no idempotents, and when J consists of idempotents corresponding to isolated vertices. In the first case, the blowdown map from X_Λ to X_A identifies the function fields $k(y_1, \dots, y_n)$ of the two toric varieties. Since the map from Theorem 11.1 does this too, the two maps coincide. In the second case, both maps project away the directions corresponding to the idempotents, so they coincide. This completes the proof of the theorem. \square

12. DILOGARITHM IDENTITIES

Recall that we write $L(x)$ for the Rogers dilogarithm of x . In this section, we show how to adapt the argument from [19] to prove the following theorem:

Theorem 12.1. *Let Λ be a finite-dimensional algebra of finite representation type, with n simple modules. Then $\sum_N L(1 - v_N) = n\pi^2/6$, where the sum is taken over all indecomposable N in K_Λ .*

The main ingredient in the proof goes back to Frenkel–Szenes [29], see also [43]. Let G be the continuous functions from $\mathbb{R}_{>0}^n$ to $\mathbb{R}_{>0}$, which we think of as an abelian group written multiplicatively. Consider $G \otimes_{\mathbb{Z}} G$. Write $S^2 G$ for the subgroup of $G \otimes_{\mathbb{Z}} G$ generated by elements of the form $a \otimes b + b \otimes a$.

The following is essentially [29, Proposition 3.1], but slightly reformulated for our convenience. In particular, their functions f_i only take one argument, but the same approach proves the version where the functions take multiple arguments.

Proposition 12.2 ([29, Proposition 3.1]). *Suppose that f_1, \dots, f_r are differentiable functions in G , whose range lies in $(0, 1)$. Suppose further that $\sum_{i=1}^r f_i \otimes (1 - f_i) \in S^2 G$. Then $\sum_{i=1}^r L(f_i(y_1, \dots, y_n))$ does not depend on the values of y_1, \dots, y_r .*

In order to apply the proposition, let us consider $\sum_N v_N \otimes (1 - v_N)$. We have

$$\sum_N v_N \otimes (1 - v_N) = \sum_{N, M} v_N \otimes v_M^{c(N, M)} = \sum_{N, M} c(N, M) v_N \otimes v_M.$$

Since $c(N, M) = c(M, N)$, this expression falls in $S^2 G$.

13. SMALL EXAMPLES

13.1. A_2 path algebra. Let $\Lambda = \mathbb{C}Q$ be the path algebra of the quiver $1 \xrightarrow{a} 2$. Then the category K_Λ has 5 indecomposable objects which we designate by the following shorthand.

$$\begin{aligned} P_1 &= (0 \rightarrow P_1) & P_2 &= (0 \rightarrow P_2) & S_2 &= (P_1 \xrightarrow{a} P_2) \\ \Sigma P_1 &= (P_1 \rightarrow 0) & \Sigma P_2 &= (P_2 \rightarrow 0) \end{aligned}$$

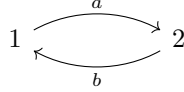
Then the u -equations are as follows.

$$\begin{aligned} u_{P_1} + u_{S_2} u_{\Sigma P_1} &= 1 & u_{P_2} + u_{\Sigma P_1} u_{\Sigma P_2} &= 1 & u_{S_2} + u_{P_1} u_{\Sigma P_2} &= 1 \\ u_{\Sigma P_1} + u_{P_1} u_{P_2} &= 1 & u_{\Sigma P_2} + u_{P_2} u_{S_2} &= 1 \end{aligned}$$

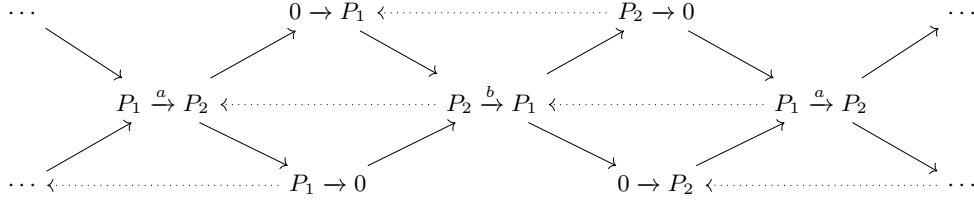
The \widehat{F} -polynomials are as follows.

$$\widehat{F}_{P_1} = u_{P_1} + u_{S_2} u_{\Sigma P_1} \quad \widehat{F}_{P_2} = u_{P_1} u_{P_2} + u_{P_2} u_{S_2} u_{\Sigma P_1} + u_{\Sigma P_1} u_{\Sigma P_2} \quad \widehat{F}_{S_2} = u_{P_2} u_{S_2} + u_{\Sigma P_2}$$

13.2. **A_2 preprojective algebra Π_{A_2} .** The algebra Π_{A_2} is the path algebra of the quiver



modulo the relations $ab = 0$ and $ba = 0$. The category $K_{\Pi_{A_2}}$ has 6 indecomposable objects. Its Auslander–Reiten quiver is periodic and is depicted below, with the dotted arrows representation the action of τ .



We compute the u -equations and \hat{F} -polynomials. We use the following shorthand for the objects of $K_{\Pi_{A_2}}$:

$$\begin{aligned} P_1 &= (0 \rightarrow P_1) & P_2 &= (0 \rightarrow P_2) & S_1 &= (P_2 \xrightarrow{b} P_1) \\ S_2 &= (P_1 \xrightarrow{a} P_2) & \Sigma P_1 &= (P_1 \rightarrow 0) & \Sigma P_2 &= (P_2 \rightarrow 0) \end{aligned}$$

Then the u -equations are as follows.

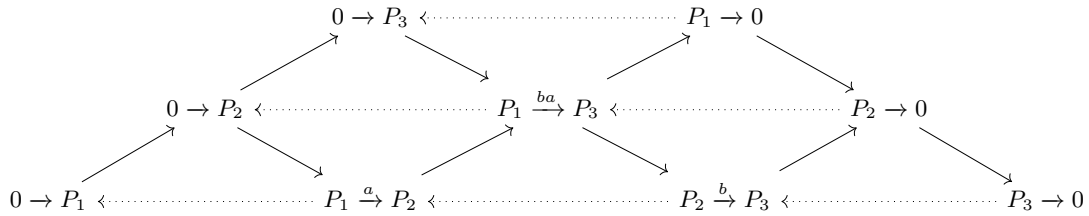
$$\begin{aligned} u_{P_1} + u_{S_2} u_{\Sigma P_1} u_{\Sigma P_2} &= 1 & u_{P_2} + u_{S_1} u_{\Sigma P_1} u_{\Sigma P_2} &= 1 \\ u_{S_1} + u_{P_2} u_{S_2}^2 u_{\Sigma P_1} &= 1 & u_{S_2} + u_{P_1} u_{S_1}^2 u_{\Sigma P_2} &= 1 \\ u_{\Sigma P_1} + u_{P_1} u_{P_2} u_{S_1} &= 1 & u_{\Sigma P_2} + u_{P_1} u_{P_2} u_{S_2} &= 1 \end{aligned}$$

The \hat{F} -polynomials are as follows.

$$\begin{aligned} \hat{F}_{P_1} &= u_{P_1} u_{P_2} u_{S_2} + u_{P_1} u_{S_1} u_{\Sigma P_2} + u_{S_2} u_{\Sigma P_1} u_{\Sigma P_2} & \hat{F}_{P_2} &= u_{P_1} u_{P_2} u_{S_1} + u_{P_2} u_{S_2} u_{\Sigma P_1} + u_{S_1} u_{\Sigma P_1} u_{\Sigma P_2} \\ \hat{F}_{S_1} &= u_{P_1} u_{S_1} + u_{S_2} u_{\Sigma P_1} & \hat{F}_{S_2} &= u_{P_2} u_{S_2} + u_{S_1} u_{\Sigma P_2} \end{aligned}$$

We can check using computer algebra software that the ideal generated by the $\hat{F} = 1$ equations is equal to the ideal generated by the u -equations. In other words, the varieties $\tilde{\mathcal{M}}_{\Pi_{A_2}}$ and $\tilde{\mathcal{U}}_{\Pi_{A_2}}$ are equal.

13.3. **A_3 path algebra.** Let $\Lambda = \mathbb{C}Q$ be the path algebra of the quiver $1 \xrightarrow{a} 2 \xrightarrow{b} 3$. Then the category K_{Λ} has 9 indecomposable objects and its Auslander–Reiten quiver looks like this (with the dotted arrows representing the action of τ).



We use the following shorthand for the objects of $K_{\Pi_{A_2}}$:

$$\begin{aligned} P_1 &= (0 \rightarrow P_1) & P_2 &= (0 \rightarrow P_2) & P_3 &= (0 \rightarrow P_3) \\ S_2 &= (P_1 \xrightarrow{a} P_2) & I_2 &= (P_1 \xrightarrow{ba} P_3) & S_3 &= (P_2 \xrightarrow{b} P_3) \\ \Sigma P_1 &= (P_1 \rightarrow 0) & \Sigma P_2 &= (P_2 \rightarrow 0) & \Sigma P_3 &= (P_3 \rightarrow 0) \end{aligned}$$

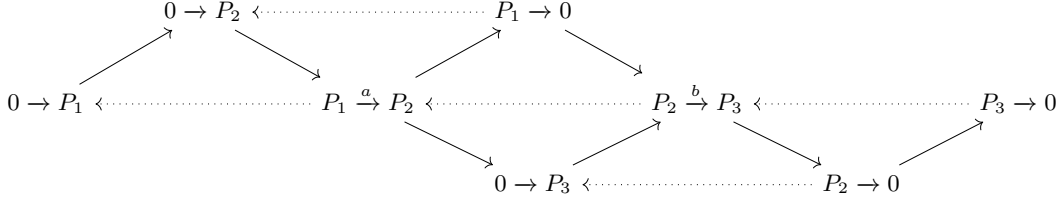
Then the u -equations are as follows.

$$\begin{aligned} u_{P_1} + u_{S_2} u_{I_2} u_{\Sigma P_1} &= 1 & u_{P_2} + u_{I_2} u_{S_3} u_{\Sigma P_1} u_{\Sigma P_2} &= 1 & u_{P_3} + u_{\Sigma P_1} u_{\Sigma P_2} u_{\Sigma P_3} &= 1 \\ u_{S_2} + u_{P_1} u_{S_3} u_{\Sigma P_2} &= 1 & u_{I_2} + u_{P_1} u_{P_2} u_{\Sigma P_2} u_{\Sigma P_3} &= 1 & u_{S_3} + u_{P_2} u_{S_2} u_{\Sigma P_3} &= 1 \\ u_{\Sigma P_1} + u_{P_1} u_{P_2} u_{P_3} &= 1 & u_{\Sigma P_2} + u_{P_2} u_{P_3} u_{S_2} u_{I_2} &= 1 & u_{\Sigma P_3} + u_{P_3} u_{I_2} u_{S_3} &= 1 \end{aligned}$$

The \widehat{F} -polynomials are as follows.

$$\begin{aligned}\widehat{F}_{P_1} &= u_{P_1} + u_{S_2} u_{I_2} u_{\Sigma P_1} \\ \widehat{F}_{P_2} &= u_{P_1} u_{P_2} + u_{P_2} u_{S_2} u_{I_2} u_{\Sigma P_1} + u_{I_2} u_{S_3} u_{\Sigma P_1} u_{\Sigma P_2} \\ \widehat{F}_{P_3} &= u_{P_1} u_{P_2} u_{P_3} + u_{P_2} u_{P_3} u_{S_2} u_{I_2} u_{\Sigma P_1} + u_{P_3} u_{I_2} u_{S_3} u_{\Sigma P_1} u_{\Sigma P_2} + u_{\Sigma P_1} u_{\Sigma P_2} u_{\Sigma P_3} \\ \widehat{F}_{S_2} &= u_{P_2} u_{S_2} + u_{S_3} u_{\Sigma P_2} \\ \widehat{F}_{I_2} &= u_{P_2} u_{P_3} u_{S_2} u_{I_2} + u_{P_3} u_{I_2} u_{S_3} u_{\Sigma P_2} + u_{\Sigma P_2} u_{\Sigma P_3} \\ \widehat{F}_{S_3} &= u_{P_3} u_{I_2} u_{S_3} + u_{\Sigma P_3}\end{aligned}$$

13.4. **A_3 with a relation.** Let Λ be the path algebra of the quiver $1 \xrightarrow{a} 2 \xrightarrow{b} 3$ modulo the relation $ba = 0$. Then the category K_Λ has 8 indecomposable objects and its Auslander–Reiten quiver looks like this (with the dotted arrows still representing the action of τ).



We use the following shorthand for the objects of K_Λ :

$$\begin{aligned}P_1 &= (0 \rightarrow P_1) & P_2 &= (0 \rightarrow P_2) & P_3 &= (0 \rightarrow P_3) \\ S_2 &= (P_1 \xrightarrow{a} P_2) & S_3 &= (P_2 \xrightarrow{b} P_3) \\ \Sigma P_1 &= (P_1 \rightarrow 0) & \Sigma P_2 &= (P_2 \rightarrow 0) & \Sigma P_3 &= (P_3 \rightarrow 0)\end{aligned}$$

Then the u -equations are as follows.

$$\begin{aligned}u_{P_1} + u_{S_2} u_{\Sigma P_1} &= 1 & u_{P_2} + u_{S_3} u_{\Sigma P_1} u_{\Sigma P_2} &= 1 & u_{P_3} + u_{\Sigma P_2} u_{\Sigma P_3} &= 1 \\ u_{S_2} + u_{P_1} u_{S_3} u_{\Sigma P_2} &= 1 & u_{S_3} + u_{P_2} u_{S_2} u_{\Sigma P_3} &= 1 \\ u_{\Sigma P_1} + u_{P_1} u_{P_2} &= 1 & u_{\Sigma P_2} + u_{P_2} u_{P_3} u_{S_2} &= 1 & u_{\Sigma P_3} + u_{P_3} u_{S_3} &= 1\end{aligned}$$

The \widehat{F} -polynomials are as follows.

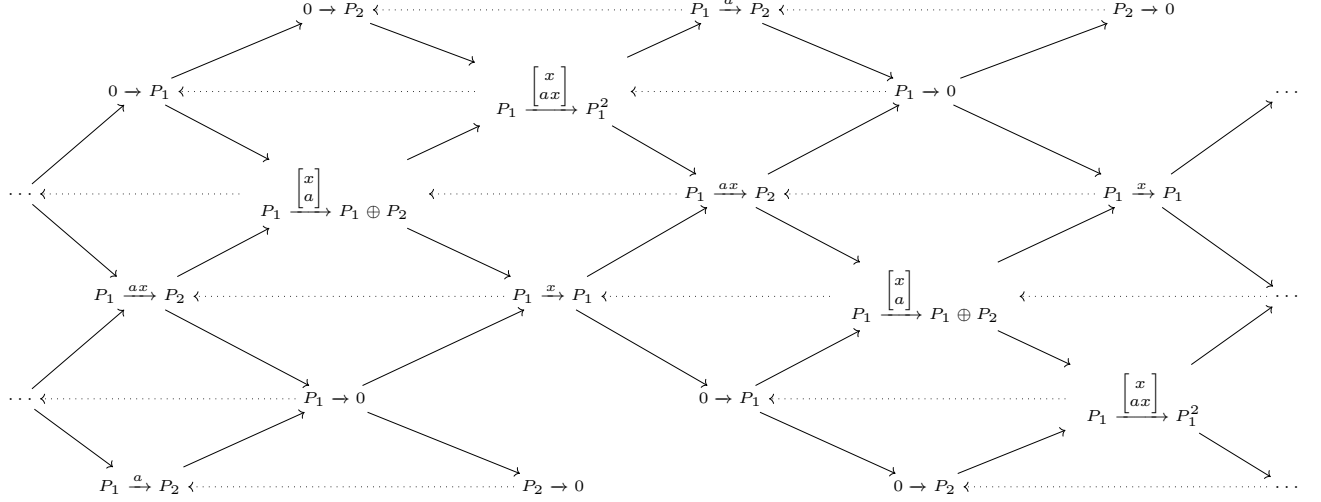
$$\begin{aligned}\widehat{F}_{P_1} &= u_{P_1} + u_{S_2} u_{\Sigma P_1} \\ \widehat{F}_{P_2} &= u_{P_1} u_{P_2} + u_{P_2} u_{S_2} u_{\Sigma P_1} + u_{S_3} u_{\Sigma P_1} u_{\Sigma P_2} \\ \widehat{F}_{P_3} &= u_{P_2} u_{P_3} u_{S_2} + u_{P_3} u_{S_3} u_{\Sigma P_2} + u_{\Sigma P_2} u_{\Sigma P_3} \\ \widehat{F}_{S_2} &= u_{P_2} u_{S_2} + u_{S_3} u_{\Sigma P_2} \\ \widehat{F}_{S_3} &= u_{P_3} u_{S_3} + u_{\Sigma P_3}\end{aligned}$$

We leave the following as an instructive exercise for the reader to illustrate Theorem 11.1. Calculate the v_X for A_3 with no relation, and then check that the above u -equations and \widehat{F} -equations for Λ , the path algebra with a relation, are satisfied by \bar{v}_Y for $Y \in K_\Lambda$, where $\bar{v}_{P_3} = v_{P_3} v_{I_2}$, $\bar{v}_{\Sigma P_1} = v_{I_2} v_{\Sigma P_1}$, and the other \bar{v}_Y agree with the rational function for the corresponding object of K_{A_3} .

13.5. **A_2 with a loop at one vertex.** Consider the path algebra of the quiver

$$x \curvearrowright 1 \xrightarrow{a} 2$$

modulo the relation $x^2 = 0$. The Auslander–Reiten quiver of K_Λ is periodic; a part of it containing a fundamental domain is depicted below.



We use the following shorthand for the objects of K_Λ :

$$\begin{aligned} P_1 &= (0 \rightarrow P_1) & P_2 &= (0 \rightarrow P_2) & S_1 &= (P_1 \xrightarrow{x} P_1) \\ S_2 &= (P_1 \xrightarrow{a} P_2) & I_1 &= (P_1 \rightarrow P_1^2) & 12 &= (P_1 \xrightarrow{ax} P_2) \\ 112 &= (P_1 \rightarrow P_1 \oplus P_2) & \Sigma P_1 &= (P_1 \rightarrow 0) & \Sigma P_2 &= (P_2 \rightarrow 0) \end{aligned}$$

Then the u -equations are as follows.

$$\begin{aligned} u_{P_1} + u_{S_1} u_{S_2}^2 u_{I_1}^2 u_{112}^2 u_{\Sigma P_1}^2 &= 1 & u_{P_2} + u_{S_1} u_{S_2} u_{12} u_{\Sigma P_1}^2 u_{\Sigma P_2} &= 1 \\ u_{S_1} + u_{P_1} u_{P_2} u_{S_1}^2 u_{S_2}^2 u_{I_1}^2 u_{112}^2 u_{\Sigma P_1} &= 1 & u_{S_2} + u_{P_1}^2 u_{P_2} u_{S_1} u_{112} u_{\Sigma P_2} &= 1 \\ u_{I_1} + u_{P_1}^2 u_{S_1}^2 u_{12} u_{112} u_{\Sigma P_1}^2 u_{\Sigma P_2}^2 &= 1 & u_{12} + u_{P_1}^2 u_{P_2} u_{S_1}^2 u_{I_1} u_{112}^2 u_{\Sigma P_1} u_{\Sigma P_2} &= 1 \\ u_{112} + u_{P_1} u_{S_1}^2 u_{S_2} u_{I_1} u_{112}^2 u_{\Sigma P_1}^2 u_{\Sigma P_2} &= 1 & u_{\Sigma P_1} + u_{P_1}^2 u_{P_2} u_{S_1} u_{I_1}^2 u_{12} u_{112}^2 &= 1 \\ u_{\Sigma P_2} + u_{P_2} u_{S_2} u_{I_1}^2 u_{12} u_{112} &= 1 \end{aligned}$$

The \hat{F} -polynomials are as follows.

$$\begin{aligned} \hat{F}_{P_1} &= u_{P_1}^2 u_{S_1} u_{112} + u_{P_1} u_{S_1} u_{S_2} u_{I_1} u_{12} u_{112} u_{\Sigma P_1} + u_{S_1} u_{S_2}^2 u_{I_1}^2 u_{112}^2 u_{\Sigma P_1}^2 \\ \hat{F}_{P_2} &= u_{P_1}^2 u_{P_2} u_{S_1} u_{112} + u_{S_1} u_{S_2} u_{12} u_{\Sigma P_1}^2 u_{\Sigma P_2} + u_{P_1} u_{P_2} u_{S_1} u_{S_2} u_{I_1} u_{12} u_{112} u_{\Sigma P_1} + u_{P_2} u_{S_1} u_{S_2}^2 u_{I_1}^2 u_{112}^2 u_{\Sigma P_1}^2 \\ \hat{F}_{S_1} &= u_{P_1} u_{S_1} u_{112} + u_{S_1} u_{S_2} u_{I_1} u_{12} u_{112} u_{\Sigma P_1} \\ \hat{F}_{S_2} &= u_{\Sigma P_2} + u_{P_2} u_{S_2} u_{I_1}^2 u_{12} u_{112} \\ \hat{F}_{I_1} &= u_{S_1} u_{12} u_{\Sigma P_1}^2 u_{\Sigma P_2}^2 + u_{P_1} u_{P_2} u_{S_1} u_{I_1} u_{12} u_{112} u_{\Sigma P_1} u_{\Sigma P_2} + u_{P_1}^2 u_{P_2}^2 u_{S_1} u_{I_1}^2 u_{12} u_{112}^2 + \\ &\quad + 2u_{P_2} u_{S_1} u_{S_2}^2 u_{I_1}^2 u_{112}^2 u_{\Sigma P_1}^2 u_{\Sigma P_2} + u_{P_1} u_{P_2}^2 u_{S_1} u_{S_2} u_{I_1}^2 u_{112}^2 u_{\Sigma P_1} + u_{P_2}^2 u_{S_1} u_{S_2}^2 u_{I_1}^2 u_{112}^2 u_{\Sigma P_1}^2 \\ \hat{F}_{12} &= u_{S_1} u_{12} u_{\Sigma P_1} u_{\Sigma P_2} + u_{P_1} u_{P_2} u_{S_1} u_{I_1} u_{12} u_{112} + u_{P_2} u_{S_1} u_{S_2} u_{I_1}^2 u_{112}^2 u_{\Sigma P_1} \\ \hat{F}_{112} &= u_{P_1} u_{S_1} u_{12} u_{112} u_{\Sigma P_1} u_{\Sigma P_2} + u_{P_1}^2 u_{P_2} u_{S_1} u_{I_1} u_{12} u_{112}^2 + u_{S_1} u_{S_2} u_{I_1} u_{12} u_{112}^2 u_{\Sigma P_1} u_{\Sigma P_2} + \\ &\quad + u_{P_1} u_{P_2} u_{S_1} u_{S_2} u_{I_1}^2 u_{112}^2 u_{\Sigma P_1} + u_{P_2} u_{S_1} u_{S_2}^2 u_{I_1}^2 u_{112}^2 u_{\Sigma P_1}^2 \end{aligned}$$

In this example again, a computer check shows that $\tilde{\mathcal{M}}_\Lambda = \tilde{\mathcal{U}}_\Lambda$.

13.6. Pellytopes. Let A be the path algebra of the quiver $1 \leftarrow 2 \leftarrow \dots \leftarrow n$, modulo the ideal generated by the relations that the composition of any two consecutive arrows is zero.

There are $3n - 1$ elements of $\text{ind } K_\Lambda$: $P_1, \dots, P_n = S_n; \Sigma P_1, \dots, \Sigma P_n; S_1, \dots, S_{n-1}$. It will turn out to be more convenient to view P_n as part of the family of simples, rather than the family of projectives, so we will consistently write S_n instead of P_n (though we will still write ΣP_n). The u -equations are as follows:

$$\begin{aligned} u_{P_i} + u_{S_{i-1}} u_{\Sigma P_i} u_{\Sigma P_{i+1}} &= 1 \\ u_{\Sigma P_i} + u_{P_{i-1}} u_{P_i} u_{S_i} &= 1 \\ u_{S_i} + u_{P_{i+1}} u_{S_{i+1}} u_{\Sigma P_i} u_{S_{i-1}} &= 1 \end{aligned}$$

where i runs from 1 to $n - 1$ in the first equation, and from 1 to n in the second and third, and we drop the variables $u_{S_0}, u_{S_{n+1}}, u_{P_0}, u_{P_n}, u_{P_{n+1}}$ whenever they arise. These are exactly the u -equations of [12].

We have numbered the vertices of our quiver so that the g -vectors of the objects of $\text{ind } K_\Lambda$ agree with [12]. The g -vector of S_i is $\mathbf{e}_i - \mathbf{e}_{i+1}$ for $1 \leq i \leq n-1$, while the g -vector of $S_n = P_n$ is \mathbf{e}_n . The g -vector of P_i is of course \mathbf{e}_i , while the g -vector of ΣP_i is $-\mathbf{e}_i$.

The F -polynomials are given by $F_{P_i} = 1 + y_{i+1} + y_i y_{i+1}$ for $1 \leq i \leq n-1$, and $F_{S_i} = 1 + y_i$ for $1 \leq i \leq n$.

The \hat{F} -polynomials are

$$\begin{aligned}\hat{F}_{P_i} &= u_{P_i} u_{P_{i+1}} u_{S_{i+1}} + u_{\Sigma P_{i+1}} u_{S_i} u_{P_i} + u_{\Sigma P_{i+1}} u_{\Sigma P_i} u_{S_{i-1}} \\ \hat{F}_{S_i} &= u_{P_i} u_{S_i} + u_{\Sigma P_i} u_{S_{i-1}},\end{aligned}$$

where i runs from 1 to $n-1$ in the first equation and from 1 to n in the second. Again, we drop u_{S_0}, u_{P_n} .

The solutions v_X are given by:

$$\begin{aligned}v_{P_i} &= \frac{F_{S_{i+1}}}{F_{P_i}} \\ v_{\Sigma P_i} &= \frac{y_i F_{S_{i-1}}}{F_{P_{i-1}}} \text{ for } 2 \leq i \leq n \\ v_{\Sigma P_1} &= \frac{y_1}{1 + y_1} \\ v_{S_i} &= \frac{F_{P_i}}{F_{S_{i+1}} F_{S_i}}\end{aligned}$$

We drop the terms $F_{P_n}, F_{S_{n+1}}$ when they appear. (We could also use the formulas given above for them, and then decree that $y_{n+1} = 0$.)

Now consider the \hat{F} -equations (or equivalently u -equations) for Λ , the path algebra of the same A_n quiver, with no relations. We set u_{ij} to correspond to M_{ij} , the indecomposable representation supported at vertices k with $i < k < j$. This accounts for all u_{ij} with $0 \leq i, i \leq j-2, j \leq n+1$. We let $u_{-1,i}$ correspond to ΣP_i . The index set $\text{ind } K_\Lambda$ is thereby identified with the diagonals of an $(n+3)$ -gon, with vertices numbered -1 to $n+1$.

To find solutions to the Pellytope u -equations in terms of those from Λ (which we denote v_{ij}), as in Section 11, we observe that for any representation of A_n which is neither simple nor projective, its projective resolution, when tensored by the radical square zero quotient of the path algebra, splits into a direct sum of an indecomposable projective and a shifted indecomposable projective. Concretely, for $i \leq j-2$, we have the presentation

$$\hat{P}_j \rightarrow \hat{P}_{i+1} \rightarrow M_{ij} \rightarrow 0.$$

(Here we write \hat{P}_i for the projective Λ -module with simple top S_i , keeping the notation P_i for the corresponding projective A -module.)

If $i = j-2$, when we tensor the presentation of $M_{j-2,j}$ by A , it becomes a minimal projective presentation of S_{j-1} . However, if $i \leq j-3$, the result of tensoring the presentation of M_{ij} by A is a complex of the form $P_j \rightarrow P_{i+1}$. Since $\text{Hom}(P_j, P_{i+1})$ is 0, the map must be zero, so the complex splits as $(0 \rightarrow P_{i+1}) \oplus (P_j \rightarrow 0)$. The result is that each $u_{i,j}$ with $i \neq -1, j \neq n+1$, and $j-i > 2$, appears in the product for P_{i+1} and that for ΣP_j .

The solution to the Pellytope u -equations in terms of the solutions for Λ is therefore given by:

$$\begin{aligned}v_{S_i} &= v_{i-1,i+1} \\ v_{P_i} &= \prod_{j=i+2}^{n+1} v_{i-1,j} \\ v_{\Sigma P_i} &= \prod_{\ell=-1}^{i-3} v_{\ell,i}\end{aligned}$$

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