

Multi-Period Trading via Convex Optimization

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September 11, 2017

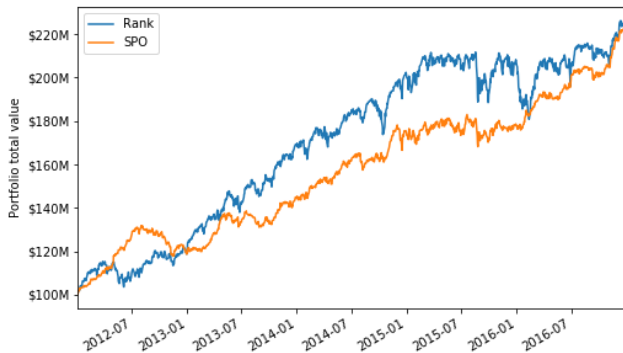
Some trading strategies

- ▶ traditional
 - ▶ buy and hold
 - ▶ hold and rebalance
 - ▶ rank assets and long/short
 - ▶ stat arb
 - ▶ momentum/reversion
- ▶ academic
 - ▶ stochastic control
 - ▶ dynamic programming
- ▶ optimization based

Example: Traditional versus optimization-based

- ▶ S&P 500, daily realized returns/volumes, 2012–2016
- ▶ initial allocation \$100M uniform on S&P 500
- ▶ simulated (noisy) market return forecasts
- ▶ rank ('long-short') trading
 - ▶ rank assets by return forecast
 - ▶ buy top 10, sell bottom 10; 1% daily turnover
- ▶ **single-period optimization (SPO)**
 - ▶ empirical factor risk model
 - ▶ forecasts of transaction and holding cost
 - ▶ hyper-parameters adjusted to match rank trading return

Example: Traditional versus optimization-based



- ▶ rank: return 16.78%, risk 13.91%
- ▶ SPO: return 16.25%, risk 9.08%

Transaction and holding cost

- ▶ **normalized transaction cost** (dollar cost/ v_t) is $\phi_t^{\text{trade}}(z_t)$
- ▶ **normalized holding cost** (dollar cost/ v_t) is $\phi_t^{\text{hold}}(z_t)$
- ▶ these are separable across assets, zero for cash account
- ▶ self-financing condition:

$$\mathbf{1}^T z_t + \phi_t^{\text{trade}}(z_t) + \phi_t^{\text{hold}}(w_t + z_t) = 0$$

- ▶ this determines cash 'trade' $(z_t)_{n+1}$ in terms of asset holdings and trades $(w_t)_{1:n}, (z_t)_{1:n}$

Single asset transaction cost model

- ▶ trading dollar amount x in an asset incurs cost

$$a|x| + b\sigma \frac{|x|^{3/2}}{V^{1/2}} + cx$$

- ▶ a, b, c are transaction cost model parameters
 - ▶ σ is one-period volatility
 - ▶ V is one-period volume
- ▶ a standard model used by practitioners
- ▶ variations: quadratic term, piecewise-linear, ...
- ▶ same formula for normalized trades, with $V \mapsto V/v_t$

Single asset holding cost model

- ▶ holding x costs $s(x)_- = s \max\{-x, 0\}$
- ▶ $s > 0$ is shorting cost rate
- ▶ variations: quadratic term, piecewise-linear, ...
- ▶ same formula for normalized portfolio (weights)

Simulation

- ▶ simulation: for $t = 1, \dots, T$,
 - ▶ (arbitrary) trading policy chooses asset trades $(z_t)_{1:n}$
 - ▶ determine cash trade $(z_t)_{n+1}$ from self-financing condition
 - ▶ update portfolio weights and value
- ▶ **backtest**
 - ▶ use realized past returns, volumes
 - ▶ evaluate candidate trading policies
- ▶ **stress test**
 - ▶ use challenging (but plausible) data
- ▶ **model calibration**
 - ▶ adjust model parameters so simulation tracks real portfolio

Outline

Introduction

Model

Single-period optimization

Multi-period optimization

Estimated portfolio return

$$\hat{R}_t^p = \hat{r}_t^T (w_t + z_t) - \hat{\phi}_t^{\text{trade}}(z_t) - \hat{\phi}_t^{\text{hold}}(w_t + z_t)$$

- ▶ quantities with $\hat{}$ are estimates or forecasts
(based on data available at time t)
- ▶ asset return forecast \hat{r}_t is most important
- ▶ transaction cost estimates depend on estimates of bid-ask spread, volume, volatility
- ▶ holding cost is typically known

Single-period optimization problem

$$\begin{aligned} & \text{maximize} && \hat{R}_t^p - \gamma^{\text{risk}} \psi_t(w_t + z_t) \\ & \text{subject to} && z_t \in \mathcal{Z}_t, \quad w_t + z_t \in \mathcal{W}_t, \\ & && \mathbf{1}^T z_t + \hat{\phi}_t^{\text{trade}}(z_t) + \hat{\phi}_t^{\text{hold}}(w_t + z_t) = 0 \end{aligned}$$

- ▶ z_t is variable; w_t is known
- ▶ ψ_t is risk measure, $\gamma^{\text{risk}} > 0$ risk aversion parameter
- ▶ objective is risk-adjusted estimated net return
- ▶ \mathcal{Z}_t are trade constraints, \mathcal{W}_t hold constraints

Single-period optimization problem

- ▶ self-financing constraint can be approximated as $\mathbf{1}^T z_t = 0$
(slightly over-estimates updated cash balance)

$$\begin{aligned} & \text{maximize} && \hat{r}_t^T (w_t + z_t) \\ & && - \gamma^{\text{risk}} \psi_t(w_t + z_t) \\ & && - \hat{\phi}_t^{\text{trade}}(z_t) \\ & && - \hat{\phi}_t^{\text{hold}}(w_t + z_t) \\ & \text{subject to} && \mathbf{1}^T z_t = 0, \quad z_t \in \mathcal{Z}_t, \quad w_t + z_t \in \mathcal{W}_t \end{aligned}$$

- ▶ a **convex optimization problem** provided risk, trade, and hold functions/constraints are

Traditional quadratic risk measure

- ▶ $\psi_t(x) = x^T \Sigma_t x$
- ▶ Σ_t is an estimate of return covariance
- ▶ factor model risk $\Sigma_t = F_t \Sigma_t^f F_t^T + D_t$
 - ▶ $F_t \in \mathbf{R}^{n \times k}$ is factor exposure matrix
 - ▶ $F_t^T w_t$ are factor exposures
 - ▶ Σ_t^f is factor covariance
 - ▶ D_t is diagonal ('idiosyncratic') asset returns
- ▶ **variation:** $\psi_t(x) = (x^T \Sigma_t x - (\sigma^{\text{tar}})^2)_+$
 - ▶ $(\sigma^{\text{tar}})^2$ is target risk

Robust risk measures

- ▶ worst case quadratic risk: $\psi_t(x) = \max_{i=1,\dots,M} x^T \Sigma_t^{(i)} x$
 - ▶ $\Sigma^{(i)}$ are scenario or market regime covariances

- ▶ worst case over correlation changes:

$$\psi_t(x) = \max_{\Delta} x^T (\Sigma + \Delta) x, \quad |\Delta_{ij}| \leq \kappa (\Sigma_{ii} \Sigma_{jj})^{1/2}$$

$\kappa \in [0, 1]$ is a parameter, say $\kappa = 0.05$

- ▶ can express as

$$\psi_t(x) = x^T \Sigma x + \kappa \left(\Sigma_{11}^{1/2} |x_1| + \dots + \Sigma_{nn}^{1/2} |x_n| \right)^2$$

Return forecast risk

- forecast uncertainty: any return forecast of form

$$\hat{r} + \delta, \quad |\delta| \leq \rho \in \mathbf{R}^{n+1}$$

is plausible; ρ_i is forecast return spread for asset i

- worst case return forecast is

$$\min_{|\delta| \leq \rho} (\hat{r}_t + \delta)^T (w_t + z_t) = \hat{r}_t^T (w_t + z_t) - \rho^T |w_t + z_t|$$

- same as using nominal return forecast, with a return forecast risk term $\psi_t(x) = \rho^T |x|$

Holding constraints

long only	$w_t + z_t \geq 0$
leverage limit	$\ (w_t + z_t)_{1:n}\ _1 \leq L^{\max}$
capitalization limit	$(w_t + z_t) \leq \delta C_t / v_t$
weight limits	$w^{\min} \leq w_t + z_t \leq w^{\max}$
minimum cash balance	$(w_t + z_t)_{n+1} \geq c_{\min} / v_t$
factor/sector neutrality	$(F_t)_i^T (w_t + z_t) = 0$
liquidation loss limit	$T^{\text{liq}} \hat{\phi}_t^{\text{trade}}((w_t + z_t) / T^{\text{liq}}) \leq \delta$
concentration limit	$\sum_{i=1}^K (w_t + z_t)_{[i]} \leq \omega$

Trading constraints

turnover limit $\|(z_t)_{1:n}\|_1/2 \leq \delta$

limit to trading volume $|(z_t)_{1:n}| \leq \delta(\hat{V}_T/v_t)$

transaction cost limit $\hat{\phi}^{\text{trade}}(z_t) \leq \delta$

Convexity

- ▶ objective terms and constraints above are convex, as are many others
- ▶ consequences of convexity: we can
 - ▶ (globally) solve, reliably and fast
 - ▶ add many objective terms and constraints
 - ▶ rapidly develop using domain-specific languages
- ▶ nonconvexities are not needed or easily handled, e.g.,
 - ▶ quantized positions
 - ▶ minimum trade sizes
 - ▶ target leverage (e.g., $\|(x_t + w_t)_{1:n}\|_1 = L^{\text{tar}}$)

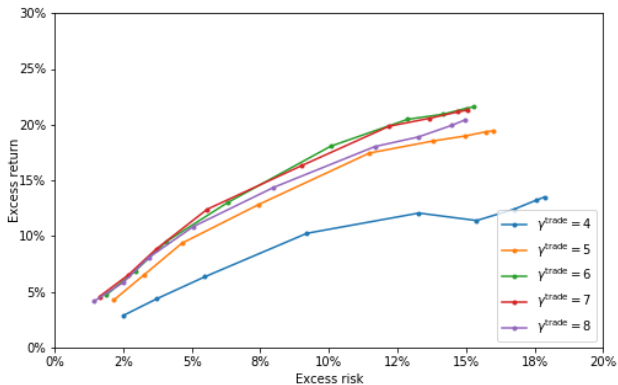
Using single-period optimization

- ▶ constraints and objective terms are inspired by estimates of the real values, e.g., of transaction or hold costs
- ▶ we add positive (hyper) parameters that scale the terms, e.g., γ^{trade} , γ^{hold}
- ▶ these are **knobs** we turn to get what we want
 - ▶ absolute value term in $\hat{\phi}^{\text{trade}}$ discourages small trades
 - ▶ 3/2-power term in $\hat{\phi}^{\text{trade}}$ discourages large trades
 - ▶ shorting cost discourages holding short positions
 - ▶ liquidation cost discourages holding illiquid positions
- ▶ we simulate/back-test to choose hyper-parameter values
- ▶ exact same (meta-) story in control, machine learning, ...

Example

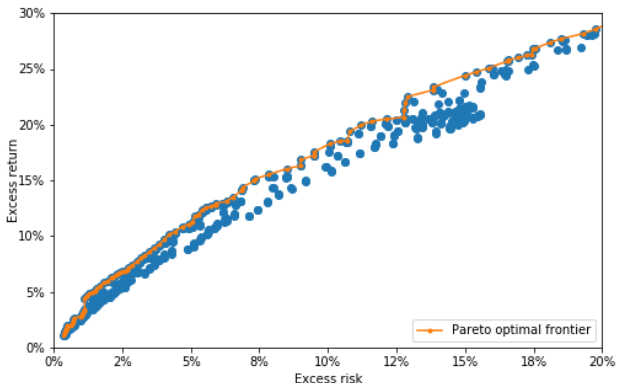
- ▶ S&P 500, daily realized returns, volumes, 2012–2016
- ▶ initial allocation \$100M uniform on S&P 500
- ▶ simulated (noisy) market return forecasts
- ▶ risk model: empirical factor model with 15 factors
- ▶ volume, volatility estimated as average of last 10 values
- ▶ vary hyper-parameters γ^{risk} , γ^{trade} , γ^{hold} over ranges

Example: Risk-return trade-off



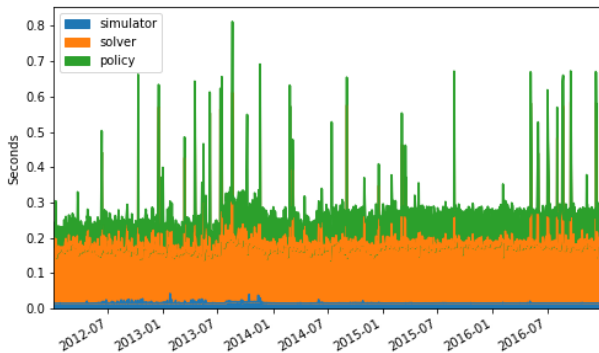
Example: Pareto optimal frontier

- grid search over 410 hyper-parameter combinations



Example: Timing

- execution time, generic CVXPY, single-thread ECOS solver



Outline

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Multi-period optimization

Idea

- ▶ at period t , optimize over sequence of portfolio weights

$$w_{t+1}, \dots, w_{t+H-1}$$

subject to $\mathbf{1}^T w_\tau = 1, \tau = t+1, \dots, t+H-1$

- ▶ H is the (planning) horizon
- ▶ execute trades $z_t = w_{t+1} - w_t$
- ▶ need forecasts over the horizon, e.g.,

$$\hat{r}_{\tau|t}, \quad \tau = t, \dots, t+H-1$$

forecast of market return in period τ made at period t

- ▶ can exploit differing short- and long-term forecasts

Multi-period optimization

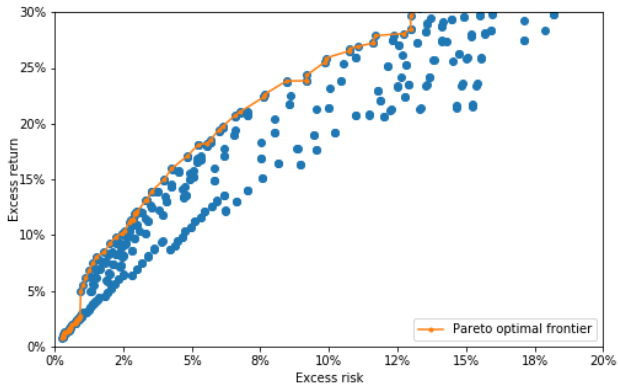
$$\begin{aligned} \text{maximize} \quad & \sum_{\tau=t+1}^{t+H} \left(\hat{r}_{\tau|t}^T w_{\tau} - \gamma^{\text{risk}} \psi_{\tau}(w_{\tau}) \right. \\ & \quad \left. - \gamma^{\text{hold}} \hat{\phi}_{\tau}^{\text{hold}}(w_{\tau}) \right. \\ & \quad \left. - \gamma^{\text{trade}} \hat{\phi}_{\tau}^{\text{trade}}(w_{\tau} - w_{\tau-1}) \right) \\ \text{subject to} \quad & \mathbf{1}^T w_{\tau} = 1, \quad w_{\tau} - w_{\tau-1} \in \mathcal{Z}_{\tau}, \quad w_{\tau} \in \mathcal{W}_{\tau}, \\ & \tau = t+1, \dots, t+H \end{aligned}$$

- ▶ reduces to single-period optimization for $H = 1$
- ▶ computational cost scales linearly in horizon H
- ▶ same idea widely used in **model predictive control**

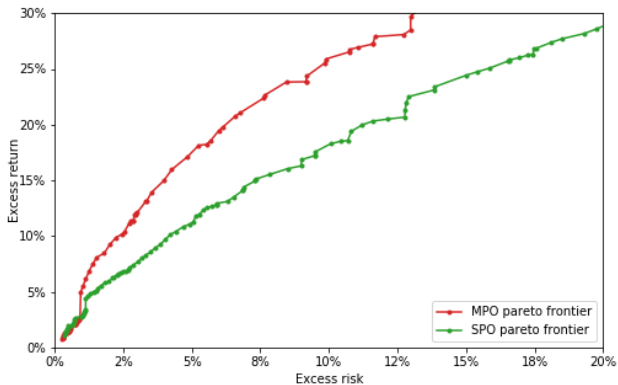
Example

- ▶ same data as single-period example
- ▶ $H = 2$, so we have forecasts for current and next periods
- ▶ grid search over 390 hyper-parameter combinations

Example: Pareto frontier



Example: Multi- and single-period comparison



Conclusions

convex optimization to choose trades

- ▶ idea traces to Markowitz (1952), model predictive control
- ▶ gives an organized way to parametrize good trading strategies
- ▶ works with any forecasts
- ▶ handles a wide variety of practical constraints and costs

Is it optimal?

- ▶ if we assume (say) $\log(\mathbf{1} + r_t) \sim \mathcal{N}(\mu, \Sigma)$ are independent, the multi-period trading problem is a convex stochastic control problem
- ▶ multi-period optimization is almost an optimal strategy (Boyd, Mueller, O'Donoghue, Wang, 2014)
- ▶ but real returns are not log-normal, or independent, or stationary, or even a stochastic process

References

- ▶ *Active Portfolio Management: A Quantitative Approach*, Grinold & Kahn
- ▶ *Convex Optimization*, Boyd & Vandenberghe
- ▶ *Multi-Period Trading via Convex Optimization*, Boyd et al., Foundations & Trends in Optimization

- ▶ `github.com/cvxgrp/cvxportfolio`