Question 15 :

D'après l'énoncé on sait que :

- $P(X=x \mid Y=+1) = f^+$
- $P(X=x \mid Y=-1) = f^-$
- $\pi_+ = P(Y=+1)$
- $\pi_- = P(Y=-1)$

$\pi^+ + \pi^- = 1$ d'où $\pi^+ = 1 - \pi^-$

- Formule de Bayes : $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$
- $P(A \cap B) = P(A \mid B) \times P(B) = P(B \mid A) \times P(A)$

$$P(Y=+1 \mid X=x) = \frac{\overbrace{P(Y=+1 \cap X=x)}^{①}}{P(X=x)} \quad \text{(d'après Bayes)}$$

De plus $P(X=x) = \underbrace{P(X=x \cap Y=+1)}_{②} + \underbrace{P(X=x \cap Y=-1)}_{③}$

① : $P(Y=+1 \cap X=x) = P(X=x \mid Y=+1) \times P(Y=+1) = f^+ \times \pi^+$

② : $P(X=x \cap Y=+1) = P(X=x \mid Y=+1) \times P(Y=+1) = f^+ \times \pi^+$

③ : $P(X=x \cap Y=-1) = P(X=x \mid Y=-1) \times P(Y=-1) = f^- \times \pi^-$

d'où
$$P(Y=+1 \mid X=x) = \frac{①}{② + ③} = \frac{f^+ \times \pi^+}{(f^+ \times \pi^+) + (f^- \times \pi^-)}$$
$$= \boxed{\frac{f^+ \times \pi^+}{(f^+ \times \pi^+) + (f^- \times (1 - \pi^+))}}$$

De manière analogue au calcul de $P(Y=+1 \mid X=x)$ on peut écrire que :

$$P(Y=-1 \mid X=x) = \frac{f^- \times \pi^-}{(f^+ \times \pi^+) + (f^- \times \pi^-)} = \boxed{\frac{f^- \times (1 - \pi^+)}{(f^+ \times \pi^+) + (f^- \times (1 - \pi^+))}}$$

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Question 16

On sait d'après la question précédente que :

$$\cdot\, P(Y=+1 \mid X=x) = \frac{f^+ \times \pi^+}{(f^+ \times \pi^+) + (f^- \times \pi^-)}$$

$$\cdot\, P(Y=-1 \mid X=x) = \frac{f^+ \times \pi^+}{(f^+ \times \pi^+) + (f^- \times \pi^-)}$$

d'où :

$$\frac{P(Y=+1 \mid X=x)}{P(Y=-1 \mid X=x)} = \frac{\dfrac{f^+ \times \pi^+}{(f^+ \times \pi^+) + (f^- \times \pi^-)}}{\dfrac{f^- \times \pi^-}{(f^+ \times \pi^+) + (f^- \times \pi^-)}}$$

$$= \frac{f^+ \times \pi^+}{(f^+ \times \pi^+) + (f^- \times \pi^-)} \times \frac{(f^+ \times \pi^+) + (f^- \times \pi^-)}{f^- \times \pi^-}$$

$$= \frac{f^+ \times \pi^+}{f^- \times \pi^-} = \boxed{\frac{f^+ \times \pi^+}{f^- \times (1-\pi^+)}}$$

d'après l'énoncé :

$$\cdot\, f^+ = \underbrace{\frac{1}{(2\pi)^{p/2}\sqrt{\det \Sigma}}}_{A} \exp\left\{\underbrace{-\frac{1}{2}(x-\mu^+)^T \Sigma^{-1} (x-\mu^+)}_{B}\right\}$$

$$\cdot\, f^- = \frac{1}{(2\pi)^{p/2}\sqrt{\det \Sigma}} \exp\left\{\underbrace{-\frac{1}{2}(x-\mu^-)^T \Sigma^{-1} (x-\mu^-)}_{C}\right\}$$

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Par suite :

$$\log\left(\frac{P(Y=+1 \mid X=x)}{P(Y=-1 \mid X=x)}\right) = \log\left(\frac{f^{+} \times \pi^{+}}{f^{-} \times (1-\pi^{+})}\right)$$

$$= \log\left(\frac{(A\ e^{B}) \times \pi^{+}}{(A\ e^{C}) \times (1-\pi^{+})}\right) = \log\left(\frac{e^{B} \times \pi^{+}}{e^{C} \times (1-\pi^{+})}\right)$$

$$= \log(e^{B} \times \pi^{+}) - \log(e^{C} \times (1-\pi^{+}))$$

$$= \left[\log(e^{B}) + \log(\pi^{+})\right] - \left[\log(e^{C}) + \log(1-\pi^{+})\right]$$

$$= B + \log(\pi^{+}) - \left[C + \log(1-\pi^{+})\right]$$

$$= B + \log(\pi^{+}) - C - \log(1-\pi^{+})$$

Donc :

$$\log\left(\frac{P(Y=+1 \mid X=x)}{P(Y=-1 \mid X=x)}\right) = B + \log(\pi^{+}) - C - \log(1-\pi^{+})$$

$$= -\frac{1}{2}(x-\mu^{+})^{T}\Sigma^{-1}(x-\mu^{+}) + \log(\pi^{+})$$

$$+ \frac{1}{2}(x-\mu^{-})^{T}\Sigma^{-1}(x-\mu^{-}) - \log(1-\pi^{+})$$

En developpant on a :

$$= -\frac{1}{2}\left((x^{T}\Sigma^{-1}x) \overbrace{-(x^{T}\Sigma^{-1}\mu_{+}) - (\mu_{+}^{T}\Sigma^{-1}x)}^{-2(x^{T}\Sigma^{-1}\mu_{+})} + (\mu_{+}^{T}\Sigma^{-1}\mu_{+})\right) + \log(\pi^{+})$$

$$+\frac{1}{2}\left((x^{T}\Sigma^{-1}x) \underbrace{-(x^{T}\Sigma^{-1}\mu_{-}) - (\mu_{-}^{T}\Sigma^{-1}x)}_{-2(x^{T}\Sigma^{-1}\mu_{-})} + (\mu_{-}^{T}\Sigma^{-1}\mu_{-})\right) - \log(1-\pi^{+})$$

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$$= -\frac{1}{2}(x^T\Sigma^{-1}x) + (x^T\Sigma^{-1}\mu_+) + \frac{1}{2}(\mu_+^T\Sigma^{-1}\mu_+) + \log(\pi^+)$$
$$+ \frac{1}{2}(x^T\Sigma^{-1}x) - (x^T\Sigma^{-1}\mu_-) - \frac{1}{2}(\mu_-^T\Sigma^{-1}\mu_-) - \log(1-\pi^+)$$

donc
$$\log\left(\frac{P(Y=-1 \mid X=x)}{P(Y=+1 \mid X=x)}\right) = \boxed{\log(\pi^+) - \log(1-\pi^+) + [x^T\Sigma^{-1}(\mu_+-\mu_-)] - \frac{1}{2}\mu_+^T\Sigma^{-1}\mu_+ + \frac{1}{2}\mu_-^T\Sigma^{-1}\mu_-}$$

Question 17 :

$$\text{Classifier} = \begin{cases} +1 \text{ si } P(Y=+1 \mid X=x) > P(Y=-1 \mid X=x) \\ -1 \text{ sinon} \end{cases}$$

$$\text{Classifier} = \begin{cases} +1 \text{ si } \frac{P(Y=-1 \mid X=x)}{P(Y=+1 \mid X=x)} > 1 \\ -1 \text{ sinon} \end{cases}$$

$$\text{Classifier} = \begin{cases} +1 \text{ si } \log\left(\frac{P(Y=-1 \mid X=x)}{P(Y=+1 \mid X=x)}\right) > 0 \\ -1 \text{ sinon} \end{cases}$$

Donc $\log\left(\frac{P(Y=-1 \mid X=x)}{P(Y=+1 \mid X=x)}\right) > 0$

$$\iff \log(\pi^+) - \log(1-\pi^+) + [x^T\Sigma^{-1}(\mu_+-\mu_-)] - \frac{1}{2}\mu_+^T\Sigma^{-1}\mu_+ + \frac{1}{2}\mu_-^T\Sigma^{-1}\mu_- > 0$$

$$\iff [x^T\Sigma^{-1}(\mu_+-\mu_-)] > \frac{1}{2}\mu_+^T\Sigma^{-1}\mu_+ - \frac{1}{2}\mu_-^T\Sigma^{-1}\mu_- + \log(1-\pi^+) - \log(\pi)$$

avec $\pi^+ = \frac{m}{n}$

$$\iff [x^T\Sigma^{-1}(\mu_+-\mu_-)] > \frac{1}{2}\mu_+^T\Sigma^{-1}\mu_+ - \frac{1}{2}\mu_-^T\Sigma^{-1}\mu_- + \log\left(1-\frac{m}{n}\right) - \log\left(\frac{m}{n}\right)$$

donc
$$\boxed{\text{Classifier} = \begin{cases} +1 \text{ si } x^T\Sigma^{-1}(\mu_+-\mu_-) > \frac{1}{2}\mu_+^T\Sigma^{-1}\mu_+ - \frac{1}{2}\mu_-^T\Sigma^{-1}\mu_- + \log\left(1-\frac{m}{n}\right) - \log\left(\frac{m}{n}\right) \\ -1 \text{ sinon} \end{cases}}$$

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