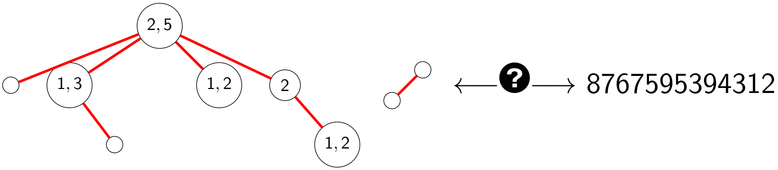
Basis of totally primitive elements of **WQSym**

Thanks for coming! These slides are made in such a way that you can jump around using the links in green (ex: Go to main result) or in the bottom-right of every slide.
You will also find some clickable ❓ where I can give you more details, don't hesitate to ask !



Problematic, Main result and Steps

- 1999 Hivert define **WQSym** the Hopf algebra on
  packed words/surjections/ordered set partitions,
- 2001 Duchanp-Hivert-Thibon conjecture auto-duality of **WQSym**,
- 2005 Foissy proves this auto-duality with bidendriform structure.

Main Result

An explicit bidendriform morphism from **WQSym** to **WQSym***.

Steps to construct the automorphism:

- Basis $\mathbb{P}$ of totally primitive elements of **WQSym**,
- Basis $\mathbb{O}$ of totally primitive elements of **WQSym***,
- Bijection between biplan forests.

Packed words

Definition

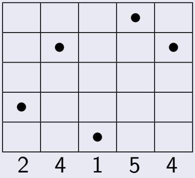
A word over the alphabet $\mathbb{N}_{>0}$ is packed if all the letters from 1 to its maximum *m* appears at least once.

Packed words of size 0, 1, 2 and 3

- ϵ
- 1
- 12 21 11
- 123 132 213 231 312 321
  122 212 221 112 121 211 111

The function *pack*

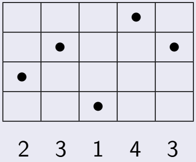
$24154 \notin$ **PW**



remove empty lines

$\rightarrow$ pack $\rightarrow$

$pack(24154) = 23143 \in$ **PW**



WQSym, the Hopf algebra on packed words

WQSym

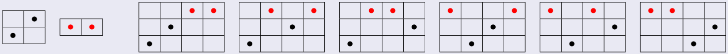
- $\mathbb{R}_{3112} + \mathbb{R}_{212} - 3\mathbb{R}_{212341} - \frac{5}{3}\mathbb{R}_{111}$
- $\mathbb{R}_{12} \cdot \mathbb{R}_{11} = \mathbb{R}_{1233} + \mathbb{R}_{1323} + \mathbb{R}_{1332} + \mathbb{R}_{3123} + \mathbb{R}_{3132} + \mathbb{R}_{3312}$
- $\Delta(\mathbb{R}_{24231}) = \mathbb{R}_{\epsilon} \otimes \mathbb{R}_{24231} + \mathbb{R}_{121} \otimes \mathbb{R}_{21} + \mathbb{R}_{1312} \otimes \mathbb{R}_{1} + \mathbb{R}_{24231} \otimes \mathbb{R}_{\epsilon}$

- Formal sums of packed words
- An associative and unitary product $\cdot$
- A coassociative and counitary coproduct Δ
- The Hopf relation $\Delta(a \cdot b) = \Delta(a) \cdot \Delta(b)$

Basis $\mathbb{R}$ of **WQSym**

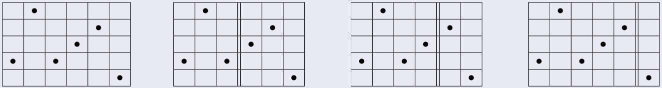
Product on $\mathbb{R}$:
Shifted shuffle

$$\mathbb{R}_{12} \cdot \mathbb{R}_{11} = \mathbb{R}_{1233} + \mathbb{R}_{1323} + \mathbb{R}_{1332} + \mathbb{R}_{3123} + \mathbb{R}_{3132} + \mathbb{R}_{3312}$$



Reduced coproduct on $\mathbb{R}$:
Deconcatenation forbidding cut between two equal letters

$$\tilde{\Delta}(\mathbb{R}_{252341}) = \mathbb{R}_{121} \otimes \mathbb{R}_{231} + \mathbb{R}_{1312} \otimes \mathbb{R}_{21} + \mathbb{R}_{14123} \otimes \mathbb{R}_{1}$$



Bidendriform bialgebra

Example of half products

- $\mathbb{R}_{12} \cdot \mathbb{R}_{11} = \mathbb{R}_{1233} + \mathbb{R}_{1323} + \mathbb{R}_{1332} + \mathbb{R}_{3123} + \mathbb{R}_{3132} + \mathbb{R}_{3312}$
- $\mathbb{R}_{12} \prec \mathbb{R}_{11} = \mathbb{R}_{1332} + \mathbb{R}_{3132} + \mathbb{R}_{3312}$
- $\mathbb{R}_{12} \succ \mathbb{R}_{11} = \mathbb{R}_{1233} + \mathbb{R}_{1323} + \mathbb{R}_{3123}$

Example of half coproducts

- $\tilde{\Delta}(\mathbb{R}_{24\mathbf{2553}1}) = \mathbb{R}_{121} \otimes \mathbb{R}_{\mathbf{3}321} + \mathbb{R}_{121\mathbf{33}} \otimes \mathbb{R}_{21} + \mathbb{R}_{131\mathbf{44}2} \otimes \mathbb{R}_{1}$
- $\Delta_{\prec}(\mathbb{R}_{24\mathbf{2553}1}) = \mathbb{R}_{121\mathbf{33}} \otimes \mathbb{R}_{21} + \mathbb{R}_{131\mathbf{44}2} \otimes \mathbb{R}_{1}$
- $\Delta_{\succ}(\mathbb{R}_{24\mathbf{2553}1}) = \mathbb{R}_{121} \otimes \mathbb{R}_{\mathbf{3}321}$

Refinement of the Hopf definition

Refinement of associativity and coassociativity: 3 and 3 equations
Refinement of the Hopf relation: 4 equations

Totally primitive elements

Primitive elements Prim(A)

P is a primitive element $\iff \tilde{\Delta}(P) = 0$

Ex : $P = \mathbb{R}_{121\mathbf{3}} - \mathbb{R}_{2\mathbf{3}21}$ $\quad \tilde{\Delta}(P) = \mathbb{R}_{121} \otimes \mathbb{R}_{\mathbf{1}} - \mathbb{R}_{\mathbf{121}} \otimes \mathbb{R}_{1} = 0$

Totally primitive element TPrim(A)

P is a totally primitive element $\iff \Delta_{\prec}(P) = \Delta_{\succ}(P) = 0$

Ex : $\mathbb{R}_{12443} - \mathbb{R}_{21443} - \mathbb{R}_{23441} + \mathbb{R}_{32441}$

Theorem [Foissy]

If A is a bidendriform bialgebra then A is freely generated by TPrim(A) as a dendriform algebra.

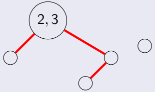
Corollary

WQSym is self-dual.

The basis $\mathbb{P}$ indexed by red biplan forests

Example of expansion from $\mathbb{P}$ to $\mathbb{R}$

$\mathbb{P}$



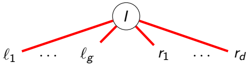
$$= \mathbb{R}_{235541} - \mathbb{R}_{245531} - \mathbb{R}_{244531} - \mathbb{R}_{245431} - \mathbb{R}_{254431} + \mathbb{R}_{325541} - \mathbb{R}_{425531} - \mathbb{R}_{524431} + \mathbb{R}_{352541} - \mathbb{R}_{452531} + \mathbb{R}_{355241} - \mathbb{R}_{455231} + \mathbb{R}_{344521} + \mathbb{R}_{345421} + \mathbb{R}_{354421} + \mathbb{R}_{534421}$$

Bijection between packed words and red biplan forests. ❓

$$\mathbb{P}_{\bigcirc} := \mathbb{R}_1,$$

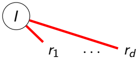
$$\mathbb{P}_{t_1,\dots,t_k} := (\dots(\mathbb{P}_{t_k} \prec \dots) \prec \mathbb{P}_{t_2}) \prec \mathbb{P}_{t_1},$$

$\mathbb{P}$



$$:= \langle \mathbb{P}_{\ell_1}, \mathbb{P}_{\ell_2}, \dots, \mathbb{P}_{\ell_k}; \Phi_I(\mathbb{P}_{r_1,\dots,r_d}) \rangle$$

$\mathbb{P}$



$$:= \Phi_I(\mathbb{P}_{r_1,\dots,r_d}).$$

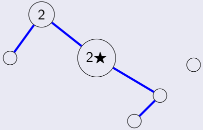
Theorem

$(\mathbb{P}_f)_{f \in \mathfrak{F}_n}$ for **WQSym$_n$**,
$(\mathbb{P}_t)_{t \in \mathfrak{T}_n}$ for Prim$_n$,
$(\mathbb{P}_t)_{t \in \mathfrak{P}_n}$ for TPrim$_n$.

The basis $\mathbb{O}$ indexed by blue biplan forests

Example of expansion from $\mathbb{O}$ to $\mathbb{Q}$

$\mathbb{O}$



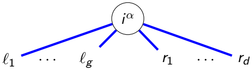
$$= \mathbb{Q}_{531442} + \mathbb{Q}_{521443} + \mathbb{Q}_{512443} - \mathbb{Q}_{534142} - \mathbb{Q}_{524143} - \mathbb{Q}_{514243} - \mathbb{Q}_{514432} - \mathbb{Q}_{524431} - \mathbb{Q}_{514423} + \mathbb{Q}_{541432} + \mathbb{Q}_{542431} + \mathbb{Q}_{541423}$$

Bijection between packed words and blue biplan forests. ❓

$$\mathbb{O}_{\circ} := \mathbb{Q}_1,$$

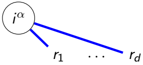
$$\mathbb{O}_{t_1,\ldots,t_k} := (\ldots(\mathbb{O}_{t_k} \prec \ldots) \prec \mathbb{O}_{t_2}) \prec \mathbb{O}_{t_1},$$

$\mathbb{O}$



$$:= \langle \mathbb{O}_{l_1}, \mathbb{O}_{l_2}, \ldots, \mathbb{O}_{l_k}; \Psi_i^\alpha(\mathbb{O}_{r_1,\ldots,r_d})\rangle$$

$\mathbb{O}$



$$:= \Psi_i^\alpha(\mathbb{O}_{r_1,\ldots,r_d}).$$

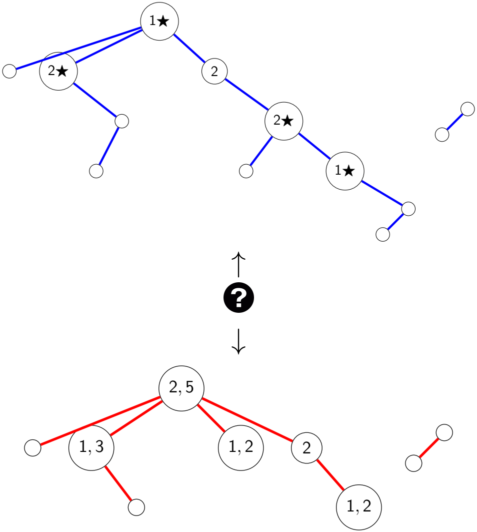
Theorem

$(\mathbb{O}_f)_{f\in\mathfrak{F}^*{}_n}$ for $\mathbf{WQSym}_n^*$

$(\mathbb{O}_t)_{t\in\mathfrak{T}^*{}_n}$ for Prim_n^*,

$(\mathbb{O}_t)_{t\in\mathfrak{P}^*{}_n}$ for TPrim_n^*.

Bijection between red and blue biplan forests

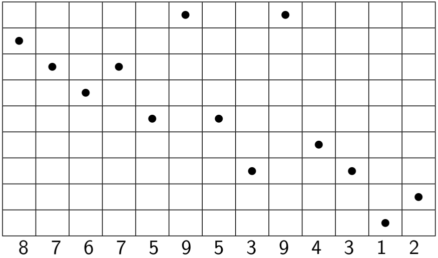


⟵ ? ⟶ 8967647523314

⟵ ? ⟶ 8767595394312

Algorithm on 8767595394312

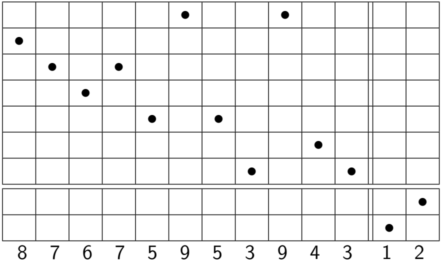
$F(8767595394312)$



Algorithm on 8767595394312

$F(8767595394312)$

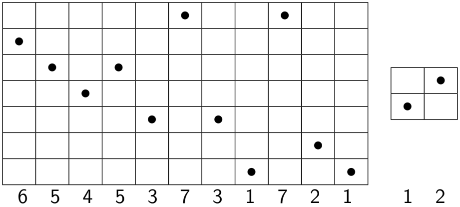
Global descent factorization



Algorithm on 8767595394312

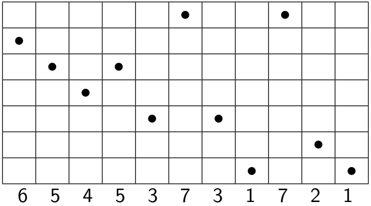
$F(8767595394312) =$
$T(65453731721)\,T(12)$

Global descent factorization + packing



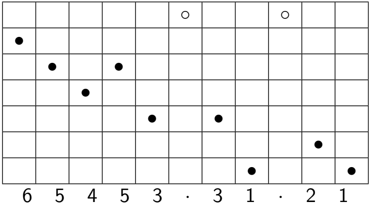
Algorithm on 8767595394312

$T(65453731721)\,T(12)$



$T(65453731721)\,T(12)$

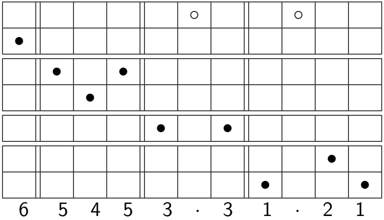
Remove the letters of maximum value



Algorithm on 8767595394312

Global descent factorization

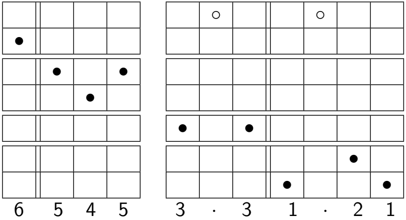
$T(65453731721)\,T(12)$



Algorithm on 8767595394312

Two groups of factors

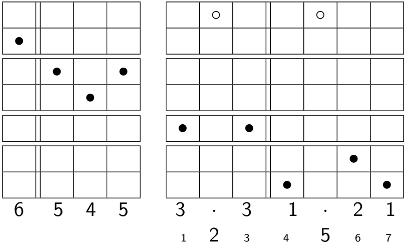
$T(65453731721)\,T(12)$



Algorithm on 8767595394312

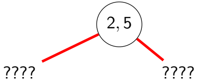
Positions of maximums

$T(65453731721)\,T(12)$

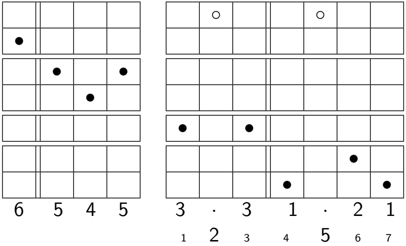


Algorithm on 8767595394312

$T(65453731721)\,T(12) =$

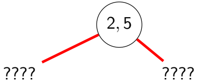


$T(12)$

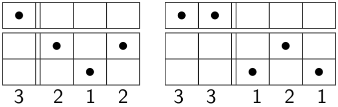


Algorithm on 8767595394312

$T(65453731721)\,T(12) =$



$T(12)$



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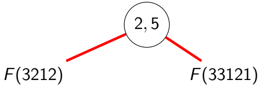
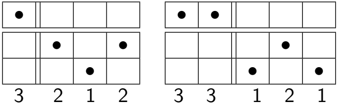


$T(12)$



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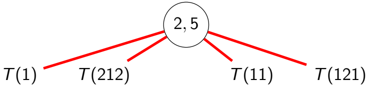
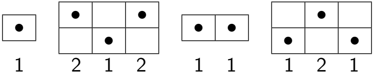
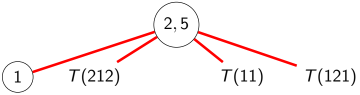




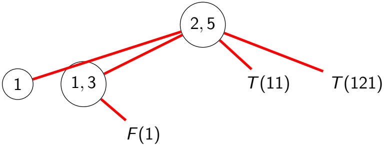
$T(12)$

Algorithm on 8767595394312



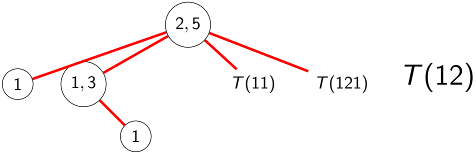
$T(12)$

Algorithm on 8767595394312

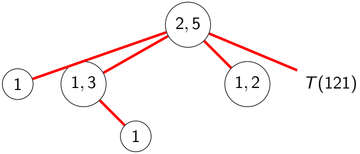


$T(12)$

Algorithm on 8767595394312

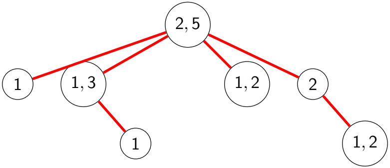


Algorithm on 8767595394312



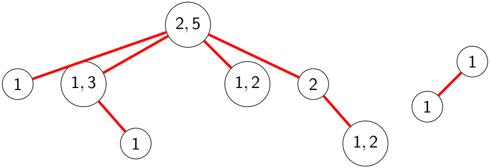
$T(12)$

Algorithm on 8767595394312

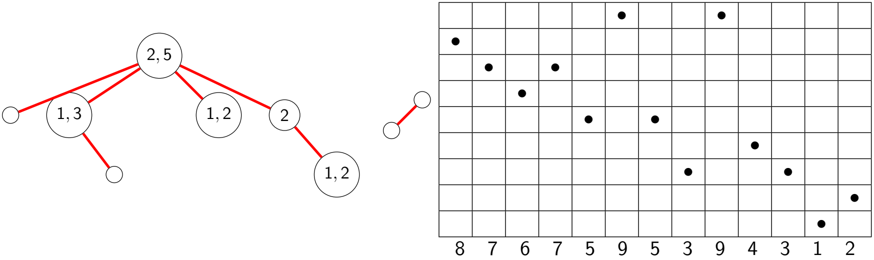


$T(12)$

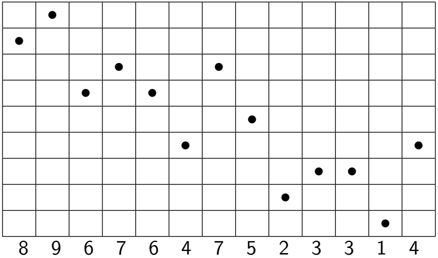
Algorithm on 8767595394312



Algorithm on 8767595394312



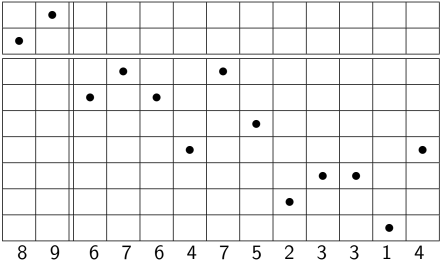
$F^*(8967647523314)$



Algorithm on 8967647523314

$F^*(8967647523314)$

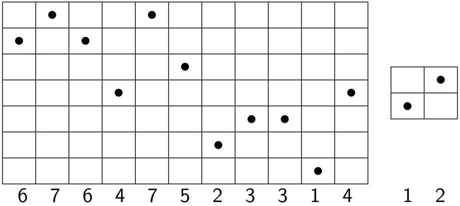
Global descent factorization



Algorithm on 8967647523314

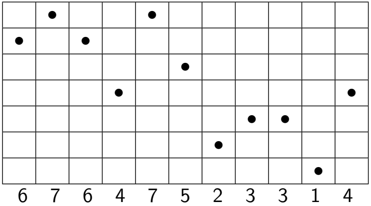
$F^*(8967647523314) = T^*(67647523314)T^*(12)$

Global descent factorization + packing + swaping



Algorithm on 8967647523314

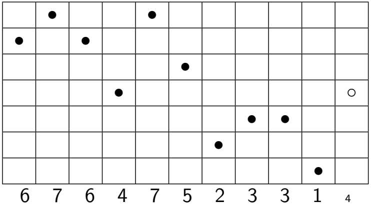
$T^*(67647523314)\,T^*(12)$



Algorithm on 8967647523314

Remove the last letter

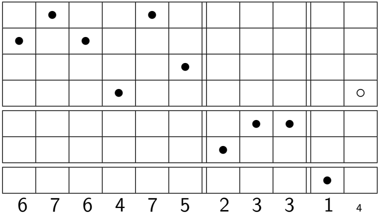
$T^*(67647523314)\,T^*(12)$



Algorithm on 8967647523314

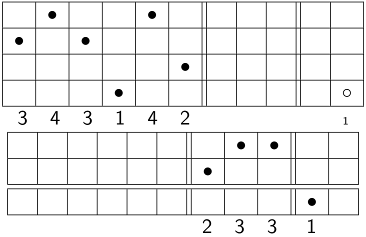
$T^*(67647523314)\,T^*(12)$

Global descent factorization



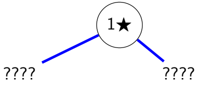
Two groups of factors

$T^*(67647523314)\,T^*(12)$



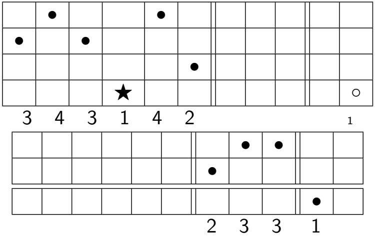
Algorithm on 8967647523314

$$T^*(67647523314)\,T^*(12) =$$

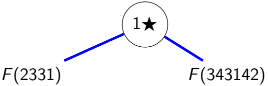


$T^*(12)$

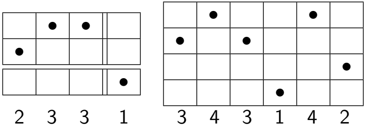
Is the value of last letter in the rest of the word?

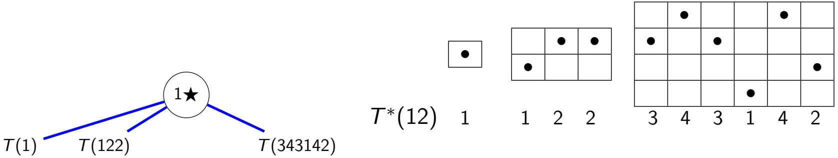


Algorithm on 8967647523314



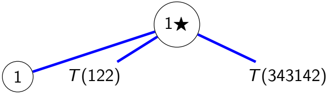
$T^*(12)$



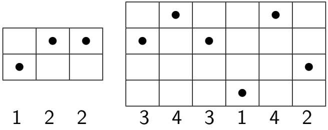


1★
T(1)
T(122)
T(343142)
T*(12)
1
1 2 2
3 4 3 1 4 2

Algorithm on 8967647523314



$T^*(12)$



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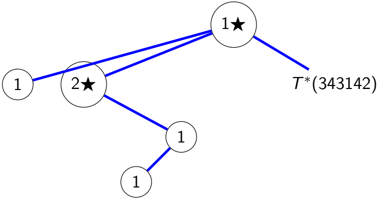
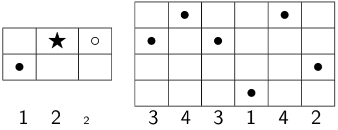
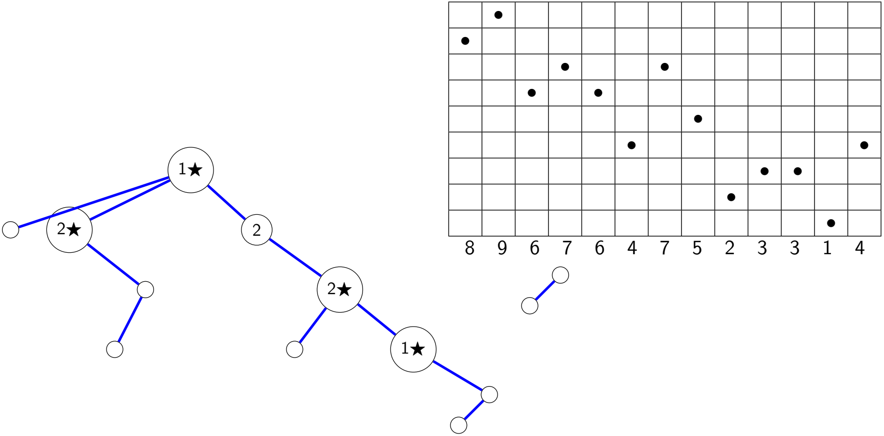


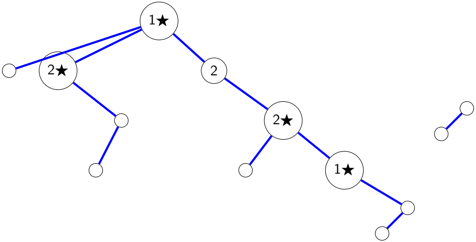


$T^*(12)$

Algorithm on 8967647523314



Bijection between red and blue biplan forests

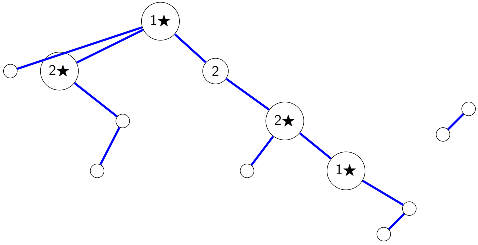


8967647523314

F

$[1-9]^{13}$

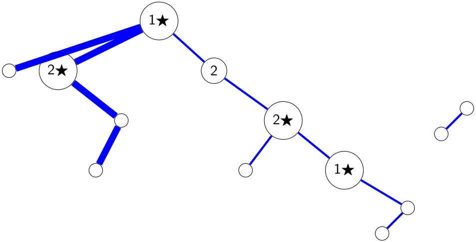
Bijection between red and blue biplan forests



89 67647523314

T_1 T_2 $[3-9]^{11}$ $[\mathbf{1},\mathbf{2}]^{\mathbf{2}}$

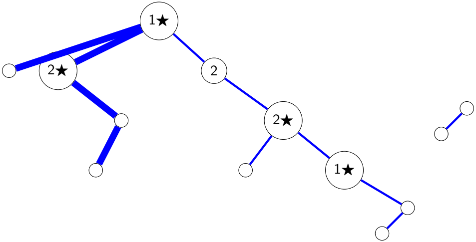
Bijection between red and blue biplan forests



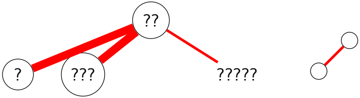
89 676475**2331**4

T_1 T_2 $[3-9]^{11}$ $[1,2]^2$

Bijection between red and blue biplan forests

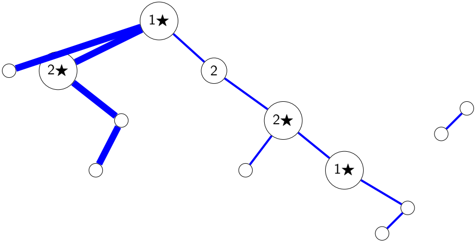


89 676475**2331**4

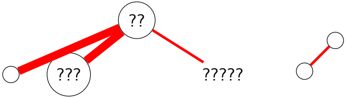


$[\mathbf{6}-\mathbf{8}]^{\mathbf{4}}[3-5,9]^{7}12$

Bijection between red and blue biplan forests

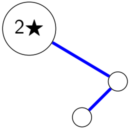


89 676475**233**14



$8[\mathbf{6},\mathbf{7}]^{\mathbf{3}}[3-5,9]^{7}12$

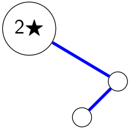
Bijection between red and blue biplan forests



122

$[1,2]^3$

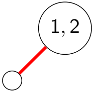
Bijection between red and blue biplan forests



122

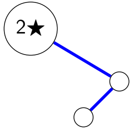
$[1, 2]^3$

$[1, 2]^3$



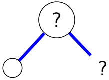
122

Bijection between red and blue biplan forests

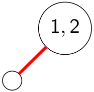


122

$[1, 2]^3$

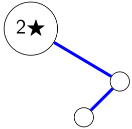


$[1, 2]^3$



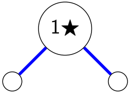
122

Bijection between red and blue biplan forests



122

$[1,2]^3$



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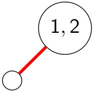

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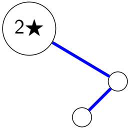
Bijection between red and blue biplan forests



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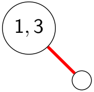

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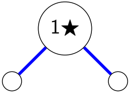

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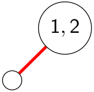

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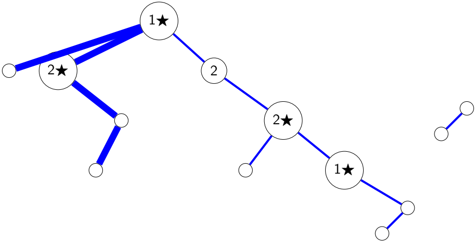
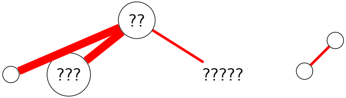
Bijection between red and blue biplan forests

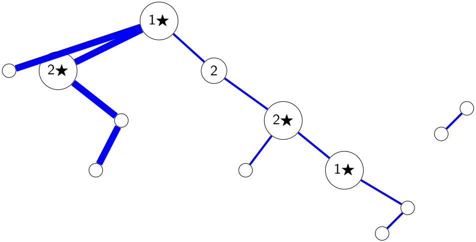


89 676475**233**14

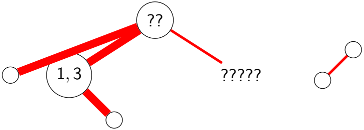


$8[\mathbf{6},\mathbf{7}]^{\mathbf{3}}[3-5,9]^{7}12$

Bijection between red and blue biplan forests

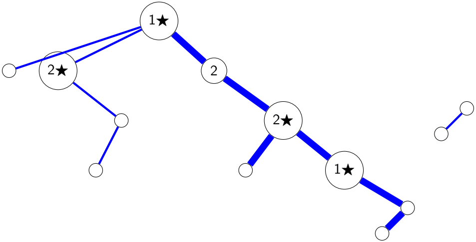


89 676475**233**14

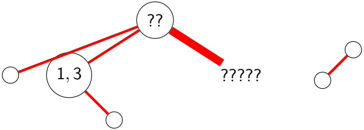


$8\mathbf{767}[3-5,9]^{7}12$

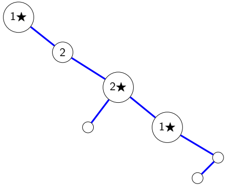
Bijection between red and blue biplan forests



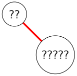
89 **676475**2331**4**



$8767[\mathbf{3}-\mathbf{5},\mathbf{9}]^{\mathbf{7}}12$

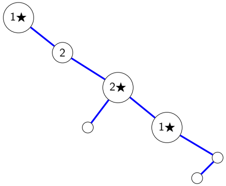


3431421= *pack*(6764754)

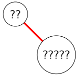


$[1-4]^7 = pack([3-5,9]^7)$

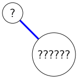
Bijection between red and blue biplan forests



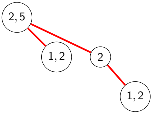
3431421



$[1-4]^7$

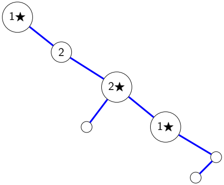


$[1-4]^7$

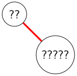


3431421

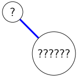
Bijection between red and blue biplan forests



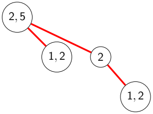
$$3431421 = \textit{pack}(6764754)$$



$$[1-4]^7 = \textit{pack}([3-5,9]^7)$$

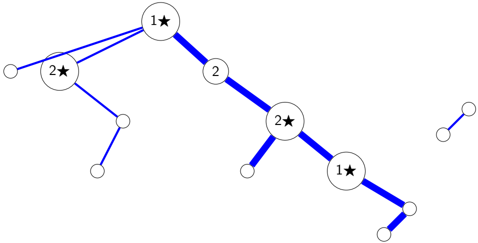


$$[1-4]^7 = \textit{pack}([4-7]^7)$$

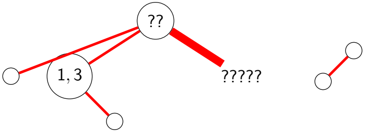


$$3431421 = \textit{pack}(5953943)$$

Bijection between red and blue biplan forests

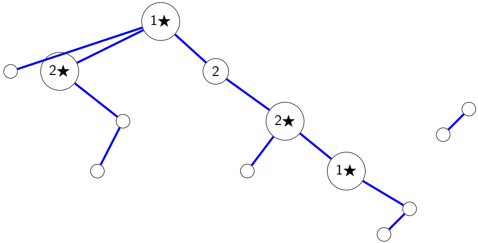


89 **676475**2331**4**

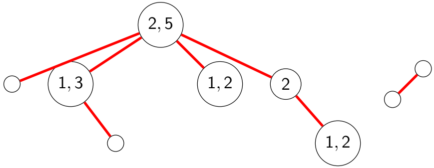


$8767[\mathbf{3}-\mathbf{5},\mathbf{9}]^{\mathbf{7}}12$

Bijection between red and blue biplan forests



89**67647**2331**4**



8767**5953943**12