Advanced Machine Learning

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Lab 2: Hidden Markov Models

The purpose of the lab is to put in practice some of the concepts covered in the lectures. To do so, you are asked to model the behavior of a robot that walks around a ring. The ring is divided into 10 sectors. At any given time point, the robot is in one of the sectors and decides with equal probability to stay in that sector or move to the next sector. You do not have direct observation of the robot. However, the robot is equipped with a tracking device that you can access. The device is not very accurate though: If the robot is in the sector i, then the device will report that the robot is in the sectors [i - 2, i + 2] with equal probability.

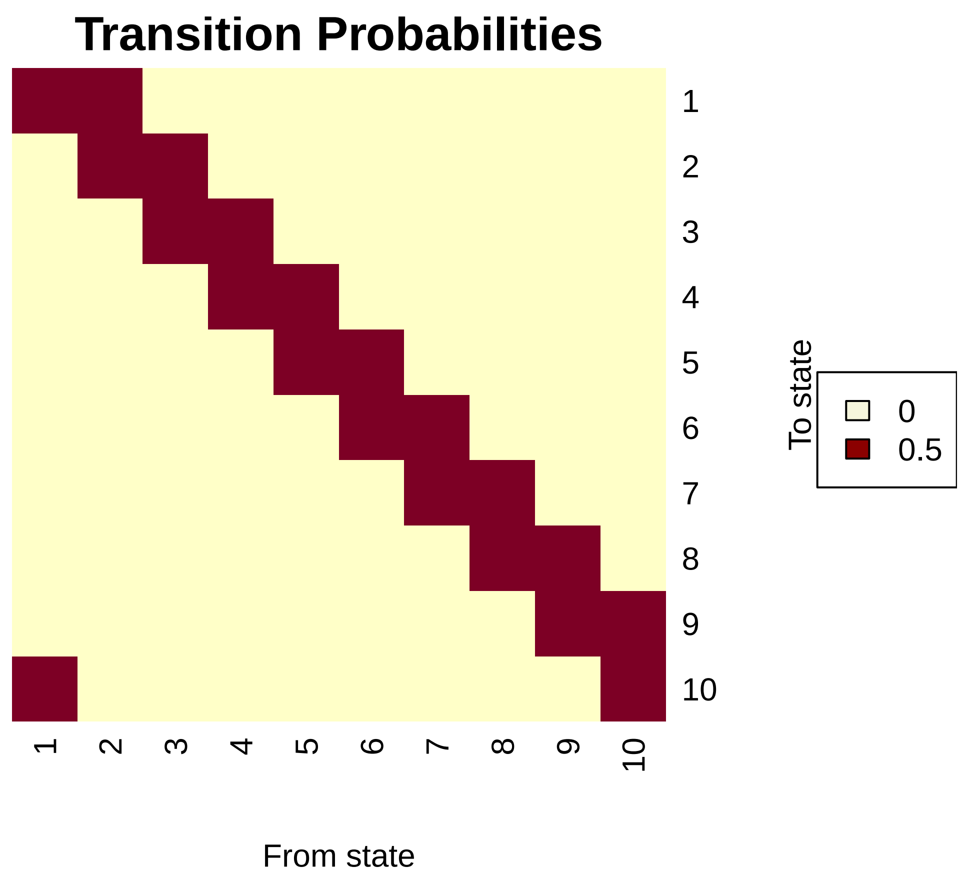
Question 1

Build a hidden Markov model (HMM) for the scenario described above.

```r
library(HMM)

set.seed(123)

n_states = 10
n_symbols = 10
# Transition probability : stay at current state 0.5 or move to the next
transProbs = matrix(0, n_states, n_states, )

for(i in 1:n_states){
  transProbs[i,i] = 0.5 #Stay
  transProbs[i,(i+n_symbols)%%(n_states)+1] = 0.5 # Move to the next state
  #All the weird indexing is to go around the issues of the 1-index in R
}
print("Transition Probabilities Matrix:")
```

```
## [1] "Transition Probabilities Matrix:"
```

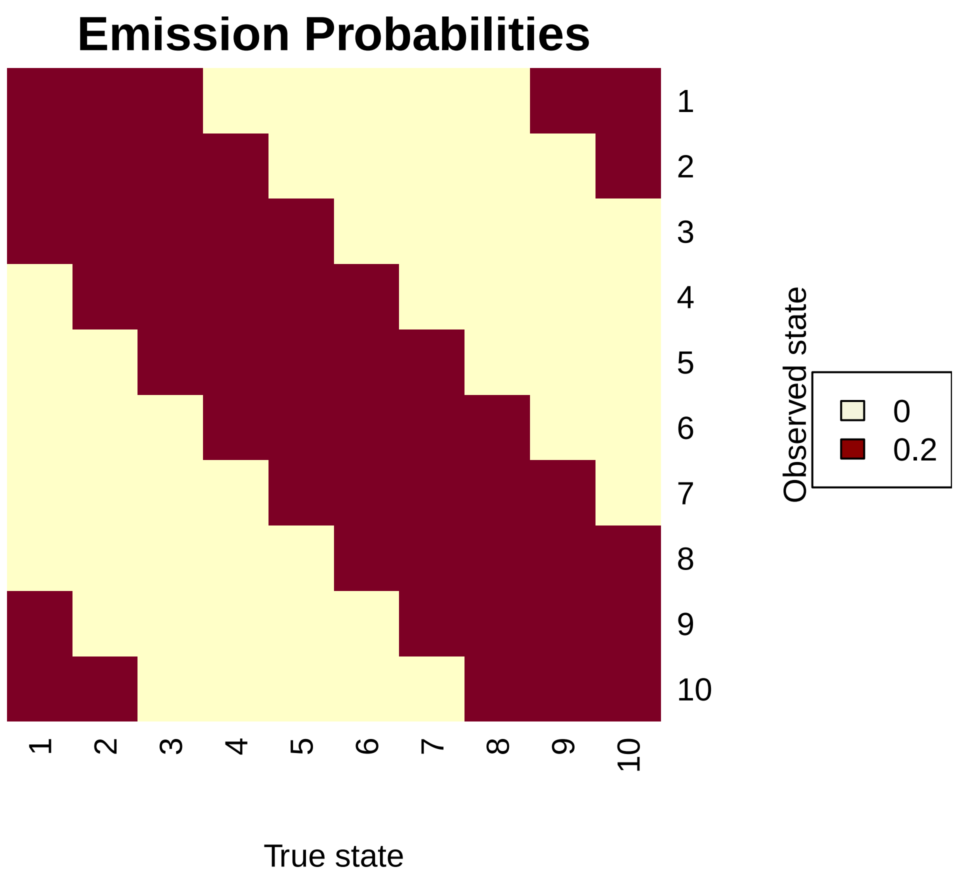
```r
heatmap(transProbs, Rowv = NA, Colv = NA, revC = T,
        main = "Transition Probabilities", xlab = "From state", ylab="To state")
legend(x="right", legend=c("0", "0.5"),fill=c("beige", "darkred"))
```



```r
# Emission probability : the sensor inaccuracy ?
emissionProbs = matrix(0, n_states, n_symbols)

for(i in 1:n_states){
  for(j in -2:2){
    emissionProbs[i, (i+j+n_symbols-1)%%(n_symbols)+1] = 0.2
    #All the weird indexing is to go around thes issues of the 1-index in R
  }
}
print("Emission Probabilities Matrix:")
```

```
## [1] "Emission Probabilities Matrix:"
```

```r
heatmap(t(emissionProbs), Rowv = NA, Colv = NA, revC = T,main = "Emission Probabilities", xlab = "True
legend(x="right", legend=c("0", "0.2"),fill=c("beige", "darkred"))
```



```
myHMM = initHMM(States = 1:10, Symbols = 1:10,
                startProbs = NULL ,
                transProbs = transProbs,
                emissionProbs = emissionProbs)
```

Question 2:

Simulate the HMM for 100 time steps.

```
sim = simHMM(myHMM, 100)
```

Question 3:

Discard the hidden states from the sample obtained above. Use the remaining observations to compute the filtered and smoothed probability distributions for each of the 100 time points. Compute also the most probable path.

```
#Discarding the hidden states
obs = sim$observation

get_probs = function(myHMM, obs){
  # This function takes in a HMM and observations
  # And returns the  distribution, the smoothed distribution, and the most probable path
```

```r
  log_forwards = forward(hmm = myHMM, observation = obs)
  forwards = exp(log_forwards) # alpha(z_t)
  # Normalize on the marginal z_t level (margin = 2)
  filtered = prop.table(forwards, margin = 2)

  log_backwards = backward(myHMM, obs)
  backwards = exp(log_backwards) # beta(z_t)

  log_alpha_beta = log_forwards+log_backwards
  alpha_beta = exp(log_alpha_beta)

  # Normalize on the marginal z_t level (margin = 2)
  smoothed = prop.table(alpha_beta)
  #sum(smoothed) # =1

  most_probable_path = viterbi(myHMM, obs)

  return(list("filtered" = filtered,
              "smoothed" = smoothed,
              "most_probable_path" = most_probable_path))
}

res = get_probs(myHMM, obs)
```

Question 4:

Compute the accuracy of the filtered and smoothed probability distributions, and of the most probable path. That is, compute the percentage of the true hidden states that are guessed by each method.

Hint: Note that the function forward in the HMM package returns probabilities in log scale. You may need to use the functions exp and prop.table in order to obtain a normalized probability distribution. You may also want to use the functions apply and which.max to find out the most probable states. Finally, recall that you can compare two vectors A and B elementwise as A==B, and that the function table will count the number of times that the different elements in a vector occur in the vector.

```r
get_accuracies = function(sim, get_probs_result){
  # Given a simulation run (sim) and the output from the get_probs() function
  # Returns the accuracy of the filtered dist., smoothed dist., and most probable path.

  #Filtered distribution accuracy
  filtered_count = table(sim$states == apply(get_probs_result$filtered, 2, which.max))
  filtering_accuracy = filtered_count[2]/sum(filtered_count)
  names(filtering_accuracy) = "Filtering accuracy"

  #Smoothed distribution accuracy
  smoothed_count = table(sim$states == apply(get_probs_result$smoothed, 2, which.max))
  smoothing_accuracy = smoothed_count[2]/sum(smoothed_count)
  names(smoothing_accuracy) = "Smoothing accuracy"

  #Most probable path accuracy
  viterbi_count = table(sim$states == get_probs_result$most_probable_path)
  viterbi_accuracy = viterbi_count[2]/sum(viterbi_count)
  names(viterbi_accuracy) = "Viterbi accuracy"

  return(list("filtering_accuracy" = filtering_accuracy,
              "smoothing_accuracy"= smoothing_accuracy,
              "viterbi_accuracy"= viterbi_accuracy))
}

acc = get_accuracies(sim, res)
print(acc)
```

```
## $filtering_accuracy
## Filtering accuracy
##               0.53
##
## $smoothing_accuracy
## Smoothing accuracy
##               0.64
##
## $viterbi_accuracy
## Viterbi accuracy
##             0.36
```

Question 5:

Repeat the previous exercise with different simulated samples. In general, the smoothed distributions should be more accurate than the filtered distributions. Why ? In general, the smoothed distributions should be more accurate than the most probable paths, too. Why ?

```r
sim2 = simHMM(myHMM, 100)

res2 = get_probs(myHMM, sim2$observation)

acc2 = get_accuracies(sim2, res2)
print(acc2)
```

```
## $filtering_accuracy
## Filtering accuracy
##               0.59
##
## $smoothing_accuracy
## Smoothing accuracy
##               0.71
##
## $viterbi_accuracy
## Viterbi accuracy
##             0.48
```

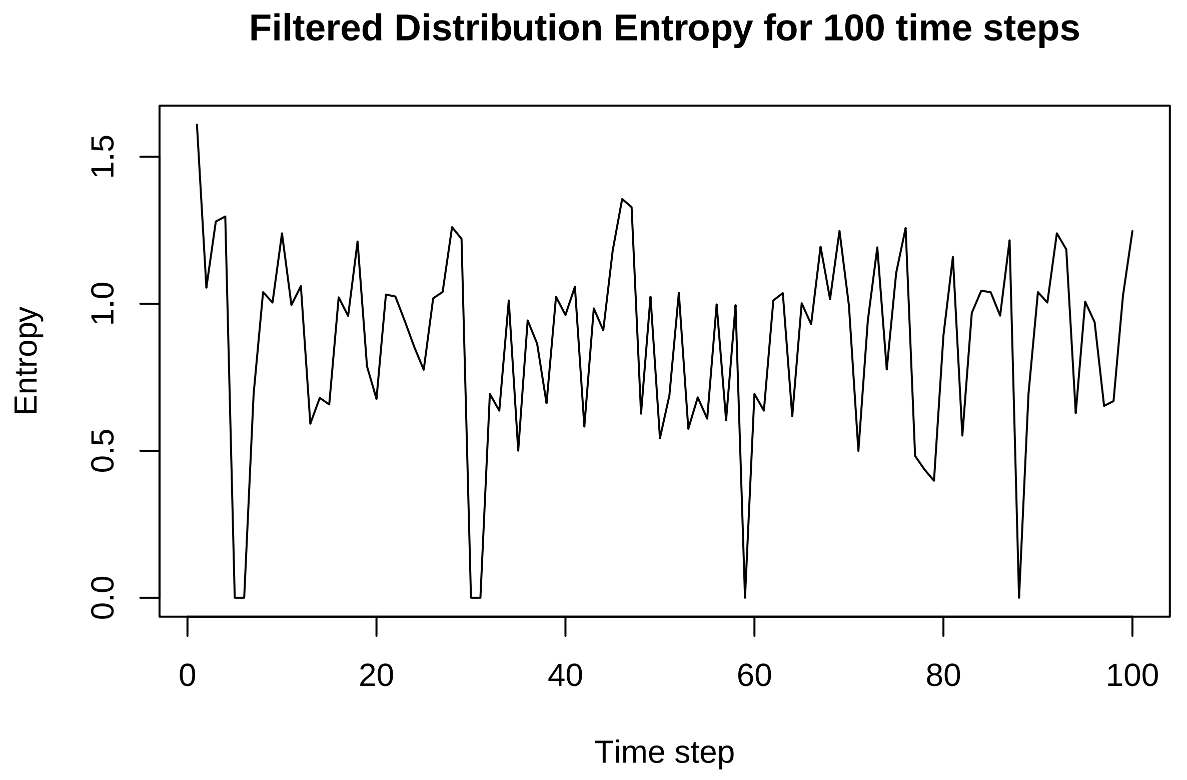
The smoothed distribution should be more accurate than the filtering because it combines information from both the forward and backward probabilities, while the filtering only has the forward probabilities.

The smoothed distribution should also be more accurate than the most probable path because the most probable path has an extra constraint that it has to output a possible path, thus, the smoothed distribution has more flexibility to make predictions, even if the overall path might have inconsistencies.

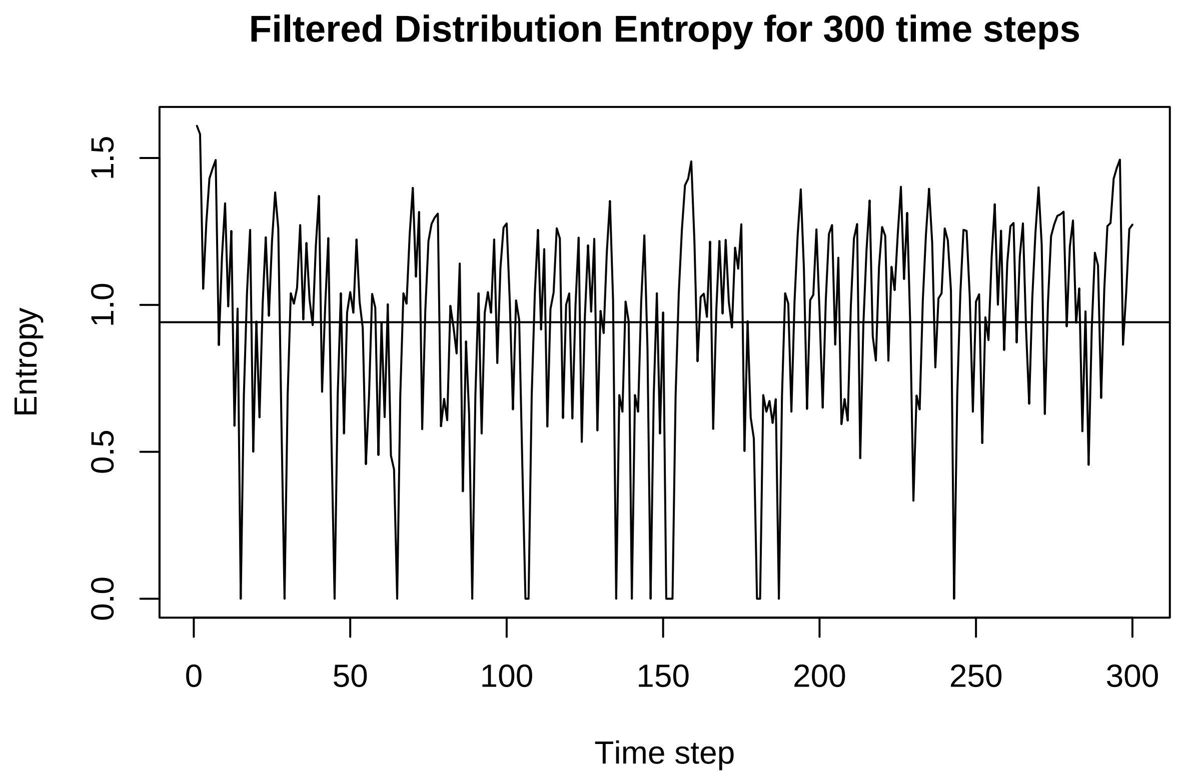
Question 6:

Is it always true that the later in time (i.e., the more observations you have received) the better you know where the robot is ? Hint: You may want to compute the entropy of the filtered distributions with the function entropy.empirical of the package entropy.

```r
library(entropy)

filt_entropies = apply(res$filtered, 2, entropy.empirical)
plot(filt_entropies, type='l', main = "Filtered Distribution Entropy for 100 time steps",
     xlab = "Time step", ylab="Entropy")
```



```r
# Simulating for 300 time steps:
sim300 = simHMM(myHMM, 300)
res300 = get_probs(myHMM, sim300$observation)
filt300_entropies = apply(res300$filtered, 2, entropy.empirical)
mean300 = mean(filt300_entropies)
plot(filt300_entropies, type="l", , main = "Filtered Distribution Entropy for 300 time steps",
     xlab = "Time step", ylab="Entropy")
abline(a=mean300, b = 0)
```



It does not seems like the entropy reduces the later in time, meaning that we do not know better where the robot is over time. This is probably because the emission probabilities stay constant in time.

Question 7:

Consider any of the samples above of length 100. Compute the probabilities of the hidden states for the time step 101.

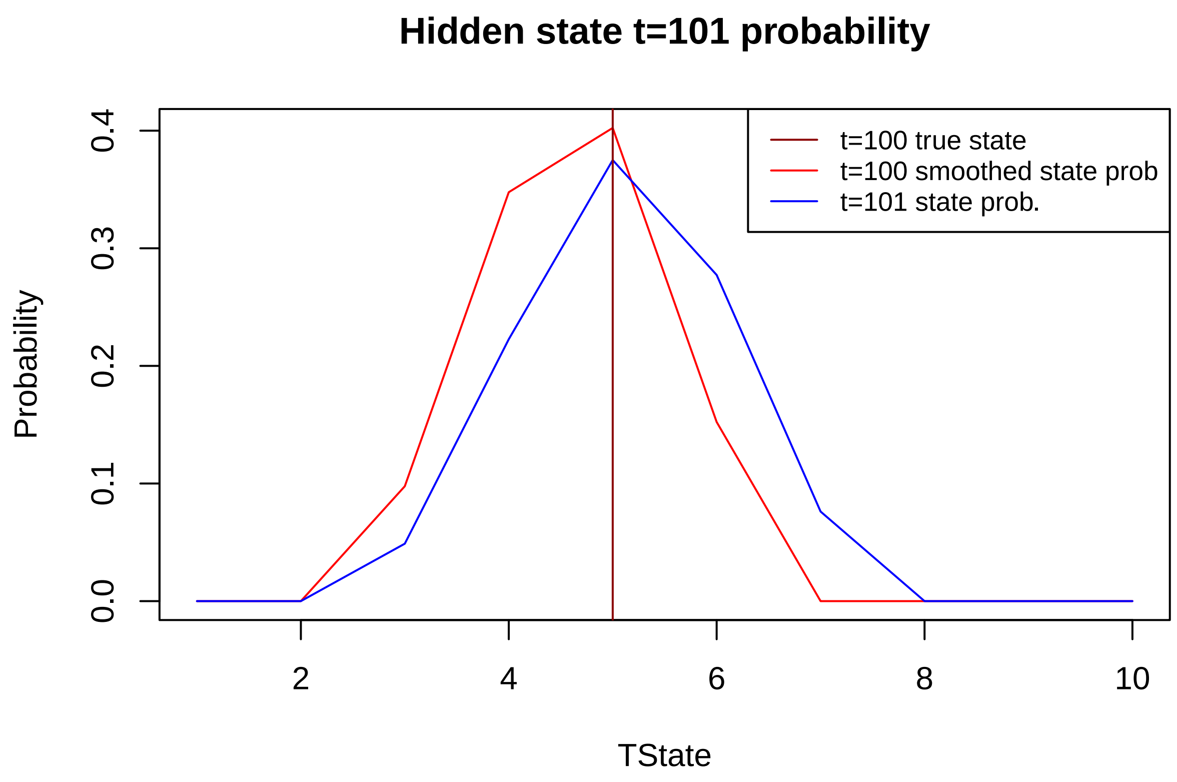
```r
#Prediction of state 101 = smoothed state 100 * transition probability
pred101 = prop.table(res$smoothed[,100]%*%transProbs)
print("state prediction probabilities:")
```

```
## [1] "state prediction probabilities:"
```

```r
print(pred101)
```

```
##      [,1] [,2]       [,3]      [,4]  [,5]      [,6]       [,7] [,8] [,9] [,10]
## [1,]    0    0 0.04883721 0.2226744 0.375 0.2773256 0.07616279    0    0     0
```

```r
plot(x = c(1:10),prop.table(res$smoothed[,100]), type = "l", col = "red",
     , main = "Hidden state t=101 probability", xlab = "TState", ylab="Probability")
lines(x = c(1:10),pred101, type = "l", col = "blue")
abline(v = sim$states[100], col="darkred")
legend("topright", legend=c("t=100 true state",
                            "t=100 smoothed state prob",
                            "t=101 state prob."),
       col=c("darkred", "red", "blue"), lty=c("solid","solid","solid"), cex=0.8)
```


Hidden state t=101 probability
Probability
TState
t=100 true state
t=100 smoothed state prob
t=101 state prob.