

ID m° 1.

Ex 1. Rocq

Ex 2. Rocq

Ex 3. IF  $\Gamma \vdash A$  is provable in NJ, take some proof tree in NJ for it, it is a valid proof tree in NJ + R, thus  $\Gamma \vdash A$  is provable in NJ + R.

Conversely, take a proof tree  $\pi$  of  $\Gamma \vdash A$  in NJ + R, by admissibility of R, every occurrence of R in  $\pi$  can be replaced by part of a tree that doesn't use R. Doing this starting from the leaves yield a valid proof tree for  $\Gamma \vdash A$  in NJ.

Ex 4.

1. Take a proof tree of  $\Gamma, A, B \vdash C$ . Replace each occurrence of

$$\frac{}{\Gamma, A, B, \Delta \vdash A} \text{Ax}$$

$$\frac{\frac{}{\Gamma, A \wedge B, \Delta \vdash A \wedge B} \text{Ax}}{\Gamma, A \wedge B, \Delta \vdash A} \wedge \text{eg}$$

and

$$\frac{}{\Gamma, A, B, \Delta \vdash B} \text{Ax}$$

with

$$\frac{\frac{}{\tilde{\Delta} \vdash A \wedge B} \text{Ax}}{\tilde{\Delta} \vdash B} \wedge \text{ed},$$

also replacing every sequent  $\Gamma, A, B, \Delta \vdash D$  with  $\Gamma, A \wedge B, \Delta \vdash D$ .

2. By induction on the size of  $\Gamma$ , we apply Q1.

3. No, derivable rules cannot remove hypotheses (with the rules in NJ).

Ex 5.

$$\begin{array}{l}
 1. \quad \frac{\Xi, \Gamma \vdash A}{\Xi, \Delta \vdash B} \text{ admissible iff } (\Xi, \Gamma \vdash A \text{ implies } \Xi, \Delta \vdash B) \\
 \text{iff } (\Xi \vdash \wedge \Gamma \Rightarrow A \text{ implies } \Xi \vdash \wedge \Delta \Rightarrow B) \quad \text{Ex h.} \\
 \text{iff } \Xi \vdash (\wedge \Gamma \Rightarrow A) \Rightarrow (\wedge \Delta \Rightarrow B) \\
 \text{iff } \Xi \vdash (\Delta \Gamma \Rightarrow A \wedge \wedge \Delta \Rightarrow B) \quad \text{currying}
 \end{array}$$

2. Similarly to Q1, we know

$$\Xi, \Gamma \vdash (\wedge \Delta \Rightarrow A) \Rightarrow B$$

is provable, thus, by repeated currying, writing  $\Delta = F_1, \dots, F_n$ , the sequent

$$\Xi, \Gamma \vdash (F_1 \Rightarrow \dots \Rightarrow F_n \Rightarrow A) \Rightarrow B$$

is provable; take  $\pi$  a proof tree for it.

$$\begin{array}{c}
 \frac{\frac{\pi}{\Xi, \Gamma \vdash \Phi \Rightarrow B}}{\Xi, \Gamma \vdash B} \quad \frac{\frac{\text{assumption}}{\Xi, \Gamma, \Delta \vdash A}}{\Xi, \Gamma \vdash F_1 \Rightarrow \dots \Rightarrow F_n \Rightarrow A} \xrightarrow{\text{repeated } \Rightarrow_I} \xrightarrow{\Rightarrow_E}
 \end{array}$$

$$\frac{\pi_2}{\Gamma \vdash A}$$

$$\frac{\pi_3}{\Delta \vdash B}$$

Ex 6.

1.

$$\frac{\frac{\Gamma \vdash A}{\Gamma, \Delta \vdash A} \text{ wk} \quad \frac{\Delta \vdash B}{\Gamma, \Delta \vdash B} \text{ wk}}{\Gamma, \Delta \vdash A \wedge B} \text{ \wedge I}$$

2.

$$\frac{\frac{\Gamma \vdash \neg A}{\Gamma, \Delta \vdash \neg A} \text{ wk} \quad \frac{\Delta \vdash A}{\Gamma, \Delta \vdash A} \text{ wk}}{\Gamma, \Delta \vdash \perp} \neg E$$

$$\frac{\frac{\Gamma \vdash A \vee B}{\Gamma, \Delta, \Sigma \vdash A \vee B} \text{ wk} \quad \frac{\frac{\Delta \vdash A \vdash C}{\Gamma, \Delta, \Sigma, A \vdash C} \text{ wk} \quad \frac{\Sigma \vdash B \vdash C}{\Gamma, \Delta, \Sigma, B \vdash C} \text{ wk}}{\Gamma, \Delta, \Sigma \vdash C} \vee E$$

$$\frac{\frac{\Gamma \vdash A \Rightarrow B}{\Gamma, \Sigma \vdash A \Rightarrow B} \text{wb}}{\Gamma, \Sigma \vdash B} \quad \frac{\frac{\Delta \vdash B}{\Gamma, \Delta \vdash B} \text{wR}}{\Rightarrow E.}$$

Ex 7.

Define  $C_T := \perp \Rightarrow \perp$  and  $C_{\neg A} := A \Rightarrow \perp$ .

$$\frac{\frac{\frac{}{\Gamma, \perp \vdash \perp} \text{Ax}}{\Gamma \vdash C_T} \Rightarrow I}{\Gamma \vdash C_T} \text{ "TI"}$$

$$\frac{\Gamma \vdash C_{\neg A} \quad \Gamma \vdash A}{\Gamma \vdash \perp} \Rightarrow E. \quad \neg E.$$

$$\frac{\Gamma, A \vdash \perp}{\Gamma \vdash C_{\neg A}} \Rightarrow I \quad \neg I$$

Ex 8. Rocq

Ex 9. Rocq

$\neg_R A$  is equivalent to  $\neg A$  iff  $\vdash \neg R$  is provable.