Algebraic and combinatorial aspects of category theory

Lecture #1. 18/11/25

Algebraic structure $\overset{\text{monad}}{T}(X) \to X$

intuition: $T(X)$ contains the derived operations

Example for a binary operation on X, we can consider

$$\{\text{binary trees with leaves in } X\} \to X.$$

Being created

$$\begin{tikzcd}
T(X) \arrow[d] & T(X \times Y) \arrow[l, "T\pi_1"'] \arrow[r, "T\pi_2"] \arrow[d] & T(Y) \arrow[d] & T\text{-}\mathbf{Alg} \arrow[d, "\text{forgetful functor}"] \\
X & X \times Y \arrow[l] \arrow[r] & Y & \mathbf{Set}
\end{tikzcd}$$

Category: objects, morphisms, composition (unitary and associative).

Notations $A \in \mathcal{C}$ iff A is an object in $\mathcal{C}$

$\mathcal{C}(A,B)$ is the set of morphisms $A \to B$.

Examples

	objects	morphisms	
a set X	$x \in X$	id_x	
a poset $(X, \leq)$	$x \in X$	$x \leq y$	(also works for preordered sets)
$\mathbf{Set}$	set X	maps	
$\mathbf{Mon}$	monoid X	homomorphisms	
	$\vdots$		

Given a graph G, then G^* is the category where objects are nodes in G and morphisms $A \to B$ the set of paths from A to B; composition is path concatenation.

> Example: BM. one object $*$ and $\mathcal{C}(*, *) = M$ with $h \circ g = h \cdot g$.

Example: objects are trees and morphisms are $T_1 \xrightarrow{T} T_2$ when you obtain T_2 by replacing T_1's leaves with copies of T.

<u>A note on size</u> "small" $\begin{cases} \text{objects form a set} \\ \text{morphisms form a set (from A to B fixed)} \end{cases}$

"locally small" $\begin{cases} \text{objects form a ~~set~~ class} \\ \text{morphisms form a set (from A to B fixed)} \end{cases}$

Functor $F : \mathcal{C} \to \mathcal{D}$ that commutes with composition and identity

Notation $F_{A,B} : \mathcal{C}(A, B) \longrightarrow \mathcal{D}(F(A), F(B))$

The axioms of functors can be seen as the fact that the two following diagrams commute.

$$\begin{tikzcd}
\mathcal{C}(B,C) \times \mathcal{C}(A,C) \arrow[r, "- \circ_{\mathcal{C}} -"] \arrow[d, "F_{B,C} \times F_{A,B}"'] & \mathcal{C}(A,C) \arrow[d, "F_{A,C}"] \\
\mathcal{D}(FB, FC) \times \mathcal{D}(FA, FB) \arrow[r, "- \circ_{\mathcal{D}} -"'] & \mathcal{D}(FA, FC)
\end{tikzcd}$$

$\iff Ff \circ Fg = F(f \circ g)$

$$F(id_A) = id_{FA} \iff \begin{tikzcd}
 & \mathcal{C}(A,A) \arrow[d, "F_{A,A}"] \\
1 \arrow[ur, "id_A"] \arrow[r, "id_{F(A)}"'] & \mathcal{D}(FA, FA)
\end{tikzcd}$$

Examples

- A monotone map is a functor between two posets seen as categories
- Forgetful functor $\mathbf{Mon} \longrightarrow \mathbf{Set}$, $\mathbf{Ring} \longrightarrow \mathbf{Mon}$
  $(M, \cdot, 1_M) \mapsto M$

Example A functor $F : BM \longrightarrow BN$ is exactly a monoid homomorphism $f : M \longrightarrow N$.

Example The "bang" functor $\mathcal{C} \xrightarrow{!} \mathbb{1}$

category with one object and only the id morphism.

Functors compose !!

$\mathbf{Cat}$ is the category of small categories. It forms a locally small category

$\mathbf{CAT}$ is the category of locally small categories. It forms a very large category.

$\mathbf{Set}, \mathbf{Grp}, \mathbf{Gph}, \mathbf{Cat} \in \mathbf{CAT}$.

A terminal object $T \in \mathcal{C}$ is defined such that for every object $A \in \mathcal{C}$, there is a unique morphism $A \longrightarrow T$

An initial object $\perp \in \mathcal{C}$ is defined such that for every object $A \in \mathcal{C}$, there is a unique morphism $\perp \longrightarrow A$.

In $\mathbf{Mon}$, $\mathbf{Grp}$ and $\mathbf{Field}$, $\mathbb{1}$ is both terminal and initial.

In $\mathbf{Set}$, $\emptyset$ is initial ($\emptyset \xrightarrow{\exists!} A$) and $\mathbb{1}$ is terminal ($A \xrightarrow{\exists!} \mathbb{1}$).

A product of A and $B \in \mathscr{C}$ is an object P with $\overbrace{\begin{array}{l} \pi_1 : P \to A \\ \pi_2 : P \to B \end{array}}^{\text{implicite}}$

such that for every $A \leftarrow Q \to B$, there exists a unique $\langle f, g\rangle : Q \to P$

such that

$$\begin{tikzcd}
& Q \arrow[dl, bend right, "f"'] \arrow[d, "\langle f,g\rangle"] \arrow[dr, bend left, "g"] & \\
A & P \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & B
\end{tikzcd}$$

commutes.

In Set, take $\langle f, g\rangle : Q \longrightarrow A \times B$

$q \longmapsto (f(q), g(q))$.

> **Exercise** $B \times A \cong A \times B$ and the unique morphism is
>
> $\langle \pi_2, \pi_1 \rangle : (a,b) \mapsto (b,a)$.

Exercise: If P and Q are both products of A and B then there exists a unique isomorphism from P to Q.

$$\begin{tikzcd}
& P \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[dd, bend left, "\exists! g"] & \\
A & & B \\
& Q \arrow[ul, "\pi_1"] \arrow[ur, "\pi_2"'] \arrow[uu, bend left, "\exists! f"] &
\end{tikzcd}$$

Then

$$\begin{tikzcd}
& P \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[dd, bend left, "id_P"] & \\
A & & B \\
& P \arrow[ul, "\pi_1"] \arrow[ur, "\pi_2"'] \arrow[uu, bend left, "f \circ g"] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
& Q \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[dd, bend left, "id_Q"] & \\
A & & B \\
& Q \arrow[ul, "\pi_1"] \arrow[ur, "\pi_2"'] \arrow[uu, bend left, "g \circ f"] &
\end{tikzcd}$$

imply $f \circ g = id_P$ and $g \circ f = id_Q$.

$$
\begin{tikzcd}
 & P \arrow[dl] \arrow[dr] & \\
A & Q \arrow[l] \arrow[r] \arrow[u, "g"] & B \\
 & P \arrow[ul] \arrow[ur] \arrow[u, "f"] \arrow[uu, bend right=90, "id_P"'] &
\end{tikzcd}
$$

// break //

In a poset seen as a category, the (binary) product is the (binary) meet.

Monads

$T(X)$ = terms with free variables in X, up to some equations

Free monoid monad: define the free monoid of $X \in Set$ by induction:

$$
\frac{}{X \vdash x}\,(x \in X) \qquad \frac{}{X \vdash 1} \qquad \frac{X \vdash M \qquad X \vdash N}{X \vdash M \cdot N}
$$

with the equations

$$
\frac{}{(M \cdot N) \cdot P \sim M \cdot (N \cdot P)} \qquad \frac{}{M \sim M \cdot 1} \qquad \frac{}{M \sim 1 \cdot M} \qquad \frac{M \sim M' \qquad N \sim N'}{M \cdot N \sim M' \cdot N'}
$$

Then $T(X) = \{X \vdash M\} / (\sim \cup \sim^{t})^{*}$ ⟶ reflexive & transitive

⟶ symmetric

Equivalently, $T(X) = X^{*} = \sum_{n \in \mathbb{N}} X^{n}$.

Applying $f : X \to Y$ induces a map

$$Tf : X^* = TX \longrightarrow Y^* = TY$$
$$(x_1, \dots, x_n) \longmapsto (f(x_1), \dots, f(x_n)).$$

Also, $Tf \circ Tg = T(f \circ g)$ and $T\,id_X = id_{TX} = id_{X^*}$

Basic idea of a monad: a set X with a map $a : TX \to X$.

There is an inclusion $X \to T(X)$:

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, equal] & TX \arrow[d, "a"] \\ & X \end{tikzcd}$$

& naturality of η and μ

and compositionnality

$$\begin{tikzcd} TTX \arrow[rrr, "\mu"] \arrow[ddddd, "Ta"'] & & & TX \arrow[ddddd, "a"] \\ & ((x_1^1, \dots, x_{n_1}^1), \dots, (x_1^k, \dots, x_{n_k}^k)) \arrow[r, mapsto] \arrow[ddd, mapsto] & (x_1^1, \dots, x_{n_k}^k) \arrow[d, mapsto] & \\ & & a(x_1, \dots, x_1^k, \dots, x_{n_k}^k) \arrow[d, equal] & \\ & & & \\ & (a(x_1^1, \dots), \dots, a(x_1^k, \dots)) \arrow[r] & a(a(x_1, \dots), \dots, a(x_1^k, \dots)) & \\ TX \arrow[rrr, "a"'] & & & X \end{tikzcd}$$

A T-algebra is a map $a : T(X) \to X$ such that

$$\begin{tikzcd} TTX \arrow[r, "\mu"] \arrow[d, "Ta"'] & TX \arrow[d, "a"] \\ TX \arrow[r, "a"'] & X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, equal] & TX \arrow[d, "a"] \\ & X \end{tikzcd}$$

There is thus a category T-Alg: objects are T-algebras

$f : X \to Y$ induces
$$\begin{tikzcd} TX \arrow[r, "Tf"] \arrow[d, "a_X"'] & TY \arrow[d, "a_Y"] \\ X \arrow[r] & Y \end{tikzcd}$$

When T is the free monoid functor, then $\mathrm{Mon} \cong T\text{-Alg}$.

$$\begin{tikzcd} \mathrm{Mon} \arrow[rr, shift left, "L"] \arrow[dr, "U"'] & & T\text{-Alg} \arrow[ll, shift left, "R"] \arrow[dl, "V"] \\ & \mathrm{Set} & \end{tikzcd}$$

For R, take $1 := a()$
take $x \cdot y := a(x,y)$

We have
$$a(x, a()) = a(a(x), a()) \overset{(\mu)}{=} a(x) = x$$

For L, define $a(x_1, \dots, x_n) := ((x_1 \cdot x_2) \cdot x_3 \cdots) \cdot x_n$.

Then we have to extend this map of objects to a functor

$L : \mathrm{Mon} \to T\text{-Alg}$
$(M, \cdot, 1) \mapsto (M, (x_1, \dots, x_n) \mapsto (x_1 \cdot x_2) \cdot \cdots x_n)$
$f : (M, \times, 1') \to (N, \cdot, 1) \mapsto Lf : (M, \dots) \to (N, \dots)$
$\qquad x \mapsto f(x)$

and $R : 1\text{-}\mathcal{A}lg \longrightarrow \mathcal{M}on$

$$(M, a) \longmapsto (M, a(-,-), a())$$

$$f : (M, a) \to (N, a') \longmapsto Rf : (M, a(-,-), a()) \to (N, a'(-,-), a'())$$
$$x \longmapsto f(x)$$

Indeed,

$$Rf(\underbrace{a(x,y)}_{x \cdot y}) = f(a(x,y)) = a'(f(x), f(y)) = Rfx \cdot Rfy.$$

and $Rf(1) = f(a()) = a()$.

Also,

$$\begin{tikzcd}
TLM \arrow[r, "TLf"] \arrow[d, "(*)^n"'] & TLN \arrow[d, "(\cdot)^n"] \\
LM \arrow[r, "Lf"'] & LN
\end{tikzcd}$$

by induction on n (as case $n=1$ is verified with $f(1) = 1$ and case $n > 1$ is verified with $f(x * y) = f(x) \cdot f(y)$

A morphism $f : X \to Y$ in ...

$\mathcal{M}on$

$$\begin{tikzcd}
X \times X \arrow[rr, "f \times f"] \arrow[d, "- * -"'] \arrow[drr, phantom, "f(x*y) = f(x) \cdot f(y)"] & & Y \times Y \arrow[d, "- \cdot -"] \\
X \arrow[rr, "f"] & & Y \\
& 1 \arrow[ul, "1_X"] \arrow[ur, "1_Y"'] &
\end{tikzcd}$$

$1\text{-}\mathcal{A}lg$

$$\begin{tikzcd}
TX \arrow[r, "Tf"] \arrow[d, "a"'] & TY \arrow[d, "a'"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

A monad in a category $\mathcal{C}$ is a $\underset{\text{endo}}{\text{functor}}$ $T : \mathcal{C} \to \mathcal{C}$ with

- unit $\eta_X : X \to TX$ and • multiplication $\mu_X : TTX \to TX$

such that $\eta : id \Rightarrow T$ and $\mu : TT \Rightarrow T$ are natural transformations, such that both diagrams commute.

$$\begin{tikzcd} TX \arrow[dr, equal] & TTX \arrow[l, "\eta_{TX}"'] \arrow[r, "T\eta_X"] \arrow[d, "\mu_X"] & TX \arrow[dl, equal] \\ & TX & \end{tikzcd} \qquad \begin{tikzcd} TTTX \arrow[r, "T\mu_X"] \arrow[d, "\mu_{TX}"'] \arrow[dr, phantom, "\text{associativity}"] & TTX \arrow[d, "\mu_X"] \\ TTX \arrow[r, "\mu_X"] & TX \end{tikzcd}$$

A T-algebra is an object $A \in \mathcal{C}$ and $a : TX \to X$ such that

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, equal] & TX \arrow[d, "a"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} FFX \arrow[r, "\mu_X"] \arrow[d, "Fa"'] & FX \arrow[d, "a"] \\ FX \arrow[r, "a"'] & X \end{tikzcd}$$

A natural transformation $\alpha : F \Rightarrow G$ where $F, G : \mathcal{C} \to \mathcal{D}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "F"{name=U}] \arrow[r, bend right=50, "G"'{name=D}] & \mathcal{D} \arrow[from=U, to=D, Rightarrow, "\alpha"] \end{tikzcd}$$

is a set of morphisms $\alpha_X : FX \to GX$ such that, for every $f : X \to Y$, the square

$$\begin{tikzcd} FX \arrow[r, "Ff"] \arrow[d, "\alpha_X"'] & FY \arrow[d, "\alpha_Y"] \\ GX \arrow[r, "Gf"'] & GY \end{tikzcd}$$

commutes.

Exercise: Find $\mathcal{T}$ on Graph such that $\mathcal{C}at \cong \mathcal{T}\text{-}\mathcal{A}lg$.

$$\mathcal{T} : \mathcal{G}raph \to \mathcal{G}raph$$
$$G \longmapsto G^*$$
$$f : G \to H \longmapsto f^* : G^* \to H^*$$

Then, a $\mathcal{T}$-algebra is a map $G^* \to G$ in $\mathcal{G}raph$.

$$\begin{tikzcd}
\mathcal{C}at \arrow[rr, bend left, "L"] \arrow[dr] & & \mathcal{T}\text{-}\mathcal{A}lg \arrow[ll, bend left, "R"] \arrow[dl] \\
& \mathcal{S}et &
\end{tikzcd}$$

$$R : \mathcal{T}\text{-}\mathcal{A}lg \longrightarrow \mathcal{C}at$$
$$(G, a : G^* \to G) \longmapsto \left(\begin{array}{ll} \text{obj} : & \text{vertices of } G \\ \text{morphisms} : & \text{paths in } G \\ \text{composition} : & p \circ q = a(p \cdot q) \\ \text{identity} : & id = a() \end{array} \right)$$

$$\big(f : (G, a) \to (H, b)\big) \longmapsto Rf : \begin{array}{l} g \longmapsto f(g) \\ g \overset{*}{-} h \longmapsto f(g) \overset{*}{-} f(h) \end{array}$$

and
$$L : \mathcal{C}at \longrightarrow \mathcal{T}\text{-}\mathcal{A}lg$$
$$\mathcal{C} \longmapsto \big(|\mathcal{C}|, a : (x_1, \dots, x_n) \mapsto x_1 \circ \dots \circ x_n\big)$$
$$F : \mathcal{C} \to \mathcal{D} \longmapsto LF : x \mapsto F(x)$$

$$\begin{tikzcd} G \arrow[r, "\eta_G"] \arrow[dr, equal] & G^* \arrow[d, "a"] \\ & G \end{tikzcd} \qquad \begin{tikzcd} G^{**} \arrow[r, "\text{concatenation}"] \arrow[d, "a^*"'] & G^* \arrow[d, "a"] \\ G^* \arrow[r, "a"'] & G \end{tikzcd}$$

Lecture #2 25/11/2025.

Last-time: products, monads, algebras, finally equivalence between T-Alg and some other categories

"For any monad T, the forgetful functor $T\text{-Alg} \to \mathscr{C}$, $(X,a) \mapsto X$ creates products"

We say $U : \mathcal{E} \to \mathcal{B}$ creates binary products if

any product $UX \leftarrow P \to UY$ in $\mathcal{B}$ has a unique antecedant $X \leftarrow \bar{P} \to Y$ in $\mathcal{E}$ which is also a product in $\mathcal{E}$.

This is different from product preservation (U product = product), or product reflexion (if $U(X \leftarrow P \to Y)$ is a product then so is $X \leftarrow P \to Y$).

Proposition: For any monad $T : \mathscr{C} \to \mathscr{C}$, the forgetful functor T-Alg creates binary products.

Let $TA \xrightarrow{r} A$ and $TB \xrightarrow{s} B$ in T-Alg.

We want a T-algebra $T(A \times B) \to A \times B$.

$$
\begin{tikzcd}
TA \arrow[d, "t"'] & T(A\times B) \arrow[l, "T\pi_1"'] \arrow[r, "T\pi_2"] \arrow[d, dashed, "\exists!"] & TB \arrow[d, "s"] \\
A & A\times B \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & B
\end{tikzcd}
$$

We need to check the monad laws apply correctly to $T(A\times B)$.

$$
\begin{tikzcd}
TT(A\times B) \arrow[rr, "Tf"] \arrow[dr, "TT\pi_1"'] \arrow[dd, "\mu_{A\times B}"'] \arrow[rrd, phantom, "T\text{déf.}"] & & T(A\times B) \arrow[dr, "T\pi_1"] \arrow[dd] & \\
& TTA \arrow[rr, "Tr"] \arrow[dd, "\mu_A"'] \arrow[dl, phantom, "\text{naturality}"] \arrow[rrd, phantom, "\text{déf.}"] & & TA \arrow[dd, "r"] \\
T(A\times B) \arrow[rr, "Tf"] \arrow[dr, "T\pi_1"'] \arrow[r, phantom, "\text{déf}"] & & A\times B \arrow[dr] \arrow[d, phantom, "\text{déf.}"] & \\
& TA \arrow[rr, "r"'] & & A
\end{tikzcd}
$$

So we have compositionnality.

$$\begin{tikzcd}
A \arrow[d, "\eta_A"'] \arrow[dd, bend right=60, equal] \arrow[dr, phantom, "\text{naturality}"] & A\times B \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "\eta_{A\times B}"] \arrow[dd, bend left=60, equal] \arrow[dr, phantom, "\text{nat.}"] & B \arrow[d, "\eta_B"] \arrow[dd, bend left=60, equal] \\
TA \arrow[d, "r"'] \arrow[dr, phantom, "\text{def.}"] & T(A\times B) \arrow[l, "T\pi_1"] \arrow[r, "T\pi_2"'] \arrow[d, "f"] \arrow[dr, phantom, "\text{def}"] & TB \arrow[d, "t"] \\
A & A\times B \arrow[l, "\pi_1"] \arrow[r] & B
\end{tikzcd}$$

It remains to show that $T(A\times B)$ is a product in T-Alg.

$$\begin{tikzcd}
 & TP \arrow[d, "s"] \arrow[ddl, "Tg"'] \arrow[ddr, "Tp"] \arrow[ddrr, "Th"] & & \\
 & P \arrow[ddl, "g"'] \arrow[ddr, "\exists!\, p"] \arrow[ddrr, "h"] & & \\
TA \arrow[d, "r"'] & & T(A\times B) \arrow[ll, "T\pi_1"] \arrow[r, "T\pi_2"] \arrow[d, "f"] & B \arrow[d, "t"] \\
A \arrow[rr, phantom, "\text{def}"] & & A\times B \arrow[ll, "\pi_1"] \arrow[r, "\pi_2"'] & TB
\end{tikzcd}$$

(This diagram is in $\mathcal{C}$ but talks about T-Alg).

This is where we start with combinatorial constructions.

$\boxed{\text{Pre Sheaves}}$

A presheaf is a functor $\mathcal{C}^{op} \to Set$.

The opposite category $\mathcal{C}^{op}$ has the same objects and morphisms from A to B are $\mathcal{C}(B,A)$ and we use reverse composition in $\mathcal{C}$

$$
\begin{tikzcd}
\mathcal{C}^{op}(B,C) \times \mathcal{C}^{op}(A,B) \arrow[r, "-\circ_{\mathcal{C}^{op}}-"] \arrow[d, "\cong"] & \mathcal{C}^{op}(A,C) \arrow[d, equal] \\
\mathcal{C}^{op}(A,B) \times \mathcal{C}^{op}(B,C) \arrow[d, equal] & \mathcal{C}(C,A) \\
\mathcal{C}(B,A) \times \mathcal{C}(C,B) \arrow[ur, "-\circ_{\mathcal{C}}-"] &
\end{tikzcd}
$$

Define $\mathcal{G}$: $\begin{tikzcd} {[0]} \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] & {[1]} \end{tikzcd}$. Presheaves F on $\mathcal{G}$ are exactly directed multigraphs

$$
\begin{tikzcd}
\underbrace{F[0]}_{V} & \underbrace{F[1]}_{E} \arrow[l, bend right, "Fs"'] \arrow[l, bend left, "Ft"]
\end{tikzcd}
$$

Define $PSh(\mathcal{C}) = \hat{\mathcal{C}} = [\mathcal{C}^{op}, Set]$. So $\mathcal{G}ph := \hat{\mathcal{G}}$.

((Remark $\mathcal{G}$ is small but $\hat{\mathcal{G}}$ is only locally small.

Graph morphisms should make

$$
\begin{tikzcd}
F[1] \arrow[r, "f_{[1]}"] \arrow[d, "Fs"'] & G[1] \arrow[d, "Gs"] \\
F[0] \arrow[r, "f_{[0]}"'] & G[0]
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
F[1] \arrow[r, "f_{[1]}"] \arrow[d, "Ft"'] & G[1] \arrow[d, "Gt"] \\
F[0] \arrow[r, "f_{[0]}"'] & G[0]
\end{tikzcd}
$$

commute, which we will write as

$$\begin{tikzcd}
F[1] \arrow[r, "f_{[1]}"] \arrow[d, shift right, "Fs"'] \arrow[d, shift left, "Ft"] & G[1] \arrow[d, shift right, "Gs"'] \arrow[d, shift left, "Gt"] \\
F[0] \arrow[r, "f_{[0]}"'] & G[0]
\end{tikzcd}$$

should <u>serially</u> commute.

A <u>coproduct</u> is of the form

$$\begin{tikzcd}
X \arrow[r, "\iota_1"] \arrow[dr, "f"'] & X+Y \arrow[d, "\exists!"] & Y \arrow[l, "\iota_2"'] \arrow[dl, "g"] \\
& A &
\end{tikzcd}$$

In Set, it can be constructed as $\{0\} \times X \sqcup \{1\} \times Y$.

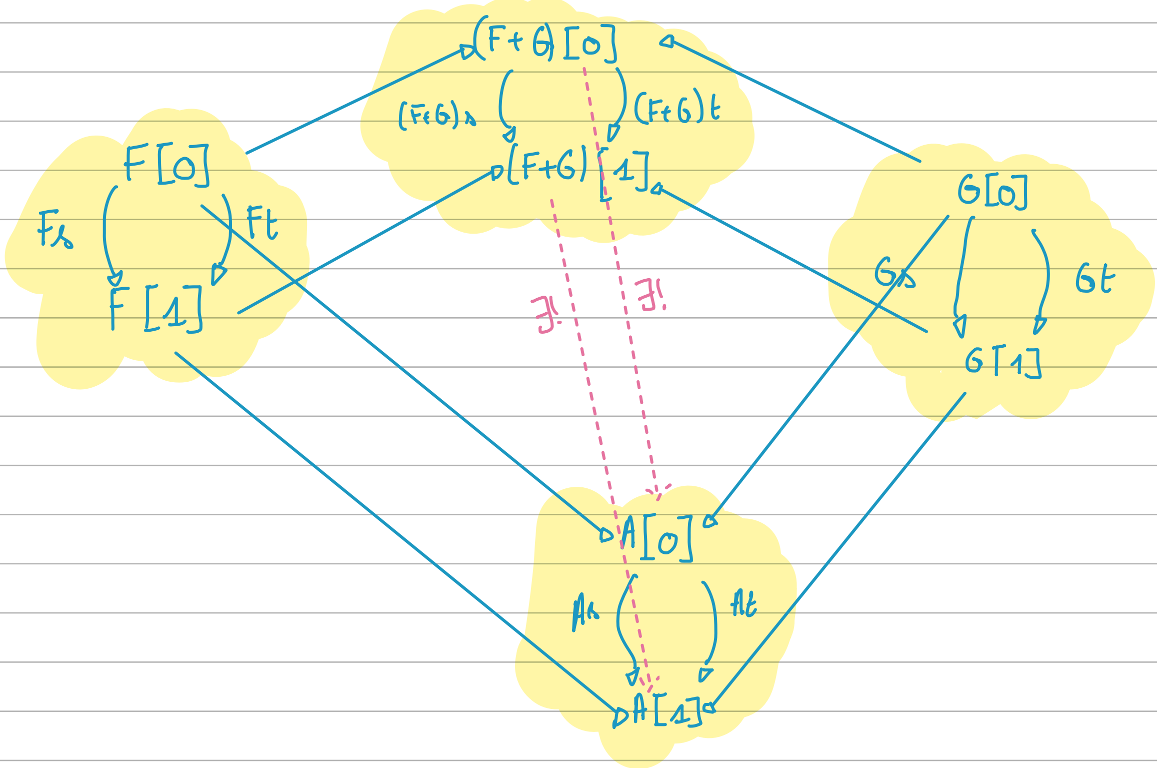
How to compute a coproduct of graphs?

$$\begin{tikzcd}
& (F+G)[0] \arrow[d, shift right, "(F+G)s"'] \arrow[d, shift left, "(F+G)t"] & \\
F[0] \arrow[ur] \arrow[d, shift right, "Fs"'] \arrow[d, shift left, "Ft"] & (F+G)[1] & G[0] \arrow[ul] \arrow[d, shift right, "Gs"'] \arrow[d, shift left, "Gt"] \\
F[1] \arrow[ur] & & G[1] \arrow[ul]
\end{tikzcd}$$

should sequentially commute. So we should have

that
$$\begin{aligned}
(F+G)[0] &= F[0] \sqcup G[0] \\
(F+G)[1] &= F[1] \sqcup G[1] \\
(F+G)\,s &= Fs + Gs \\
(F+G)\,t &= Ft + Gt
\end{aligned}$$

Also



Let us talk about colimits. Colimits are the universal cocones.

A diagram in $\mathcal{C}$ is a functor $F : \mathbb{I} \to \mathcal{C}$ for some small $\mathbb{I}$.

A cocone "$\lambda : D \to c$" is a natural transformation from D to the constant functor $c \in \mathcal{C}$.

A cocone morphism is of the form
$$\begin{tikzcd} & D \arrow[dl, "\lambda"'] \arrow[dr, "\lambda'"] & \\ c \arrow[rr, "k"'] & & c' \end{tikzcd}$$
where $k \in \mathcal{C}(c, c')$.

We thus define a category Cocone (D) of cocones over D.

A colimit of D is the initial object of Cocone (D) :

$\forall c,$
$$\begin{tikzcd} & D \arrow[dl, "\lambda"'] \arrow[dr] & \\ \operatorname{colim} D \arrow[rr, "\exists!"'] & & c \end{tikzcd}$$

$2 := 1 + 1$

Example: for coproduct, use the diagram

$$\begin{tikzcd} \bullet \arrow[loop below, "id"] & \bullet \arrow[loop below, "id"] \end{tikzcd}$$

for initial object, use the empty diagram .

Indeed, the universal property of colim(empty diagram) tells us that

$$\forall c, \qquad \mathrm{colim}(\emptyset) \xrightarrow{\exists!} c$$

$$\begin{tikzcd}
C \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[d, "i_2"] \arrow[ddr, bend left, "v"] & \\
A \arrow[r, "i_1"'] \arrow[drr, bend right, "u"'] & A +_C B \arrow[dr, dashed, "\overline{\varphi}"] & \\
 & & D
\end{tikzcd}$$

$A +_C B := A + B \,/\, f(c) = g(c)$.

$$\begin{cases} i_1 : A \longrightarrow A +_C B \\ i_2 : B \longrightarrow A +_C B \end{cases}$$

$$\begin{aligned} \varphi : A + B &\longrightarrow D \\ a &\longmapsto u(a) \\ b &\longmapsto v(b) \end{aligned}$$

and $\overline{\varphi} : A +_C B \longrightarrow D$
exists as $\varphi(f(c)) = \varphi(g(c))$
$u(f(c)) = v(g(c))$

$$\begin{tikzcd}
2 \arrow[r, "i_0"] \arrow[d, "!"'] & 2 + 3 \arrow[d] \\
1 \arrow[r] & 1 + (2 + 3) \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

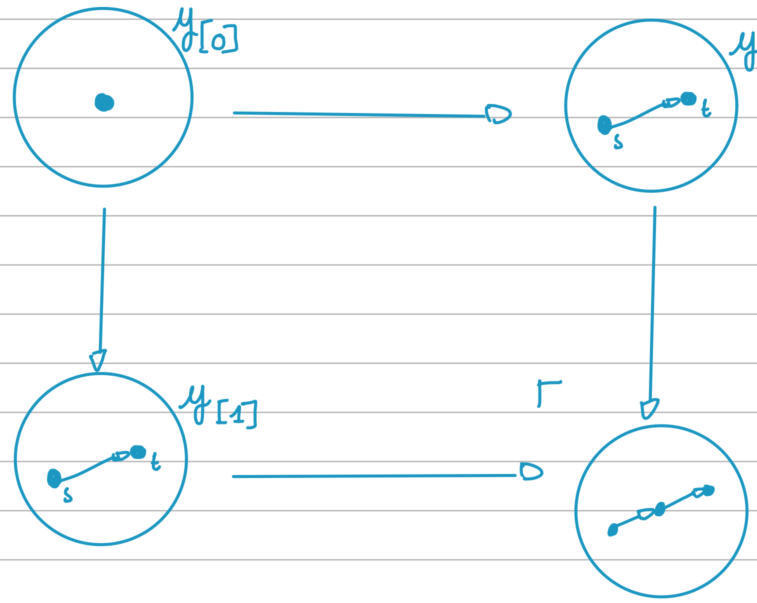
$1 + (2+3) \,/\, i_0(0_2) = !(0_2) = !(1_2) = i_0(1_2)$

$$\|$$

$$1 + 3 = 4$$

The pushout of $\overset{\bullet}{\underset{}{\mathcal{Y}_{[0]}}}$, $\overset{\bullet_s \to \bullet_t}{\mathcal{Y}_{[1]}}$ in Graphs along $y_t : \bullet \longmapsto \bullet_t$
$y_s : \bullet \longmapsto \bullet_s$

is $\bullet \to \bullet \to \bullet$



Lecture #3

02/12/2025

Exercise In Set, the equalizer of $A \underset{g}{\overset{f}{\rightrightarrows}} B$ is

$$E := \{ a \in A \mid f(a) = g(a) \}$$

with $i : E \to A$ the inclusion arrow. For any $x : X \to A$ such that $x \circ f = x \circ g$, then $x(X) \subseteq E$ and so there is an unique inclusion arrow

$$\begin{aligned} m : X &\longrightarrow E \\ u &\longmapsto x(u) \end{aligned}$$

such that

$$\begin{tikzcd} X \arrow[d, dotted, "\exists!\, m"'] \arrow[dr, "x"] & & \\ E \arrow[r, "i"'] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

composes.

Exercise • $[\mathscr{C},\mathscr{D}]$ is a category $\big(F\circ G(X) = F(G(X))$

$F\circ G(f) = F(G(f))$

$Id(X) = X \quad Id(f) = f \big)$

• $ob([\mathscr{C},\mathscr{D}]) \subseteq ob(\mathscr{D})^{ob(\mathscr{C})} \times Mor(\mathscr{D})^{Mor(\mathscr{C})}$ and so $ob([\mathscr{C},\mathscr{D}])$ is a set as $\mathscr{C}$ and $\mathscr{D}$ are small.

For some $F, G \in [\mathscr{C},\mathscr{D}]$, $\quad Hom(F,G) \subseteq \prod_{X\in\mathscr{C}} Hom(FX, GX)$ is a set.

↳ small ↳ $\mathscr{D}$ small

• For some $F, G \in [\mathscr{C},\mathscr{D}]$, $\quad Hom(F,G) \subseteq \prod_{X\in\mathscr{C}} Hom(FX, GX)$ is a set.

↳ small ↳ in $\mathscr{D}$ locally small

• The functor $F : [\mathscr{C},\mathscr{D}] \longrightarrow [ob(\mathscr{C}),\mathscr{D}]$ is defined as

$(U : \mathscr{C} \to \mathscr{D}) \longmapsto \bar{U} : X \longmapsto UX$

$\bar{U} : id_X \longmapsto id_{UX}$

$(\eta : U \Rightarrow V) \longmapsto \bar{\eta} : \bar{U} \Rightarrow \bar{V}$

$\bar{U}X = UX \longrightarrow VX = \bar{V}X$

$$\begin{tikzcd} UX \arrow[r, "\eta_X"] \arrow[d, "Uf"'] & VX \arrow[d, "Vf"] \\ UY \arrow[r, "\eta_Y"'] & VY \end{tikzcd}$$

Take a product

$$FA \xLeftarrow{\pi_1} X \xRightarrow{\pi_2} FB$$

in $[ob(\mathscr{C}),\mathscr{D}]$, then

$$A = GFA \xLeftarrow{G\pi_1} GX \xRightarrow{G\pi_2} GFB = B$$

with $G : [ob(\mathscr{C}),\mathscr{D}] \longrightarrow [\mathscr{C},\mathscr{D}]$ the inclusion map.

Take $P \in [\mathscr{C},\mathscr{D}]$ with $\eta : P \Rightarrow A$ $\mu : P \Rightarrow B$ commutes

$$\begin{tikzcd} & P \arrow[dl, Rightarrow, "\eta"'] \arrow[dr, Rightarrow, "\mu"] & \\ A & GX \arrow[l, Rightarrow, "G\pi_1"] \arrow[r, Rightarrow, "G\pi_1"'] & B \end{tikzcd}$$

in $[\mathscr{C},\mathscr{D}]$

Then we have that

$$\begin{tikzcd}
 & FB \arrow[dl, Rightarrow, "F\eta"'] \arrow[d, dotted, Rightarrow, "\gamma"] \arrow[dr, Rightarrow, "F\mu"] & \\
FA & X \arrow[l, Rightarrow, "\pi_1"] \arrow[r, Rightarrow, "\pi_1"'] & FB
\end{tikzcd}$$

commutes in $[ob(\mathcal{C}), \mathcal{D}]$ and so

$$\begin{tikzcd}
 & P \arrow[dl, Rightarrow, "\eta"'] \arrow[d, Rightarrow, "G\gamma"] \arrow[dr, Rightarrow, "\mu"] & \\
A & GX \arrow[l, Rightarrow, "G\pi_1"] \arrow[r, Rightarrow, "G\pi_1"'] & B
\end{tikzcd}$$ in $[\mathcal{C}, \mathcal{D}]$

commutes. Unicity is by $Nat(P, GX) \hookrightarrow Nat(FP, X)$.

Given a graph G, we define $el(G)$ the category

- objects pairs (c, x) with $c \in \mathcal{G}$ and $x \in X(c)$
- morphisms $f : (c, x) \longrightarrow (c', y)$ st $f : c \to d$ and $X(f)(y) = x$.

$$el(X) \xrightarrow{\pi_X} \mathcal{G} \xrightarrow{y} \hat{\mathcal{G}} = \mathcal{G}raph.$$

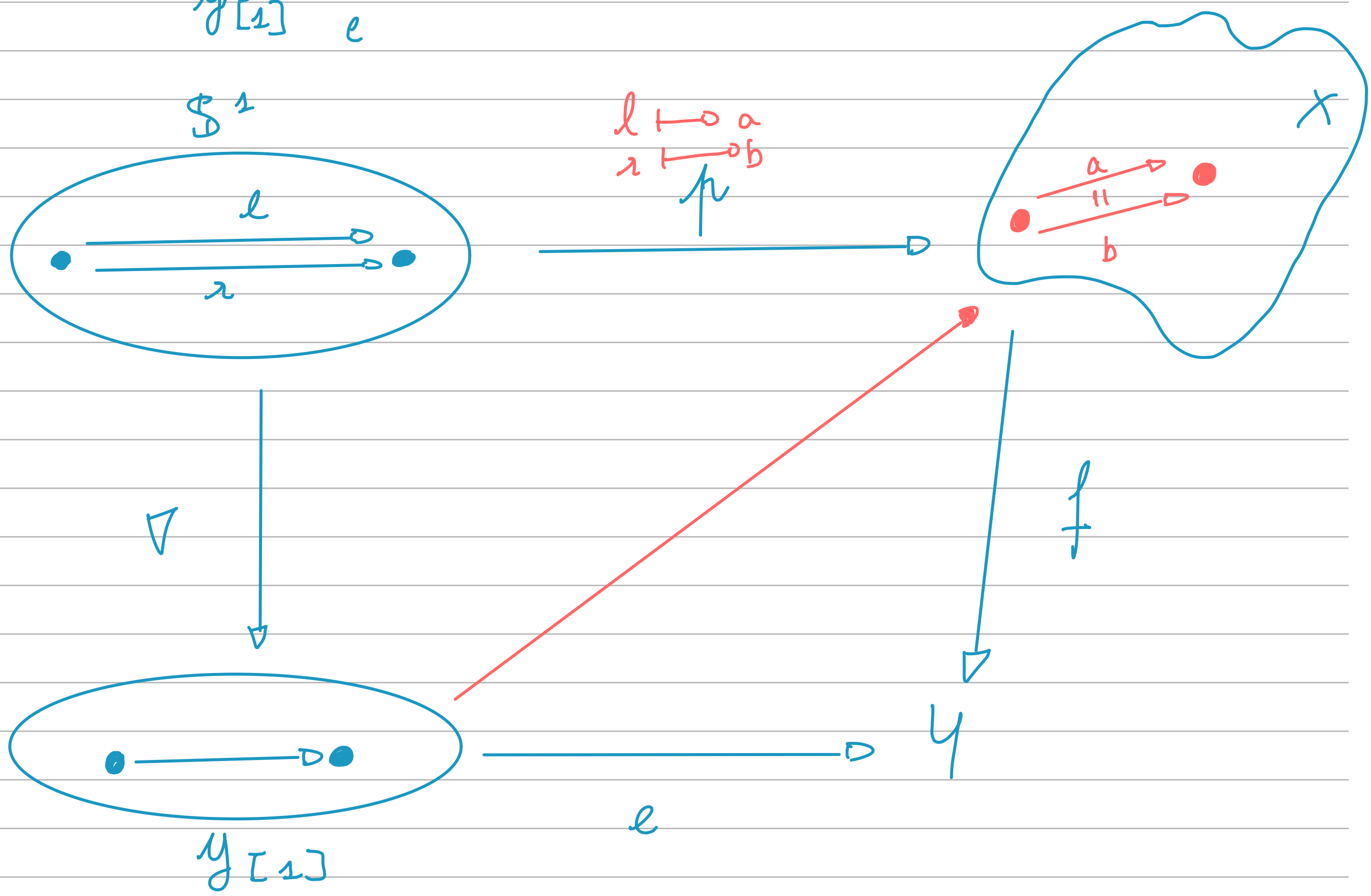
Proposition (co-Yonneda) Every presheaf $X \in \hat{\mathcal{C}}$ is the colimit of

$$el(X) \xrightarrow{\pi_X} \mathcal{C} \longrightarrow \hat{\mathcal{C}}$$

for a small $\mathcal{C}$.

A functor $F : \mathcal{C} \to \mathcal{D}$ is faithful if, for every $F_{AB} : \mathcal{C}(A,B) \longrightarrow \mathcal{D}(A,B)$ is injective.

$$\begin{tikzcd}
S^1 \arrow[r, "p"] \arrow[d, "\nabla"'] & X \arrow[d, "f"] \\
y[1] \arrow[r, "e"'] \arrow[ur, dotted] & Y
\end{tikzcd}$$



Exercise

$$\begin{tikzcd}
y[0] + y[0] \arrow[r, "{[y_s, y_t]}"] \arrow[d, "{[y_s, y_t]}"'] & y[1] \arrow[d] \\
y[1] \arrow[r] & S^1 \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

a b a b

A graph is simple if $\begin{tikzcd} \cdot \arrow[r, shift left, "e"] \arrow[r, shift right, "e'"'] & \cdot \end{tikzcd}$ implies $e=e'$

Equivalently if $X \to T$ is faithful.

Surjective / Injective functions in Set form a factorization system

$$\begin{tikzcd} 1 \arrow[r, "a"] \arrow[dr, "f(a)"'] & A \arrow[r, "s"] \arrow[d, "f"] & X \arrow[d, "\iota_1"] \\ & B \arrow[r, "h"'] \arrow[ur, "f(a)\mapsto s(a)"] & X+Y \arrow[u, bend right] \end{tikzcd}$$

$*$ $**$

We have that

$$\begin{tikzcd} A \arrow[r, "s"] \arrow[d, "f"'] & X \arrow[d, "\iota_1"] \\ B \arrow[r, "h"'] & X+Y \end{tikzcd}$$

commutes, and so we immedialty have the required commutations $(*)$ and $(**)$.

Injective / surjective too

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\iota_1"'] \arrow[dr, phantom, "*"] & A+B \arrow[d, "{[A,k]}"] \\ X+Y \arrow[r, "{[g,h]}"', "**"] \arrow[ur, "{[f,\iota_1 h]}"] & A \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\text{bijective on objects}"] \arrow[rr, bend right, "F"'] & M_F \arrow[r, "\text{fully faithful}"] & \mathcal{D} \end{tikzcd}$$

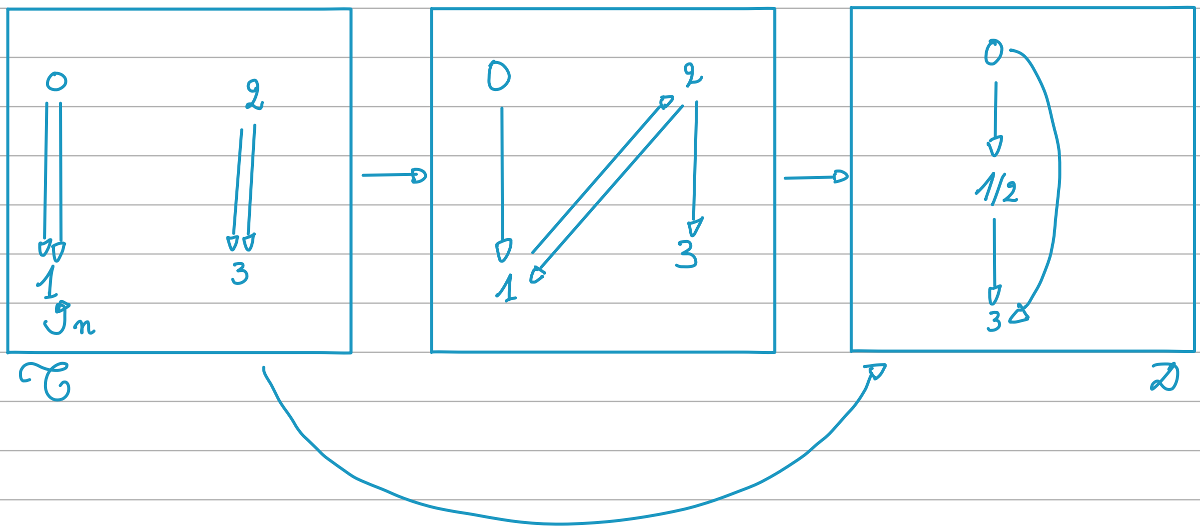
M_F : objects: $ob(\mathcal{C})$

morphisms:

$\mathcal{C}(A,B) / \{f,g \mid Ff = Fg\}$.

or alternatively,

$Hom_{M_F}(A,B) := Hom_{\mathcal{D}}(FA, FB)$



<u>Theorem</u>. $\left({}^{\pitchfork}(\mathcal{J}^{\pitchfork}), {}^{\pitchfork}\mathcal{J}\right)$ is a factorization system where $\mathcal{J}$ is a set of morphisms in $\mathcal{C}$ (for some "suitable" $\mathcal{C}$).

<u>Exercise</u> Design a $\mathcal{J}$ such that the full subcategory of graphs X with $!_X \in \mathcal{J}^{\pitchfork}$ is equivalent to the category of undirected graph.

$$\mathcal{J} := \left\{ \begin{array}{l} \sigma : \mathbb{S}^1 \longrightarrow y_{[1]} \\ u : y_{[1]} \longrightarrow \mathbb{S}^1_{\circlearrowleft} \end{array} \right\}$$

Where $\mathbb{S}^1_{\circlearrowleft}$: 

$$\begin{tikzcd} \omega \arrow[rr, "y"] \arrow[dr, bend right, "R"'] & \arrow[d, phantom, "\Downarrow"] & \hat{\omega} \arrow[dl, bend left, "R^*"] \\ & \mathcal{G}raph & \end{tikzcd}$$

$R(n) := 0 \to 1 \to \cdots \to n-1$

$R(n \le p) :=$ prefix $R(n) \to R(p)$

$R^*(X)(n) = \mathrm{Graph}(R(n), X) =$ paths of length n

$R^*(X)(n \le p)$: inclusion to length p

Comma category

Given $\mathcal{C} \xrightarrow{F} \mathcal{E} \xleftarrow{G} \mathcal{D}$

define F/G with

- <u>objects</u>: $(C \in \mathcal{C}, D \in \mathcal{D}, f \in \mathcal{E}(FC, GD))$
- <u>morphisms</u>: $(C, D, f) \xrightarrow{(u,v)} (C', D', f')$ such that

$$\begin{tikzcd}
FC \arrow[r, "u"] \arrow[d, "f"'] & FC' \arrow[d, "g"] \\
GD \arrow[r, "v"'] & GD'
\end{tikzcd}$$

This forms a category:

$$\begin{tikzcd}
FC \arrow[r, equal] \arrow[d, "f"'] & FC \arrow[d, "f"] \\
GD \arrow[r, equal] & GD
\end{tikzcd}
\qquad
\begin{tikzcd}
FC \arrow[r, "u"] \arrow[d, "f"'] \arrow[rr, bend left, "u' \circ u"] & FC' \arrow[r, "u'"] \arrow[d, "g"] & FC'' \arrow[d, "h"] \\
GD \arrow[r, "v"'] \arrow[rr, bend right, "v' \circ v"'] & GD' \arrow[r, "v'"'] & GD''
\end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Graph} \arrow[r] & \mathrm{Set} & \mathbb{1} \arrow[l, "A"'] \\ G \arrow[r, mapsto] & E(G) = G[1] \end{tikzcd}$$

Then the comma category E/A is equivalent to the category of labelled graphs.

$$\begin{tikzcd} X[1] \arrow[rr, "{f[1]}"] \arrow[dr] & & Y[1] \arrow[dl] \\ & A & \end{tikzcd}$$

$$\begin{tikzcd} A^* \arrow[rr] \arrow[dr, "R"'] & & \widehat{A^*} \\ & \mathrm{Graph}/A \arrow[ur, "R^*"'] & \end{tikzcd}$$

$$R^*(X,\ell)(a_1 \dots a_n) = \mathrm{Graph}/_A\big(([n+1],[a_1 \dots a_n]),(X,\ell)\big).$$

Lecture #3 09/12/2025

$$\begin{tikzcd} \mathrm{Mnd}_f(\mathrm{Set}) \arrow[rr, "\sim"] \arrow[dr, "-\text{-Alg}"'] & & \mathrm{Law} \\ & \mathcal{CAT} \arrow[ur, "\mathrm{Mod}"'] & \end{tikzcd}$$

$\mathrm{Mod}(\mathbb{L}) \hookrightarrow [\mathbb{L}, \mathrm{Set}]$

Lawvere theory: small category w/ finite products freely gen$^{\text{ed}}$ by an object

Def. An algebraic signature is an object O with a map $a : O \to \mathbb{N}$.

Def. $\Sigma^*(X) = \{ e \mid X \vdash_\Sigma e \}$

$$\frac{}{X \vdash_\Sigma [x]}\ \mathrm{var}\quad x \in X \qquad\qquad \frac{X \vdash_\Sigma e_i \quad \forall i \in [\![1, a(o)]\!]}{X \vdash_\Sigma o(e_1, \dots, e_{a(o)})}\ o \in O$$

$\boxed{\text{Syntactic category } \mathbb{L}_\Sigma}$

objects: $\mathbb{N}$

morphisms: $m \to n$ is a morphism $\in \Sigma^*(m)$ $\begin{cases} n\text{-tuple } \langle M_1, \dots, M_n \rangle \\ \text{each with scope } \underbrace{x_1, \dots, x_m} \end{cases}$

$\hookrightarrow$ seen as an assignment

composition is done by substitution:

$$\begin{tikzcd} m \arrow[r, "M"] \arrow[rr, bend right, "P"'] & n \arrow[r, "N"] & p \end{tikzcd}$$

$$P = \langle N_1[M], \dots, N_n[M] \rangle.$$

identity is $\langle x_1, \dots, x_n \rangle$.

Check right identity axiom:

for every $\langle M_1, \ldots, M_m \rangle$, we have

$$M_i[\langle x_1, \ldots, x_n \rangle] = M_i$$

as we substitute x_j with x_j

and so

$$\begin{tikzcd} m \arrow[r, "M"] \arrow[d, equal] & n \\ n \arrow[ur, "M"'] & \end{tikzcd}$$

Left identity: for any $M : m \longrightarrow n$, then

$\forall i,\quad x_i[M] = M_i$ and so $\langle x_1[M], \ldots, x_n[M] \rangle = \langle M_1, \ldots, M_n \rangle = M$

Associativity:

given $m \xrightarrow{M} n \xrightarrow{N} p \xrightarrow{P} q$

$$(P \circ (N \circ M))_i = P_i[\langle N_1[M], \ldots, N_p[M] \rangle]$$

and $((P \circ N) \circ M)_i = P_i[N][M]$

Both are equal as, for the first one, we substitute in the substitution, and this is exactly the same as substituting in the result

Substitution lemma.

Proposition $\mathbb{L}_\Sigma$ has binary products

- $m \times n = m +_{\mathbb{N}} n$
- initial : $0_{\mathbb{N}}$

Proof simple enough.

Proposition : every object is a finite power of 1.

$\hookrightarrow$ $0 = 1^0$ and $n = 1^n = n + \dots + n$

$\hookrightarrow$ nullary product in $\mathbb{L}_\Sigma$ $\hookrightarrow$ n-ary product in $\mathbb{L}_\Sigma$.

Exercise: Show that $\mathscr{C}$ has finite products iff it has binary products and a terminal object

"$\Rightarrow$". If $\mathscr{C}$ has finite product, it has ~~binary product.~~ nullary products

this means there is a unique morphism

$$A \longrightarrow \prod_0 X_i$$

and so it is terminal.

"$\Leftarrow$" By induction, $\mathscr{C}$ has n-ary products.

- terminal objects are nullary products
- for $I = \underbrace{\{1, \dots, n\}}_{I'} \sqcup \{n+1\}$, we have

$$\prod_{i \in I} X_i := \left(\prod_{i \in I'} X_i\right) \times X_{n+1}$$

and this is indeed a product of $(X_i \mid i \in I)$:

$$
\begin{tikzcd}
 & A \arrow[dl, dashed, "\exists ! g"'] \arrow[dr, dashed, "{\langle g, f_{n+1}\rangle} \text{ unique}"] \arrow[ddl, bend right=60, "f_i"'] \arrow[ddr, bend left=60, "f_{n+1}"] & \\
\prod_{i\in I'} X_i \arrow[d] & & \left(\prod_{i\in I'} X_i\right) \times X_{n+1} \arrow[ll, "\pi_1"'] \arrow[d, "\pi_2"] \\
X_i & & X_{n+1}
\end{tikzcd}
$$

Def: $A^I = \prod_{i\in I} A$ for $A \in \mathcal{C}$.

"heterogeneous" as $I \in Set$: $\mathcal{C}(X, A^I) \cong Set(I, \mathcal{C}(X, A))$.

Def A category $\mathcal{C}$ is skeletal if all isomorphic objects are equal.

Exm finite sets Set_f is equivalent to the skeletal category of finite cardinals $\mathbb{F}$

$$
\begin{tikzcd}
\mathbb{F} \arrow[r, hook, shift left, "\varepsilon"] & Set_f \arrow[l, shift left, "|-|"]
\end{tikzcd}
$$

Choose a bijection $\varepsilon_X : |X| \xrightarrow{\sim} X$ for every set $X \in Set$.

Functoriality of $|-|$:

$$
\begin{tikzcd}
|X| \arrow[r, "\varepsilon_X", "\sim"'] \arrow[d, dashed, "|f|"'] & X \arrow[d, "f"] \\
|Y| \arrow[r, "\sim", "\varepsilon_Y"'] & Y
\end{tikzcd}
$$

where $|f| := \varepsilon_Y^{-1} \circ f \circ \varepsilon_X$.

$$\begin{tikzcd}
{|X|} \arrow[r, "\varepsilon_X", "\sim"'] \arrow[d, "{|f|}"'] \arrow[dd, bend right=60, "{|g\circ f|}"'] & X \arrow[d, "f"] \arrow[dd, bend left=60, "g\circ f"] \\
{|Y|} \arrow[r, "\sim", "\varepsilon_Y"'] \arrow[d, "{|g|}"'] & Y \arrow[d, "g"] \\
{|Z|} \arrow[r, "\sim", "\varepsilon_Z"'] & Z
\end{tikzcd}$$

$\eta_n : n \longrightarrow |n|$ is an equality hence natural

$$\begin{tikzcd}
\mathbb{F}' \arrow[rr, bend left] \arrow[rr, phantom, "\overset{\eta}{\Downarrow} \quad \overset{\varepsilon}{\Uparrow}"] & & Set_f \arrow[ll, bend left, "{|-|}"]
\end{tikzcd}$$

Def: embedding : injective on objects and faithful

Def: $F : \mathcal{C} \longrightarrow \mathcal{D}$ is essentially surjective if $\forall D \in \mathcal{D}$ is $\cong$ to FC for some $C \in \mathcal{C}$.

Exm $\{\bullet\} \longrightarrow \{\bullet \cong \circ\}$ is ess. surj.

Def A Lawvere theory is a small, skeletal category w/ finite products and whose objects are finite powers of a single generating object.

A morphism of Lawvere theory is a functor preserving finite products and the generating object.

It forms a category Law.

$\mathbb{L}_{\Sigma_\emptyset}$ has objects $m \in \mathbb{N}$ and morphisms $\mathbb{L}_{\Sigma_0}(m,n) = Set(n,m)$

as $m \longrightarrow_{\mathbb{L}_{\Sigma_\emptyset}} n$ corresponds to $n \longrightarrow_{Set} \underbrace{\Sigma^*_\emptyset(m)}_{= m}$

$\langle x_{i_1}, \dots, x_{i_n} \rangle$ with $\# \{ i_1, \dots, i_n \} \leq m$

Thus, $\mathbb{L}_{\Sigma_\emptyset} \cong \mathbb{F}^{op}$

<u>Exercise:</u> Let $F : \mathscr{C} \to \mathscr{D}$. Suppose F is essentially surjective on objects and fully faithful.

Define $G : \mathscr{D} \to \mathscr{C}$ by: for $D \in \mathscr{D}$, $GD := C$ st. $\varepsilon_D : FC \cong D$.

For $f : D \to D'$,

$$Gf : \begin{array}{ccc} GD & \xrightarrow{g} & GD' \\ \| & & \| \\ C & & C' \end{array}$$

where $g : C \to C'$ is the unique morphism st $Fg = f : D \to D'$.

If $D \xrightarrow{f} D' \xrightarrow{f'} D''$ then

$$\begin{array}{ccccc} GD & \xrightarrow{g} & GD' & \xrightarrow{g'} & GD'' \\ \| & & & & \| \\ C & & & & C' \end{array}$$

and there is a unique morphism st $F(g' \circ g) = f' \circ f$.

Identity is the same way.

We have $\eta_A : GFA \xrightarrow{\sim} A$ as $FA \cong FA$

and $\varepsilon_B : FGB \xrightarrow{\sim} B$.

Both are natural as

$$\begin{tikzcd} FGB \arrow[r, "\sim"] \arrow[d, "FGf"'] & B \arrow[d, "f"] \\ FGB' \arrow[r, "\sim"'] & B' \end{tikzcd} \qquad \begin{tikzcd} GFA \arrow[r, "\sim"] \arrow[d, "f"'] & A \arrow[d, "f"] \\ GFA' \arrow[r, "\sim"'] & A' \end{tikzcd}$$

thus F is an equivalence of category.

Exercise whiskering & interchange (from lecture #1)

$$\begin{tikzcd} \mathcal{A} \arrow[r, "X"] & \mathcal{B} \arrow[r, bend left, "Y"] \arrow[r, bend right, "Y'"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{C} \arrow[r, "Z"] & \mathcal{D} \end{tikzcd}$$

Define $\alpha \circ_0 X : Y \circ X \Longrightarrow Y' \circ X$

for $A \in \mathcal{A}$, $YXA \longrightarrow Y'XA$

$(\alpha \circ_0 X)_A := \alpha_{XA}$

$$\begin{tikzcd} YXA \arrow[r, "\alpha_{XA}"] \arrow[d, "YXf"'] \arrow[dr, phantom, "nat"] & Y'XA \arrow[d, "Y'Xf"] \\ YXA' \arrow[r, "\alpha_{XA'}"'] & Y'XA' \end{tikzcd}$$

Naturality is by naturality of α and functor X.

Define $Z \circ_0 \alpha : Z \circ Y \Longrightarrow Z \circ Y'$

for $B \in \mathcal{B}$, $ZYB \longrightarrow ZY'B$

$(Z \circ_0 \alpha)_B := Z\alpha_B$

$$\begin{tikzcd} YB \arrow[r, "\alpha_B"] \arrow[d, "Yf"'] \arrow[dr, phantom, "nat"] & Y'B \arrow[d, "Zf"] \\ YB' \arrow[r, "\alpha_{B'}"'] & Y'B' \end{tikzcd}$$

apply Z

$$\begin{tikzcd} ZYB \arrow[r, "Z\alpha_B"] \arrow[d, "ZYf"'] \arrow[dr, phantom, "Z(nat)"] & ZY'B \arrow[d, "ZY'f"] \\ ZYB' \arrow[r, "Z\alpha_{B'}"'] & ZY'B' \end{tikzcd}$$

We have that

$$((\beta \circ_0 Y) \circ (Z \circ_0 \alpha))_B = (\beta \circ_0 Y')_B \circ (Z \circ_0 \alpha)_B$$
$$= \beta_{Y'B} \circ Z\alpha_B$$

and $$((Z' \circ_0 \alpha) \circ (\beta \circ_0 Y))_B = (Z' \circ_0 \alpha)_B \circ (\beta \circ_0 Y)_B$$
$$= Z'\alpha_B \circ \beta_{YB}$$

$$\begin{tikzcd} ZYB \arrow[r, "Z\alpha_B"] \arrow[d, "\beta_{YB}"'] \arrow[dr, phantom, "naturality"] & ZY'B \arrow[d, "\beta_{Y'B}"] \\ Z'YB \arrow[r, "Z'\alpha_B"'] & Z'Y'B \end{tikzcd}$$

and we conclude by naturality of β.

__Exercise__
$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\alpha \Downarrow"] & \mathscr{D} \end{tikzcd}$$
Show that if every $\alpha_C : FC \xrightarrow{\sim} GC$. then $\alpha : F \overset{\sim}{\Rightarrow} G$

Define $\beta : G \Longrightarrow F$ by $\beta_C := \alpha_C^{-1}$

We have

$$\begin{tikzcd} GC \arrow[r, "\beta_C = \alpha_C^{-1}"] \arrow[d, "Gf"'] & FC \arrow[d, "Ff"] \\ GC' \arrow[r, "\beta_{C'} = \alpha_{C'}^{-1}"'] & FC' \end{tikzcd}$$
by naturality of $\alpha : F \Rightarrow G$.

Then, $\alpha \circ \beta : G \Longrightarrow G$ is, on object C, $(\alpha \circ \beta)_C = \alpha_C \circ \alpha_C^{-1} = id_C$

and $\beta \circ \alpha : F \Longrightarrow F$ is, on object C, $(\beta \circ \alpha)_C = \alpha_C^{-1} \circ \alpha_C = id_C$,

thus $\alpha \circ \beta = id : G \Rightarrow G$ and $\beta \circ \alpha = id : F \Rightarrow F$.

We can conclude that $\alpha : F \overset{\sim}{\Rightarrow} G$.

__Exercise__ If F is fully faithful, then it reflects isomorphisms.

Take $f : C \longrightarrow C'$ in $\mathscr{C}$

such that $Ff : FC \longrightarrow FC'$ is an isomorphism.

Then, we know by fully faithfullness, there is a unique $g : C \longrightarrow C'$

such that $Fg : FC' \longrightarrow FC$.

Now, $F(f \circ g) = Ff \circ Fg = id_{FC'} = F id_{C'}$

and by fully faithfullness, $f \circ g = id_{c'}$.

Similarly, $F(g \circ f) = F id_c$ thus $g \circ f = id_c$.

We can conclude that f is an isomorphism.

Going back to Lawvere theories

- $\mathbb{F}^{op}$ is initial in Law
- Def. A model of $\mathbb{L}$ is a finite product preserving $\mathbb{L} \to Set$.

  It forms a full subcategory $\underbrace{[\mathbb{L}, Set]_{fp}}_{\text{finite product preserving}} \hookrightarrow [\mathbb{L}, Set]$.

<u>Rk</u> $M : \mathbb{L} \to Set$ a model

then $M(n) = M(1^n) = (\underbrace{M(1)}_{\text{carrier}})^n$

<u>Recap</u> A model is uniquely defined by a set X and maps $X^n \to X$.

Interlude : adjuctions

Prop. For any T-Algebra $a : TA \to A$, the map

$$T\text{-}Alg(TC, A) \xrightarrow{U} \mathcal{C}(TC, A) \xrightarrow{\mathcal{C}(\eta_C, A)} \mathcal{C}(C, A)$$

is bijective, i.e.

$$\begin{tikzcd} C \arrow[r, "\eta_C"] \arrow[dr, "f"'] & TC \arrow[d, dashed, "\tilde{f}"] \\ & A \end{tikzcd}$$

Adjunction : a functor $F : \mathcal{A} \longrightarrow \mathcal{C}$ with

- $\forall C$, $F_0(C)$ exists (we do not assume F_0 is a functor)
- $\eta_C : C \longrightarrow UF_0 C$

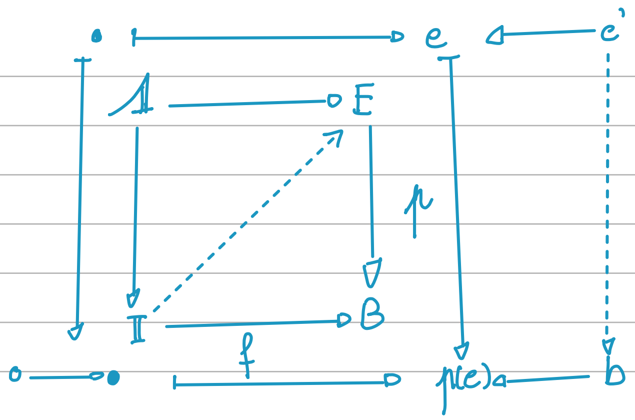
such that, for all $A \in \mathcal{A}$, and $f : C \longrightarrow UA$, there is a unique $\tilde{f} : F_0 C \longrightarrow$ such that

$$\begin{tikzcd} C \arrow[r, "\eta_C"] \arrow[dr, "f"'] & UF_0 C \arrow[d, dashed, "U\tilde{f}"] \\ & UA \end{tikzcd}$$

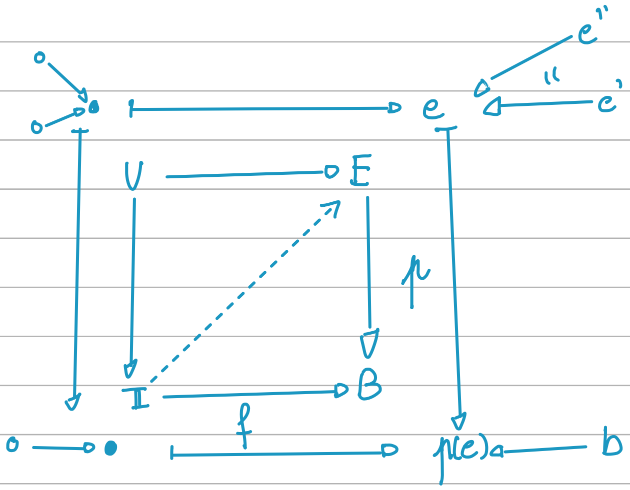
Lecture #4 16/12/2025

<u>Exercise</u> : A discrete fibration is a functor $p : E \to B$ such that any morphism $f : b \to p(e)$ has a unique antecedent $e' \to e$.

$$J^{\pitchfork} = \left\{ f \;\middle|\; \forall u, v,\ \forall j \in J,\ \exists l,\ \begin{tikzcd} \cdot \arrow[r, "u"] \arrow[d, "j"'] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r, "v"'] \arrow[ur, dashed, "l"] & \cdot \end{tikzcd} \right\}$$



but we don't have unicity of $e \to e'$, so we add $V \to \mathbb{I}$ to $\mathcal{J}$.



$\mathcal{J} = \{ \bullet \mapsto (\circ \to \bullet),$
$\left(\begin{smallmatrix}\circ \searrow \\ \circ \nearrow\end{smallmatrix}\bullet\right) \mapsto (\circ \to \bullet) \}.$

Lawvere theories from monads

$\mathcal{K}\ell^T$ is the category with $ob(\mathcal{K}\ell^T) = ob(\mathcal{C})$
and $\mathcal{K}\ell^T(A, B) = T\text{-}\mathcal{A}lg(TA, TB)$.

Factorisation of the functor $F^T : \mathcal{C} \to T\text{-}\mathcal{A}lg$ as id on objects & fully faithful

$$\begin{tikzcd} \mathcal{C} \arrow[r, "io_T"] \arrow[rr, bend right, "F^T"'] & \mathcal{K}\ell^T \arrow[r, "ff_T"] & T\text{-}\mathcal{A}lg \end{tikzcd}$$

$$\mathbb{L}_\Sigma(m, n) = (\mathbb{L}_1(m, 1))^n = (\Sigma^*(m))^n = \mathcal{S}et(n, \Sigma^*(m)).$$

From Monads to Lawvere theories

Define $T^{\circ}_{\mathbb{L}}(X) = \sum_{n\in\mathbb{N}} \mathbb{L}(n,1)\times X^{n}$

... but

$$\begin{tikzcd} & 1 & \\ 2 \arrow[ur, "o"] \arrow[dr, "{[x,x]}"'] & & 1 \arrow[ul, "o\Delta"'] \arrow[dl, "x"] \\ & X & \end{tikzcd}$$

where $\Delta : 1 \to 2$

we want to identify

$(2, o, (x,x))$ and $(1, o\Delta, (x))$

$(m, h, v) \sim (n, k, w)$ when

$$\begin{tikzcd} & X \arrow[dl, "v"'] \arrow[dr, "w"] & \\ m & & n \arrow[ll, "f"] \end{tikzcd} \qquad \begin{tikzcd} m \arrow[rr, "f"] \arrow[dr, "v"'] & & n \arrow[dl, "w"] \\ & X & \end{tikzcd}$$

commute

Define $T_{\mathbb{L}}(X) = \left(\sum_{n\in\mathbb{N}} \mathbb{L}(n,1)\times X^{n}\right)/\sim$

written $\int^{n} \mathbb{L}(n,1)\times X^{n}$ __COEND__

$$\begin{tikzcd} \mathbb{F} \arrow[rr, hook] \arrow[dr, "{\mathbb{L}(-,1)}"'] & \arrow[d, phantom, "\Rightarrow \lambda_{\mathbb{L}}"] & Set \\ & Set \arrow[ur, "T_{\mathbb{L}}"'] & \end{tikzcd}$$

__Exercise.__ Every monad yield a Hom-based monad.

$$C \xrightarrow{\eta_C} TC$$

$$\begin{tikzcd} TC \arrow[r, "T\eta_C"] \arrow[dr, equal, "ax"] \arrow[dr, bend right=60, "\eta_C^{\dagger}"'] & TTC \arrow[d, "\mu_C"] \\ & TC \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "\eta_C"] \arrow[d, "f"'] \arrow[dr, phantom, "nat"] & TC \arrow[d, "Tf"] \\ TD \arrow[r, "\eta_{TD}"] \arrow[d, equal] \arrow[dr, phantom, "ax"] & TTD \arrow[dl, "\mu_D"] \\ TD & \end{tikzcd}$$

$$
\begin{tikzcd}
TC \arrow[r, "Tf"] \arrow[rr, bend left=40, "f^\dagger"] \arrow[rr, phantom, bend left=20, "\text{def}"] \arrow[dd, "T(g^\dagger f)"'] \arrow[ddr, phantom, "\text{def}"] & TTD \arrow[r, "\mu_D"] \arrow[d, "TTg"] \arrow[ddl, "Tg^\dagger"'] \arrow[dr, phantom, "\text{nat}"] & TD \arrow[d, "Tg"] \arrow[dd, bend left=60, "g^\dagger"] \\
 & TTTE \arrow[r, "\mu_{TE}"'] \arrow[dl, "T\mu_E"] \arrow[dr, phantom, "\text{monad}"] & TTE \arrow[d, "\mu_E"] \\
TTE \arrow[rr, "\mu_E"'] & & TE
\end{tikzcd}
$$

T(def)

def

Lecture # 5

06/01/2026

Let us show that $T_{\mathbb{L}}$ is a monad.

$\leadsto$ hom-based monads

$$
\begin{tikzcd}
\mathbb{F} \arrow[r, hook] \arrow[dr, "\mathbb{L}(-,1)"'] \arrow[r, phantom, shift right=6, "\overset{\lambda_{\mathbb{L}}}{\Rightarrow}"] & \mathrm{Set} \arrow[d, "T_{\mathbb{L}}"] \\
 & \mathrm{Set}
\end{tikzcd}
$$

$(g^\dagger \circ f)^\dagger = g^\dagger \circ f^\dagger$

Define the category $Mnd(\mathcal{C})$: $(F, \eta, \mu) \xrightarrow{\alpha} (G, \eta', \mu')$

iff $\alpha : F \Rightarrow G$ and

$$
\begin{tikzcd}
 & id_{\mathcal{C}} \arrow[dl, "\eta"'] \arrow[dr, "\eta'"] & \\
F \arrow[rr, "\alpha"'] & & G
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
FF \arrow[r, "\alpha \circ_0 \alpha"] \arrow[d, "\mu"'] & GG \arrow[d, "\mu'"] \\
F \arrow[r, "\alpha"'] & G
\end{tikzcd}
$$

$$\begin{array}{ccc}
T_{\mathbb{K}}\, X & \longrightarrow & T_{\mathbb{L}}\, X \\
\int^{n} \mathbb{K}(n,1) \times X^{n} & \longmapsto & \int^{n} \mathbb{L}(n,1) \times X^{n} \\
[n, h, v] & \longmapsto & [n, Fh, v]
\end{array}$$

1) check well defined :

if

$$\begin{tikzcd}
 & X & \\
n \arrow[ur, "v"] \arrow[rr, "f"'] & & p \arrow[ul, "w"'] \\
n \arrow[dr, "h"'] & & p \arrow[ll, "If"'] \arrow[dl, "k"] \\
 & 1 &
\end{tikzcd}$$

$[n, h, v] \sim [p, k, w]$

then

$$\begin{tikzcd}
 & X & \\
n \arrow[ur, "v"] \arrow[rr, "f"'] & & p \arrow[ul, "w"'] \\
n = Fn \arrow[dr, "Fh"'] & & Fp = p \arrow[ll, "FIf = Jf"'] \arrow[dl, "Fk"] \\
 & F1 = 1 &
\end{tikzcd}$$

$[n, Fh, v] \sim [p, Fk, w]$

as

$$\begin{tikzcd}
 & \mathbb{F}^{op} \arrow[dl, "I"'] \arrow[dr, "J"] & \\
\mathbb{K} \arrow[rr, "F"'] & & \mathbb{L}
\end{tikzcd}$$

$$n \xrightarrow{v} X \qquad n \xrightarrow{h} 1$$

2) check naturality

$$\begin{tikzcd}
{[n,h,v]} \atop {1 \xleftarrow{h} n \xrightarrow{v} X} \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & {[n,Fh,v]} \atop {1 \xleftarrow{Fh} n \xrightarrow{v} X} \arrow[ddd, mapsto] \\
 & T_{\mathbb{K}}(X) \arrow[r, "T_F(X)"] \arrow[d, "T_{\mathbb{K}}(f)"'] & T_{\mathbb{L}}(X) \arrow[d, "T_{\mathbb{L}}(f)"] & \\
 & T_{\mathbb{K}}(Y) \arrow[r, "T_F(Y)"'] & T_{\mathbb{L}}(Y) & \\
1 \xleftarrow{h} n \xrightarrow{v} X \xrightarrow{f} Y \atop {[n, h, f \circ v]} \arrow[rrr, mapsto] & & & 1 \xleftarrow{Fh} n \xrightarrow{v} X \xrightarrow{f} Y \atop {[n, Fh, f \circ v]}
\end{tikzcd}$$

the naturality square commute.

The simplex ⟨3⟩ with the spine in pink.