

# Assignment # 4

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Q1 We have, for  $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ , for  $n=2$ .

$$U[1] = U \otimes \mathbb{1} = \begin{pmatrix} a \mathbb{1}_2 & b \mathbb{1}_2 \\ c \mathbb{1}_2 & d \mathbb{1}_2 \end{pmatrix} = \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix}$$

and

$$U[2] = \mathbb{1} \otimes U = \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} = \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{pmatrix}.$$

For  $n=3$ , we have

$$U[1] = U \otimes (\overbrace{\mathbb{1}_2 \otimes \mathbb{1}_2}^{\mathbb{1}_4}) = \begin{pmatrix} a & 0 & 0 & 0 & b & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & a & 0 & 0 & 0 & b \\ c & 0 & 0 & 0 & d & 0 & 0 & 0 \\ 0 & c & 0 & 0 & 0 & d & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 & d & 0 \\ 0 & 0 & 0 & c & 0 & 0 & 0 & d \end{pmatrix}$$

and

$$U[2] = \mathbb{1}_2 \otimes (\overbrace{U \otimes \mathbb{1}_2}^{U[1]_{n=1}}) = \begin{pmatrix} a & 0 & b & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & b & 0 & 0 & 0 & 0 \\ c & 0 & d & 0 & 0 & 0 & 0 & 0 \\ 0 & c & 0 & d & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & a & 0 & b \\ 0 & 0 & 0 & 0 & c & 0 & d & 0 \\ 0 & 0 & 0 & 0 & 0 & c & 0 & d \end{pmatrix}.$$

For CNOT [3, 1] with  $n=3$ , we have

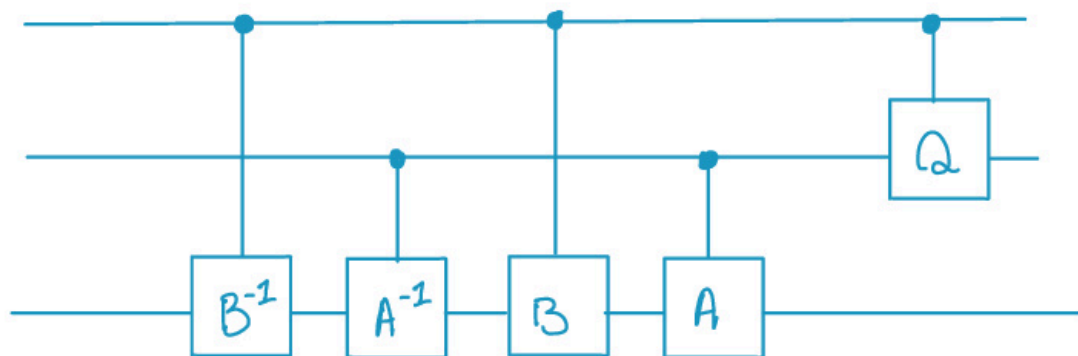
$$\text{CNOT}[3, 1] |x y 0\rangle = |x y 0\rangle \quad x, y \in \{0, 1\}$$

$$\text{CNOT}[3, 1] |x y 1\rangle = |\bar{x} y 1\rangle$$

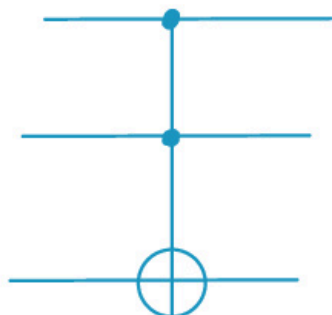
Thus,

$$\text{CNOT}[3, 1] = \begin{matrix} & |000\rangle & |001\rangle & |010\rangle & |011\rangle & |100\rangle & |101\rangle & |110\rangle & |111\rangle \\ \begin{matrix} |000\rangle \\ |001\rangle \\ |010\rangle \\ |011\rangle \\ |100\rangle \\ |101\rangle \\ |110\rangle \\ |111\rangle \end{matrix} & \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \end{matrix}$$

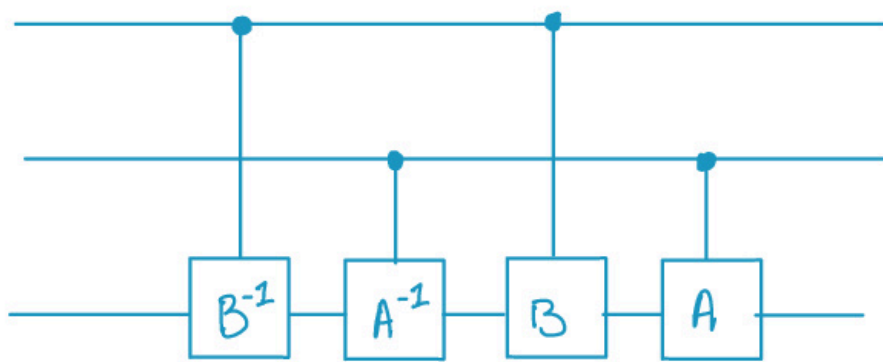
Q2. We define  $Q := \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}$ . We have that:



is equivalent to the Toffoli gate  $\text{CCNOT}[1, 2, 3]$ :



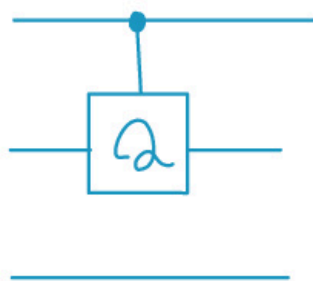
To see that, we can compute the unitary matrix of



and get the following matrix:

$$U = \begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 0 & i \\ & & & & & & & i & 0 \end{pmatrix}.$$

The unitary matrix of



is the diagonal matrix

$$B = \text{diag}(1, \dots, 1, -i, -i).$$

Thus the unitary matrix for the whole circuit is:

$$CNOT = \begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & 1 & & & \\ & & & & 1 & & \\ & & & & & 1 & \\ & 0 & & & & & 0 \\ & & & & & & 1 \\ & & & & & 0 & 1 \\ & & & & & 1 & 0 \end{pmatrix}.$$

END of ASSIGNMENT #4