Assignment # 4

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Q1 We have, for $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, for $n = 2$,

$$U[1] = U \otimes \mathbb{1} = \begin{pmatrix} a\mathbb{1}_2 & b\mathbb{1}_2 \\ c\mathbb{1}_2 & d\mathbb{1}_2 \end{pmatrix} = \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix}$$

and

$$U[2] = \mathbb{1} \otimes U = \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} = \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{pmatrix}.$$

For $n = 3$, we have

$$U[1] = U \otimes (\overbrace{\mathbb{1}_2 \otimes \mathbb{1}_2}^{\mathbb{1}_4}) = \begin{pmatrix} a & 0 & 0 & 0 & b & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & a & 0 & 0 & 0 & b \\ c & 0 & 0 & 0 & d & 0 & 0 & 0 \\ 0 & c & 0 & 0 & 0 & d & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 & d & 0 \\ 0 & 0 & 0 & c & 0 & 0 & 0 & d \end{pmatrix}$$

and

$$U[2] = \mathbb{1}_2 \otimes \overbrace{(U \otimes \mathbb{1}_2)}^{U[1]_{n=1}} = \begin{pmatrix} a & 0 & b & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & b & 0 & 0 & 0 & 0 \\ c & 0 & d & 0 & 0 & 0 & 0 & 0 \\ 0 & c & 0 & d & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & a & 0 & b \\ 0 & 0 & 0 & 0 & c & 0 & d & 0 \\ 0 & 0 & 0 & 0 & 0 & c & 0 & d \end{pmatrix}.$$

For CNOT [3, 1] with $n = 3$, we have

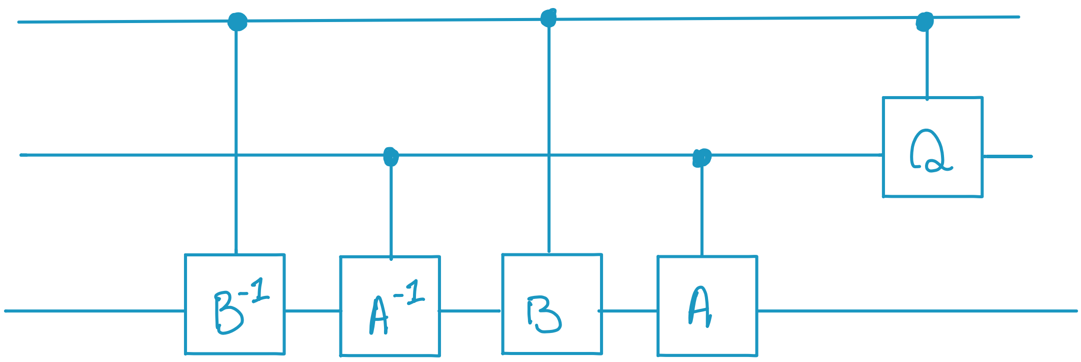
$$\text{CNOT}[3,1] \, |x \, y \, 0\rangle = |x \, y \, 0\rangle \qquad x, y \in \{0,1\}$$

$$\text{CNOT}[3,1] \, |x \, y \, 1\rangle = |\bar{x} \, y \, 1\rangle$$

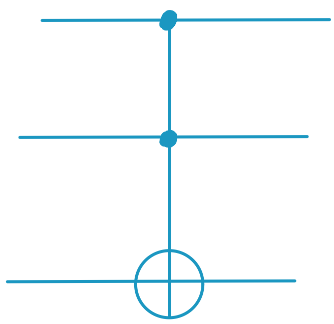
Thus,

$$\text{CNOT}[3,1] = \begin{array}{c} \\ |000\rangle \\ |001\rangle \\ |010\rangle \\ |011\rangle \\ |100\rangle \\ |101\rangle \\ |110\rangle \\ |111\rangle \end{array} \!\!\begin{array}{c} \begin{array}{cccccccc} |000\rangle & |001\rangle & |010\rangle & |011\rangle & |100\rangle & |101\rangle & |110\rangle & |111\rangle \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

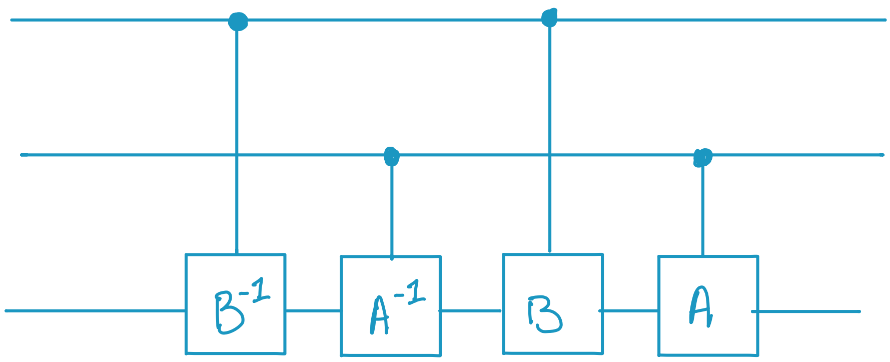
Q2. We define $Q := \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}$. We have that :



is equivalent to the Toffoli gate CCNOT [1, 2, 3] :



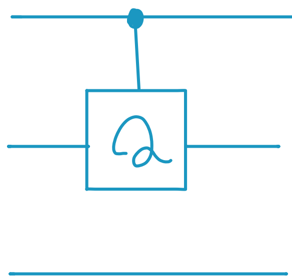
To see that, we can compute the unitary matrix of



and get the following matrix:

$$U = \begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & \text{\Large 0} & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & \text{\Large 0} & & & 1 & & \\ & & & & & & 0 & i \\ & & & & & & i & 0 \end{pmatrix}.$$

The unitary matrix of



is the diagonal matrix

$$B = \mathrm{diag}(1, \ldots, 1, -i, -i).$$

Thus the unitary matrix for the whole circuit is:

$$\mathrm{CNOT} = \begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & 0 & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & 0 & & & & 1 & & \\ & & & & & & 0 & 1 \\ & & & & & & 1 & 0 \end{pmatrix}.$$

END of ASSIGNMENT #4