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Proof & Programs

TD n° 5.

Exercise 1.

We will show DNE implies EM:

$$
\dfrac{
  \dfrac{
    \dfrac{}{\neg(A\vee\neg A)\vdash\neg(A\vee\neg A)}\,ax
    \qquad
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{}{\Gamma\vdash\neg(A\vee\neg A)}\,ax
          \qquad
          \dfrac{\dfrac{}{\Gamma\vdash A}\,ax}{\Gamma\vdash A\vee\neg A}\,\vee i\ell
        }{\Gamma := \neg(A\vee\neg A), A\vdash\bot}
      }{\neg(A\vee\neg A)\vdash\neg A}\,\neg i
    }{\neg(A\vee\neg A)\vdash A\vee\neg A}\,\vee i r
  }{\dfrac{\neg(A\vee\neg A)\vdash\bot}{\vdash\neg\neg(A\vee\neg A)}\,\neg i}\,\neg e
  \qquad
  \overbrace{\vdash\neg\neg(A\vee\neg A)\Rightarrow A\vee\neg A}^{\text{DNE}}
}{\vdash A\vee\neg A}\,\Rightarrow_E
$$

we can assume the context is empty by weakening

We will show CP implies DNE:

$$
\dfrac{
  \overbrace{\vdash(\neg A\Rightarrow\neg\neg\neg A)\Rightarrow\neg\neg A\Rightarrow A}^{\text{CP}}
  \qquad
  \dfrac{
    \dfrac{
      \dfrac{}{\neg A,\neg\neg A\vdash\neg A}\,ax
      \qquad
      \dfrac{}{\neg A,\neg\neg A\vdash\neg\neg A}\,ax
    }{\neg A,\neg\neg A\vdash\bot}\,\neg_E
  }{\vdash\neg A\Rightarrow\neg\neg\neg A}\,\Rightarrow_I,\neg i
}{\vdash\neg\neg A\Rightarrow A}\,\Rightarrow_E
$$

We will show EM implies CP:

$$\dfrac{\dfrac{\Gamma \vdash \overbrace{A \vee \neg A}^{\text{EM}} \qquad \dfrac{}{\Gamma, A \vdash A}\,\text{ax} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, \neg A \vdash \neg A \Rightarrow \neg B}\,\text{ax} \quad \dfrac{}{\Gamma, \neg A \vdash \neg A}\,\text{ax}}{\Gamma, \neg A \vdash \neg B}\,{\Rightarrow_E} \quad \dfrac{}{\Gamma, \neg A \vdash B}\,\text{ax}}{\Gamma, \neg A \vdash \bot}\,\neg_E}{\Gamma, \neg A \vdash A}\,\bot_E}{\Gamma := \neg A \Rightarrow \neg B,\ B \vdash A}\,\vee_E}{\vdash (\neg A \Rightarrow \neg B) \Rightarrow B \Rightarrow A}\,{\Rightarrow_I} \times 2$$

Thus EM, DNE and CP are equivalent.

We have that DNE implies PL:

$$\dfrac{\dfrac{(A \Rightarrow B) \Rightarrow A \vdash \overbrace{\neg\neg A \Rightarrow A}^{\text{DNE}} \qquad \dfrac{\dfrac{\dfrac{}{\Gamma \vdash \neg A}\,\text{ax} \quad \dfrac{\dfrac{}{\Gamma \vdash (A \Rightarrow B) \Rightarrow A}\,\text{ax} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, A \vdash A}\,\text{ax} \quad \dfrac{}{\Gamma, A \vdash \neg A}\,\text{ax}}{\Gamma, A \vdash \bot}\,\neg_E}{\Gamma, A \vdash B}\,\bot_E}{\Gamma \vdash A \Rightarrow B}\,{\Rightarrow_I}}{\Gamma \vdash A}\,{\Rightarrow_E}}{\Gamma := (A \Rightarrow B) \Rightarrow A,\ \neg A \vdash \bot}\,\neg_E}{(A \Rightarrow B) \Rightarrow A \vdash \neg\neg A}\,\neg_i}{(A \Rightarrow B) \Rightarrow A \vdash A}\,{\Rightarrow_E}}{\vdash ((A \Rightarrow B) \Rightarrow A) \Rightarrow A}\,{\Rightarrow_I}$$

Finally, we prove that PL implies DNE:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, A \vdash A \Rightarrow \bot$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, A \vdash A$}
\RightLabel{$\Rightarrow_E$}
\BinaryInfC{$\Gamma, A \vdash \bot$}
\RightLabel{$\neg_I$}
\UnaryInfC{$\Gamma \vdash \neg A$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash \neg\neg A$}
\RightLabel{$\neg_e$}
\BinaryInfC{$\Gamma := A \Rightarrow \bot, \neg\neg A \vdash \bot$}
\RightLabel{$\bot_E$}
\UnaryInfC{$A \Rightarrow \bot, \neg\neg A \vdash A$}
\RightLabel{$\Rightarrow_I$}
\UnaryInfC{$\neg\neg A \vdash (A \Rightarrow \bot) \Rightarrow A$}
\AxiomC{$\neg\neg A \vdash \overbrace{((A \Rightarrow \bot) \Rightarrow A) \Rightarrow A}^{PL}$}
\RightLabel{$\Rightarrow_E$}
\BinaryInfC{$\neg\neg A \vdash A$}
\RightLabel{$\Rightarrow_I$}
\UnaryInfC{$\vdash \neg\neg A \Rightarrow A$}
\end{prooftree}
$$

Thus all the rules are equivalent.
However, neither of these rules are provable, in general, for NJ (as $\vdash A \vee B$ implies either $\vdash A$ or $\vdash B$ and we apply it to EM).

$$
\begin{tikzcd}
 & PL \arrow[dl, Leftrightarrow] & \\
DNE \arrow[rr, Rightarrow] & & EM \arrow[dll, Rightarrow] \\
CP \arrow[u, Rightarrow] & &
\end{tikzcd}
$$

Exercise 2.

We have:

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$P \vdash P, Q$}
\UnaryInfC{$P \vdash Q, P$}
\UnaryInfC{$\vdash P \Rightarrow Q, P$}
\UnaryInfC{$\vdash P, P \Rightarrow Q$}
\UnaryInfC{$\vdash P \vee (P \Rightarrow Q)$}
\end{prooftree}
$$

and:

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$\neg A \Rightarrow A \vdash \neg A \Rightarrow A, A$}
\AxiomC{}
\UnaryInfC{$\neg A \Rightarrow A, A \vdash A, \bot$}
\UnaryInfC{$\neg A \Rightarrow A, A \vdash \bot, A$}
\UnaryInfC{$\neg A \Rightarrow A \vdash \neg A, A$}
\BinaryInfC{$\neg A \Rightarrow A \vdash A, A$}
\UnaryInfC{$\neg A \Rightarrow A \vdash A$}
\UnaryInfC{$\vdash (\neg A \Rightarrow A) \Rightarrow A$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$\neg(P\Rightarrow Q), P \vdash P, \bot$}
\UnaryInfC{$\neg(P\Rightarrow Q), P \vdash \bot, P$}
\UnaryInfC{$\neg(P\Rightarrow Q), \vdash \neg P, P$}
\UnaryInfC{$\neg(P\Rightarrow Q) \vdash P, \neg P$}
\UnaryInfC{$\neg(P\Rightarrow Q) \vdash P \vee \neg P$}
\AxiomC{}
\UnaryInfC{$\neg(P\Rightarrow Q), P \vdash P$}
\AxiomC{}
\UnaryInfC{$\Gamma \vdash \neg(P\Rightarrow Q)$}
\AxiomC{}
\UnaryInfC{$\Gamma, P \vdash \neg P$}
\AxiomC{}
\UnaryInfC{$\Gamma, P \vdash P$}
\BinaryInfC{$\Gamma, P \vdash \bot$}
\UnaryInfC{$\Gamma, P \vdash Q$}
\UnaryInfC{$\Gamma \vdash P\Rightarrow Q$}
\BinaryInfC{$\Gamma := \neg(P\Rightarrow Q), \neg P \vdash \bot$}
\UnaryInfC{$\neg(P\Rightarrow Q), \neg P \vdash P$}
\TrinaryInfC{$\neg(P\Rightarrow Q) \vdash P$}
\AxiomC{}
\UnaryInfC{$\neg(P\Rightarrow Q), Q \vdash \neg(P\Rightarrow Q)$}
\AxiomC{}
\UnaryInfC{$\neg(P\Rightarrow Q), Q, P \vdash Q$}
\UnaryInfC{$\neg(P\Rightarrow Q), Q \vdash P\Rightarrow Q$}
\BinaryInfC{$\neg(P\Rightarrow Q), Q \vdash \bot$}
\UnaryInfC{$\neg(P\Rightarrow Q) \vdash \neg Q$}
\BinaryInfC{$\neg(P\Rightarrow Q) \vdash P \wedge \neg Q$}
\UnaryInfC{$\vdash \neg(P\Rightarrow Q) \Rightarrow P \wedge \neg Q$}
\end{prooftree}
$$

proof of the excluded middle in NC.

and finally we have by ex 1 that EM $\Rightarrow$ CP and we know NJ $\subseteq$ NC thus we simply have to prove EM:

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$P \vdash P, \bot$}
\UnaryInfC{$P \vdash \bot, P$}
\UnaryInfC{$\vdash \neg P, P$}
\UnaryInfC{$\vdash P, \neg P$}
\UnaryInfC{$\vdash P \vee \neg P$}
\end{prooftree}
$$

Exercise 3.

Q1. If we have shown that every rule of NC, after applying $(\cdot)^{Ko}$ is admissible, then a simple induction on the proof of $\Gamma \vdash_{NC} A, \Delta$ shows that $\Gamma^{Ko}, \neg\Delta^{Ko} \vdash_{NJ} A^{Ko}$.

For example, if we are in the situation

$$
\begin{prooftree}
\AxiomC{π}
\noLine
\UnaryInfC{$\Gamma \vdash_{NC} A, \Delta$}
\AxiomC{π'}
\noLine
\UnaryInfC{$\Gamma \vdash_{NC} B, \Delta$}
\RightLabel{$\wedge_i$}
\BinaryInfC{$\Gamma \vdash_{NC} A \wedge B, \Delta$}
\end{prooftree}
$$

Then by induction hypothesis, we have that

$$\Gamma^{K_0}, \neg\Delta^{K_0} \vdash_{NJ} A^{K_0} \quad \text{and} \quad \Gamma^{K_0}, \neg\Delta^{K_0} \vdash_{NJ} B^{K_0}$$

and by admissibility of $(\wedge_I)^{K_0}$, we have

$$\frac{\Gamma^{K_0}, \neg\Delta^{K_0} \vdash_{NJ} A^{K_0} \qquad \Gamma^{K_0}, \neg\Delta^{K_0} \vdash_{NJ} B^{K_0}}{\Gamma^{K_0}, \neg\Delta^{K_0} \vdash_{NJ} (A \wedge B)^{K_0}} (\wedge_I)^{K_0}.$$

Q2. For conjunction introduction, we have:

$$\dfrac{\dfrac{}{\Gamma^{K_0}, \neg\Delta^{K_0}, \neg(A^{K_0} \wedge B^{K_0}) \vdash \neg(A^{K_0} \wedge B^{K_0})}\text{ax} \qquad \dfrac{\dfrac{\Gamma^{K_0}, \neg\Delta^{K_0} \vdash A^{K_0} \quad \Gamma^{K_0}, \neg\Delta^{K_0} \vdash B^{K_0}}{\Gamma^{K_0}, \neg\Delta^{K_0} \vdash A^{K_0} \wedge B^{K_0}}\wedge_i}{\Gamma^{K_0}, \neg\Delta^{K_0}, \neg(A^{K_0} \wedge B^{K_0}) \vdash A^{K_0} \wedge B^{K_0}}\text{wk}}{\dfrac{\Gamma^{K_0}, \neg\Delta^{K_0}, \neg(A^{K_0} \wedge B^{K_0}) \vdash \bot}{\Gamma^{K_0}, \neg\Delta^{K_0} \vdash \neg\neg(A^{K_0} \wedge B^{K_0})}\neg_i}\neg_e$$

for $\wedge E1$, $\wedge E2$, $\top I$, $\bot E$ we use the similar trick to remove the $\neg\neg$ in the conclusion with $\neg_i$ followed by $\neg e$, and we conclude by applying the NJ rule after a weakening.

For exchange, we have: $B^{K_0} = \neg\neg\varphi$

$$
\dfrac{
\dfrac{}{\Psi \vdash \neg A^{K_0}}\,\text{ax}
\quad
\dfrac{
\dfrac{\dfrac{\Gamma^{K_0}, \neg\Delta^{K_0}, \neg B^{K_0}, \neg\Sigma^{K_0} \vdash A^{K_0}}{\Psi, \neg B^{K_0} \vdash A^{K_0}}\,\text{wk \& xch}}{\Psi \vdash \neg\neg\neg\varphi \Rightarrow A^{K_0}}\,{\Rightarrow_I}
\quad
\dfrac{\dfrac{\dfrac{}{\Psi, \neg\neg\varphi \vdash \neg\neg\varphi}\,\text{ax} \quad \dfrac{}{\Psi, \neg\neg\varphi \vdash \neg\varphi}\,\text{ax}}{\Psi, \neg\neg\varphi}\,\neg e}{\Psi \vdash \neg\neg\neg\varphi}\,\neg i
}{\Psi \vdash A^{K_0}}
}{
\dfrac{
\dfrac{\Psi := \Gamma^{K_0}, \neg(\Delta \cup \Sigma)^{K_0}, \neg A^{K_0}, \neg\varphi \vdash \bot}{\Gamma^{K_0}, \neg(\Delta \cup \Sigma)^{K_0}, \neg A^{K_0} \vdash B^{K_0}}\,\neg i
}{\Gamma^{K_0}, \neg\Delta^{K_0}, \neg A^{K_0}, \neg\Sigma^{K_0} \vdash B^{K_0}}
}\,\neg e
$$

For contraction:

$$\dfrac{\Gamma \vdash \Delta', A, A, \Sigma}{\Gamma \vdash \Delta', A, \Sigma}$$

we have if $\Delta' = B, \Delta$: $\dfrac{\Gamma^{K_0}, \neg\Delta^{K_0}, \neg A^{K_0}, \neg A^{K_0}, \neg\Sigma^{K_0} \vdash B^{K_0}}{\Gamma^{K_0}, \neg\Delta^{K_0}, \neg A^{K_0}, \neg\Sigma^{K_0} \vdash B^{K_0}}$

and if $\Delta' = \cdot$, we rewrite A^{K_0} as $\neg\neg\varphi$, adding $\neg\varphi$ into the hypothesis, which is equivalent to $\neg\neg\varphi = \neg A^{K_0}$, thus we conclude by structural rules.

Exercise 4.

First, by induction on A, we show

$$A^{K_0} \vdash A \qquad \text{and} \qquad A \vdash A^{K_0}.$$

If $A = X$ then

$$\dfrac{\dfrac{\text{see ex 2}}{\neg\neg X \vdash X \cup \neg X} \qquad \dfrac{}{\neg\neg X, X \vdash X}\,ax \qquad \dfrac{\dfrac{\dfrac{}{\cdots \vdash \neg X}\,ax \quad \dfrac{}{\cdots \vdash \neg\neg X}\,ax}{\neg\neg X, \neg X \vdash \bot}}{\neg\neg X, \neg X \vdash X}\,\bot_E}{\neg\neg X \vdash X}\,\vee e \qquad (*)$$

and

$$\dfrac{\dfrac{\dfrac{}{\cdots \vdash X}\,ax \quad \dfrac{}{\cdots \vdash \neg X}\,ax}{X, \neg X \vdash \bot}\,\neg e}{X \vdash \neg\neg X}\,\neg i$$

If $A = B \wedge C$ then, by ih, $\neg\neg B^{K_0} \vdash B$ and $C^{k_0} \vdash C$,
So $\vdash \neg\neg B^{K_0} \Rightarrow B$ by reversibility of $\Rightarrow_I$ (and same for C).

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\cdots \vdash B^{k_0} \wedge C^{k_0}}\,ax}{\cdots \vdash B^{k_0}}\,\wedge ie \quad \dfrac{}{\cdots \vdash \neg B^{K_0}}\,ax}{\gamma, \neg(B \wedge C), \neg B^{K_0}, B^{K_0} \wedge C^{k_0} \vdash \bot}\,\neg e}{\gamma, \neg(B \wedge C), \neg B^{K_0} \vdash \neg\gamma'}\,\neg i$$

$$\dfrac{\dfrac{\vdash \neg\neg B^{K_0} \Rightarrow B}{\gamma, \neg(B \wedge C) \vdash \neg\neg B^{k_0} \Rightarrow B} \qquad \dfrac{\dfrac{\gamma, \neg(B \wedge C), \neg B^{K_0} \vdash \neg\gamma' \quad \dfrac{}{\cdots \vdash \neg\neg\gamma' = \gamma}\,ax}{\gamma, \neg(B \wedge C), \neg B^{k_0} \vdash \bot}}{\gamma, \neg(B \wedge C) \vdash \neg\neg B^{K_0}}}{\gamma, \neg(B \wedge C) \vdash B}\,\Rightarrow_E \qquad \text{similar}$$

$$\dfrac{\dfrac{\text{ex 2}}{\gamma \vdash (B \wedge C) \cup \neg(B \wedge C)} \qquad \dfrac{}{\gamma, B \wedge C \vdash B \wedge C}\,ax \qquad \dfrac{\cdots}{\gamma, \neg(B \wedge C) \vdash B \wedge C}\,\wedge i}{\underbrace{\neg\neg(B^{k_0} \wedge C^{K_0})}_{\gamma = \neg\neg\gamma'} \vdash B \wedge C}$$

The use of EM turned out to be useless in this case.

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\underbrace{\dfrac{\vdash B^{k_0} \Rightarrow B}{\Gamma \vdash B^{k_0} \Rightarrow B}}_{\text{by ih \& reversibility of } \Rightarrow I}
\qquad
\dfrac{\dfrac{}{\Gamma \vdash B \wedge C}\,\text{ax}}{\Gamma \vdash B}\,\wedge e l
}{\Gamma \vdash B^{k_0}}
\qquad
\dfrac{\text{similarly}}{\Gamma \vdash C^{k_0}}
}{\Gamma \vdash B^{k_0} \wedge C^{k_0}}\,\wedge i
}{\Gamma := B \wedge C,\ \neg(B^{k_0} \wedge C^{k_0}) \vdash \bot}\,\neg e
}{B \wedge C \vdash \neg\neg(B^{k_0} \wedge C^{k_0})}\,\neg i
$$

All the other cases are similar.