TD n° 1

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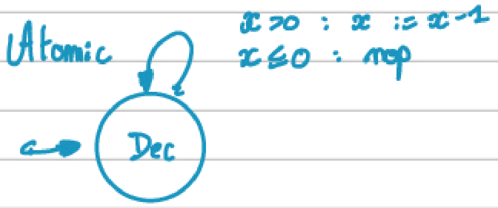
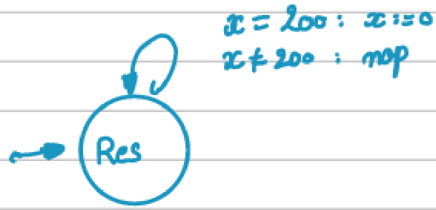
Modelling Concurrent Systems

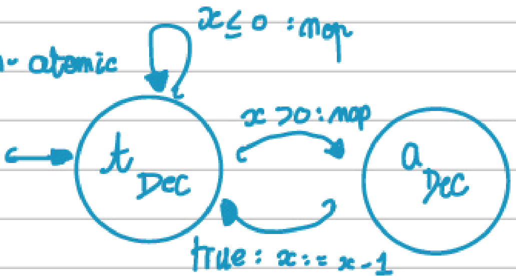
Exercise 1.

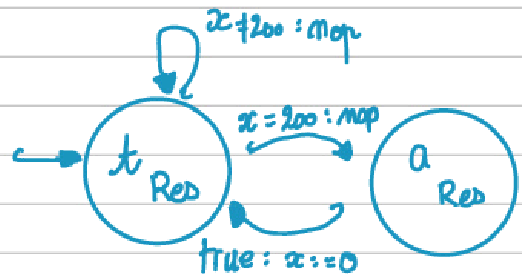
Q1. Atomic





Non-atomic



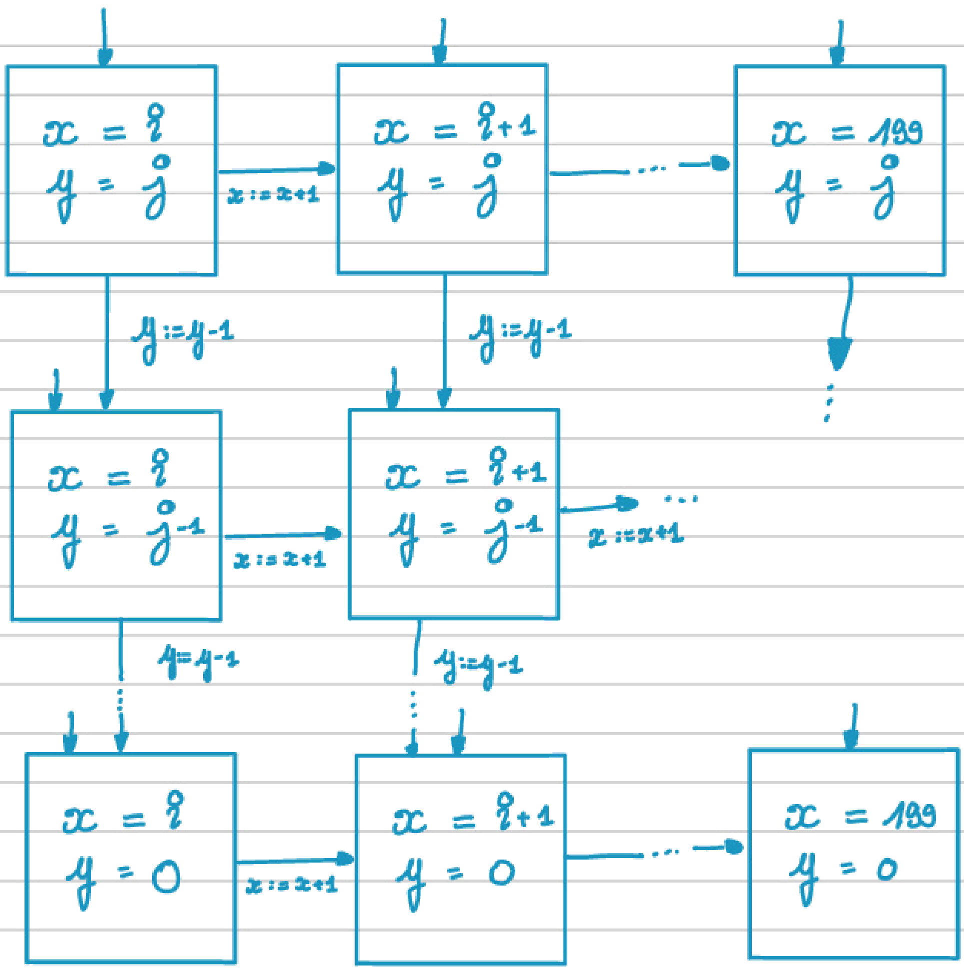


Q2. $TS(A_{Inc}) \,|||\, TS(A_{Dec})$

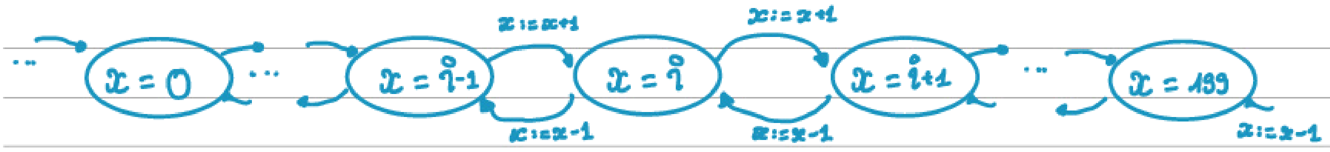
On déduit la mémoire avec $TS(A_{Inc}) ||| TS(A_{Dec})$.

$\forall i \; \forall j$

$\boxed{\begin{array}{l} x = i \\ y = j \end{array}}$



TS($A_{Inc} \,|||\, A_{Dec}$) :

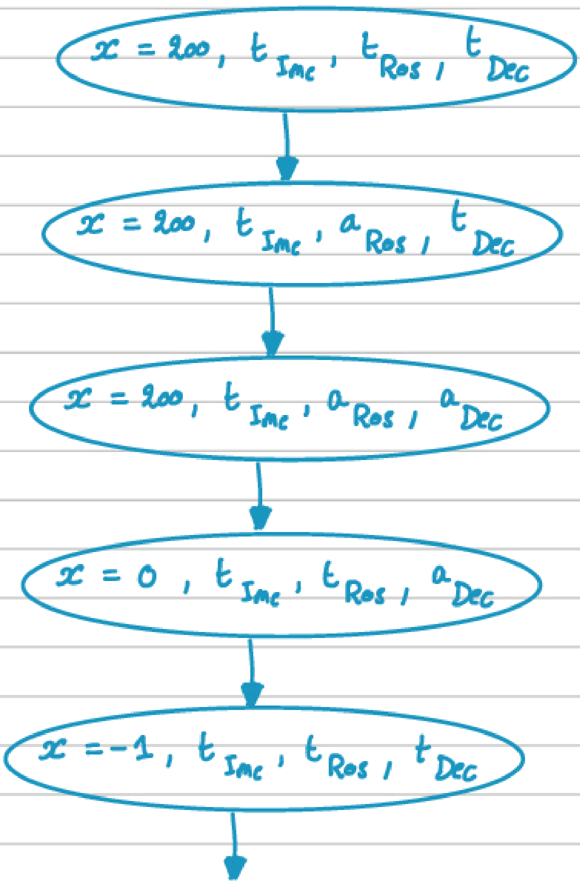


Q3. Les effets préservent l'invariant :

- pour $x := x+1$ si $x < 200$ et $0 \leq x \leq 200$ alors $0 \leq x+1 \leq 200$
- pour $x := x-1$ si $x > 0$ et $0 \leq x \leq 200$ alors $0 \leq x-1 \leq 200$
- pour $x := 0$ si $x = 200$ et $0 \leq x \leq 200$ alors $0 \leq 0 \leq 200$

D'où l'invariant est invariant.

Q4.

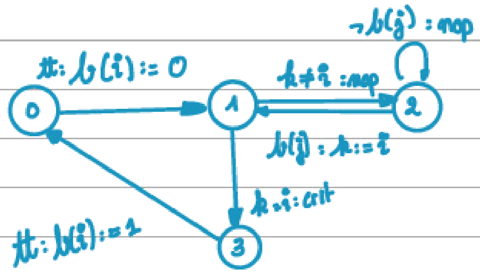


Kaboom ! on a cassé l'invariant!

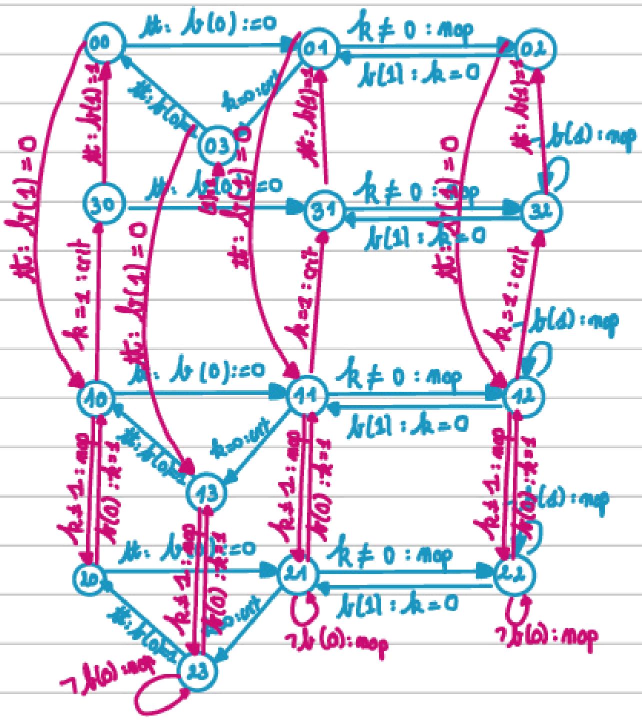
Exercice 2.

Q1.

Q1.



Q2.



Q3. The state 33 is unreachable. Thus, we ensure mutual exclusion.

Exercise 3.

In ex 1 / Q2,

we have a state $x = 1$, $x' = 0$

we don't have that in

$TS(PG_1 ||| PG_2)$.

TD n° 2

Linear Time Properties (and a bit of Modelling)

I. Safeness and Invariance

Exercise 1. Invariance is safe.

Define $P_{bad} := \{ \hat{\sigma} \in (2^{AP})^* \mid \exists i.\ \hat{\sigma}(i) \not\models \varphi \}$.

Then,

$$\begin{aligned}
\sigma \in P &\iff \forall i,\quad \sigma(i) \models \varphi \\
&\iff \forall \hat{\sigma} \leq_{fini} \sigma,\quad \forall i \leq \text{length } \hat{\sigma},\quad \hat{\sigma}(i) \models \varphi \\
&\iff \forall \hat{\sigma} \leq_{fini} \sigma,\quad \text{non}(\exists i.\ \hat{\sigma}(i) \not\models \varphi) \\
&\iff \forall \hat{\sigma} \leq_{fini} \sigma,\quad \hat{\sigma} \notin P_{bad}.
\end{aligned}$$

Thus P is a safety property.

Exercise 2

1. $\emptyset$ inv. safety
2. $\{\sigma \mid \forall i, \sigma(i) \models x = 0\}$ inv. safety
3. $\{\sigma \mid \forall i.\ \sigma(i) \models \neg(x=0) \wedge \neg(x>1)\} = \{\emptyset\}^\omega$ inv. safety
4. $\{\sigma \mid \sigma(0) \models x = 0\}$ safety $P_{bad} := \{\hat{\sigma} \mid \hat{\sigma}(0) \not\models x = 0\}$.
5. $\{\sigma \mid \sigma(0) \models \neg(x=0)\}$ safety $P_{bad} := \{\hat{\sigma} \mid \hat{\sigma}(0) \models x = 0\}$
6. $\{\sigma \mid \sigma(0) \models x=0$ and $\exists i.\ \sigma(i) \models x > 1\}$ neither
7. $\{\sigma \mid \exists N\ \forall i \geq N,\ \sigma(i) \models \neg(x > 1)\}$ neither
8. $\{\sigma \mid \forall N\ \exists i \geq N,\ \sigma(i) \models x > 1\}$ neither
9. $(2^{AP})^*$ inv. safety

II Operations on Safety Properties

Exercise 3. Characterization of safety properties

Q1. We have that

$$P \text{ safety property} \iff \exists P_{bad}, \forall \sigma, \ (\sigma \in P \iff \mathrm{Pref}\,\sigma \cap P_{bad} = \emptyset)$$
$$\iff \exists P_{bad}, \forall \sigma, \ (\sigma \in P^c \iff \mathrm{Pref}\,\sigma \cap P_{bad} \neq \emptyset)$$
$$(*) \iff \forall \sigma, \ (\sigma \in P^c \iff \mathrm{Pref}\,\sigma \cap P = \emptyset)$$

For $(*)$, "$\Leftarrow$" we take $P_{bad} := (2^{AP})^* \setminus P$ and we have the required property.

"$\Rightarrow$" We have $\sigma \in \mathrm{Pref}\,\sigma$ and we can conclude.

Q2. We show P safety $\iff$ $P \supseteq \mathrm{cl}\,P$. We always have $\mathrm{cl}\,P \supseteq P$

"$\Rightarrow$" Let $\sigma \in P^c$, we will show $\sigma \notin \mathrm{cl}\,P$.
i.e. $\mathrm{pref}\,\sigma \not\subseteq \mathrm{pref}\,P$

we have $\hat{\sigma} \in \mathrm{pref}\,\sigma$ and $\hat{\sigma} \notin \mathrm{pref}\,P$ by (1)

"$\Leftarrow$" Let $\sigma \in P^c$. we will show $\exists \hat{\sigma} \in \mathrm{pref}\,\sigma$ such that $\hat{\sigma} \notin \mathrm{pref}\,P$
i.e. $\mathrm{pref}\,\sigma \not\subseteq \mathrm{pref}\,P$.

Exercise 4. Union & intersections

Q1. For $\sigma \in (P \cup Q)^c = P^c \cap Q^c$, there exists $\hat{\sigma}_P \leq_{fini} \sigma$ st $\hat{\sigma}_P \cdot (2^{AP})^\omega \cap P = \emptyset$
and $\hat{\sigma}_Q \leq_{fini} \sigma$ st $\hat{\sigma}_Q \cdot (2^{AP})^\omega \cap Q = \emptyset$.

Let $\hat{\sigma} \leq \hat{\sigma}_P$ and $\hat{\sigma} \leq \hat{\sigma}_Q$. We have

$$\hat{\sigma} \cdot (2^{AP})^\omega \cap (P \cup Q) = (\hat{\sigma} \cdot (2^{AP})^\omega \cap P) \cup (\hat{\sigma} \cdot (2^{AP})^\omega \cap Q)$$
$$= (\hat{\sigma}_P \cdot (2^{AP})^\omega \cap P) \cup (\hat{\sigma}_Q \cdot (2^{AP})^\omega \cap Q)$$
$$= \emptyset$$

Thus $P \cup Q$ is a safety property.

Q2. $(P \cap Q)_{bad} ::= P_{bad} \cup Q_{bad}$

$\sigma \in P \cap Q \iff \forall \hat{\sigma} \subseteq_{fini} \sigma \quad \hat{\sigma} \notin P_{bad}$ and $\hat{\sigma} \notin Q_{bad}$

$\iff \forall \hat{\sigma} \subseteq_{fini} \sigma \quad \hat{\sigma} \notin (P \cap Q)_{bad}$

III Safety properties and Transition Systems.

Exercise 5. Finite traces

If $TS \models P$ then, $Tr_{fin}(TS) \cap P_{bad}$

$= pref(Tr^{\omega}(TS)) \cap P_{bad}$

TD n° 3

Safety and Liveness Properties

I. Liveness properties.

Exercise 1. (Closure of liveness properties)

$$cl(P) = (2^{AP})^\omega \iff \forall \sigma \in (2^{AP})^\omega \quad pref\, \sigma \subseteq pref\, P$$
$$\iff \forall \hat{\sigma} \in (2^{AP})^* \quad \hat{\sigma} \in pref\, P$$
$$\iff \forall \hat{\sigma} \in (2^{AP})^* \quad \exists \sigma \in P,\ \hat{\sigma} \leq \sigma$$
$$\iff P \text{ liveness property}$$

Exercise 2 (Unions & intersections).

We proved that $cl(P \cup Q) = cl(P) \cup cl(Q)$
and $cl(P \cap Q) = cl(P) \cap cl(Q)$

Thus $P \cup Q$ and $P \cap Q$ are liveness properties.

Even better: if P <u>or</u> Q is liveness, then $P \cup Q$ is liveness.

II Topology on infinite words

Exercise 3 (Σ^ω as a topological space)

We need to prove that $\Omega\Sigma^\omega$ is stable under arbitrary unions and finite intersections.

1) Let $\mathcal{U} \subseteq \wp(\Sigma^*)$. Define $\bar{\mathcal{U}} := \bigcup \mathcal{U}$. We have that

$$ext(\bar{\mathcal{U}}) = \bigcup_{u \in \bar{\mathcal{U}}} ext(u) = \bigcup_{U \in \mathcal{U}} \bigcup_{u \in U} ext(u)$$

Thus $\Omega\Sigma^\omega$ is stable under arbitrary unions.

2) Let $U, V \subseteq \Sigma^*$.

We have that

$$ext(U) \cap ext(V) = \bigcup_{u \in U} \bigcup_{v \in V} ext(u) \cap ext(v)$$

For some $u \in U$, $v \in V$, we have the 3 following cases:

a) $u \nleq v$ and $v \nleq u$ thus $ext(u) \cap ext(v) = \emptyset$

b) $u \leq v$ thus $ext(u) \cap ext(v) = ext(u)$

c) $v \leq u$ thus $ext(u) \cap ext(v) = ext(v)$

Define

$$W := \{u \in U \mid \exists v \in V,\ u \leq v\} \cup \{v \in V \mid \exists u \in U,\ v \leq u\}$$

and we have $ext(W) = ext(U) \cap ext(V)$.

By induction, $\Omega\Sigma^\omega$ is stable under finite intersections.

Exercise 4. (Open sets)

Let $P \subseteq \Sigma^\omega$.

P is open iff $\exists U \subseteq \Sigma^* \quad P = \bigcup_{u \in U} ext(u)$

iff $\begin{cases} \forall \sigma \in P, \exists u \in \Sigma^*, \ \sigma \in ext(u) \\ \forall u \in \Sigma^*, \ ext(u) \subseteq P \end{cases}$

iff $\forall \sigma \in P, \exists u \in \Sigma^*, \sigma \in ext(u) \subseteq P$

iff $\forall \sigma \in P, \exists \hat{\sigma} \in \Sigma^*, \hat{\sigma} \leq \sigma$ and $ext(\hat{\sigma}) \subseteq P$.

Exercise 5. (Density and liveness)

P is dense iff for any non-empty open set U', $P \cap U' \neq \emptyset$

iff for any non-empty $U \subseteq \Sigma^*$, $P \cap ext(U) \neq \emptyset$

iff for any $u \in \Sigma^*$, $P \cap ext(u) \neq \emptyset$

iff for any $\hat{\sigma} \in \Sigma^*$, $\exists \sigma \in P$, $\hat{\sigma} \subseteq \sigma$.

iff P is a liveness.

III. Decomposition theorem.

Exercise 6. Let $P \subseteq (2^{AP})^\omega$.

Define $P_{safe} := cl(P)$ which is a safety property as $cl(P_{safe}) = cl^2(P) = cl(P)$

and $P_{live} = P \cup P_{safe}^C$ which is a liveness property as P_{live} is

topologically dense: if $U \subseteq \Sigma^*$, and $ext(U) \cap P = \emptyset$ then $P \subseteq ext(U)^C$
and thus $cl(P) \subseteq ext(U)^C$ so we can conclude $ext(U) \subseteq cl(P)^C$. (closed)

Thus proving the decomposition theorem.

Exercise 7.

Q1. $cl(P_1) = P_1$ thus liveness and $P_{safe} = \Sigma^\omega$

Q2. $cl(P_2) = \{\sigma \mid \sigma(0) = a\}$ thus not liveness and not safety $P_{liveness} = \Sigma^\omega \setminus \{a^\omega\}$.

Q3. $cl(P_3) = \{a^\omega\} = P_3$ thus liveness and $P_{safe} = \Sigma^\omega$

Q4. $cl(P_4) = \{\sigma \mid \sigma \text{ contains } \leq 1 \text{ } b\text{'s}\} = P_4 \cup \{a^\omega\}$ thus not liveness and not safety $P_{live} = \Sigma^\omega \setminus \{a^\omega\}$.

Q5. $cl(P_5) = \Sigma^\omega$ thus liveness and $P_{safe} = \Sigma^\omega$

Q6 $cl(P_6) = \Sigma^\omega$ thus liveness and $P_{safe} = \Sigma^\omega$

Q7. $cl(P_7) = \Sigma^\omega$ thus liveness and $P_{safe} = \Sigma^\omega$

Q8. $cl(P_8) = \Sigma^\omega$ thus liveness and $P_{safe} = \Sigma^\omega$.