

# Linear optimization.

**Example 1.**   ▷ A *vertex cover* in a graph  $G = (V, E)$  is a set of vertices  $X \subseteq V$  such that, for every edge  $xy \in E$ ,  $x \in X$  or  $y \in X$ . We will write  $\tau(G)$  for the minimum size of a vertex cover of  $G$ .

▷ A *matching* is a set of disjoint edges. We will denote  $\nu(G)$  is the maximum size of a matching.

For the pentagonal graph  $G_\Delta$ , we have that that  $\nu(G_\Delta) = 2$  and  $\tau(G_\Delta) = 3$ .

Computing a vertex cover of minimum size is **NP**-hard; and, finding a matching of maximal size is very tricky but polynomial (Edmond's theorem). It is obvious that we have  $\nu \leq \tau$ .

We will consider the *fractional relaxation* of these problems.

Consider a variable  $x_v$  for every vertex  $v$  and ask that

▷  $\forall uv \in E, x_u + x_v \geq 1$ ;

▷  $\forall u \in V, x_u \geq 0$ ;

such that  $\sum_{v \in V} x_v$  is minimal, which we will write  $\tau^*(G)$ . This is called the *parameter fractional vertex cover*. We have that  $\tau^*(G_\Delta) = \frac{5}{2}$ .

For matching, we put a weight  $y_e$  for every edge  $e$  such that

▷  $\forall v \in V, \sum_{e \ni v} y_e \leq 1$ ;

▷  $y_e \geq 0$ ,

such that  $\sum_{e \in E} y_e$  is maximized, which we will write  $\nu^*$ . We have that  $\nu^*(G_\Delta) = \frac{5}{2}$ .

The fact that  $\nu^*(G_\Delta) = \tau^*(G_\Delta)$  is a more general fact:

$$\nu \leq \nu^* = \tau^* \leq \tau.$$

**Remark 1.**  $\triangleright$  The problem of linear programming is in **P**. Linear solver programs can be done in polynomial time. Thus, computing a released solution is possible and useful.

- $\triangleright$  *Duality*: Dual fractional parameters are equal, parameters come by pairs.

**Remark 2 (Why is linear programming tractable?).**  $\triangleright$  There is an efficient algorithm called the *simplex algorithm*, but it is not in **P**.

- $\triangleright$  There is a polynomial algorithm (using ellipsoids) but it is not useful in practice.
- $\triangleright$  There is an algorithm which is both in **P** and efficient using interaction-point methods.

## 1 The simplex algorithm.

Let (P) be the following linear problem:

$$(P) : \text{maximize } 5x_1 + 4x_2 + 3x_3 \text{ with } \begin{cases} 2x_1 + 3x_2 + x_3 & \leq 5 \\ 4x_1 + 3x_2 + 2x_3 & \leq 11 \\ 3x_1 + 4x_2 + 2x_3 & \leq 8 \\ x_1, x_2, x_3 & \geq 0 \end{cases}.$$

We can try to increase  $x_1$ , or  $x_2$ , or  $x_3$  but what's the next step? Introduce new variables called *slack variables*  $x_4, x_5, x_6$  (one for each constraint).

We can transform (P) with:

$$(D_0) : \begin{cases} x_4 = 5 - 2x_1 - 3x_2 - x_3 \\ x_5 = 11 - 4x_1 - x_2 - 2x_3 \\ x_6 = 8 - x_1 - 4x_2 - 2x_3 \\ z = 5x_1 + 4x_2 + 3x_3 \end{cases}.$$

The problem of maximizing the objective function of (P) is equivalent to maximize  $z$  under the constraints of the inequalities transformed into equalities and with  $x_1, \dots, x_6 \geq 0$ .

The problem  $(D_0)$  is the *(initial) dictionary*.

To a dictionary, we associate a solution by setting to 0 the non-basic variables  $x_1, x_2, x_3$  and getting solutions for the basic variables  $x_4, x_5, x_6$ . In our case, we would have  $x_4 = 5$ ,  $x_5 = 11$  and  $x_6 = 8$ , for an objective of 0.

! If one of these values of the solution (here, 5, 11, 8), would be negative, there would be a problem. We could have an empty domain (this is the problem of solving the decision problem associated to (P)). We will see later how to solve this problem.

We can try by hand to increase  $z$ . If we try to increase  $x_1$ , we have that the highest limitation for  $x_1$  is  $\frac{5}{2}$  (we can see that using  $x_4, x_5, x_6 \geq 0$  by solving for  $x_1$  with  $x_2 = x_3 = 0$ , this constraint is from the one on  $x_4$ ). We increase it, but... what next?

Now, this is the idea of Dantzig: **Pivot**. We will call  $x_1$  the *entering variable* and  $x_4$  the *leaving variable*. We will then exchange the role of  $x_1$  and  $x_4$  and substitute:

$$(D_1) : \begin{cases} x_1 = \frac{5}{2} - \frac{x_4}{2} - \frac{3x_2}{2} - \frac{x_3}{2} \\ x_5 = 1 + 2x_4 + 5x_2 \\ x_6 = \frac{1}{2} + \frac{3}{2}x_4 - \frac{x_2}{2} \\ z = \frac{25}{4} - \frac{5x_4}{2} - \frac{7x_2}{2} + \frac{x_3}{2} \end{cases}.$$

We observe that  $(D_1)$  is equivalent to  $(D_0)$  and (P) when  $x_1, \dots, x_6 \geq 0$ . We can now iterate the process, choosing a new entering variable.

To increase  $z$ , we only have one choice: increasing  $x_3$ . To find the leaving variable, we have that  $x_3 \leq 1$  with the constraint on  $x_6$ . We should then pivot  $x_3$  and  $x_6$ :

$$(D_2) \begin{cases} x_1 = 2 - x_4 - 2x_2 + x_6 \\ x_5 = 1 + 2x_4 + 5x_2 \\ x_3 = 1 + 3x_4 + x_2 - 2x_6 \\ z = 13 - x_4 - 3x_2 - x_6 \end{cases}.$$

Now, we have no choice for an entering variable. This means we are on an **optimal solution** and the simplex algorithm stops.

The solutions of  $(D_2)$  with  $x_1, \dots, x_6 \geq 0$  are equivalent to  $(P)$ . This means that  $z \leq 13$ . The solution associated to  $(D_2)$  is

$$s_2 = (2, 0, 1, 0, 1, 0).$$

$x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6$

The optimal for  $(P)$  is  $(2, 0, 1)$  for an objective of 13.

**!** The linear problem  $(D_2)$  contains the certificate that  $\text{OPT} \leq 13$ . In  $z = 13 - x_4 - 3x_2 - x_6$ , the variables  $x_4$  and  $x_6$  are slack and thus correspond to the constraints

- ▷  $x_4 \rightarrow 2x_1 + 3x_2 + x_3 \leq 5 \ (\times 1)$ ;
- ▷  $x_6 \rightarrow 3x_1 + 4x_2 + 2x_3 \leq 8 \ (\times 1)$ ;

thus, the objective is

$$\text{obj} \leq 5x_1 + 7x_2 + 3x_3 \leq 13,$$

which we obtain by summing the two rows. We get back the original objective functions. The certificate of optimality is: a non-negative combination of constraints is larger than the objective function.

**Intuition: What are pivots?** The simplex moves from every vertex to another (adjacent) vertex of the polyhedron.

**How many steps?** Consider a polyhedron  $P$  with  $n$  vertex and  $m$  facets. The *skeleton* is the graph obtain from vertices of  $P$  and edges

of  $P$ . An upper bound of the number of pivots is the diameter (*i.e.* the distance between the furthest vertices) of  $G$ .

For the cube, we get that the dimension is 3 with a diameter 3 and 6 facets. For the  $K_4$  the complete graph with 4 vertices, we have a dimension of 3, with 4 facets and we have a diameter of 1.

It is conjectured that the diameter is bounded by the number of facets minus the dimension (Hirsch conjecture).

If this conjecture is true, we have that there exists a sequence of pivots with length bounded by  $n + m - n = m$ .

This is false but polynomial versions are still open.