

Algebraic and combinatorial aspects of category theory

Lecture #1.

18/11/25

Algebraic structure $T(X) \xrightarrow{\text{monad}} X$

intuition: $T(X)$ contains the derived operations

Example for a binary operation on X , we can consider

{binary trees with leaves in $X\} \rightarrow X$.

Being created

$$\begin{array}{ccccc}
 T(X) & \xleftarrow{T\pi_1} & T(X \times Y) & \xrightarrow{T\pi_2} & T(Y) \\
 \downarrow & & \downarrow & & \downarrow \\
 X & \xleftarrow{\quad} & X \times Y & \xrightarrow{\quad} & Y
 \end{array}
 \qquad \qquad \qquad
 \begin{array}{c}
 T\text{-Alg} \\
 \text{forgetful} \\
 \text{functor} \\
 \downarrow \\
 \text{Set}
 \end{array}$$

Category: objects, morphisms, composition (unitary and associative).

Notations $A \in \mathcal{C}$ iff A is an object in \mathcal{C}

$\mathcal{C}(A, B)$ is the set of morphisms $A \rightarrow B$.

Examples	objects	morphisms
a set X	$x \in X$	id_x
a poset (X, \leq)	$x \in X$	$x \leq y$ (also works for preordered sets)
<u>Set</u>	<u>set</u> X	<u>maps</u>
<u>Mon</u>	<u>monoid</u> X	<u>homomorphisms</u>
	:	

Given a graph G , then G^* is the category where objects are nodes in G and morphisms $A \rightarrow B$ the set of paths from A to B composition is path concatenation

Example: BM. one object $*$ and $\mathcal{C}(*, *) = M$
with $h \circ g = h \cdot g$.

Example: objects are trees and morphisms are $T_1 \xrightarrow{T} T_2$ when you obtain T_2 by replacing T_1 's leaves with copies of T .

at most one size

- "small" { objects form a set
- { morphisms form a set (from A to B fixed)

"locally small" { objects form a set class

- { morphisms form a set (from A to B fixed)

Functor $F: \mathcal{C} \rightarrow \mathcal{D}$ that commutes with composition and identity

Notation $F_{A,B}: \mathcal{C}(A, B) \longrightarrow \mathcal{D}(F(A), F(B))$

The axioms of functors can be seen as the fact that the two following diagrams commute.

$$\begin{array}{ccc}
 \mathcal{C}(B, C) \times \mathcal{C}(A, C) & \xrightarrow{- \circ \mathcal{C} -} & \mathcal{C}(A, C) \\
 \downarrow F_{B,C} \times F_{A,B} & & \downarrow F_{A,C} \\
 \mathcal{D}(FB, FC) \times \mathcal{D}(FA, FC) & \xrightarrow{- \circ \mathcal{D} -} & \mathcal{D}(FA, FC)
 \end{array}$$

$\Leftrightarrow Ff \circ Fg = F(f \circ g)$

$$\begin{array}{ccc}
 F(id_A) = id_{FA} & \Leftrightarrow & \begin{array}{c} id_A \xrightarrow{\quad} \mathcal{C}(A, A) \\ \downarrow F_{A,A} \\ \mathcal{D}(FA, FA) \end{array} \\
 & & \downarrow id_{F(A)} \xrightarrow{\quad} \mathcal{D}(FA, FA)
 \end{array}$$

Examples

- A monotone map is a functor between two posets seen as categories
- Forgetful functor $\text{Mon} \rightarrow \text{Set}$, $\text{Ring} \rightarrow \text{Mon}$
 $(M, \cdot, 1_M) \mapsto M$

Example A functor $F: \text{BM} \rightarrow \text{BN}$ is exactly a monoid homomorphism $f: M \rightarrow N$.

Example The "bang" functor $\mathcal{G} \xrightarrow{!} \mathbf{1}$
↑ category with one object and only the id morphism.

Functors compose !!

Cat is the category of small categories. It forms a locally small category

CAT is the category of locally small categories. It forms a very large category.

$\text{Set}, \text{Grp}, \text{Gph}, \text{Cat} \in \text{Cat}$.

A terminal object $T \in \mathcal{G}$ is defined such that for every object $A \in \mathcal{G}$, there is a unique morphism $A \rightarrow T$

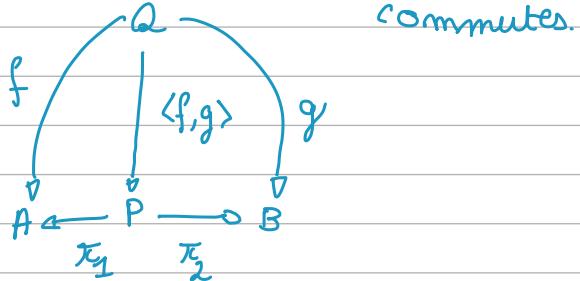
An initial object $I \in \mathcal{G}$ is defined such that for every object $A \in \mathcal{G}$, there is a unique morphism $I \rightarrow A$.

In Mon , Grp and Field , $\mathbf{1}$ is both terminal and initial.

In Set , \emptyset is initial ($\emptyset \xrightarrow{\exists!} A$) and $\mathbf{1}$ is terminal ($A \xrightarrow{\exists!} \mathbf{1}$).

A product of A and $B \in \mathcal{C}$ is an object P with $\pi_1 : P \rightarrow A$
 $\pi_2 : P \rightarrow B$
such that for every $A \leftarrow Q \rightarrow B$, there exists a unique $\langle f, g \rangle : Q \rightarrow P$

such that



commutes.

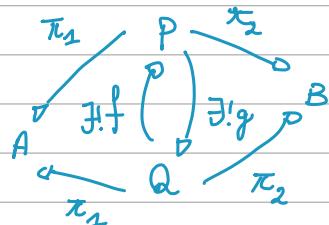
In Set, take $\langle f, g \rangle : Q \rightarrow A \times B$
 $q \mapsto (f(q), g(q))$.

Exercise $B \times A \cong A \times B$ and the unique morphism is

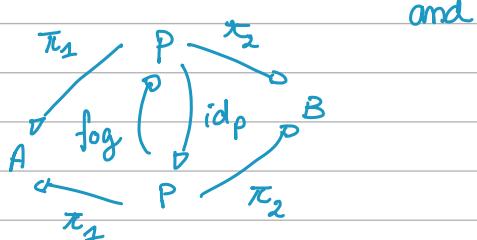
$$\langle \pi_2, \pi_1 \rangle : (a, b) \mapsto (b, a).$$

Exercise: If P and Q are both products of A and B then

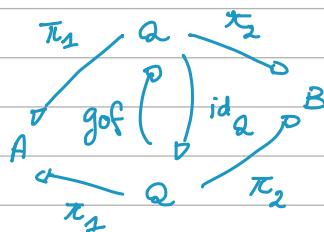
there exists a unique isomorphism from P to Q .



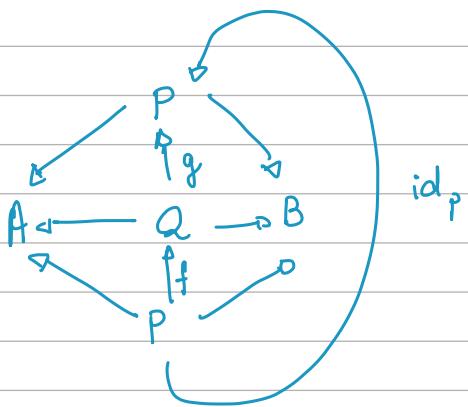
Then



and



imply $f \circ g = \text{id}_P$ and $g \circ f = \text{id}_Q$.



//break //

Monads

$A(X)$ = terms with free variables in X , up to some equations

Free monoid monad : define the free monoid of $X \in \text{Set}$ by induction:

$$\frac{x \in X}{X \vdash x} \quad \frac{}{X \vdash 1} \quad \frac{X \vdash M \quad X \vdash N}{X \vdash M \cdot N}$$

with the equations

$$\frac{(M \cdot N) \cdot P \sim M \cdot (N \cdot P)}{M \sim M \cdot 1} \quad \frac{M \sim 1 \cdot M}{M \cdot N \sim M' \cdot N'} \quad \frac{M \sim M'}{N \sim N'}$$

Then $T(X) = \{X \vdash M\} / (\sim \cup \sim^+)^*$

Equivalently, $T(X) = X^* = \sum_{n \in \mathbb{N}} X^n$.

Applying $f: X \rightarrow Y$ induces a map

$$Tf: X^* = TX \longrightarrow Y^* = TY$$

$$(x_1, \dots, x_m) \longmapsto (f(x_1), \dots, f(x_m)).$$

Also, $Tf \circ Tg = T(f \circ g)$ and $T\text{id}_X = \text{id}_{TX} = \text{id}_{X^*}$

Basic idea of a monad: a set X with a map $a: TX \rightarrow X$.

There is an inclusion $X \rightarrow T(X)$:

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & TX \\ & \parallel & \downarrow a \\ X & & X \end{array}$$

& naturality of η and μ

and compositionality

$$\begin{array}{ccccc} TTX & \xrightarrow{\mu} & TX & & \\ \uparrow \text{Ta} & & \uparrow a & & \\ ((x_1^1, \dots, x_{n_1}^1), \dots, (x_k^1, \dots, x_{n_k}^k)) & \longleftarrow & (x_1^1, \dots, x_{n_k}^k) & & \\ \downarrow & & \downarrow & & \\ (\alpha(x_1^1, \dots), \dots, \alpha(x_1^k, \dots)) & \longrightarrow & \alpha(\alpha(x_1^1, \dots), \dots, \alpha(x_1^k, \dots)) & \xrightarrow{\alpha} & X \\ & & & & \downarrow a \\ & & & & X \end{array}$$

A T -algebra is a map $a: T(X) \rightarrow X$ such that

$$\begin{array}{ccc} TTX & \xrightarrow{\mu} & TX \\ \downarrow Ta & & \downarrow a \\ TX & \xrightarrow{a} & X \end{array}$$

$$X \xrightarrow{\eta_X} TX$$

η_X

There is thus a category $T\text{-Alg}$: objects are T -algebras

$$f: X \rightarrow Y \text{ induces } TX \xrightarrow{Tf} TY$$

$$\begin{array}{ccc} a_X & \downarrow & a_Y \\ X & \xrightarrow{f} & Y \end{array}$$

When T is the free monoid functor, then $\text{Mon} \cong 1\text{-Alg}$.

$$\begin{array}{ccc} \text{Mon} & \xrightarrow{L} & T\text{-Alg} \\ & \searrow u \quad \swarrow v & \\ & \text{Set} & \end{array}$$

for R , take $1 := a()$
take $x \cdot y := a(x, y)$

We have

$$a(x, a(y)) = a(a(x), a(y)) \stackrel{(u)}{=} a(x) = x$$

For L , define $a(x_1, \dots, x_n) := (x_1 \cdot x_2) \cdot x_3 \cdots \cdot x_n$.

Then we have to extend this map of objects to a functor

$L: \text{Mon} \rightarrow 1\text{-Alg}$

$$(M, \cdot, 1) \mapsto (M, (x_1, \dots, x_n) \mapsto (x_1 \cdot x_2) \cdot \dots \cdot x_n)$$

$$f: (M, \cdot, 1) \rightarrow (N, \cdot, 1) \mapsto Lf: (M, \dots) \rightarrow (N, \dots)$$

$$x \mapsto f(x)$$

and $R: 1\text{-Alg} \longrightarrow \text{Mon}$

$$(M, \alpha) \longmapsto (M, \alpha(-, -), \alpha(1))$$

$$f: (M, \alpha) \rightarrow (N, \alpha') \longmapsto Rf: (M, \alpha(-, -), \alpha(1)) \xrightarrow{x \longmapsto f(x)} (N, \alpha'(-, -), \alpha'(1))$$

Indeed,

$$Rf(\underbrace{\alpha(x, y)}_{x \cdot y}) = f(\alpha(x, y)) = \alpha'(f(x), f(y)) = Rfx \cdot Rfy.$$

$$\text{and } Rf(1) = f(\alpha(1)) = \alpha(1).$$

Also, $TLM \xrightarrow{Tlf} TLN$

$$\begin{array}{ccc} (\times)^m & & (\cdot)^n \\ \downarrow & & \downarrow \\ LM & \xrightarrow{Lf} & LN \end{array}$$

by induction on n (as case $n=1$ is verified with $f(1)=1$ and case $n>1$ is verified with $f(x \cdot y) = f(x) \cdot f(y)$)

A morphism $f: X \rightarrow Y$ in ...

Mon

$$X \times X \xrightarrow{f \times f} Y \times Y$$

$$\begin{array}{ccc} \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \\ \downarrow 1_X & \nearrow 1_Y & \end{array}$$

$f(x \cdot y) = f(x) \cdot f(y)$

1-Alg

$$TX \xrightarrow{Tf} TY$$

$$\begin{array}{ccc} a & \downarrow & a' \\ X & \xrightarrow{f} & Y \end{array}$$

A monad in a category \mathcal{C} is a functor $T: \mathcal{C} \rightarrow \mathcal{C}$ with
an endo

- unit $\eta_x: x \rightarrow TX$

- multiplication $\mu_x: TTX \rightarrow TX$

such that $\eta: id \Rightarrow T$ and $\mu: TT \Rightarrow T$ are natural transformations, such that both diagrams commute.

$$\begin{array}{ccc} TX & \xleftarrow{\eta_{TX}} & TTX \xrightarrow{T\eta_X} TX \\ \downarrow \quad \downarrow \mu_x & \nearrow \quad \searrow & \downarrow \mu_{TX} \\ TX & & TTX \xrightarrow{\mu_X} TX \end{array}$$

$$TTTX \xrightarrow{T\mu_X} TTX \xrightarrow{\mu_X} TX$$

associativity

A T -algebra is an object $A \in \mathcal{C}$ and $a: TX \rightarrow A$ such that

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & TX \\ \downarrow \quad \downarrow a & \nearrow & \downarrow Fa \\ X & \xrightarrow{a} & A \end{array}$$

$$\begin{array}{ccc} FFX & \xrightarrow{\mu_X} & FX \\ \downarrow Fa & \downarrow a & \downarrow a \\ FX & \xrightarrow{a} & A \end{array}$$

A natural transformation $\alpha: F \Rightarrow G$ where $F, G: \mathcal{C} \rightarrow \mathcal{D}$

$$\begin{array}{ccc} F & & G \\ \alpha \swarrow & \downarrow \alpha & \searrow \alpha \\ F & & G \end{array}$$

is a set of morphisms $\alpha_x: FX \rightarrow GX$ such that, for every $f: x \rightarrow y$,

the square

$$\begin{array}{ccc} FX & \xrightarrow{Ff} & FY \\ \downarrow \alpha_x & & \downarrow \alpha_y \\ GX & \xrightarrow{Gf} & Gy \end{array}$$

commutes.

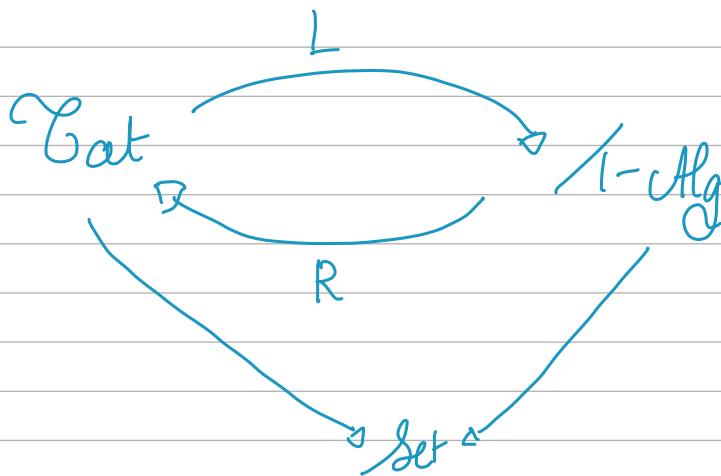
Exercise Find 1-on Graph such that $\text{Cat} \cong 1\text{-Alg}$.

1: $\text{Graph} \rightarrow \text{Graph}$

$$G \mapsto G^*$$

$$f: G \rightarrow H \mapsto f^*: G^* \rightarrow H^*$$

Then, a 1-algebra is a map $G^* \rightarrow G$ in Graph.



$R: 1\text{-Alg} \longrightarrow \text{Cat}$

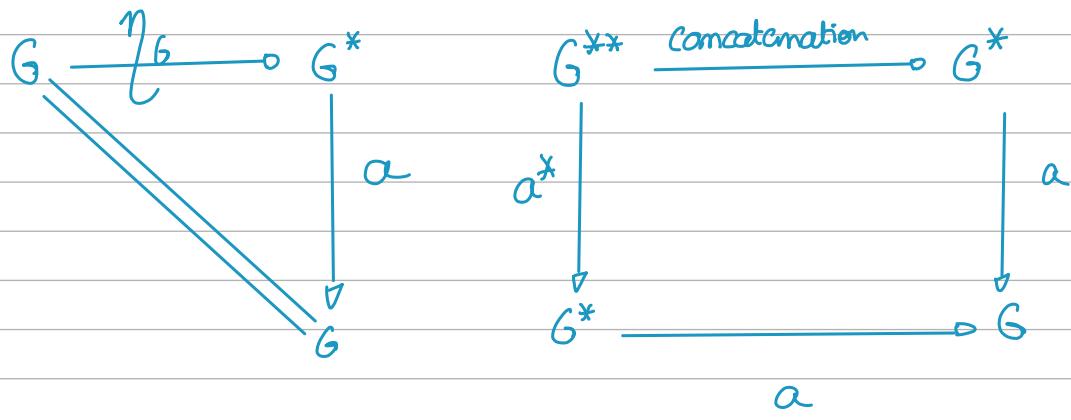
$(G, a: G^* \rightarrow G) \mapsto \left(\begin{array}{l} \text{obj: vertices of } G \\ \text{morphisms: paths in } G \\ \text{composition: } p \circ q = a(p \cdot q) \\ \text{identity: } \text{id} = a(\text{id}) \end{array} \right)$

$(f: (G, a) \rightarrow (H, b)) \mapsto Rf: g \mapsto f(g)$
 $g \dashv h \mapsto f(g) \dashv f(h)$

and $L: \text{Cat} \longrightarrow 1\text{-Alg}$

$\mathcal{C} \longleftrightarrow (\{\mathcal{C}\}, a: (x_1, \dots, x_n) \mapsto x_1 \circ \dots \circ x_m)$

$F: \mathcal{C} \rightarrow \mathcal{D} \longleftrightarrow LF: x \mapsto F(x)$



Lecture #2

25/11/2025

Last time: products, monads, algebras, finally equivalence between $T\text{-Alg}$ and some other categories

"For any monad T , the forgetful functor $T\text{-Alg} \xrightarrow{\alpha} \mathcal{G}$ creates products"

We say $U: \mathcal{E} \rightarrow \mathcal{B}$ creates binary products if

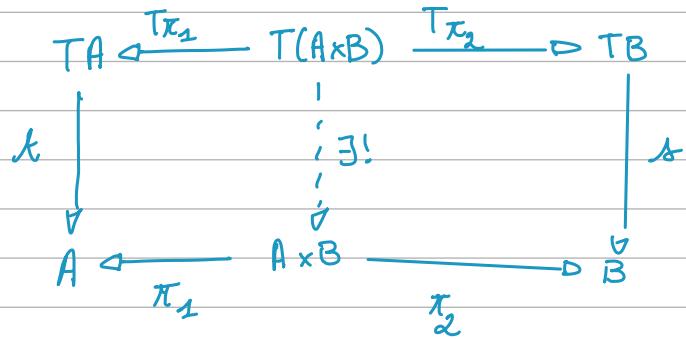
any product $Ux \leftarrow P \rightarrow Uy$ in \mathcal{B} has
a unique antecedent $x \leftarrow \bar{P} \rightarrow y$ in \mathcal{E} which
is also a product in \mathcal{E} .

This is different from product preservation (U product = product),
or product reflexion (if $U(x \leftarrow P \rightarrow y)$ is a product then so is
 $x \leftarrow P \rightarrow y$).

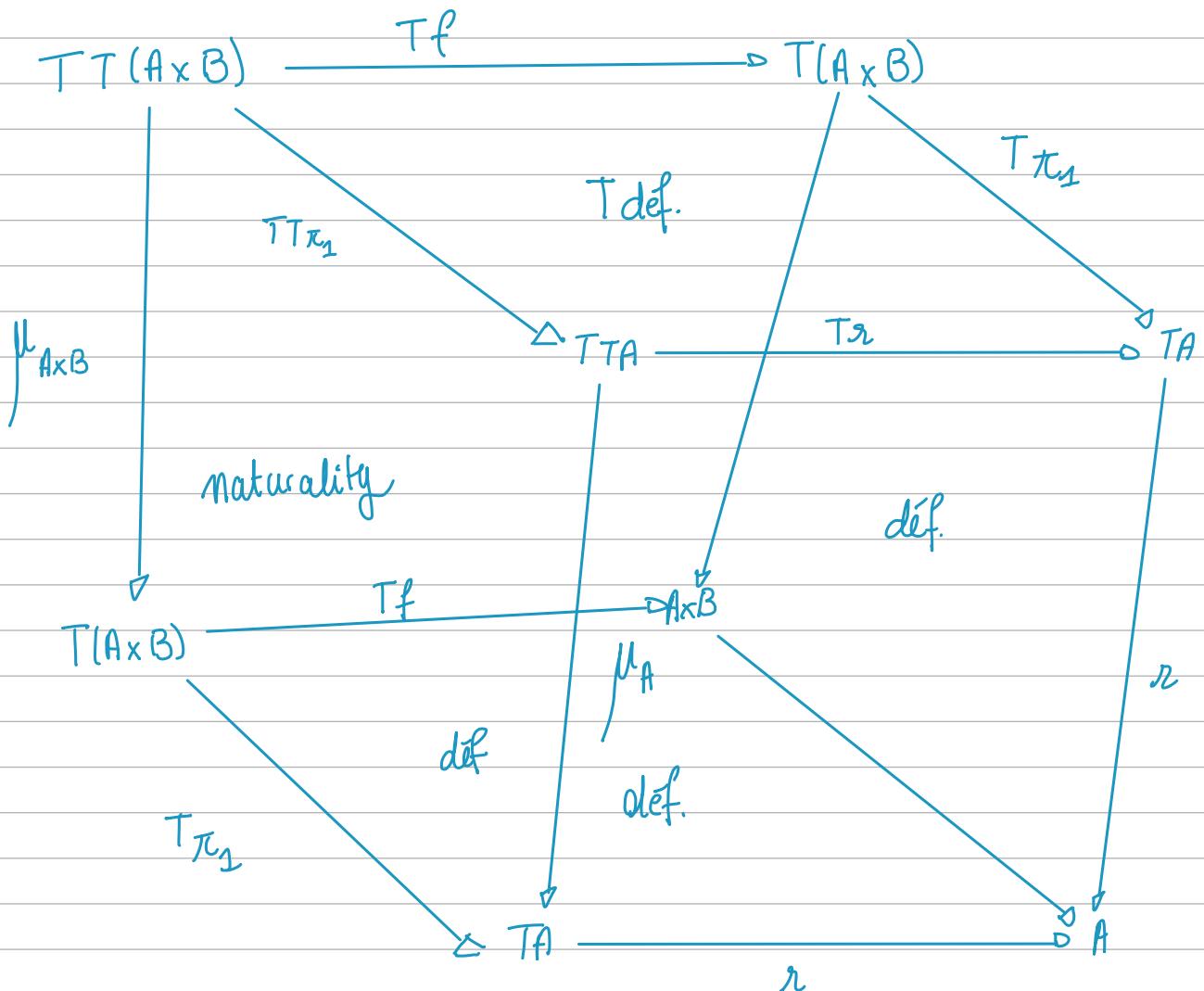
Proposition: For any monad $T: \mathcal{G} \rightarrow \mathcal{G}$, the forgetful functor $T\text{-Alg}$ creates binary products.

Let $TA \xrightarrow{\pi} A$ and $TB \xrightarrow{\delta} B$ in $T\text{-Alg}$.

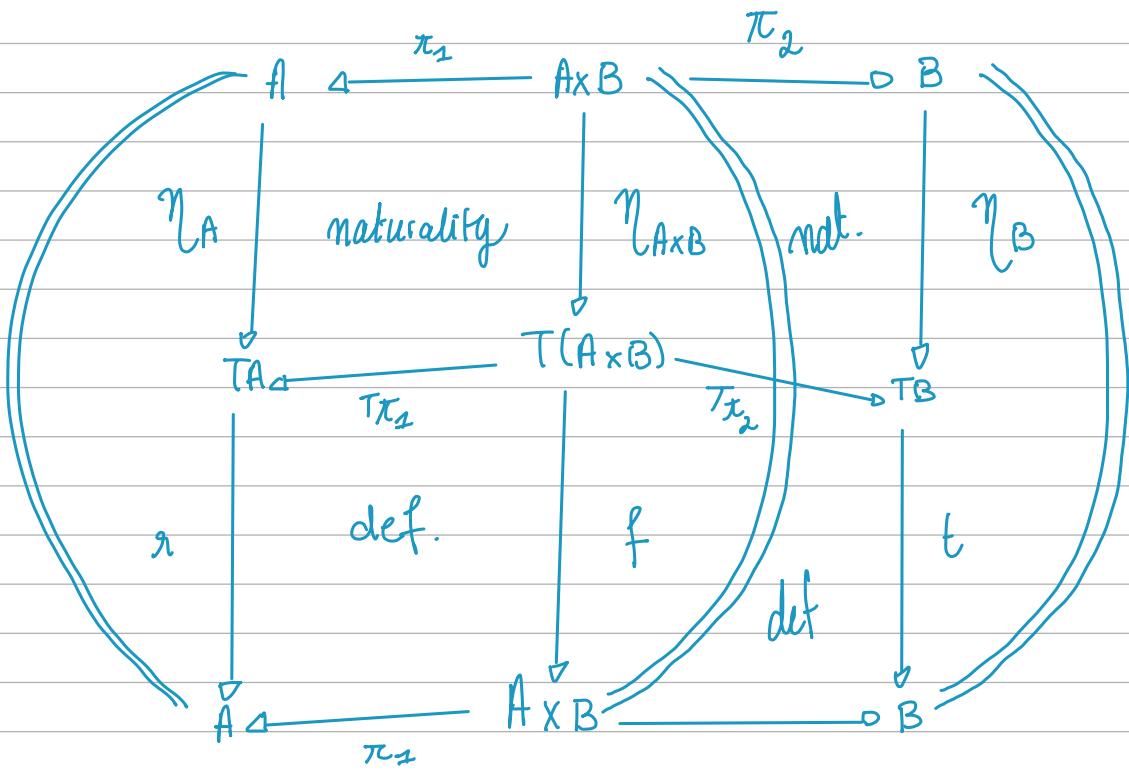
We want a T -algebra $T(A \times B) \rightarrow A \times B$.



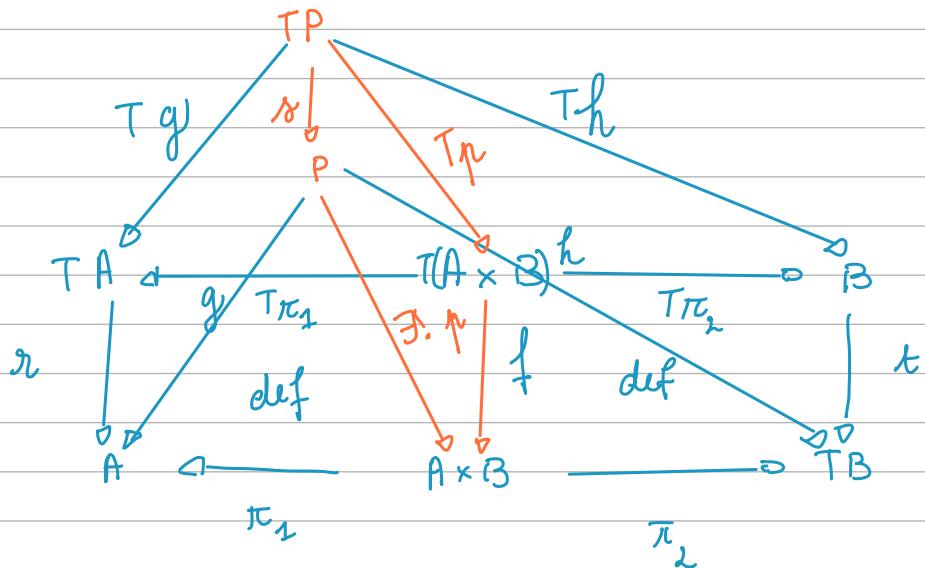
We need to check the monad laws apply correctly to $T(A \times B)$.



So we have compositionality.



It remains to show that $T(A \times B)$ is a product in $T\text{-ctg}$.



(This diagram
is in §
but talks about
T-Alg).

This is where we start with combinatorial constructions.

Pre Sheaves

A presheaf is a functor $\mathcal{C}^{\text{op}} \rightarrow \text{Set}$.

The opposite category \mathcal{C}^{op} has the same objects and morphisms from A to B as $\mathcal{C}(B, A)$ and we use reverse composition in \mathcal{C}

$$\begin{array}{ccc} \mathcal{C}^{\text{op}}(B, C) \times \mathcal{C}^{\text{op}}(A, B) & \xrightarrow{-\circ \text{id}} & \mathcal{C}^{\text{op}}(A, C) \\ \text{id} \times \text{id} \swarrow & \text{id} \searrow & \parallel \\ \mathcal{C}^{\text{op}}(A, B) \times \mathcal{C}^{\text{op}}(B, C) & & \mathcal{C}(C, A) \\ \parallel & & \xrightarrow{-\circ \text{id}} \\ \mathcal{C}(B, A) \times \mathcal{C}(C, B) & & \end{array}$$

Define $\mathcal{G}: [0] \xrightarrow{\delta} [1]$. Presheaves F on \mathcal{G} are exactly directed multigraphs

$$\begin{array}{ccc} F[0] & \xleftarrow{F\delta} & F[1] \\ \cup & \curvearrowright & \cup \\ V & FE & E \end{array}$$

Define $\text{Psh}(\mathcal{G}) = \hat{\mathcal{G}} = [\mathcal{C}^{\text{op}}, \text{Set}]$. So $\mathcal{G}_{\text{ph}} := \hat{\mathcal{G}}$.

(Remark \mathcal{G} is small but $\hat{\mathcal{G}}$ is only locally small.

Graph morphisms should make

$$\begin{array}{ccc} F[1] & \xrightarrow{f[1]} & G[1] \\ \downarrow F\delta & & \downarrow G\delta \\ F[0] & \xrightarrow{f[0]} & G[0] \end{array}$$

$$\begin{array}{ccc} F[1] & \xrightarrow{f[1]} & G[1] \\ \downarrow G\delta \text{ and } FE & & \downarrow GE \\ F[0] & \xrightarrow{f[0]} & G[0] \end{array}$$

commute, which we will write as

$$\begin{array}{ccc} F[1] & \xrightarrow{f[1]} & G[1] \\ F_\Delta \left(\begin{array}{c} \nearrow \\ \downarrow \end{array} \right) F_t & & G_\Delta \left(\begin{array}{c} \searrow \\ \downarrow \end{array} \right) G_t \\ F[0] & \xrightarrow{f[0]} & G[0] \end{array}$$

should serially commute.

A coproduct is of the form

$$\begin{array}{ccccc} X & \xrightarrow{\pi_1} & X+Y & \xleftarrow{\pi_2} & Y \\ & \searrow f & \downarrow \exists! & \swarrow g & \\ & A & & & \end{array}$$

In Set, it can be constructed as $\{0\} \times X \sqcup \{1\} \times Y$.

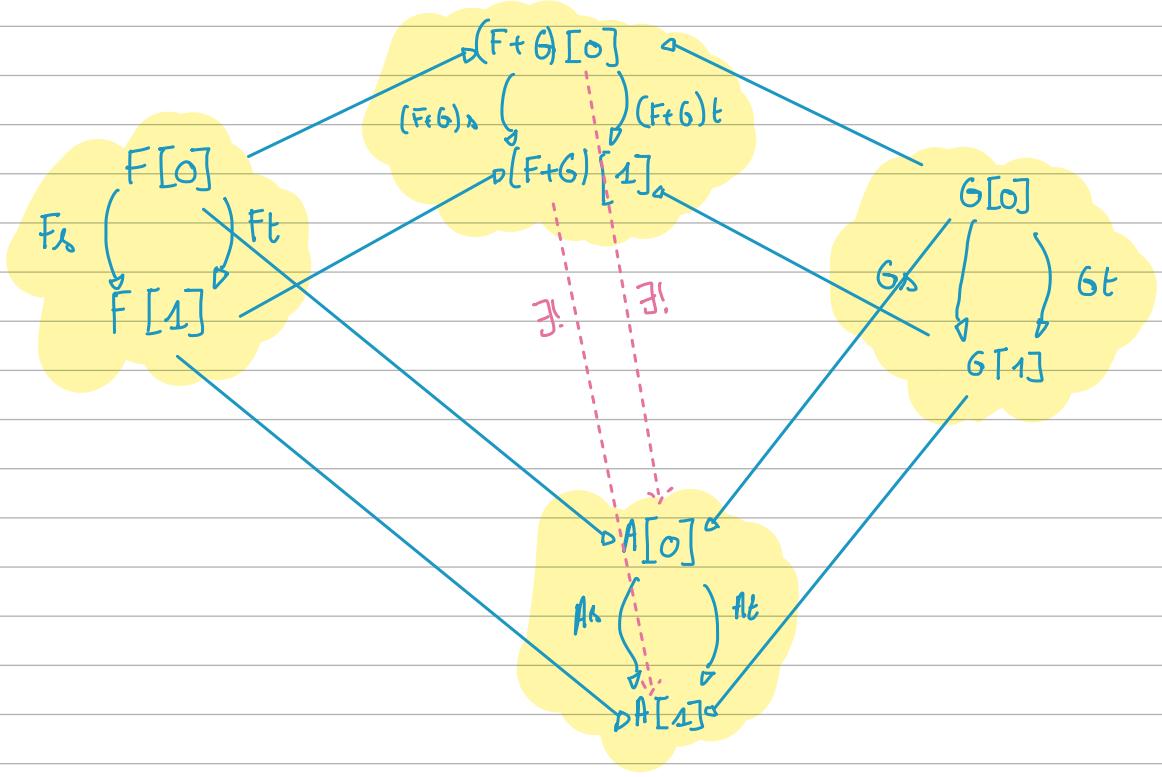
How to compute a coproduct of graphs?

$$\begin{array}{ccccc} & & (F+G)[0] & & \\ & \nearrow & \downarrow & \searrow & \\ F_\Delta \left(\begin{array}{c} \nearrow \\ \downarrow \end{array} \right) F_t & & (F+G)[1] & & G_\Delta \left(\begin{array}{c} \nearrow \\ \downarrow \end{array} \right) G_t \\ F[0] & & & & G[0] \\ \searrow & & \downarrow & & \searrow \\ F[1] & & & & G[1] \end{array}$$

should sequentially commute. So we should have

$$\begin{aligned} (F+G)[0] &= F[0] \sqcup G[0] \\ (F+G)[1] &= F[1] \sqcup G[1] \\ (F+G) \Delta &= F \Delta + G \Delta \\ (F+G) t &= F t + G t \end{aligned}$$

Also

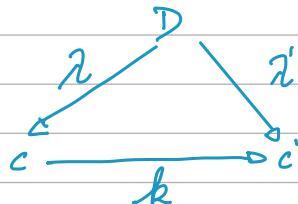


Let us talk about colimits. Colimits are the universal cones.

A diagram in \mathcal{C} is a functor $F: \mathbb{I} \rightarrow \mathcal{C}$ for some small \mathbb{I} .

A cocone " $\lambda: D \rightarrow c$ " is a natural transformation from D to the constant functor $c \in \mathcal{C}$.

A cocone morphism is of the form

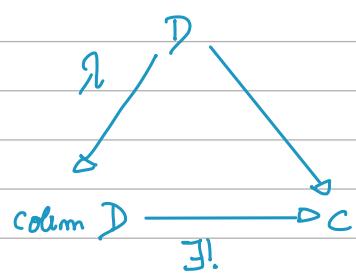


where $k \in \mathcal{C}(c, c')$.

We thus define a category $\text{Cocone}(\mathbb{D})$ of cocones over \mathbb{D} .

A colimit of \mathbb{D} is the initial object of $\text{Cocone}(\mathbb{D})$:

$\forall c,$



$$2 := 1 + 1$$

Example: for coproduct, use the diagram

1

16

for initial obj, use the empty diagram .

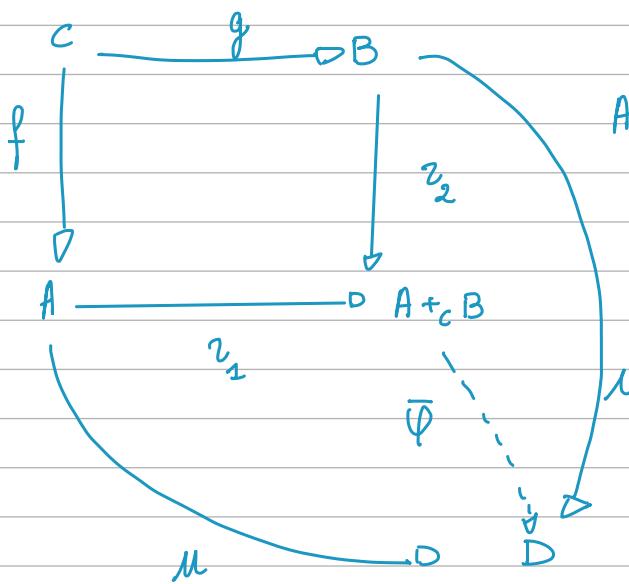
Indeed, the universal property of $\text{colim}(\text{empty diagram})$ tells us that

Hc,

$$\text{colim}(\emptyset) \xrightarrow{\exists!}$$

3!

3



$$A \dagger_c B := A + B \quad / \quad f(c) = g(c).$$

$$\begin{cases} \varrho_1 : A \longrightarrow A +_C B \\ \varrho_2 : B \longrightarrow A +_C B \end{cases}$$

$$\varphi : A + B \rightarrow B$$

$$\begin{array}{ccc} a & \xrightarrow{\hspace{1cm}} & u(a) \\ b & \xrightarrow{\hspace{1cm}} & v(b) \end{array}$$

and $\Psi : A +_c B \longrightarrow D$
exists as $\Psi(f(c)) = \Psi(g(c))$
 $\mu(f(c)) = \nu(g(c))$

$$2 \xrightarrow{i_0} 2 + 3$$

$$! \downarrow \quad \downarrow \\ 1 \longrightarrow 1 + (2 + 3) / \begin{matrix} 1_0(0) = 1_0(2) = 1_0(1) \\ 2 \\ 2 \end{matrix}$$

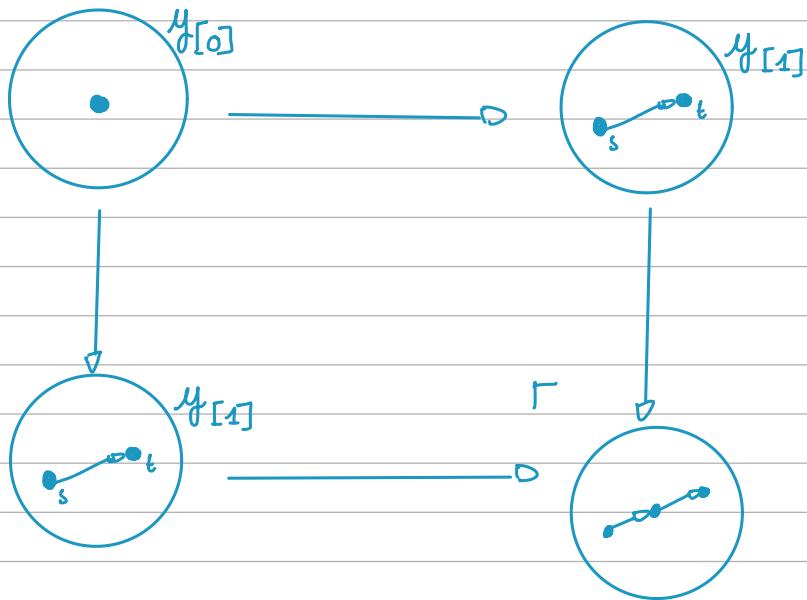
$$1 + 3 = 4$$

The pushout of $y_{[0]}, y_{[1]}$ in Graphs along y_t :

$$y_t: \bullet \xrightarrow{t} \bullet_t$$

$$y_s: \bullet \xrightarrow{s} \bullet_s$$

is $\bullet \rightarrow \bullet \rightarrow \bullet$



Lecture #3

02/12/2025

Exercise In Set, the equalizer of $A \xrightarrow{\begin{smallmatrix} f \\ g \end{smallmatrix}} B$ is

$$E := \{a \in A \mid f(a) = g(a)\}$$

with $i: E \rightarrow A$ the inclusion arrow. For any $x: X \rightarrow A$ such that $x \circ f = x \circ g$, then $x(X) \subseteq E$ and so there is an unique inclusion arrow

$$\begin{aligned} m: X &\longrightarrow E \\ u &\longmapsto x(u) \end{aligned}$$

such that

$$\begin{array}{ccccc} & X & & & \\ & \downarrow m & & & \\ \exists! m: & E & \xrightarrow{i} & A & \xrightarrow{\begin{smallmatrix} f \\ g \end{smallmatrix}} B \\ & & & \searrow x & \\ & & & & \end{array}$$

composed.

Exercise • $[\mathcal{C}, \mathcal{D}]$ is a category

$$(F \circ G)(x) = F(G(x))$$

$$F \circ G(f) = F(G(f))$$

$$\text{Id}(x) = x \quad \text{Id}(f) = f$$

- $\text{ob}([\mathcal{C}, \mathcal{D}]) \subseteq \text{ob}(\mathcal{D})$ $\text{ob}(\mathcal{C}) \times \text{Mor}(\mathcal{D})^{\text{Mor}(\mathcal{C})}$
and so $\text{ob}([\mathcal{C}, \mathcal{D}])$ is a set
as \mathcal{C} and \mathcal{D} are small.

For some $F, G \in [\mathcal{C}, \mathcal{D}]$, $\text{Hom}(F, G) \subseteq \prod_{X \in \mathcal{C}} \text{Hom}(FX, GX)$ is a set.
 \hookrightarrow small

- For some $F, G \in [\mathcal{C}, \mathcal{D}]$, $\text{Hom}(F, G) \subseteq \prod_{X \in \mathcal{C}} \text{Hom}(FX, GX)$ is a set.
 \hookrightarrow in \mathcal{D} locally small
 \hookrightarrow small

- The functor $F: [\mathcal{C}, \mathcal{D}] \longrightarrow [\text{ob}(\mathcal{C}), \mathcal{D}]$ is defined as

$$(U: \mathcal{C} \rightarrow \mathcal{D}) \mapsto \bar{U}: X \mapsto UX$$

$$(\eta: U \Rightarrow V) \mapsto \bar{\eta}: \bar{U} \Rightarrow \bar{V}$$

$$UX \xrightarrow{\eta \circ VX} VX \quad \bar{U}X \xrightarrow{\bar{\eta} \circ \bar{V}X} \bar{V}X$$

$$\begin{array}{ccc} UX & \xrightarrow{\eta \circ VX} & VX \\ \downarrow U\eta & \searrow V\eta & \\ \bar{U}X & \xrightarrow{\bar{\eta} \circ \bar{V}X} & \bar{V}X \end{array}$$

Take a product

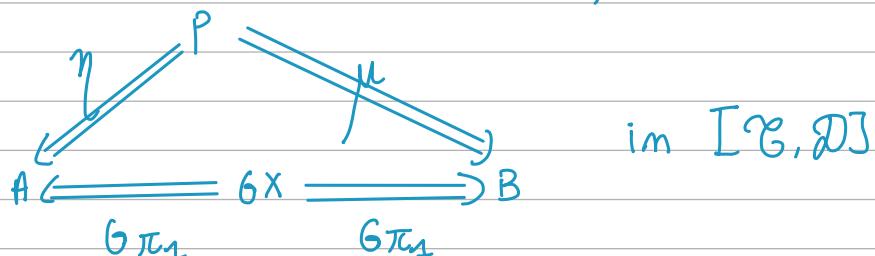
$$FA \xleftarrow{\pi_1} X \xrightarrow{\pi_2} FB$$

in $[\text{ob}(\mathcal{C}), \mathcal{D}]$, then

$$A = GFA \xleftarrow{G\pi_1} GX \xrightarrow{G\pi_2} GF B = B$$

with $G: [\text{ob}(\mathcal{C}), \mathcal{D}] \longrightarrow [\mathcal{C}, \mathcal{D}]$ the inclusion map.

Take $P \in [\mathcal{C}, \mathcal{D}]$ with $\eta: P \Rightarrow A$ $\mu: P \Rightarrow B$ commutes



Then we have that

$$\begin{array}{ccccc}
 & & FB & & \\
 & F\eta \swarrow & \downarrow \gamma & \searrow F\mu & \\
 FA & \xleftarrow{\pi_1} & X & \xrightarrow{\pi_1} & FB
 \end{array}$$

commutes in $[ob(\mathcal{C}), \mathcal{D}]$ and so

$$\begin{array}{ccccc}
 & P & & & \\
 \eta \swarrow & \downarrow G\gamma & \searrow \mu & & \\
 A & \xleftarrow{G\pi_1} & GX & \xrightarrow{G\pi_1} & B
 \end{array}
 \quad \text{in } [\mathcal{C}, \mathcal{D}]$$

commutes. Unicity is by $\text{Nat}(P, GX) \hookrightarrow \text{Nat}(FP, X)$.

Given a graph G , we define $el(G)$ the category

- objects pairs (c, x) with $c \in \mathcal{G}$ and $x \in X(c)$
- morphisms $f : (c, x) \longrightarrow (c', y)$ st $f : c \rightarrow c'$ and $x(f) = y$.

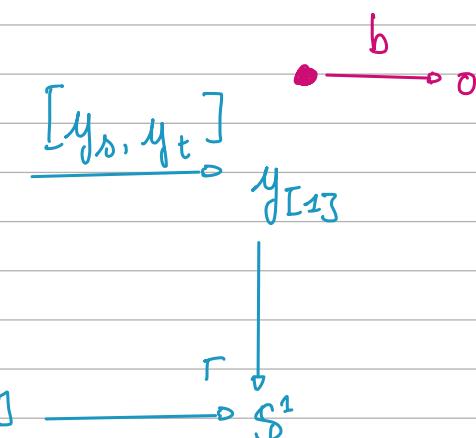
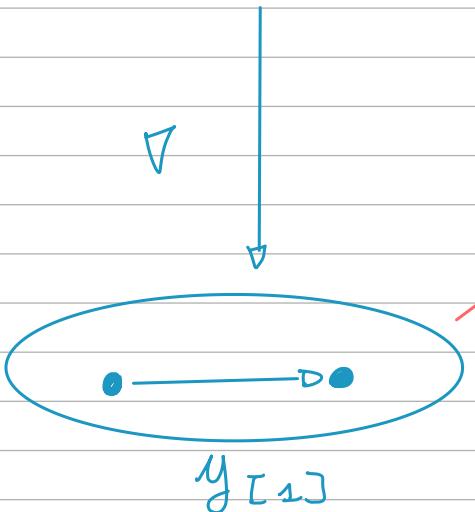
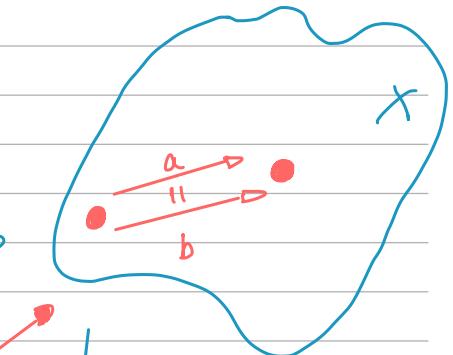
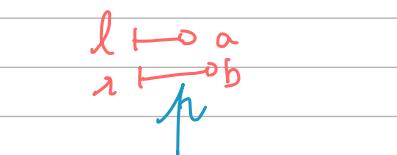
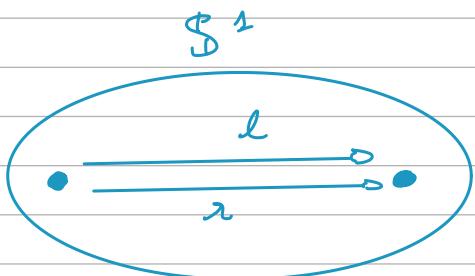
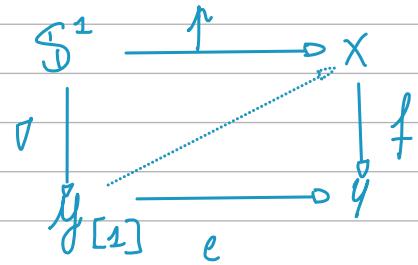
$$el(X) \xrightarrow{\pi_X} \mathcal{G} \xrightarrow{Y} \hat{\mathcal{G}} = \text{Graph.}$$

Proposition (co-Yoneda) Every presheaf $X \in \hat{\mathcal{C}}$ is the colimit of

$$el(X) \xrightarrow{\pi_X} \mathcal{G} \longrightarrow \hat{\mathcal{C}}$$

for a small \mathcal{G} .

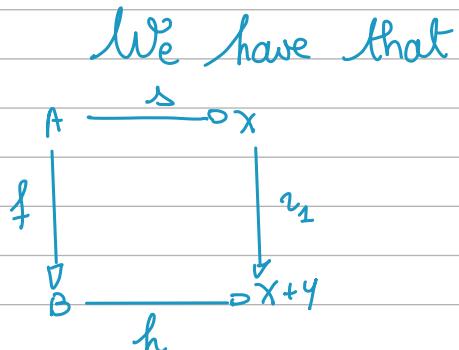
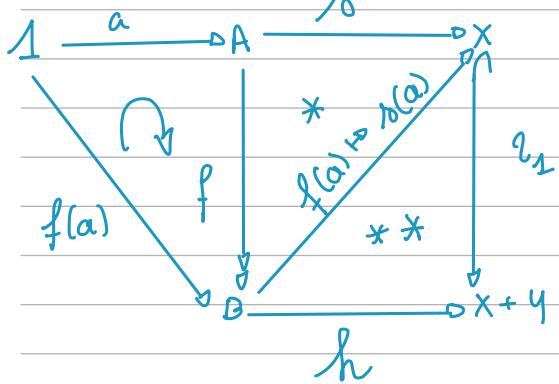
A functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is faithful if, for every $F_{AB}: \mathcal{C}(A, B) \longrightarrow \mathcal{D}(F(A), F(B))$
is injective.



A graph is simple if $\frac{e}{e'} \Rightarrow$ implied $e = e'$

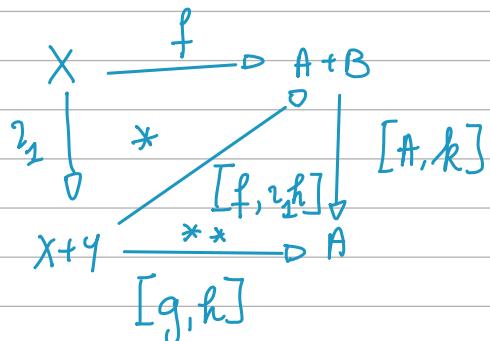
Equivalently if $X \rightarrow T$ is faithful.

Surjective / Injective functions in Set form a factorization system



commutes, and so we immediately have the required commutations (*) and (**).

Injective / surjective too



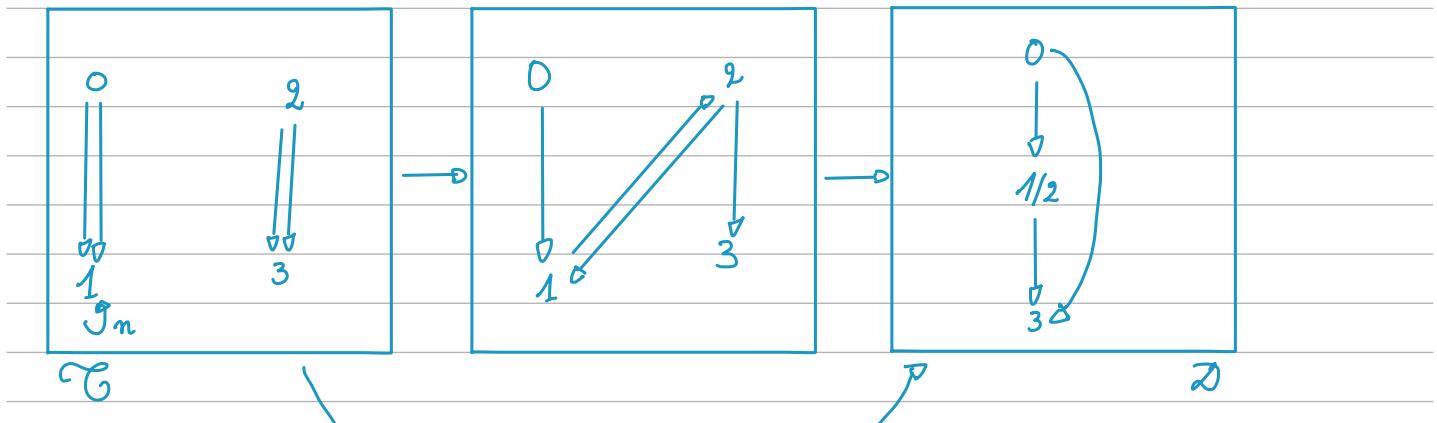
\mathcal{C} $\xrightarrow{\text{bijective on objects}}$ M_F $\xrightarrow{\text{fully faithful}}$ \mathcal{D}

F $M_F : \underline{\text{objects}}: \underline{\text{ob}}(\mathcal{C})$
 $\underline{\text{morphisms}}:$

$\mathcal{C}(A, B) / \{f, g \mid Ff = Fg\}.$

or alternatively,

$\mathbf{Hom}_{M_F}(A, B) := \mathbf{Hom}_{\mathcal{D}}(FA, FB)$

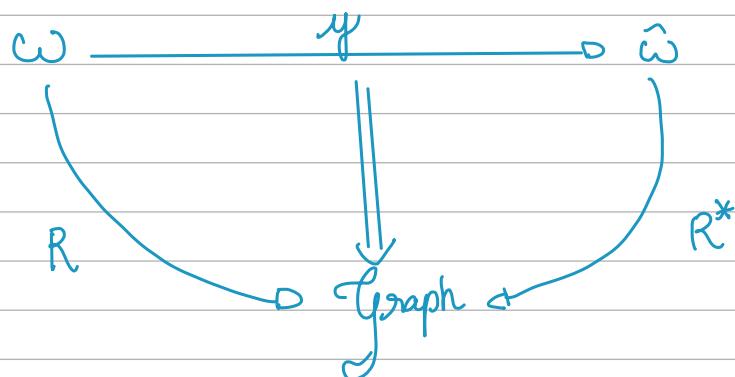


Theorem. $(\wedge(\mathcal{J}^\perp), \wedge\mathcal{J})$ is a factorization system where \mathcal{J} is a set of morphisms in \mathcal{C} (for some "suitable" \mathcal{C}).

Exercise Design a \mathcal{S} such that the full subcategory of graphs X with $!x \in \mathcal{J}^\perp$ is equivalent to the category of undirected graph.

$$\mathcal{J} := \left\{ \begin{array}{l} \square : S^1 \longrightarrow Y_{[1]} \\ U : Y_{[1]} \longrightarrow S^1 \end{array} \right\}$$

Where $S^1 \cup : \bullet \rightrightarrows \bullet$



$$R(n) := 0 \rightarrow_1 \rightarrow \cdots \rightarrow_{n-1}$$

$$R(n \leq p) := \text{prefix } R(n) \rightarrow R(p)$$

$$R^*(X)(n) = \text{Graph}(R(n), X) = \text{paths of length } n$$

$R^*(X)(n \leq p)$: inclusion to length p

Comma category

Given $\mathcal{C} \xrightarrow{F} \mathcal{E} \leftarrow G \mathcal{D}$

define F/G with

- objects: $(C \in \mathcal{C}, D \in \mathcal{D}, f \in E(FC, GD))$

- morphisms: $(C, D, f) \xrightarrow{(u, v)} (C', D', f')$ such that

$$\begin{array}{ccc} FC & \xrightarrow{u} & FC' \\ \downarrow f & & \downarrow g \\ GD & \xrightarrow{\nu} & GD' \end{array}$$

$$\mu' \circ u$$

This forms a category:

$$\begin{array}{ccc} FC & \xrightarrow{u} & FC' & \xrightarrow{u'} & FC'' \\ \downarrow f & & \downarrow f & & \downarrow g \\ GD & \xrightarrow{\nu} & GD' & \xrightarrow{\nu'} & GD'' \end{array}$$

$$\mu' \circ \nu$$

$$\mu'' \circ \nu'$$

$$\mu'' \circ \nu' \circ \mu$$

$$\begin{array}{ccccc} \text{Graph} & \longrightarrow & \text{Set} & \leftarrow & \mathbb{A} \\ & & & & | \\ G & \longleftarrow & E(G) = G[1] & & \end{array}$$

Then the comma category \mathbb{E}/\mathbb{A} is equivalent to the category of labelled graphs.

$$\begin{array}{ccc} X[1] & \xrightarrow{f[1]} & Y[1] \\ & \searrow & \swarrow \\ & \mathbb{A} & \end{array}$$

$$\begin{array}{ccc} \mathbb{A}^* & \longrightarrow & \widehat{\mathbb{A}}^* \\ & \searrow R & \swarrow R^* \\ & \text{Graph}/\mathbb{A} & \end{array}$$

$R^*(x, l)(a_1 \dots a_n)$
 $= \text{Graph}/\mathbb{A}((\underline{[n+1]}, \underline{[a_1 \dots a_n]}), (x, l))$