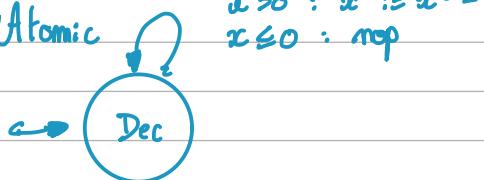
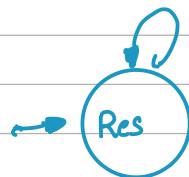


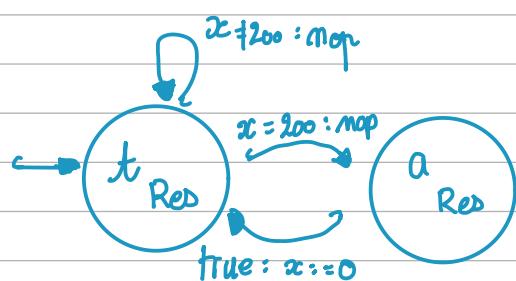
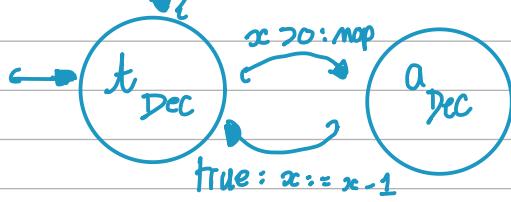
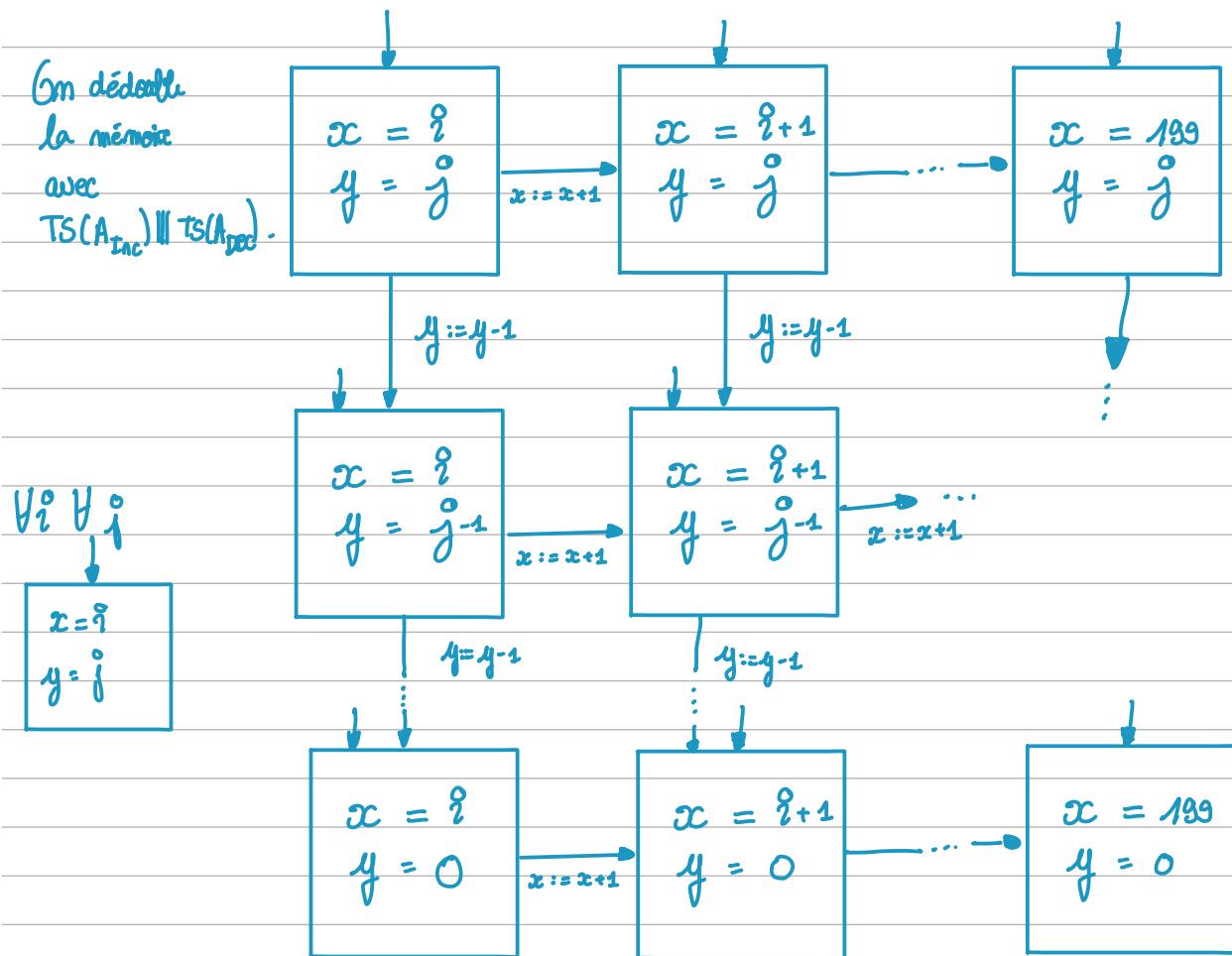
Modelling Concurrent Systems

Exercise 1.

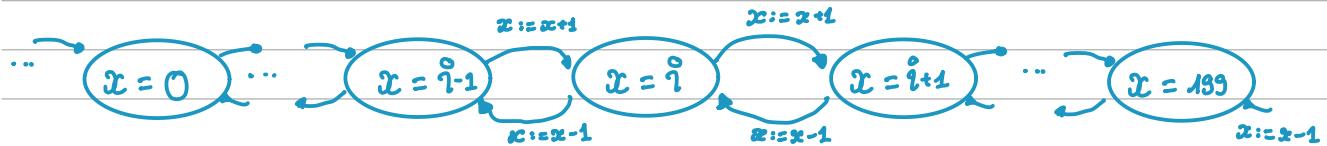
Q1. Atomic


 $x = 200 : x := 0$
 $x \neq 200 : \text{nopl}$


CP non-atomic

Q2. $TS(A_{\text{Inc}}) \parallel TS(A_{\text{Dec}})$ 

TS ($A_{Inc} \parallel A_{Dec}$):

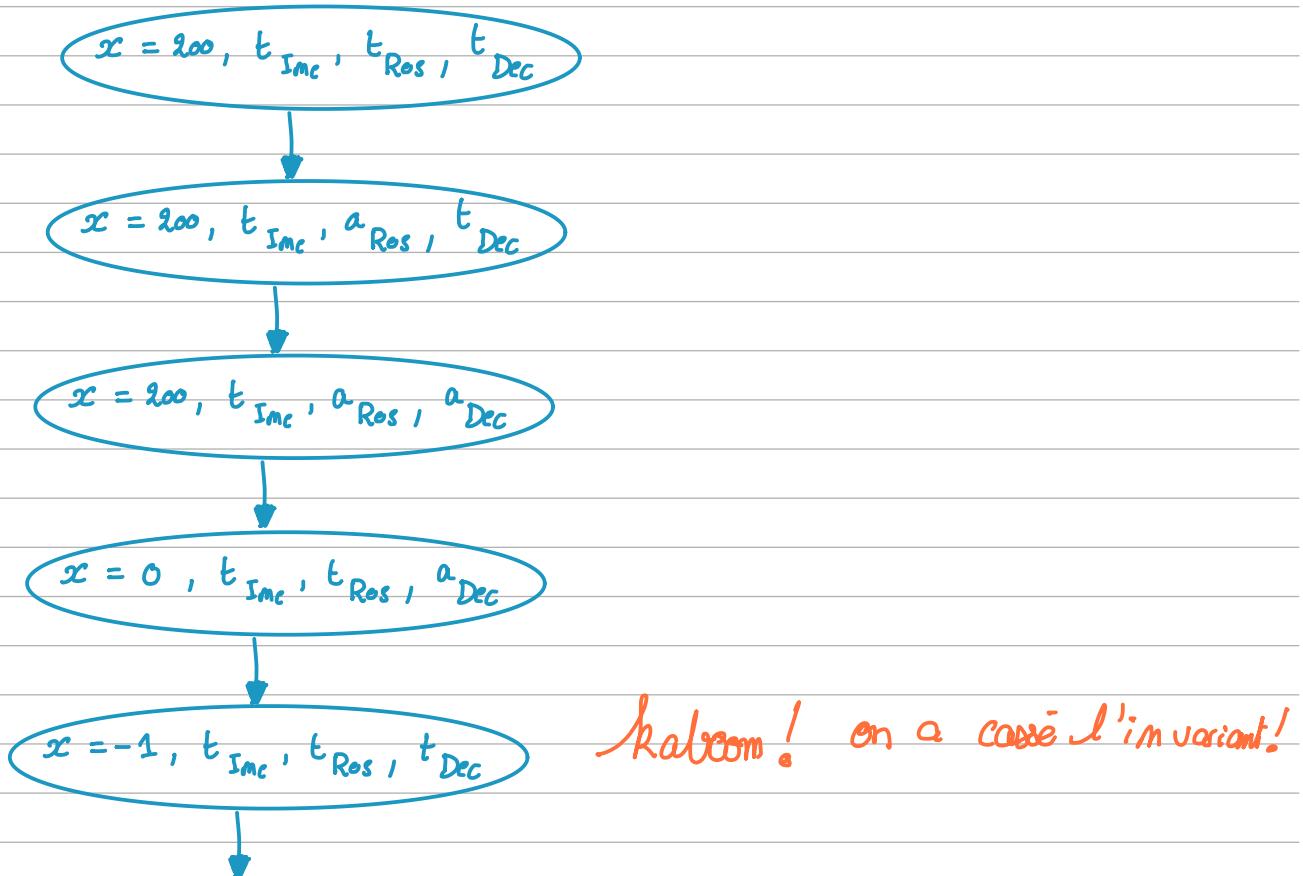


Q3. Les effets préseruent l'invariant :

- pour $x := x+1$ si $x < 200$ et $0 \leq x \leq 999$ alors $0 \leq x+1 \leq 200$
- pour $x := x-1$ si $x > 0$ et $0 \leq x \leq 999$ alors $0 \leq x-1 \leq 200$
- pour $x := 0$ si $x = 200$ et $0 \leq x \leq 999$ alors $0 \leq 0 \leq 200$

D'où l'invariant est invariant.

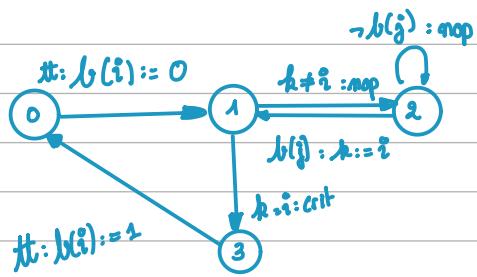
Q4.



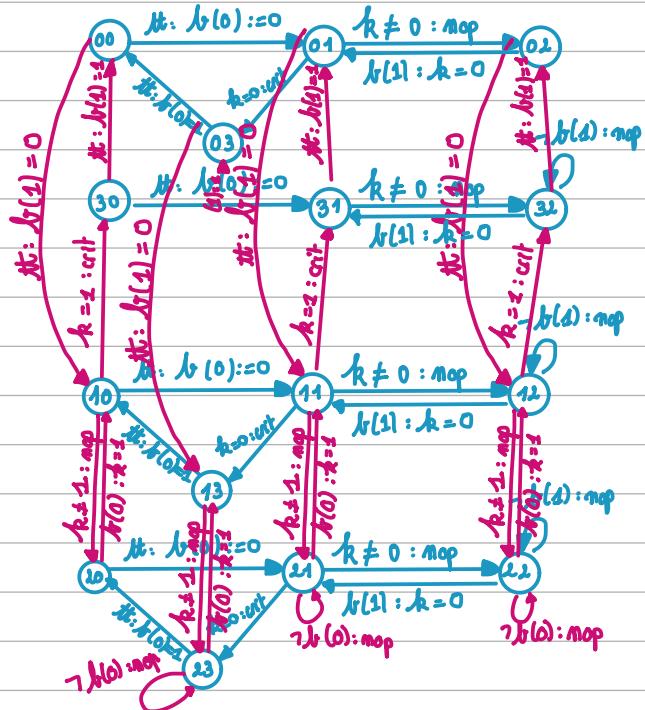
Exercice 2.

Q1.

Q1.



Q2.



Q3. The state 33 is unreachable. Thus, we ensure mutual exclusion.

Exercise 3.

In ex 1 / Q2,

we have a state $\begin{matrix} x = 1 \\ x^1 = 0 \end{matrix}$

We don't have that in

$T S(PG_1 \parallel\!\!\!|| PG_2)$.

TD n° 2

Linear Time Properties (and a bit of modelling.)

I. Safeness and Invariance

Exercise 1. Invariance is safe.

Define $P_{\text{bad}} := \{ \hat{\tau} \in (\mathcal{Z}^{\text{AP}})^* \mid \exists i. \hat{\tau}(i) \not\models \psi \}$.

Then,

$$\begin{aligned} \tau \in P &\Leftrightarrow \forall i. \tau(i) \models \psi \\ &\Leftrightarrow \forall \hat{\tau} \subseteq_{\text{fini}} \tau, \forall i \leq \text{length } \hat{\tau}, \hat{\tau}(i) \models \psi \\ &\Leftrightarrow \forall \hat{\tau} \subseteq_{\text{fini}} \tau, \text{non}(\exists i. \hat{\tau}(i) \not\models \psi) \\ &\Leftrightarrow \forall \hat{\tau} \subseteq_{\text{fini}} \tau, \hat{\tau} \notin P_{\text{bad}}. \end{aligned}$$

Thus P is a safety property.

Exercise 2.

1. \emptyset inv. safety

2. $\{ \tau \mid \forall i. \tau(i) \models x = 0 \}$ inv. safety

3. $\{ \tau \mid \forall i. \tau(i) \models \neg(x = 0) \wedge \neg(x > 1) \} = \{ \emptyset \}^\omega$ inv. safety

4. $\{ \tau \mid \tau(0) \models x = 0 \}$ safety $P_{\text{bad}} := \{ \hat{\tau} \mid \hat{\tau}(0) \not\models x = 0 \}$.

5. $\{ \tau \mid \tau(0) \models \neg(x = 0) \}$ safety $P_{\text{bad}} := \{ \hat{\tau} \mid \hat{\tau}(0) \models x = 0 \}$

6. $\{ \tau \mid \tau(0) \models x = 0 \text{ and } \exists i. \tau(i) \models x > 1 \}$ neither

7. $\{ \tau \mid \exists N \forall i \geq N, \tau(i) \models \neg(x > 1) \}$ neither

8. $\{ \tau \mid \forall N \exists i \geq N, \tau(i) \models x > 1 \}$ neither

9. $(\mathcal{Z}^{\text{AP}})^*$ inv. safety

II Operations on Safety Properties

Exercise 3. Characterization of safety properties

Q1. We have that

$$\begin{aligned} P \text{ safety property} &\Leftrightarrow \exists P_{\text{bad}}, \forall \tau, (\tau \in P \Leftrightarrow \text{Pref } \tau \cap P_{\text{bad}} = \emptyset) \\ &\Leftrightarrow \exists P_{\text{bad}}, \forall \tau, (\tau \in P^c \Leftrightarrow \text{Pref } \tau \cap P_{\text{bad}} \neq \emptyset) \\ (* \Leftrightarrow) \quad \forall \tau, (\tau \in P^c \Leftrightarrow \text{Pref } \tau \cap P = \emptyset) \end{aligned}$$

For (*), " \Leftarrow " we take $P_{\text{bad}} := (2^{AP})^* \setminus P$ and we have the required property.

" \Rightarrow " We have $\tau \in \text{Pref } \tau$ and we can conclude.

Q2. We show P safety $\Leftrightarrow P \supseteq \text{cl } P$.

We always have $\text{cl } P \supseteq P$.

" \Rightarrow " Let $\tau \in P^c$, we will show $\tau \notin \text{cl } P$.

i.e. $\text{pref } \tau \not\subseteq \text{pref } P$

we have $\hat{\tau} \in \text{pref } \tau$ and $\hat{\tau} \in \text{pref } P$ by (*)

" \Leftarrow " Let $\tau \in P^c$. we will show $\exists \hat{\tau} \in \text{pref } \tau$ such that $\hat{\tau} \notin \text{pref } P$
i.e. $\text{pref } \tau \not\subseteq \text{pref } P$.

Exercise 4. Union & intersections.

Q1. For $\sigma \in (P \cup Q)^c = P^c \cap Q^c$, there exists $\hat{\sigma}_P \subseteq_{\text{fin}} \sigma$ s.t. $\hat{\sigma}_P \cdot (2^{AP})^\omega \cap P = \emptyset$
and $\hat{\sigma}_Q \subseteq_{\text{fin}} \sigma$ s.t. $\hat{\sigma}_Q \cdot (2^{AP})^\omega \cap Q = \emptyset$.

Let $\hat{\sigma} \subseteq \hat{\sigma}_P$ and $\hat{\sigma} \subseteq \hat{\sigma}_Q$. We have

$$\begin{aligned} \hat{\sigma} \cdot (2^{AP})^\omega \cap (P \cup Q) &= (\hat{\sigma} \cdot (2^{AP})^\omega \cap P) \cup (\hat{\sigma} \cdot (2^{AP})^\omega \cap Q) \\ &= (\hat{\sigma}_P \cdot (2^{AP})^\omega \cap P) \cup (\hat{\sigma}_Q \cdot (2^{AP})^\omega \cap Q) \\ &= \emptyset \end{aligned}$$

Thus $P \cup Q$ is a safety property.

$$Q2. (P \cap Q)_{bad} := P_{bad} \cup Q_{bad}$$

$$\begin{aligned}\tau \in P \cap Q &\Leftrightarrow \forall \hat{\tau} \subseteq_{\text{fini}} \tau \quad \hat{\tau} \notin P_{bad} \text{ and } \hat{\tau} \notin Q_{bad} \\ &\Leftrightarrow \forall \hat{\tau} \subseteq_{\text{fini}} \tau \quad \hat{\tau} \notin (P \cap Q)_{bad}\end{aligned}$$

III Safety properties and Transition Systems.

Exercise 5. Finite traces

$$\begin{aligned}\text{If } 1s \models P \text{ then, } \text{Tr}_{fin}(TS) \cap P_{bad} \\ = \text{pref}(\text{Tr}^\omega(TS)) \cap P_{bad}\end{aligned}$$

TD n° 3

Safety and Liveness Properties

I. Liveness properties.

Exercise 1. (Closure of liveness properties)

$$\begin{aligned} cl(P) = (\mathcal{L}^{AP})^\omega &\Leftrightarrow \forall \tau \in (\mathcal{L}^{AP})^\omega \quad pref \tau \subseteq pref P \\ &\Leftrightarrow \forall \hat{\tau} \in [\mathcal{L}^{AP}]^* \quad \hat{\tau} \in pref P \\ &\Leftrightarrow \forall \hat{\tau} \in (\mathcal{L}^{AP})^* \quad \exists \tau \in P, \hat{\tau} \leq \tau \\ &\Leftrightarrow P \text{ liveness property} \end{aligned}$$

Exercise 2 (Unions & intersections).

We proved that $cl(P \cup Q) = cl(P) \cup cl(Q)$
and $cl(P \cap Q) = cl(P) \cap cl(Q)$

thus $P \cup Q$ and $P \cap Q$ are liveness properties.

Even better: if $P \underline{\text{or}} Q$ is liveness, then $P \cup Q$ is liveness.

II Topology on infinite words

Exercise 3 (Σ^ω as a topological space)

We need to prove that $\Omega \Sigma^\omega$ is stable under arbitrary unions and finite intersections.

1) Let $\mathcal{U} \subseteq \wp(\Sigma^*)$. Define $\bar{\mathcal{U}} := \bigcup \mathcal{U}$. We have that

$$ext(\bar{\mathcal{U}}) = \bigcup_{u \in \bar{\mathcal{U}}} ext(u) = \bigcup_{U \in \mathcal{U}} \bigcup_{u \in U} ext(u)$$

thus $\Omega\Sigma^\omega$ is stable under arbitrary unions.

2) Let $U, V \subseteq \Sigma^*$.

We have that

$$\text{ext}(U) \cap \text{ext}(V) = \bigcup_{u \in U} \bigcup_{v \in V} \text{ext}(u) \cap \text{ext}(v)$$

For some $u \in U, v \in V$, we have the 3 following cases:

a) $u \notin v$ and $v \notin u$ thus $\text{ext}(u) \cap \text{ext}(v) = \emptyset$

b) $u \subseteq v$ thus $\text{ext}(u) \cap \text{ext}(v) = \text{ext}(u)$

c) $v \subseteq u$ thus $\text{ext}(u) \cap \text{ext}(v) = \text{ext}(v)$

Define

$$W := \{u \in U \mid \exists v \in V, u \subseteq v\} \cup \{v \in V \mid \exists u \in U, v \subseteq u\}$$

and we have $\text{ext}(W) = \text{ext}(U) \cap \text{ext}(V)$.

By induction, $\Omega\Sigma^\omega$ is stable under finite intersections.

Exercise 4. (Open sets)

Let $P \subseteq \Sigma^\omega$.

P is open iff $\exists U \subseteq \Sigma^* \quad P = \bigcup_{u \in U} \text{ext}(u)$

iff $\forall \tau \in P, \exists u \in \Sigma^*, \tau \in \text{ext}(u)$
 $\forall u \in \Sigma^*, \text{ext}(u) \subseteq P$

iff $\forall \tau \in P, \exists u \in \Sigma^*, \tau \in \text{ext}(u) \subseteq P$

iff $\forall \tau \in P, \exists \hat{\tau} \in \Sigma^*, \hat{\tau} \leq \tau$ and $\text{ext}(\hat{\tau}) \subseteq P$.

Exercise 5. (Density and liveness)

P is dense iff for any non-empty open set U' , $P \cap U' \neq \emptyset$

iff for any non-empty $U \subseteq \Sigma^*$, $P \cap \text{ext}(U) \neq \emptyset$

iff for any $u \in \Sigma^*$, $P \cap \text{ext}(uP) \neq \emptyset$

iff for any $\hat{\tau} \in \Sigma^*$, $\exists \tau \in P$, $\hat{\tau} \subseteq \tau$.

iff P is a liveness.

III. Decomposition theorem.

Exercise 6. Let $P \in (\mathcal{Z}^{AP})^\omega$.

Define $P_{\text{safe}} := \text{cl}(P)$ which is a safety property as $\text{cl}(P_{\text{safe}}) = \text{cl}^2(P) = \text{cl}(P)$

and $P_{\text{liveness}} = P \cup P_{\text{safe}}^C$ which is a liveness property as P_{liveness} is

topologically dense: if $U \subseteq \Sigma^*$, and $\text{ext}(U) \cap P = \emptyset$ then $P \subseteq \text{ext}(U)^C$ and thus $\text{cl}(P) \subseteq \text{ext}(U)^C$ so we can conclude $\text{ext}(U) \subseteq \text{cl}(P)^C$. (closed)

thus proving the decomposition theorem.

Exercise 7.

Q1. $\text{cl}(P_1) = P_1$ thus liveness and $P_{\text{safe}} = \Sigma^\omega$

Q2. $\text{cl}(P_2) = \{\tau \mid \tau(0) = a\}$ thus not liveness and not safety, $P_{\text{liveness}} = \Sigma^\omega \setminus \{a^\omega\}$

Q3. $\text{cl}(P_3) = \{a^\omega\} = P_3$ thus liveness and $P_{\text{safe}} = \Sigma^\omega$

Q4. $\text{cl}(P_4) = \{\tau \mid \tau \text{ contains } \leq 1 b's\} = P_4 \cup \{a^\omega\}$ thus not liveness and not safety, $P_{\text{liveness}} = \Sigma^\omega \setminus \{a^\omega\}$.

Q5. $\text{cl}(P_5) = \Sigma^\omega$ thus liveness and $P_{\text{safe}} = \Sigma^\omega$

Q6. $\text{cl}(P_6) = \sum^\omega$ thus liveness and $P_{\text{safe}} = \sum^\omega$

Q7. $\text{cl}(P_7) = \sum^\omega$ thus liveness and $P_{\text{safe}} = \sum^\omega$

Q8. $\text{cl}(P_8) = \sum^\omega$ thus liveness and $P_{\text{safe}} = \sum^\omega$.