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Tutorial n° 1.

2. Linear algebra

A. 1. $(A^\dagger)^\dagger = \overline{(\overline{A}^T)}^T = \overline{\overline{(A^{TT})}} = \overline{\overline{A}} = A$

2. $(AB)^\dagger = \overline{(AB)}^T = \overline{(B^T A^T)} = \overline{B}^T \overline{A}^T = B^\dagger A^\dagger.$
identique pour $(Av)^\dagger = v^\dagger A^\dagger.$

3. $\langle A^\dagger u, v\rangle = (A^\dagger u)^\dagger v = u^\dagger A^{\dagger\dagger} v = u^\dagger A v = \langle u, Av\rangle.$

B. 1. Si A est hermitienne, $AA^\dagger = AA = A^\dagger A$ donc A normale.
Si A est unitaire, $AA^\dagger = AA^{-1} = \mathbb{1} = A^{-1}A = A^\dagger A$ donc A normale.

2. $(UV)^\dagger = V^\dagger U^\dagger = V^{-1}U^{-1} = (UV)^{-1}$ donc UV est unitaire.

3. $(G+H)^\dagger = \overline{G+H}^T = (\overline{G}+\overline{H})^T = \overline{G}^T + \overline{H}^T = G^\dagger + H^\dagger = G+H$
donc $G+H$ est hermitienne

4. $(vv^\dagger)^2 = v\underbrace{v^\dagger v}v^\dagger = \langle v,v\rangle vv^\dagger = \|v\|^2 vv^\dagger = vv^\dagger$
car v est unitaire

$(vv^\dagger)^\dagger = v^{\dagger\dagger} v^\dagger = vv^\dagger$ donc $vv^\dagger$ est bien une matrice de projection.

Soit $\lambda \in \mathbb{C}$ et u un vecteur.

On a : $P^2 u = Pu.$

$Pu = \lambda u \implies P(Pu) = \overbrace{P(\lambda u)}^{\lambda^2 u} = \lambda u$
$\implies \lambda^2 = \lambda$

D'où $\lambda = 0$ ou $\lambda = 1.$

3. Quantum random access code.

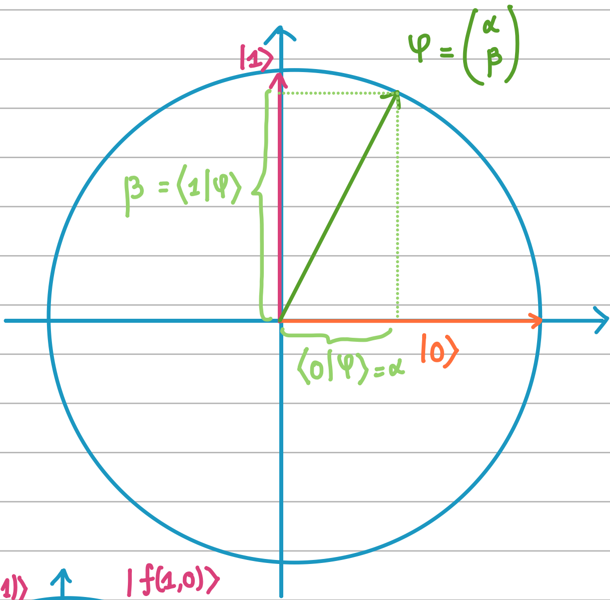
1. On a $f : \{0,1\}^2 \to \{0,1\}$ donc on a nécessairement une collision.
Sans perdre en généralité, supposons $f(0,x) = f(1,x).$

D'où, pour obtenir le 1er bit, on a :
$f(0,x) \underset{q_0}{\rightsquigarrow} 0$ et $f(1,x) \underset{q_1}{\rightsquigarrow} 1.$

Nécessairement, $q_0 + q_1 = 1$ et $q_0 \geq p$, $q_1 \geq p$,

d'où $p \leq \frac{1}{2}$.

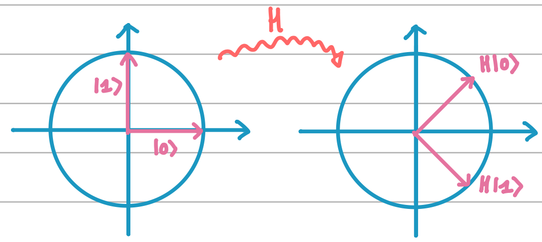
2.



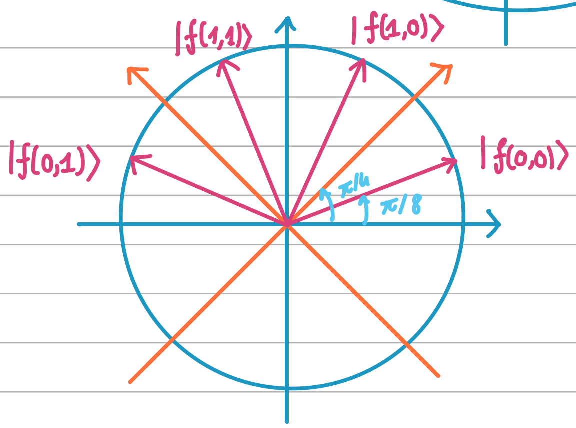
Appliquer une matrice unitaire correspond à une rotation ou une symmétrie de ce cercle unitaire (et du vecteur φ dedans).

Exemple avec la matrice de Hadamard

$$H = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$



3.



C'est une symmétrie sur l'axe x puis une rotation de 45°.

$U_1 = \mathbb{1}$ et $U_2 = R_{\pi/4}$.

La probabilité de succès est de $\cos^2(\pi/8) \simeq 0{,}86$.

4. Tensor products

1.

$$A \otimes B = \begin{pmatrix} 0 & e^{2i\pi/3} & 0 & e^{i\pi/3} \\ e^{-2i\pi/3} & 0 & -1 & 0 \\ 0 & -1 & 0 & e^{-i\pi/3} \\ e^{i\pi/3} & 0 & e^{i\pi/3} & 0 \end{pmatrix}; \quad B \otimes A = \begin{pmatrix} 0 & 0 & e^{2i\pi/3} & e^{i\pi/3} \\ 0 & 0 & -1 & e^{-i\pi/3} \\ e^{-2i\pi/3} & -1 & 0 & 0 \\ e^{-i\pi/3} & e^{i\pi/3} & 0 & 0 \end{pmatrix}$$

2.

a) $$A \otimes (\lambda B) = \begin{pmatrix} a_{11}(\lambda B) & a_{21}(\lambda B) & \cdots \\ a_{12}(\lambda B) & a_{22}(\lambda B) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} (a_{11}\lambda)B & (a_{21}\lambda)B & \cdots \\ (a_{12}\lambda)B & (a_{22}\lambda)B & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} = \lambda A \otimes B$$

d) $$(A \otimes B)^\dagger = \begin{pmatrix} a_{11}B & a_{21}B & \cdots \\ a_{12}B & a_{22}B & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}^\dagger = \begin{pmatrix} \bar{a}_{11}B^\dagger & \bar{a}_{12}B^\dagger & \cdots \\ \bar{a}_{21}B^\dagger & \bar{a}_{22}B^\dagger & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

$$= A^\dagger \otimes B^\dagger$$

Tutorial 2

I. Warm-up calculations.

1) Measurements and probabilities

$\mathbb{P}(|\varphi\rangle \text{ is measured as } |b\rangle) = |\langle b|\varphi\rangle|^2$

Q1. $\| |\varphi\rangle \| = 1$ thus $|\varphi\rangle$ is a state
measuring $|\varphi\rangle$ leads to 0 with prabilityy 1/2
measuring $|\varphi\rangle$ leads to 1 with prabilityy 1/2

Q2. $\| |\varphi\rangle \| = 1$ thus $|\varphi\rangle$ is a state
measuring $|\varphi\rangle$ leads to 0 with prabilityy 1/2
measuring $|\varphi\rangle$ leads to 1 with prabilityy 1/2

Q3. $\| |\varphi\rangle \| = 1$ thus $|\varphi\rangle$ is a state
measuring $|\varphi\rangle$ leads to 0 with prabilityy 1
measuring $|\varphi\rangle$ leads to 1 with prabilityy 0

Q4. $\| |\varphi\rangle \| = 1$ thus $|\varphi\rangle$ is a state
measuring $|\varphi\rangle$ leads to 0 with prabilityy 1/2
measuring $|\varphi\rangle$ leads to 1 with prabilityy 1/2

Q5. $\| |\varphi\rangle \| = 1$ thus $|\varphi\rangle$ is a state
measuring $|\varphi\rangle$ leads to 00 with prabilityy 1/2
measuring $|\varphi\rangle$ leads to 01 with prabilityy 0
measuring $|\varphi\rangle$ leads to 10 with prabilityy 0
measuring $|\varphi\rangle$ leads to 11 with prabilityy 1/2

Q6. $\| |\varphi\rangle \| = 1$ thus $|\varphi\rangle$ is a state
measuring $|\varphi\rangle$ leads to 00 with prabilityy 1/2
measuring $|\varphi\rangle$ leads to 01 with prabilityy 0
measuring $|\varphi\rangle$ leads to 10 with prabilityy 0
measuring $|\varphi\rangle$ leads to 11 with prabilityy 1/2

Q7. $\| |\varphi\rangle \| = 1$ thus $|\varphi\rangle$ is a state
measuring $|\varphi\rangle$ leads to 00 with prabilityy 1/4
measuring $|\varphi\rangle$ leads to 01 with prabilityy 1/2
measuring $|\varphi\rangle$ leads to 10 with prabilityy 0
measuring $|\varphi\rangle$ leads to 11 with prabilityy 1/4

Q8. $\| |\varphi\rangle \| = 1$ thus $|\varphi\rangle$ is a state

measuring $|\varphi\rangle$ leads to 00 with probability $1/4$
measuring $|\varphi\rangle$ leads to 01 with probability $1/2$
measuring $|\varphi\rangle$ leads to 10 with probability $1/8$
measuring $|\varphi\rangle$ leads to 11 with probability $1/8$

2) Partial measurements.

$$\sqrt{2}\,\langle 1|\psi\rangle = \langle 1|0\rangle\cdot\langle 1|\varphi\rangle + \langle 1|1\rangle\langle 1|\psi\rangle$$
$$= \langle 1|\psi\rangle$$
$$=$$

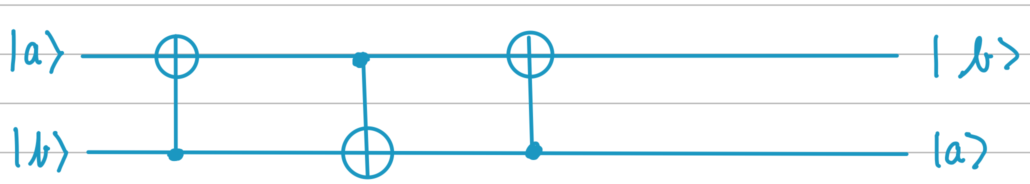
$$|\psi'\rangle := H\otimes 1\!\!1\ |\psi\rangle = \frac{1}{\sqrt{2}}\Big(|+\rangle|\varphi\rangle + |-\rangle|\psi\rangle\Big) = \frac{1}{2}\Big(|0\rangle\big(|\varphi\rangle + |\psi\rangle\big) + |1\rangle\big(|\varphi\rangle - |\psi\rangle\big)\Big)$$

We have that $\mathbb{P}(\text{measuring } |1\rangle) = \| |\varphi\rangle - |\psi\rangle \|_2^2 \times \frac{1}{4}$

$$= \big(\langle\varphi| - \langle\psi|\big)\big(|\varphi\rangle - |\psi\rangle\big) \times \frac{1}{4}$$
$$= \frac{1}{4}\Big(\langle\varphi|\varphi\rangle + \langle\psi|\psi\rangle - \langle\varphi|\psi\rangle - \langle\psi|\varphi\rangle\Big)$$
$$= \frac{1}{4}\Big(2 - \langle\varphi|\psi\rangle - \overline{\langle\varphi|\psi\rangle}\Big)$$
$$= \frac{1}{2}\Big(1 - \mathrm{Re}\big(\langle\varphi|\psi\rangle\big)\Big).$$

3) Gates

Q1.

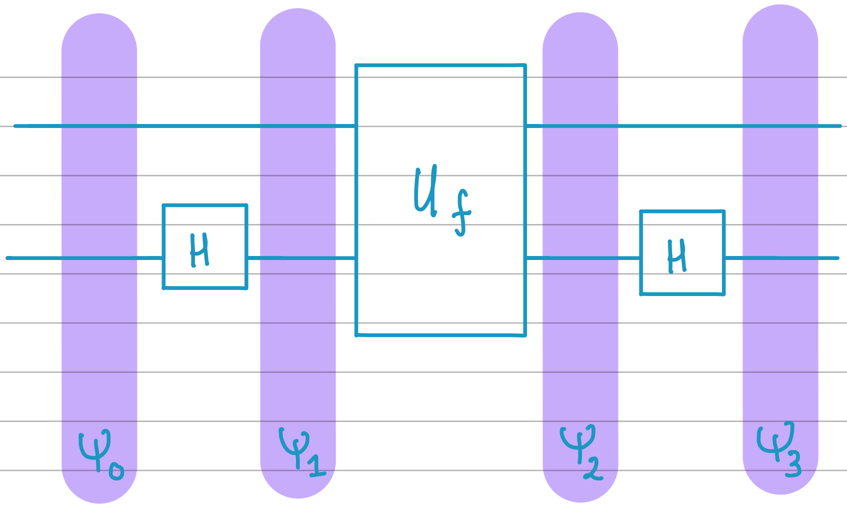


$|a\oplus b\rangle$
$|b\rangle$

$|a\oplus b\rangle$
$|b\oplus(a\oplus b)\rangle$
$\shortparallel$
$|a\rangle$

$|a\oplus b\oplus a\rangle = |b\rangle$
$|a\rangle$

Q2.



$$|\psi_0\rangle = |a\rangle|b\rangle$$

$$|\psi_1\rangle = (\mathbb{1}\otimes H)\,|a\rangle|b\rangle = \frac{1}{\sqrt{2}}|a\rangle\left(|0\rangle + (-1)^b|1\rangle\right)$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}\,U_f|a\rangle|0\rangle + \frac{(-1)^b}{\sqrt{2}}\,U_f|a\rangle|1\rangle$$
$$= \frac{1}{\sqrt{2}}|a\rangle|0\oplus f(a)\rangle + \frac{(-1)^b}{\sqrt{2}}|a\rangle|1\oplus f(a)\rangle$$
$$= \frac{1}{\sqrt{2}}|a\rangle\left(|f(a)\rangle + (-1)^b|\overline{f(a)}\rangle\right)$$

$$|\psi_3\rangle = (\mathbb{1}\otimes H)|\psi_2\rangle = \frac{1}{2}|a\rangle\left(\left(|0\rangle + (-1)^{f(a)}|0\rangle\right) + (-1)^b\left(|0\rangle + (-1)^{\overline{f(a)}}\right)\right)$$

Par étude des cas $b=0$ et $b=1$, on a bien que
$$|\psi_3\rangle = |a\rangle\,(-1)^{f(a)\,b}\,|b\rangle.$$

III. Superdense coding.

Q1. Two bits

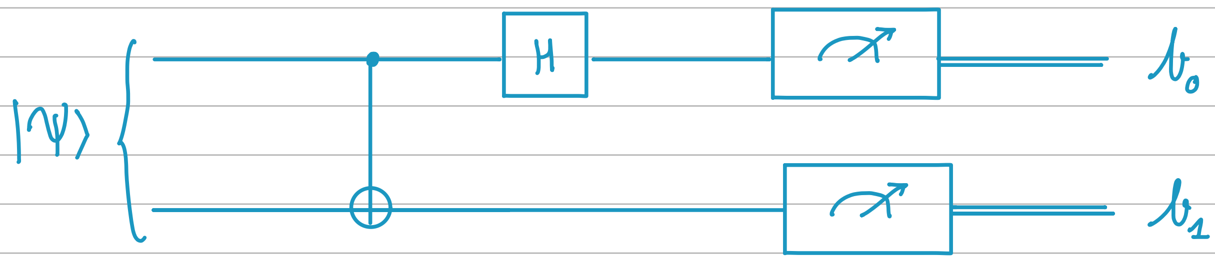
Q2.

00	⇝	$\lvert 00\rangle + \lvert 11\rangle$	$\to \lvert 00\rangle + \lvert 11\rangle$	$\to \lvert 00\rangle + \lvert 11\rangle$
01	⇝	$\lvert 00\rangle + \lvert 11\rangle$	$\to \lvert 10\rangle + \lvert 01\rangle$	$\to \lvert 10\rangle + \lvert 01\rangle$
10	⇝	$\lvert 00\rangle + \lvert 11\rangle$	$\to \lvert 00\rangle + \lvert 11\rangle$	$\to \lvert 00\rangle - \lvert 11\rangle$
11	⇝	$\lvert 00\rangle + \lvert 11\rangle$	$\to \lvert 10\rangle + \lvert 01\rangle$	$\to -\lvert 10\rangle + \lvert 01\rangle$

Tous ces états sont orthogonaux ; ils forment une base

$$\mathcal{B} = \left\{ \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \frac{|10\rangle + |01\rangle}{\sqrt{2}}, \frac{|00\rangle - |11\rangle}{\sqrt{2}}, \frac{|01\rangle - |10\rangle}{\sqrt{2}} \right\}.$$

Il suffit d'appliquer une matrice de passage de $\mathcal{B}$ dans $\mathcal{B}_{\text{Computationnelle}} = \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$.



Tutorial #3.

2. Some properties of circuits

2.1. Do circuits commute?

Q1. $A = \mathbb{1}_2 \otimes A'$ and $B = B' \otimes \mathbb{1}_2$ commute: $AB = BA = B' \otimes A$

Q2. $A = \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{X} \otimes \mathbb{1}_2$ and $B = H \otimes \mathbb{1}_2$ do not commute: $AB \neq BA$.

2.2. Are circuits ambiguous?

Q1. This is exactly 2.1/Q1.

Q2. Let $|\phi\rangle = \alpha|00\rangle + \beta|10\rangle + \gamma|01\rangle + \delta|11\rangle$.

When measuring the first qubit and then the second, we obtain that:

$$\Pr[1^{st} \text{ qubit is measured as } 0] = |\alpha|^2 + |\gamma|^2$$

$$\Pr[\text{measuring } 00] = \Pr[1^{st} \text{ qubit} \rightsquigarrow 0] \times \Pr[2^{nd} \text{ qubit} \rightsquigarrow 0 \mid 1^{st} \text{ qubit} \rightsquigarrow 0].$$

$$= (|\alpha|^2 + |\gamma|^2) \times \frac{|\alpha|^2}{|\alpha|^2 + |\gamma|^2} = |\alpha|^2$$

we can do the same for all the other cases.

When measuring the 2^{nd} qubit and then the 1^{st} qubit, we have:

$$\Pr[2^{nd} \text{ qubit} \rightsquigarrow 0] = |\alpha|^2 + |\beta|^2$$

and $\Pr[\text{measuring } 00] = \Pr[2^{nd} \text{ qubit} \rightsquigarrow 0] \times \Pr[1^{st} \text{ qubit} \rightsquigarrow 0 \mid 2^{nd} \text{ qubit} \rightsquigarrow 0]$

$$= (|\alpha|^2 + |\beta|^2) \times (|\alpha|^2 / (|\alpha|^2 + |\beta|^2))$$

$$= |\alpha|^2.$$

Q3. Let $|\Phi\rangle = \alpha|\Psi_0\rangle \otimes |0\rangle + \beta|\Psi_1\rangle \otimes |1\rangle$.

We have that:

$$\Pr[|\Phi\rangle \overset{\text{last}}{\rightsquigarrow} 0] = |\alpha|^2 \qquad + \qquad \Pr[|\Phi\rangle \overset{\text{last}}{\rightsquigarrow} 1] = |\beta|^2$$

$$(U \otimes \mathbb{1})|\Phi\rangle = \alpha(U|\Psi_0\rangle) \otimes |0\rangle + \beta(U|\Psi_1\rangle) \otimes |1\rangle.$$

$$\Pr[(U\otimes\mathbb{1})|\Phi\rangle \overset{\text{last}}{\rightsquigarrow} 0] = |\alpha|^2 \qquad + \qquad \Pr[(U\otimes\mathbb{1})|\Phi\rangle \overset{\text{last}}{\rightsquigarrow} 1] = |\beta|^2$$

3. The CHSH Game.

Q1. We have $A: \{0,1\} \to \{0,1\}$ and $B: \{0,1\} \to \{0,1\}$ two deterministic functions. With $A(x) = x$ and $B(y) = y$, we have a probability of success of $3/4$.

We know that the probability of success for any deterministic strategy is in $\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$. Suppose that we have a stategy with a sucess rate $> \frac{3}{4}$, i.e. $= 1$, that is,

$$\forall x, y, \qquad A(x) \oplus B(y) = x \wedge y.$$

This means $A(0) \oplus B(0) = A(1) \oplus B(0) = A(0) \oplus B(1) = 0$

By disjunction, we have no case such that this is true.

We can conclude that $\frac{3}{4}$ is the optimal success rate for a deterministic strategy.

Q2. (a)

$$\underbrace{\max_{n\in\mathbb{N}} \quad \max_{\lambda \text{ a random on } [n]} \quad \max_{(A_k,B_k)_{k\in[n]}} \sum_{k=0}^{n} \Pr[\lambda = k] \times \text{SuccessRate}(A_k, B_k)}$$

(b) SuccessRate(Shared Randomness) $\geqslant \max_{A,B}$ Success Rate $(A,B) = 3/4$.

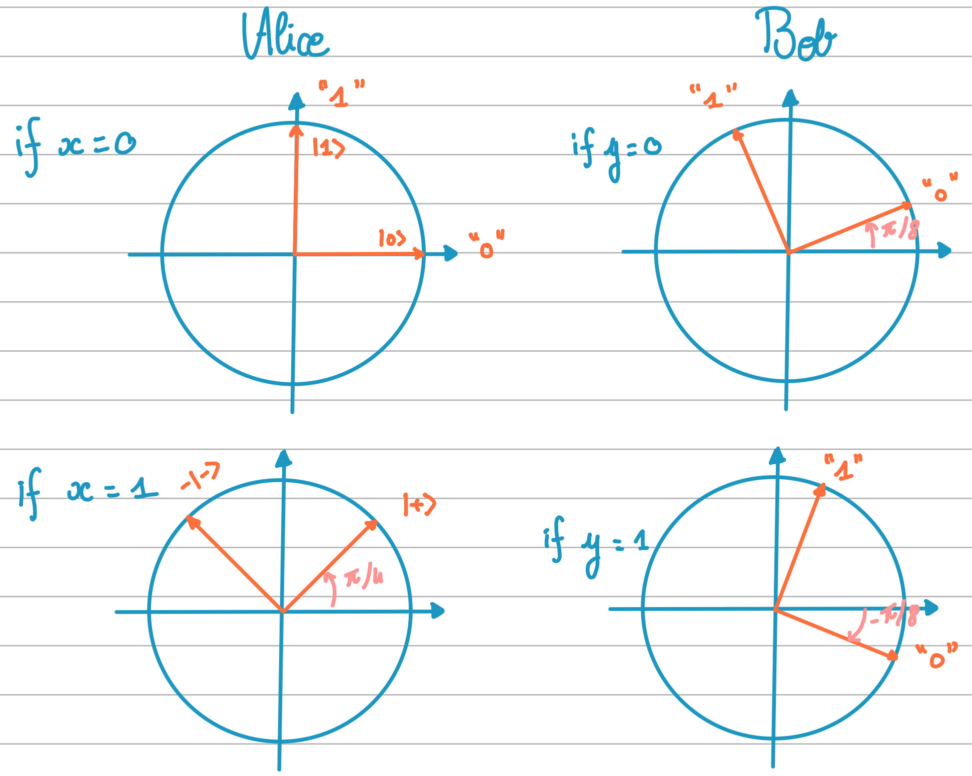
We have that

Fix $n \in \mathbb{N}$, λ a random variable over $[n]$ and $(A_k, B_k)_{k\in[n]}$.

$$\sum_{k=0}^{n} Pr[\lambda = k] \times \text{Success Rate}(A_k, B_k) \leq \sum_{k=0}^{n} Pr[\lambda = k] \cdot 3/4 = \frac{3}{4} \underbrace{\sum_{k=0}^{n} Pr[\lambda = k]}_{1} = \frac{3}{4}$$

Thus, Success Rate (Probabilistic Strategy).

Q3. We make the following measurments :



Success Rate $= \cos^2 \frac{\pi}{8} \approx 0.85$