TD n°1

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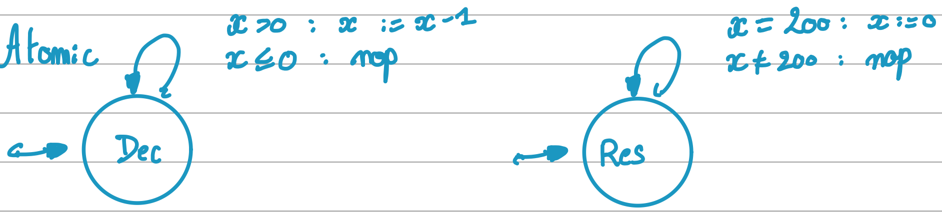
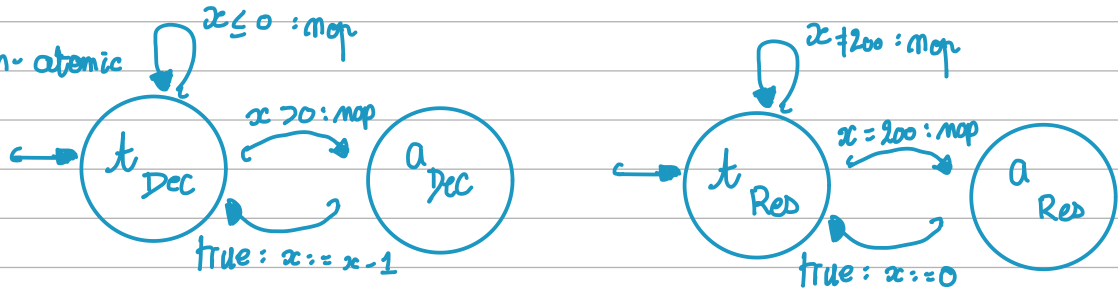
Modelling Concurrent Systems

Exercise 1.

Q1. Atomic

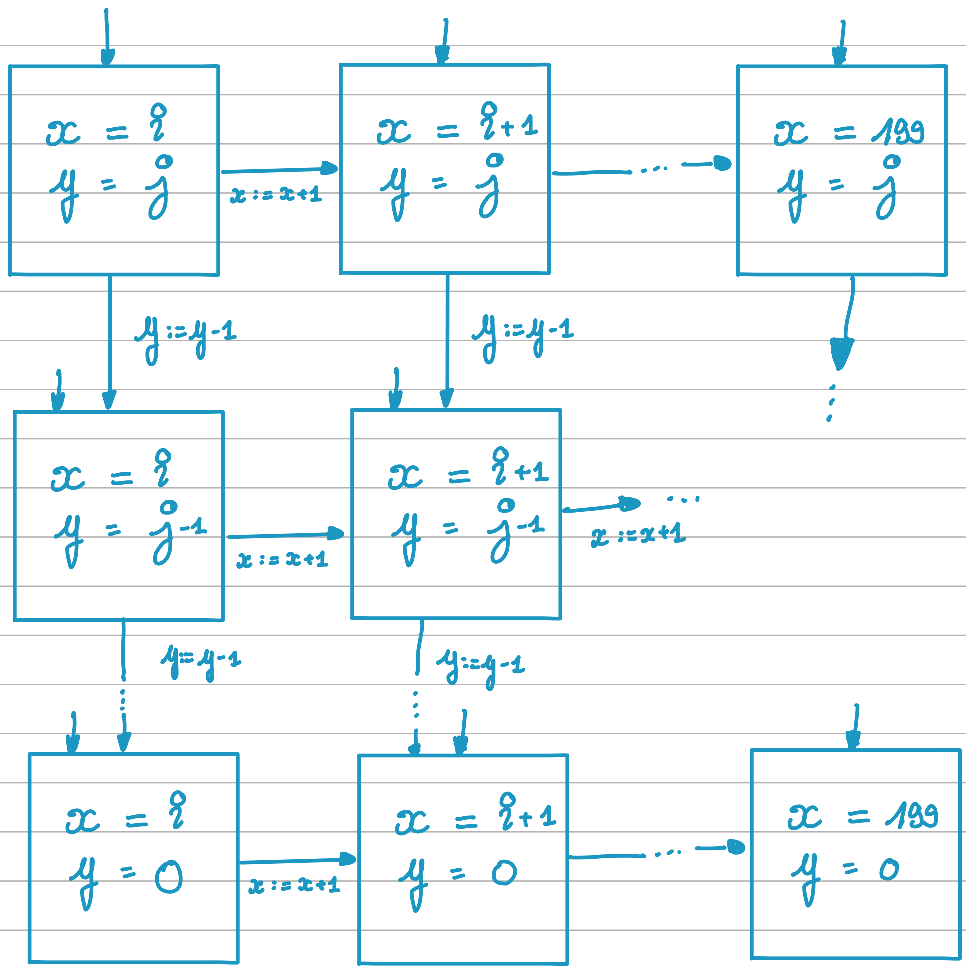


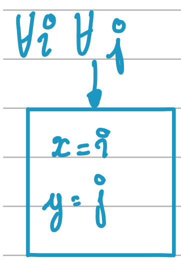
Non-atomic



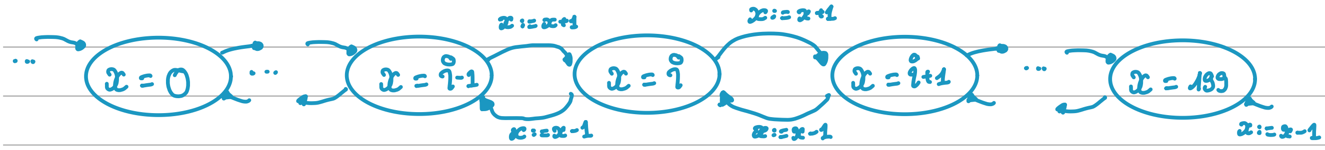
Q2. $TS(A_{Inc}) \,|||\, TS(A_{Dec})$

On déduit la mémoire avec $TS(A_{Inc}) ||| TS(A_{Dec})$.





$TS(A_{Inc} \parallel A_{Dec})$:

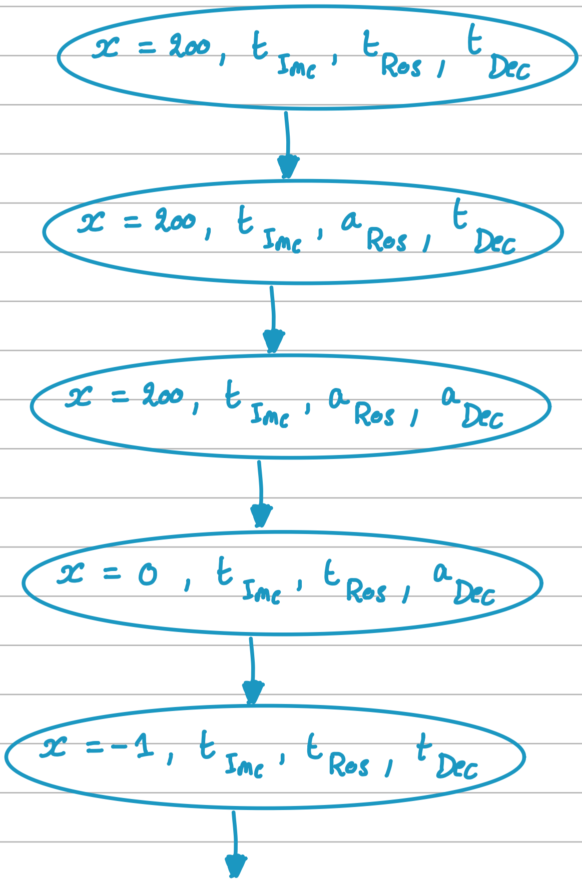


Q 3. Les effets préservent l'invariant :

- pour $x := x+1$ si $x < 200$ et $0 \leq x \leq 200$ alors $0 \leq x+1 \leq 200$
- pour $x := x-1$ si $x > 0$ et $0 \leq x \leq 200$ alors $0 \leq x-1 \leq 200$
- pour $x := 0$ si $x = 200$ et $0 \leq x \leq 200$ alors $0 \leq 0 \leq 200$

D'où l'invariant est invariant.

Q 4.

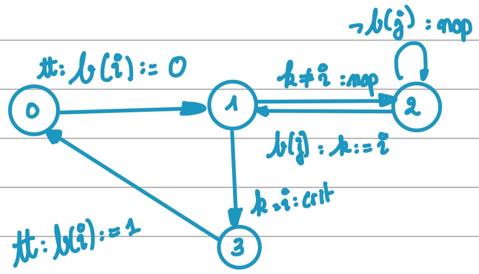


Kaboom ! on a cassé l'invariant !

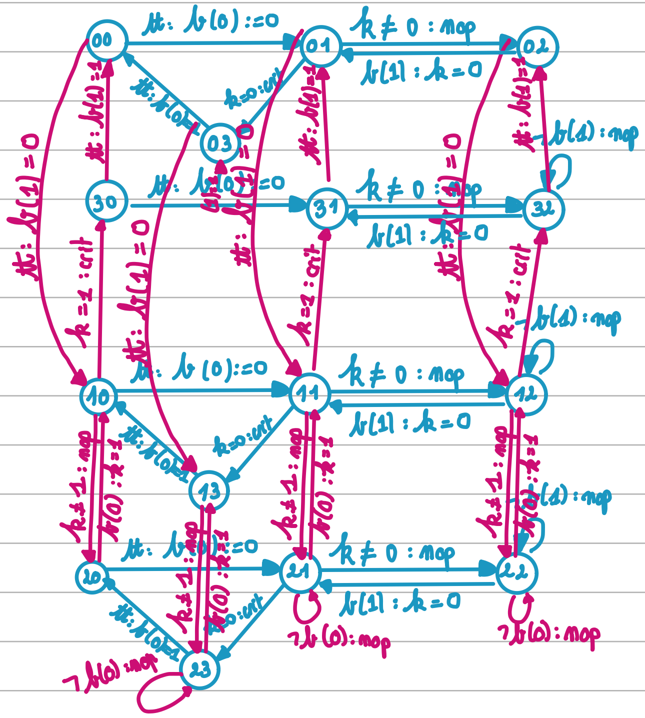
Exercise 2.

Q1.

Q1.



Q2.



Q3. The state 33 is unreachable. Thus, we ensure mutual exclusion.

Exercise 3.

In ex 1 / Q2,

we have a state $x = 1$, $x' = 0$

we don't have that in

$TS(PG_1 ||| PG_2)$.

TD n° 2

Linear Time Properties (and a bit of Modelling)

I. Safeness and Invariance

Exercise 1. Invariance is safe.

Define $P_{bad} := \{ \hat{\sigma} \in (2^{AP})^* \mid \exists i.\ \hat{\sigma}(i) \not\models \varphi \}$.

Then,

$$\begin{aligned}
\sigma \in P &\iff \forall i,\ \sigma(i) \models \varphi \\
&\iff \forall \hat{\sigma} \subseteq_{fini} \sigma,\ \forall i \leq \text{length}\ \hat{\sigma},\ \hat{\sigma}(i) \models \varphi \\
&\iff \forall \hat{\sigma} \subseteq_{fini} \sigma,\ \text{non}(\exists i.\ \hat{\sigma}(i) \not\models \varphi) \\
&\iff \forall \hat{\sigma} \subseteq_{fini} \sigma,\ \hat{\sigma} \notin P_{bad}.
\end{aligned}$$

Thus P is a safety property.

Exercise 2.

1. $\emptyset$ inv. safety
2. $\{\sigma \mid \forall i,\ \sigma(i) \models x = 0\}$ inv. safety
3. $\{\sigma \mid \forall i.\ \sigma(i) \models \neg(x=0) \wedge \neg(x > 1)\} = \{\emptyset\}^\omega$ inv. safety
4. $\{\sigma \mid \sigma(0) \models x = 0\}$ safety $P_{bad} := \{\hat{\sigma} \mid \hat{\sigma}(0) \not\models x = 0\}$.
5. $\{\sigma \mid \sigma(0) \models \neg(x=0)\}$ safety $P_{bad} := \{\hat{\sigma} \mid \hat{\sigma}(0) \models x = 0\}$
6. $\{\sigma \mid \sigma(0) \models x = 0$ and $\exists i.\ \sigma(i) \models x > 1\}$ neither
7. $\{\sigma \mid \exists N\ \forall i \geq N,\ \sigma(i) \models \neg(x > 1)\}$ neither
8. $\{\sigma \mid \forall N\ \exists i \geq N,\ \sigma(i) \models x > 1\}$ neither
9. $(2^{AP})^*$ inv. safety

II Operations on Safety Properties

Exercise 3. Characterization of safety properties

Q1. We have that

P safety property $\iff \exists P_{bad}, \forall \sigma, \ (\sigma \in P \iff \text{Pref}\,\sigma \cap P_{bad} = \emptyset)$
$\iff \exists P_{bad}, \forall \sigma, \ (\sigma \in P^c \iff \text{Pref}\,\sigma \cap P_{bad} \neq \emptyset)$
$(*) \iff \forall \sigma, \ (\sigma \in P^c \iff \text{Pref}\,\sigma \cap P = \emptyset)$

For $(*)$, "$\Leftarrow$" we take $P_{bad} := (2^{AP})^* \setminus P$ and we have the required property.

"$\Rightarrow$" We have $\sigma \in \text{Pref}\,\sigma$ and we can conclude.

Q2. We show P safety $\iff P \supseteq cl\,P$. We always have $cl\,P \supseteq P$.

"$\Rightarrow$" Let $\sigma \in P^c$, we will show $\sigma \notin cl\,P$.
i.e. $\text{pref}\,\sigma \not\subseteq \text{pref}\,P$

we have $\hat\sigma \in \text{pref}\,\sigma$ and $\hat\sigma \notin \text{pref}\,P$ by (1)

"$\Leftarrow$" Let $\sigma \in P^c$. we will show $\exists \hat\sigma \in \text{pref}\,\sigma$ such that $\hat\sigma \notin \text{pref}\,P$
i.e. $\text{pref}\,\sigma \not\subseteq \text{pref}\,P$.

Exercise 4. Union & intersections

Q1. For $\sigma \in (P \cup Q)^c = P^c \cap Q^c$, there exists $\hat\sigma_P \leq_{fini} \sigma$ st $\hat\sigma_P \cdot (2^{AP})^\omega \cap P = \emptyset$
and $\hat\sigma_Q \leq_{fini} \sigma$ st $\hat\sigma_Q \cdot (2^{AP})^\omega \cap Q = \emptyset$.

Let $\hat\sigma \leq \hat\sigma_P$ and $\hat\sigma \leq \hat\sigma_Q$. We have

$$\begin{aligned}\hat\sigma \cdot (2^{AP})^\omega \cap (P \cup Q) &= (\hat\sigma \cdot (2^{AP})^\omega \cap P) \cup (\hat\sigma \cdot (2^{AP})^\omega \cap Q) \\ &= (\hat\sigma_P \cdot (2^{AP})^\omega \cap P) \cup (\hat\sigma_Q \cdot (2^{AP})^\omega \cap Q) \\ &= \emptyset\end{aligned}$$

Thus $P \cup Q$ is a safety property.

Q2. $(P \cap Q)_{bad} ::= P_{bad} \cup Q_{bad}$

$\sigma \in P \cap Q \iff \forall \hat{\sigma} \subseteq_{fini} \sigma \quad \hat{\sigma} \notin P_{bad}$ and $\hat{\sigma} \notin Q_{bad}$

$\iff \forall \hat{\sigma} \subseteq_{fini} \sigma \quad \hat{\sigma} \notin (P \cap Q)_{bad}$

III Safety properties and Transition Systems.

Exercise 5. Finite traces

If $TS \models P$ then, $Tr_{fin}(TS) \cap P_{bad}$

$= pref(Tr^{\omega}(TS)) \cap P_{bad}$

TD n° 3

Safety and Liveness Properties

I. Liveness properties.

Exercise 1. (Closure of liveness properties)

$$cl(P) = (2^{AP})^\omega \iff \forall \sigma \in (2^{AP})^\omega \quad pref\, \sigma \subseteq pref\, P$$
$$\iff \forall \hat{\sigma} \in (2^{AP})^* \quad \hat{\sigma} \in pref\, P$$
$$\iff \forall \hat{\sigma} \in (2^{AP})^* \quad \exists \sigma \in P, \ \hat{\sigma} \subseteq \sigma$$
$$\iff P \text{ liveness property}$$

Exercise 2 (Unions & intersections).

We proved that $cl(P \cup Q) = cl(P) \cup cl(Q)$
and $cl(P \cap Q) = cl(P) \cap cl(Q)$

Thus $P \cup Q$ and $P \cap Q$ are liveness properties.

Even better: if P or Q is liveness, then $P \cup Q$ is liveness.

II Topology on infinite words

Exercise 3 (Σ^ω as a topological space)

We need to prove that $\Omega\Sigma^\omega$ is stable under arbitrary unions and finite intersections.

1) Let $\mathcal{U} \subseteq \wp(\Sigma^*)$. Define $\bar{\mathcal{U}} := \bigcup \mathcal{U}$. We have that

$$ext(\bar{\mathcal{U}}) = \bigcup_{u \in \bar{\mathcal{U}}} ext(u) = \bigcup_{U \in \mathcal{U}} \bigcup_{u \in U} ext(u)$$

Thus $\Omega\Sigma^\omega$ is stable under arbitrary unions.

2) Let $U, V \subseteq \Sigma^*$.

We have that

$$ext(U) \cap ext(V) = \bigcup_{u \in U} \bigcup_{v \in V} ext(u) \cap ext(v)$$

For some $u \in U$, $v \in V$, we have the 3 following cases:

a) $u \nsubseteq v$ and $v \nsubseteq u$ thus $ext(u) \cap ext(v) = \emptyset$

b) $u \subseteq v$ thus $ext(u) \cap ext(v) = ext(u)$

c) $v \subseteq u$ thus $ext(u) \cap ext(v) = ext(v)$

Define

$$W := \{u \in U \mid \exists v \in V,\ u \subseteq v\} \cup \{v \in V \mid \exists u \in U,\ v \subseteq u\}$$

and we have $ext(W) = ext(U) \cap ext(V)$.

By induction, $\Omega\Sigma^\omega$ is stable under finite intersections.

Exercise 4. (Open sets)

Let $P \subseteq \Sigma^\omega$.

P is open iff $\exists U \subseteq \Sigma^*$ $\quad P = \bigcup_{u \in U} ext(u)$

iff $\left| \begin{array}{l} \forall \sigma \in P,\ \exists u \in \Sigma^*,\ \sigma \in ext(u) \\ \forall u \in \Sigma^*,\ ext(u) \subseteq P \end{array} \right.$

iff $\forall \sigma \in P, \exists u \in \Sigma^*, \sigma \in ext(u) \subseteq P$

iff $\forall \sigma \in P, \exists \hat{\sigma} \in \Sigma^*, \hat{\sigma} \subseteq \sigma$ and $ext(\hat{\sigma}) \subseteq P$.

Exercise 5. (Density and liveness)

P is dense iff for any non-empty open set U', $P \cap U' \neq \emptyset$

iff for any non-empty $U \subseteq \Sigma^*$, $P \cap ext(U) \neq \emptyset$

iff for any $u \in \Sigma^*$, $P \cap ext(u) \neq \emptyset$

iff for any $\hat{\sigma} \in \Sigma^*$, $\exists \sigma \in P$, $\hat{\sigma} \subseteq \sigma$.

iff P is a liveness.

III. Decomposition theorem.

Exercise 6. Let $P \subseteq (2^{AP})^\omega$.

Define $P_{safe} := cl(P)$ which is a safety property as $cl(P_{safe}) = cl^2(P) = cl(P)$

and $P_{live} = P \cup P_{safe}^C$ which is a liveness property as P_{live} is

topologically dense: if $U \subseteq \Sigma^*$, and $ext(U) \cap P = \emptyset$ then $P \subseteq ext(U)^C$

and thus $cl(P) \subseteq ext(U)^C$ so we can conclude $ext(U) \subseteq cl(P)^C$. (closed)

Thus proving the decomposition theorem.

Exercise 7.

Q1. $cl(P_1) = P_1$ thus liveness and $P_{safe} = \Sigma^\omega$

Q2. $cl(P_2) = \{\sigma \mid \sigma(0) = a\}$ thus not liveness and not safety $P_{liveness} = \Sigma^\omega \setminus \{a^\omega\}$.

Q3. $cl(P_3) = \{a^\omega\} = P_3$ thus liveness and $P_{safe} = \Sigma^\omega$

Q4. $cl(P_4) = \{\sigma \mid \sigma$ contains ≤ 1 b's$\} = P_4 \cup \{a^\omega\}$ thus not liveness and not safety $P_{live} = \Sigma^\omega \setminus \{a^\omega\}$.

Q5. $cl(P_5) = \Sigma^\omega$ thus liveness and $P_{safe} = \Sigma^\omega$

Q6 $cl(P_6) = \Sigma^\omega$ thus liveness and $P_{safe} = \Sigma^\omega$

Q7. $cl(P_7) = \Sigma^\omega$ thus liveness and $P_{safe} = \Sigma^\omega$

Q8. $cl(P_8) = \Sigma^\omega$ thus liveness and $P_{safe} = \Sigma^\omega$.

TD n° 4

Topology.

I. Topology on ω-words : Examples.

c.f exercise 7 / td 3

Safety ⟷ closed

liveness ⟷ dense

closure ⟷ closure (c.f. TD5)

open : 1,
$P_1 = ext(b)$

II General Properties of Topological Spaces

Exercise 2 (Closed sets)

Q1. $\emptyset = X^c$ and $X = \emptyset^c$

Q2. C_i^c is open thus $\bigcup C_i^c$ is open and then $(\bigcup C_i^c)^c = \bigcap C_i$ is closed.

Q3. C_i^c is open thus $\bigcap C_i^c$ is open and then $(\bigcap C_i^c)^c = \bigcup C_i$ is closed.

Exercise 3 (Closed and open sets).

Q1. "$\Rightarrow$" Take $U := A \in \Omega X$. For all $x \in A$, we have $x \in U$ and $U \subseteq A$.
"$\Leftarrow$"

We have $A = \bigcup_{a \in A} U_a \in \Omega X$ thus A is open.

Q2.

A closed iff A^c open
iff $\forall x \in A^c, \exists U \in \Omega X,\ x \in U \wedge U \subseteq A^c$
iff $\forall x \notin A, \exists U \in \Omega X,\ x \in U \wedge U \cap A = \emptyset$

Exercise 4. (Closure)

Q1. Let us show $x \notin \bar{A}$ iff $\exists N \in \mathcal{N}_x$, $N \cap A = \emptyset$.

"$\Rightarrow$" Consider $x \in \bar{A}$, thus there exists a closed set $A \subseteq C \subseteq X$ such that $x \notin C$.
Take $N := C^c$ which is open and contains N. And, $A \cap N = \emptyset$.

"$\Leftarrow$" Consider x such that there exists $N \in \mathcal{N}_x$, with $N \cap A = \emptyset$.
Then, there exists an open set $U \in \Omega X$ such that $x \in U \subseteq N$.
Take $C := U^c$ which is closed, and $U \cap A = \emptyset$ thus $A \subseteq N$.
Also, $x \notin C$.

Q2. $\bar{A}$ is the smallest closed set containing A as closed sets are stable under arbitrary intersections and $\bar{A} := \inf \{ C \subseteq X \mid C \text{ closed } \& \ A \subseteq C \}$.
$\hookrightarrow$ for $\subseteq$ set inclusion

$\bar{A} = A \Leftrightarrow$ the smallest closed set containing A is A
$\Leftrightarrow$ A is closed.

Q3. A dense iff $\forall U \in \Omega X \setminus \{\emptyset\}$, $A \cap U \neq \emptyset$

iff $\forall C$ closed and $C \neq X$, $A \subseteq C$

iff $\bar{A} \subseteq X$

III Topology on ω-words: Properties

Exercise 5.

Suppose $v \subseteq u$, thus $u = vw$. Take $u\sigma \in ext(u)$ and we have $vw\sigma \in ext(v)$.

Suppose $ext(u) \subseteq ext(v)$. We have $ua^\omega, ub^\omega \in ext(u)$ thus $ua^\omega, ub^\omega \in ext(v)$.
And $v \in pref(ua^\omega) \cap pref(ub^\omega) = pref\ u$ thus $v \subseteq u$.

Exercise 6.

P is closed iff $\forall x \notin P$, $\exists V \in \Omega X$, $x \in V$ and $V \cap P = \emptyset$

iff $\forall x \notin P$, $\exists \mathcal{U} \subseteq \Sigma^*$, $x \in \mathrm{ext}(\mathcal{U})$ and $\mathrm{ext}(\mathcal{U}) \cap P = \emptyset$

iff $\forall x \notin P$, $\exists \hat{\sigma} \in \Sigma^*$, $\hat{\sigma} \subseteq x$ and $\mathrm{ext}(\hat{\sigma}) \cap P = \emptyset$.

IV Bases and Subbases.

Exercise 7.

Let $(V_i)_{i \in I}$ be a family of elements of ΩX.

We write $V_i := \bigcup_{j \in I_j} \mathcal{U}_{ij}$ where $\mathcal{U}_{ij} \in \mathcal{B}$.

Take $\bigcup V_i = \bigcup_{\substack{i \in I \\ j \in I_j}} \mathcal{U}_{ij} \in \Omega X$.

Let $A = \bigcup_{i \in I} A_i$ $A_i \in \mathcal{B}$ and $B = \bigcup_{j \in I} B_j$.

$$A \cap B = \bigcup_{\substack{i \in I \\ j \in J}} \underbrace{(A_i \cap B_j)}_{\text{an element of } \mathcal{B}} \in \Omega X.$$

By induction, ΩX is closed under finite $\cap$'s.

V. Metric Spaces

Exercise 8 (Open ball topology)

Q1. Take $A, B \in \mathcal{U}$. For every $a \in A$, there exists $\varepsilon_A^a > 0$ such that $B_{\varepsilon_A^a}(a) \subseteq A$.

For every $b \in B$, there exists $\varepsilon_B^b > 0$ such that $B_{\varepsilon_B^b}(b) \subseteq B$.

For every $x \in A \cap B$, take $\varepsilon^x := \min(\varepsilon_A^a, \varepsilon_B^b) > 0$

We have, $B_{\varepsilon^x}(x) \subseteq A \cap B$.

Consider $(A_i)_{i \in I}$ a family of elements of ΩX.

For every $i \in I$ and $a \in A_i$, there exists $\varepsilon_a^i > 0$ such that $B_{\varepsilon_a^i}(a) \subseteq A_i$.

Let $a \in \bigcup_{i \in I} A_i$. There exists $i \in I$ such that $a \in A_i$. We have $B_{\varepsilon_a^i}(a) \subseteq A_i \subseteq \bigcup_{i \in I} A_i$

Q2. We have $\overline{S} = \{x \in X \mid \forall N \in \mathcal{N}_x, N \cap S \neq \emptyset\}$.

We have

$x \in \overline{S} \Rightarrow \forall N \in \mathcal{N}_x, N \cap S \neq \emptyset$

$\Rightarrow \forall \varepsilon > 0, B_\varepsilon(x) \cap S \neq \emptyset$

$B_\varepsilon(x) \in \mathcal{N}_x$.

On the other hand,

$\forall \varepsilon > 0, B_\varepsilon(x) \cap S \neq \emptyset \Rightarrow$

$\forall N \in \mathcal{N}_x, \begin{cases} N \subseteq B_\varepsilon(x) \text{ for some } \varepsilon > 0 \\ N \cap S \neq \emptyset \end{cases}$

$\Rightarrow \forall N \in \mathcal{N}_x, N \cap S \neq \emptyset$

Exercise 9 (Distance on ω-words)

We trivially have $d(x,y) = 0 \iff x = y$ and $d(x,y) = d(y,x)$.

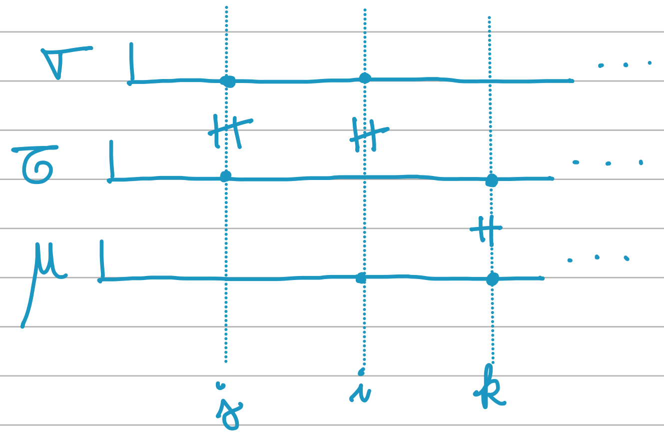
Let σ, τ, μ be three words on A^ω. We will show $d(\sigma, \mu) \leq d(\sigma, \tau) + d(\tau, \mu)$

If $\sigma = \mu$, the result is trivial. Same thing for $\sigma = \tau$ or $\tau = \mu$.

Suppose $\tau \neq \mu \neq \sigma \neq \tau$.

Let $i := \min \{ i \mid \sigma(i) \neq \mu(i) \}$, $j := \min \{ j \mid \sigma(j) \neq \tau(j) \}$
and $k := \min \{ k \mid \tau(k) \neq \mu(k) \}$.

Without loss of generality, let us suppose $j < k$ (if $j = k$ then $i = j = k$ and thus the result is trivially true)



We necessarily have $i > j$ as $\sigma|_j = \tau|_j = \mu|_j$

as $\tau|_k = \mu|_k$.

Thus, $d(\sigma, \mu) = 2^{-i} \leq 2^{-j} \leq 2^{-j} + 2^{-k}$
$\leq d(\sigma, \tau) + d(\tau, \mu)$.

We can conclude: (A^ω, d) is a metric space.

Exercise 10 (Open ball topology on ω-words).

1) Let $U \subseteq \Sigma^*$. Let us show that $ext(U)$ is open for the open-ball topology.

Let $u\sigma \in ext(U)$. Take $\varepsilon := 2^{-length(u)}$.

We have $B_\varepsilon(u\sigma) = \{ \tau \mid \min \{ i \mid (u\sigma)(i) \neq \tau(i) \} > length(u) \}$
$= ext(u)$.

Thus $ext(U) \in \mathcal{U}$.

2) Let $V \in \mathcal{U}$. We will show that $V = \text{ext}(U)$ for some $U \subseteq \Sigma^*$.

Let $v \in V$, and let $\varepsilon_v > 0$ such that $B_{\varepsilon_v}(v) \subseteq V$.

Define $U := \{ v|_{-\log_2(\varepsilon_v)} \mid v \in V \}$.

We have that:

$$\text{ext}(U) = \bigcup_{v \in V} \text{ext}(v|_{-\log_2(\varepsilon_v)})$$

$$= \bigcup_{v \in V} B_{\varepsilon_v}(v)$$

$$= V.$$

Thus the two topologies coincide.

TD n°6

I. Closure Operators

Exercise 1.

Consider the partial order $\leq$ on $L^c \subseteq L$ induced by $(L, \leq)$.

Let $S \subseteq L^c$.

We have that $\bigwedge S \in L^c$, i.e. $c(\bigwedge S) = \bigwedge S$.

Indeed, $\bigwedge S \leq c(\bigwedge S)$ by expansivity, and

$c(\bigwedge S) \leq \bigwedge S$ as, $\forall s$,

$\bigwedge S \leq s$ thus $c(\bigwedge S) \leq c(s) = s$
and finally $c(\bigwedge S) \leq \bigwedge S$.

Also, $c(\bigvee S) \in L^c$ as $c(c(\bigvee S)) = c(\bigvee s)$.

For any $s \in S$, we have $\bigvee S \geq s$ thus $c(\bigvee S) \geq c(s) = s$

Also, for some $w \in L^c$ such that $\forall s \in S$, $w \geq s$

then $w \geq \bigwedge S$ and thus $w = c(w) \geq c(\bigwedge S)$.

Exercise 2.

Q1. We have $\overline{X} := \bigcap_{X \subseteq C \text{ closed}} C$, thus $\overline{\varnothing} = \varnothing$ and $\overline{X \cup Y} = \overline{X} \cup \overline{Y}$

We always have $\overline{X} \supseteq X$ and $\overline{\overline{X}} = \overline{X}$ as $\overline{X}$ is closed and contains $\overline{X}$.

If $X \subseteq Y$ then $\overline{X} \subseteq \overline{Y}$ as $\overline{Y}$ is a closed set containing X.

Q2. Define $\mathcal{U}^c := \{ U \subseteq X \mid c(X \setminus U) = X \setminus U \}$. Then, $\mathcal{U}^c$ is a topology on X:

- $X \in \mathcal{U}^c$ as $c(\emptyset) = \emptyset$
- if $A, B \in \mathcal{U}^c$ then $A \cap B \in \mathcal{U}^c$ as
$$\begin{aligned} c(X \setminus (A \cap B)) &= c((X \setminus A) \cup (X \setminus B)) \\ &= c(X \setminus A) \cup c(X \setminus B) \\ &= (X \setminus A) \cup (X \setminus B) \\ &= X \setminus (A \cap B). \end{aligned}$$

- $\emptyset \in \mathcal{U}^c$ as $X \subseteq c(X)$ thus $c(X) = X$.

- we can apply ex 1 with $(L, \leq) = (\mathcal{P}(X), \subseteq)$ as closed sets are exactly those where $c(A) = A$.

II. Galois Connections.

Exercise 3.

1) if $u \leq v$ then $g(f(u)) \leq u \leq v$ as $f(u) \leq f(u)$ and thus $f(u) \leq f(v)$.

2) other case by duality
- $x \leq f(g(x))$ as $g(x) \leq g(x)$
- $x \leq y$ implies $f(g(x)) \leq f(g(y))$ by 1)
- $f(g(f(g(x)))) \geq x$ by ↗

  and, $f(g(f(g(x)))) \leq x$ as $f(g(x)) \leq f(g(x))$.

3) Let $S \subseteq A$, then $f(\bigwedge S) = \bigwedge f(S)$ as $f(\bigwedge S)$ is a meet of $f(S)$:

- for all $f(s) \in f(S)$, $f(s) \geq f(\bigwedge S)$ as $s \geq \bigwedge S$.
- for all $w \in B$, if $w \leq f(s)$ $\forall s \in S$, then $g(w) \leq s$ $\forall s \in S$, thus $g(w) \leq \bigwedge S$ thus $w \leq f(\bigwedge S)$.

4. by duality

5. we have $a \leq f(b)$ iff $g(a) \leq b$ iff $a \leq f'(b)$.

Then, $f'(b) \leq f(b)$ and $f(b) \leq f'(b)$, thus $f(b) = f'(b)$ $\forall b$.

6. by duality

Exercise 4:

1. "$\Rightarrow$" ex3
"$\Leftarrow$" suppose $f(\bigwedge S) = \bigwedge S$ $\forall S \subseteq B$. Define $g(a) := \bigwedge \{ b \mid f(b) \geq a \}$.
We have $g(a) \leq b$ iff $a \leq f(b)$ $\forall a, b$.

2. by duality

Exercise 5.

Q1 $\quad f_!(A) \subseteq B$ iff $\forall a,\ f(a) \in B$ iff $\forall a, \exists b,\ f(a) = b$ iff $A \subseteq f^\bullet(B)$.

Q2. $f^\bullet(\bigcup S) = \{ x \mid f(x) \in \bigcup S \} = \bigcup_{X \in S} \{ x \mid f(x) \in X \} = \bigcup f^\bullet(S)$

$f^\bullet(\bigcap S) = \{ x \mid f(x) \in \bigcap S \} = \bigcap_{X \in S} \{ x \mid f(x) \in X \} = \bigcap f^\bullet(S)$

$f^\bullet(Y \setminus A) = \{ x \mid f(x) \in Y \setminus A \} = \{ x \mid f(x) \notin A \} = X \setminus \{ x \mid f(x) \in A \} = X \setminus f^\bullet(A)$

Exercise 6. $\mathrm{pref}(A) \subseteq B$ iff $\forall a\ \forall u \leq a,\ u \in B$ iff $A \subseteq \mathrm{cl}(B)$

III Continuous functions

Exercise 7. nope!

TD n° 7

Galois Connections & Observable Properties

I Galois Connections.

c.f. TD n° 6

II. Observable properties

Exercise 5.

1) By def°, ext(u) is open and

$$A^\omega \setminus \mathrm{ext}(u) = \bigcup_{\substack{|v|=|u| \\ v \neq u}} \mathrm{ext}(v)$$

Thus it is also closed.

2) Clopen are closed under finite unions

Thus $\mathrm{ext}(U) = \bigcup_{u \in U} \mathrm{ext}(u)$ is a clopen.

3) Suppose there exists a finite $U \subseteq_{finite} \mathbb{N}^*$ such that

$$\mathrm{ext}(U) = \mathrm{ext}(\mathbb{N}_{>0}) = \{ n\,\sigma \mid n > 0, \sigma \in \mathbb{N}^\omega \}.$$

Then let $k := \max\{ \hat{v}(0) \mid \hat{v} \in U \}$ (if $U = \emptyset$ we immediatly have a contradiction).

Then, $(k+1)\, 0^\omega$ is in $\mathrm{ext}(\mathbb{N}_{>0})$ but not in $\mathrm{ext}(U)$, a contradiction.

Exercise 6.

Let K be a compact and $C \subseteq K$ be a closed subset. Consider an open cover $\mathcal{F} = \{U_i\}_{i \in I}$ of C. Then we can consider an open cover $\{X \setminus C\} \cup \mathcal{F}$ of K. By compactness there exists a finite subcover of K. We can simply remove $X \setminus C$ to obtain a finite subcover of $\mathcal{F}$ of C.

Thus C is compact.

Exercise 7.

Define $\mathcal{F} := \{ \mathrm{ext}(a) \mid a \in A \}$. If A^ω was compact then there would exist a subcover $\{\mathrm{ext}(a') \mid a' \in A'\}$ for some finite $A' \subseteq_{fin} A$. Thus there exists $a \in A \setminus A'$ and so

$$\mathrm{ext}(a) \not\subseteq \underbrace{\{\mathrm{ext}(a') \mid a' \in A'\}}_{A^\omega \subsetneq}$$

a contradiction.

Thus A^ω is not compact.

Exercise 8.

For $W \subseteq_{fin} (2^{AP})^*$ then $\xrightarrow{\text{Ex 5}}$ $\mathrm{ext}(W)$ is a clopen thus $\mathrm{ext}(W)$ is observable.

On the other hand, if P is a clopen, then P is closed in $(2^{AP})^\omega$ which is compact thus by ex 6, P is compact.

Exercise 9.

Let $x, y \in A^\omega$ with $x \neq y$. Let

$$\hat{\imath} := \max \{ i \in \mathbb{N} \mid x(i) = y(i) \}$$

or -1 if $\{\dots\}$ is empty.

Then let $\bar{x} := x(0) \dots x(\hat{\imath}+1)$
$\bar{y} := y(0) \dots y(\hat{\imath}+1)$.

Thus $ext(\bar{x}) \cap ext(\bar{y}) = \emptyset$ as they differ on the $(i+1)$th letter.

And $x \in ext(\bar{x})$, $y \in ext(\bar{y})$.

Thus A^ω is Hausdorff.

III Linear-time Modal Logic

Exercise 10.

(i) Let $\rho : \mathcal{V} \to \mathcal{P}((2^{AP})^\omega)$.

$$[\![\varphi \wedge \neg \varphi]\!]_\rho = [\![\varphi]\!]_\rho \cap ((2^{AP})^\omega \setminus [\![\varphi]\!]_\rho) = \emptyset = [\![\bot]\!]_\rho.$$

(ii) & (iii) Define $f : \sigma \mapsto \sigma \upharpoonright 1$.

Then $[\![\circ \varphi]\!]_\rho = f^\bullet([\![\varphi]\!]_\rho)$ and by ex 3, $f^\bullet$ preserves union and singletons:

$$[\![\circ(\varphi \vee \psi)]\!]_\rho = f^\bullet([\![\varphi]\!]_\rho \cup [\![\psi]\!]_\rho) = f^\bullet([\![\varphi]\!]_\rho) \cup f^\bullet([\![\psi]\!]_\rho)$$
$$= [\![(\circ \varphi) \vee (\circ \psi)]\!]$$

and

$$[\![\circ(\neg\varphi)]\!]_\rho = f^\circ\left((2^{AP})^\omega \setminus [\![\varphi]\!]_\rho\right) = (2^{AP})^\omega \setminus f^\circ([\![\varphi]\!]_\rho)$$
$$= [\![\neg \circ\varphi]\!]_\rho .$$

Exercise 11.

The set $f(\varphi) := \{ \sigma(0) \mid \sigma \in [\![\varphi]\!] \}$ can be written

as W or $2^{AP} \setminus W$ for some finite $W \subseteq 2^{AP}$.

By induction on φ:

- $\varphi \neq X$ as φ closed
- $f(a) = \{a\}$
- $f(\top) = 2^{AP}$
- $f(\bot) = \emptyset$
- $f(\varphi \wedge \psi) = f(\varphi) \cap f(\psi)$
- $f(\varphi \vee \psi) = f(\varphi) \cup f(\psi)$
- $f(\neg\varphi) = 2^{AP} \setminus f(\varphi)$
- $f(\circ\varphi) = 2^{AP}$

And $ext(2\mathbb{N}) \neq [\![\varphi]\!]$ for all closed φ as $f(\varphi) = 2\mathbb{N}$ cannot be written as W or $2^{\mathbb{N}} \setminus W$ for some finite W.

TD n° 8

Linear-Time Modal Logic and Linear Temporal Logic

II Linear Temporal Logic

Exercise 3.

1. $\{1, 2, 3, 4\}$
2. $\{3\}$
3. $\emptyset$
4. $\{1, 2, 3, 4\}$
5. $\{1, 2, 3, 4\}$
6. $\{1, 2, 3, 4\}$

Exercise 4. Knaster-Tarski Fixpoint Theorem

- Define $F := \{a \mid f(a) \leq a\}$.

  For any $m \in F$, $\mu(f) \leq m$ thus $f(\mu(f)) \leq f(m) \leq m$ $\forall m \in F$.
  so $f(\mu(f))$ is a lower bound of F.

  By definition, $f(\mu(f)) \leq \mu(f)$ and so $f(f(\mu(f))) \leq f(\mu(f))$
  thus $f(\mu(f)) \in F$.

  We conclude that $f(\mu(f)) = \mu(f)$.

  If m is a fixpoint of f then $m \in F$ and thus
  $$m \leq \mu(f).$$

- By duality (consider the complete lattice $(L, \geq)$).

Exercise 5.

We have that $\varphi \,U\, \psi \equiv \psi \vee (\varphi \wedge \circ(\varphi \,U\, \psi))$ thus $[\![\varphi \,U\, \psi]\!]_\rho$ is a fix point of Θ (as long as $X \notin vars(\varphi) \cup vars(\psi)$).

On the other hand, let P such that $[\![\Theta]\!](P) = P$. We will show that $[\![\varphi \, U \, \psi]\!]_\rho \subseteq P$. Take $\sigma \in [\![\varphi \, U \, \psi]\!]_\rho$.

Let i such that $\sigma \uparrow i \in [\![\psi]\!]_\rho$ and $\forall j < i,\ \sigma \uparrow j \in [\![\varphi]\!]_\rho$

By induction on $i \in \mathbb{N}$,

- if $i = 0$ then $\sigma \uparrow 0 = \sigma \in [\![\psi]\!]_\rho$ and thus $\sigma \in [\![\Theta]\!](P) = P$.
- if $i = 1$ then $\sigma \uparrow 1 \in [\![\psi]\!]_\rho$ and $\sigma \in [\![\varphi]\!]$ thus $\sigma \in [\![\Theta]\!](P) = P$.
- if $i = 2$ then $\sigma \uparrow 2 \in [\![\psi]\!]_\rho$ and $\sigma \uparrow 1, \sigma \in [\![\varphi]\!]$ thus $\sigma \in [\![\Theta]\!](P) = P$.
- and so on

Thus $[\![\varphi \, U \, \psi]\!]_\rho$ is the least fixpoint of $[\![\Theta]\!]_\rho(X)$.

Exercise 6.

1. $$\neg(\varphi \, W \, \psi) \equiv \neg\neg(\neg\psi \, U \neg(\varphi \vee \psi))$$
$$\equiv \neg\psi \, U \, (\neg\varphi \wedge \neg\psi)$$

2. $$\neg\psi \, W \, (\neg\varphi \wedge \neg\psi) \equiv \neg(\neg\neg\varphi \, U \neg(\neg\psi \vee (\neg\varphi \wedge \neg\psi)))$$
$$\equiv \neg(\varphi \quad U \, (\psi \wedge (\varphi \vee \psi)))$$
$$\equiv \neg(\varphi \quad U \quad \psi)$$

3. $$[\![\circ(\varphi \, U \, \psi)]\!]_\rho = \{\sigma \mid \exists i \ \ \sigma \uparrow i{+}1 \in [\![\psi]\!]_\rho \wedge \forall j < i, \sigma \uparrow j+1 \in [\![\varphi]\!]_\rho\}$$
$$= [\![\circ\varphi \ U \ \circ\psi]\!]_\rho$$

4. $$[\![\neg\Box\neg\varphi]\!]_\rho = \{\sigma \mid \forall i,\ \sigma \uparrow i \notin [\![\varphi]\!]_\rho\}^C$$
$$= \{\sigma \mid \exists i,\ \sigma \uparrow i \in [\![\varphi]\!]_\rho\} = [\![\Diamond\varphi]\!]_\rho$$

5. $$[\![\varphi \vee \circ\Diamond\varphi]\!]_\rho = \{\sigma \mid \sigma \in [\![\varphi]\!]_\rho \text{ or } \forall i,\ \sigma \uparrow i{+}1 \in [\![\varphi]\!]_\rho\}$$
$$= \{\sigma \mid \forall i,\ \sigma \uparrow i \in [\![\varphi]\!]_\rho\} = [\![\Diamond\varphi]\!]_\rho.$$

Exercise 7.

We have that $[\![\Box\varphi]\!]_\rho$ is a fixpoint of $[\![\varphi_\Box]\!]_\rho(X)$.

Let P such that $[\![\varphi_\Box]\!]_\rho(P) = P$. Let us show that $P \subseteq [\![\Box\varphi]\!]_\rho$.

Let $\sigma \in [\![\varphi_\Box]\!]_\rho(P) = P$. Then, $\sigma \in [\![\varphi]\!]_\rho$ and $\sigma \upharpoonright 1 \in P$

and thus, by induction, for all i, $\sigma \upharpoonright i \in [\![\varphi]\!]_\rho$.

Thus $\sigma \in [\![\Box\varphi]\!]_\rho$.

Exercise 8.

1. $$[\![\top \, \mathcal{U} \, \varphi]\!]_\rho = \{\sigma \mid \exists i,\ \sigma \upharpoonright i \in [\![\varphi]\!]_\rho,\ \underbrace{\forall j < i\ \sigma \upharpoonright j \in [\![\top]\!]_\rho}\}$$

$$= \{\sigma \mid \exists i,\ \sigma \upharpoonright i \in [\![\varphi]\!]_\rho\} \quad \text{true}$$

$$= [\![\Diamond\varphi]\!]_\rho$$

2. $$\varphi \, \mathcal{W} \perp \equiv \neg(\neg\varphi \ \mathcal{U} \ \neg(\varphi \vee \perp))$$

$$\equiv \neg(\neg\varphi \ \mathcal{U} \ \neg\varphi)$$

$$[\![\varphi \, \mathcal{W} \perp]\!]_\rho = \{\sigma \mid \forall i,\ \sigma \upharpoonright i \in [\![\varphi]\!]_\rho \text{ and } \exists j < i\ \sigma \upharpoonright i \in [\![\varphi]\!]_\rho\}$$

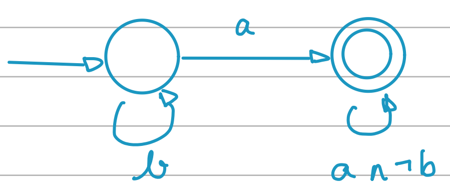
$$= \{\sigma \mid \forall i\ \sigma \upharpoonright i \in [\![\varphi]\!]_\rho\}$$

TD n° 9

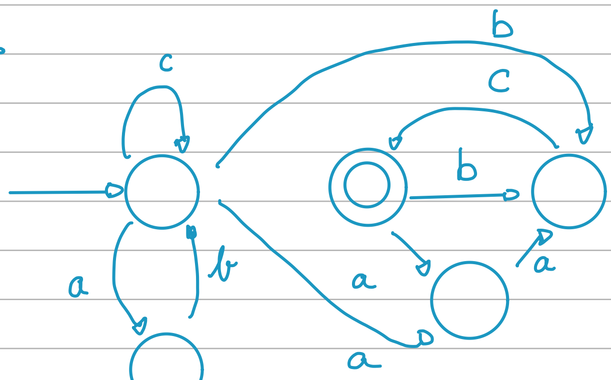
Linear Temporal Logic and Büchi Automata

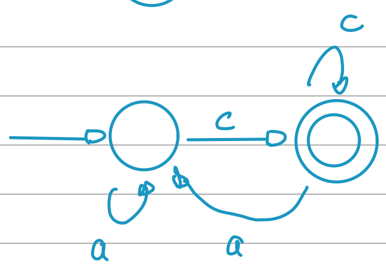
II Constructing Büchi Automata

Exercise 4:



Exercise 5.





III Operations on Büchi Automata.

Exercise 6. Let $\mathcal{A} = (Q, \delta, I, F)$.

Suppose $\mathcal{A}$ does not have any transitions into I and that $F \cap I = \emptyset$ (add an initial step and "simulate" ε-transitions).

Then, take $\mathcal{A}_\omega = (Q, \delta', I, I)$ where

$$\delta'(q,a) = \begin{cases} \delta(q,a) & \text{if } \delta(q,a) \in F \\ \delta(q,a) & \text{otherwise} \end{cases}$$

Exercise 7.

1) Take $\mathcal{A}_1 \sqcup \mathcal{A}_2$ and that's it

2) Take $\mathcal{A} \odot \mathcal{A}_1 := (Q_{\mathcal{A}} \sqcup Q_{\mathcal{A}_1}, \delta, I_{\mathcal{A}}, F_{\mathcal{A}_1})$

where $\delta(q \in Q_{\mathcal{A}}, a) = \begin{cases} \delta_{\mathcal{A}}(q,a) & \text{if } q \notin F_{\mathcal{A}} \\ \delta_{\mathcal{A}}(q,a) \cup I_{\mathcal{A}_1} \end{cases}$

$$\delta(q \in Q_{\mathcal{A}_1}, a) = \delta_{\mathcal{A}_1}(q,a).$$

Exercise 8.

1) is false as inverting an NBA' final and non-final states doesn't yield the complement:

for example, the empty automaton

Exercise 9.

$\mathcal{A}_1 = (Q_1, \delta_1, I_1, F_1)$ $\qquad$ $\mathcal{A}_2 = (Q_2, \delta_2, I_2, F_2)$

$\mathcal{A}_2^g = (Q_1, \delta_1, I_1, \{F_1\})$ $\qquad$ $\mathcal{A}_2^g = (Q_2, \delta_2, I_2, \{F_2\})$

$\mathcal{A}^g = (Q_1 \times Q_2, \delta_1 \times \delta_2, I_1 \times I_2, Q_1 \times F_2 \cup F_1 \times Q_2)$

$\wr$

$\mathcal{A} = \dots$

IV Decomposition theorem

Exercise 10.

1) Define $A = \mathcal{L}_\omega(E_1 F_1^\omega + \dots + E_n F_n^\omega)$

Then, $\operatorname{Pref} A = \mathcal{L}(E_1 F_1^* P_1 + \dots + E_n F_n^* P_n)$

where $\mathcal{L}(P_i) = \operatorname{Pref} \mathcal{L}(F_i)$

2) Take $\sigma \in \Sigma^\omega$,

$$\sigma \in \mathrm{cl}(U) \iff \mathrm{pref}(\sigma) \subseteq U^c$$
$$\iff \mathrm{pref}(\sigma) \cap U = \emptyset$$

3) Pref P is regular thus take $P_{bad} := (\operatorname{Pref} P)^c$ which is regular (Q1)

4)

TD n° 10

Büchi's Theorem and Automata.

I. Büchi's Theorem.

Exercise 1.

Suppose $u \sim_{\mathcal{A}} u'$. Let $q, q' \in Q$.

If $q \xrightarrow{vuw} q'$ then $q \xrightarrow{v} q_1 \xrightarrow{u} q_2 \xrightarrow{w} q'$

and then, as $u \sim_{\mathcal{A}} u'$, $q \xrightarrow{v} q_1 \xrightarrow{u'} q_2 \xrightarrow{w} q'$.

Similarly if $q \xrightarrow{vu'w} q'$.

If $q \xrightarrow[F]{vuw} q'$ then

either $q \xrightarrow[F]{v} q_1 \xrightarrow{u} q_2 \xrightarrow{w} q' \rightsquigarrow q \xrightarrow[F]{v} q_1 \xrightarrow{u'} q_2 \xrightarrow{w} q'$

either $q \xrightarrow{v} q_1 \xrightarrow[F]{u} q_2 \xrightarrow{w} q' \rightsquigarrow q \xrightarrow{v} q_1 \xrightarrow[F]{u'} q_2 \xrightarrow{w} q'$

either $q \xrightarrow{v} q_1 \xrightarrow{u} q_2 \xrightarrow[F]{w} q' \rightsquigarrow q \xrightarrow{v} q_1 \xrightarrow{u'} q_2 \xrightarrow[F]{w} q'$

Similarly if $q \xrightarrow[F]{vu'w} q'$.

Exercise 2.

Q1. The equivalence class of $u \in \Sigma^*$ is exactly

$$\bigcap \{ S(q,q') \mid u \in S(q,q'), q,q' \in Q \}$$
$$\cap \bigcap \{ S^F(q,q') \mid u \in S^F(q,q'), q,q' \in Q \}$$
$$\cap \bigcap \{ S(q,q')^c \mid u \notin S(q,q'), q,q' \in Q \}$$
$$\cap \bigcap \{ S^F(q,q')^c \mid u \notin S^F(q,q'), q,q' \in Q \}.$$

There is only a finite choice of those sets, so $\sim_{\mathcal{A}}$ is of finite index.

Q2. This is the intersection & complement of regular languages as,

$S(q,q') = \mathcal{L}(\mathcal{A}_{q,q'})$ and $S^F(q,q') = \mathcal{L}(\mathcal{A}^F_{q,q'})$

where $\mathcal{A}^F_{q,q'} = (Q', \Sigma, \delta', Q_0', F')$

$Q' = \{0,1\} \times Q$, $Q_0' = \{(0,q)\}$

$F' = \{(1,q')\}$, $\delta'(i,q,a) = \begin{cases} \{i\} \times \delta(q,a) & \text{if } q \notin F \\ \{1\} \times \delta(q,a) & \text{if } q \in F. \end{cases}$

Exercise 3.

Let $w \in U \cdot V^\omega \cap \mathcal{L}_\omega(\mathcal{A}) \neq \emptyset$.

Write $w = u v_1 \dots v_n \dots$ where $u \in U$ and $v_i \in V$ $\forall i$.

Then let $w' \in U V^\omega$ and let us show $w' \in \mathcal{L}_\omega(\mathcal{A})$.

Write $w' = u' v_1' \dots v_n' \dots$ with $u' \in U$ and $v_i' \in V$ $\forall i$.

Then $u \sim_{\mathcal{A}} u'$ and $\forall i$, $v_i \sim_{\mathcal{A}} v_i'$.

So,

$Q_0 \ni q_0 \xrightarrow{u} q_1 \xrightarrow{v_1} q_2 \xrightarrow{v_2} q_3 \to \dots$ where an infinite amount of them go through a final state

induces

$Q_0 \ni q_0 \xrightarrow{u'} q_1 \xrightarrow{v_1'} q_2 \xrightarrow{v_2'} q_3 \to \dots$ where an infinite amount of them go through a final state

So $w' \in \mathcal{L}_\omega(\mathcal{A})$.

Exercise 4.

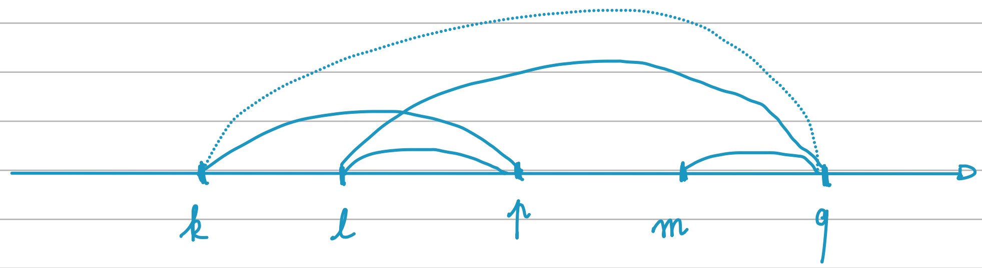
Relation $\equiv_\sigma$ is obviously reflexive and symmetric.

And if $k \equiv_\sigma l$ and $l \equiv_\sigma m$ then $k \equiv_\sigma^p l$ and $l \equiv_\sigma^q m$

for some $p \geq k, l$ and $q \geq l, m$.

So, $\sigma[k:p] \sim \sigma[l:p]$ and $\sigma[l:q] \sim \sigma[m:q]$

and so $\sigma[k:q] \sim \sigma[l:q] \sim \sigma[m:q]$.



The equivalence class of $k \in \mathbb{N}$ is of the form

$$\{ l \mid \exists n, \ \sigma[k:n] \sim \sigma[l:n] \}$$

As $\sim$ is a congruence, we can take the smallest n possible, and, having that, we use the hypothesis that $\sim$ is of finite index.

We can deduce that $\equiv_\sigma$ is of finite index.

II Deterministic Büchi Automata.

Exercise 5.

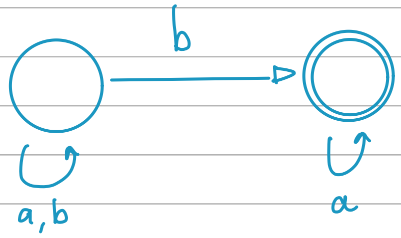
Q1. Take $\sigma \in \mathcal{L}_\omega(\mathcal{A})$.

Then, we get $\underset{\in Q_0}{q_0} \xrightarrow{\sigma[0:n_1]} \underset{\in F}{q_1} \xrightarrow{\sigma[n_1:n_2]} \underset{\in F}{q_2} \longrightarrow \dots$

and so $\forall k$, $q_0 \xrightarrow{\sigma[0:n_k]} q_k \in F$ thus $\sigma[0:n_k] \in \mathcal{L}(\mathcal{A})$.

So, we have that $\exists^{\infty} i$, $\sigma[0:i+1] \in \mathcal{L}(\mathcal{A})$ and we can conclude $\sigma \in pref(\mathcal{L}(\mathcal{A}))$.

2.



$\forall i$, $(ab)^i \in \mathcal{L}(\mathcal{A})$

but $(ab)^\omega \notin \mathcal{L}_\omega(\mathcal{A})$.

3. Suppose $\mathcal{A}$ is a (complete) DBA. We write $\delta : \Sigma \times Q \to Q$ for the transition function.

Take $\sigma \in pref(\mathcal{L}(\mathcal{A}))$.

Then, $\exists^{\infty} i.$ $\sigma[0: i+1] \in \mathcal{L}(\mathcal{A})$, so we can construct by induction a sequence $(q_n) \in Q^{\mathbb{N}}$ such that

$$q_i \xrightarrow{\sigma[n_i : n_{i+1}+1]} q_{i+1}$$ where n_i is the ith "i"

as we have a DBA.

Q4. For $L \subseteq \Sigma^\omega$,

L is the language of a DBA $\Rightarrow L = pref(\mathcal{L}(\mathcal{A}))$

where $\mathcal{A}$ is a DBA such that $\mathcal{L}_\omega(\mathcal{A}) = L$.

If $L = pref(\mathcal{L}(\mathcal{A}))$ for some DFA $\mathcal{A}$, then $\mathcal{A}$ is a DBA and $L = \mathcal{L}_\omega(\mathcal{A})$.

TD n° 11

HML and bisimulations

I. HML-formulae

Exercise 1

	TS_1	TS_2	$TS_3(1)$	$TS_3(2)$	TS_4
TS_1		$\langle\bullet\rangle^2(\neg b \wedge \langle\bullet\rangle a)$	$[\bullet]\langle\bullet\rangle a$	$[\bullet]\langle\bullet\rangle a$	$\langle\bullet\rangle\langle\bullet\rangle(a \wedge b)$
TS_2			$\langle\bullet\rangle\langle\bullet\rangle(a \wedge \neg b)$	$\langle\bullet\rangle\langle\bullet\rangle(a \wedge \neg b)$	$\langle\bullet\rangle^2(a \wedge \neg b \wedge \langle\bullet\rangle(a \wedge \neg b))$
$TS_3(1)$				$[\bullet]\langle\bullet\rangle a$	$\langle\bullet\rangle^2(a \wedge \neg b)$
$TS_3(2)$					$\langle\bullet\rangle^2(a \wedge \neg b)$
TS_4					

Exercise 2.

1) $\llbracket \neg[\alpha]\neg\varphi \rrbracket = S \setminus \{s \mid \forall s',\ s \xrightarrow{\alpha} s' \text{ implies } s' \notin \llbracket\varphi\rrbracket\}$
$= \{s \mid \exists s',\ s \xrightarrow{\alpha} s' \text{ and } s' \in \llbracket\varphi\rrbracket\}$
$= \llbracket \langle\alpha\rangle\varphi \rrbracket$

2) $\llbracket \neg\langle\alpha\rangle\neg\varphi \rrbracket = \llbracket \neg\neg[\alpha]\neg\neg\varphi \rrbracket = \llbracket [\alpha]\varphi \rrbracket$

3) $\llbracket \langle\alpha\rangle(\varphi \vee \psi) \rrbracket = \{s \mid \exists s'\ s \xrightarrow{\alpha} s' \text{ and } (s' \in \llbracket\varphi\rrbracket \text{ or } s' \in \llbracket\psi\rrbracket)\}$
$= \{s \mid \exists s'\ s \xrightarrow{\alpha} s' \text{ and } s' \in \llbracket\varphi\rrbracket\}$
$\cup \{s \mid \exists s'\ s \xrightarrow{\alpha} s' \text{ and } s' \in \llbracket\psi\rrbracket\}$
$= \llbracket \langle\alpha\rangle\varphi \rrbracket \cup \llbracket \langle\alpha\rangle\psi \rrbracket.$

4) We use 1 and 3. $\neg(\neg\varphi \vee \neg\psi) \equiv \varphi \wedge \psi$

5) $[\![\langle\alpha\rangle\bot]\!] = \{s \mid \exists s' \ s \xrightarrow{\alpha} s' \text{ and } s' \in \underbrace{[\![\bot]\!]}_{\emptyset}\}$

$= \emptyset = [\![\bot]\!]$

6) We apply 2 with 5. $\neg\top \equiv \bot$. and $\neg\bot \equiv \top$.

Exercise 3.

1) $[\![\langle\bullet\rangle\varphi]\!] = \{\sigma \mid \exists\sigma', \ \overbrace{\sigma \xrightarrow{\bullet} \sigma'}^{\sigma' = \sigma\uparrow 1} \text{ and } \sigma' \in [\![\varphi]\!]\} = \{\sigma \mid \sigma\uparrow 1 \in [\![\varphi]\!]\}$

$[\![[\bullet]\varphi]\!] = \{\sigma \mid \forall\sigma', \ \underbrace{\sigma \xrightarrow{\bullet} \sigma'}_{\sigma' = \sigma\uparrow 1} \text{ implies } \sigma' \in [\![\varphi]\!]\} = \{\sigma \mid \sigma\uparrow 1 \in [\![\varphi]\!]\}$

2)

Suppose having an LML-formula φ, let us construct an HML formula Φ st $[\![\varphi]\!] = [\![\Phi]\!] = P$.

By induction on φ, we only consider

- Case $\varphi = \circ\varphi'$ then $\Phi = \langle\bullet\rangle\Phi'$

On the other hand, we transform $\Phi = \langle\bullet\rangle\Phi'$ into $\varphi = \circ\varphi'$
$\Phi = [\bullet]\Phi'$ into $\varphi = \circ\varphi'$

3) if $\alpha \sim \beta$ and $\alpha \neq \beta$, then let $i = \min\{i \mid \alpha(i) \neq \beta(i)\}$ which is finite, then

$\alpha \Vdash \langle\bullet\rangle^i\left(\bigwedge_{a \in \alpha(i)} a\right) \wedge \left(\bigwedge_{a \notin \alpha(i)} \neg a\right)$ $\beta \not\Vdash \langle\bullet\rangle^i\left(\bigwedge_{a \in \alpha(i)} a\right) \wedge \left(\bigwedge_{a \notin \alpha(i)} \neg a\right)$

so $\alpha \not\sim \beta$, a contradiction.

II Bisimulations

Exercise 4.

1) $\forall (s_1, s_0) \in R^{-1}$, $L(s_0) = L(s_1)$ as $(s_0, s_1) \in R$.

$$
\begin{tikzcd}
s_1 \arrow[r, dash, "R^{-1}"] \arrow[d, "\alpha"'] & s_0 \arrow[d, dotted, "\alpha"] \\
s_1' \arrow[r, dotted, dash, "R^{-1}"'] & s_0'
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
s_0 \arrow[r, dash, "R"] \arrow[d, dotted, "\alpha"'] & s_1 \arrow[d, "\alpha"] \\
s_0' \arrow[r, dotted, dash, "R"'] & s_1'
\end{tikzcd}
$$

and similarly for the other one

2) $\forall (s_0, s_2) \in T \circ R \quad L(s_0) = L(s_2)$ as $(s_0, s_1) \in R$.

$\qquad\qquad\qquad\qquad \text{''} \quad L(s_1) \quad \text{''} \qquad (s_1, s_2) \in T$ for some s_1.

$$
\begin{tikzcd}
s_0 \arrow[r, dash, "T \circ R"] \arrow[d, "\alpha"'] & s_2 \arrow[d, dotted, "\alpha"] \\
s_0' \arrow[r, dotted, dash, "T \circ R"'] & s_2'
\end{tikzcd}
\qquad
\begin{tikzcd}
s_0 \arrow[r, dash, "R"] \arrow[d, "\alpha"'] & s_1 \arrow[r, dash, "T"] \arrow[d, dotted, "\alpha"'] & s_2 \arrow[d, dotted, "\alpha"] \\
s_0' \arrow[r, dotted, dash, "R"'] & s_1' \arrow[r, dotted, dash, "T"'] & s_2'
\end{tikzcd}
$$

and similarly for the other one.

Exercise 5.

Consider $Refl := \{(s,s) \mid s \in S\}$.

This is a bisimulation.

Then, $s \sim s$ as $s R s$.

Exercise 6.