

## Exercise 1. Lumberjack.

Q1.  $\begin{cases} w + p \leq 100 \\ 10w + 50p \leq 4000 \\ w, p \geq 0 \end{cases}$

Q2.

75h

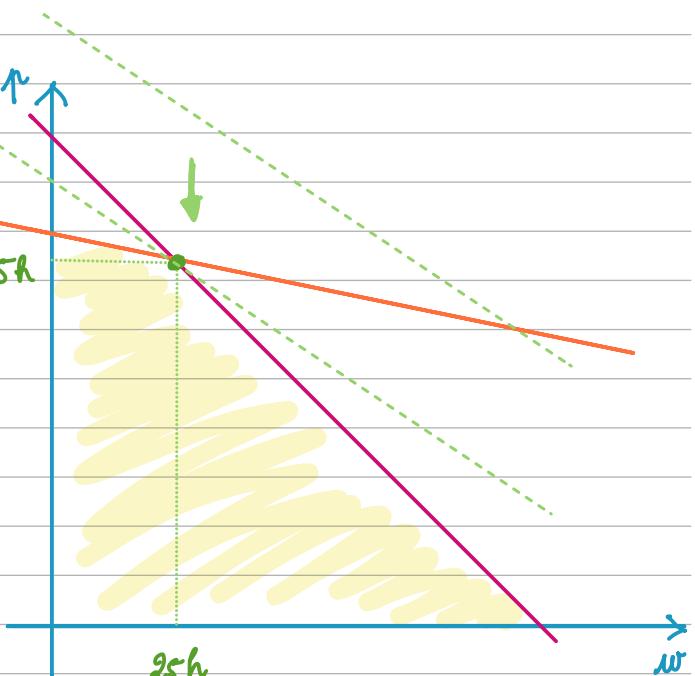
Maximize  $50w + 120p =: y$



find a line

$$p = \frac{y}{120} - \frac{50}{120}w$$

with the highest y-intercept.



Q3. The best strategy is 25h with the wood cut and 75h re-seeded, for a profit of 10250 k\$.

## Exercise 2. Student diet problem

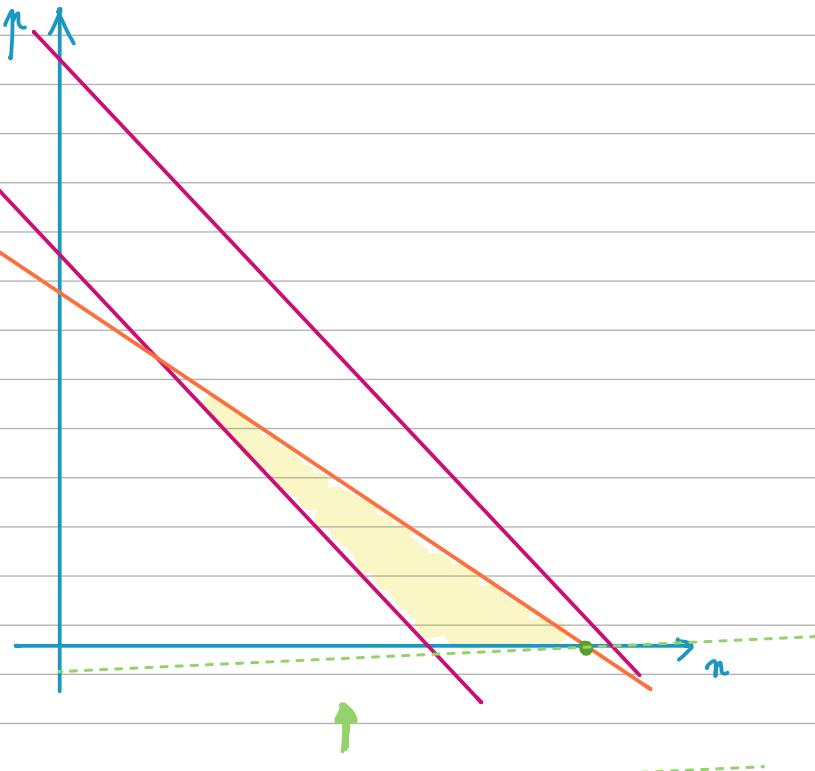
Q1.  $\begin{cases} 60n + 700p + 400r \leq 3500 \\ 60n + 700p + 400r \geq 2500 \quad (1) \\ 21n + 20p \leq 120 \\ 21n + 20p \geq 80 \quad (2) \\ n, p, r \geq 0 \end{cases}$

Minimize  $0,5n + 3,5p + 0,5r$

Q2. Instead of drawing in 3D, as we are minimizing a strictly increasing function, we consider two cases: if (1) is an equality or if (2) is an equality.

Case 1:  $60n + 700p + 400r = 2500$

$$\Leftrightarrow r = 6,25 - 1,75p - 1,15n$$



Dans ce cas, l'optimal est  $n = 125/23$ ,  $p = 0$  et  $\lambda = 0$  (\*)

avec un prix de 2,92€ environ.

Dans l'autre cas, on trouve  $n = \frac{80}{21}$ ,  $p = 0$  et  $\lambda = 25/4$ , avec un prix de 5,03€ environ. (†).

On conclut que (\*) est l'optimal.

### Exercice 3. Bank allocation.

$c$ : crédit client,  $v$ : crédit voiture  $p$ : prêt.

$$\begin{aligned} v + p &\geq 60\% \times (v + p + c) \Rightarrow \\ p &\geq 40\% (v + p + c) \Rightarrow \\ 6\% c + 4\% v + 2\% p &\leq (long.c) 3,2\% \Rightarrow \end{aligned} \quad \left\{ \begin{array}{l} v + c + p \leq 10^6 \\ 0,3v + 0,3p - 0,6c \geq 0 \\ 0,6p - 0,4v - 0,4c \geq 0 \\ 0,028c + 0,0008v - 0,0012p \leq 0 \\ v, p, c \leq 0 \end{array} \right.$$

Maximise  $0,06c + 0,04v + 0,02p$

We can compute the solution by case by case analysis.

### Exercise 6. Independent Set Problem

Variables:  $x_{uv}$  for every vertex  $u \in V$

Constraints: for any edge  $uv \in E$ ,  $x_u + x_v \leq 1$

for any vertex  $u \in V$ ,  $1 \geq x_u \geq 0$

Maximize  $\sum_{u \in V} x_u =$  size of the independent set

### Exercise 7. Dominating Set Problem

Variables  $x_u \quad u \in V$

Constraints  $\forall u \in V, 0 \leq x_u \leq 1$

$\forall u \in V, \sum_{v \in N[u]} x_v \geq 1$

$\hookrightarrow$  closed neighborhood

Minimize  $\sum_{u \in V} x_u$

### Exercise 8. N-queens problem

Variables:  $x_{i,j}, i, j \in [1, N]$

Constraints:  $\forall i, \sum x_{i,j} \leq 1$

$\sum x_{j,i} \leq 1$

$\sum x_{i,i+j} \leq 1$

$\sum x_{i,i-j} \leq 1$

Maximize  $\sum_{i,j} x_{i,j}$

$\forall i, j \quad 0 \leq x_{i,j} \leq 1$

# 1D n° 2 Linear Programming and the simplex method.

## Exercise 1. Computer production

Q1.

$$\left\{ \begin{array}{l} n + d \leq 10000 \text{ M} \\ n + 2d \leq 15000 \text{ M} \\ 4n + 3d \leq 38000 \text{ M} \\ n, d \geq 0 \end{array} \right.$$

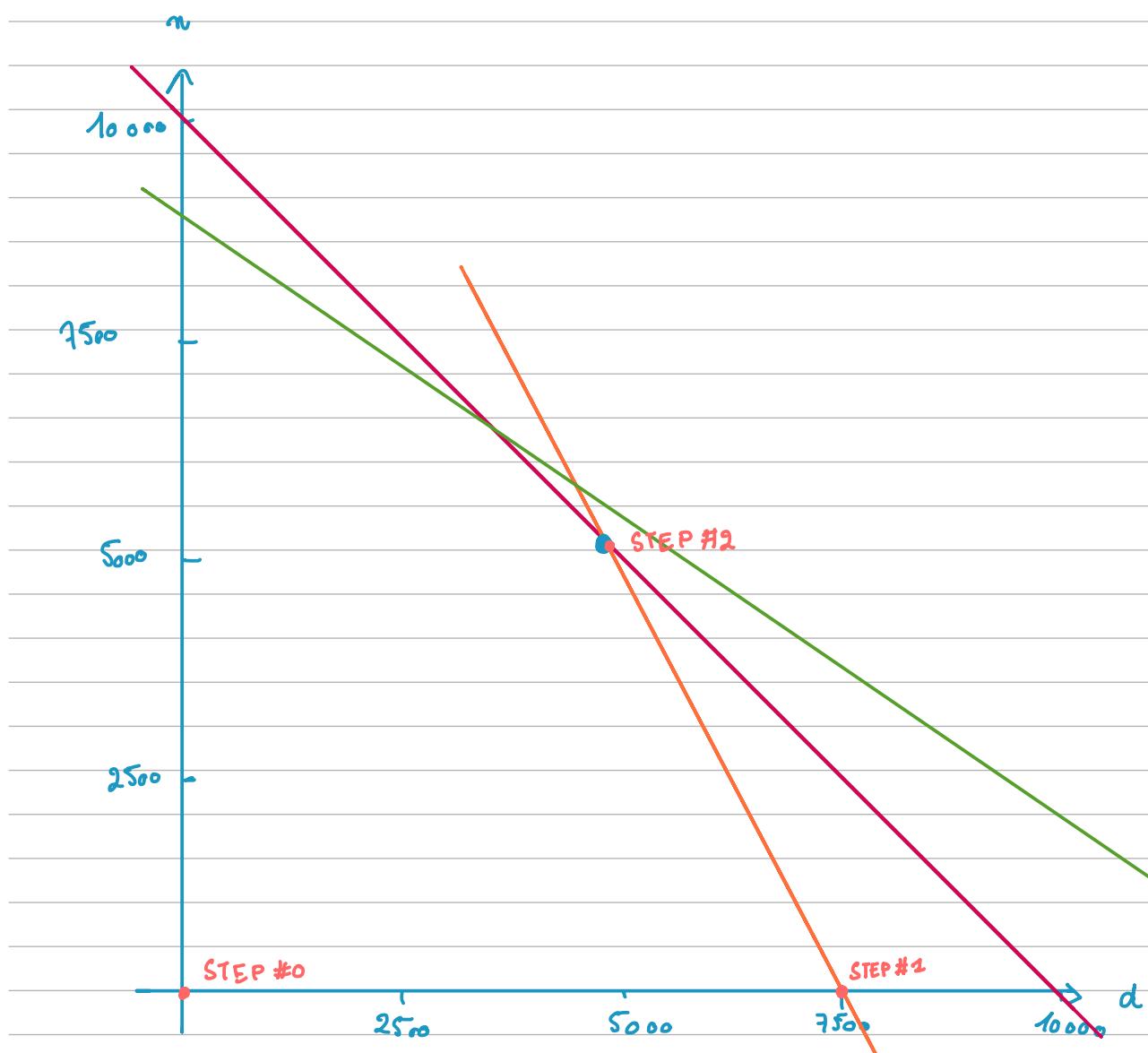
$$\max 750n + 1000d$$

$$n = 9500 - \frac{3}{4}d$$

$$\frac{3}{4}d = 9500$$

Q2.

L'optimal est  $n=5000$  et  $d=5000$ .



$$\begin{cases} u = 10 - n - d \\ v = 15 - n - 2d \\ w = 38 - 6n - 3d \\ y = 75n + 100d \end{cases}$$

**STEP #1** Choice for pivot :  $d$  enters and  $v$  leaves.

We have  $d = 7,5 - \frac{n}{2} - \frac{v}{2}$  thus, we substitute

$$\begin{cases} u = 2,5 - \frac{n}{2} + \frac{v}{2} \\ d = 7,5 - \frac{n}{2} - \frac{v}{2} \\ w = 15,5 - 2,5n - 2,5v \\ y = 75n + 25m - 50v \end{cases}$$

**STEP #2** Choice for pivot :  $m$  enters and  $u$  leaves

Solution:  $5000 = m = d$  for a reward of 87500

Exercise 2. Multiple optimal solutions



# TD m<sup>o</sup> 4

Exercice 1. Le cas non faisable / non faisable.

Dual	Primal	Empty domain	Unbounded	Optimal solution
Empty domain	Yes	Yes	No	
Unbounded	Yes	No	No	
Optimal solution	No	No	No	Yes

Considérons  $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \leq \begin{pmatrix} -1 \\ -1 \end{pmatrix}$ ,  $\max x+y$

son dual est  $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \geq \begin{pmatrix} -1 \\ -1 \end{pmatrix}$ ,  $\min x+y$

Exercice 2.

$$\text{Q1. } \begin{cases} \max \sum_{i,j} c_{ij} x_{ij} \\ (a_i) : \sum_{j=1}^n x_{ij} \leq 1 \\ (t_j) : \sum_{i=1}^m x_{ij} \leq 1 \end{cases}$$

Solution optimale :  $x_{13} = x_{24} = x_{32} = x_{41} = 1$

pour une affectation agent compétence de 20

$$\text{Q2. We compute } 4 \times (a_1) + 4 \times (a_2) + 5 \times (a_3) + 3 \times (a_4) \\ + (t_1) + (t_2) + 2 \cdot (t_4)$$

and we obtain that

25	15	16	30	4
20	25	16	36	4
30	36	25	49	5
16	16	9	25	3

1    1    0    2

Exercice 3.

Q1. A vertex  $v$  is a point of  $P$  that cannot be written as  $\lambda v_1 + (1-\lambda)v_2 = v$  for some  $\lambda \in [0,1]$ ,  $v_1, v_2 \in P$ . Such values  $(\lambda, v_1, v_2)$  provide an easy certificate.

Q2. Define  $\varepsilon := \min\left(\min\{x_i, \frac{1}{2} - x_i\} \mid i \in I^-\right) \cup \left\{\min\{x_i, x_j - \frac{1}{2}\} \mid i \in I^+\right\}$

Then it is easy to see that  $x + \varepsilon y$  is a point of  $P$ .

Q3. We have that  $v^+ := x + \varepsilon y \in P$  and  $v^- := x - \varepsilon y \in P$

but  $\frac{1}{2}v^+ + \frac{1}{2}v^- = x$  thus it is only possible if  $\varepsilon$  is undefined.

Q4. We solve the LP problem :

$$\min \sum_{i \in V} c(i)x_i$$

$$\text{such that } \forall ij \in E, x_i + x_j \leq 1 \\ \forall i \in V, x_i \geq 0$$

Take the solution and put  $\frac{1}{2}-0.1$ .

Exercise 6.

Q1. (P)

$$\max 0$$

such that

$$\forall v \in V, w_v \geq 0$$

$$\forall v \in V, \sum_{u \in E} w_u - \sum_{u' \in E} w_{u'} \geq 0$$

$$\sum_{v \in V} w_v = 1.$$

$$Ax \leq 0$$

$$x \geq 0$$

$$\sum x = 1$$

Q2. (D)

$$\min z_y$$

such that

$$y \geq 0$$

$$-Ay + z_y 1 \geq 0$$

Q3 0 is a solution of (P)

thus (P) is non-empty.

## Exercise 5.

Q1. Consider the instance  $T_1 = \{x\}$ ,  $T_2 = \{y\}$  and  $T_3 = \{x, y\}$ .

Q2.

$$\begin{array}{|c|c|} \hline \text{(P)} & \begin{array}{l} \max 0 \\ \forall j \in [1, m], \sum_{i \in T_j} x_i = 1 \\ x_1, \dots, x_n \geq 0 \end{array} & \text{(Q)} & \begin{array}{l} \min y \\ A^T y \geq 0 \\ y \geq 0 \end{array} \\ \hline \end{array}$$

$$A = \left( \mathbb{1}_{i \in T_j} \right)$$