

TD n° 1.

Ex 1. Rccq

Ex 2. Rccq

Ex 3. If  $\Gamma \vdash A$  is provable in NJ, take some proof tree in NJ for it, if it is a valid proof tree in  $NJ + R$ , thus  $\Gamma \vdash A$  is provable in  $NJ + R$ .

Conversely, take a proof tree  $\pi$  of  $\Gamma \vdash A$  in  $NJ + R$ , by admissibility of  $R$ , every occurrence of  $R$  in  $\pi$  can be replaced by part of a tree that doesn't use  $R$ . Doing this starting from the leaves yield a valid proof tree for  $\Gamma \vdash A$  in NJ.

Ex 4.

1. Take a proof tree of  $\Gamma, A, B \vdash C$ . Replace each occurrence of

$$\frac{}{\Gamma, A, B, D \vdash A} Ax$$

$$\frac{\Gamma, A \wedge B, \Delta \vdash A \wedge B}{\Gamma, A \wedge B, \Delta \vdash A} Ax$$
$$\frac{\Gamma, A \wedge B, \Delta \vdash A}{\Gamma, A \wedge B, \Delta \vdash C} Neg$$

and

$$\frac{}{\Gamma, A, B, D \vdash B} Ax$$

with

$$\frac{\tilde{\Delta} \vdash A \wedge B}{\tilde{\Delta} \vdash B} Ax$$
$$\frac{\tilde{\Delta} \vdash A \wedge B}{\tilde{\Delta} \vdash C} Ned,$$

also replacing every sequent  $\Gamma, A, B, \Delta \vdash D$  with  $\Gamma, A \wedge B, \Delta \vdash D$ .

2. By induction on the size of  $\Gamma$ , we apply Q1.

3. No, derivable rules cannot remove hypotheses (with the rules in NJ).

Ex 5.

1.  $\frac{\Sigma, \Gamma \vdash A}{\Sigma, \Delta \vdash B}$  admissible iff
  - ( $\Sigma, \Gamma \vdash A$  implies  $\Sigma, \Delta \vdash B$ )  $\rightarrow$  Ex h.
  - iff ( $\Sigma + \Delta \Gamma \Rightarrow A$  implies  $\Sigma \vdash \Delta \Rightarrow B$ )
  - iff  $\Sigma \vdash (\Delta \Gamma \Rightarrow A) \Rightarrow (\Delta \Rightarrow B)$   $\rightarrow$  currying
  - iff  $\Sigma + (\Delta \Gamma \Rightarrow A) \wedge \Delta \Rightarrow B$

2. Similarly to Q1, we know

$$\Sigma, \Gamma + (\Delta \Rightarrow A) \Rightarrow B$$

is provable, thus, by repeated currying, writing  $\Delta = F_1, \dots, F_n$ , the sequent

$$\Sigma, \Gamma \vdash (F_1 \Rightarrow \dots \Rightarrow F_n \Rightarrow A) \Rightarrow B$$

is provable; take  $\Pi$  a proof tree for it.

$$\frac{\begin{array}{c} \triangle \Pi \\ \hline \Sigma, \Gamma \vdash \Phi \Rightarrow B \end{array}}{\Sigma, \Gamma \vdash B} \text{ assumption}$$

$$\frac{\Sigma, \Gamma, \Delta \vdash A \quad \frac{\Sigma, \Gamma + F_1 \Rightarrow \dots \Rightarrow F_n \Rightarrow A}{\Phi} \text{ repeated } \Rightarrow_I}{\Sigma, \Gamma \vdash B} \Rightarrow_E$$

Ex 6.

$$1. \frac{\frac{\frac{\triangle \Pi_1}{\Gamma \vdash A} \text{ wr}}{\Gamma, \Delta \vdash A} \text{ wr} \quad \frac{\triangle \Pi_2}{\Delta \vdash B} \text{ wr}}{\Gamma, \Delta \vdash A \wedge B} \wedge I$$

$$2. \frac{\frac{\Gamma \vdash \neg A}{\Gamma, \Delta \vdash \neg A} \text{ wr} \quad \frac{\Delta \vdash A}{\Gamma, \Delta \vdash A} \text{ wr}}{\Gamma, \Delta \vdash \perp} \neg E$$

$$\frac{\Gamma \vdash A \vee B \quad \frac{\Delta, A \vdash C \quad \frac{\Sigma, B \vdash C}{P, \Delta, \Sigma, B \vdash C} \text{ wr}}{P, \Delta, \Sigma \vdash C} \text{ wr}}{P, \Delta, \Sigma \vdash A \vee B} \vee E$$

$$\frac{\frac{\Gamma + A \Rightarrow B}{\Gamma, \Sigma \vdash A \Rightarrow B} \text{ wb} \quad \frac{\Delta + B}{\Gamma, \Delta \vdash B} \text{ wr}}{\Gamma, \Sigma \vdash B} \Rightarrow E.$$

Ex 7.

Define  $C_{\perp} := \perp \Rightarrow \perp$  and  $C_{\neg A} := A \Rightarrow \perp$ .

$$\frac{\frac{\Gamma, \perp \vdash \perp}{\Gamma \vdash C_{\perp}} \text{ Ax}}{\Gamma \vdash C_{\perp}} \Rightarrow_I \text{ "T I"}$$

$$\frac{\Gamma \vdash C_{\neg A} \quad \Gamma \vdash A}{\Gamma \vdash \perp} \Rightarrow_E \neg E.$$

$$\frac{\Gamma, A \vdash \perp}{\Gamma \vdash C_{\neg A}} \Rightarrow_I \neg I$$

Ex 8. Rocq

Ex 9. Rocq

$\neg_R A$  is equivalent to  $\neg A$  iff  $\vdash \neg R$  is provable.