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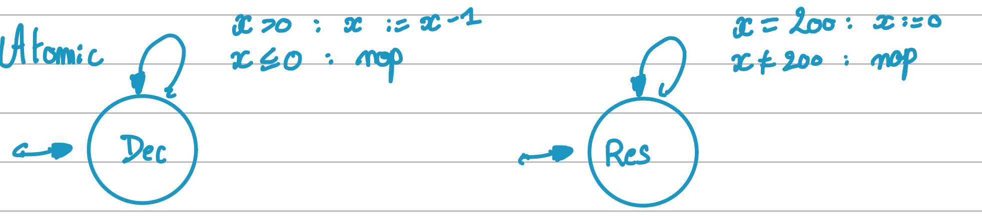
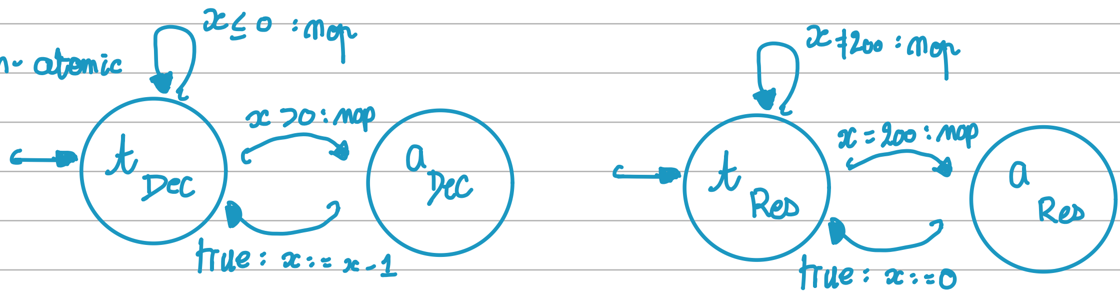
Modelling Concurrent Systems

Exercise 1.

Q1. Atomic

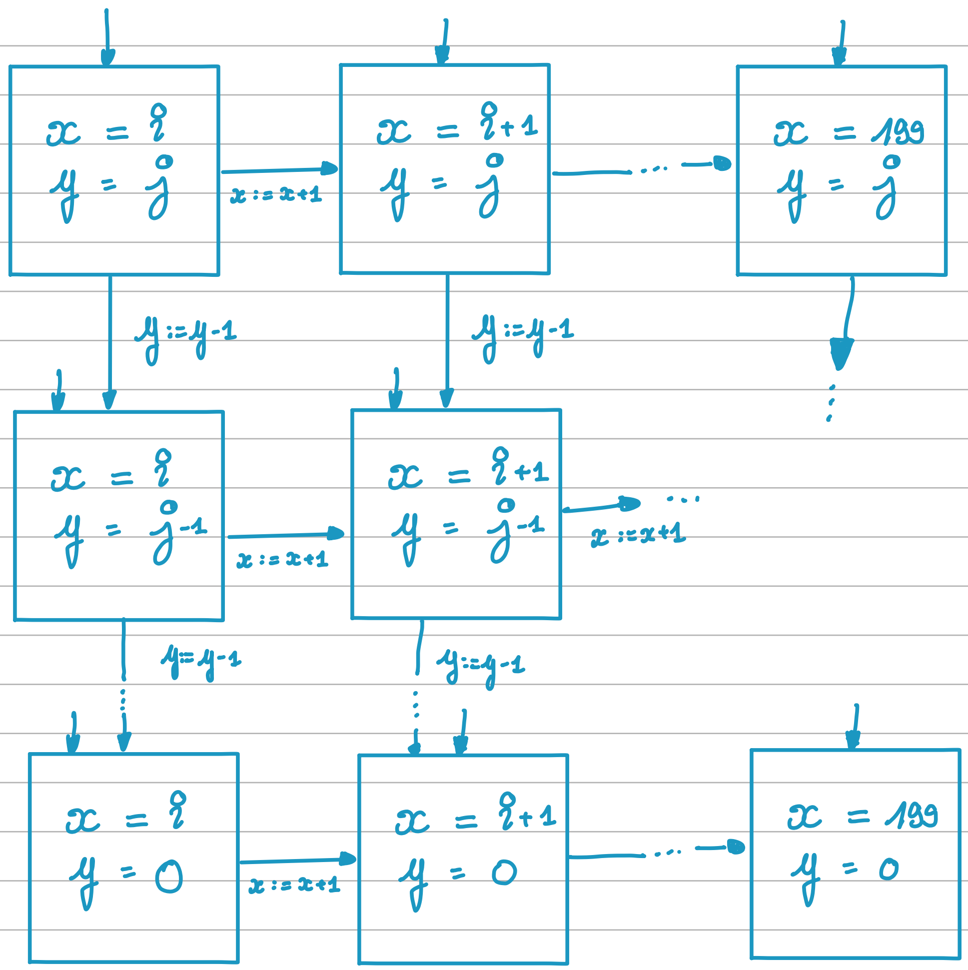


Non-atomic

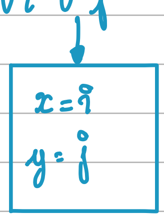


Q2. $TS(A_{Inc}) \,|||\, TS(A_{Dec})$

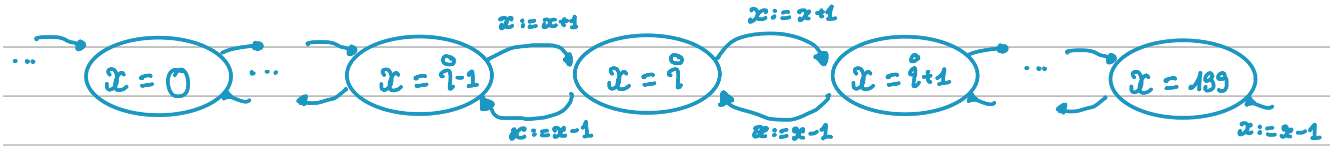
On déduit la mémoire avec $TS(A_{Inc}) ||| TS(A_{Dec})$.



$\forall i \; \forall j$



$TS(A_{Inc} \parallel\!| A_{Dec})$:

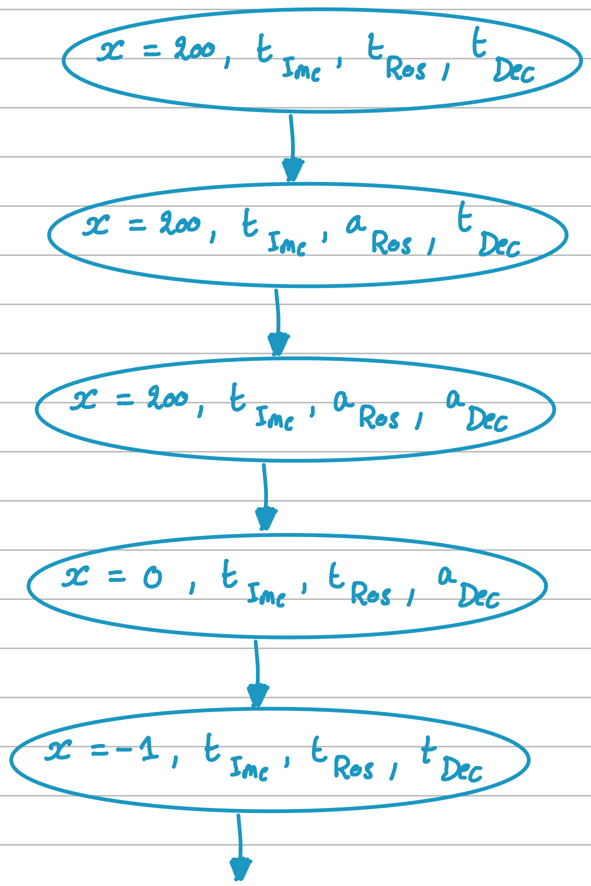


Q3. Les effets préservent l'invariant :

- pour $x := x+1$ si $x < 200$ et $0 \leq x \leq 200$ alors $0 \leq x+1 \leq 200$
- pour $x := x-1$ si $x > 0$ et $0 \leq x \leq 200$ alors $0 \leq x-1 \leq 200$
- pour $x := 0$ si $x = 200$ et $0 \leq x \leq 200$ alors $0 \leq 0 \leq 200$

D'où l'invariant est invariant.

Q4.

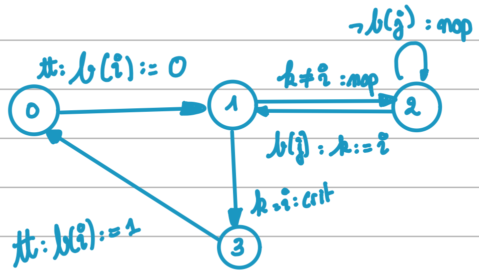


Kaboom! on a cassé l'invariant!

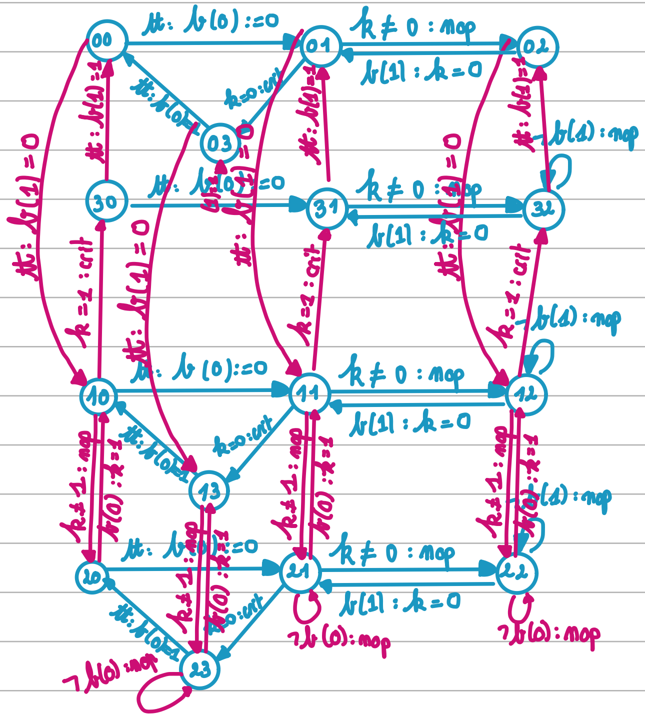
Exercise 2.

Q1.

Q1.



Q2.



Q3. The state 33 is unreachable. Thus, we ensure mutual exclusion.

Exercise 3.

In ex 1 / Q2,

we have a state $x = 1$, $x' = 0$

we don't have that in

$TS(PG_1 \mid\mid\mid PG_2)$.

TD n° 2

Linear Time Properties (and a bit of Modelling)

I. Safeness and Invariance

Exercise 1. Invariance is safe.

Define $P_{bad} := \{ \hat{\sigma} \in (2^{AP})^* \mid \exists i.\ \hat{\sigma}(i) \not\models \varphi \}$.

Then,

$$\begin{aligned}
\sigma \in P &\iff \forall i,\ \sigma(i) \models \varphi \\
&\iff \forall \hat{\sigma} \subseteq_{fini} \sigma,\ \forall i \leq \text{length } \hat{\sigma},\ \hat{\sigma}(i) \models \varphi \\
&\iff \forall \hat{\sigma} \subseteq_{fini} \sigma,\ \text{non}(\exists i.\ \hat{\sigma}(i) \not\models \varphi) \\
&\iff \forall \hat{\sigma} \subseteq_{fini} \sigma,\ \hat{\sigma} \notin P_{bad}.
\end{aligned}$$

Thus P is a safety property.

Exercise 2.

1. $\emptyset$ inv. safety
2. $\{\sigma \mid \forall i,\ \sigma(i) \models x = 0\}$ inv. safety
3. $\{\sigma \mid \forall i.\ \sigma(i) \models \neg(x=0) \wedge \neg(x>1)\} = \{\emptyset\}^\omega$ inv. safety
4. $\{\sigma \mid \sigma(0) \models x = 0\}$ safety $P_{bad} := \{\hat{\sigma} \mid \hat{\sigma}(0) \not\models x = 0\}$.
5. $\{\sigma \mid \sigma(0) \models \neg(x=0)\}$ safety $P_{bad} := \{\hat{\sigma} \mid \hat{\sigma}(0) \models x = 0\}$
6. $\{\sigma \mid \sigma(0) \models x=0$ and $\exists i.\ \sigma(i) \models x > 1\}$ neither
7. $\{\sigma \mid \exists N\ \forall i \geq N,\ \sigma(i) \models \neg(x > 1)\}$ neither
8. $\{\sigma \mid \forall N\ \exists i \geq N,\ \sigma(i) \models x > 1\}$ neither
9. $(2^{AP})^*$ inv. safety

II Operations on Safety Properties

Exercise 3. Characterization of safety properties

Q1. We have that

P safety property $\iff \exists P_{bad}, \forall \sigma, \ (\sigma \in P \iff \mathrm{Pref}\,\sigma \cap P_{bad} = \emptyset)$

$\iff \exists P_{bad}, \forall \sigma, \ (\sigma \in P^c \iff \mathrm{Pref}\,\sigma \cap P_{bad} \neq \emptyset)$

$(*) \iff \forall \sigma, \ (\sigma \in P^c \iff \mathrm{Pref}\,\sigma \cap P = \emptyset)$

For $(*)$, "$\Leftarrow$" we take $P_{bad} := (2^{AP})^* \setminus P$ and we have the required property.

"$\Rightarrow$" We have $\sigma \in \mathrm{Pref}\,\sigma$ and we can conclude.

Q2. We show P safety $\iff$ $P \supseteq \mathrm{cl}\,P$. We always have $\mathrm{cl}\,P \supseteq P$.

"$\Rightarrow$" Let $\sigma \in P^c$, we will show $\sigma \notin \mathrm{cl}\,P$.

i.e. $\mathrm{pref}\,\sigma \not\subseteq \mathrm{pref}\,P$

we have $\hat{\sigma} \in \mathrm{pref}\,\sigma$ and $\hat{\sigma} \notin \mathrm{pref}\,P$ by (1)

"$\Leftarrow$" Let $\sigma \in P^c$. we will show $\exists \hat{\sigma} \in \mathrm{pref}\,\sigma$ such that $\hat{\sigma} \notin \mathrm{pref}\,P$

i.e. $\mathrm{pref}\,\sigma \not\subseteq \mathrm{pref}\,P$.

Exercise 4. Union & intersections

Q1. For $\sigma \in (P \cup Q)^c = P^c \cap Q^c$, there exists $\hat{\sigma}_P \leq_{fini} \sigma$ st $\hat{\sigma}_P \cdot (2^{AP})^\omega \cap P = \emptyset$
and $\hat{\sigma}_Q \leq_{fini} \sigma$ st $\hat{\sigma}_Q \cdot (2^{AP})^\omega \cap Q = \emptyset$.

Let $\hat{\sigma} \leq \hat{\sigma}_P$ and $\hat{\sigma} \leq \hat{\sigma}_Q$. We have

$$\hat{\sigma} \cdot (2^{AP})^\omega \cap (P \cup Q) = (\hat{\sigma} \cdot (2^{AP})^\omega \cap P) \cup (\hat{\sigma} \cdot (2^{AP})^\omega \cap Q)$$

$$= (\hat{\sigma}_P \cdot (2^{AP})^\omega \cap P) \cup (\hat{\sigma}_Q \cdot (2^{AP})^\omega \cap Q)$$

$$= \emptyset$$

Thus $P \cup Q$ is a safety property.

Q2. $(P \cap Q)_{bad} ::= P_{bad} \cup Q_{bad}$

$\sigma \in P \cap Q \iff \forall \hat{\sigma} \subseteq_{fini} \sigma \quad \hat{\sigma} \notin P_{bad}$ and $\hat{\sigma} \notin Q_{bad}$

$\iff \forall \hat{\sigma} \subseteq_{fini} \sigma \quad \hat{\sigma} \notin (P \cap Q)_{bad}$

III Safety properties and Transition Systems.

Exercise 5. Finite traces

If $TS \models P$ then, $Tr_{fin}(TS) \cap P_{bad}$

$= pref(Tr^{\omega}(TS)) \cap P_{bad}$

TD n° 3

Safety and Liveness Properties

I. Liveness properties.

Exercise 1. (Closure of liveness properties)

$$\begin{aligned}
cl(P) = (2^{AP})^\omega &\iff \forall \sigma \in (2^{AP})^\omega \quad \mathrm{pref}\,\sigma \subseteq \mathrm{pref}\,P \\
&\iff \forall \hat{\sigma} \in (2^{AP})^* \quad \hat{\sigma} \in \mathrm{pref}\,P \\
&\iff \forall \hat{\sigma} \in (2^{AP})^* \quad \exists \sigma \in P,\ \hat{\sigma} \leq \sigma \\
&\iff P \text{ liveness property}
\end{aligned}$$

Exercise 2 (Unions & intersections).

We proved that $cl(P \cup Q) = cl(P) \cup cl(Q)$
and $cl(P \cap Q) = cl(P) \cap cl(Q)$

Thus $P \cup Q$ and $P \cap Q$ are liveness properties.

Even better: if P <u>or</u> Q is liveness, then $P \cup Q$ is liveness.

II Topology on infinite words

Exercise 3 (Σ^ω as a topological space)

We need to prove that $\Omega\Sigma^\omega$ is stable under arbitrary unions and finite intersections.

1) Let $\mathcal{U} \subseteq \wp(\Sigma^*)$. Define $\bar{\mathcal{U}} := \bigcup \mathcal{U}$. We have that

$$ext(\bar{\mathcal{U}}) = \bigcup_{u \in \bar{\mathcal{U}}} ext(u) = \bigcup_{U \in \mathcal{U}} \bigcup_{u \in U} ext(u)$$

Thus $\Omega\Sigma^\omega$ is stable under arbitrary unions.

2) Let $U, V \subseteq \Sigma^*$.

We have that

$$ext(U) \cap ext(V) = \bigcup_{u \in U} \bigcup_{v \in V} ext(u) \cap ext(v)$$

For some $u \in U$, $v \in V$, we have the 3 following cases:

a) $u \nleq v$ and $v \nleq u$ thus $ext(u) \cap ext(v) = \emptyset$

b) $u \leq v$ thus $ext(u) \cap ext(v) = ext(u)$

c) $v \leq u$ thus $ext(u) \cap ext(v) = ext(v)$

Define

$$W := \{u \in U \mid \exists v \in V,\ u \leq v\} \cup \{v \in V \mid \exists u \in U,\ v \leq u\}$$

and we have $ext(W) = ext(U) \cap ext(V)$.

By induction, $\Omega\Sigma^\omega$ is stable under finite intersections.

Exercise 4. (Open sets)

Let $P \subseteq \Sigma^\omega$.

P is open iff $\exists U \subseteq \Sigma^*$ $P = \bigcup_{u \in U} ext(u)$

iff $\begin{cases} \forall \sigma \in P,\ \exists u \in \Sigma^*,\ \sigma \in ext(u) \\ \forall u \in \Sigma^*,\ ext(u) \subseteq P \end{cases}$

iff $\forall \sigma \in P, \exists u \in \Sigma^*, \sigma \in ext(u) \subseteq P$

iff $\forall \sigma \in P, \exists \hat{\sigma} \in \Sigma^*, \hat{\sigma} \leq \sigma$ and $ext(\hat{\sigma}) \subseteq P$.

Exercise 5. (Density and liveness)

P is dense iff for any non-empty open set U', $P \cap U' \neq \emptyset$

iff for any non-empty $U \subseteq \Sigma^*$, $P \cap ext(U) \neq \emptyset$

iff for any $u \in \Sigma^*$, $P \cap ext(u) \neq \emptyset$

iff for any $\hat{\sigma} \in \Sigma^*$, $\exists \sigma \in P$, $\hat{\sigma} \subseteq \sigma$.

iff P is a liveness.

III. Decomposition theorem.

Exercise 6. Let $P \subseteq (2^{AP})^\omega$.

Define $P_{safe} := cl(P)$ which is a safety property as $cl(P_{safe}) = cl^2(P) = cl(P)$

and $P_{live} = P \cup P_{safe}^C$ which is a liveness property as P_{live} is topologically dense: if $U \subseteq \Sigma^*$, and $ext(U) \cap P = \emptyset$ then $P \subseteq ext(U)^C$ and thus $cl(P) \subseteq ext(U)^C$ so we can conclude $ext(U) \subseteq cl(P)^C$. (closed)

Thus proving the decomposition theorem.

Exercise 7.

Q1. $cl(P_1) = P_1$ thus liveness and $P_{safe} = \Sigma^\omega$

Q2. $cl(P_2) = \{\sigma \mid \sigma(0) = a\}$ thus not liveness and not safety $P_{liveness} = \Sigma^\omega \setminus \{a^\omega\}$.

Q3. $cl(P_3) = \{a^\omega\} = P_3$ thus liveness and $P_{safe} = \Sigma^\omega$

Q4. $cl(P_4) = \{\sigma \mid \sigma \text{ contains} \leq 1 \ b\text{'s}\} = P_4 \cup \{a^\omega\}$ thus not liveness and not safety $P_{live} = \Sigma^\omega \setminus \{a^\omega\}$.

Q5. $cl(P_5) = \Sigma^\omega$ thus liveness and $P_{safe} = \Sigma^\omega$

Q6 $cl(P_6) = \Sigma^\omega$ thus liveness and $P_{safe} = \Sigma^\omega$

Q7. $cl(P_7) = \Sigma^\omega$ thus liveness and $P_{safe} = \Sigma^\omega$

Q8. $cl(P_8) = \Sigma^\omega$ thus liveness and $P_{safe} = \Sigma^\omega$.

TD n° 4

Topology.

I. Topology on ω-words : Examples.

c.f exercise 7 / td 3

Safety ⟷ closed

liveness ⟷ dense

closure ⟷ closure (c.f. TD5)

open: $1,$
$P_1 = ext(b)$

II General Properties of Topological Spaces

Exercise 2 (Closed sets)

Q1. $\emptyset = X^c$ and $X = \emptyset^c$

Q2. C_i^c is open thus $\bigcup C_i^c$ is open and then $(\bigcup C_i^c)^c = \bigcap C_i$ is closed.

Q3. C_i^c is open thus $\bigcap C_i^c$ is open and then $(\bigcap C_i^c)^c = \bigcup C_i$ is closed.

Exercise 3 (Closed and open sets).

Q1. "⟹" Take $U := A \in \Omega X$. For all $x \in A$, we have $x \in U$ and $U \subseteq A$.

"⟸"

We have $A = \bigcup_{a \in A} U_a \in \Omega X$ thus A is open.

Q2.

A closed iff A^c open
iff $\forall x \in A^c, \exists U \in \Omega X,\ x \in U \wedge U \subseteq A^c$
iff $\forall x \notin A, \exists U \in \Omega X,\ x \in U \wedge U \cap A = \emptyset$

Exercise 4. (Closure)

Q1. Let us show $x \notin \bar{A}$ iff $\exists N \in \mathcal{N}_x$, $N \cap A = \emptyset$.

"$\Rightarrow$" Consider $x \in \bar{A}$, thus there exists a closed set $A \subseteq C \subseteq X$ such that $x \notin C$.
Take $N := C^c$ which is open and contains N. And, $A \cap N = \emptyset$.

"$\Leftarrow$" Consider x such that there exists $N \in \mathcal{N}_x$, with $N \cap A = \emptyset$.
Then, there exists an open set $U \in \Omega X$ such that $x \in U \subseteq N$.
Take $C := U^c$ which is closed, and $U \cap A = \emptyset$ thus $A \subseteq N$.
Also, $x \notin C$.

Q2. $\bar{A}$ is the smallest closed set containing A as closed sets are stable under arbitrary intersections and $\bar{A} := \inf \{ C \subseteq X \mid C \text{ closed } \& \ A \subseteq C \}$.
$\hookrightarrow$ for $\subseteq$ set inclusion

$\bar{A} = A \Leftrightarrow$ the smallest closed set containing A is A
$\Leftrightarrow$ A is closed.

Q3. A dense iff $\forall U \in \Omega X \setminus \{\emptyset\}$, $A \cap U \neq \emptyset$

iff $\forall C$ closed and $C \neq X$, $A \subseteq C$

iff $\bar{A} \subseteq X$

III Topology on ω-words: Properties

Exercise 5.

Suppose $v \subseteq u$, thus $u = vw$. Take $u\sigma \in \mathrm{ext}(u)$ and we have $vw\sigma \in \mathrm{ext}(v)$.

Suppose $\mathrm{ext}(u) \subseteq \mathrm{ext}(v)$. We have $ua^\omega, ub^\omega \in \mathrm{ext}(u)$ thus $ua^\omega, ub^\omega \in \mathrm{ext}(v)$.
And $v \in \mathrm{pref}(ua^\omega) \cap \mathrm{pref}(ub^\omega) = \mathrm{pref}\, u$ thus $v \subseteq u$.

Exercise 6.

P is closed iff $\forall x \notin P$, $\exists V \in \Omega X$, $x \in V$ and $V \cap P = \emptyset$

iff $\forall x \notin P$, $\exists U \subseteq \Sigma^*$, $x \in ext(U)$ and $ext(U) \cap P = \emptyset$

iff $\forall x \notin P$, $\exists \hat{\sigma} \in \Sigma^*$, $\hat{\sigma} \subseteq x$ and $ext(\hat{\sigma}) \cap P = \emptyset$.

IV Bases and Subbases.

Exercise 7.

Let $(V_i)_{i \in I}$ be a family of elements of ΩX.

We write $V_i := \bigcup_{j \in I_j} U_{ij}$ where $U_{ij} \in \mathcal{B}$.

Take $\bigcup V_i = \bigcup_{\substack{i \in I \\ j \in I_j}} U_{ij} \in \Omega X$.

Let $A = \bigcup_{i \in I} A_i$ $A_i \in \mathcal{B}$ and $B = \bigcup_{j \in I} B_j$.

$$A \cap B = \bigcup_{\substack{i \in I \\ j \in J}} (\underbrace{A_i \cap B_j}_{\text{an element of } \mathcal{B}}) \in \Omega X.$$

By induction, ΩX is closed under finite $\cap$'s.

V. Metric Spaces

Exercise 8 (Open ball topology)

Q1. Take $A, B \in \mathcal{U}$. For every $a \in A$, there exists $\varepsilon_A^a > 0$ such that $B_{\varepsilon_A^a}(a) \subseteq A$.

For every $b \in B$, there exists $\varepsilon_B^b > 0$ such that $B_{\varepsilon_B^b}(b) \subseteq B$.

For every $x \in A \cap B$, take $\varepsilon^x := \min(\varepsilon_A^a, \varepsilon_B^b) > 0$

We have, $B_{\varepsilon^x}(x) \subseteq A \cap B$.

Consider $(A_i)_{i \in I}$ a family of elements of ΩX.

For every $i \in I$ and $a \in A_i$, there exists $\varepsilon_a^i > 0$ such that $B_{\varepsilon_a^i}(a) \subseteq A_i$.

Let $a \in \bigcup_{i \in I} A_i$. There exists $i \in I$ such that $a \in A_i$. We have $B_{\varepsilon_a^i}(a) \subseteq A_i \subseteq \bigcup_{i \in I} A_i$

Q2. We have $\overline{S} = \{x \in X \mid \forall N \in \mathcal{N}_x, N \cap S \neq \emptyset\}$.

We have

$$x \in \overline{S} \Rightarrow \forall N \in \mathcal{N}_x, N \cap S \neq \emptyset$$

$$\Rightarrow \forall \varepsilon > 0, B_\varepsilon(x) \cap S \neq \emptyset$$

$B_\varepsilon(x) \in \mathcal{N}_x$.

On the other hand,

$$\forall \varepsilon > 0, B_\varepsilon(x) \cap S \neq \emptyset \Rightarrow$$

$$\forall N \in \mathcal{N}_x, \begin{cases} N \subseteq B_\varepsilon(x) \text{ for some } \varepsilon > 0 \\ N \cap S \neq \emptyset \end{cases}$$

$$\Rightarrow \forall N \in \mathcal{N}_x, N \cap S \neq \emptyset$$

Exercise 9 (Distance on ω-words)

We trivially have $d(x,y) = 0 \iff x = y$ and $d(x,y) = d(y,x)$.

Let σ, τ, μ be three words on A^ω. We will show $d(\sigma, \mu) \leq d(\sigma, \tau) + d(\tau, \mu)$

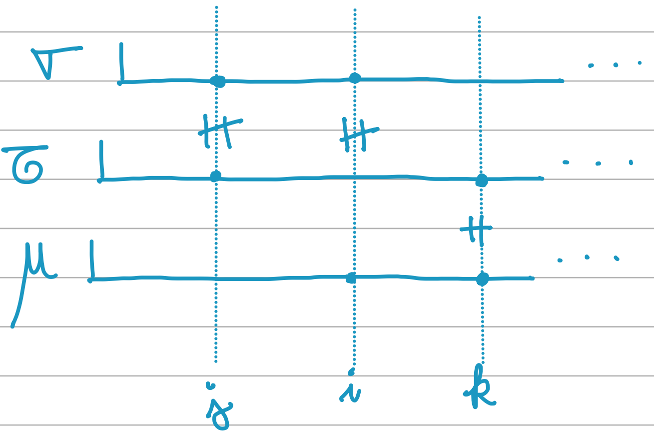
If $\sigma = \mu$, the result is trivial. Same thing for $\sigma = \tau$ or $\tau = \mu$.

Suppose $\tau \neq \mu \neq \sigma \neq \tau$.

Let $i := \min \{ i \mid \sigma(i) \neq \mu(i) \}$, $j := \min \{ j \mid \sigma(j) \neq \tau(j) \}$

and $k := \min \{ k \mid \tau(k) \neq \mu(k) \}$.

Without loss of generality, let us suppose $j < k$ (if $j = k$ then $i = j = k$ and thus the result is trivially true)



We necessarily have $i > j$ as $\sigma|_j = \tau|_j = \mu|_j$

as $\tau|_k = \mu|_k$.

Thus, $d(\sigma, \mu) = 2^{-i} \leq 2^{-j} \leq 2^{-j} + 2^{-k}$
$\leq d(\sigma, \tau) + d(\tau, \mu)$.

We can conclude: (A^ω, d) is a metric space.

Exercise 10 (Open ball topology on ω-words).

1) Let $U \subseteq \Sigma^*$. Let us show that $\text{ext}(U)$ is open for the open-ball topology.

Let $u\sigma \in \text{ext}(U)$. Take $\varepsilon := 2^{-\text{length}(u)}$.

We have $B_\varepsilon(u\sigma) = \{ \tau \mid \min\{ i \mid (u\sigma)(i) \neq \tau(i) \} > \text{length}(u) \}$
$= \text{ext}(u)$.

Thus $\text{ext}(U) \in \mathcal{U}$.

2) Let $V \in \mathcal{U}$. We will show that $V = ext(U)$ for some $U \subseteq \Sigma^*$.

Let $v \in V$, and let $\varepsilon_v > 0$ such that $B_{\varepsilon_v}(v) \subseteq V$.

Define $U := \{ v|_{-\log_2(\varepsilon_v)} \mid v \in V \}$.

We have that:

$$
\begin{aligned}
ext(U) &= \bigcup_{v \in V} ext(v|_{-\log_2(\varepsilon_v)}) \\
&= \bigcup_{v \in V} B_{\varepsilon_v}(v) \\
&= V.
\end{aligned}
$$

Thus the two topologies coincide.