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Proof & Programs

TD n° 5.

Exercise 1.

We will show DNE implies EM:

$$\dfrac{\dfrac{\dfrac{}{\neg(A\vee\neg A)\vdash\neg(A\vee\neg A)}\,ax \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma\vdash\neg(A\vee\neg A)}\,ax \qquad \dfrac{\dfrac{}{\Gamma\vdash A}\,ax}{\Gamma\vdash A\vee\neg A}\,vil}{\Gamma:=\neg(A\vee\neg A), A\vdash\bot}}{\neg(A\vee\neg A)\vdash\neg A}\,\neg i}{\neg(A\vee\neg A)\vdash A\vee\neg A}\,vir}{\dfrac{\neg(A\vee\neg A)\vdash\bot}{\vdash\neg\neg(A\vee\neg A)}\,\neg i}\,\neg e \qquad \overbrace{\vdash\neg\neg(A\vee\neg A)\Rightarrow A\vee\neg A}^{DNE}}{\vdash A\vee\neg A}\,\Rightarrow_E$$

we can assume the context is empty by weakening

We will show CP implies DNE:

$$\dfrac{\overbrace{\vdash(\neg A\Rightarrow\neg\neg\neg A)\Rightarrow\neg\neg A\Rightarrow A}^{CP} \qquad \dfrac{\dfrac{\dfrac{}{\neg A,\neg\neg A\vdash\neg A}\,ax \qquad \dfrac{}{\neg A,\neg\neg A\vdash\neg\neg A}\,ax}{\neg A,\neg\neg A\vdash\bot}\,\neg_E}{\vdash\neg A\Rightarrow\neg\neg\neg A}\,\Rightarrow_I,\neg i}{\vdash\neg\neg A\Rightarrow A}\,\Rightarrow_E$$

We will show EM implies CP:

$$
\begin{prooftree}
\AxiomC{$\Gamma \vdash \underbrace{A \vee \neg A}_{EM}$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, A \vdash A$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, \neg A \vdash \neg A \Rightarrow \neg B$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, \neg A \vdash \neg A$}
\RightLabel{$\Rightarrow_E$}
\BinaryInfC{$\Gamma, \neg A \vdash \neg B$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, \neg A \vdash B$}
\RightLabel{$\neg_E$}
\BinaryInfC{$\Gamma, \neg A \vdash \bot$}
\RightLabel{$\bot_E$}
\UnaryInfC{$\Gamma, \neg A \vdash A$}
\RightLabel{$\vee_E$}
\TrinaryInfC{$\Gamma := \neg A \Rightarrow \neg B, B \vdash A$}
\RightLabel{$\Rightarrow_I \times 2$}
\UnaryInfC{$\vdash (\neg A \Rightarrow \neg B) \Rightarrow B \Rightarrow A$}
\end{prooftree}
$$

Thus EM, DNE and CP are equivalent.

We have that DNE implies PL:

$$
\begin{prooftree}
\AxiomC{$(A \Rightarrow B) \Rightarrow A \vdash \underbrace{\neg\neg A \Rightarrow A}_{DNE}$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash \neg A$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash (A \Rightarrow B) \Rightarrow A$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, A \vdash A$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, A \vdash \neg A$}
\RightLabel{$\neg_E$}
\BinaryInfC{$\Gamma, A \vdash \bot$}
\RightLabel{$\bot_E$}
\UnaryInfC{$\Gamma, A \vdash B$}
\RightLabel{$\Rightarrow_I$}
\UnaryInfC{$\Gamma \vdash A \Rightarrow B$}
\RightLabel{$\Rightarrow_E$}
\BinaryInfC{$\Gamma \vdash A$}
\RightLabel{$\neg_E$}
\BinaryInfC{$\Gamma := (A \Rightarrow B) \Rightarrow A, \neg A \vdash \bot$}
\RightLabel{$\neg_i$}
\UnaryInfC{$(A \Rightarrow B) \Rightarrow A \vdash \neg\neg A$}
\RightLabel{$\Rightarrow_E$}
\BinaryInfC{$(A \Rightarrow B) \Rightarrow A \vdash A$}
\RightLabel{$\Rightarrow_I$}
\UnaryInfC{$\vdash ((A \Rightarrow B) \Rightarrow A) \Rightarrow A$}
\end{prooftree}
$$

Finally, we prove that PL implies DNE:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{$\neg_I$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, A \vdash A \Rightarrow \bot$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, A \vdash A$}
\RightLabel{$\Rightarrow_E$}
\BinaryInfC{$\Gamma, A \vdash \bot$}
\RightLabel{$\neg_I$}
\UnaryInfC{$\Gamma \vdash \neg A$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash \neg\neg A$}
\RightLabel{$\neg_e$}
\BinaryInfC{$\Gamma := A \Rightarrow \bot, \neg\neg A \vdash \bot$}
\RightLabel{$\bot_E$}
\UnaryInfC{$A \Rightarrow \bot, \neg\neg A \vdash A$}
\RightLabel{$\Rightarrow_I$}
\UnaryInfC{$\neg\neg A \vdash (A \Rightarrow \bot) \Rightarrow A$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{$\neg\neg A \vdash \overbrace{((A \Rightarrow \bot) \Rightarrow A) \Rightarrow A}^{PL}$}
\AxiomC{$\neg\neg A \vdash (A \Rightarrow \bot) \Rightarrow A$}
\RightLabel{$\Rightarrow_E$}
\BinaryInfC{$\neg\neg A \vdash A$}
\RightLabel{$\Rightarrow_I$}
\UnaryInfC{$\vdash \neg\neg A \Rightarrow A$}
\end{prooftree}
$$

Thus all the rules are equivalent.
However, neither of these rules are provable, in general, for NJ (as $\vdash A \vee B$ implies either $\vdash A$ or $\vdash B$ and we apply it to EM).

$$
\begin{tikzcd}
 & PL \\
DNE \arrow[ur, Leftrightarrow] \arrow[r, Rightarrow] & EM \arrow[dl, Rightarrow] \\
CP \arrow[u, Rightarrow] &
\end{tikzcd}
$$

Exercise 2.

We have:

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$P \vdash P, Q$}
\UnaryInfC{$P \vdash Q, P$}
\UnaryInfC{$\vdash P \Rightarrow Q, P$}
\UnaryInfC{$\vdash P, P \Rightarrow Q$}
\UnaryInfC{$\vdash P \vee (P \Rightarrow Q)$}
\end{prooftree}
$$

and:

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$\neg A \Rightarrow A \vdash \neg A \Rightarrow A, A$}
\AxiomC{}
\UnaryInfC{$\neg A \Rightarrow A, A \vdash A, \bot$}
\UnaryInfC{$\neg A \Rightarrow A, A \vdash \bot, A$}
\UnaryInfC{$\neg A \Rightarrow A \vdash \neg A, A$}
\BinaryInfC{$\neg A \Rightarrow A \vdash A, A$}
\UnaryInfC{$\neg A \Rightarrow A \vdash A$}
\UnaryInfC{$\vdash (\neg A \Rightarrow A) \Rightarrow A$}
\end{prooftree}
$$

$$
\dfrac{
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg(P \Rightarrow Q), P \vdash P, \bot}}{\neg(P \Rightarrow Q), P \vdash \bot, P}}{\neg(P \Rightarrow Q) \vdash \neg P, P}}{\neg(P \Rightarrow Q) \vdash P, \neg P}}{\neg(P \Rightarrow Q) \vdash P \vee \neg P}
\quad
\dfrac{}{\neg(P \Rightarrow Q), P \vdash P}
\quad
\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \neg(P \Rightarrow Q)} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, P \vdash \neg P} \quad \dfrac{}{\Gamma, P \vdash P}}{\Gamma, P \vdash \bot}}{\Gamma, P \vdash Q}}{\Gamma \vdash P \Rightarrow Q}}{\Gamma := \neg(P \Rightarrow Q), \neg P \vdash \bot}}{\neg(P \Rightarrow Q), \neg P \vdash P}
}{\neg(P \Rightarrow Q) \vdash P}
$$

$$
\dfrac{\dfrac{\dfrac{}{\neg(P \Rightarrow Q), Q \vdash \neg(P \Rightarrow Q)} \quad \dfrac{\dfrac{}{\neg(P \Rightarrow Q), Q, P \vdash Q}}{\neg(P \Rightarrow Q), Q \vdash P \Rightarrow Q}}{\neg(P \Rightarrow Q), Q \vdash \bot}}{\neg(P \Rightarrow Q) \vdash \neg Q}
$$

$$
\dfrac{\dfrac{\neg(P \Rightarrow Q) \vdash P \qquad \neg(P \Rightarrow Q) \vdash \neg Q}{\neg(P \Rightarrow Q) \vdash P \wedge \neg Q}}{\vdash \neg(P \Rightarrow Q) \Rightarrow P \wedge \neg Q}
$$

proof of the excluded middle in NC.

and finally we have by ex 1 that $EM \Rightarrow CP$ and we know $NJ \subseteq NC$ thus we simply have to prove EM:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P \vdash P, \bot}}{P \vdash \bot, P}}{\vdash \neg P, P}}{\vdash P, \neg P}}{\vdash P \vee \neg P}
$$

Exercise 3.

Q1. If we have shown that every rule of NC, after applying $(.)^{Ko}$ is admissible, then a simple induction on the proof of $\Gamma \vdash_{NC} A, \Delta$ shows that $\Gamma^{Ko}, \neg \Delta^{Ko} \vdash_{NJ} A^{Ko}$.

For example, if we are in the situation

$$
\dfrac{\begin{matrix} \pi \\ \Gamma \vdash_{NC} A, \Delta \end{matrix} \qquad \begin{matrix} \pi' \\ \Gamma \vdash_{NC} B, \Delta \end{matrix}}{\Gamma \vdash_{NC} A \wedge B, \Delta} \wedge_i
$$

Then by induction hypothesis, we have that

$$\Gamma^{K_0}, \neg\Delta^{K_0} \vdash_{NJ} A^{K_0} \quad \text{and} \quad \Gamma^{K_0}, \neg\Delta^{K_0} \vdash_{NJ} B^{K_0}$$

and by admissibility of $(\wedge_I)^{K_0}$, we have

$$\dfrac{\Gamma^{K_0}, \neg\Delta^{K_0} \vdash_{NJ} A^{K_0} \qquad \Gamma^{K_0}, \neg\Delta^{K_0} \vdash_{NJ} B^{K_0}}{\Gamma^{K_0}, \neg\Delta^{K_0} \vdash_{NJ} (A \wedge B)^{K_0}} (\wedge_I)^{K_0}.$$

Q2. For conjunction introduction, we have:

$$\dfrac{\dfrac{}{\Gamma^{K_0}, \neg\Delta^{K_0}, \neg(A^{K_0} \wedge B^{K_0}) \vdash \neg(A^{K_0} \wedge B^{K_0})} ax \qquad \dfrac{\dfrac{\Gamma^{K_0}, \neg\Delta^{K_0} \vdash A^{K_0} \quad \Gamma^{K_0}, \neg\Delta^{K_0} \vdash B^{K_0}}{\Gamma^{K_0}, \neg\Delta^{K_0} \vdash A^{K_0} \wedge B^{K_0}} \wedge i}{\Gamma^{K_0}, \neg\Delta^{K_0}, \neg(A^{K_0} \wedge B^{K_0}) \vdash A^{K_0} \wedge B^{K_0}} wk}{\dfrac{\Gamma^{K_0}, \neg\Delta^{K_0}, \neg(A^{K_0} \wedge B^{K_0}) \vdash \bot}{\Gamma^{K_0}, \neg\Delta^{K_0} \vdash \neg\neg(A^{K_0} \wedge B^{K_0})} \neg i} \neg e$$

for $\wedge E1$, $\wedge E2$, $\top I$, $\bot E$ we use the similar trick to remove the $\neg\neg$ in the conclusion with $\neg i$ followed by $\neg e$, and we conclude by applying the NJ rule after a weakening.

For exchange, we have: $B^{K_0} = \neg\neg\varphi$

$$
\dfrac{
\dfrac{\dfrac{\Gamma^{K_0}, \neg\Delta^{K_0}, \neg B^{K_0}, \neg\Sigma^{K_0} \vdash A^{K_0}}{\Psi, \neg B^{K_0} \vdash A^{K_0}}\,\text{wk \& xch}}{\Psi \vdash \neg\neg\neg\varphi \Rightarrow A^{K_0}}\Rightarrow_I
\qquad
\dfrac{\dfrac{\dfrac{}{\Psi, \neg\neg\varphi \vdash \neg\neg\varphi}\text{ax} \quad \dfrac{}{\Psi, \neg\neg\varphi \vdash \neg\varphi}\text{ax}}{\Psi, \neg\neg\varphi}\neg e}{\Psi \vdash \neg\neg\neg\varphi}\neg i
}{\Psi \vdash A^{K_0}}
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\Psi \vdash \neg A^{K_0}}\text{ax} \qquad \Psi \vdash A^{K_0}}{\Psi := \Gamma^{K_0}, \neg(\Delta \cup \Sigma)^{K_0}, \neg A^{K_0}, \neg\varphi \vdash \bot}\neg e}{\Gamma^{K_0}, \neg(\Delta \cup \Sigma)^{K_0}, \neg A^{K_0} \vdash B^{K_0}}\neg i}{\Gamma^{K_0}, \neg\Delta^{K_0}, \neg A^{K_0}, \neg\Sigma^{K_0} \vdash B^{K_0}}
$$

For contraction:

$$\dfrac{\Gamma \vdash \Delta', A, A, \Sigma}{\Gamma \vdash \Delta', A, \Sigma}$$

we have if $\Delta' = B, \Delta$: $\dfrac{\Gamma^{K_0}, \neg\Delta^{K_0}, \neg A^{K_0}, \neg A^{K_0}, \neg\Sigma^{K_0} \vdash B^{K_0}}{\Gamma^{K_0}, \neg\Delta^{K_0}, \neg A^{K_0}, \neg\Sigma^{K_0} \vdash B^{K_0}}$

and if $\Delta' = \cdot$, we rewrite A^{K_0} as $\neg\neg\varphi$, adding $\neg\varphi$ into the hypothesis, which is equivalent to $\neg\neg\varphi \Rightarrow \neg A^{K_0}$, thus we conclude by structural rules.

Exercise 4.

First, by induction on A, we show

$$A^{K_0} \vdash A \quad \text{and} \quad A \vdash A^{K_0}.$$

If $A = X$ then

$$\dfrac{\dfrac{\text{see ex 2}}{\neg\neg X \vdash X \vee \neg X} \qquad \dfrac{}{\neg\neg X, X \vdash X}\,\text{ax} \qquad \dfrac{\dfrac{\dfrac{}{\cdots \vdash \neg X}\,\text{ax} \quad \dfrac{}{\cdots \vdash \neg\neg X}\,\text{ax}}{\neg\neg X, \neg X \vdash \bot}}{\neg\neg X, \neg X \vdash X}\,\bot_E}{\neg\neg X \vdash X}\,\vee_e \qquad (*)$$

and

$$\dfrac{\dfrac{\dfrac{}{\cdots \vdash X}\,\text{ax} \quad \dfrac{}{\cdots \vdash \neg X}\,\text{ax}}{X, \neg X \vdash \bot}\,\neg e}{X \vdash \neg\neg X}\,\neg i$$

If $A = B \wedge C$ then, by ih, $\neg\neg B^{k_0} \vdash B$ and $C^{k_0} \vdash C$,
So $\vdash \neg\neg B^{k_0} \Rightarrow B$ by reversibility of $\Rightarrow_I$ (and same for C).

$$\dfrac{\dfrac{}{\cdots \vdash \neg B^{k_0}}\,\text{ax} \quad \dfrac{\dfrac{}{\cdots \vdash B^{k_0} \wedge C^{k_0}}\,\text{ax}}{\cdots \vdash B^{k_0}}\,\wedge_i e}{\gamma, \neg(B \wedge C), \neg B^{k_0}, B^{k_0} \wedge C^{k_0} \vdash \bot}\,\neg e$$

$$\dfrac{\dfrac{\gamma, \neg(B \wedge C), \neg B^{k_0}, B^{k_0} \wedge C^{k_0} \vdash \bot}{\gamma, \neg(B \wedge C), \neg B^{k_0} \vdash \neg\gamma'}\,\neg i \qquad \dfrac{}{\cdots \vdash \neg\neg\gamma' = \gamma}\,\text{ax}}{\gamma, \neg(B \wedge C), \neg B^{k_0} \vdash \bot}$$

$$\dfrac{\dfrac{\vdash \neg\neg B^{k_0} \Rightarrow B}{\gamma, \neg(B \wedge C) \vdash \neg\neg B^{k_0} \Rightarrow B} \qquad \dfrac{\gamma, \neg(B \wedge C), \neg B^{k_0} \vdash \bot}{\gamma, \neg(B \wedge C) \vdash \neg\neg B^{k_0}}}{\gamma, \neg(B \wedge C) \vdash B}\,\Rightarrow_E \qquad \text{similar}$$

$$\dfrac{\dfrac{\text{ex 2}}{\gamma \vdash (B \wedge C) \vee \neg(B \wedge C)} \qquad \dfrac{}{\gamma, B \wedge C \vdash B \wedge C}\,\text{ax} \qquad \dfrac{\gamma, \neg(B \wedge C) \vdash B \quad \text{similar}}{\gamma, \neg(B \wedge C) \vdash B \wedge C}\,\wedge_i}{\underbrace{\neg\neg(B^{k_0} \wedge C^{k_0})}_{\gamma = \neg\neg\gamma'} \vdash B \wedge C}$$

The use of EM turned out to be useless in this case.

by ih & reversibility of $\underbrace{\Rightarrow I}$

$$
\dfrac{
  \dfrac{
    \dfrac{\vdash B^{k_0} \Rightarrow B}{\Gamma \vdash B^{k_0} \Rightarrow B}
    \qquad
    \dfrac{\dfrac{}{\Gamma \vdash B \wedge C}\,ax}{\Gamma \vdash B}\,\wedge el
  }{\Gamma \vdash B^{k_0}}
  \qquad
  \dfrac{\text{similarly}}{\Gamma \vdash C^{k_0}}
}{
  \dfrac{\Gamma \vdash B^{k_0} \wedge C^{k_0}}{
    \dfrac{\Gamma := B \wedge C,\ \neg(B^{k_0} \wedge C^{k_0}) \vdash \bot}{B \wedge C \vdash \neg\neg(B^{k_0} \wedge C^{k_0})}\,\neg i
  }\,\neg e
}\,\wedge i
$$

All the other cases are similar.