TD n° 1 Modelization & linear Programming Hugo SAKOU

Exercise 1. Lumberjack.

Q1. $\begin{cases} w + p \leq 100 \\ 10w + 50p \leq 4000 \\ w, p \geq 0 \end{cases}$

maximize $50w + 120p =: y$

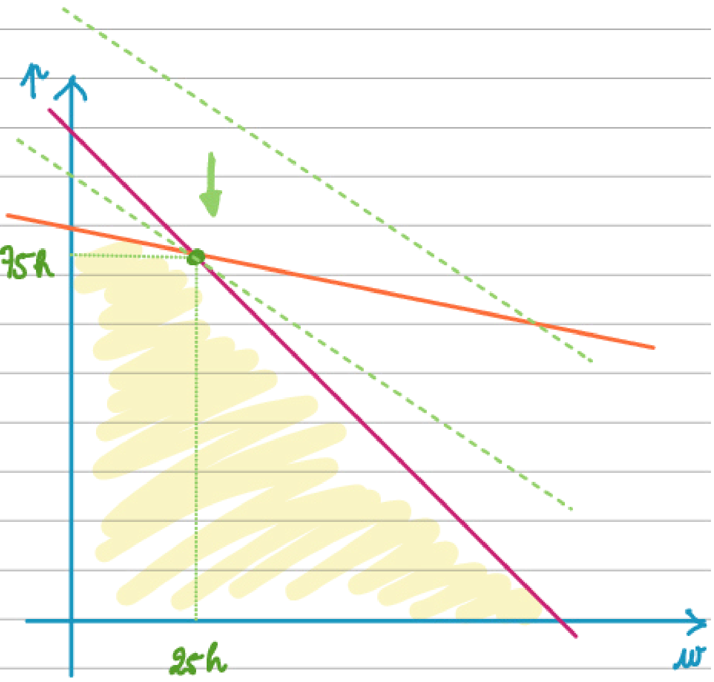
$\Updownarrow$

find a line

$$p = \frac{y}{120} - \frac{50}{120}a$$

with the highest y-intercept.

Q2.



Q3. The best strategy is 25h with the wood cut and 75h re-seeded, for a profit of 10 250 k\$.

Exercise 2. Student diet problem

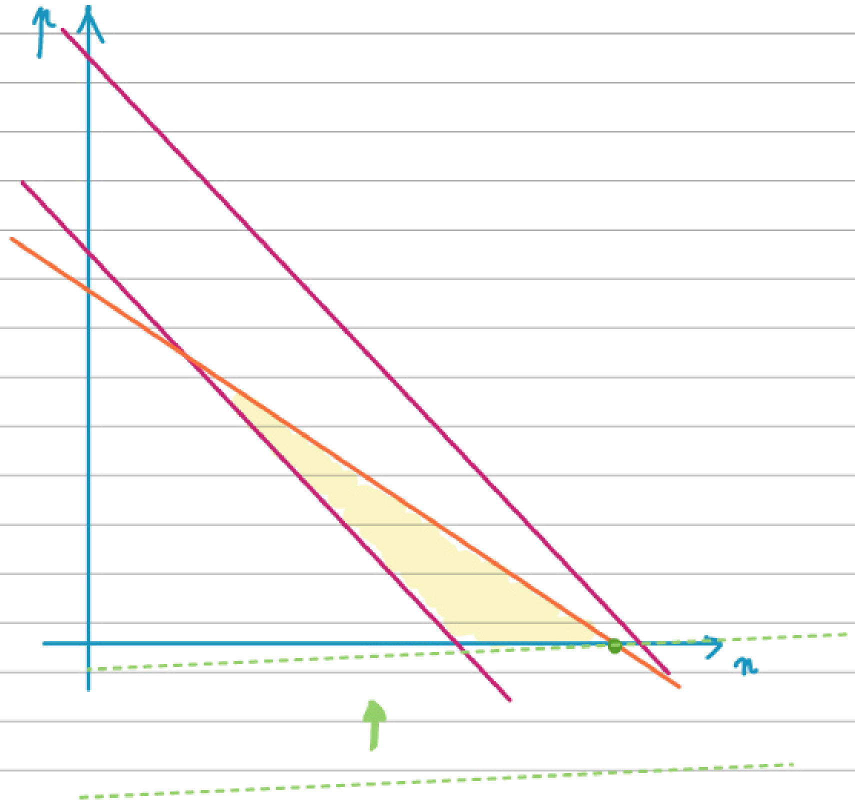
Q1.
$$\begin{cases} 460n + 700p + 400r \leq 3500 \\ 460n + 700p + 400r \geq 2500 \quad (1) \\ 21n + 20p \leq 120 \\ 21n + 20p \geq 80 \quad (2) \\ n, p, r \geq 0 \end{cases}$$

minimize $0{,}5n + 3{,}5p + 0{,}5r$

Q2. Instead of drawing in 3D, as we are minimizing a strictly increasing function, we consider two cases: if (1) is an equality or if (2) is an equality.

Case 1: $460n + 700p + 400r = 2500$

$\iff r = 6{,}25 - 1{,}75p - 1{,}15n$



Dans ce cas, l'optimal est $n = 125/23$, $p = 0$ et $r = 0$ (*)

avec un prix de 2,92 € environ.

Dans l'autre cas, on trouve $n = \frac{80}{21}$, $p = 0$ et $r = 25/4$, avec un prix de 5,03€ environ. (**).

On conclut que (*) est l'optimal.

Exercice 3. Bank allocation.

c : crédit client, v : crédit voiture p : prêt.

$v + p \geqslant 60\% \times (v+p+c) \Rightarrow$

$p \geqslant 40\% \; (v+p+6) \Rightarrow$

$6\% c + 4\% v + 2\% p \leq (v+p+c) \, 3{,}2\% \Rightarrow$

$$\begin{cases} v + c + p \leq 10^6 \\ 0{,}3 v + 0{,}3 p - 0{,}6 c \geqslant 0 \\ 0{,}6 p - 0{,}4 v - 0{,}4 p \geqslant 0 \\ 0{,}028 c + 0{,}0008 v - 0{,}0012 p \leq 0 \\ v, p, c \leq 0 \end{cases}$$

Maximize $0{,}06 c + 0{,}04 v + 0{,}02 p$

We can compute the solution by case by case analysis.

Exercise 6. Independant Set Problem

Variables: x_v for every vertex $v \in V$

Constraints: for any edge $uv \in E$, $x_u + x_v \leq 1$

for any vertex $v \in V$, $1 \geq x_v \geq 0$

Maximize $\sum_{v \in V} x_v$ = size of the independant set

Exercise 7. Dominating Set Problem

Variables x_u $u \in V$

Constraints $\forall u \in V$, $0 \leq x_u \leq 1$

$\forall u \in V$, $\sum_{v \in N[u]} x_v \geq 1$

↳ closed neighborhood

Minimize $\sum_{u \in V} x_u$

Exercise 8. N-queens problem

Variables: $x_{i,j}$, $i,j \in [\![1, N]\!]$

Constraints: $\forall i$, $\sum x_{i,j} \leq 1$

$\sum x_{j,i} \leq 1$

$\sum x_{i,i-j} \leq 1$

$\sum x_{i,i+j} \leq 1$

$\forall i,j$ $0 \leq x_{ij} \leq 1$

Maximize $\sum_{ij} x_{ij}$

TD n° 2 Linear Programming and the simplex method

Exercise 1. Computer production

Q1.

$$\begin{cases} n + d \leq 10\,000 \\ n + 2d \leq 15\,000 \\ 4n + 3d \leq 38\,000 \\ n, d \geq 0 \end{cases}$$

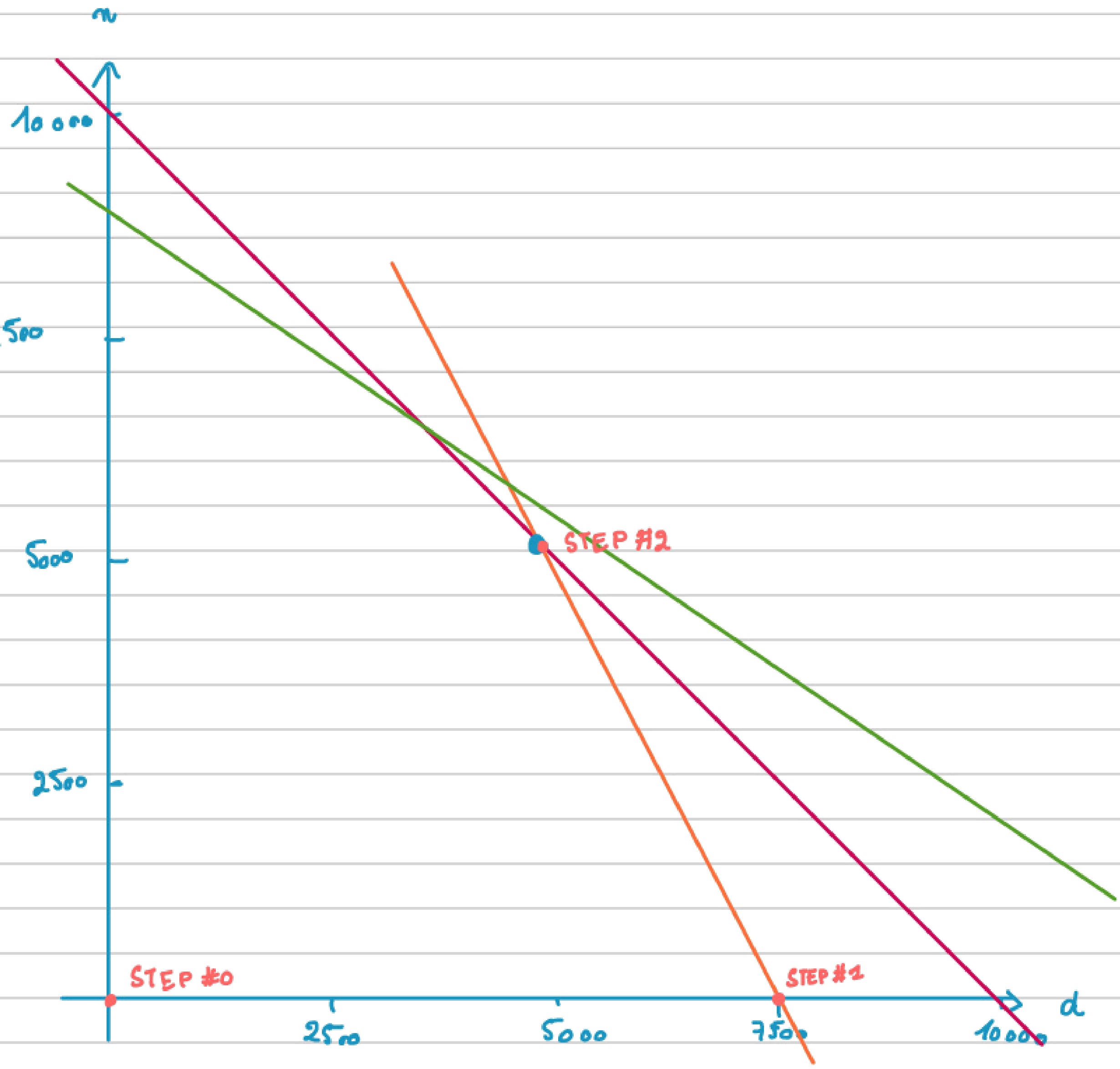
$n = 9500 - \frac{3}{4}d$

$\frac{3}{4}d = 9500$

max $750n + 1000d$

Q2.

L'optimal est $n = 5000$ et $d = 5000$.



$$\begin{cases} u = 10 - n - d \\ v = 15 - n - 2d \\ w = 38 - 4n - 3d \\ z = 75n + 100d \end{cases}$$

$\boxed{\text{STEP \#1}}$ Choice for pivot : d enters and v leaves.

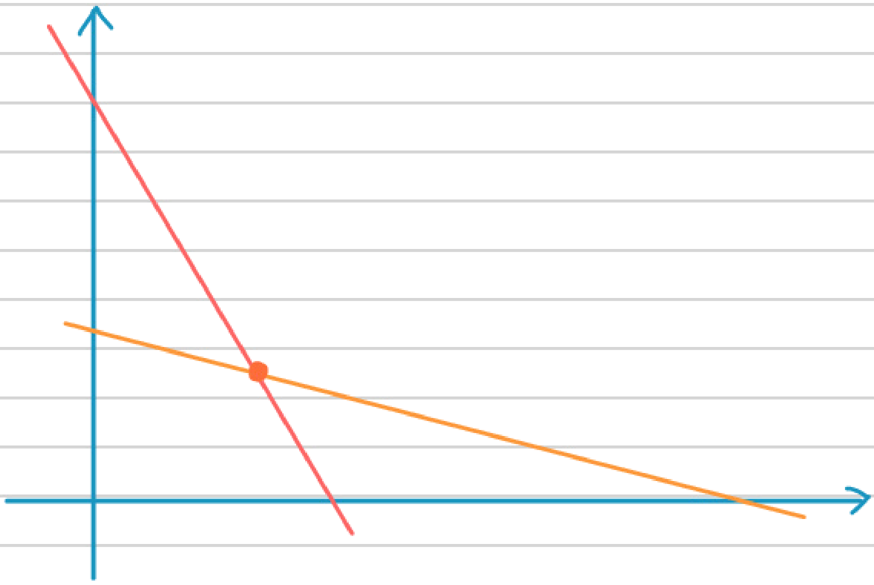
We have $d = 7.5 - \frac{n}{2} - \frac{v}{2}$ thus, we substitute

$$\begin{cases} u = 2{,}5 - \frac{n}{2} + \frac{v}{2} \\ d = 7{,}5 - \frac{n}{2} - \frac{v}{2} \\ w = 15{,}5 - 2{,}5n - 2{,}5v \\ z = 750 + 25n - 50v \end{cases}$$

$\boxed{\text{STEP \#2}}$ Choice for pivot : n enters and u leaves

Solution: $5000 = n = d$ for a reward of 87500

Exercise 2. Multiple optimal solutions



TD n° 4

Dual \ Primal	Empty domain	Unbounded	Optimal solution
Empty domain	Yes	Yes	No
Unbounded	Yes	No	No
Optimal solution	No	No	Yes

Exercice 1. Le cas non faisable / non faisable.

Considérons $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \leq \begin{pmatrix} -1 \\ -1 \end{pmatrix}$, $\max x+y$

Son dual est $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \geq \begin{pmatrix} -1 \\ -1 \end{pmatrix}$, $\min x+y$

Exercice 2.

Q1. $(P) \begin{cases} \max \sum c_{ij} x_{ij} \\ (a_i) : \sum_{j=1}^{4} x_{ij} \leq 1 \\ (t_j) : \sum_{i=1}^{4} x_{ij} \leq 1 \end{cases}$

Solution optimale : $x_{13} = x_{24} = x_{32} = x_{41} = 1$.

pour une affectation agent compétence de 20

Q2. We compute $4 \times (a_1) + 4 \times (a_2) + 5 \times (a_3) + 3 \times (a_4) + (t_1) + (t_2) + 2 \cdot (t_4)$

and we obtain that

$$\sum \begin{array}{|c|c|c|c|c} \hline 25 & 15 & 16 & 30 & 4 \\ \hline 20 & 25 & 16 & 36 & 4 \\ \hline 30 & 36 & 25 & 49 & 5 \\ \hline 16 & 16 & 9 & 25 & 3 \\ \hline 1 & 1 & 0 & 2 & \end{array} \leq 20$$

Exercise 3.

Q1. A vertex v is a point of P that ~~is~~ cannot be written as $\lambda v_1 + (1-\lambda) v_2 = v$ for some $\lambda \in [0,1]$, $x_1, x_2 \in P$. Such values (λ, x_1, x_2) provide an easy certificate.

Q2. Define $\varepsilon := \min\left(\{\min(x_i, \frac{1}{2} - x_i) \mid i \in I^-\} \cup \{\min(1-x_i, x_i - \frac{1}{2}) \mid i \in I^+\}\right)$

Then it is easy to see that $x + \varepsilon y$ is a point of P.

Q3. We have that $v^+ := x + \varepsilon y \in P$ and $v^- := x - \varepsilon y \in P$

but $\frac{1}{2} v^+ + \frac{1}{2} v^- = v$ thus it is only possible if ε is undefined.

Q4. We solve the LP problem :

$$\begin{array}{|l} \min \sum_{i \in V} c(i) x_i \\ \text{such that } \forall ij \in E,\ x_i + x_j \leq 1. \\ \qquad \forall i \in V,\ x_i \geq 0 \end{array}$$

Take the solution and put $\frac{1}{2} \to 1$.

Exercise 4.

Q1. (P)

$$\begin{array}{|l} \max 0 \\ \text{such that} \\ \forall v \in V,\ w_v \geq 0 \\ \forall v \in V,\ \sum_{uv \in E} w_u - \sum_{vu \in E} w_u \geq 0 \\ \sum_{v \in V} w_v = 1. \end{array} \quad \left. \begin{array}{l} Ax \leq 0 \\ x \geq 0 \\ \sum x = 1 \end{array} \right)$$

Q2. (D)

$$\begin{array}{|l} \min \eta \\ \text{such that} \\ \quad y \geq 0 \\ \quad -Ay + \eta \mathbb{1} \geq 0 \end{array}$$

Q3 0 is a solution of (P)

thus (P) is non-empty.

Exercise 5.

Q1. Consider the instance $T_1 = \{x\}$, $T_2 = \{y\}$ and $T_3 = \{x, y\}$.

Q2.

$$(P) \quad \begin{cases} \max\ 0 \\ \forall j \in [\![1, m]\!], \ \sum_{i \in T_j} x_i = 1 \\ x_1, \dots, x_n \geq 0 \end{cases} \qquad (D) \quad \begin{cases} \min\ y \\ A^T y \geq 0. \\ y \geq 0 \end{cases}$$

$$A = \left(\mathbb{1}_{i \in T_j} \right)$$

TD n° 6

Exercice 1.

e) The set of vertices $V_{OPT_{1/2}} \cup V_1$ is a vertex cover of G thus

$$OPT \leq OPT_{1/2} + |V_1|.$$

Indeed, $V_{OPT_{1/2}}$ tells us the vertices that can be turned into an optimal solution.

f) Assume $|V_0 \cap V_{OPT}| < |V_1 \setminus V_{OPT}|$. Let $\varepsilon > 0$. We add ε to every vertex in $V_0 \cap V_{OPT}$ and subtract ε to every vertex in $V_1 \setminus V_{OPT}$.

Let us show that we are still in the domain: let $uv \in E$,

- if $u \in V_0 \cap V_{OPT}$ and $v \in V_1 \setminus V_{OPT}$, then we have $x_u + \varepsilon + x_v - \varepsilon = x_u + x_v \geq 1$
- if $u, v \in V_0$ then impossible
- if $u, v \in V_{1/2}$ then nothing changed
- if $u \in V_0$ and $v \in V_{1/2}$, then impossible
- etc.

The solution is of value strictly smaller, absurd!

g) We exchange vertices in $V_{OPT} \cap V_0$ to $V_1 \setminus V_{OPT}$. This remains a vertex cover as we exchanged a 0 to a 1 and vice-versa.

This gives us that $V_1 \setminus V_{OPT} = \emptyset$ and thus, $OPT \geq OPT_{1/2} + |V_1|$ as we can remove the

Exercice 3 - Gomory Cut

$$\max\ 4x_1 + 3x_2 \quad \text{such that} \quad \begin{cases} 2x_1 + x_2 \leq 11 & x_3 \\ -x_1 + 2x_2 \leq 6 & x_4 \\ x_1, x_2 \in \mathbb{N} \end{cases}$$

a) (*) $x_1 + \frac{2}{5}x_3 - \frac{1}{5}x_4 = \frac{16}{5}$ thus $x_1 - x_4 \leq 16/5$
thus $x_1 - x_4 \leq 3$ (**)

argument of authority: $x_4 \in \mathbb{N}$ and thus $\frac{2}{3}x_3 + \frac{4}{5}x_4 \geq 1/5$ by (*)-(**).

b) $(*)\ x_2 + \frac{1}{5} x_3 + \frac{2}{5} x_4 = \frac{23}{5}$ thus $x_2 \leq 4$

thus $\frac{1}{5} x_3 + \frac{2}{5} \geq \frac{3}{5}$

c) do magic and it works (but it's not better than normal)

(a.k.a use the simplex)

Exercice 4.

a) we have that $nt \geq \sum \text{entries} \geq m(t+1)$ thus $n > m$.

b) the kernel of A is non-trivial

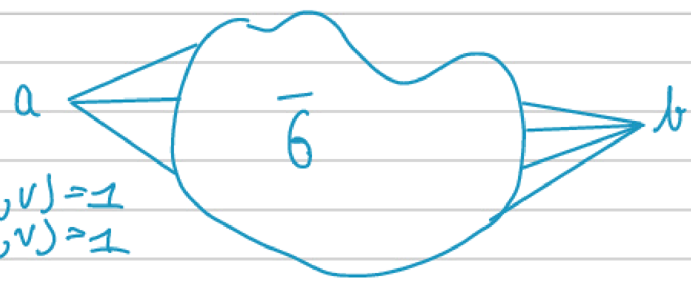
c) ...

Exercice 5. Jain's iterative rounding algorithm.

a) Consider $\bar{G} = (\bar{V}, \bar{E})$ a graph and define G to be the graph

$r(a,b) = 2$

$\forall v \notin \{a,b\} : r(a,v) = 1$
$r(b,v) = 1$

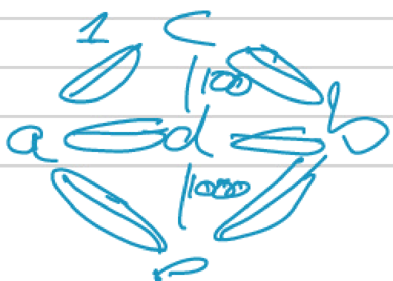


and take

$r(a,b) := 2,\quad r(u,v) := 0 \quad \forall u,v \in V^2 \setminus \{(a,b)\}.$

with weights $c_e := l(e)$ with $e \in \bar{E}$

and $c_{au}, c_{ub} := 0$.



Part II

TD n° 1 Introduction & preliminaries

Exercise 1.

We have $\|\nabla f(x) + d\|^2 < \|\nabla f(x)\|$

$$\|\nabla f(x)\|^2 + 2\nabla f(x)^T d + \|d\|^2$$

Thus $2\nabla f(x)^T d + \|d\|^2 < 0$

and finally $\nabla f(x)^T d < 0$.

Exercise 2.

a) $\nabla f(x,y) = \left(\frac{\partial f}{\partial x}(x,y) \quad \frac{\partial f}{\partial y}(x,y)\right)^T = \begin{pmatrix} 2(x+y)^2 \\ 4y(x+y^2) \end{pmatrix}$

and we have $\nabla f(z_0)^T p_0 = (2 \ 0)\begin{pmatrix} -1 \\ -1 \end{pmatrix} = -2 < 0$

b) We want to find all x, y such that $\nabla f(x,y) = 0$.

All such points are $(-y^2, y) \ \forall y$.

They're all global minima as $f(x,y) \geq 0$.

c) No: $f(-y^2, y) = f(-y^2, -y) = 0$ but $f(-y^2, 0) = y^4 \neq 0$ for $y \neq 0$

↳ halfway point of the two minima

Exercise 4.

Take $\vec{x}, \vec{y} \in H_n = \{\vec{z} \mid \prod z_i \geq 1\} = \{\vec{z} \mid \sum \log z_i \geq 0\}$.

Exercise 5.

a) $\{x \mid \alpha \leq a^T x \leq \beta\} = \{x \mid \alpha \leq a^T x\} \cap \{x \mid a^T x \leq \beta\}$

both of which are convex, thus so is the slab.

$a^T(t x_1 + (1-t)x_2) = t \underbrace{a^T x_1}_{\in [\alpha,\beta]} + (1-t) \underbrace{a^T x_2}_{\in [\alpha,\beta]} \in [\alpha, \beta].$

b) $\{x \mid \forall i \quad \alpha_i \leq x_i \leq \beta_i\}$

$= \bigcap_i \left(\mathbb{R}^{i-1} \times \underbrace{\{x_i \mid \alpha_i \leq x_i \leq \beta_i\}}_{[\alpha_i, \beta_i]} \times \mathbb{R}^{n-i-1} \right)$

c) $\{x \mid a_1^T x \leq b_1 \text{ and } a_2^T x \leq b\} = \{x \mid a_1^T x \leq b_1\} \cap \{x \mid a_2^T x \leq b_2\}$

d) $\{x \mid \|x - x_0\|_2 \leq \|x-y\|_2 \ \forall y \in S\} = \{x \mid \|x-x_0\|_2^2 \leq \|x-y\|_2^2 \ \forall y \in S\}$

$= \{x \mid \cancel{\|x\|_2^2} - 2x^T x_0 + \|x_0\|_2^2 \leq \cancel{\|x\|_2^2} - 2x^T y + \|y\|_2^2 \ \forall y \in S\}$

$= \bigcap_{y \in S} \underbrace{\{x \mid \|x_0\|_2^2 - \|y\|_2^2 \leq 2x^T(y - x_0)\}}_{\text{convex}}$

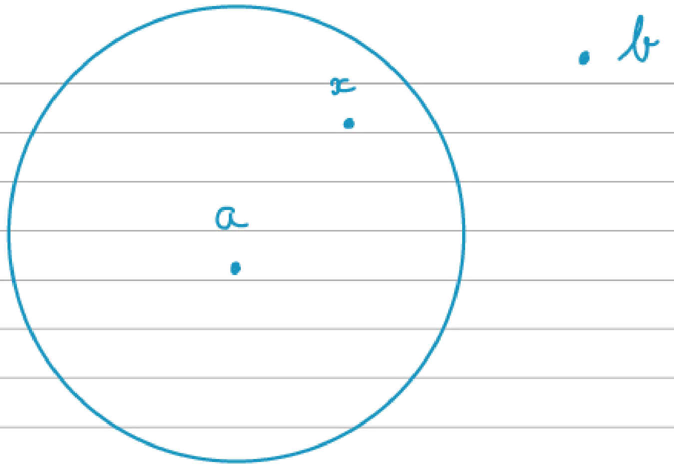
e) Take $S = \{-1, 1\}$, and $T = \{0\}$,

the set is then $(-\infty, -\frac{1}{2}] \cup [\frac{1}{2}, +\infty)$ not convex.

f) $\{x \mid x + S_1 \subseteq S_2\} = \bigcap_{s \in S_2} \{x \mid x + s \in S_1\} = \bigcap_{s \in S_2} (S_1 - s_2).$

g) This is a filled sphere of center a and of radius $\|b-a\|_2 \theta / 2$.

ball



Thus it is convex.

Exercise 6.

1) $H_f(x,y) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is positive and negative thus f is convex and concave but not strictly

2) $H_f(x,y) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ is positive and negative thus f is convex and concave but not strictly

3) $H_f(x,y) = \begin{pmatrix} e^{x} + e^{x+y} & e^{x+y} \\ e^{x+y} & e^{x+y} \end{pmatrix} \xrightarrow{C_1 \leftarrow C_1 - C_2} \begin{pmatrix} e^{x} & * \\ 0 & e^{x+y} \end{pmatrix}$ which is strictly positive thus f is (strictly) convex.

Exercise 7.

a) $f(t\vec{x} + (1-t)\vec{y}) \geq f(t\vec{x}) + f((1-t)\vec{y})$

$\geq t f(\vec{x}) + (1-t) f(\vec{y})$

b) $\max_i (t\vec{x} + (1-t)\vec{y})_i \geq \max_i t x_i + \max_i (1-t) y_i$

c) $H_f(x,y) = \frac{2}{y^3} \underbrace{\begin{pmatrix} y^2 & -xy \\ -xy & x^3 \end{pmatrix}}_{M} = \frac{2}{y^3} \begin{pmatrix} y \\ -x \end{pmatrix} \begin{pmatrix} y \\ -x \end{pmatrix}^T$ thus zMz^T is a norm.

thus is definite positive.

Exercise 8

Trivial

Exercise 9

a) Let x, y two global minimizers of f.

$$f(tx + (1-t)y) \leq t\, f(x) + (1-t)\, f(y) \leq f(x)$$

b) By convexity,

$$f(y) \geq f(x) + \nabla f(x)^T (x-y)$$

Thus, $\nabla f(x)^T (x-y) \leq f(y) - f(x) < 0.$

Exercise 10.

(i) $\Rightarrow$ (ii). Suppose f convex. We have:

$$f(tx + (1-t)y) = f(x + t(y-x)) \leq t f(x) + (1-t) f(y)$$

Thus, $f(y) - f(x) \geq \dfrac{f(x + t(y-x)) - f(x)}{t} \xrightarrow[t \to 0]{} \nabla f(x)^T (y-x)$

(ii) $\Rightarrow$ (i) Let $z := tx + (1-t)y.$

We have

$$f(x) \geq f(z) + \nabla f(z)^T (x-z) \text{ and } f(y) \geq f(z) + \nabla f(z)^T (y-z)$$

thus,

$$t\, f(x) + (1-t) f(y) \geq f(z) + \cancel{\nabla f(z)^T (tx + (1-t)y - z)}$$

$$\geq f(z)$$

TD n° 2

Gradient & Newton's method

Exercise 1.

c.f. section 2.3 of the notes

$p_k = -\nabla q(x_k) = -(A^T x_k - b) = -g_k$ is the steepest descent direction.

With the line-search method, $x_{k+1} = x_k + \alpha_k p_k = x_k - \alpha_k g_k$

$$\phi(\alpha) = q(x - \alpha g_k) = q(x + \alpha p_k)$$

$$\phi(\alpha) = \frac{1}{2}(x_k - \alpha g_k)^T A (x_k - \alpha g_k) - b^T (x_k - \alpha g_k)$$

$$= \frac{1}{2} x_k^T A x_k - \alpha (g_k^T A x_k + x_k^T A g_k) \times \frac{1}{2} + \alpha^2 g_k A g_k$$

$$- b^T x_k + \alpha b^T g_k$$

$$= \underbrace{\left(\frac{1}{2} A x_k - b\right)^T}_{g_k} x_k - \alpha \underbrace{\left(A x_k - b\right)^T}_{g_k} g_k + \alpha^2 g_k A g_k$$

$$- \alpha \, x_k^T A g_k \times \frac{1}{2}$$

$$+ \dots$$

$$\min_{\alpha > 0} \phi(\alpha) = \frac{g_k^T g_k}{g_k^T A g_k}$$

Exercise 2.

$$\|x_k - x^*\|_A^2 - \|x_{k+1} - x^*\|_A^2 = \|x_k - x^*\|_A^2 - \|x_k - \alpha_k g_k - x^*\|_A^2$$

$g_k = A(x_k - x^*)$

$$= 2\alpha_k g_k^T \underbrace{A(x_k - x^*)}_{g_k} - \alpha_k^2 \|g_k\|_A^2 \qquad (1)$$

$$g_k^T A^{-1} g_k = (x_k - x^*)^T A^{-1} A A^{-1} (x_k - x^*) = \|x_k - x^*\|_A^2 \qquad (2)$$

In (1), we have

$$\|x_k - x^*\|_A^2 - \|x_{k+1} - x^*\|_A^2 = 2\alpha_k g_k^T g_k - \alpha_k^2 \|g_k\|_A^2$$

$$= \frac{2(g_k^T g_k)^2}{g_k^T A g_k} - \left(\frac{g_k^T g_k}{g_k^T A g_k}\right)^2 g_k^T A g_k$$

$$= \frac{g_k^T g_k}{g_k^T A g_k}$$

And with (2),

$$\|x_{k+1} - x^*\|_A^2 = \|x_k - x^*\|_A^2 - \frac{(g_k^T g_k)^2}{g_k^T A g_k} = \left(1 - \frac{g_k^T g_k}{g_k^T A g_k \, g_k A^{-1} g_k}\right) \times \|x_k - x^*\|_A^2$$

Exercise 3.

Let us show that $x_1 = x_0 + \alpha_0 g_0$ is a solution where $g_0 = Ax_0 - b$

$$\alpha_0 = \frac{\|Ax_0 - b\|^2}{\|Ax_0 - b\|_A^2} = \frac{\|Ax_0 - Ax^*\|^2}{\|Ax_0 - Ax^*\|_A^2} = \frac{\lambda \|x_0 - x^*\|^2}{\lambda \|x_0 - x^*\|_A^2} = \frac{\|x_0 - x^*\|}{(x_0 - x^*)^T A (x_0 - x^*)}$$

$$= \frac{\|x_0 - x^*\|}{\lambda \|x_0 - x^*\|} = \frac{1}{\lambda}$$

$$Ax_1 - b = A(x_0 - \alpha_0 g_0) - b = A\left(x_0 - \frac{1}{\lambda} A x_0 + \frac{1}{\lambda} A x^*\right)$$

$$= A\left(x_0 - \frac{1}{\lambda} \lambda (x_0 - x^*)\right)$$

$$= Ax^* = b.$$

Exercise 4.

$$(1): f(x_k + \alpha_k p_k) \le f(x_k) + c\,\alpha_k \nabla f(x_k) p_k^T$$

$$g_k = \nabla f(x_k) = Ax_k - b_k = -p_k$$

$$(2): f(x_{k+1}) - f(x_k) = -\alpha_k \underbrace{\|g_k\|^2}_{\substack{s \\ \shortparallel \\ -g_k p_k}} + \frac{1}{2} \alpha_k^2 \underbrace{g^T A g}_{t}$$

With (1) + (2), we have:

$$-\alpha_k s + \frac{1}{2}\alpha_k^2 t \leq c\,\alpha_k g_k p_k^T = -c\,\alpha_k s$$

$$\frac{1}{2}\alpha_k t \leq (1-c)\, s \qquad \text{as} \qquad \alpha_k = s/t$$

thus $\frac{1}{2} \leq 1 - c$ iff $c \leq 1/2$.

(3): $f(x_{k+1}) \geq f(x_k) + (1-c)\,\alpha_k g_k^T p_k$

thus $-\alpha_k s + \frac{1}{2}\alpha_k^2 t \geq -(1-c)\,\alpha_k s$

thus $\frac{1}{2}\alpha_k t \geq c\,s$ as $\alpha_k = s/t$

and finally $\frac{1}{2} \geq c$