Sep 27

zero coupon Bond $(T) \rightarrow$ maturity.

$$\text{payoff}(T) = 1.$$

suppose r is rate, today's price is $P(\underset{\text{Tody}}{0}, \underset{\text{maturity}}{T})$,
then

$$P(0,T) \cdot \boxed{e^{rT}} = 1 \quad \boxed{e^{rT}} \rightarrow \text{Growth factor}$$

$$P(0,T) = e^{-rT}$$

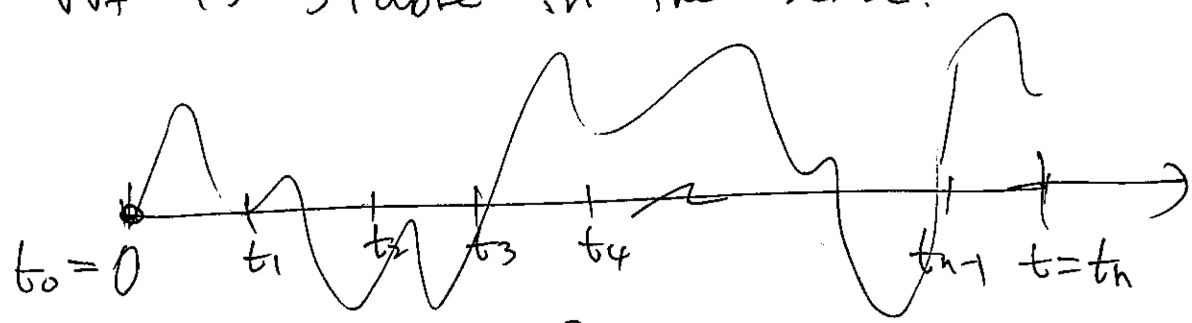
Stk ~ BSM(σ).

Suppose S_t is price at t.

W_t is Brownian Motion

$$S_t = S_0 \cdot \exp\{(r - \tfrac{1}{2}\sigma^2)t + \sigma W_t\} \rightarrow \text{EMM } Q$$

W_t is stable in the sense.



$$W_t = (W_{t_1} - \overset{=0}{W_{t_0}}) + (W_{t_2} - W_{t_1}) + \cdots + (W_{t_n} - W_{t_{n-1}})$$

$$= \Delta W_{t_1} + \Delta W_{t_2} + \cdots + \Delta W_{t_n}$$

where $\Delta W_{t_i} = W_{t_i} - W_{t_{i-1}}$

proposition

B

① $\{\Delta W_{t_1}, \Delta W_{t_2} \cdots \Delta W_{t_n}\}$ are independent

② $$\Delta W_{t_n} = W_{t_n} - W_{t_{n-1}} \sim N(0, \Delta t_n)$$

where $\Delta t_n = t_n - t_{n-1}$

Asian option (Geometric)

Geom. Asia call

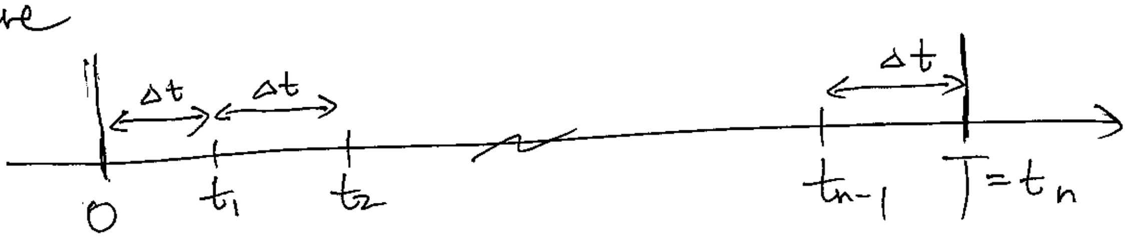
payoff for $GAC(T, K, n)$

$$\Pi_T^c = (A_T - K)^+$$

payoff for $GAP(T, K, n)$

$$\Pi_T^p = (A_T - K)^-$$

where

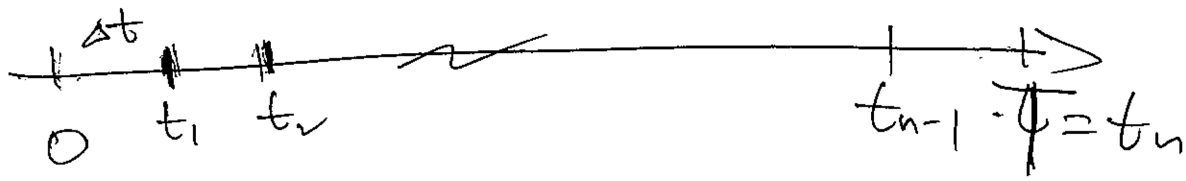


$$\Delta t = \frac{T}{n}$$

$$A_T = (S_{t_1} S_{t_2} \cdots S_{t_n})^{\frac{1}{n}}$$

Goal what's the price of GAC and GAP?

Ans $\Pi_0^c = \mathbb{E}^Q[e^{-rT}\Pi_T^c]$, $\Pi_0^p = \mathbb{E}^Q[e^{-rT}\Pi_T^p]$



$$S(t_1) = S_0 \exp\left\{ \underbrace{(r - \tfrac{1}{2}\sigma^2)}_{\mu}\Delta t + \sigma \underbrace{\Delta W_{t_1}} \right\}.$$

$$\Delta W_{t_1} \sim N(0, \Delta t) = \sqrt{\Delta t}\, Z_1$$

where $Z_1 \sim N(0,1)$, $\mu = r - \frac{1}{2}\sigma^2$

$$S(t_1) = S_0 \exp\{\mu\Delta t + \sigma\sqrt{\Delta t}\, Z_1\}$$

$$S(t_2) = S_{t_1} \exp\{\mu\Delta t + \sigma\sqrt{\Delta t}\, Z_2\}$$

$$i \to S(t_i) = S_{t_{i-1}} \exp\{\mu\Delta t + \sigma\sqrt{\Delta t}\, Z_i\}$$

$$\vdots$$

$$S(t_n) = S_{t_{n-1}} \exp\{\mu\Delta t + \sigma\sqrt{\Delta t}\, Z_n\}$$

where $\{Z_1, Z_2 \cdots Z_n\}$ are iid $N(0,1)$

$$S(t_i) = S_0 \exp\{\underline{\mu\Delta}t + \sigma\sqrt{\Delta t}\, Z_1\} \cdots \exp\{\underline{\mu\Delta}t + \sigma\sqrt{\Delta t}\, Z_i\}$$

$$S(t_i) = \underline{S_0} \exp\left\{ i\mu\Delta t + \sigma\sqrt{\Delta t} \sum_{j=1}^{i} Z_j \right\}$$

$$\prod_{i=1}^{n} S(t_i) = S_0^n \cdot \exp\left\{ \sum_{i=1}^{n} \left(i\mu\Delta t + \sigma\sqrt{\Delta t} \sum_{j=1}^{i} Z_j \right) \right\}$$

$$= S_0^n \exp\left\{ \frac{\mu T(n+1)}{2} + \sigma\sqrt{\Delta t} \sum_{i=1}^{n} \sum_{j=1}^{i} Z_j \right\}$$

$$\cancel{= S_0^n \exp\left\{ \frac{\mu T(n+1)}{2} + \sigma\sqrt{\Delta t}\ \sum \right\}}$$

$$= S_0^n \exp\left\{ \frac{\mu T(n+1)}{2} + \sigma\sqrt{\Delta t} \sum_{i=1}^{n} (n+1-i) Z_i \right\}$$ 4

$\boxed{\sum_{i=1}^{n} (n+1-i) Z_i}$ is sum of normal random variables, thus is another normal random variable.

its mean is

$$\mathbb{E} \sum_{i=1}^{n} (n+1-i) Z_i = \sum_{i=1}^{n} \mathbb{E} (n+1-i) Z_i = 0$$

$$\mathrm{Var}\left(\sum_{i=1}^{n} (n+1-i) \cdot Z_i\right)$$

$$= \sum_{i=1}^{n} \mathrm{Var}\left((n+1-i) Z_i\right)$$

$$= \sum_{i=1}^{n} (n+1-i)^2 \underbrace{\mathrm{Var}(Z_i)}_{\to 1}$$

$$= n^2 + \cdots + 1^2 = \frac{n(n+1)(2n+1)}{6}$$

Now, we have

$$\sum_{i=1}^{n} (n+1-i) Z_i \sim N\left(0, \frac{n(n+1)(2n+1)}{6}\right)$$

$$= \sqrt{\frac{n(n+1)(2n+1)}{6}} \, \hat{Z}$$

where $\hat{Z} \sim N(0,1)$

$$A_T = \left(\prod_{i=1}^{n} S_{t_i}\right)^{\frac{1}{n}}$$ P_f

$$= \left(S_0^n \cdot \exp\left\{\frac{(n+1)\mu T}{2} + \sigma\sqrt{\Delta t}\sqrt{\frac{n(n+1)(2n+1)}{6}}\hat{Z}\right\}\right)^{\frac{1}{n}}$$

$$= S_0 \cdot \exp\left\{\frac{(n+1)\mu T}{2n} + \frac{\sigma}{n}\sqrt{\frac{\cancel{n}(n+1)(2n+1)}{6}}\,\sqrt{T}\,\hat{Z}\right\}$$

Recall $\text{BSM-call}(S_0, K, 0, T, r, \sigma)$.

If $S_T = S_0 \exp\{(r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}Z\}$ ~~$\sigma W_T\}$~~

then $\mathbb{E}[e^{-rT}(S_T - K)^+] = \text{BSM\_Call}(S_0, K, 0, T, r, \sigma)$

Now I want

$$\Pi_0^c = \mathbb{E}[e^{-rT}(A_T - K)^+]$$

where $A_T = S_0 \cdot \exp\{(\hat{r} - \frac{1}{2}\hat{\sigma}^2)T + \hat{\sigma}\sqrt{T}\hat{Z}\}$ ~~$\hat{\sigma} W_T\}$~~

where $\hat{\sigma} = \frac{\sigma}{n}\sqrt{\frac{(n+1)(2n+1)}{6}}$

$$\hat{r} - \frac{1}{2}\hat{\sigma}^2 = \frac{(n+1)\mu}{2n}$$

$$\Pi_0^c = e^{\hat{r}T - rT}\,\underbrace{\mathbb{E}[e^{-\hat{r}T}(A_T - K)^+]}$$

$$= e^{(\hat{r}-r)T}\,\text{BSM\_Call}(S_0, K, 0, T, \hat{r}, \hat{\sigma})$$

§ Calibration under BSM

Recall

$$\text{BSM-Call}(S_0, K, 0, T, r, \sigma) = \text{Price}$$

Reality is σ is unknown.

Calibration is to find σ, given

$$\{S_0, K, 0, T, r, \text{Price}\}$$

from Market quote

The computed σ is called implied volatility.

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ex $N = 15$.

Market quotes

$$\{C_0^*, C_1^* \cdots C_{15}^*\}$$

I want find σ s.t. theoretical prices

$$\{C_0, C_1 \cdots C_{15}\}$$

are "close" to market price.