Oct 04

§ Binomial option pricing

Recall

$BSM(S_0 \overset{\text{Init}}{=}, \text{vol} = \sigma, \text{rate} = r)$

Then it implies stock under EMM $\mathbb{Q}$

$$\frac{dS_t}{S_t} = r\,dt + \sigma\, dW_t, \quad S_0$$

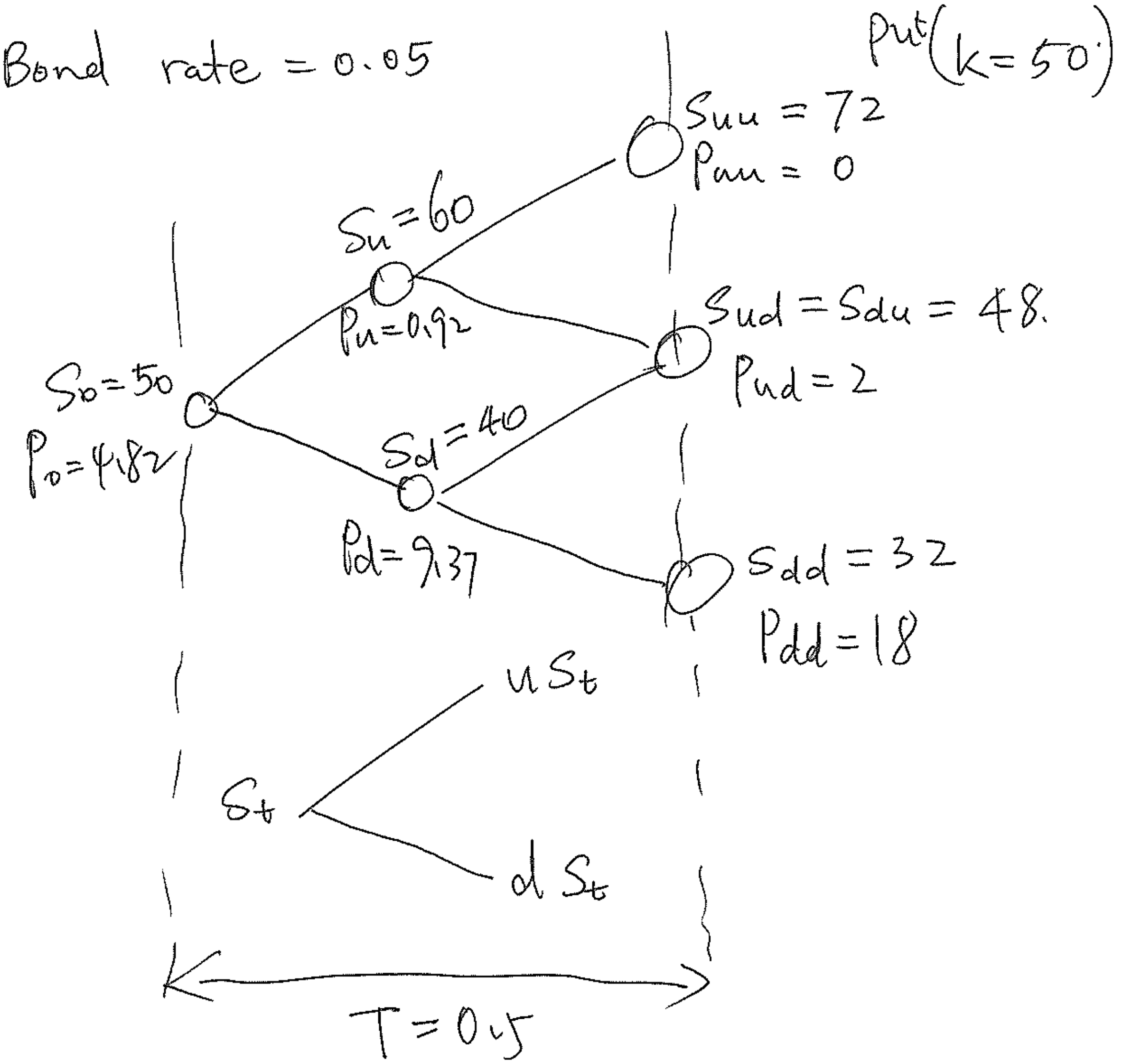
or $S_t = S_0 \exp\{(r - \frac{1}{2}\sigma^2)t + \sigma W_t\}$.

We consider Binomial Tree model, denoted by

★ BinTree$(S_0 = 50, N = 2, T = 0.5, \underline{u = 1.2}, \underline{d = 0.8}, r = 0.05)$

① Bond rate $= 0.05$

②



★ Option types

① Eu Call ($T = 0.5$, $K = 50$)

$$\text{payoff}\big|_T = (S_T - K)^+$$

② Eu Put (T, K)

$$\text{payoff}\big|_T = (S_T - K)^-$$

③ Am Call (T, K)

$$\text{payoff}\big|_t = (S_t - K)^+ , \quad t \le T$$

④ Am Put (T, K)

$$\text{payoff}\big|_t = (S_t - K)^- , \quad t \le T.$$

Q.

$$\begin{tikzcd}[row sep=small] & uS_t \\ S_t \arrow[ur, "q"] \arrow[dr, "1-q"'] & & & r \\ & dS_t \end{tikzcd}$$

what's the EMM?

Recall martingale

$\{X_t : t \in T\}$ is a mtgl. process, if.

$$\mathbb{E}\left[X_{t+h} \mid X_t\right] = X_t$$

Recall mtgl (discrete)

$\{X_t : t \in \mathbb{N}\}$ is a mtgl. process if

$$\mathbb{E}\left[X_{t+1} \mid X_t\right] = X_t$$

ex

$$\begin{tikzcd}[column sep=huge]
 & 1000 = X_1(u) \\
X_0 = 100 \arrow[ur, "q=\frac{1}{2}"] \arrow[dr, "\frac{1}{2}=1-q"'] & \\
 & 99 = X_1(d)
\end{tikzcd}$$

① if $\mathbb{Q} = \{q, 1-q\} = (\frac{1}{2}, \frac{1}{2})$, then it's not mtgl. b/c

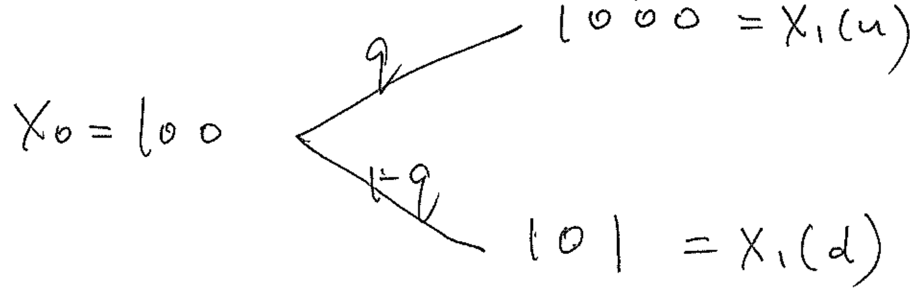
$$\mathbb{E}[X_1] = 1000 \cdot \frac{1}{2} + 99 \cdot \frac{1}{2} = \frac{1099}{2} \neq X_0$$

② Is there mtgl measure?
(i.e. find $\mathbb{Q}$ s.t. $\mathbb{E}^{\mathbb{Q}}[X_1] = X_0$?)

$$1000q + 99(1-q) = 100$$

$$q = \frac{1}{901}$$

ex

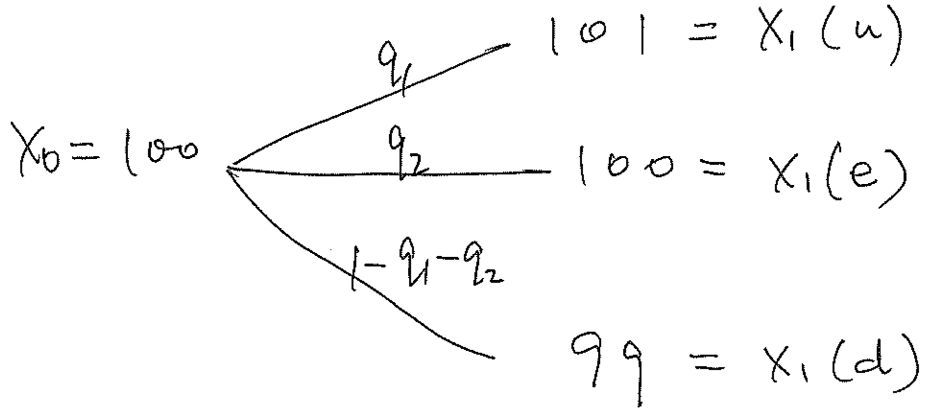


Q. Find mtgl measure?

Sol
$$\mathbb{E}[X_1] = 1000q + 101(1-q) = 100$$

$q =$ Does not exist.

ex



Q Find mtgl. meas?

Sol
$$Q = (q_1 = 0,\ q_2 = 1,\ 1 - q_1 - q_2 = 0)$$

or
$$Q = (q_1 = \tfrac{1}{2}\ \ q_2 = 0,\ q_3 = \tfrac{1}{2})$$

⋯

or
$$Q = (q_1,\ q_2 = 1 - q_1 - q_3,\ q_3 = q_1)$$

where $q_1 \in [0, \tfrac{1}{2}]$

~~**Def** $(S_t : t \in \mathbb{N})$ is stock price, r is rate~~

~~$\mathbb{Q}$ is EMM if~~

$$\cancel{\mathbb{E}[S_{t+1} | S_t] = e^r}$$

Def $T = \{0, \Delta t, 2\Delta t, \cdots\} = \{n\Delta t : n = 0, 1, 2 \cdots\}$

stock price : $S_{n\Delta t}$

Bond rate : r.

$\mathbb{Q}$ is EMM if

$$\mathbb{E}\left[S_{(n+1)\Delta t} \,\middle|\, S_{n\Delta t}\right] = e^{r\Delta t} S_{n\Delta t}$$

Or

$$S_{n\Delta t} = \mathbb{E}\Big[\underbrace{e^{-r\Delta t} S_{(n+1)\Delta t}}_{\text{Discounted Stock price.}} \,\Big|\, S_{n\Delta t}\Big]$$

Rk. Discounted Stock price is a mtgl. w.r.t. EMM $\mathbb{Q}$

Q Find EMM for

$$\begin{tikzcd}[row sep=small] & u S_t \\ S_t \arrow[ur, "q"] \arrow[dr, "1-q"'] & \\ & d S_t \end{tikzcd} \quad , \quad r.$$

Soln

$$u \cdot q + (1-q) d = e^{r\Delta t}, \qquad \Delta t = \frac{T}{n}$$

$$\boxed{q = \frac{e^{r\Delta t} - d}{u - d}}$$

Constructing Bin.Eu Opt.

set_up_param:

$M = N+1$, # of terminal nodes

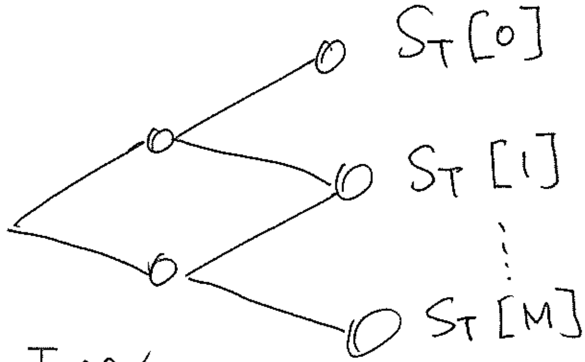
$$\left.\begin{array}{l} u = 1+p_u \\ d = 1-p_d \end{array}\right\} \text{up/down factor}$$

$$\left.\begin{array}{l} q_u = \dfrac{e^{r\Delta t}-d}{u-d} \\ q_d = 1-q_u \end{array}\right\} \text{EMM.}$$

$\Delta t = \dfrac{T}{N}$: Time length of one period

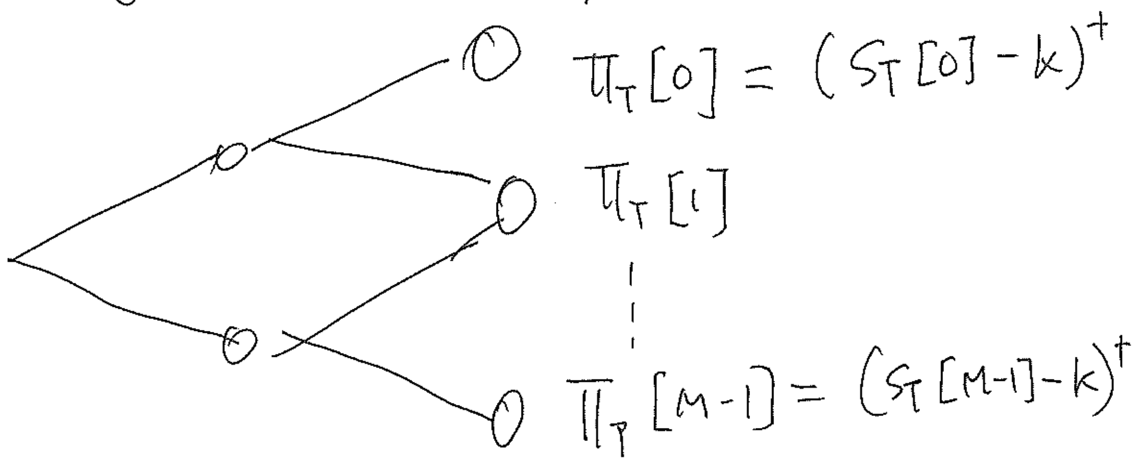
$df = e^{-r\Delta t}$, discount factor

Init_stock_tree : Compute $\{S_T[i],\ i=0\cdots M-1\}$



Init_payoff_Tree

<u>ex</u> Eu Call ($T=0.5$, $K=50$).



Traverse - tree (Backward)

$$\Pi_N[\,] \to \Pi_{N-1}[\,] \to \Pi_{N-2}[\,] \rightsquigarrow \Pi_0[0]$$

$$\begin{tikzcd}[row sep=small] & \Pi_i[j] \\ \Pi_{i-1}[j] \arrow[ur, "q"] \arrow[dr, "1-q"'] & \\ & \Pi_i[j+1] \end{tikzcd}$$

$$\Pi_{i-1}[j] = e^{-r\Delta t}\left(q\,\Pi_i[j] + (1-q)\,\Pi_i[j+1]\right)$$

Q Given BSM(σ), rate $= r$

We have BSM formula for pricing

EuCall(T, K), Eu Put (T, K)

Arithmetic Asian option $\Big\}$ has no formula
Barrier option

☆ $CRR(S_0 = 50,\ N=2,\ T=0.5,\ \sigma=0.3,\ r=0.05)$

$= BinTree(S_0=50,\ N=2,\ T=0.5,\ u = e^{\sigma\sqrt{\Delta t}},\ d=\frac{1}{u},\ r=0.05)$

$\xrightarrow{N\to\infty} BSM(S_0=50,\ T=0.5,\ \sigma=0.3,\ r=0.05)$

where $\Delta t = \dfrac{T}{N}$

$$S_t \to S_{t+\Delta t}(u) = e^{\sigma\sqrt{\Delta t}}\, S_t$$

$$S_t \to S_{t+\Delta t}(d) = e^{-\sigma\sqrt{\Delta t}}\, S_t$$

ex

Q: what is price of ~~Call~~ Put ($T=0.5$, $k=50$)

Q1 for $BSM(S_0=50,\ r=0.05,\ \sigma=0.3) = ?$

Q2 for $CRR(S_0=50,\ N=200,\ T=0.5,\ \sigma=0.3,\ r=0.05) = ?$