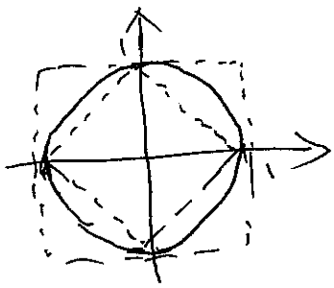
§. Basic Monte Carlo.

(Ref) [Gla 03] Chap 1.

Q. what is def. of MC?

ex Can you approximate π?

Method 1



$\pi = \pi \cdot 1^2 =$ Area of unit circle.

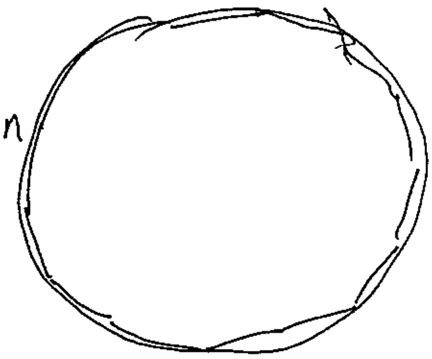
B/c $2 \le$ Area of unit circle ≤ 4.

Thus my approximation is

$$\pi \approx \frac{4+2}{2} = 3$$

unit circle

To improve, we imbed n-polygon for $B_1(0)$

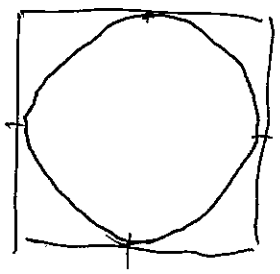


$\pi \approx$ Area of n-polygon $\triangleq A_n$

we expect.

$$\lim_{n \to \infty} A_n = \pi$$

Method 2



~~Suppose I shoot n~~

Suppose ① I make n-shots to the rectangular, and m shots fall in B_1, ② shots are uniformly distributed, then

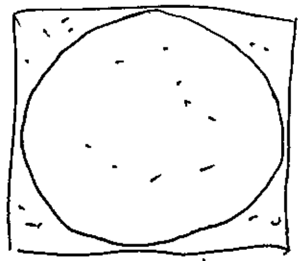
$$\pi \approx \frac{m}{n} \cdot 4$$

ex Implement MC for the approximation of π

(S1) Draw N random points uniformly in rectangle $(-1,1)\times(-1,1)$, i.e.

$$\{Z_i = (X_i, Y_i) : \begin{array}{l} X_i \sim U(-1,1) \\ Y_i \sim U(-1,1) \\ \text{i.i.d.} \\ 1 \le i \le N \end{array}\}$$

(S2) Identify the number N_1 of pts inside the B_1



(S3) $$\hat{\pi} = \underset{\substack{\uparrow \\ \text{Rect. Area}}}{4} \cdot \frac{N_1}{N}$$

Rk π is deterministic value.
But its approximation $\hat{\pi}$ is random.

Def A random estimator (approximation) $\hat{\alpha}$ to a deterministic value α is called MC,

① Bias $= \mathbb{E}[\hat{\alpha}] - \alpha$

② $MSE(\hat{\alpha}) = \mathbb{E}[(\hat{\alpha} - \alpha)^2]$.

prop $\mathrm{MSE}(\hat{\alpha}) = |\mathrm{Bias}|^2 + \mathrm{Var}(\hat{\alpha})$

pf $\mathrm{MSE}(\hat{\alpha}) = \mathbb{E}[(\hat{\alpha}-\alpha)^2]$

$= \mathbb{E}[\beta^2]$ where $\beta = \hat{\alpha}-\alpha$

$= \mathrm{Var}\,\beta + (\mathbb{E}[\beta])^2$

$= \mathrm{Var}(\hat{\alpha}-\alpha) + (\mathbb{E}[\hat{\alpha}-\alpha])^2$

$= \mathrm{Var}(\hat{\alpha}) + (\underbrace{\mathbb{E}[\hat{\alpha}]-\alpha}_{\mathrm{Bias}})^2$

$\square$

draft

$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$

$\mathrm{Var}(X) = \mathrm{Var}(X+c)$

$\mathbb{E}[X+c] = \mathbb{E}[X]+c$

Rk "The smaller MSE, the better MC."

Def MC is called <u>unbiased</u>, if

$$\mathrm{Bias} = 0$$

i.e. $\mathbb{E}[\hat{\alpha}] = \alpha$.

ex In "approx. π by $\hat{\pi}$",

① Find 1^{st}, 2^{nd} moments of $\hat{\pi}$

② Find Bias, & MSE of $\hat{\pi}$

Def m^{th} moment is ~~$\mathbb{E}[X^m]$~~ $\mathbb{E}[X^m]$

Soln

① $\mathbb{E}[\hat{\pi}] = \mathbb{E}\left[4 \cdot \frac{N_1}{N}\right] = \frac{4}{N}\mathbb{E}[N_1]$

$= \frac{4}{N}\mathbb{E}\left[\sum_{i=1}^{N} I_{\{|Z_i|<1\}}\right]$

$= \frac{4}{N}\sum_{i=1}^{N}\mathbb{E}\left[I_{\{|Z_i|<1\}}\right]$

$= \frac{4}{N}\left(\mathbb{E}[I_{\{|Z_1|<1\}}] + \mathbb{E}[I_{\{|Z_2|<1\}}] + \cdots + \mathbb{E}[I_{\{|Z_N|<1\}}]\right)$

$= \cancel{\frac{4}{N} \cdot N} \cdot \mathbb{E}[I_{\{|Z_1|<1\}}]$

$= 4 \cdot \mathbb{E}[I_{\{|Z_1|<1\}}]$

$= 4 \cdot \mathbb{P}(|Z_1|<1) = 4 \cdot \frac{\text{Circ Area}}{\text{Rec Area}}$

$= \pi$

Recall Indicator

$I_A = \begin{cases} 1 & \text{if } A \text{ occur} \\ 0 & \text{otherwise} \end{cases}$

$\mathbb{E}[X] + \mathbb{E}[Y] = \mathbb{E}[X+Y]$

$\mathbb{E}[I_A] = \mathbb{P}(A)$

$Var(X \cdot c) = c^2 Var(X)$

$Var(X+Y) = Var(X) + Var(Y)$ if $X \perp Y$

$\cancel{Var(\hat{\pi}) =}$

$\mathbb{E}[\hat{\pi}^2] = (\mathbb{E}[\hat{\pi}])^2 + Var(\hat{\pi}) = \pi^2 + \underbrace{Var(\hat{\pi})}_{?}$

$Var(\hat{\pi}) = Var\left(4 \cdot \frac{N_1}{N}\right) = \frac{4^2}{N^2} Var(N_1)$

$= \frac{16}{N^2} \cdot Var\left(\sum_{i=1}^{N} I_{\{|Z_i|<1\}}\right)$

$= \frac{16}{N^2} \cdot \sum_{i=1}^{N} Var\left(I_{\{|Z_i|<1\}}\right)$

$= \frac{16}{N^2} \cdot N \cdot Var\left(I_{\{|Z_1|<1\}}\right)$

$= \frac{16}{N} \cdot \mathbb{P}(|Z_1|<1)\ \mathbb{P}(|Z_1| \geq 1)$

$= \frac{4 \cancel{16}}{N} \frac{\pi}{\cancel{4}} \cdot \left(1 - \frac{\pi}{4}\right) = \boxed{\frac{\pi(4-\pi)}{N}}$

② $\text{Bias} = \mathbb{E}[\hat{\pi}] - \pi = 0$

~~$Var(\hat{\pi})$~~

$$MSE(\hat{\pi}) = |\text{Bias}|^2 + Var(\hat{\pi})$$
$$= 0^2 + \frac{\pi(4-\pi)}{N} = \frac{\pi(4-\pi)}{N}$$

[5]

Def Given a series of estimator of α say $\{\alpha_n, n \in \mathbb{N}\}$, we say (α_n) is **consistent** if

$$\boxed{\operatorname*{p-lim}_{n \to \infty} \alpha_n = \alpha.}$$

(We explain it later).

prop If $\boxed{MSE(\hat{\alpha}_n)} \to 0$ then $\hat{\alpha}_n$ is consistent.

ex. If N is fixed, then

$$\text{Bias}(\hat{\pi}) = 0.$$

prop suppose n ppl. do the same unbiased approximation, their out comes are

$$\{\hat{\alpha}^{(i)} : 1 \le i \le n\}$$

$$\frac{\sum_{i=1}^{n} \hat{\alpha}^{(i)}}{n} \longrightarrow \alpha \quad \text{almost surely}.$$

Pf Law of Large number (LLN)

ex Let $\hat{\pi}_N$ be estimator with N total points.
Then, $\{\hat{\pi}_N : N \in \mathbb{N}\}$ is a consistent estimator of π. b/c

$$MSE(\hat{\pi}_N) = \frac{\pi(4-\pi)}{N} \xrightarrow[\text{as } N\to\infty]{} 0$$

Def

① "$p\text{-}\lim_{n\to\infty} \alpha_n = \alpha$" or "$\alpha_n \xrightarrow[n\to\infty]{} \alpha$ in prob"

if $\lim_{n\to\infty} \mathbb{P}(|\alpha_n - \alpha| > \varepsilon) = 0$, $\forall \varepsilon > 0$

② "$\lim_{n\to\infty} \alpha_n = \alpha$" or "$\alpha_n \xrightarrow[n\to\infty]{} \alpha$ a.s"

if $\mathbb{P}(\lim_{n\to\infty} \alpha_n = \alpha) = 1$

③ "$L_2\text{-}\lim_{n\to\infty} \alpha_n = \alpha$" or "$\alpha_n \xrightarrow[n\to\infty]{} \alpha$ in L^2" if

$$\lim_{n\to\infty} \mathbb{E}|\alpha_n - \alpha|^2 = 0$$

Rk Read "wiki"

ex Justify the statement

"$p\text{-}\lim_{n\to\infty} \alpha_n = \alpha$" implies "$L_2\text{-}\lim_{n\to\infty} \alpha_n = \alpha$".

Ans No, b/c following counter-example

Let $\mathbb{P} = U([0,1])$ $\longrightarrow$ probability

$\Omega = [0,1]$ $\longrightarrow$ sample space

$\alpha_n : [0,1] \to \mathbb{R}$ s.t.

$$\alpha_n(\omega) = \begin{cases} n & \text{if } 0 < \omega < \frac{1}{n^2} \\ 0 & \text{otherwise} \end{cases}$$

$\alpha : [0,1] \to \mathbb{R}$ s.t.

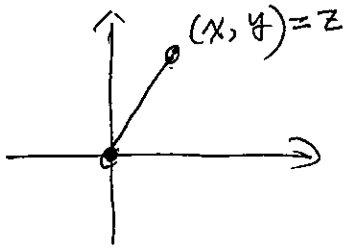
$\alpha(\omega) \equiv 0.$

$$\alpha_2(\omega) = \begin{cases} 2 & \text{if } 0 < \omega < \frac{1}{4} \\ 0 & \text{otherwise} \end{cases}$$

Draft

$$\mathbb{R}^2 = \mathbb{R}\times\mathbb{R} = \{(\underline{x, y}) : x\in\mathbb{R}\ ,\ y\in\mathbb{R}\}$$

$z\in\mathbb{R}^2$. means a point on $\mathbb{R}^2$-plane



$$|z| = \sqrt{|z_x|^2 + |z_y|^2}$$

$$I_{\{|z|<1\}} = \begin{cases} 1 & \text{if } \ |z|<1 \\ 0 & \text{if } \ |z|\geq 1 \end{cases}$$

If $$X = \begin{cases} a_1 , & p_1 \\ a_2 & p_2 \\ \vdots & \\ a_n & p_n \end{cases}$$

Then $$\mathbb{E}[X] = \sum_{i=1}^{n} a_i\, p_i$$

$$Var(X) = \underbrace{\sum_{i=1}^{n} a_i^2 \cdot p_i}_{E[X^2]} - \left(E[X]\right)^2$$

If $$X = I_A = \begin{cases} 1 & \mathbb{P}(A) \\ 0 & 1-\mathbb{P}(A) \end{cases}$$

$$\mathbb{E}[I_A] = \mathbb{E}[X] = 1\cdot\mathbb{P}(A) + 0\cdot(1-\mathbb{P}(A)) = \mathbb{P}(A)$$

$$\mathbb{E}[I_A^2] = 1^2\cdot\mathbb{P}(A) + 0^2(1-\mathbb{P}(A)) = \mathbb{P}(A)$$

$$Var(I_A) = \mathbb{P}(A) - \mathbb{P}^2(A) = \mathbb{P}(A)\,(1-\mathbb{P}(A))$$