

CAS MA575: Assignment #5

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1.

It is obvious that $\frac{\partial f(x)}{\partial x} = (\frac{\partial f(x)}{\partial x_1}, \dots, \frac{\partial f(x)}{\partial x_d})^T$, and $\frac{\partial(\frac{\partial f(x)}{\partial x_1})}{\partial x} = (\frac{\partial^2 f(x)}{\partial x_1^2}, \dots, \frac{\partial^2 f(x)}{\partial x_1 \partial x_d})^T$

Therefore, $\frac{\partial}{\partial x} \left(\frac{\partial f(x)}{\partial x} \right)^T = \frac{\partial}{\partial x} \left(\frac{\partial f(x)}{\partial x_1}, \dots, \frac{\partial f(x)}{\partial x_d} \right) = \left[\frac{\partial(\frac{\partial f(x)}{\partial x_1})}{\partial x}, \dots, \frac{\partial(\frac{\partial f(x)}{\partial x_d})}{\partial x} \right] =$
 $\left[\left(\frac{\partial^2 f(x)}{\partial x_1^2}, \dots, \frac{\partial^2 f(x)}{\partial x_1 \partial x_d} \right)^T, \dots, \left(\frac{\partial^2 f(x)}{\partial x_d \partial x_1}, \dots, \frac{\partial^2 f(x)}{\partial x_d^2} \right)^T \right] = H$

2.(a)

From the definition of covariance matrix, we can see $cov(Y) = (a_{ij})_{p \times p}$ is a $p \times p$ matrix, and $a_{ij} = cov(y_i, y_j)$.

$Y - E(Y) = (y_1 - E(y_1), \dots, y_p - E(y_p))^T \Rightarrow \{Y - E(Y)\}\{Y - E(Y)\}^T =$
 $(b_{ij})_{p \times p}, b_{ij} = (y_i - E(y_i))(y_j - E(y_j))$

Therefore, $E[\{Y - E(Y)\}\{Y - E(Y)\}^T] = (c_{ij})_{p \times p}, c_{ij} = E[\{y_i - E(y_i)\}\{y_j - E(y_j)\}^T] = cov(y_i, y_j) = a_{ij} \Rightarrow E[\{Y - E(Y)\}\{Y - E(Y)\}^T] = cov(Y)$

We set $Y - E(Y)$ as $A^T, E[\{y_i - E(y_i)\}\{y_j - E(y_j)\}^T] = E(A^T A)$, because for any vector $x, x^T E(A^T A)x = E(x^T A^T A x) = E((Ax)^T (Ax)) \geq 0 \Rightarrow cov(Y)$ is positive semidefinite.

2.(b)

$$\begin{aligned} cov(AY) &= E[\{AY - E(AY)\}\{AY - E(AY)\}^T] \\ &= E[\{A(Y - E(Y))\}\{A(Y - E(Y))\}^T] \\ &= E[A\{Y - E(Y)\}\{Y - E(Y)\}^T A^T] \\ &= AE[\{Y - E(Y)\}\{Y - E(Y)\}^T] A^T \\ &= Acov(Y)A^T \end{aligned}$$

3.(a)

$$\begin{aligned}
cov(X + Z, Y) &= E[\{X + Z - E(X + Z)\}\{Y - E(Y)\}^T] \\
&= E[\{[X - E(X)] + [Z - E(Z)]\}\{Y - E(Y)\}^T] \\
&= E[\{X - E(X)\}\{Y - E(Y)\}^T] + E[\{Z - E(Z)\}\{Y - E(Y)\}^T] \\
&= cov(X, Y) + cov(Z, Y)
\end{aligned}$$

(b)

$$\begin{aligned}
cov(AX, Y) &= E[\{AX - E(AX)\}\{Y - E(Y)\}^T] \\
&= E[A\{X - E(X)\}\{Y - E(Y)\}^T] \\
&= AE[\{X - E(X)\}\{Y - E(Y)\}^T] \\
&= A \sum_{XY}
\end{aligned}$$

(c)

$$\begin{aligned}
cov(Y, X) &= E[\{Y - E(Y)\}\{X - E(X)\}^T] \\
&= E[(\{X - E(X)\}\{Y - E(Y)\}^T)^T] \\
&= E[\{Y - E(Y)\}\{X - E(X)\}^T]^T \\
&= \sum_{XY}^T
\end{aligned}$$

(d)

$$\begin{aligned}
cov(Y, AX) &= E[\{Y - E(Y)\}\{AX - E(AX)\}^T] \\
&= E[\{Y - E(Y)\}\{X - E(X)\}^T A^T] \\
&= E[\{Y - E(Y)\}\{X - E(X)\}^T]^T A^T \\
&= \sum_{XY}^T A^T
\end{aligned}$$