CAS MA575: Assignment #5

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1.

It is obvious that $\frac{\partial f(x)}{\partial x} = (\frac{\partial f(x)}{\partial x_1}, ..., \frac{\partial f(x)}{\partial x_d})^T$, and $\frac{\partial(\frac{\partial f(x)}{\partial x_1})}{\partial x} = \big(\frac{\partial^2 f(x)}{\partial x_1^2}, ..., \frac{\partial^2 f(x)}{\partial x_1 \partial x_d}\big)^T$

Therefore, $\frac{\partial}{\partial x}\left(\frac{\partial f(x)}{\partial x}\right)^T = \frac{\partial}{\partial x}\left(\frac{\partial f(x)}{\partial x_1}, ..., \frac{\partial f(x)}{\partial x_d}\right) = \big[\frac{\partial(\frac{\partial f(x)}{\partial x_1})}{\partial x}, ..., \frac{\partial(\frac{\partial f(x)}{\partial x_d})}{\partial x}\big] = \big[\big(\frac{\partial^2 f(x)}{\partial x_1^2}, ..., \frac{\partial^2 f(x)}{\partial x_1 \partial x_d}\big)^T, ...\big(\frac{\partial^2 f(x)}{\partial x_d \partial x_1}, ..., \frac{\partial^2 f(x)}{\partial x_d^2}\big)^T\big] = H$

2.(a)

From the definition of covariance matrix, we can see $cov(Y) = (a_{ij})_{p\times p}$ is a $p \times p$ matrix, and $a_{ij} = cov(y_i, y_j)$.

$Y - E(Y) = \Big(y_1 - E(y_1), ..., y_p - E(y_p)\Big)^T \Rightarrow \{Y - E(Y)\}\{Y - E(Y)\}^T = (b_{ij})_{p\times p}, b_{ij} = (y_i - E(y_i))(y_j - E(y_j))$

Therefore, $E\big[\{Y - E(Y)\}\{Y - E(Y)\}^T\big] = (c_{ij})_{p\times p}, c_{ij} = E\big[\{y_i - E(y_i)\}\{y_j - E(y_j)\}^T\big] = cov(y_i, y_j) = a_{ij} \Rightarrow E\big[\{Y - E(Y)\}\{Y - E(Y)\}^T\big] = cov(Y)$

We set $Y - E(Y)$ as A^T,$E\big[\{y_i - E(y_i)\}\{y_j - E(y_j)\}^T\big] = E(A^T A)$, becasue for any vector $x, x^T E(A^T A)x = E(x^T A^T Ax) = E\big((Ax)^T(Ax)\big) \geq 0 \Rightarrow cov(Y)$ is positive semidefinite.

2.(b)

$$
\begin{aligned}
cov(AY) &= E\big[\{AY - E(AY)\}\{AY - E(AY)\}^T\big] \\
&= E\big[\{A\big(Y - E(Y)\big)\}\{A\big(Y - E(Y)\big)\}^T\big] \\
&= E\big[A\{Y - E(Y)\}\{Y - E(Y)\}^T A^T\big] \\
&= AE\big[\{Y - E(Y)\}\{Y - E(Y)\}^T\big]A^T \\
&= Acov(Y)A^T
\end{aligned}
$$

3.(a)

$$
\begin{aligned}
cov(X+Z,Y) &= E\big[\{X+Z-E(X+Z)\}\{Y-E(Y)\}^T\big] \\
&= E\big[\{[X-E(X)]+[Z-E(Z)]\}\{Y-E(Y)\}^T\big] \\
&= E\big[\{X-E(X)\}\{Y-E(Y)\}^T\big] + E\big[\{Z-E(Z)\}\{Y-E(Y)\}^T\big] \\
&= cov(X,Y) + cov(Z,Y)
\end{aligned}
$$

(b)

$$
\begin{aligned}
cov(AX,Y) &= E\big[\{AX-E(AX)\}\{Y-E(Y)\}^T\big] \\
&= E\big[A\{X-E(X)\}\{Y-E(Y)\}^T\big] \\
&= AE\big[\{X-E(X)\}\{Y-E(Y)\}^T\big] \\
&= A\sum\nolimits_{XY}
\end{aligned}
$$

(c)

$$
\begin{aligned}
cov(Y,X) &= E\big[\{Y-E(Y)\}\{X-E(X)\}^T\big] \\
&= E\big[\big(\{X-E(X)\}\{Y-E(Y)\}^T\big)^T\big] \\
&= E\big[\{Y-E(Y)\}\{X-E(X)\}^T\big]^T \\
&= \sum\nolimits_{XY}^T
\end{aligned}
$$

(d)

$$
\begin{aligned}
cov(Y,AX) &= E\big[\{Y-E(Y)\}\{AX-E(AX)\}^T\big] \\
&= E\big[\{Y-E(Y)\}\{X-E(X)\}^T A^T\big] \\
&= E\big[\{Y-E(Y)\}\{X-E(X)\}^T\big]^T A^T \\
&= \sum\nolimits_{XY}^T A^T
\end{aligned}
$$