CAS MA575: Assignment #7

Hu Hengchang(U34497427)
hhc@bu.edu

Boston University — November 12, 2018

1.

$$A = \begin{pmatrix} 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \text{solve this equation } |A - \lambda E| = 0$$

$\Rightarrow \lambda_1 = \frac{1}{2}, \lambda_2 = 1, \lambda_3 = \frac{3}{2} \Rightarrow$ corresponding $h_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, h_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, h_3 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \Rightarrow Q = \begin{pmatrix} h_1 & h_2 & h_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}, Q^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \Lambda = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{3}{2} \end{pmatrix} \Rightarrow A = Q\Lambda Q^T$

$A^{-1} = Q\Lambda^{-1}Q^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{2}{3} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \Rightarrow A^{-1} = \begin{pmatrix} \frac{4}{3} & -\frac{2}{3} & 0 \\ -\frac{2}{3} & \frac{4}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$A^{\frac{1}{2}} = Q\Lambda^{\frac{1}{2}}Q^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{\frac{3}{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}+1}{2\sqrt{2}} & \frac{\sqrt{3}-1}{2\sqrt{2}} & 0 \\ \frac{\sqrt{3}-1}{2\sqrt{2}} & \frac{\sqrt{3}+1}{2\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A^{-\frac{1}{2}} = Q\Lambda^{-\frac{1}{2}}Q^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{\frac{2}{3}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{6}+3\sqrt{2}}{6} & \frac{\sqrt{6}-3\sqrt{2}}{6} & 0 \\ \frac{\sqrt{6}-3\sqrt{2}}{6} & \frac{\sqrt{6}+3\sqrt{2}}{6} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2.

(a)

$\hat{\beta} = (X^TX)^{-1}X^TY \Rightarrow E(\hat{\beta}) = (X^TX)^{-1}X^TE(Y) = (X^TX)^{-1}X^TX\beta = \beta$
and $Var(\hat{\beta}) = (X^TX)^{-1}X^TVar(Y)X(X^TX)^{-1} = \sigma^2(X^TX)^{-1} \Rightarrow \hat{\beta} \sim N(\beta, \sigma^2(X^TX)^{-1})$
$a^T(\hat{\beta}-\beta) \sim N(0, \sigma^2 a^T(X^TX)^{-1}a) \Rightarrow \frac{a^T(\hat{\beta}-\beta)}{\sigma\sqrt{a^T(X^TX)^{-1}a}} \sim N(0,1)$

From the lecture $\frac{(n-p)\hat{\sigma}^2}{\sigma^2} \sim \chi^2_{n-p} \Rightarrow \frac{a^T(\hat{\beta}-\beta)}{\hat{\sigma}\sqrt{a^T(X^TX)^{-1}a}} = \frac{\frac{a^T(\hat{\beta}-\beta)}{\sigma\sqrt{a^T(X^TX)^{-1}a}}}{\sqrt{\frac{(n-p)\hat{\sigma}^2}{\sigma^2}/(n-p)}} \sim t_{n-p}$

(b)

Since $T = \frac{a^T(\hat{\beta}-\beta)}{\hat{\sigma}\sqrt{a^T(X^TX)^{-1}a}} \sim t_{n-p}$, when $T > t_{1-\alpha}(n-p)$, we reject the null hypothesis, i.e H_0.

3.

$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & -1 \end{pmatrix} \Rightarrow \hat{\beta} = (X^TX)^{-1}X^TY = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{6}{7} \\ \\ \frac{1}{3} \end{pmatrix}$

From Question 2, we can get that $(\hat{\beta}-\beta) \sim N(0, \Sigma)$, where $\Sigma = \sigma^2(X^TX)^{-1}$. Since Σ is variance matrix so it is psd $\Rightarrow \Sigma^{-\frac{1}{2}}(\hat{\beta}-\beta) \sim N(0, I_{2\times 2}) \Rightarrow (\Sigma^{-\frac{1}{2}}(\hat{\beta}-\beta))^T(\Sigma^{-\frac{1}{2}}(\hat{\beta}-\beta)) = (\hat{\beta}-\beta)\Sigma^{-1}(\hat{\beta}-\beta) \sim \chi^2_2$.

Meanwhile, $\frac{(n-2)\hat{\sigma}^2}{\sigma^2} \sim \chi^2_{n-2}$, and $\Sigma^{-1} = \frac{1}{\sigma^2}(X^TX) = \frac{1}{\sigma^2}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, $\hat{\sigma}^2 = \frac{(Y-X\hat{\beta})^T(Y-X\hat{\beta})}{(n-p)} = \frac{1}{12}$. Therefore, $\frac{(\hat{\beta}-\beta)^T\Sigma^{-1}(\hat{\beta}-\beta)/2}{\frac{(n-2)\hat{\sigma}^2}{\sigma^2}/(n-2)} \sim F_{(2,1)} \Rightarrow (12\beta_1^2 + 12\beta_2^2 - 24\beta_1 + 6\beta_2 - 12\beta_1\beta_2 + 13) \sim F_{(2,1)}$. Therefore, the confidence region is $(12\beta_1^2 + 12\beta_2^2 - 24\beta_1 + 6\beta_2 - 12\beta_1\beta_2 + 13) \leq F_{(2,1)}(1-\alpha) = 199.5$