UTRECHT UNIVERSITY

Department of Physics

Theoretical Physics master thesis

D-brane gauge theories with spontaneous supersymmetry braking through freely acting orbifolds

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Abstract

In the context of String Theory, freely acting orbifolds have proven to be an effective method of spontaneously breaking supersymmetry (cite). The effects on the spectrum of the closed string in type IIB String Theory have been studied in detail in (cite), and this thesis aims to explore the effects of the SUSY breaking in the open string spectrum. Here we first show how the open string spectrum is affected in general by the orbifold action, and we calculate the full orbifold projection on a specific example of D1/D5 brane system. This system is closely linked to black hole solutions of the low energy supergravity, and in the last section we give predictions as to how the orbifold projection acts on the low energy worldvolume CFT and thus the black hole theormodynamics in the system with broken supersymmetry.

Contents

1. Introduction

Physics aims to describe the dynamics between all the fundamental constutuents of nature. In the one hand, we can use Quantum Field Theory to describe Particle Physics, and in the other we can use General Relativity to describe astronomical interactions. These two theories are fundamentally different in the sense that the first is a quantum theory, while the second one is not, and the most naive attempts to convert it to quantum language fail fundamentally.

String theory is a paradigm change to the way Particle Physics is built, in the sense that now the fundamental objects are no longer point-like, but extended one dimensional *strings*. Among an impressive list of results that were derived not long after String Theory was invented, the most notable one might be that this fully quantum theory is a theory of gravity. Thus, being a promising candidate for a unifying theory of physics.

One of the issues of String Theory is that in order to have a consistent theory (no tachyons) we need to add supersymmetry in the sense of fermionic excitations of the string. It turns out that the low energy effective theory of this system is Super Gravity (SUGRA), which is the supersymmetric extension of Einstein's General Relativity. But, as we all know, supersymmetry is not actually a low energy symmetry of nature in our universe.

At this point, we can look for ways of breaking the supersymmetry of String Theory. The one considered for this thesis is compactification by freely acting orbifolds. In essence, there is a discrete symmetry in the compact dimensions that gets quotiented away, projecting a part of the spectrum that has a nontrivial group charge.

String Theories in orbifolded backgrounds have been studied extensively (cite), with a focus on the closed string spectrum and the low energy SUGRA (cite). It was discovered that the orbifold projection can be regarded as a

Higgs-like mechanism for the charged fields.

In this thesis we will focus on the open string spectrum, which has not yet beef fully understood in orbifold backgrounds. A full description of the spectrum will be given in a general orbifold for some specific examples, namely the D1/D5 system.

The main goal is to calculate the central charge of the effective CFT in the infrared (IR) of this D1/D5 system, that is dual to a certain black hole solution of the corresponding SUGRA. This process is still not well understood but a prediction can be made based on the projections of the spectrum.

1.1 Outline

This thesis will be organized as follows. In Chapter 2 we will briefly describe general aspects of String Theory relevant for delevoping the later calculations. In Chapter 3 we will describe the massless spectrum of type IIB String Theory from a group theoretical point of view. In Chapter 4 we will use the group theoretical description to understand how the orbifold modifies the spectrum and thus breaks supersymmetry.

Lastly, in Chapters 5 and 6 we will give a dynamical description to the spectrum found in previous chapters, with the goal of calculating thermodynamic quantities of the black hole that describes the D1/D5 system in the orbifold background.

1.2 Conventions

2. Preliminaries

In this chapter we will present some basic concepts to give a foundation to this thesis. Only the necessary steps will be presented in order to give context for future chapters. For a more extensive review on the topic of Superstring Theory, the reader can refer to [1], [2].

2.1 Type IIB string theory

String Theory is, in its most simple realization, one of the most straight forward generalizations of the quantum theory behind Particle Physics. Instead of the fundamental object being a particle, which classically traces a 1D world-line when it propagates through space-time, a string describes a 2D world-sheet. This world-sheet is parametrized by coordinates $\sigma^\alpha = (\sigma, \tau)$, through the embeddings $X^\mu(\sigma, \tau)$.

Superstring theory, as the name implies, also has fermionic degrees of freedom, that give rise to world-sheet SUSY ψ^μ and $\tilde{\psi}^\mu$. This world-sheet SUSY, as it turns out, gives rise to space-time SUSY when the String Theory is treated with carefully. This procedure is known as the GSO projection, and will be an integral part in future chapters.

Consider the following action,

$$S = \frac{1}{4\pi} \int_{\mathcal{M}} d^2z \left\{ \frac{1}{\alpha'} \partial X^\mu \bar{\partial} X_\mu + \psi^\mu \bar{\partial} \psi_\mu + \tilde{\psi}^\mu \partial \tilde{\psi}_\mu \right\}$$

with $\mathcal{M}$ being a complex cylinder, $z \in \mathbb{C}$, $z + 2\pi \sim z$.

We can start by solving the classical equations of motion, which can be read as $\partial\bar{\partial} X^\mu = \bar{\partial}\psi^\mu = \partial\tilde{\psi}^\mu = 0$. These mean the fields can be written in terms of holomorphic and anti-holomorphic functions as follows,

$$
\begin{aligned}
X^{\mu} &= X_L^{\mu}(z) + X_R^{\mu}(\bar{z}) \\
\psi^{\mu} &= \psi^{\mu}(z) \\
\tilde{\psi}^{\mu} &= \tilde{\psi}^{\mu}(\bar{z})
\end{aligned}
$$

Now, to find a mode expansion we have to impose boundary conditions for these fields. In order to find the closed string spectrum, we impose the following periodicity conditions. For the bosonic field these are $X^{\mu}(z+2\pi) = X^{\mu}(z)$, while the fermions can close up to a $\pm$ sign. This allows for two sectors in the spectrum, called Rammond (R) and Naveu-Schwarz (NS), given by the periodicity conditions,

$$
\begin{aligned}
\mathrm{R} : \psi^{\mu}(z+2\pi) &= +\psi^{\mu}(z) \\
\mathrm{NS} : \psi^{\mu}(z+2\pi) &= -\psi^{\mu}(z)
\end{aligned}
$$

and the same for $\tilde{\psi}^{\mu}$. Al fields can then be expressed in terms of Fourier modes. The bosonic modes will be called a_n^{μ}, $\tilde{a}_n^{\mu}$, $n \in \mathbb{Z}$, while the fermionic modes are b_r^{μ}, $\tilde{b}_r^{\mu}$, with r being in $\mathbb{Z}$ in the NS sector, or in $\mathbb{Z}+1/2$ in the R sector.

Focusing on one half of the spectrum, say the left moving spectrum, we identify the NS vacuum to be a space-time scalar $|0\rangle_R$, while the R vacuum is degenerate under the action of b_0^{μ}. The zero-modes of the R sector generate the $D = 10$ Clifford algebra, so the R vacuum is characterized by a spinor, $|a\rangle_R$. This spinor can be characterized by $SO(2)$ eigenvalues $|s_0, s_1, s_2, s_3, s_4\rangle$, $s_i = \pm 1/2$, forming a 32 dimensional complex space. This is a Dirac spinor

32 of $SO(1,9)$.

The closed string spectrum will consist of both the right and left moving spectrum along with the level matching condition. Focusing again on the right part of the spectrum, we can find the following mass formula,

$$M^2 = \frac{1}{\alpha'}\left(\sum_{n,r} a{-}n \cdot a_n + rb_{-r} \cdot b_r - a\right) \tag{2.1}$$

whith $a = -1/2$ in the NS sector and $a = 0$ in the R sector. We can extract useful information from this. First of all, there is a tachyon $|0\rangle_R$ with mass $M^2 = -1/2\alpha'$. We will assign fermion number $(-1)^F = -1$ to this state. The only massless states are then $|a\rangle_R$ and $b^{\mu}_{-1/2}|0\rangle_R$. The vector $b^{\mu}_{-1/2}|0\rangle_R$ has $(-1)^F = 1$, while for the fermion $\mathbf{32} = \mathbf{16}_s \oplus \mathbf{16}_c$[1] we have a choice. We can choose the value of $(-1)^F$ for one chirality, and the other will automatically adopt the oposite one.

All this discussion of fermion numbers is related to the GSO projection. In order to make the theory well defined, we require the spectrum to only contain states that have an even fermion number. Thus, the tachyon is projected, and at the massless level, only one chirality of the fermion remains. The same reasoning follows for the left moving spectrum, leading to the two different type II Superstring Theories. Type IIA has both chiralities in the spectrum, while IIB has only one (conventionally the $\mathbf{16}_s$). To write out the spectrum, it is common to go to lightcone gauge, in which the massless excitations will form representations of the little group of $SO(1,9)$, namely, $SO(8)$. The massless spectrum of type IIB string theory can then be written as,

$$(\mathbf{8}_v, \mathbf{8}_v) \oplus (\mathbf{8}_v, \mathbf{8}_s) \oplus (\mathbf{8}_s, \mathbf{8}_v) \oplus (\mathbf{8}_s, \mathbf{8}_s)$$

[1]This is the typical decomposition into Weyl spinors, for details refer to Appendix A

The takeaway from this short discussion should be the halving of the spectrum via GSO projection, and the description of the spectrum of type IIB. In future chapters, GSO projection will be used to calculate the spectrum of the D1/D5 brane system.

2.2 Orbifolds

Intuitively, orbifolds can be understood as a generalization or manifolds where we allow for the existence of singular points. Mathematically, they are defined by a differentiable manifold $\mathcal{M}$, and a symmetry group G inside $\mathcal{M}$. Then the quotient $\mathcal{M}/G$ is an orbifold.

As an example, we can consider $\mathbb{R}^2$, whith reflections $(x, y) \to (-x, -y)$ identified. This is a realization of the orbifold $\mathbb{R}^2/\mathbb{Z}_2$. This particular orbifold has a conical singularity at the fixed point at the origin, so the orbifold is actually not a manifold.

In what will follow we will consider freely acting orbifolds, which are orbifolds by construction as a quotient between a manifold and a symmetry group, but don't have any fixed points. The resulting structure then turns out to be a manifold again. In relation to string theory, freely acting orbifolds are interesting because

2.3 D-branes

When we introduced String Theory, we assumed periodic boundary conditions to represent a closed string propagating through spacetime. But this choice could be extended to non-periodic boundary conditions. Strings can indeed end on hypersurfaces that are called D-branes.

Consider the bosonic part coming from the world-sheet action given by equation 2.1. Instead of imposing periodic boundary conditions, we can instead consider the string endings fixed on a surface of dimeension p, so that $\partial_\sigma X^a(\sigma, \tau)|_{\sigma=0,\pi} = 0$, $\partial_\tau X^i(\sigma, \tau)|_{\sigma=0,\pi} = 0$, with $a = 0, ...p$, $i = p + 1, ..., D$.

If we express the coordinates in terms of the modes a_n^μ, the Neumann and Dirichlet conditions imply a relation between left and right moving modes as $a_n^a = \overline{a}_n^a$ and $a_n^i = -\overline{a}_n^i$. The spectrum then, can be constructed from the set of, for instance, left moving oscillators, as the right moving degrees of freedom are related by the boundary conditions.

The same procedure can be performed to the fermionic oscillations, and we find the same conclusion. The take away from the splitting into Neumann+Dirichlet boundary conditions, is that the full Lorentz group is now not a symmetry of the spectrum, because rotations between a and i indices will mix different boundary conditions. The splitting can be schematically represented as $SO(1,9) \to SO(1,p) \times SO(9-p)$. In terms of the representation theory of the spectrum, in the closed string we found that the stated arranged themselves into representations of the Lorentz group $SO(1,9)$, but now in turn, they will be arranged into representations of the subgroups mentioned.

To work a quick example, we can think of a D9 brane covering all directions of the background space. In this case, the open string spectrum is just the left moving part of what we presented as the closed string spectrum. The massless spectrum is then given by the states $b^\mu_{-1/2}|0\rangle$ and the Rammond vacuum $|a\rangle$, which in lightcone gauge they fall into the $\mathbf{8}_v$ and $\mathbf{8}_s$ of $SO(1,9)$.

All the rest of the single Dp-brane spectrums can be computed from the D9 spectrum by dimensional reduction. The procedure is best understood from the vector. The index $\mu = 0, ..., 9$ splits into two indices $a = 0, ..., p$ and $i = p+1, ..., 10$, so that a vector $b^\mu_{-1/2}|0\rangle \to b^a{}_{-1/2} \otimes b^i{}_{-1/2}|0\rangle$. The procedure for spinor representations is outlined in detail in Appendix A. In summary, if we wanted to calculate the D5 spectrum with the data from the D9 spectrum, the dimensional reduction would go as follows,

$$
\begin{aligned}
SO(1,9) &\to SO(1,5) \times SO(4) \\
\mathbf{10} &\to (\mathbf{6},\mathbf{1}) \oplus (\mathbf{1},\mathbf{4}) \\
\mathbf{16}_s &\to (\mathbf{4}_s,\mathbf{2}_s) \oplus (\mathbf{4}_c,\mathbf{2}_c)
\end{aligned}
$$

2.4 Supersymmetry

Esto es una prueba de compilación

3. Open string spectrum

In this chapter we will describe the spectrum of D-brane systems in the context of type IIB String Theory. We start by discussing single brane spectrums, and then move on to general brane configurations, to conclude with the main example of this thesis, the D1/D5 brane system.

3.1 Dp-Dp spectrum

Consider a single D9-brane. This equates to considering Neumann boundary conditions in all directions of a string. The NS vacuum is a scalar $|0\rangle$, while the R sector is a spinor $|a\rangle$, given by $SO(2)$ eigenvalues s_0, s_1, s_2, s_3, s_4. The massless spectrum can be found from the mass formula $\alpha' M^2 = N + 1/2$. This leads to the modes $b^{\mu}_{-1/2}|0\rangle$ and $|a\rangle$.

The representation theory of the massless spectrum turns out to be straight forward. There is a vector and a fermion in D = 10, composed by $b^{\mu}_{-1/2}|0\rangle$ and $|a\rangle$ respectively.

To obtain the spectrum of an arbitrary Dp-brane we can perform dimensional reduction over the D9 spectrum we already constructed. By dimensional reduction we mean splitting the D9 symmetry group $SO(1,9)$ into the transverse and world-volume symmetry groups of the lower dimensional Dp-brane, namely, $SO(1,p) \times SO(9-p)$.

Starting with a vector in the $\mathbf{10}$ of $SO(1,9)$, it decomposes into a $(\mathbf{p},\mathbf{1}) \oplus (\mathbf{1},\mathbf{9}-\mathbf{p})$ of $SO(1,p) \times SO(9-p)$. The original R vacua was a Majorana-Weyl spinor of $SO(1,9)$, and depending on p it will decompose into the corresponding irreducible spinors of $SO(1,p) \times SO(9-p)$. The dimensional reduction of spinors is discussed in detail in A.

This is already the full spectrum of a single Dp-brane. Extending it to a stack of Q_p Dp-branes adds a Chan-Paton factor to both string ends, which labels the adjoint representation of $U(Q_p)$.

We can count on-shell degrees of freedom to make sure the spectrum is supersymmetric. For a explicit example, we can consider a single D5-brane. By on-shell we refer to adopting light-cone gauge, meaning that $SO(1,5) \to SO(4)$, effectively identifying 2 degrees of freedom for the vector, and specifying a s_0 eigenvalue for the spinor.

$$\begin{aligned}
(\mathbf{6},\mathbf{1}) &\to (\mathbf{4},\mathbf{1} \\
(\mathbf{1},\mathbf{4}) &\to (\mathbf{1},\mathbf{4}) \\
(\mathbf{4}_s,\mathbf{2}_s) &\to (\mathbf{2}_s,\mathbf{2}_s) \\
(\mathbf{4}_c,\mathbf{2}_c) &\to (\mathbf{2}_c,\mathbf{2}_c)
\end{aligned}$$

Now if we count degrees of freedom in the right side, we find that there are 8 matching bosonic and fermionic degrees of freedom, meaning that there can be 8 on-shell supercharges in the theory. A general discussion can be made for any p to find the same 8 possible on-shell supercharges.

3.2 Dp-Dp′ spectrum

Now consider two D branes of arbitrary dimensions p, p'. In general, some of the directions will have the same boundary conditions, i.e. DD or NN, while some will have mixed boundary conditions, i.e. DN or ND. This is illustrated in Figure 3.1.

As we did for the single D brane, the boundary conditions will lead to a relation between left and right moving modes.

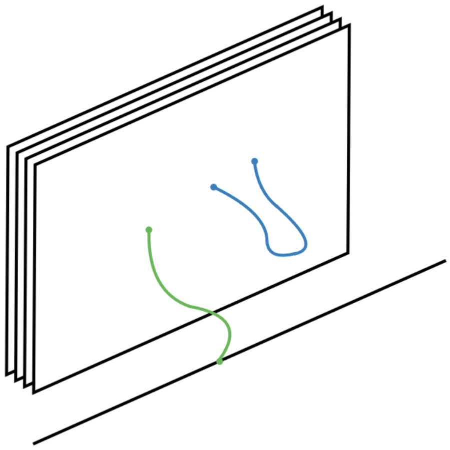

Figure 3.1: Schematic representation of a stack of Dp branes and a single Dp′ brane with strings attached. The blue string will have the same boundary conditions in all directions while the green string will have mixed conditions in some. Figure taken from [3].

$$\text{NN: } X^\mu(z,\bar{z}) = x^\mu - i\alpha' p^\mu \ln(z\bar{z}) + i\sqrt{\frac{\alpha'}{2}} \sum_{m\neq 0} \frac{a_m^\mu}{m}\left(z^{-m} + \bar{z}^{-m}\right) \tag{3.1}$$

$$\text{DN, ND: } X^\mu(z,\bar{z}) = i\sqrt{\frac{\alpha'}{2}} \sum_{r\in\mathbb{Z}+1/2} \frac{a_r^\mu}{r}\left(z^{-r} \pm \bar{z}^{-r}\right) \tag{3.2}$$

$$\text{DD: } \quad X^\mu(z,\bar{z}) = -i\frac{\delta X^\mu}{2\pi} \ln(z/\bar{z}) + i\sqrt{\frac{\alpha'}{2}} \sum_{m\neq 0} \frac{a_m^\mu}{m}\left(z^{-m} - \bar{z}^{-m}\right) \tag{3.3}$$

Notice that the moding for the ND/DN directions gets shifted by 1/2 to accomodate the mixed boundary conditions. This property is also apparent on the fermions

Once again, we aim to calculate the massless spectrum. Thus, we want to find the zero point energy in both the R and the NS sector. As per usual, the R vacuum is massless, while the zero point energy for the NS sector is,

$$-\frac{1}{2} + \frac{\nu}{8} \tag{3.4}$$

3.3 D1/D5 spectrum

The D1/D5 brane system will be the main example treated in this thesis. It deserves a propper introduction, so first of all let's define it in detail. Consider a stack of Q_5 D5 branes covering the 5, 6, 7, 8, 9 directions. Next, consider a stack of Q_1 D1 branes laying inside the D5 stack in the 5 direction. Table 3.1 gives a summary of what was described.

	0	1	2	3	4	5	6	7	8	9
D1	-	.	.	.	.	-	.	.	.	.
D5	-	.	.	.	.	-	-	-	-	-

Table 3.1: Schematic representation of the D1/D5 system.

As we can see, the number of mixed directions is $\nu = 4$, so from equation 3.4, the zero point energy of the NS sector of 1-5 and 5-1 string vanishes. Then, the massless spectrum of excitations of these strings will be both the R vacuum and the NS vacuum. Take for instance a 1-5 string, meaning that the endpoint at $\sigma = 0$ lays in the D1 brane, while the one at $\sigma = \pi$ lays in the D5 brane. The bassless modes will be generated by,

$$\text{NS: } b_0^i, \quad i = 6,7,8,9 \tag{3.5}$$

$$\text{R: } b_0^M, \quad M = 0,1,2,3,4,5 \tag{3.6}$$

Both of these sectors generate Clifford algebras separately of the respective subgroups of $SO(1,9)$. At first glance, we could think the important subgroup is $SO(1,5) \times SO(4) \subset SO(1,9)$, but as it was mentionied previously, we want to focus on the theory on the intersection world-volume, so we want to look at the subgroup $SO(1,1) \times SO(4)_I \times SO(4)_E \subset SO(1,9)$, where the subindices indicate the directions 1, 2, 3, 4 for I and 6, 7, 8, 9 for E.

According to this splitting, the Clifford algebras of Equations 3.5 generate spinors of $SO(1,5)$ and $SO(4)_I$ respectively. The $SO(1,5)$ spinor then splits into various $SO(1,1) \times SO(4)_E$ spinors.

Let's consider the R vacuum given by the $SO(1,5)$ spinor $|s_0, s_1, s_2\rangle$. The GSO projection picks one $SO(1,5)$ chirality, and in this case we decide it picks the $\mathbf{4}_s$.

We pick a basis where the first eigenvalue s_0 is associated to the 0, 5 directions, while the other are associated to the 1, 2, 3, 4 directions. This is convenient because the spinor splits as $|s_0\rangle \otimes |s_1, s_2\rangle$ under $SO(1,9) \to SO(1,1) \times SO(4)_E$. Now, counting chiralities in $SO(1,1)$ and $SO(4)_E$ we can see that this spinor fills the representations $(\mathbf{1}_s, \mathbf{2}_s) \oplus (\mathbf{1}_c, \mathbf{2}_c)$.

A similar, albeit easier calculation, shows that the NS vacuum is characterized by a $SO(4)_I$ spinor in the $\mathbf{2}_s$.

At last, the spectrum of the 1-5 string can be obtained by noticing that in the NS sector all states are $SO(4)_I$ singlets, while in the R sector they are $SO(1,1) \times SO(4)_E$ singlets, i.e,

$$\text{NS: } (\mathbf{1}_s, \mathbf{2}_s, \mathbf{1}) \oplus (\mathbf{1}_c, \mathbf{2}_c, \mathbf{1}) \tag{3.7}$$

$$\text{R: } (\mathbf{1}, \mathbf{1}, \mathbf{2}_s) \tag{3.8}$$

Note that the 5-1 strings will produce the same zero modes, so that the vacuum will be identical.

The only missing piece at this point it the 1-1 and 5-5 massless spectrums. We already know that these come from the dimensional reduction of the $\mathbf{10}$ and the $\mathbf{16}_s$ of $SO(1,9)$. It is straight forward to see that the vector decomposes as $(\mathbf{2}, \mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{4}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{1}, \mathbf{4})$. The procedure to decompose the spinor is very similar to what we did with the $\mathbf{4}_s$ of $SO(1,5)$ previously. Now we need to split the $\mathbf{16}_s$ via the eigenvalues s_0 in $SO(1,1)$, s_1, s_2 in $SO(4)_E$ and s_3, s_4 in $SO(4)_I$. Again, counting chiralities we end up with,

$$\mathbf{16}_s \to (\mathbf{1}_s, \mathbf{2}_s, \mathbf{2}_s) \oplus (\mathbf{1}_s, \mathbf{2}_c, \mathbf{2}_c) \oplus (\mathbf{1}_c, \mathbf{2}_c, \mathbf{2}_s) \oplus (\mathbf{1}_c, \mathbf{2}_s, \mathbf{2}_c)$$

4. Orbifolds

4.1 Orbifold compactification

4.2 Orbifold group action on the spectrum

Representations of the rotation group -> action by the orbifold group (discrete SO(4) rotations). So non-trivial charges.

5. D-brane gauge theories

5.1 Gauge theory on a single brane and dimensional reduction

5.2 Gauge theory of the D1/D5 system

5.3 Coulomb and Higgs branch

6. Infrared limit and Black Hole thermodynamics

6.1 IR SCFT = black hole theormodynamics

6.2 Predictions of IR limit in orbifold context

7. Conclusions

Appendices

A. Spinors in various dimensions

In this appendix we will juntify some claims about spinors in general even dimensions that are used throughout the thesis.

It is well known that the Dirac representation is not irreducible in even dimensions, in which a chirality projection exists into the two different Weyl basis. There is also always a Majorana condition that can induces a real structure in the spinor spaces of all dimensions, but is only truncates the degrees of freedom of Weyl spinors in some dimensions. All these topics will be covered in the following sections in detail with the main objective of describing irreducible spinors in all even dimensions $D \leq 10$.

A.1 Weyl Spinors in $D = 2, 4, 6, 8, 10$

We start with the Clifford algebra,

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}, \tag{A.1}$$

and let the metric be $\eta^{\mu\nu} = \text{diag}\,(-1, 1, \ldots, 1)$ the $D = 2k + 2$ dimensional Minkowski metric.

We first make a change of basis for the algebra elements so that the fundamental representation can be directly extracted from the algebra. Take the following linear combinations,

$$\Gamma^{0\pm} = \frac{1}{2}\left(\pm\Gamma^0 + \Gamma^1\right) \tag{A.2}$$

$$\Gamma^{a\pm} = \frac{1}{2}\left(\Gamma^{2a} \pm i\Gamma^{2a+1}\right), \quad a = 1, \ldots, k \tag{A.3}$$

Now the Clifford algebra A.1 can be stated in terms of the new operators as,

$$\begin{aligned} &\left\{\Gamma^{a+}, \Gamma^{b-}\right\} = \delta^{ab} \\ &\left\{\Gamma^{a+}, \Gamma^{b+}\right\} = \left\{\Gamma^{a-}, \Gamma^{b-}\right\} = 0. \end{aligned} \tag{A.4}$$

We can see that these operators are raising and lowering operators for $k+1$ different eigenvalues. We can write an arbitrary basis element of this representation as $|s_0, \ldots, s_k\rangle$, with $s_i = \pm 1$. An arbitrary spinor in this representation is then in the complex span of this basis, which has 2^{k+1} complex components. This is what is called a Dirac spinor, or $\mathbf{2^{k+1}}_{\text{Dirac}}$.

These eigenvalues s_i are actually eigenvalues of rotations in planes given by the grouping of A.2. To see this explicitly, we need to recover the Lorentz algebra from the Clifford algebra. Define,

$$\Sigma^{\mu v} = -\frac{i}{4}\left[\Gamma^\mu, \Gamma^v\right] \tag{A.5}$$

The elements $\Sigma^{\mu v}$ define the Lorentz algebra os $SO(1, 2k+1)$. The generators $\Sigma^{2a,2a+1}$ commute and have eigenvalues proportional to s_a when acting on the basis element $|s_0, \ldots, s_k\rangle$. To be precise, the operator,

$$S_a \equiv i^{\delta_{a,0}} \Sigma^{2a,2a+1} = \Gamma^{a+}\Gamma^{a-} - \frac{1}{2} \tag{A.6}$$

has eigenvalue s_a.

Next, we can define the chirality operator,

$$\Gamma = i^{-k}\Gamma^0\Gamma^1 \dots \Gamma^{d-1} \tag{A.7}$$

which has the properties,

$$(\Gamma)^2 = 1, \quad \{\Gamma, \Gamma^\mu\} = 0, \quad [\Gamma, \Sigma^{\mu v}] = 0 \tag{A.8}$$

From the first property, we see that Γ has eigenvalues ± 1. From the rest we see that we can split the basis $|s_0, \dots, s_k\rangle$ into two subspaces according to the eigenvalues of Γ. We can rewrite the chirality operator in terms of the rotation generators S_a as follows,

$$\Gamma = 2^{k+1} S_0 S_1 \dots S_k \tag{A.9}$$

which allows us to identify the two chiralities as $+1$ when the product of the s_a is positive and -1 when it is negative. The two subspaces defined by the chirality operator are called Weyl representations and are labeled as $\mathbf{2^k}_s$ and $\mathbf{2^k}_c$ for the $+1$ and -1 subspaces respectively. So we can finally state that the dirac representation splits into Weyl representations as,

$$\mathbf{2^{k+1}}_{\text{Dirac}} = \mathbf{2^k}_s \oplus \mathbf{2^k}_c \tag{A.10}$$

A.2 Majorana condition

A.3 Table of irreducible spinors in even dimensions

Bibliography

[1] J. Polchinski, *String theory I-II. volume II: Superstring theory and beyond*. Cambridge University Press, 1998.

[2] C. V. Johnson, *D-branes*. Cambridge University Press, 2023.

[3] K. C. Stemerdink, "Black Holes from Branes: Various string theoretical constructions," Ph.D. dissertation, 2022. DOI: `10.33540/1379`.