INTRODUCTION TO LANGUAGE THEORY AND COMPILING

Basic definitions: alphabet, word, language

Definition 1.4 (Alphabet)**.** An *alphabet* is a *finite* set of symbols. We will usually denote alphabets by Σ.

Definition 1.6 (Word)**.** A *word* on an alphabet Σ is a *finite* (and possibly empty) sequence of symbols from Σ. We use the symbol ε to denote the *empty word*, i.e., the empty sequence (that contains no symbol).

Definition 1.8 (Language)**.** A *language* on an alphabet Σ is a (possibly empty or infinite) set of words on Σ.

The Σ^* notation

Let Σ be an alphabet. Since any alphabet is a set, we can also regard Σ as a *language*, which contains only words of one character. Then, we can write Σ^*, which contains *all the words (including the empty one) that are made up of characters from* Σ. This notation will be used very often in the rest of these notes.

Operations on words and languages

Definition 1.17 (Concatenation of two words)**.** Given two words $w = w_1 w_2 \cdots w_n$ and $v = v_1 v_2 \cdots v_\ell$, the *concatenation* of w and v, denoted $w \cdot v$, is the word:

$$w \cdot v = w_1 w_2 \cdots w_n v_1 v_2 \cdots v_\ell$$

By convention, $\varepsilon \cdot w = w \cdot \varepsilon = w$, for all words w. In particular $\varepsilon \cdot \varepsilon = \varepsilon$.

1. For all languages L, for all natural numbers n, L^n is the language containing all words obtained by taking n words from L an concatenating them:

$$L^n = \{w_1 w_2 \cdots w_n \mid \text{for all } 1 \leq i \leq n : w_i \in L\}$$

2. For all languages L, the *Kleene closure* of L, denote L^* is the language containing all words made up of an arbitrary number of concatenations of words from L:

$$L^* = \{w_1 w_2 \cdots w_n \mid n \geq 0 \text{ and for all } 1 \leq i \leq n : w_i \in L\}$$

3. A variation on the Kleene closure is L^+ which is the language containing all words made up of an arbitrary and *strictly positive* number of concatenations of words from L:

$$L^+ \quad = \{w_1 w_2 \cdots w_n \mid n \geq 1 \text{ and for all } 1 \leq i \leq n : w_i \in L\}$$

Regular languages

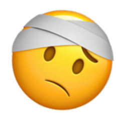
Definition 2.1 (Regular languages)**.** Let us fix an alphabet Σ. Then, a language L is regular iff:

a finite number of times

1. either $L = \emptyset$;
2. or $L = \{\varepsilon\}$;
3. or $L = \{a\}$ for some $a \in \Sigma$;
4. or $L = L_1 \cup L_2$;
5. or $L = L_1 \cdot L_2$;
6. or $L = L_1^*$

where L_1 and L_2 are regular languages on Σ.

Using the same ideas, we can prove that all finite languages are regular.



Definition 2.3 (Regular expressions)**.** Given a finite alphabet Σ, the following are regular expressions on Σ:

1. The constant $\emptyset$. It denotes the language $L(\emptyset) = \emptyset$.

2. The constant ε. It denotes the language $L(\varepsilon) = \{\varepsilon\}$.

3. All constants $a \in \Sigma$. Each constant $a \in \Sigma$ denotes the language $L(a) = \{a\}$.

4. All expressions of the form $r_1 + r_2$, where r_1 and r_2 are regular expressions on Σ. Each expression $r_1 + r_2$ denotes the language $L(r_1 + r_2) = L(r_1) \cup L(r_2)$.

5. All expressions of the form $r_1 \cdot r_2$, where r_1 and r_2 are regular expressions on Σ. Each expression $r_1 \cdot r_2$ denotes the language $L(r_1 \cdot r_2) = L(r_1) \cdot L(r_2)$.

6. All expressions of the form r^*, where r is a regular expression on Σ. Each expression r^* denotes the language $L(r^*) = \big(L(r)\big)^*$.

In addition, parenthesis are allowed in regular expressions to group subexpressions (with their usual semantics).

Theorem 2.1. *For all regular languages L, there is a regular expression r s.t. $L(r) = L$. For all regular expressions r, $L(r)$ is a regular language.*

Definition 2.5 (Finite automaton). A finite automaton is a tuple:

$$A = \langle Q, \Sigma, \delta, q_0, F \rangle$$

where:

1. Q is a finite set of states;
2. Σ is the (finite) input alphabet;
3. $\delta : Q \times (\Sigma \cup \{\varepsilon\}) \mapsto 2^Q$ is the transition function;
4. $q_0 \in Q$ is the initial state;
5. $F \subseteq Q$ is the set of accepting states.