Graph Theory

The Basics

The degree of a vertex

$\delta(G) := \min\{\, d(v) \mid v \in V \,\}$ is the *minimum degree* of G, the number $\Delta(G) := \max\{\, d(v) \mid v \in V \,\}$ its *maximum degree*.

Proposition 1.3.1. *Every graph G contains a path of length $\delta(G)$*

Proof. Let $x_0 \ldots x_k$ be a longest path in G. Then all the neighbours of x_k lie on this path (Fig. 1.3.4). Hence $k \geqslant d(x_k) \geqslant \delta(G)$. If $i < k$ is minimal with $x_i x_k \in E(G)$, then $x_i \ldots x_k x_i$ is a cycle of length at least $\delta(G)+1$. □

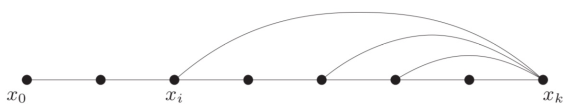


Fig. 1.3.4. A longest path $x_0 \ldots x_k$, and the neighbours of x_k

A seperator in $G(V,E)$ is $S \subseteq V$ s.t. $G - S$ is disconnected

$\delta(G) \geq |\min S|$

Trees and forests

Lemma 1. *Every tree T on at least 2 vertices has a leaf.*

Proof. Assume this is not the case. Then all the vertices of the tree have degree at least 2. In this case we can find a cycle in T greedily by traversing the tree until we visit the same vertex twice. This will happen because T is finite. □

Lemma 2. *Any tree T contains exactly $|V(T)| - 1$ edges.*

Proof. We show the claim by induction on $|V|$. The base $|V| = 1$ is easy to check to be true. Let ℓ be a leaf in T. The vertex ℓ exists by the previous lemma. Remove ℓ from T and let T' be the resulting graph. The graph T' is connected and acyclic and therefore a tree. Hence by the induction hypothesis $|E(T')| = |V(T')| - 1$ and therefore $|E(T)| = |V(T)| - 1$. □

Lemma 3. *Any tree $T = (V, E)$ contains a 1/2-balanced separator of size* 1.

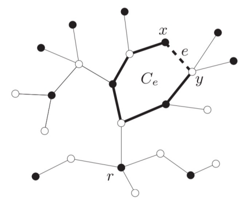
An α-balanced seperator is $S \subseteq V(G)$ s.t. every connected component in $G-S$ contains $\leq \alpha |V(G)|$ vertices

Proof. We find the required separator greedily. Let v be any vertex of T. Check if every connected component of $T - \{v\}$ is of size at most $1/2|V|$. If this is the case, we are done. If this is not the case, let u be a neighbor of v in the unique component of size greater than $1/2|V|$ in $T - \{v\}$. Repeat for u the same steps as for v. The algorithm stops as T is finite and we never go back to a vertex we visited. □

Bipartite graphs

Proposition 1.6.1. *A graph is bipartite if and only if it contains no odd cycle.*

Proof. Let $G = (V, E)$ be a graph without odd cycles; we show that G is bipartite. Clearly a graph is bipartite if all its components are bipartite or trivial, so we may assume that G is connected. Let T be a spanning tree in G, pick a root $r \in T$, and denote the associated tree-order on V by $\leqslant_T$. For each $v \in V$, the unique path rTv has odd or even length. This defines a bipartition of V; we show that G is bipartite with this partition.



Let $e = xy$ be an edge of G. If $e \in T$, with $x <_T y$ say, then $rTy = rTxy$ and so x and y lie in different partition classes. If $e \notin T$ then $C_e := xTy + e$ is a cycle (Fig. 1.6.3), and by the case treated already the vertices along xTy alternate between the two classes. Since C_e is even by assumption, x and y again lie in different classes. $\square$

Matching in bipartite graphs

Theorem 2.1.2. (Hall 1935)
G contains a matching of A if and only if $|N(S)| \geqslant |S|$ for all $S \subseteq A$.