Introduction to cryptography

3. Hashing

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Cryptographic hash functions

$$h : \{0,1\}^* \rightarrow \{0,1\}^n$$

Input message →  → Digest

- Applications
  - *Signatures*: $\text{sign}_{\text{RSA}}(h(M))$ instead of $\text{sign}_{\text{RSA}}(M)$
  - *Key derivation*: master key K to derived keys $(K_i = h(K\|i))$
  - *Bit commitment, predictions*: $h(\text{what I know})$
  - *Message authentication*: $h(K\|M)$
  - ...

Generalized: extendable output function (XOF)

$$h \; : \; \{0,1\}^* \rightarrow \{0,1\}^\infty$$

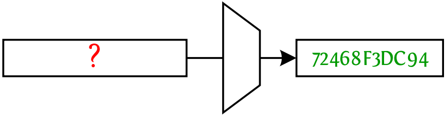
"XOF: a function in which the output can be extended to any length."

[Ray Perlner, SHA 3 workshop 2014]

- Applications
  - *Signatures*: full-domain hashing, mask generating function
  - *Key derivation*: as many/long derived keys as needed
  - *Stream cipher*: $C = P \oplus h(K \| \text{nonce})$

Preimage resistance

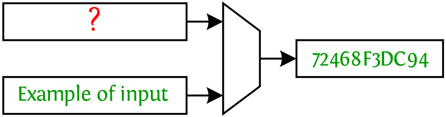
- Given $y \in \mathbf{Z}_2^n$, find $x \in \mathbf{Z}_2^*$ such that $h(x) = y$



- If h is a random function, about 2^n attempts are needed
- **Example**: given derived key $K_1 = h(K\|1)$, find master key K

Second preimage resistance

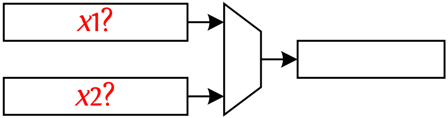
- Given $x \in \mathbf{Z}_2^*$, find $x' \neq x$ such that $h(x') = h(x)$



- If h is a random function, about 2^n attempts are needed
- **Example**: signature forging
  - Given M and $\mathrm{sign}(h(M))$, find $M' \neq M$ with equal signature

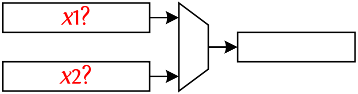
Collision resistance

- Find $x_1 \neq x_2$ such that $h(x_1) = h(x_2)$



- If h is a random function, about $2^{n/2}$ attempts are needed
  - Birthday paradox: among 23 people, probably two have same birthday
  - Scales as $\sqrt{|\text{range}|} = 2^{n/2}$

Collision resistance (continued)



- **Example**: “secretary” signature forging
  - Set of good messages $\{M_i^{\text{good}}\}$
  - Set of bad messages $\{M_i^{\text{bad}}\}$
  - Find $h(M_i^{\text{good}}) = h(M_j^{\text{bad}})$
  - Boss signs M_i^{good}, but valid also for M_j^{bad}

[Yuval, 1979]

Other requirements

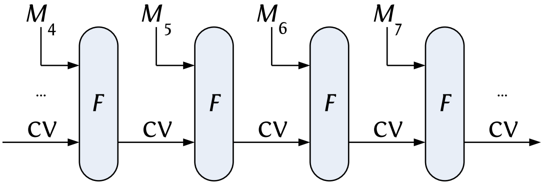
- Security claims by listing desired properties
  - Collision resistant
  - (Second-) preimage resistant
    - Multi-target preimage resistance
    - Chosen-target forced-prefix preimage resistance
  - Correlation-free
  - Resistant against length-extension attacks
  - ...
- But ever-growing list of desired properties
- A good hash function should behave like a **random mapping**...

Security requirements summarized

- Hash or XOF h with n-bit output
- Modern security requirements
  - h behaves like a random mapping
  - ... up to security strength s
- Classical security requirements, derived from it

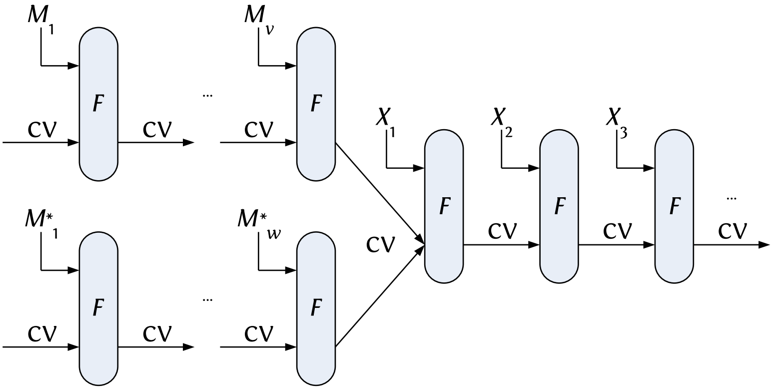
Preimage resistance	$2^{\min(n,s)}$
Second-preimage resistance	$2^{\min(n,s)}$
Collision resistance	$2^{\min(n/2,s)}$

Iterated functions



- All practical hash functions are iterated
  - Message M cut into blocks $M_1, \ldots, M_l$
  - q-bit chaining value
- Output is function of final chaining value

Internal collisions!



- Different inputs M and M^* giving the same chaining value
- Messages $M \| X$ and $M^* \| X$ always collide for any string X

Does not occur in a random mapping!

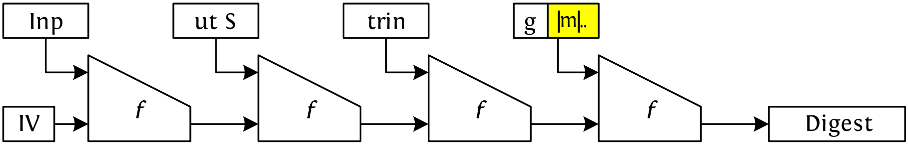
Examples of hash functions

- MD5: $n = 128$
  - Published by Ron Rivest in 1992
  - Successor of MD4 (1990)
- SHA-1: $n = 160$
  - Designed by NSA, standardized by NIST in 1995
  - Successor of SHA-0 (1993)
- SHA-2: family supporting multiple lengths
  - Designed by NSA, standardized by NIST in 2001
  - SHA-224, SHA-256, SHA-384 and SHA-512
- SHA-3: based on KECCAK
  - Designed by Bertoni, Daemen, Peeters and VA in 2008
  - Standardized by NIST in 2015
  - SHA3-{224, 256, 384, 512}, SHAKE{128, 256}, ParallelHash{128, 256}, ...
- Other SHA-3 finalists
  - Blake (Aumasson et al.), Grøstl (Gauravaram et al.), JH (Wu), Skein (Ferguson et al.)

Attacks on MD5, SHA-0 and SHA-1

- 2004: SHA-0 broken (Joux et al.)
- 2004: MD5 broken (Wang et al.)
- 2005: practical attack on MD5
  (Lenstra et al., and Klima)
- 2005: SHA-1 theoretically broken
  (Wang et al.)
- 2006: SHA-1 broken further
  (De Cannière and Rechberger)
- 2016: freestart collision on SHA-1
  (Stevens, Karpman and Peyrin)
- 2017: actual collision on SHA-1
  (Stevens, Bursztein, Karpman, Albertini and Markov)

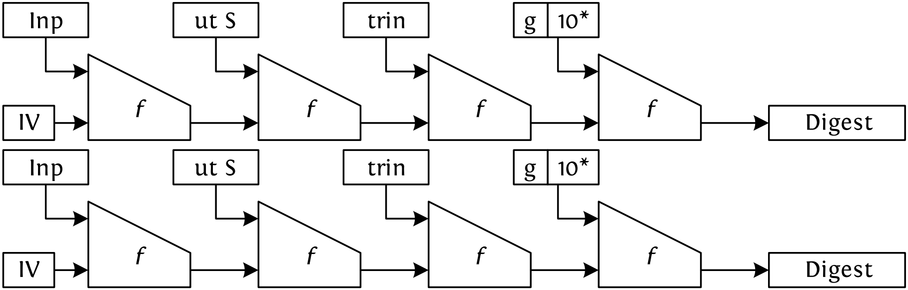
Merkle-Damgård



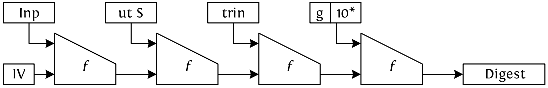
- Uses a compression function from $n + m$ bits to n bits
- Instances: MD5, SHA-1, SHA-2 ...
- Merkle-Damgård strengthening: encode the length at the end

[Merkle, CRYPTO'89], [Damgård, CRYPTO'89]

Merkle-Damgård: preserving collision resistance



Merkle-Damgård: length extension



Recurrence (modulo the padding):

- $h(M_1) = f(\mathrm{IV}, M_1) = \mathrm{CV}_1$
- $h(M_1 \| \ldots \| M_i) = f(\mathrm{CV}_{i-1}, M_i) = \mathrm{CV}_i$

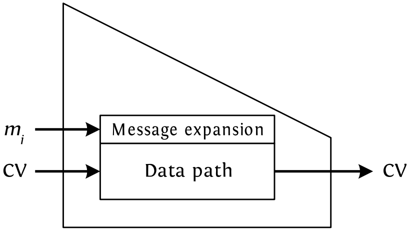
Forgery on naïve message authentication code (MAC):

- $\mathrm{MAC}(M) = h(\mathrm{Key} \| M) = \mathrm{CV}$
- $\mathrm{MAC}(M \| \mathrm{suffix}) = f(\mathrm{CV} \| \mathrm{suffix})$

Solution: **HMAC**

$$\mathrm{HMAC}(M) = h(\mathrm{Key}_{\mathrm{out}} \| h(\mathrm{Key}_{\mathrm{in}} \| M))$$

Towards defining Davies-Meyer

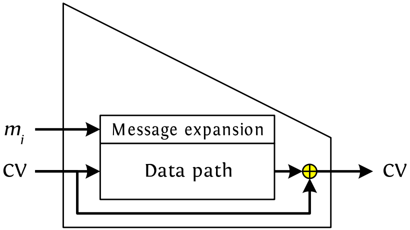


Compression function based on a block cipher $f(\mathrm{CV}, m_i) = E_{m_i}(\mathrm{CV})$:

- the block m_i is input as the BC's "key",
- the old chaining value CV is input as the BC's input block,
- the BC's output block becomes the new chaining value.

Problem: not preimage resistant!
How? Fix m_i, then compute input $\mathrm{CV} = E_{m_i}^{-1}(\text{output CV})$.

Davies-Meyer



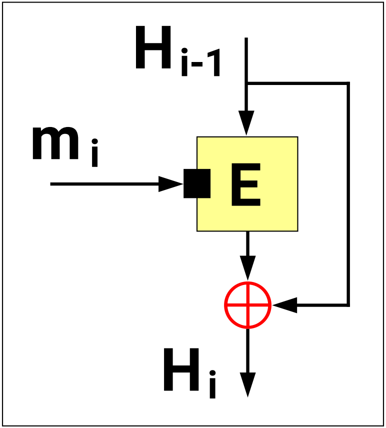
Compression function based on a block cipher $f(\mathrm{CV}, m_i) = E_{m_i}(\mathrm{CV}) \oplus \mathrm{CV}$:

- the block m_i is input as the BC's "key",
- the old chaining value CV is input as the BC's input block,
- the BC's output block is XORed to the old chaining value, yielding the new chaining value.

Problem fixed!

Why? One cannot find a preimage by computing $E_{m_i}^{-1}$, as the output block is unknown.

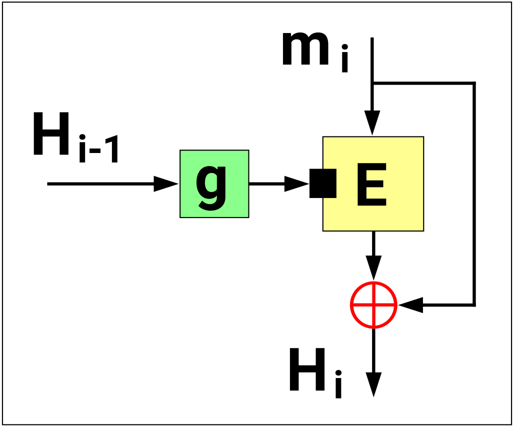
Other constructions using block ciphers (1/3)



Davies-Meyer

[Matyas et al., IBM Tech. D. B., 1985], [Quisquater et al., Eurocrypt'89]

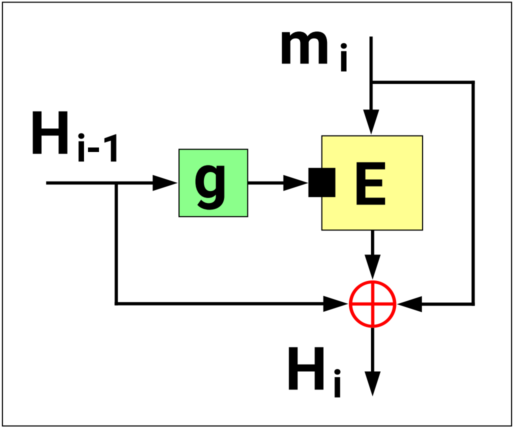
Other constructions using block ciphers (2/3)



Matyas-Meyer-Oseas

[Matyas et al., IBM Tech. D. B., 1985]

Other constructions using block ciphers (3/3)



Miyaguchi-Preneel

[Miyaguchi et al., NTT Rev., 1990], [Preneel, PhD th., 1993]

Inside SHA-1

- Uses Davies-Meyer with
  - data path $n = 160 = 5 \times 32$
  - message expansion $m = 512 = 16 \times 32$
- State initialized with $(A, B, C, D, E) =$

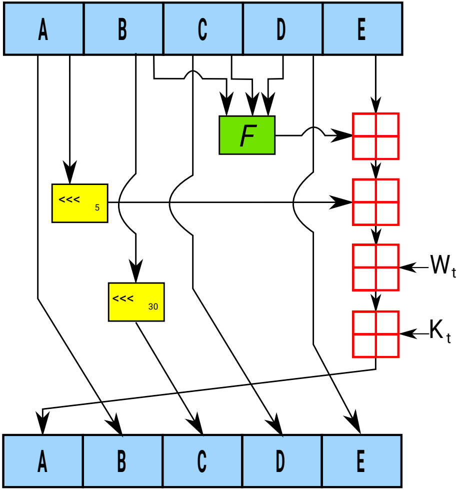
$$(\texttt{67452301}, \texttt{EFCDAB89}, \texttt{98BADCFE}, \texttt{10325476}, \texttt{C3D2E1F0})$$

- Message block $(w_0, \ldots, w_{15})$ expanded as

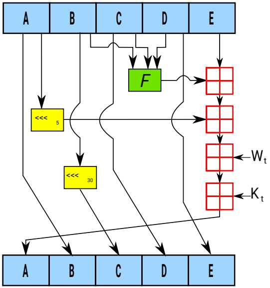
$$w_t = (w_{t-3} \oplus w_{t-8} \oplus w_{t-14} \oplus w_{t-16}) \lll 1 \quad (16 \leq t \leq 79)$$

- Data path with 80 steps...

Inside SHA-1: data path



Inside SHA-1: data path details



$0 \leq t \leq 19$	$f(B,C,D) = (B \odot C) \oplus (\bar{B} \odot D)$	$K_t = 5A827999$
$20 \leq t \leq 39$	$f(B,C,D) = B \oplus C \oplus D$	$K_t = 6ED9EBA1$
$40 \leq t \leq 59$	$f(B,C,D) = (B \odot C) \oplus (B \odot D) \oplus (C \odot D)$	$K_t = 8F1BBCDC$
$60 \leq t \leq 79$	$f(B,C,D) = B \oplus C \oplus D$	$K_t = CA62C1D6$

Collision in SHA-1

- February 23, 2017: first collision on SHA-1 published
- Estimated complexity: $2^{63} \ll 2^{80}$

[Stevens, Bursztein, Karpman, Albertini and Markov]

Collision in SHA-1

$$\text{SHA-1}(P \| M_1^{(1)} \| M_2^{(1)} \| S) = \text{SHA-1}(P \| M_1^{(2)} \| M_2^{(2)} \| S)$$

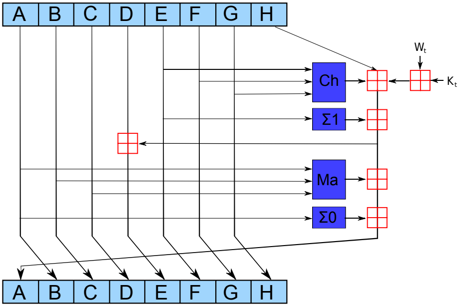
CV_0	4e	a9	62	69	7c	87	6e	26	74	d1	07	f0	fe	c6	79	84	14	f5	bf	45
$M_1^{(1)}$			7f	46	dc	93	a6	b6	7e	01	3b	02	9a	aa	1d	b2	56	0b		
			45	ca	67	d6	88	c7	f8	4b	8c	4c	79	1f	e0	2b	3d	f6		
			14	f8	6d	b1	69	09	01	c5	6b	45	c1	53	0a	fe	df	b7		
			60	38	e9	72	72	2f	e7	ad	72	8f	0e	49	04	e0	46	c2		
$CV_1^{(1)}$	8d	64	d6	17	ff	ed	53	52	eb	c8	59	15	5e	c7	eb	34	f3	8a	5a	7b
$M_2^{(1)}$			30	57	0f	e9	d4	13	98	ab	e1	2e	f5	bc	94	2b	e3	35		
			42	a4	80	2d	98	b5	d7	0f	2a	33	2e	c3	7f	ac	35	14		
			e7	4d	dc	0f	2c	c1	a8	74	cd	0c	78	30	5a	21	56	64		
			61	30	97	89	60	6b	d0	bf	3f	98	cd	a8	04	46	29	a1		
CV_2	1e	ac	b2	5e	d5	97	0d	10	f1	73	69	63	57	71	bc	3a	17	b4	8a	c5

CV_0	4e	a9	62	69	7c	87	6e	26	74	d1	07	f0	fe	c6	79	84	14	f5	bf	45
$M_1^{(2)}$			73	46	dc	91	66	b6	7e	11	8f	02	9a	b6	21	b2	56	0f		
			f9	ca	67	cc	a8	c7	f8	5b	a8	4c	79	03	0c	2b	3d	e2		
			18	f8	6d	b3	a9	09	01	d5	df	45	c1	4f	26	fe	df	b3		
			dc	38	e9	6a	c2	2f	e7	bd	72	8f	0e	45	bc	e0	46	d2		
$CV_1^{(2)}$	8d	64	c8	21	ff	ed	52	e2	eb	c8	59	15	5e	c7	eb	36	73	8a	5a	7b
$M_2^{(2)}$			3c	57	0f	eb	14	13	98	bb	55	2e	f5	a0	a8	2b	e3	31		
			fe	a4	80	37	b8	b5	d7	1f	0e	33	2e	df	93	ac	35	00		
			eb	4d	dc	0d	ec	c1	a8	64	79	0c	78	2c	76	21	56	60		
			dd	30	97	91	d0	6b	d0	af	3f	98	cd	a4	bc	46	29	b1		
CV_2	1e	ac	b2	5e	d5	97	0d	10	f1	73	69	63	57	71	bc	3a	17	b4	8a	c5

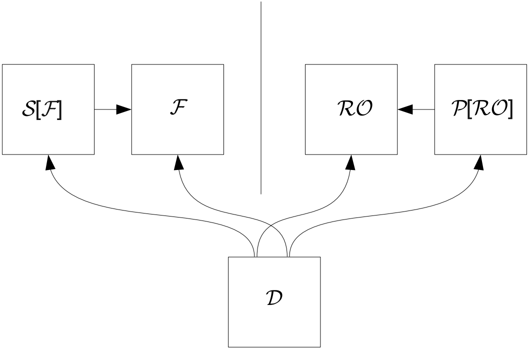
From SHA-1 to SHA-2

Changes from SHA-1 to SHA-2:

- Two compression functions
  - SHA-{224, 256}: $n = 256 = 8 \times 32$ and $m = 512 = 16 \times 32$
  - SHA-{384, 512}: $n = 512 = 8 \times 64$ and $m = 1024 = 16 \times 64$
- Non-linear message expansion
- Stronger data path mixing



Generic security: indifferentiability [Maurer et al. (2004)]



Applied to hash functions in [Coron et al. (2005)]

- distinguishing mode-of-use from ideal function ($\mathcal{RO}$)
- covers adversary with access to primitive $\mathcal{F}$ at left
- additional interface, covered by a *simulator* at right

Consequences of indifferentiability

Theorem 2. *Let $\mathcal{H}$ be a hash function, built on underlying primitive π, and RO be a random oracle, where $\mathcal{H}$ and RO have the same domain and range space. Denote by $\boldsymbol{Adv}^{\mathrm{pro}}_{\mathcal{H}}(q)$ the advantage of distinguishing $(\mathcal{H}, \pi)$ from (RO, S), for some simulator S, maximized over all distinguishers $\mathcal{D}$ making at most q queries. Let* atk *be a security property of $\mathcal{H}$. Denote by $\boldsymbol{Adv}^{\mathrm{atk}}_{\mathcal{H}}(q)$ the advantage of breaking $\mathcal{H}$ under* atk*, maximized over all adversaries $\mathcal{A}$ making at most q queries. Then:*

$$\boldsymbol{Adv}^{\mathrm{atk}}_{\mathcal{H}}(q) \leq \boldsymbol{Pr}^{\mathrm{atk}}_{RO}(q) + \boldsymbol{Adv}^{\mathrm{pro}}_{\mathcal{H}}(q), \tag{1}$$

where $\boldsymbol{Pr}^{\mathrm{atk}}_{RO}(q)$ denotes the success probability of a generic attack against $\mathcal{H}$ under atk*, after at most q queries.*

[Andreeva, Mennink, Preneel, ISC 2010]

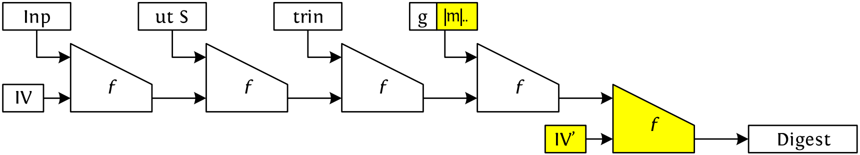
Limitations of indifferentiability

- Only about the mode
  - No security proof with a concrete primitive
- Only about single-stage games [Ristenpart et al., Eurocrypt 2011]
  - Example: hash-based storage auditing

$$Z = h(\mathsf{File} \| C)$$

Making Merkle-Damgård indifferentiable

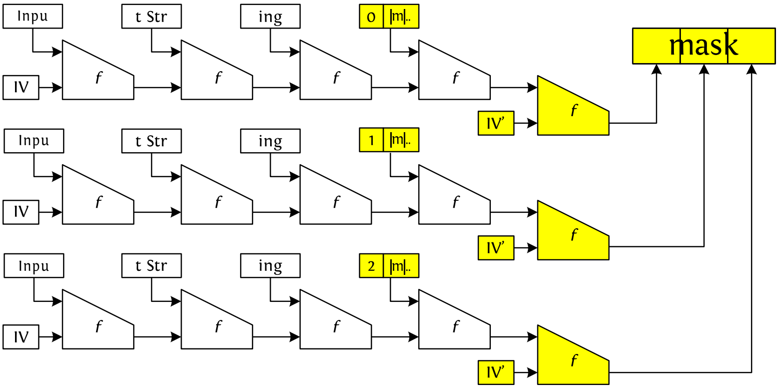
Enveloped Merkle-Damgård



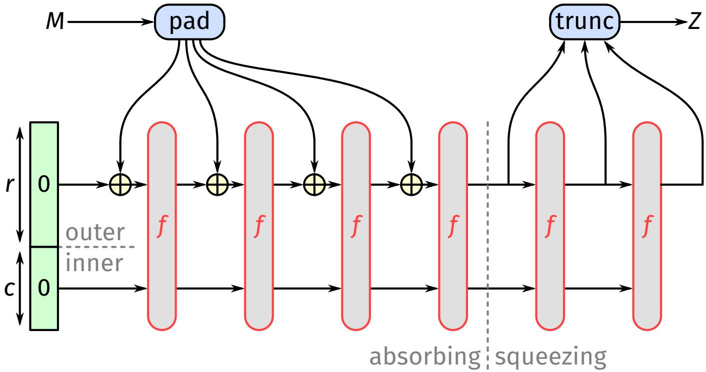
[Bellare and Ristenpart, Asiacrypt 2006]

Making Merkle-Damgård suitable for XOFs

Mask generating function construction “MGF1”



The sponge construction



- Calls a b-bit permutation f, with $b = r + c$
  - r bits of *rate*
  - c bits of *capacity* (security parameter)
- Natively implements a XOF

Generic security of the sponge construction

Theorem (Bound on the $\mathcal{RO}$-differentiating advantage of sponge)

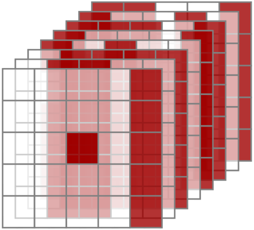
$$\mathrm{Adv} \leq \frac{t^2}{2^{c+1}}$$

Adv: *differentiating advantage of random sponge from random oracle*
t: *time complexity (# calls to f)*   c: *capacity*   [Eurocrypt 2008]

Preimage resistance	$2^{\min(n,c/2)}$
Second-preimage resistance	$2^{\min(n,c/2)}$
Collision resistance	$2^{\min(n/2,c/2)}$
Any other attack	$2^{\min(\mathcal{RO},c/2)}$ (*)

(*) This means the minimum between $2^{c/2}$ and the complexity of the attack on a random oracle.

KECCAK-f



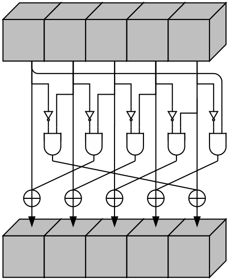
- The seven permutation army:
  - 25, 50, 100, 200, 400, 800, 1600 bits
  - toy, lightweight, fastest
  - standardized in [FIPS 202]
- Repetition of a simple round function
  - that operates on a 3D state
  - (5×5) lanes
  - up to 64-bit each

KECCAK-*f* in pseudo-code

```
KECCAK-F[b](A) {
  forall i in 0…nr-1
    A = Round[b](A, RC[i])
  return A
}

Round[b](A,RC) {
  θ step
  C[x] = A[x,0] xor A[x,1] xor A[x,2] xor A[x,3] xor A[x,4], forall x in 0…4
  D[x] = C[x-1] xor rot(C[x+1],1),                           forall x in 0…4
  A[x,y] = A[x,y] xor D[x],                                  forall (x,y) in (0…4,0…4)

  ρ and π steps
  B[y,2*x+3*y] = rot(A[x,y], r[x,y]),                        forall (x,y) in (0…4,0…4)

  χ step
  A[x,y] = B[x,y] xor ((not B[x+1,y]) and B[x+2,y]),         forall (x,y) in (0…4,0…4)

  ι step
  A[0,0] = A[0,0] xor RC

  return A
}
```

https://keccak.team/keccak_specs_summary.html

χ, the nonlinear mapping in Keccak-f



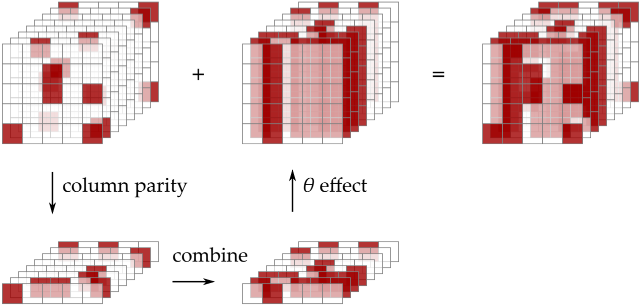
- "Flip bit if neighbors exhibit 01 pattern"
- Operates independently and in parallel on 5-bit rows
- Cheap: small number of operations per bit
- Algebraic degree 2, inverse has degree 3

θ, mixing bits

- Compute parity $c_{x,z}$ of each column
- Add to each cell parity of neighboring columns:

$$b_{x,y,z} = a_{x,y,z} \oplus c_{x-1,z} \oplus c_{x+1,z-1}$$

- Cheap: two XORs per bit

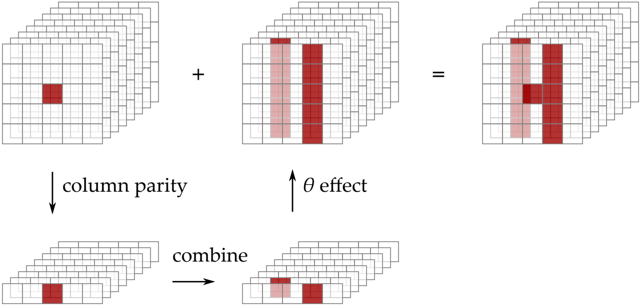


θ, mixing bits – effect on a single bit

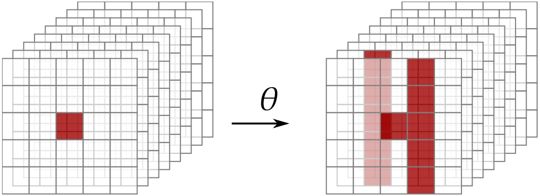
- Compute parity $c_{x,z}$ of each column
- Add to each cell parity of neighboring columns:

$$b_{x,y,z} = a_{x,y,z} \oplus c_{x-1,z} \oplus c_{x+1,z-1}$$

- Cheap: two XORs per bit

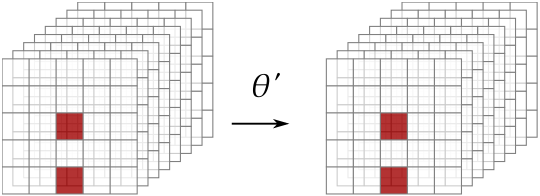


Diffusion of θ



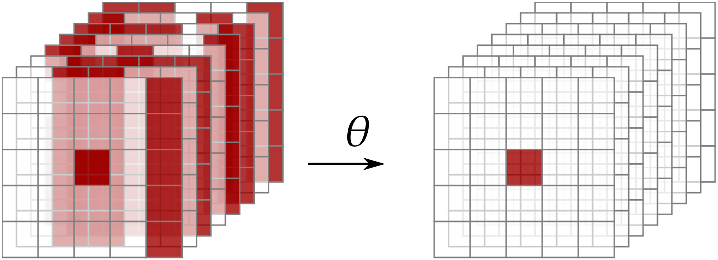
$$1 + \left(1 + y + y^2 + y^3 + y^4\right)\left(x + x^4 z\right)$$
$$\left(\bmod \left\langle 1 + x^5, 1 + y^5, 1 + z^w \right\rangle\right)$$

Diffusion of θ (kernel)



$$1 + \left(1 + y + y^2 + y^3 + y^4\right)\left(x + x^4 z\right)$$
$$\left(\bmod \left\langle 1 + x^5, 1 + y^5, 1 + z^w \right\rangle\right)$$

Diffusion of θ^{-1}



$$1 + \left(1 + y + y^2 + y^3 + y^4\right) \mathbf{Q},$$

$$\text{with } \mathbf{Q} = 1 + \left(1 + x + x^4 z\right)^{-1} \bmod \left\langle 1 + x^5, 1 + z^w \right\rangle$$

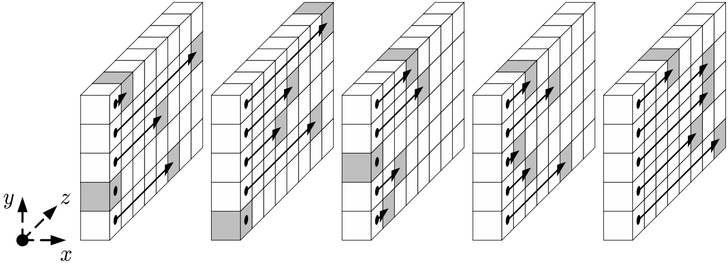
- **Q** is dense, so:
  - Diffusion from single-bit output to input very high
  - Increases resistance against LC/DC and algebraic attacks

ρ for inter-slice dispersion

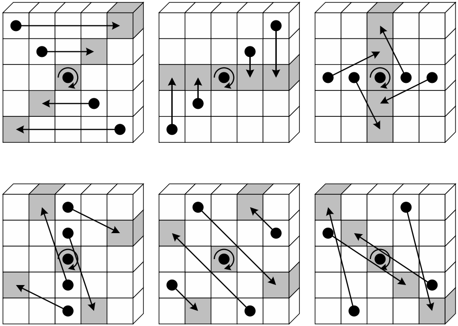
- We need diffusion between the slices ...
- ρ: cyclic shifts of lanes with offsets

$$i(i+1)/2 \bmod 2^{\ell}, \text{ with } \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}^{i-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

- Offsets cycle through all values below 2^{ℓ}



π for disturbing horizontal/vertical alignment



$$a_{x,y} \leftarrow a_{x',y'} \text{ with } \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

ι to break symmetry

- XOR of round-dependent constant to lane in origin
- Without ι, the round mapping would be symmetric
  - invariant to translation in the z-direction
  - susceptible to *rotational* cryptanalysis
- Without ι, all rounds would be the same
  - susceptibility to *slide* attacks
  - defective cycle structure
- Without ι, we get simple fixed points (000 and 111)

KECCAK-f summary

- Round function:

$$\mathsf{R} = \iota \circ \chi \circ \pi \circ \rho \circ \theta$$

- Number of rounds: $12 + 2\ell$
  - KECCAK-$f[25]$ has 12 rounds
  - KECCAK-$f[1600]$ has 24 rounds

NIST FIPS 202 (August 2015)

- Four drop-in replacements to SHA-2
- Two *extendable output functions* (XOF)

XOF	**SHA-2 drop-in replacements**
$\textsc{Keccak}[c=256](M\|11\|11)$	
	first 224 bits of $\textsc{Keccak}[c=448](M\|01)$
$\textsc{Keccak}[c=512](M\|11\|11)$	
	first 256 bits of $\textsc{Keccak}[c=512](M\|01)$
	first 384 bits of $\textsc{Keccak}[c=768](M\|01)$
	first 512 bits of $\textsc{Keccak}[c=1024](M\|01)$
SHAKE128 and **SHAKE256**	**SHA3-224** to **SHA3-512**

- Toolbox for building other functions

NIST SP 800-185 (December 2016)

Customized SHAKE (**cSHAKE**)

- $H(x) = \text{cSHAKE}(x, \text{name}, \text{customization string})$
- E.g., $\text{cSHAKE128}(x, N, S) = \textsc{Keccak}[c = 256](\text{encode}(N, S) \| x \| \texttt{00})$
- $\text{cSHAKE128}(x, N, S) \triangleq \text{SHAKE128}$ when $N = S =$ “”

KMAC: message authentication code (no need for HMAC-SHA-3!)

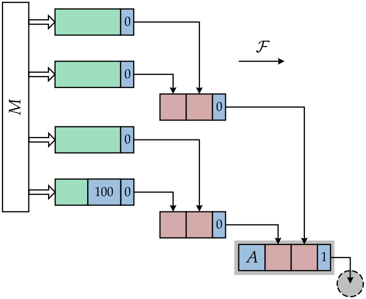
$$\text{KMAC}(K, x, S) = \text{cSHAKE}(\text{encode}(K) \| x, \text{"KMAC"}, S)$$

TupleHash: hashing a sequence of strings $\mathbf{x} = x_n \circ x_{n-1} \circ \cdots \circ x_1$

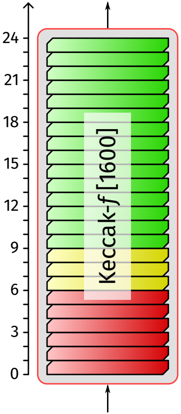
$$\text{TupleHash}(\mathbf{x}, S) = \text{cSHAKE}(\text{encode}(\mathbf{x}), \text{"TupleHash"}, S)$$

NIST SP 800-185 (December 2016)

ParallelHash: faster hashing with parallelism



Status of KECCAK cryptanalysis



- Preimage attacks up to 4 rounds
  [He et al., ToSC 2021] [Wang et al., IACR ePrint 2022/977]
- Collision attacks up to 6 rounds
  [Song et al., CRYPTO 2017] [Guo et al., ASIACRYPT 2022]
  [Zhang et al., CRYPTO 2024]
- Structural distinguishers
  - 7 rounds (practical time)
    [Huang et al., EUROCRYPT 2017]
  - 8 rounds (2^{122} time)
    [Huang et al., IEICE 2019]
  - 9 rounds (2^{64} time) – SymSum
    [Suryawanshi et al., AFRICACRYPT 2020]
- Lots of third-party cryptanalysis available at:
  `https://keccak.team/third_party.html`

What is SymSum?

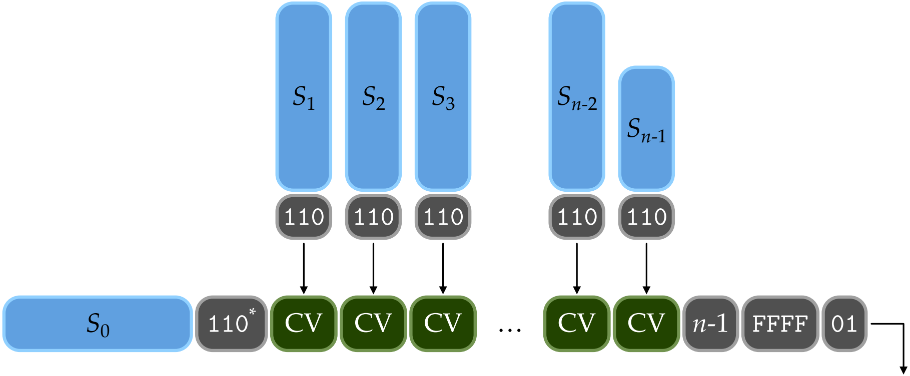
A SymSum structural distinguisher produces:

- a set S of self-symmetric input strings, i.e., $a\|a\|b\|b\|c\|c\| \ldots$, with $|a| = |b| = |c| = 32$ bits,
- such that

$$\bigoplus_{m \in S} H(m) \quad \text{is self-symmetric}$$

[Saha et al., ToSC 2017] [Suryawanshi et al., AFRICACRYPT 2020]

KangarooTwelve



An arrow means “hash with **TurboSHAKE**” (12 rounds).

[Viguier et al., draft-irtf-cfrg-kangarootwelve]