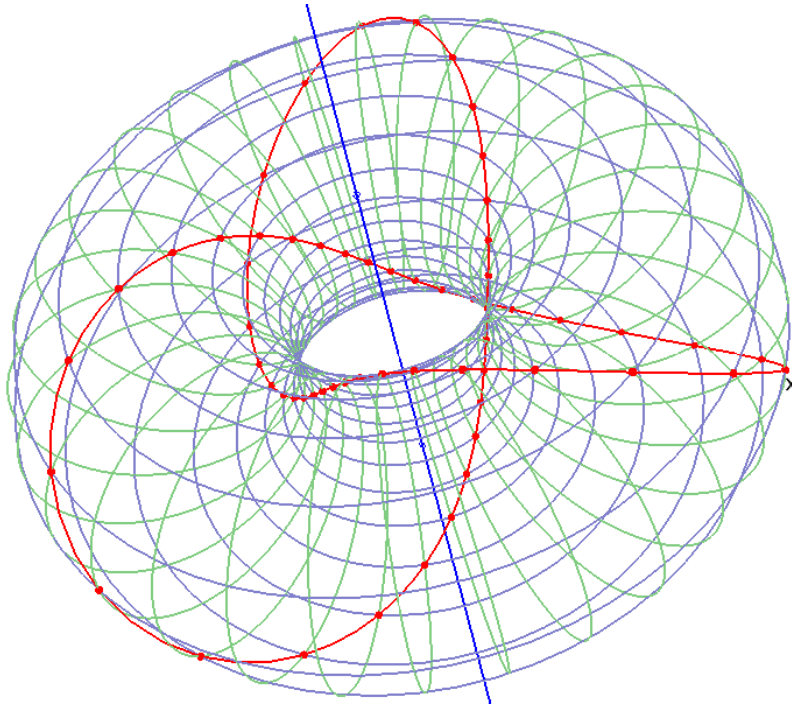# CGA in Practice 2:
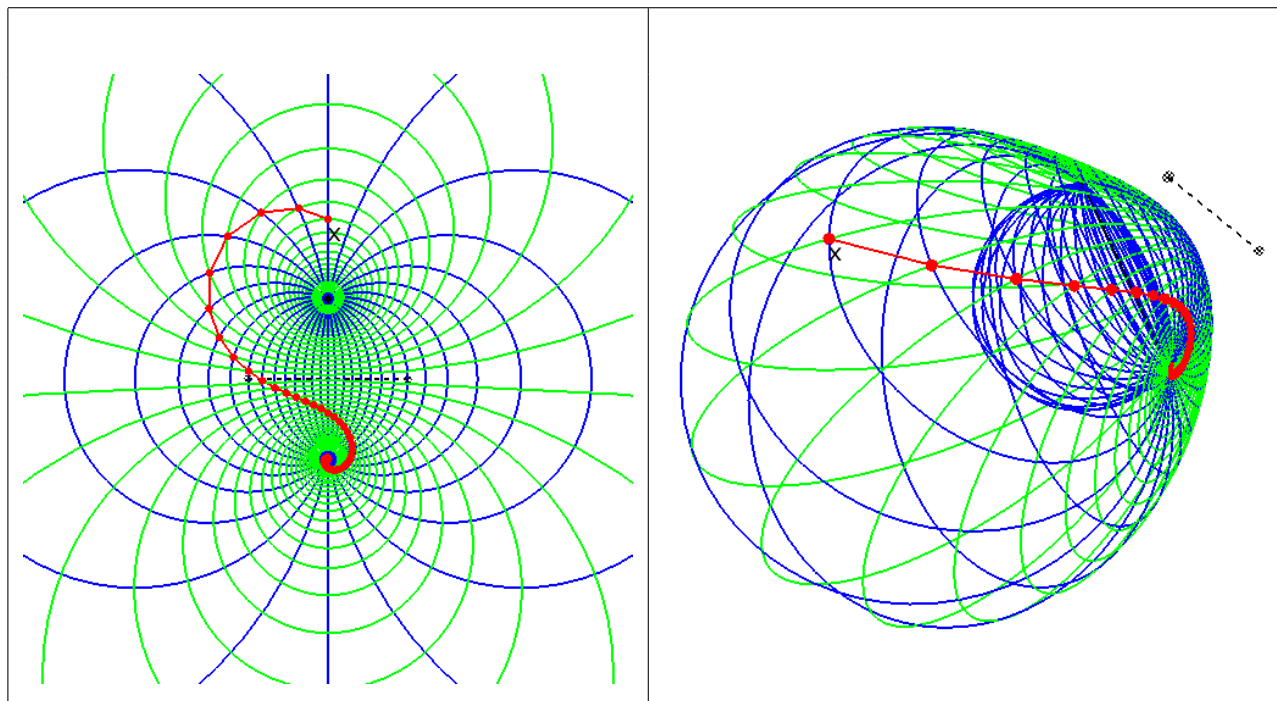# The Construction of 3D Conformal Motions

Leo Dorst (`L.Dorst@uva.nl`)
Intelligent Systems Laboratory, Informatics Institute,
University of Amsterdam, The Netherlands

AGACSE, Barcelona, July 2015
XVII Summer school, Santander, 2016
AGACSE, Campinas, July 2018

# 1   The Structure of 3D Conformal Motions

Spoilers (slightly simplified):

- All 3D conformal motions can be characterized by two point pairs.

- The motions are decomposable into two perpendicular circular motions.

- All trajectories reside on some (generalized) Dupin cyclide.

- Chasles decomposition is a special case: screws on a cylinder.

2 CGA: Representing Eculidean $\mathbb{E}^3$ in Minkowski $\mathbb{R}^{4,1}$

Conformal motions preserve local angles.
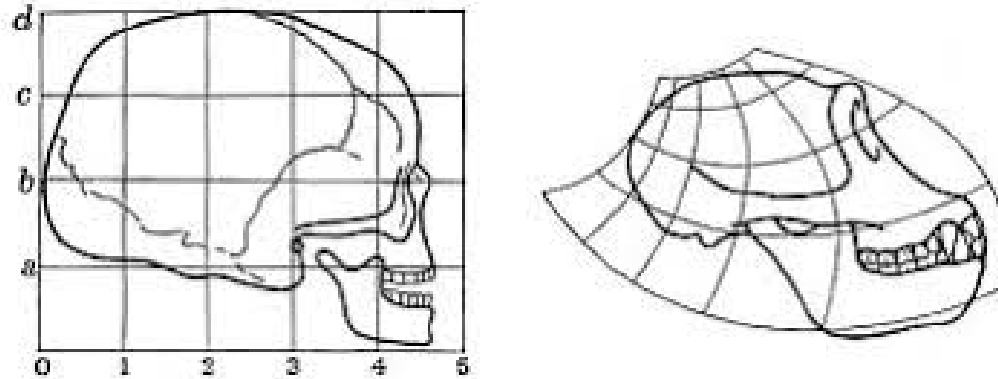
In $\mathbb{R}^{4,1}$, conformal motions of $\mathbb{E}^3$ become rotors (proper orthogonal transformations), under the mapping:

(weighted) point at location $\mathbf{p}$ $\longleftrightarrow$ $p = \alpha\,(\mathbf{p} + n_o + \frac{1}{2}\mathbf{p}^2 n_\infty)$

where $\mathbf{p}$ is Euclidean, and $n_o = \frac{1}{2}(e_+ + e_-)$ and $n_\infty = e_- - e_+$.

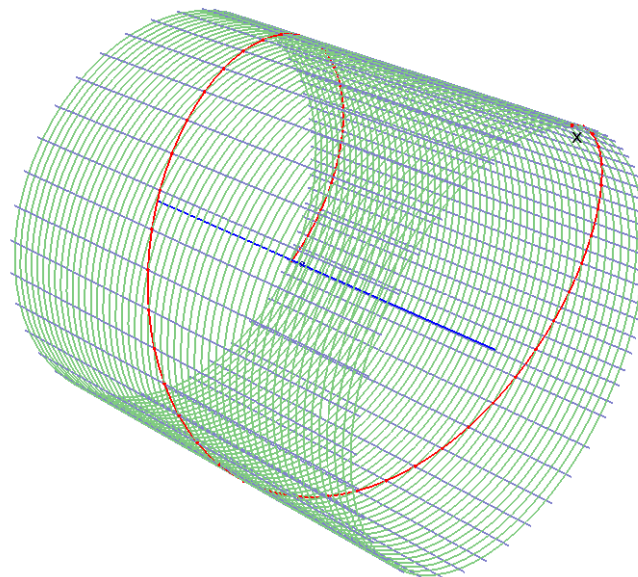The blades formed by $\wedge$ contain lines, planes, spheres, circles, point pairs, tangent vectors etc.

The 'rounds' can be real, imaginary or null.

3   The Core Idea: Orthogonal Decomposition of CGA Rotors

$$
\begin{aligned}
\text{rotor of } \mathbb{R}^{4,1} &= \exp(\text{bivector}) \\
&= \exp(\text{sum of 2-blades}) \\
&= \exp(\text{sum of two commuting 2-blades}) \\
&= \exp(B_1 + B_2) \ \text{ with } B_1 B_2 = B_2 B_1 \\
&= \exp(B_1)\, \exp(B_2) \\
&= \lim_{n \to \infty} \Big(\exp(B_1/n)\, \exp(B_2/n) \Big)^n
\end{aligned}
$$

Conformal motion under rotor of $\mathbb{R}^{4,1}$
then becomes a 'simultaneous motion'
of two orthogonal primitive motions.

Example: Chasles' Theorem
for rigid mody motions:
orthogonal rotation and translation.

4   The 2-Blades of CGA and Their Geometry

The 2-blades have straightforward interpretations. The real ones contain points, the others can be interpreted through meet and join.

imaginary point pair	tangent vector	real point pair
dual line	direction vector	flat point

5 The 2-Blades of CGA and Their Geometry

The 2-blades have straightforward interpretations. The real ones contain points, the others can be interpreted through meet and join.

imaginary point pair	tangent vector	real point pair
$\underline{T}_{\mathbf{c}}[(n_o - \frac{1}{2}\rho^2 n_\infty) \wedge \mathbf{u}]$	$\underline{T}_{\mathbf{c}}[n_o \wedge \mathbf{u}]$	$\underline{T}_{\mathbf{c}}[(n_o + \frac{1}{2}\rho^2 n_\infty) \wedge \mathbf{u}]$
dual line	direction vector	flat point
$\underline{T}_{\mathbf{c}}[\mathbf{u} \wedge \mathbf{v}]$	$\mathbf{u} \wedge n_\infty$	$\underline{T}_{\mathbf{c}}[n_o \wedge n_\infty]$

6 Orbits of simple rotor $R = \exp(\textbf{2-blade})$

'conformal rotation'	**transversion**	**'conformal scaling'**
imaginary point pair	*tangent vector*	*real point pair*
rotation	**translation**	**(isotropic) scaling**
dual line	*direction vector*	*flat point*
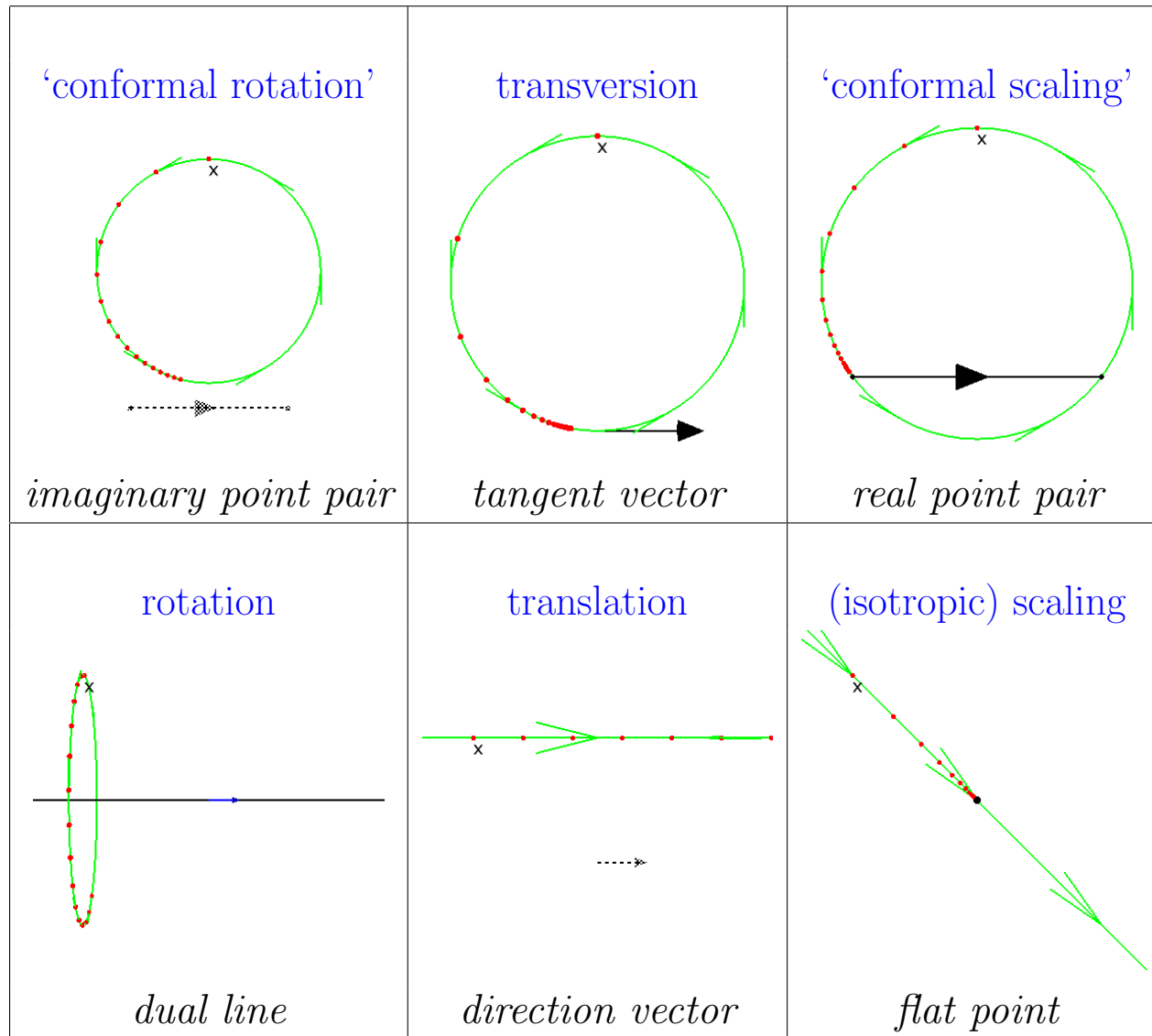
`GAViewer: simplerotors()`

7   Orthogonal decomposition of point pairs

$$
\begin{aligned}
\text{real circle} &= \text{real sphere} \cap \text{orthogonal plane} \\
\text{imaginary point pair} &= (\text{real circle})^* \\
&= (\text{real sphere})^* \wedge (\text{orthogonal plane})^*
\end{aligned}
$$

imaginary point pair	null point pair (tangent vector)	real point pair
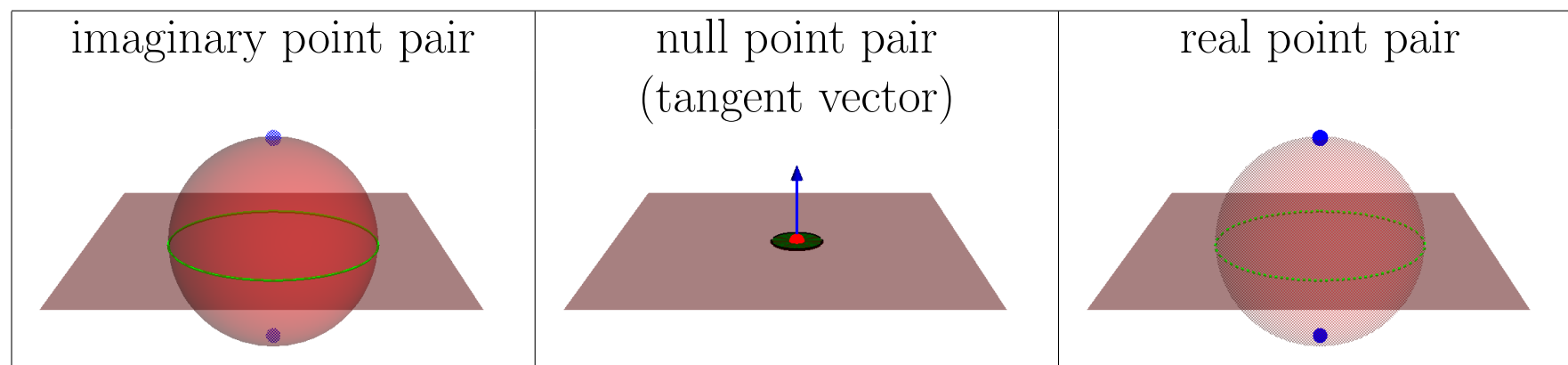
Dual sphere σ, dual plane π, such that $\sigma \cdot \pi = 0$:

$$
PP = \sigma\,\pi =
\begin{cases}
\text{with } \sigma^2 > 0 \text{ for imaginary point pair} \\
\text{with } \sigma^2 = 0 \text{ for tangent vector (normal to plane)} \\
\text{with } \sigma^2 < 0 \text{ for real point pair}
\end{cases}
$$

8   The Core Idea: Orthogonal Decomposition of CGA Rotors

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\begin{aligned}
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&= \exp(B_1 + B_2) \ \text{ with } B_1 B_2 = B_2 B_1 \\
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Conformal motion under rotor of $\mathbb{R}^{4,1}$
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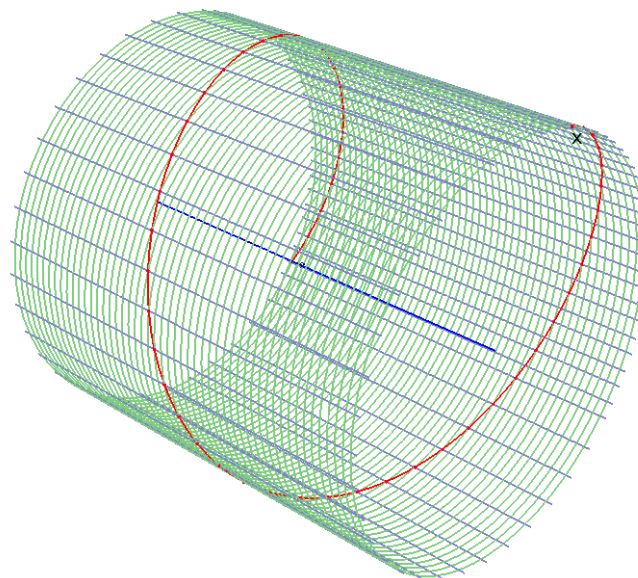
9   Bivector Split and Logarithm

The split of an $\mathbb{R}^{4,1}$ bivector into 2 commuting blades has been solved, in the process of computing the logarithm of an $\mathbb{R}^{4,1}$ rotor [1].
Always possible (except when simple), almost always unique.

Given rotor, to write: $R = e^{-(B_+ + B_-)/2} = e^{-B_+/2}\, e^{-B_-/2} = e^{-B_-/2}\, e^{-B_+/2}$.

1. $S = 2(\langle R \rangle_4 - \langle R \rangle_0)\, \langle R \rangle_2 = \sinh(B_+) + \sinh(B_-)$, but how to split?.

2. The 'bivector split' of any bivector S of $\mathbb{R}^{4,1}$ can be computed as: $S_\pm = \frac{1}{2} S\left(1 \pm \|S\|^2/S^2\right)$, with $\|S\| = \sqrt[4]{(2\langle S^2 \rangle_0 - S^2)\, S^2}$, and for $\|S\| \neq 0$. When $\|S\| = 0$, no split, or no unique split, see CA2GC.

3. $S_\pm = \sinh(B_\pm)$ thus found; $C_\pm = \cosh(B_\pm) = -\langle R^2 \rangle_2/S_\pm$. Then $B_\pm = \operatorname{atanh2}(S_\pm, C_\pm)$

4. Done: $\operatorname{Log}(R) = -\frac{1}{2}B_+ - \frac{1}{2}B_-$.

[1] Dorst & Valkenburg, *Square Root and Logarithm of Rotors in 3D Conformal Geometric Algebra Using Polar Decomposition*, in: Guide to GA in Practice 2011.

10   Commutation of 2-blades, geometrically

Two round point pairs commute if the associated spheres are perpendicular.
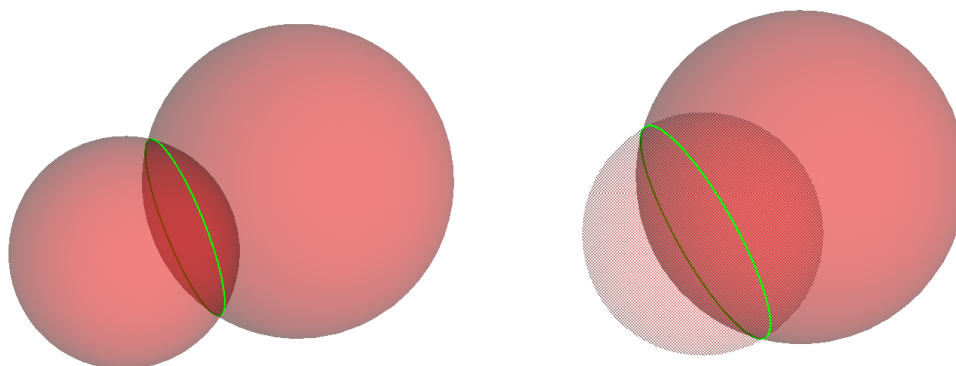
- imaginary point pairs: perpendicular intersection of real spheres, always possible.

- real and imaginary point pairs: an imaginary sphere must be intersected equatorially by a real sphere (special case: equal radius 'coincident')

- dual lines (also imaginary) work with anything: coincident flat point, coincident ipp, etc.

11   Orthogonal Orbits and the Commutation of Point Pairs

Let us get a feeling for the orthogonal circular orbits of commuting point pairs.
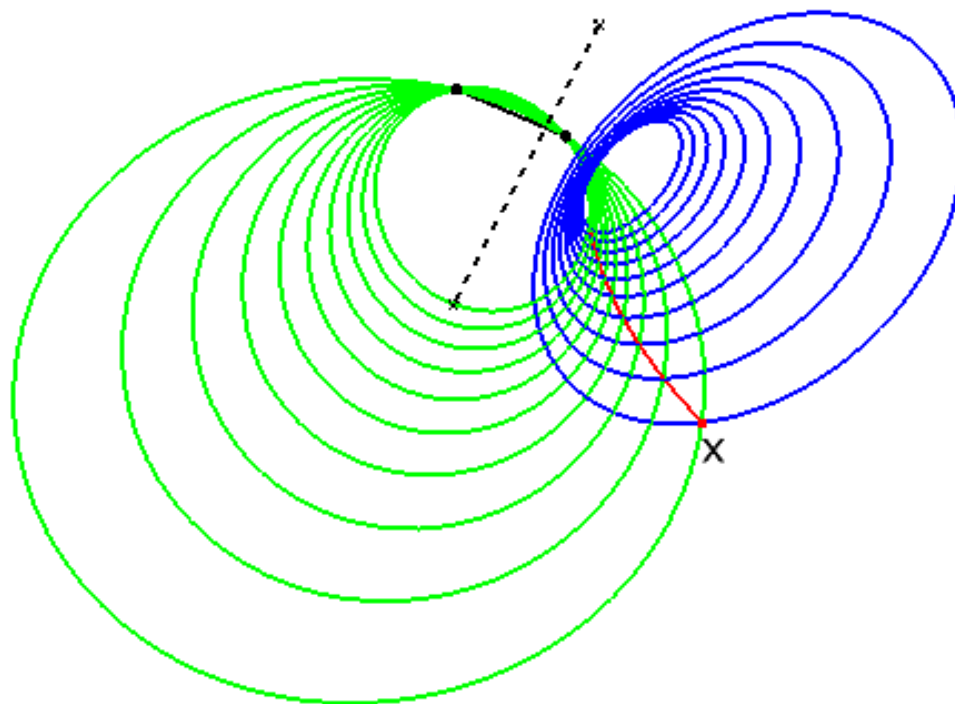
GAViewer: logs-and-knots/pperp()
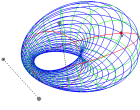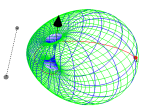
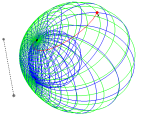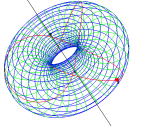## 12 Orbit of a general rotor; cyclides
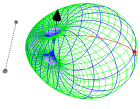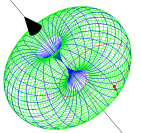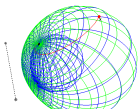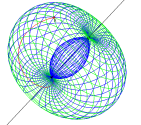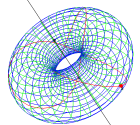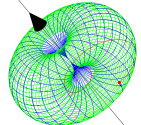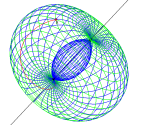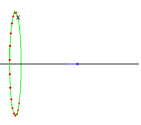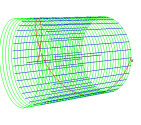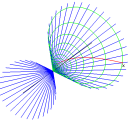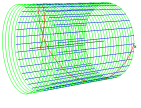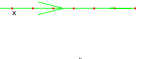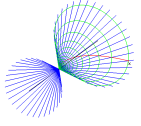
The orbit of a rotor can be seen as the simultaneous following of the orbits of its constituent commuting simple rotors: arcs of orthogonal circles.

A torus has orthogonal circles; spherical inversion is conformal; so inversion of a torus has orthogonal circles: Dupin cyclides.
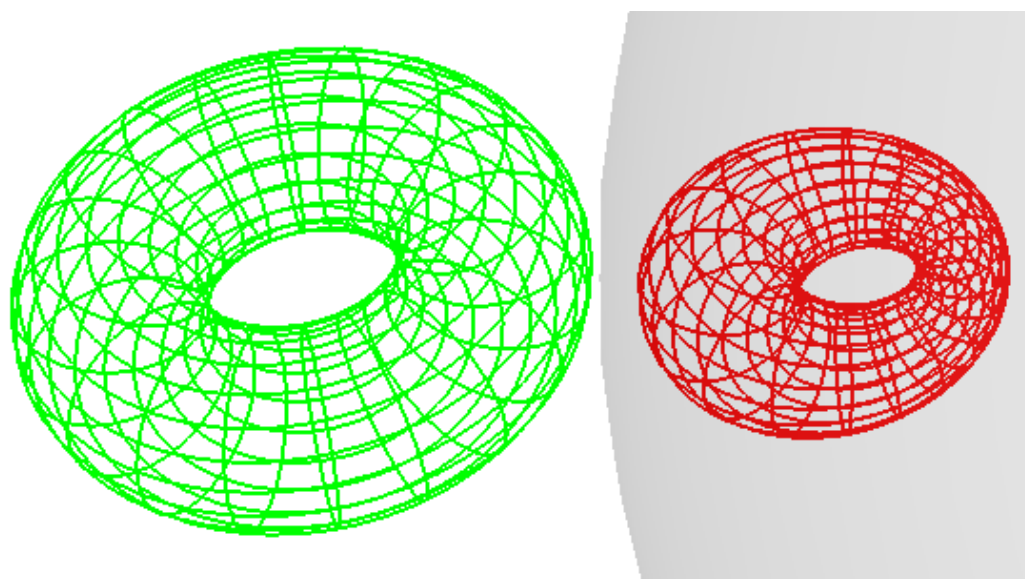
GAViewer: logs-and-knots/DEMOdupin()

There are some degenerate cases involving lines (as circles): cones (co-pointal) and cylinders (parallel). Their inversions should be included.

	imaginary point pair $B^2 < 0$	tangent vector $B^2 = 0$	real point pair $B^2 > 0$	dual line $B^2 < 0$ $n_\infty \cdot \wedge B = 0$	direction vector $B^2 = 0$ $n_\infty \wedge B = 0$	flat point $B^2 > 0$ $n_\infty \wedge B = 0$
imaginary point pair	ring cyclide	cuspidal cyclide	spindle cyclide	ring torus	NO	NO
tangent vector	cuspidal cyclide	NO	NO	horn torus	NO	NO
real point pair	spindle cyclide	NO	NO	spindle torus	NO	NO
dual line (axis)	ring torus	horn torus	spindle torus	simple/cylinder	cylinder	cone
direction vector	NO	NO	NO	cylinder	simple	NO
flat point	NO	NO	NO	cone	NO	NO

13   Two imaginary point pairs: ring cyclides

GAViewer: logs-and-knots/cyclides()
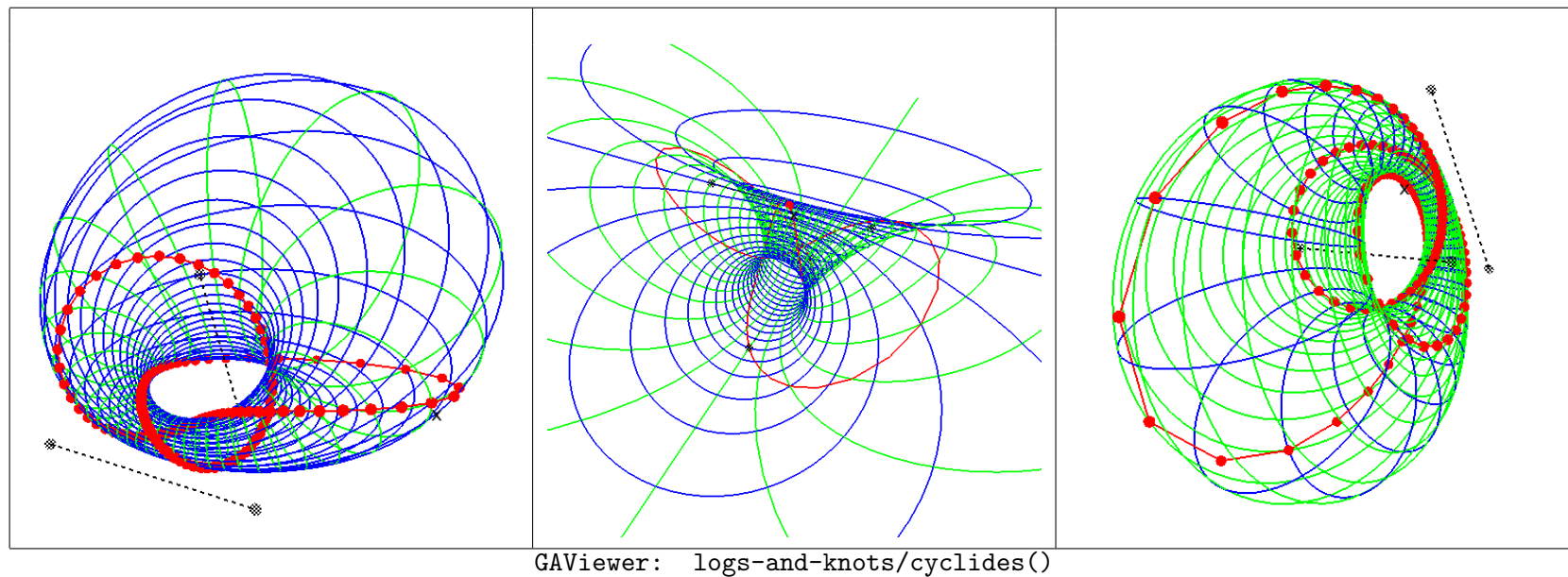
Figure 1: The motions governed by two imaginary point pairs reside on a ring cyclide. From left to right, as the point x moves the ring cyclide is turned 'inside out', with an infinite-sized parabolic ring cyclide in between.

14   Imaginary point pair plus tangent: needle/cusp cyclides

GAViewer: logs-and-knots/cyclides()

Figure 2: A needle cyclide, a parabolic needle cyclide, and a cuspidal cyclide, all generated from the same two point pairs but different x (and in varied views).

15 Real plus imaginary point pair: horn/spindle cyclides

GAViewer: logs-and-knots/cyclides()
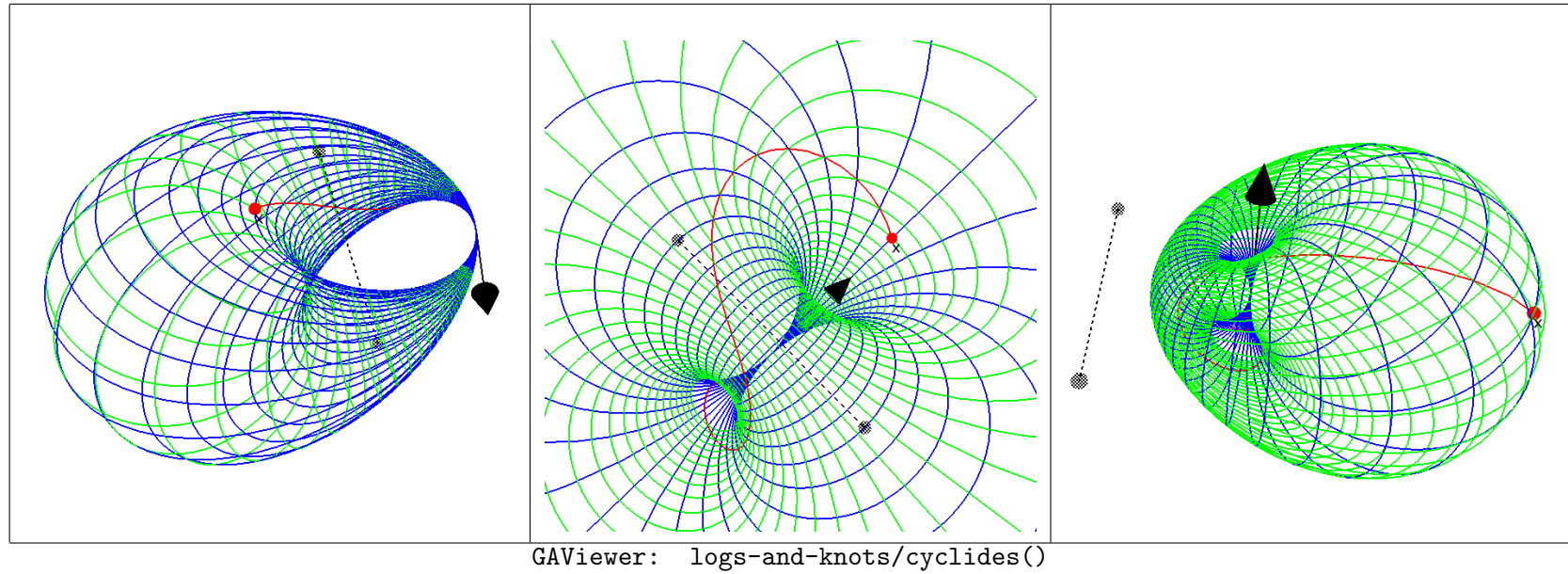
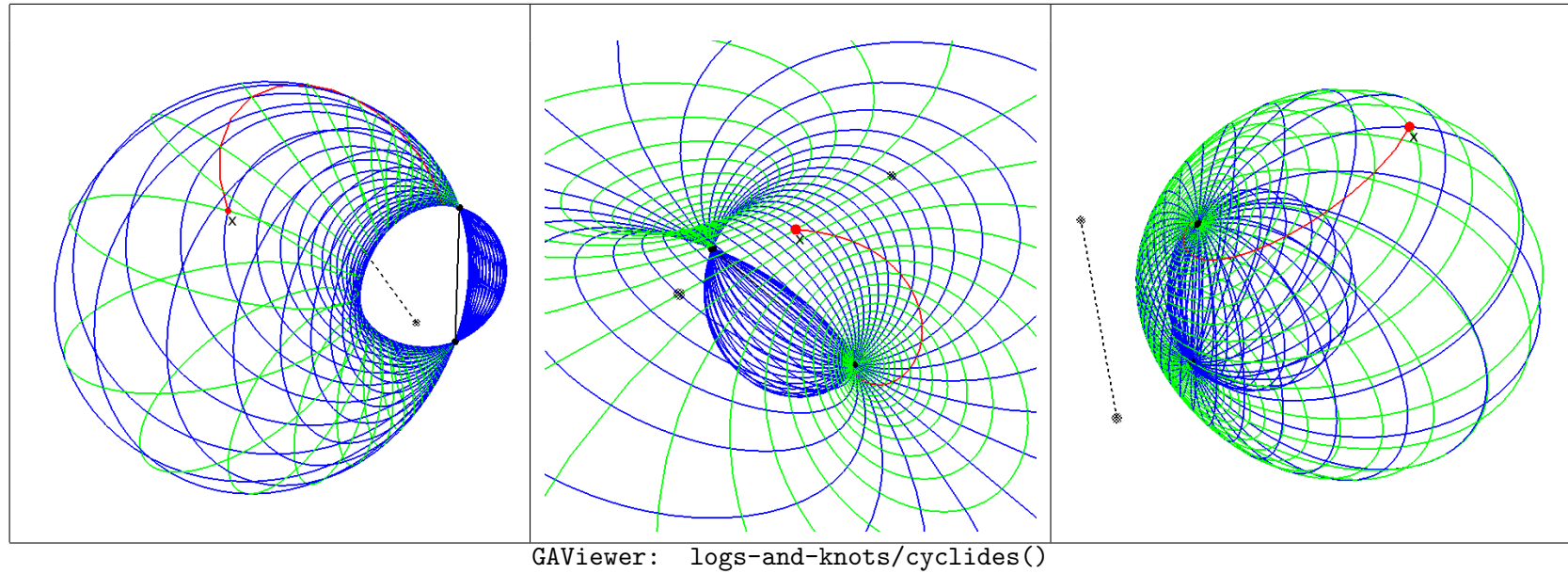Figure 3: A horn cyclide, a parabolic horn cyclide, and a spindle cyclide, all generated from the same two point pairs but different x (and in varied views).

16 Coplanar real and imaginary point pair: symmetric cyclides

GAViewer: logs-and-knots/cyclides()

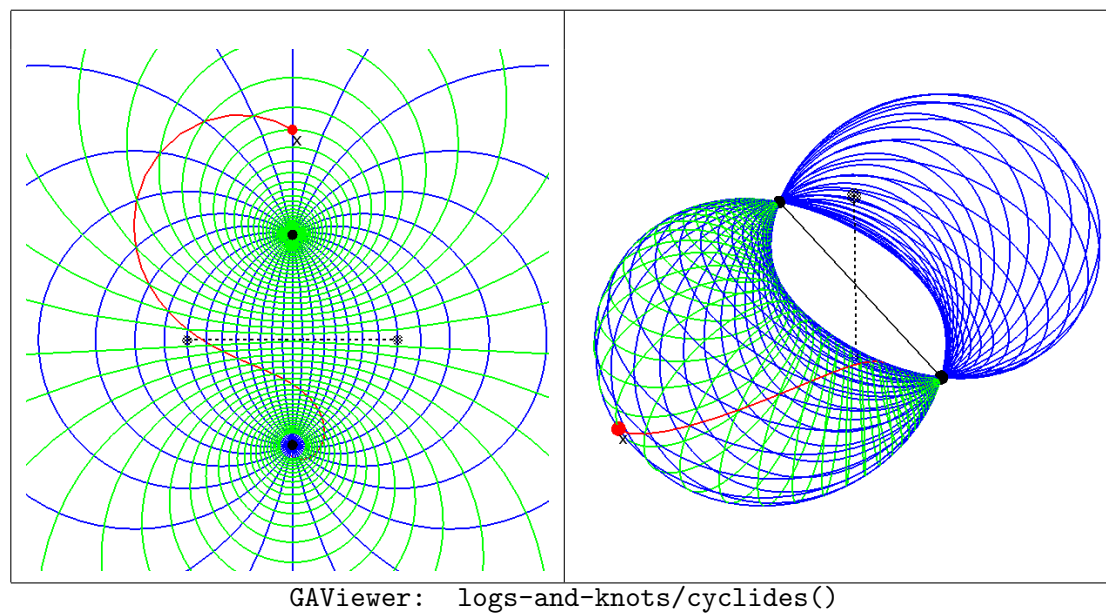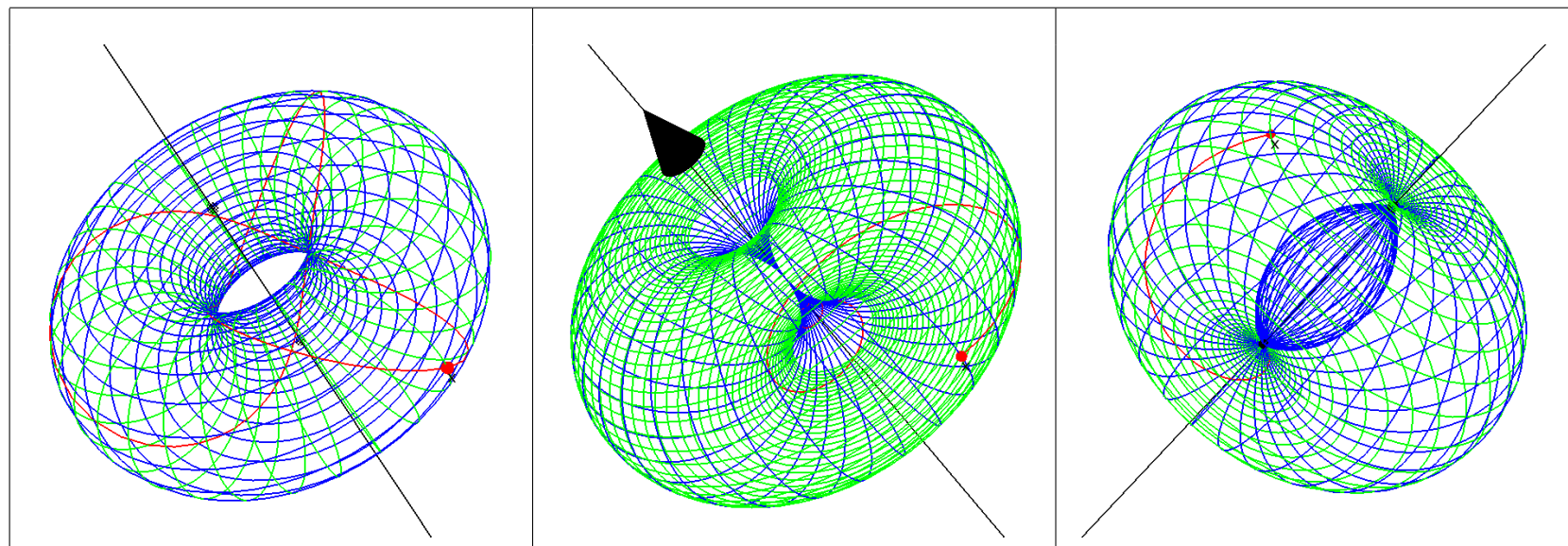Figure 4: The planar orbits of a conformal motion from a coplanar real/imaginary point pair, and spatial orbits from the same point pair.

17   Dual line plus point pair: torus-like

GAViewer: logs-and-knots/cyclides()

Figure 5: The various types of torus.

18 Dual line plus direction or flat point: cylinder, cone

GAViewer: logs-and-knots/bivsplit3D()
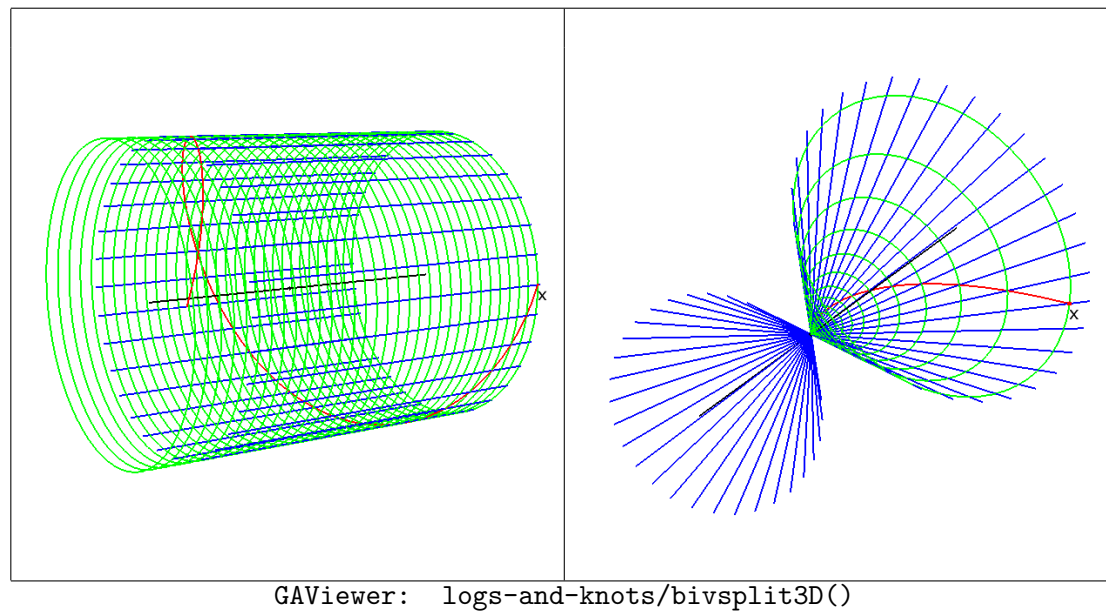
Figure 6: A cone and a cylinder as special cases.

19 Knots by well-chosen relative speeds on two imaginary point pairs

GAViewer: logs-and-knots/cyclides()

Figure 7: The (3,2) and (2,3) knots on ring cyclides. The switch between (3,2) and (2,3) knots is here made by moving the point x 'to the other side' of the point pairs.

20 Naturally simple shapes: orbits of conformally carried circles

GAViewer: logs-and-knots/cyclides() with circle probe x = (no+ni/2) e1 (e2+e3)

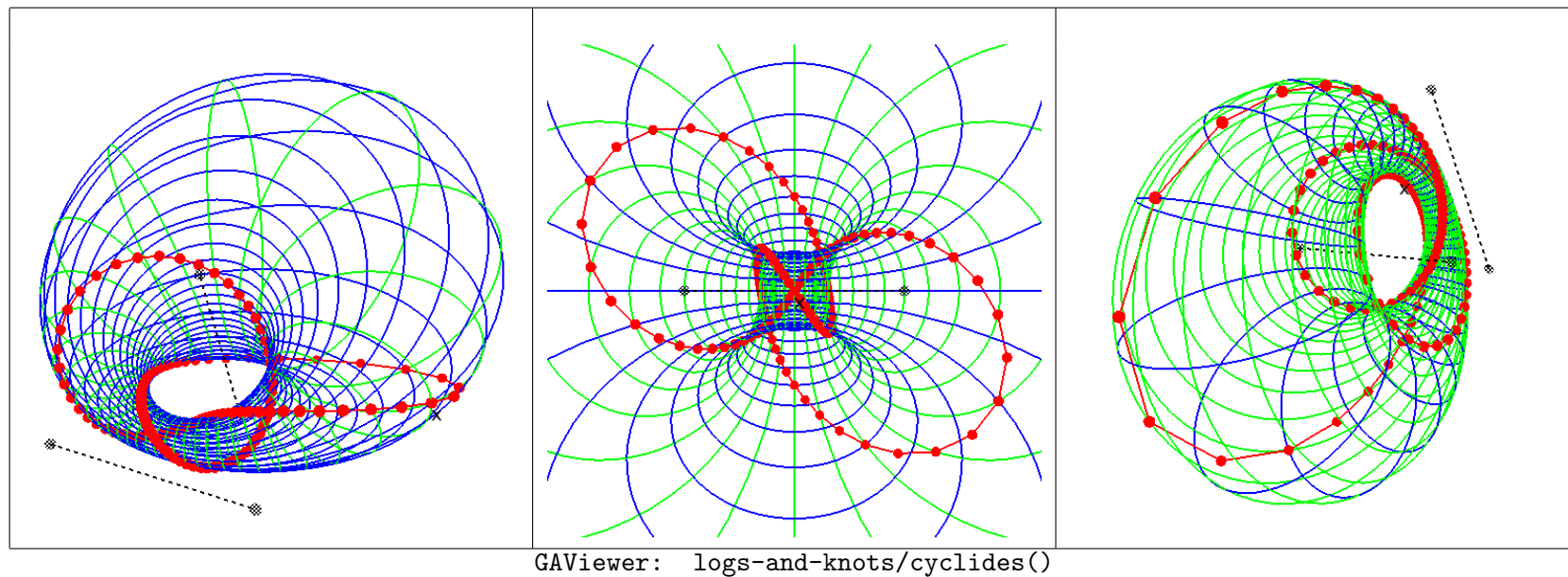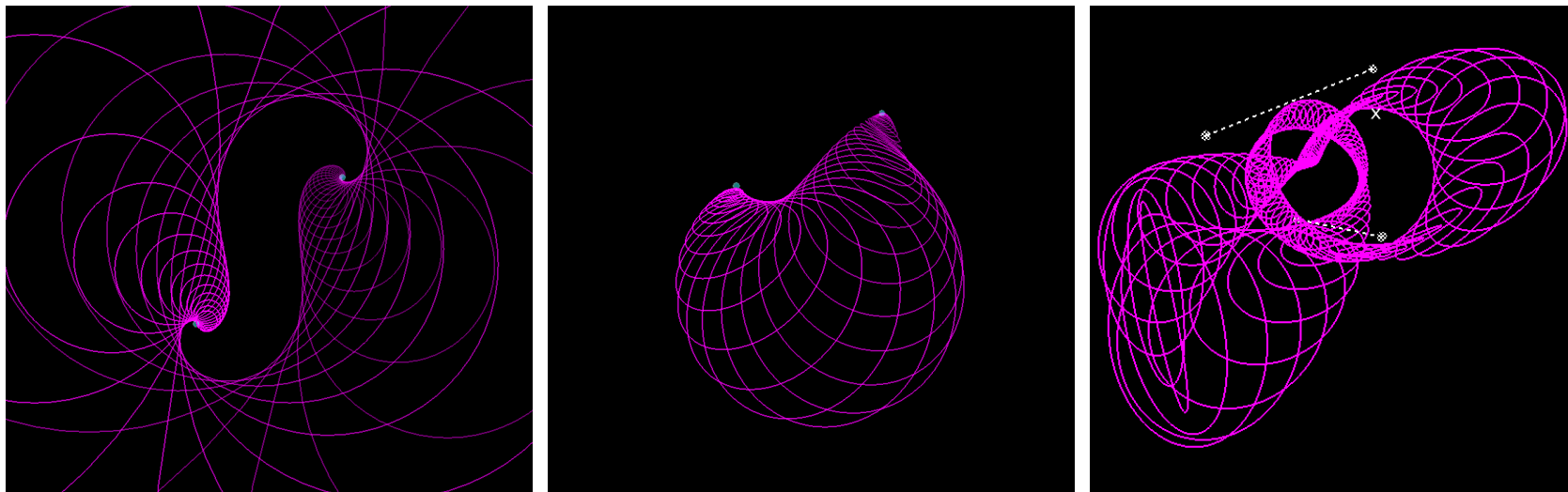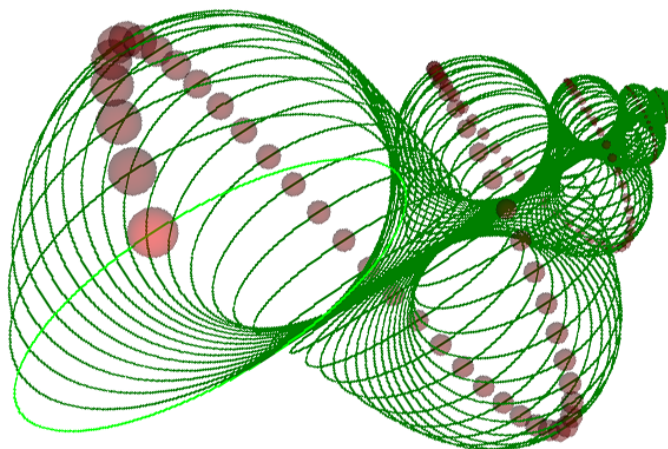# 21 Conclusions on 'The Construction of 3D Conformal Motions'

- **coherent and compact** characterization of conformal motions

- **more managable** than usual decomposition in translation, transversion, rotation, scaling (with complicated commutation rules)

- **interesting primitive shapes** that are simple to generate

- application: natural blending of orthogonal circular arcs (Colapinto).

Mathematics in Computer Science

CrossMark
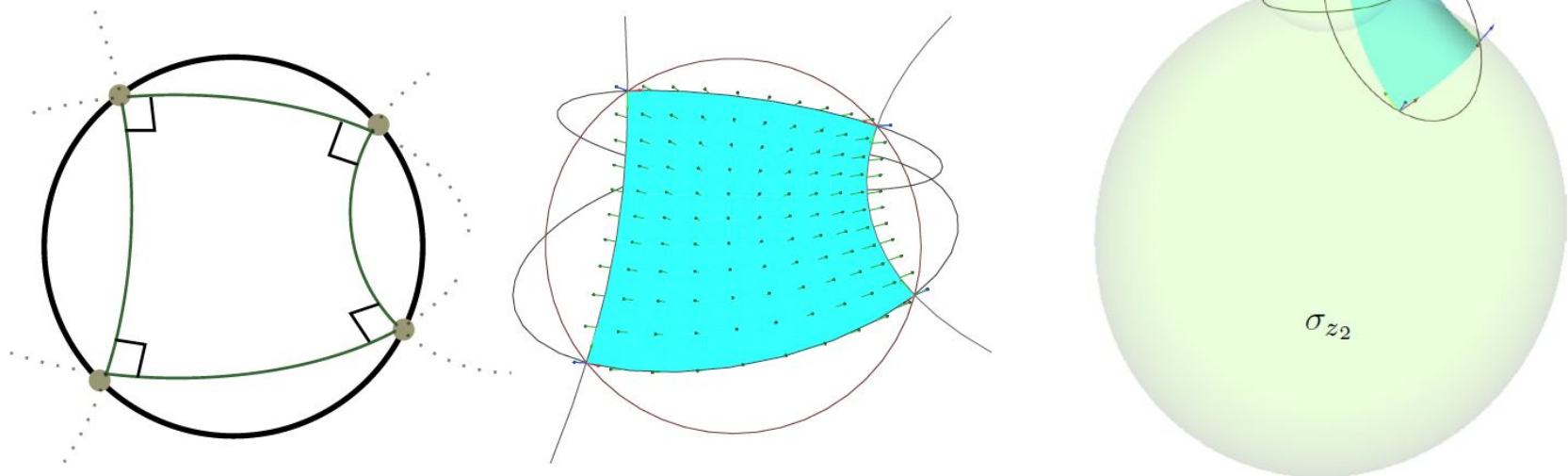
The Construction of 3D Conformal Motions

Leo Dorst

Abstract This paper exposes a very geometrical yet directly computational way of working with conformal motions in 3D. With the increased relevance of conformal structures in architectural geometry, and their traditional use in CAD, its results should be useful to designers and programmers. In brief, we exploit the fact that any 3D conformal motion is governed by two well-chosen point pairs: the motion is composed of (or decomposed into) two specific orthogonal circular motions in planes determined by those point pairs. The resulting orbit of a point is an equiangular spiral on a Dupin cyclide. These results are compactly expressed and programmed using conformal geometric algebra (CGA), and this paper can serve as an introduction to its usefulness. Although the point pairs come in different kinds (imaginary, real, tangent vector, direction vector, axis vector and 'flat point'), causing the great variety of conformal motions, all are unified both algebraically and computationally as 2-blades in CGA, automatically producing properly parametrized simple rotors by exponentiation. An additional advantage of using CGA is its covariance: conformal motions for other primitives such as circles are computed using exactly the same formulas, and hence the same software operations, as motions of points. This generates an interesting class of easily generated shapes, like spatial circles moving conformally along a knot on a Dupin cyclide.