Ex. 2

$D_{t+1} = D_t e^{\bar{g} - \frac{1}{2}\sigma_g^2 + g_{t+1}}$ - dividend process

$g_{t+1} = \rho_g g_t + \sigma_g \varepsilon_g(t+1)$ - stoch. component (ε_g iid)

$R_{t+1} = \frac{D_{t+1} + P_{t+1}}{P_t}$ - gross return

$E[R_{t+1}] = e^{\bar{k} + k_t}$ - expected return

$PD_t = P_t / D_t \quad pd_t = \log PD_t$

ⓐ $k_t = 0 \ \forall t \Rightarrow e^{\bar{k}} = const.$; $\rho_g = 0 \Rightarrow E[g_{t+1}] = 0$

$R_{t+1} = \frac{D_{t+1} + P_{t+1}}{P_t} \Rightarrow P_t = \frac{D_{t+1} + P_{t+1}}{R_{t+1}}$

$E[R_{t+1}] = e^{\bar{k} + k_t} = e^{\bar{k}}$; $D_{t+1} = P_t e^{\bar{g} - \frac{1}{2}\sigma_g^2 + g_{t+1}} = D_t e^{\bar{g}}$ (since volatility of a constant = 0)

Thus, we have:

$P_t = \frac{D_t e^{\bar{g}} + E_t[P_{t+1}]}{e^{\bar{k}}}$, where $E_t[P_{t+1}] = \frac{D_t e^{\bar{g}} + E[P_{t+2}]}{e^{\bar{k}}} \Rightarrow$

$\Rightarrow \frac{D_t}{e^{\bar{k}-\bar{g}}} \frac{D_t}{e^{\bar{k}}(e^{\bar{k}-\bar{g}})} + \frac{E[P_{t+2}]}{e^{2\bar{k}}}$

Applying the brute force approach and substituting further $E[P_{t+n}]$ we get the following equation:

$P_t = D_t e^{-(\bar{k}-\bar{g})} \sum_{k=0}^{N-1} e^{-(\bar{k}-\bar{g})k} + \left(\frac{1}{e^{\bar{k}}}\right)^N E_t[P_{t+N}]$

Assuming no bubble condition and taking the lim gives: $\lim_{N\to+\infty} \left(\frac{1}{e^{\bar{k}}}\right)^N E_t[P_{t+N}] = 0$. Thus, we have:

$P_t = D_t e^{-(\bar{k}-g)} \sum_{k=0}^{\infty} e^{-(\bar{k}-\bar{g})k}$

Since we assume that the price is finite $\bar{g} < \bar{k}$ has to hold and we have the following equation in the limit: $P_t = D_t e^{\bar{g}-\bar{k}} \frac{1}{1 - e^{(\bar{g}-\bar{k})}} = \frac{D_t}{e^{\bar{k}-\bar{g}} - 1} \approx \frac{D_t}{k-g} \Rightarrow$ the PD ratio is constant

The results are not consistent with the emperical result. According to the emperical results, the volatility is not constant and the price volatility is 5 to 10 times higher than the dividend volatility. The condition under which such difference is possible (assuming constant volatility):

$\sqrt{5} < \frac{1}{e^{k-g}-1} < \sqrt{10}$

$2.236 < \frac{1}{e^{k-g}-1} < 3.162$

$0,447 > e^{k-g}-1 > 0.316$

$1,447 > e^{k-g} > 1,316$

$0,369 > k-g > 0,275 \Rightarrow$ the condition that is highly unlikely to hold in real life.

ⓑ $\rho_g \neq 0 \Rightarrow k_{t+1} = \rho_k k_t + \sigma_k \varepsilon_k(t+1)$

We need to find a function that $PD_t = PD(g_t; k_t)$ that solves a fixed point: $E[R_{t+1}] = E_t\left[\frac{D_{t+1}}{D_t} \frac{1 + PD_{t+1}}{PD_t}\right]$

Equivalently, $e^{\bar{k}+k_t} = E_t[\exp(\bar{g} - \frac{1}{2}\sigma_g^2 + g_{t+1} + \log(1 + e^{pd_{t+1}}) - pd_t]$

Let's first find log-returns:

$\log(R_{t+1}) = \log(P_{t+1} + D_{t+1}) - \log(P_t) = p_{t+1} + \log(1 + \frac{D_{t+1}}{P_{t+1}}) - p_t = p_{t+1} - p_t + \log(1 + \exp(d_{t+1} - p_{t+1}))$

Let's apply 1st order Taylor approximation to the function $f(x) = \log(1+e^x)$ around $\bar{x} = \bar{d} - \bar{p}$:

$f(x) \cdot f(\bar{x}) + \frac{f'(\bar{x})}{1!}(x - \bar{x})$

$\log(1 + \exp(d_{t+1} - p_{t+1})) \approx \log(1 + \exp(\bar{d} - \bar{p})) + \frac{\exp(\bar{d}-\bar{p})}{\exp(\bar{d}-\bar{p}) + 1}(d_{t+1} - p_{t+1} - (\bar{d} - \bar{p}))$.

Thus we have the approximation for the function $\log(1+e)$

Let's use the approximation for $\log(1+e^{pd_{t+1}})$ and plug it into the initial equation:

$e^{\bar{k}+k_t} = E_t\left[\exp(\bar{g} - \frac{1}{2}\sigma_g^2 + g_{t+1} + \log(1 + e^{\bar{pd}}) + \frac{e^{\bar{pd}}}{e^{\bar{pd}}+1}(pd_{t+1} - \bar{pd}) - pd_t)\right]$

(c) $pd_t = pd(g_t; k_t) = A + Bg_t + Ck_t$

We need to approximize the following function: $\log(1+\exp(A+Bg_{t+1}+Ck_{t+1}))$ around $\bar{pd} = A$

$$\log(1+\exp(A+Bg_t+Ck_t)) = \log(1+e^A) + \frac{e^A}{1+e^A}(A+B_{t+1}+Ck_{t+1}-A)$$

Thus, we get:

$$e^{\bar{k}+k_t} = E_t\left[\exp\left(\bar{g} - \frac{1}{2}\sigma_g^2 + g_{t+1} + \log(1+e^A) + \frac{e^A}{1+e^A}(A+Bg_{t+1}+Ck_{t+1}-A) - (A+Bg_t+Ck_t)\right)\right]$$

$$e^{\bar{k}+k_t} = \exp(\bar{g} - \frac{1}{2}\sigma_g^2 + \log(1+e^A) - A - Bg_t - Ck_t)E_t\left[\exp(g_{t+1} + \frac{e^A}{1+e^A}(Bg_{t+1}+Ck_{t+1})\right] =$$
$$\exp(\bar{g} - \frac{1}{2}\sigma_g^2 + \log(1+e^A) - A - Bg_t - Ck_t)E_t\left[\exp(g_{t+1}(1+\frac{e^A}{1+e^A}B) + \frac{e^A}{1+e^A}Ck_{t+1})\right]$$

Let's plug in g_{t+1} and k_{t+1}:

$$e^{\bar{k}+k_t} = \exp(\bar{g} - \frac{1}{2}\sigma_g^2 + \log(1+e^A) - A - Bg_t - Ck_t)E_t\left[\exp((\rho_g g_t + \sigma_g\varepsilon_g(t+1))(1+\frac{e^A}{1+e^A}B) + \frac{e^A}{1+e^A}C(\rho_k k_t + \sigma_k\varepsilon_k(t+1)))\right] =$$

$$= \exp(\bar{g} - \frac{1}{2}\sigma_g^2 + \log(1+e^A) - A - Bg_t - Ck_t + \rho_g g_t(1+\frac{e^A}{1+e^A}B) + \rho_k k_t\frac{e^A}{1+e^A}C)E_t\left[\exp(\sigma_g\varepsilon_g(t+1)(1+\frac{e^A}{1+e^A}B) + \sigma_k\varepsilon_k(t+1)C\frac{e^A}{1+e^A}\right]$$

Since we know that the random variables are normaly distributed (iid), we can use the moment generating function: $E_t[e^{\lambda\varepsilon_g(t+1)}] = e^{\frac{\lambda^2}{2}}$

In order to compute an expectation:

$$E_t[\exp(\sigma_g\varepsilon_g(t+1)(1+\frac{e^A}{1+e^A}B) + \sigma_k\varepsilon_k(t+1)C\frac{e^A}{1+e^A}] = \exp(\frac{1}{2}\sigma_g^2(1+\frac{e^A}{1+e^A}B)^2 + \frac{1}{2}\sigma_k^2C^2\frac{e^{2A}}{(1+e^A)^2})$$

Thus, we have:

$$\bar{k}+k_t = \bar{g} - \frac{1}{2}\sigma_g^2 + \log(1+e^A) - A + g_t(\rho_g(1+\frac{e^A}{1+e^A}B) - B) + k_t(\rho_k\frac{e^A}{1+e^A}C - C) + \frac{1}{2}(\sigma_g^2(1+\frac{e^A}{1+e^A}B)^2 + \sigma_k^2C^2\frac{e^{2A}}{(1+e^A)^2})$$

Matching the coefficients gives:

$$\begin{cases} \bar{k} = \bar{g} - \frac{1}{2}\sigma_g^2 + \log(1+e^A) - A + \frac{1}{2}(\sigma_g^2(1+\frac{e^A}{1+e^A}B)^2 + \sigma_k^2C^2\frac{e^{2A}}{(1+e^A)^2}) \\ k_t: \rho_k\frac{e^A}{1+e^A}C - C = 1 \\ g_t: \rho_g(1+\frac{e^A}{1+e^A}B) - B = 0 \end{cases}$$

From the system we have:

$$\begin{cases} C = \dfrac{1}{\rho_k\frac{e^A}{1+e^A} - 1} = \dfrac{1+e^A}{e^A(\rho_k-1)-1} \\ B(\frac{e^A}{1+e^A}\rho_g - 1) = -\rho_g \Rightarrow B = \dfrac{-\rho_g(1+e^A)}{\rho_g e^A - 1 - e^A} = \dfrac{-\rho_g(1+e^A)}{e^A(\rho_g-1)-1} = \rho_g\dfrac{1+e^A}{e^A(1-\rho_g)+1} \\ A = \bar{g} - \bar{k} - \frac{1}{2}\sigma_g^2 + \log(1+e^A) + \frac{1}{2}(\sigma_g^2(1+\frac{e^A}{1+e^A}B)^2 + \sigma_k^2C^2\frac{e^{2A}}{(1+e^A)^2}) \end{cases}$$

(d) $\rho_g = \rho_k = \sigma_k = 0$

Plugging it into previous solution gives:

$$\begin{cases} B = 0 \\ C = -1 \\ \bar{k} = \bar{g} - \frac{1}{2}\sigma_g^2 + \log(1+e^A) - A + \frac{1}{2}\sigma_g = \bar{g} + \log(1+e^A) - A = \bar{g} - \log(\frac{e^A}{1+e^A}) \end{cases}$$

It follows that $pd_t = A$. It matches the results in point (a):

$$\log(\frac{e^A}{1+e^A}) = \bar{g} - \bar{k}$$

$$e^{\bar{g}-\bar{k}} = \frac{e^A}{1+e^A} = 1 + \frac{1}{e^A} \Rightarrow e^A = \frac{1}{e^{\bar{k}-\bar{g}}-1} = PD_t$$

For the stochastic process to be mean-reverting, it has to be stationary, which means the distribution doesn't depend on t. In our case the autocorrelation is represented by ρ_k and ρ_g. It can be expressed as $\rho_k = \frac{Cov(k_t; k_{t+1})}{Var(k_t)}$ (WSS process). It can be noticed that in order to have stationry process, ρ has to be strictly less than one in absolute value. If $\rho = 0 \Rightarrow$ the process is a random walk and if $\rho = 1 \Rightarrow$ unit root. Thus, $\rho_g \neq 1$; $\rho_k \neq 1$ and either ρ_k or $\rho_g \neq 0$:

1) Let's suppose $\rho_k \in (0,1)$ and $\rho_g = 0$. Thus, we have:

$$pd_{t+1} = A + Bg_{t+1} + Ck_{t+1} = A + B\sigma_g\varepsilon_g(t+1) + C(\rho_k k_t + \sigma_k\varepsilon_k(t+1)) = A + \rho_k k_t C + \overbrace{(\sigma_k\varepsilon_k(t+1)C + \sigma_g\varepsilon_g(t+1)B)}^{\text{i.i.d.}}$$

Since k_t is the only parameter that depends on the past information and $\rho_k \in (0,1) \Rightarrow$ the process is stationary.

2) Now let's suppose $\rho_g \in (0,1)$ and $\rho_k = 0$.

$$pd_{t+1} = A + B(\rho_g g_t + \sigma_g\varepsilon_g(t+1)) + C\sigma_k\varepsilon_k(t+1) = A + \rho_g g_t B + (\sigma_g\varepsilon_g(t+1)B + \sigma_k\varepsilon_k(t+1)C)$$

Since g_t is the only parameter that depends on the past information and $\rho_g \in (0,1) \Rightarrow$ the process is stationary.

Since empirically dividend growth is iid. ($\rho_g \approx 0$). If $\rho_k \in (0;1)$, it will guarantee that the pd_t is stochastic, mean-reverting process $\Rightarrow$

the expected return $E_t[R_{t+1}]$ has to be predictable.