PS6: Problem 1

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$R^e = \alpha + BF + \varepsilon$ $E[F] = \mu_F$ $V[F] = \Omega_F$ $E[\varepsilon|F] = 0$ $V[\varepsilon] = \Omega_\varepsilon$

a) $\Omega_\varepsilon = 0$ Show that the absence of arbitrag implies $\exists\, \lambda$ s.t. $\alpha = B\lambda$

Let's create the following portfolio:

$w'R^e = w'\alpha + w'BF + w'\varepsilon$

The absence of arbitrage implies that there exist w such that $w'B = 0$ which means that the return of the portfolio without systematic risk should be equal to the risk-free rate. It's also given that $\Omega_\varepsilon = 0 \Rightarrow w'E(R) = R_0 \Rightarrow$
$\Rightarrow w'(\alpha - 1R_0) = 0 \Rightarrow w'\alpha = 0 \Rightarrow$ zero premium after subtraction of R_0.

Thus, $w'B = 0$ (no exposure to the risk factor) and $w'\alpha = 0$, it means that w' is orthogonal to B and is orthogonal to $\alpha \Rightarrow \alpha$ and B are pointing in the same direction $\Rightarrow$
$\Rightarrow$ the vectors are linear combinations of each other $\Rightarrow \alpha = B\lambda$ is another linear combination and it also implies no arbitrage.

b) $F_k = R^e_{F_k} = w_k^T R^e$ $\forall k$ Show that the absence of arbitrage implies that $\alpha = 0$

No arbitrage implies the APT: $R^e_i = (B_i + \lambda)$ for a traded factor $F = R^e_{F_k}$

Since $B_F = 1 \Rightarrow$ the APT implies $R^e_F = R^e_F + \lambda \Rightarrow \lambda$ has to be 0 for no arbitrage and APT simplifies $R^e_i = B_i R^e_F$. Since when there is no arbitrage $\lambda = 0 \Rightarrow \alpha = B\lambda = B\cdot 0 = 0$

Thus, the absence of arbitrage also implies $\alpha = 0$

c) $\Omega_\varepsilon \neq 0$; $\alpha \neq 0$; $V[R] = \Sigma = B\Omega^F B^T + \Omega_\varepsilon$

$r^e = R^e - BR^e_F$ – pure alpha bet excess return $\Rightarrow$

Replicating the pure alpha-bet returns:

diagonal matrix $\overset{N\times N}{W_\alpha}\ \overset{N\times L}{R^e} = r^e = R^e - BR^e_F = R^e - BW^T R^e$

$W_\alpha = (1 - BW^T)$ – weights in various assets to replicate pure-bet portfolios.

Decomposition into mean-variance efficient factor portfolio and portfolio of pure alpha-bets:

$R_p = X_F R^e_F + Y(R^e - BR^e_F)$

$$\left.\begin{aligned} E[R_p] &= X_F \mu^e_F + Y\alpha_i \\ V[R_p] &= X_F' \Omega_F X_F + Y'\Omega_\varepsilon Y \end{aligned}\right\} \Rightarrow$$

$\Rightarrow \max E[R_p] - \frac{a}{2} V[R_p] = X_F \mu^e_F + Y\alpha_i - \frac{a}{2}(X_F' \Omega_F X_F + Y'\Omega_\varepsilon Y)$

$$\left.\begin{aligned} &\text{FOC}_{X_F}: \mu^e_F - aX_F'\Omega_F \Rightarrow X_F = \frac{\mu^e_F}{a}\Omega^{-1}_F \\ &\text{FOC}_{Y}: \alpha_i - aY'\Omega_\varepsilon \Rightarrow Y = \frac{\alpha}{a}\Omega^{-1}_\varepsilon \end{aligned}\right\}$$ Max Sharpe Ratio portfolio weights X_F in factor portfolio; Y in α-bets.

Sharpe Ratio:

$$SR_p = \sqrt{SR_F^2 + IR^2} = \sqrt{\mu_F^{e\prime} \Omega_F^{-1} \mu_F^e + \alpha' \Omega_\varepsilon^{-1} \alpha}$$

$$SR_F = \mu_F^e / \sigma_F$$

$$IR = \alpha_i / \sigma_i$$

If $\alpha = 0 \Rightarrow IR = 0 \Rightarrow$ α-bets don't improve the Sharpe Ratio $\Rightarrow$
It's optimal to invest in K-factor portfolio.