PS10

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Ex. 1

$P_{t+1} = P_t + \mu + \sigma \varepsilon_{t+1}$; $r_t = 0$; n_{-1} - initial position ; $\frac{\lambda}{2} u_t$ - trading cost

$E\left[\sum_{t=0}^{T} \rho^t \left\{ n_t \mu - \frac{\lambda}{2}(n_t - n_{t-1})^2 - \frac{\gamma}{2} n_t^2 \sigma^2 \right\}\right]$ - total discounted wealth

(1) $V(k, n_{k-1}) = \max_{n_t, t>k} E\left[\sum_{t=k}^{T} \rho^{t-k} \left\{ n_t \mu - \frac{\lambda}{2}(n_t - n_{t-1})^2 - \frac{\gamma}{2} n_t^2 \sigma^2 \right\}\right]$ - value function at time $k \in [0, T]$

The value function at time T:

$$V(T, n_{T-1}) = \max_{n_T} \left\{ n_T \mu - \frac{\lambda}{2}(n_T - n_{T-1})^2 - \frac{\gamma}{2} n_T^2 \sigma^2 \right\}$$

FOC_{n_T}: $\mu - \lambda(n_T - n_{T-1}) - \gamma n_T \sigma^2 = 0$

$\mu - \lambda n_T + \lambda n_{T-1} - \gamma n_T \sigma^2 = 0$

$\mu - n_T(\lambda + \gamma\sigma^2) + \lambda n_{T-1} = 0$

$n_T^* = \frac{\mu - \lambda n_{T-1}}{\lambda + \gamma \sigma^2} \Rightarrow \lambda n_{T-1} = (\lambda + \gamma\sigma^2) n_T - \mu$

Let's plug n_T^* into the value function:

$$V(T, n_{T-1}) = n_T \mu - \frac{\lambda}{2}(n_T^2 - 2 n_T n_{T-1} + n_{T-1}^2) - \frac{\gamma}{2} n_T^2 \sigma^2$$
$$= n_T \mu - \frac{\lambda}{2} n_T^2 + \lambda n_T n_{T-1} - \frac{\lambda}{2} n_{T-1}^2 - \frac{\gamma}{2} n_T^2 \sigma^2$$
$$= n_T\left(\mu - \frac{n_T}{2}(\lambda + \gamma\sigma^2) + \lambda n_{T-1}\right) - \frac{\lambda}{2} n_{T-1}^2$$
$$= n_T\left(\mu - \frac{n_T}{2}(\lambda + \gamma\sigma^2) + (\lambda + \gamma\sigma^2) n_T - \mu\right) - \frac{\lambda}{2} n_{T-1}^2$$
$$= n_T\left(\left(n_T - \frac{n_T}{2}\right)(\lambda + \gamma\sigma^2)\right) - \frac{\lambda}{2} n_{T-1}^2$$
$$= n_T\left(\frac{n_T}{2}(\lambda + \gamma\sigma^2)\right) - \frac{\lambda}{2} n_{T-1}^2$$
$$= \frac{1}{2}\left(\frac{\mu + \lambda n_{T-1}}{\lambda + \gamma\sigma^2}\right)\left(\frac{\mu + \lambda n_{T-1}}{\lambda + \gamma\sigma^2}(\lambda + \gamma\sigma^2)\right) - \frac{\lambda}{2} n_{T-1}^2$$
$$= \frac{1}{2}\left(\frac{(\mu + \lambda n_{T-1})^2}{\lambda + \gamma\sigma^2}\right) - \frac{\lambda}{2} n_{T-1}^2$$
$$= \frac{1}{2}\left(\frac{\mu^2 + 2\lambda n_{T-1}\mu + \lambda^2 n_{T-1}^2}{\lambda + \gamma\sigma^2}\right) - \frac{\lambda}{2} n_{T-1}^2$$
$$= \frac{1}{2}\left(\frac{\lambda^2}{\gamma + \sigma^2} - \lambda\right) n_{T-1}^2 + \frac{\mu\lambda}{\lambda + \gamma\sigma^2} n_{T-1} + \frac{1}{2}\frac{\mu^2}{\lambda + \gamma\sigma^2}$$
$$= -\frac{1}{2}\left(\lambda - \frac{\lambda^2}{\gamma + \sigma^2}\right) n_{T-1}^2 + \frac{\mu\lambda}{\lambda + \gamma\sigma^2} n_{T-1} + \frac{1}{2}\frac{\mu^2}{\lambda + \gamma\sigma^2}$$
$$= -\frac{1}{2} Q_T n_{T-1}^2 + q_T n_{T-1} + C_T$$

Thus, $Q_T = \lambda - \frac{\lambda^2}{\gamma + \sigma^2}$

$q_T = \frac{\mu\lambda}{\lambda + \gamma\sigma^2}$

$C_T = \frac{1}{2}\frac{\mu^2}{\lambda + \gamma\sigma}$

(2) $$V(t, n_{t-1}) = \max_{n_t}\{n_t\mu - \frac{\lambda}{2}(n_t - n_{t-1})^2 - \frac{\gamma}{2}n_t^2\sigma^2 + \rho E_t[V(t+1, n_t)]\}$$

Using the solution obtained in the previous point:

$$V(t, n_{t-1}) = \max_{n_t}\{n_t\mu - \frac{\lambda}{2}(n_t - n_{t-1})^2 - \frac{\gamma}{2}n_t^2\sigma^2 + \rho(-\frac{1}{2}Q_{t+1}n_t^2 + q_{t+1}n_t + C_{t+1})\} \quad \forall t$$

FOC_n : $\mu - \lambda(n_t - n_{t-1}) - \gamma n_t\sigma^2 - \rho Q_{t+1}n_t + \rho q_{t+1} = 0$

$\mu - n_t(\lambda + \gamma\sigma^2 + \rho Q_{t+1}) + \lambda n_{t-1} + \rho q_{t+1} = 0$

$$n_t^* = \frac{\mu + \lambda n_{t-1} + \rho q_{t+1}}{\lambda + \gamma\sigma^2 + \rho Q_{t+1}} \Rightarrow \lambda n_{t-1} = n_t^*(\lambda + \gamma\sigma^2 + \rho Q_{t+1}) - \mu - \rho q_{t+1}$$

Let's plug the solution into the value function:

$$V(t, n_{t-1}) = n_t\mu - \frac{\lambda}{2}(n_t^2 - 2n_tn_{t-1} + n_{t-1}^2) - \frac{\gamma}{2}n_t^2\sigma^2 - \rho\frac{1}{2}Q_{t+1}n_t^2 + \rho q_{t+1}n_t + \rho C_{t+1}$$

$$= n_t\mu - \frac{\lambda}{2}n_t^2 + \lambda n_tn_{t-1} - \frac{\lambda}{2}n_{t-1}^2 - \frac{\gamma}{2}n_t^2\sigma^2 - \frac{1}{2}\rho Q_{t+1}n_t^2 + \rho q_{t+1}n_t + \rho C_{t+1}$$

$$= n_t(\mu - \frac{1}{2}n_t(\lambda + \gamma\sigma^2 + \rho Q_{t+1}) + \lambda n_{t-1} + \rho q_{t+1}) - \frac{\lambda}{2}n_{t-1}^2 + \rho C_{t+1}$$

$$= n_t(\mu - \frac{1}{2}n_t(\lambda + \gamma\sigma^2 + \rho Q_{t+1}) + n_t(\lambda + \gamma\sigma^2 + \rho Q_{t+1}) - \mu - \rho q_{t+1} + \rho q_{t+1}) - \frac{\lambda}{2}n_{t-1}^2 + \rho C_{t+1}$$

$$= n_t((n_t - \frac{1}{2}n_t)(\lambda + \gamma\sigma^2 + \rho Q_{t+1})) - \frac{\lambda}{2}n_{t-1}^2 + \rho C_{t+1}$$

$$= n_t(\frac{1}{2}n_t(\lambda + \gamma\sigma^2 + \rho Q_{t+1})) - \frac{\lambda}{2}n_{t-1}^2 + \rho C_{t+1}$$

$$= \left(\frac{\mu + \lambda n_{t-1} + \rho q_{t+1}}{\lambda + \gamma\sigma^2 + \rho Q_{t+1}}\right)\left(\frac{1}{2}\frac{\mu + \lambda n_{t-1} + \rho q_{t+1}}{\lambda + \gamma\sigma^2 + \rho Q_{t+1}}(\lambda + \gamma\sigma^2 + \rho Q_{t+1})\right) - \frac{\lambda}{2}n_{t-1}^2 + \rho C_{t+1}$$

$$= \frac{1}{2}\frac{(\mu + \lambda n_{t-1} + \rho q_{t+1})^2}{\lambda + \gamma\sigma^2 + \rho Q_{t+1}} - \frac{\lambda}{2}n_{t+1}^2 + \rho C_{t+1}$$

$$= \frac{1}{2}\left(\frac{(\mu^2 + \lambda^2 n_{t-1}^2 + \rho^2 q_{t+1}^2 + 2\mu\lambda n_{t-1} + 2\mu\rho q_{t+1} + 2\lambda n_{t-1}\rho q_{t+1})}{\lambda + \gamma\sigma^2 + \rho Q_{t+1}}\right) - \frac{\lambda}{2}n_{t-1}^2 + \rho C_{t+1}$$

$$= -\frac{1}{2}\left(\lambda - \frac{\lambda^2}{\lambda + \gamma\sigma^2 + \rho Q_{t+1}}\right)n_{t-1}^2 + \frac{\lambda(\mu + \rho q_{t+1})}{\lambda + \gamma\sigma^2 + \rho Q_{t+1}}n_{t-1} + \frac{1}{2}\frac{(\mu + \rho q_{t+1})^2}{\lambda + \gamma\sigma^2 + \rho Q_{t+1}} + \rho C_{t+1}$$

$$= -\frac{1}{2}Q_t n_{t-1}^2 + q_t n_{t-1} + C_t$$

Thus,
$$Q_t = \lambda - \frac{\lambda^2}{\lambda + \gamma\sigma^2 + \rho Q_{t+1}}$$

$$q_t = \frac{\lambda(\mu + \rho q_{t+1})}{\lambda + \gamma\sigma^2 + \rho Q_{t+1}}$$

$$C_t = \frac{1}{2}\frac{(\mu + \rho q_{t+1})^2}{\lambda + \gamma\sigma^2 + \rho Q_{t+1}} + \rho C_{t+1}$$

.b

(3) Since we have the value function $V(t; n_{t-1})$, aim_t is a portfolio that maximizes the value function at time $t+1$:

$$V(t+1, aim_t) = -\frac{1}{2}Q\, aim_t^2 + q\, aim_t + C$$

FOC: $-Q_{t+1}aim_t + q_{t+1} = 0 \Rightarrow$

$$aim_t = \frac{q_{t+1}}{Q_{t+1}} = \frac{\lambda(\mu + \rho q_{t+2})}{\lambda + \gamma\sigma^2 + \rho Q_{t+2}} \cdot \left(\frac{1}{\lambda} - \frac{\lambda + \gamma\sigma^2 + \rho Q_{t+2}}{\lambda^2}\right)$$

$$= \frac{\mu + \rho q_{t+2}}{\lambda + \gamma \sigma^2 + \rho Q_{t+2}} - \frac{\mu + \rho q_{t+2}}{\lambda}$$

$$= \frac{\mu + \rho q_{t+2}}{\gamma \sigma^2 + \rho Q_{t+2}}$$

$$n_t^* = \frac{1}{\lambda + \gamma \sigma^2 + \rho Q_{t+1}} \left(\mu + \lambda n_{t-1} + \rho q_{t+1} \right)$$

$$= \frac{\gamma \sigma^2 + \rho Q_{t+1}}{\lambda + \gamma \sigma^2 + \rho Q_{t+1}} \left(\frac{\mu + \lambda n_{t-1} + \rho q_{t+1}}{\gamma \sigma^2 + \rho Q_{t+1}} \right)$$

$$= \frac{\gamma \sigma^2 + \rho Q_{t+1}}{\lambda + \gamma \sigma^2 + \rho Q_{t+1}} \cdot \frac{\mu + \rho q_{t+1}}{\gamma \sigma^2 + \rho Q_{t+1}} + \frac{\lambda}{\lambda + \gamma \sigma^2 + \rho Q_{t+1}} n_{t-1}$$

$$= \tau_t \, aim_{t-1} + (1 - \tau_t) n_{t-1}$$

Thus, $\tau_t = \frac{\gamma \sigma^2 + \rho Q_{t+1}}{\lambda + \gamma \sigma^2 + \rho Q_{t+1}}$

For 1.4 – 1.6 See Python code