PS6: Problem 1

Tuesday, March 29, 2022 2:01 PM

$R^e = \alpha + BF + \varepsilon \quad E[F] = \mu_F \quad V[F] = \Omega_F \quad E[\varepsilon | F] = 0 \quad V[\varepsilon] = \Omega_\varepsilon$

(a) $\Omega_\varepsilon = 0$ Show that the absence of arbitrag implies $\exists \lambda$ s.t. $\alpha = B\lambda$

Let's create the following portfolio:

$w'R^e = w'\alpha + w'BF + w'\varepsilon$

The absence of arbitrage implies that there exist w such that $w'B = 0$ which means that the return of the portfolio without systematic risk should be equal to the risk-free rate. It's also given that $\Omega_\varepsilon = 0 \Rightarrow w'E(R) = R_0 \Rightarrow$
$\Rightarrow w'(\alpha - 1R_0) = 0 \Rightarrow w'\alpha = 0 \Rightarrow$ zero premium after subtraction of R_0.

Thus, $w'B = 0$ (no exposure to the risk factor) and $w'\alpha = 0$, it means that w' is orthogonal to B and is orthogonal to $\alpha \Rightarrow \alpha$ and B are pointing in the same direction $\Rightarrow$
$\Rightarrow$ the vectors are linear combinations of each other $\Rightarrow \alpha = B\lambda$ is another linear combination and it also implies no arbitrage.

(b) $F_k = R^e_{F_k} = w_k^T R^e \ \forall k$ Show that the absence of arbitrage implies that $\alpha = 0$

No arbitrage implies the APT: $R^e_i = (B_i + \lambda)$ for a traded factor $F = R^e_{F_k}$.

Since $B_F = 1 \Rightarrow$ the APT implies $R^e_F = R^e_F + \lambda \Rightarrow \lambda$ has to be 0 for no arbitrage and APT simplifies $R^e_i = B_i R^e_F$. Since when there is no arbitrage $\lambda = 0 \Rightarrow \alpha = B\lambda = B \cdot 0 = 0$

Thus, the absence of arbitrage also implies $\alpha = 0$

(c) $\Omega_\varepsilon \neq 0$; $\alpha \neq 0$, $V[R] = \Sigma = B\Omega^F B^T + \Omega_\varepsilon$

$r^e = R^e - BR^e_F$ - pure alpha bet excess return $\Rightarrow$

Replicating the pure alpha-bet returns:

diagonal matrix $\underset{N\times N}{W_\alpha} \underset{N\times 1}{R^e} = r^e = R^e - BR^e_F = R^e - BW^T R^e$

$W_\alpha = (1 - BW^T)$ - weights in various assets to replicate pure-bet portfolios.

Decomposition into mean-variance efficient factor portfolio and portfolio of pure alpha-bets:

$R_p = X_F R^e_F + Y(R^e - BR^e_F)$

$$\left.\begin{aligned} E[R_p] &= X_F \mu^e_F + Y\alpha_i \\ V[R_p] &= X_F' \Omega_F X_F + Y'\Omega_\varepsilon Y \end{aligned}\right\} \Rightarrow$$

$\Rightarrow \max E[R_p] - \frac{a}{2} V[R_p] = X_F \mu^e_F + Y\alpha_i - \frac{a}{2}(X_F' \Omega_F X_F + Y'\Omega_\varepsilon Y)$

$$\left.\begin{aligned} &FOC_{X_F}: \ \mu^e_F - aX_F'\Omega_F \Rightarrow X_F = \frac{\mu^e_F}{a}\Omega^{-1}_F \\ &FOC_{Y}: \ \alpha_i - aY'\Omega_\varepsilon \Rightarrow Y = \frac{\alpha}{a}\Omega^{-1}_\varepsilon \end{aligned}\right\}$$ Max Sharpe Ratio portfolio weights X_F in factor portfolio; Y in α-bets.

Sharpe Ratio:

$$SR_p = \sqrt{SR_F^2 + IR^2} = \sqrt{\mu_F^{e\prime} \Sigma_F^{-1} \mu_F^e + \alpha' \Sigma_\varepsilon^{-1} \alpha}$$

$$SR_F = \mu_F^e / \sigma_F$$

$$IR = \alpha_i / \sigma_i$$

If $\alpha = 0 \Rightarrow IR = 0 \Rightarrow$ α-bets don't improve the Sharpe Ratio $\Rightarrow$ It's optimal to invest in K-factor portfolio.

(d) Ω_ε diagnal means $E(\delta_i \delta_j) = 0$

as $N \to \infty$, for pure-alpha bet portfolio $R_\alpha = w_\alpha R^e = \frac{1}{N}\sum_i^N \text{sign}[\alpha_i] r_i$

$E(R_\alpha) = \frac{1}{N}\sum_i |\alpha_i|$ $\quad Var[R_\alpha] = \frac{1}{N^2}\sum_i \delta_{\varepsilon i}^2 \le \frac{1}{N}\max_i \delta_{\varepsilon i}^2$

$SR_p = \sqrt{SR_F^2 + \sum_i IR_i^2}$

$\sum_i IR_i^2 = \sum_i \frac{\alpha_i^2}{\delta_i^2}$ $\qquad \lim_{n\to\infty} \sum_i IR_i^2 = \infty$ if $\alpha \neq 0$.

$\lim_{n\to\infty} SR_p = \infty$

$\therefore$ In order to avoid arbitrage, most $\alpha_i = 0$, so that SR_p is finite.

Thus, APT holds for most assets.

(e). $Cov(R^e, f) = Cov(\alpha + Bf + \varepsilon, f)$ $\qquad E(\varepsilon | f) = 0$

$= Cov(Bf, f) + Cov(\varepsilon, f)$

$= Cov(Bf, f) = B\,Var(f)$

$\therefore B = Cov(R^e, f)\,Var(f)^{-1}$

Now, we assume Ω_ε is diagonal

$Var(R_f^e) = Var(w^T R^e)$

$= w^T Var(R^e) w$

$= w^T Var(\alpha + Bf + \varepsilon) w$

$= w^T Var(Bf) w + w^T \Omega_\varepsilon w$

$= w^T B \Omega_f B^T w + w^T \Omega_\varepsilon w$

$= w^T Cov(R^e, f) \Omega_f^{-1} Cov(R^e, f)^T w + w^T \Omega_\varepsilon w$

$= Cov(R_f^e, f) \Omega_f^{-1} Cov(R_f^e, f)^T + w^T \Omega_\varepsilon w$

$= \Omega_f + w^T \Omega_\varepsilon w$

when $n \to \infty$, $Var(R_f^e) = \Omega_f$

however, $w^T \Omega_\varepsilon w = \sum_i w_i^T w_i \sigma_i^2 > 0$.

$\therefore$ contradiction

For $Cov(\varepsilon_i, \varepsilon_j) \neq 0$,

This means we can't depict assets well in APT when $n \to \infty$, there are always some factor we didn't consider.