EX1

(a) $W_t = \frac{\Sigma^{-1}(\mu - R_0)}{1\Sigma^{-1}(\mu - R_0)}$ $\Sigma = \begin{bmatrix} 1 & 0.2 & 0.2 \\ 0.2 & 1 & 0.2 \\ 0.2 & 0.2 & 1 \end{bmatrix}$

$W_t = \begin{bmatrix} 0.6086 \\ 0.2029 \\ 0.1885 \end{bmatrix}$.

$\mu_t = \mu' W_t = 0.1073$

$\delta_t^2 = W_t' \Sigma W_t = \sum_i W_i^2 \delta_i^2 + \sum_{i \neq j} W_i W_j \delta_i \delta_j \times 0.2 = 0.0165$.

$\delta_t = 0.128$

$SR = \frac{\mu_t - R_0}{\delta_t} = 0.447$

(b) $\delta_z^2 = W_z' \Sigma W_t = 0$.

$W_z' \Sigma W_t = \frac{W_z' \Sigma \Sigma^{-1}(\mu - R_0)}{1\Sigma^{-1}(\mu - R_0)} = \frac{W_z'(\mu - R_0)}{1\Sigma^{-1}(\mu - R_0)} = 0$. $W_z'\mu = W_z' R_0 = R_0$.

$\mu_z = R_0$

As a m-v portfolio, $W_z = \lambda\Sigma^{-1}1 + \gamma\Sigma^{-1}\mu$. from Lecture 2

with all param $\lambda, \gamma, A, B, \Delta$ in Lecture 2.

we get $W_z = (1.4236, -0.1945, -0.2291)'$, $\mu_z = R_0$

$\delta_z^2 = W_z' \Sigma^{-1} W_z = 0.0241$

$SR = 0$

(c) ① $\mu_p = R_0 + W'(\mu - R_0 1)$. $\delta_p^2 = W' \Sigma W$

$\max(\mu_p - \frac{a}{2}\delta_p^2) = \max_W (R_0 + W'(\mu - R_0 1) - \frac{a}{2} W' \Sigma W)$.

first order: $\mu - R_0 1 = a\Sigma W$

$\therefore W_0 = \frac{1}{a}\Sigma^{-1}(\mu - R_0 1)$

only invest in tang portfolio

for $W_t = \frac{\Sigma^{-1}(\mu - R_0 1)}{1\Sigma^{-1}(\mu - R_0 1)}$ $W_0 = \frac{1\Sigma^{-1}(\mu - R_0 1)}{a} W_{tan}$. and riskfree asset

② By leverage constrain: $X_t + X_z \le m$

$\therefore$ the question can be formulated as

$$\max E(R_0 + X_t(R_t - R_0) + X_z(R_z - R_0)) - \frac{a}{2} Var(X_t R_t + X_z R_z)$$

$$s.t. \quad X_t + X_z \le m$$

$$L = R_0 + X_t(\mu_t - R_0) + X_z(\mu_z - R_0) - \frac{a}{2}(X_t^2 \delta_t^2 + X_z^2 \delta_z^2) + \lambda(m - X_t - X_z)$$

$$\frac{\partial L}{\partial X_t} = \mu_t - R_0 - a\delta_t^2 X_t - \lambda = 0 \qquad X_t = \frac{1}{a\delta_t^2}(\mu_t - R_0 - \lambda)$$

$$\frac{\partial L}{\partial X_z} = \mu_z - R_0 - a\delta_z^2 X_z - \lambda = 0 \quad \Rightarrow \quad X_z = \frac{1}{a\delta_z^2}(-\lambda)$$

$$\lambda(m - X_z - X_t) = 0$$

Case A:

$$X_z + X_t = m$$

$$\frac{1}{a\delta_t^2}(\mu_t - R_0 - \lambda) + \frac{1}{a\delta_z^2}(-\lambda) = m$$

$$\delta_z^2(\mu_t - R_0 - \lambda) + \delta_t^2(-\lambda) = am\delta_t^2\delta_z^2$$

$$(\delta_t^2 + \delta_z^2)\lambda = \delta_z^2(\mu_t - R_0) - am\delta_t^2\delta_z^2$$

$$\lambda = \frac{\delta_z^2(\mu_t - R_0) - am\delta_t^2\delta_z^2}{\delta_t^2 + \delta_z^2}$$

$$\therefore X_z = \frac{-\mu_t + R_0 + am\delta_t^2}{a(\delta_t^2 + \delta_z^2)}$$

$$X_t = \frac{\mu_t - R_0}{a(\delta_t^2 + \delta_z^2)} + \frac{m\delta_z^2}{\delta_t^2 + \delta_z^2}$$

$$W_{risk} = W_z X_z + W_t X_t$$

$$= \frac{\Sigma^{-1}(\mu - R_0)}{1\Sigma^{-1}(\mu - R_0)}\left(\frac{\mu_t - R_0}{a(\delta_t^2 + \delta_z^2)} + \frac{m\delta_z^2}{\delta_t^2 + \delta_z^2}\right) + (\lambda\Sigma^{-1}1 + \gamma\Sigma^{-1}\mu) \cdot \frac{-\mu_t + R_0 + am\delta_z^2}{a(\delta_t^2 + \delta_z^2)}$$

where λ, γ are same from (b),

$W_{p1} = (1-m)W_0 + W_{risk}$. $\quad W_1, W_2, W_3$ is in W_{risk}

case B. $X_t + X_z < m$

$\lambda = 0 \quad X_t = \frac{1}{a\delta_t^2}(u_t - R_0) \quad X_z = 0.$

$\therefore$ all risky assets are from tangency portfolio.

$W_{risk} = X_t W_t = \frac{\Sigma^{-1}(u - R_0 1)}{1\Sigma^{-1}(u - R_0)} \cdot \frac{(u_t - R_0)}{a\delta_t^2} = \frac{\Sigma^{-1}(u - R_0 1)}{1\Sigma^{-1}(u - R_0)} \left(\frac{C - BR_0}{B - AR_0} - R_0\right) \cdot \frac{(B - AR_0)^2}{a(C - 2R_0B + R_0^2 A)}$

$= \frac{1}{a}\Sigma^{-1}(u - R_0 1)$

$W_{p2} = \left(1 - \frac{1}{a\delta_t^2}(u_t - R_0)\right)W_0 + \frac{1}{a}\Sigma^{-1}(u - R_0 1).$

$\therefore W_p = \begin{cases} W_{p1} & a < a^* \\ W_{p2} & a \geq a^* \end{cases}$

(d). from (c),

$X_t + X_z < m \qquad \frac{1}{a\delta_t^2}(u_t - R_0) < m$

$a > \frac{u_t - R_0}{m\delta_t^2} \qquad a^* = \frac{u_t - R_0}{m\delta_t^2} = 2.676$

when $a > a^*$, $X_z = 0$, portfolio not constrained.

$a < a^*$, $X_z = \frac{-u_t + R_0 + am\delta_t^2}{a(\delta_t^2 + \delta_z^2)}$, it will invest in zero-beta portfolio

(e). $a < a^*$, sharp ratio falls as a falls. because it will invest on zero-beta portfolio which has lower SR.