1.(a) For optimal portfolio,

$\triangle$ $\max E(R_p) - \frac{\gamma}{2} Var(R_p)$ s.t. $w'1 = 1$

$= \max\{R_0 + E[\sum_{i=1}^{N} w_i (R_i - R_0)] - \frac{\gamma}{2} Var[\sum_{i=0}^{N} (w_i R_i)]$

$= \max\{R_0 + E[\sum_{i=1}^{N} w_i (R_i - R_0)] - \frac{\gamma}{2} \sum_{i=0}^{N} \sum_{j=0}^{N} cov(w_i R_i, w_j R_j)$ ← R_0 represent risk-free asset

$= \max\{R_0 + \sum_{i=1}^{N} [w_i (u_i - R_0)] - \frac{\gamma}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} cov(w_i R_i, w_j R_j)$ $Var(R_0) = 0$

$= \max\{R_0 + \sum_{i=1}^{N} (w_i (u_i - R_0)) - \frac{\gamma}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j cov(R_i, R_j)$

$\triangle$ first order condition:

$\frac{\partial U(w)}{\partial w_i} = (u_i - R_0) - 2 \cdot \frac{\gamma}{2} \sum_{j=1}^{N} w_j cov(R_i, R_j) = 0$

$u_i - R_0 = \gamma \sum_{j=1}^{N} w_j cov(R_i, R_j)$

$= \gamma \sum_{j=1}^{N} cov(R_i, w_j R_j) = \gamma cov(R_i, R_p)$

for $R_0 = r_f$, $R_p = \sum_{j=0}^{N} w_j R_j$,

$u_i - r_f = \gamma cov(R_i, R_p)$

(b)

From (a), we have $u_i - r_f = \gamma cov(R_i, R_p) = \gamma \sum_{j=0}^{N} cov(R_i, R_j) \cdot w_j$

in vector form, $u - r_f 1 = \gamma \Sigma \cdot w$, Σ is cov matrix, w is weight vec.

$w'(u - r_f 1) = \gamma w' \Sigma w = \gamma \delta_p^2$

$w'(u - r_f 1) = u_p - r_f = \gamma \delta_p^2$

$u_i - r_f = \gamma cov(R_i, R_p)$

$(u_i - r_f) \gamma \delta_p^2 = \gamma cov(R_i, R_p)(u_p - r_f)$

$\therefore u_i - r_f = \frac{u_p - r_f}{\delta^2} cov(R_i, R_p)$

$= \beta_{i,p}(u_p - r_f)$

(c)

From (b), we have $u_i = R_f + \beta_{i,p}(u_p - R_f)$

thus, we can set $R_i = \beta_0 + \beta_1 (R_p - R_f) + \varepsilon_i$.

$E(\varepsilon_i)=0$, $Cov(R_p, \varepsilon_i)=0$ $\because R_f$ is stable $\therefore Cov(R_p-R_f, \varepsilon_i)=0$

$$\beta_1 = \frac{Cov(R_i, R_p-R_f)}{Var(R_p-R_f)} = \frac{Cov(R_i, R_p)}{Var(R_p)} = \frac{Cov(R_i, R_p)}{\delta_p^2}$$

$$\therefore R_i = \beta_0 + \frac{Cov(R_i, R_p)}{\delta_p^2}(R_p-R_f)+\varepsilon_i$$

$$E(R_i) = \mu_i = E(\beta_0) + \frac{Cov(R_i, R_p)}{\delta_p^2}(\mu_p - R_f) + E(\varepsilon_i)$$

$$= E(\beta_0) + \frac{Cov(R_i, R_p)}{\delta_p^2}(\mu_p - R_f) + 0$$

from (b), $\mu_i = R_f + \frac{Cov(R_i, R_p)}{\delta_p^2}(\mu_p - R_f)$

$\therefore$ $E(\beta_0) = R_f$

$\therefore$ we can get $R_i = R_f + \beta_i(R_p - R_f) + \varepsilon_i$

(d) By definition, $SR_p = \frac{\mu_p - R_f}{\delta_p}$. Thus, $\mu_p = SR_p \cdot \delta_p + R_f$.

From lecture, mean-var efficient portfolio is combined by risk-free asset and tangency portfolio.

Assume R_t is return of tangency portfolio, and we invest $W<1$ in it.

$R_{p_1} = W_1 R_t + (1-W_1)R_f$, $R_{p_2} = W_2 R_t + (1-W_2)R_f$

$$Var(R_{p_1}) = Var(W_1 R_t + (1-W_1)R_f)$$

$$= Var(W_1 R_t) = W_1^2 \cdot Var(R_t) = W_1^2 \delta_t^2$$

$$Var(R_{p_2}) = W_2^2 Var(R_t) = W_2^2 \delta_t^2$$

$$\therefore SR_{p_1} = \frac{\mu_{p_1} - R_f}{\delta_{p_1}} = \frac{E(W_1 R_t + (1-W_1)R_f) - R_f}{\delta_{p_1}}$$

$$= \frac{W_1 \mu_t - W_1 R_f}{W_1 \delta_t} = \frac{\mu_t - R_f}{\delta_t}$$

$$SR_{p_2} = \frac{E(W_2 R_2 + (1-W_2)R_f) - R_f}{\delta_{p_2}} = \frac{W_2 \mu_t - W_2 R_f}{W_2 \delta_t} = \frac{\mu_t - R_f}{\delta_t}$$

$\therefore SR_{p_1} = SR_{p_2} \longrightarrow$ all mean-variance efficient portfolios have same SR