ECE4634
Digital Communications
Fall 2007

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Lecture # 21 – Introduction to Signal-Space Approach to Modulation

Analog and Digital Communications

Overview

- We have described three families of modulation schemes known as Phase Shift Keying, Frequency Shift Keying, and Amplitude Shift Keying
  - Each of these can be binary or *M*-ary
  - We have also looked at combinations of these (QAM)
- We would like to examine the performance of all of the modulation schemes discussed
- To do so, we need a method of representing any modulation scheme known as the *signal space method*
- The signal space concept will allow us to design the optimal receiver and determine its performance

Key Ideas from I/Q Representation of Signals

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$$s(t) = x(t)\cos(2\pi f_c t) - y(t)\sin(2\pi f_c t)$$

- We can represent bandpass signals independent of carrier frequency.
- The idea of quadrature sets up a coordinate system for looking at common modulation types.
- The coordinate system is sometimes called a signal constellation diagram.
- In-phase (Real part of complex baseband) maps to *x*-axis and Quadrature (imaginary part of complex baseband) maps to the *y*-axis

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Example of Signal Constellation Diagram: QPSK

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$$s_i(t) = A\cos\left(2\pi f_c t + \theta_i\right)$$

$$\theta_i = \begin{cases} \dfrac{\pi}{4} & b_i b_{i+1} = 00 \\ \dfrac{3\pi}{4} & b_i b_{i+1} = 10 \\ -\dfrac{3\pi}{4} & b_i b_{i+1} = 11 \\ -\dfrac{\pi}{4} & b_i b_{i+1} = 01 \end{cases}$$

$$s_i(t) = x_i(t)\cos\left(2\pi f_c t\right) - y_i(t)\sin\left(2\pi f_c t\right)$$

$$x_i(t) = \begin{cases} \dfrac{A\sqrt{2}}{2} & b_i = 0 \\ -\dfrac{A\sqrt{2}}{2} & b_i = 1 \end{cases} \qquad y_i(t) = \begin{cases} \dfrac{A\sqrt{2}}{2} & b_{i+1} = 0 \\ -\dfrac{A\sqrt{2}}{2} & b_{i+1} = 1 \end{cases}$$

Example of Signal Constellation Diagram: QPSK

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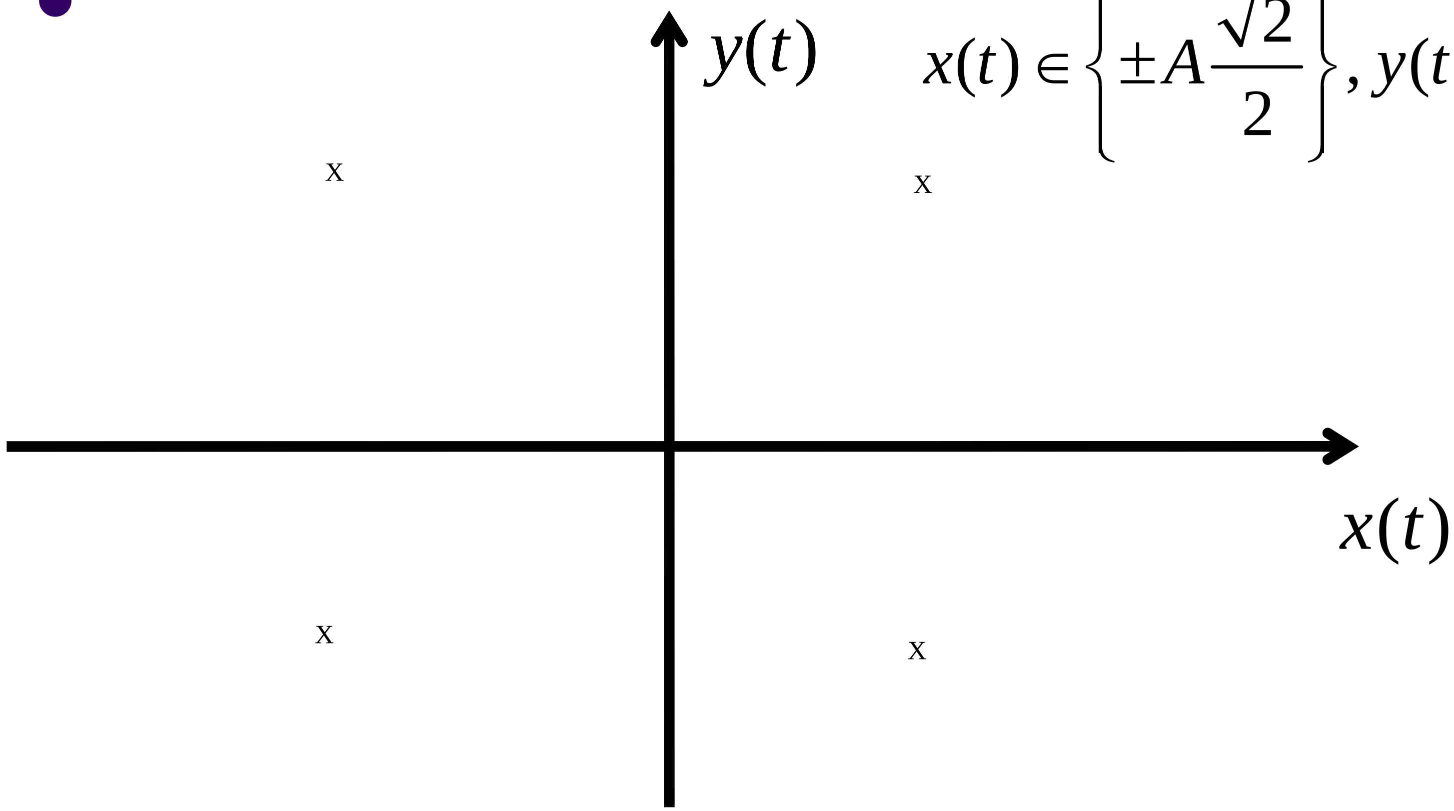
- $$x(t) \in \left\{ \pm A\frac{\sqrt{2}}{2} \right\}, y(t) \in \left\{ \pm A\frac{\sqrt{2}}{2} \right\}$$



Interpretation of Signal Constellation Diagram

- Axis are labeled with *x(t)* and *y(t)*
  - In-phase/quadrature or real/imaginary
- Possible symbols are plotted as points
- Symbol amplitude is proportional to distance from origin
- Probability of mistaking one signal for another is related to the distance between signal points
- Decisions are made on the received signal based on the distance of the received signal (in the I/Q plane) to the signal points in the constellation

A New Way of Viewing Modulation

- The I/Q representation of modulation is very convenient for some modulation types.
- We will examine an even more general way of looking at modulation using signal spaces.
- By choosing an appropriate set of axes for our signal constellation, we will be able to:
  - Design modulation types which have desirable properties
  - Construct optimal receivers for a given modulation
  - Analyze the performance of modulation types using very general techniques.

Basis Functions for a Signal Set

- For any modulation scheme one of *M* signals (often termed symbols) is transmitted during each symbol interval

$$\{s_1(t),\ldots,s_M(t)\}$$

- We would like to create a set of $K \le M$ signals that can be used to build any symbol in my set
  - If K << M this will be more efficient than having to generate *M* different signals
- Specifically, we say that the functions $\{f_1(t),\ldots,f_K(t)\}$ ($K \le M$) form *a complete orthonormal basis* for the signal set if
  - Any signal can be described by a linear combination:

$$s_i(t) = \sum_{k=1}^{K} s_{i,k} f_k(t), i = 1,\ldots,M$$

  - The basis functions are orthogonal to each other:

$$\int_a^b f_i(t) f_j^*(t)dt = 0, \forall i \ne j$$

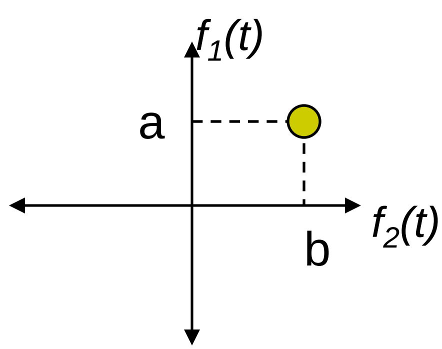
  - The basis functions are normalized:

$$\int_a^b |f_k(t)|^2 dt = 1, \forall k$$

Note: The time interval [a,b] is the symbol time

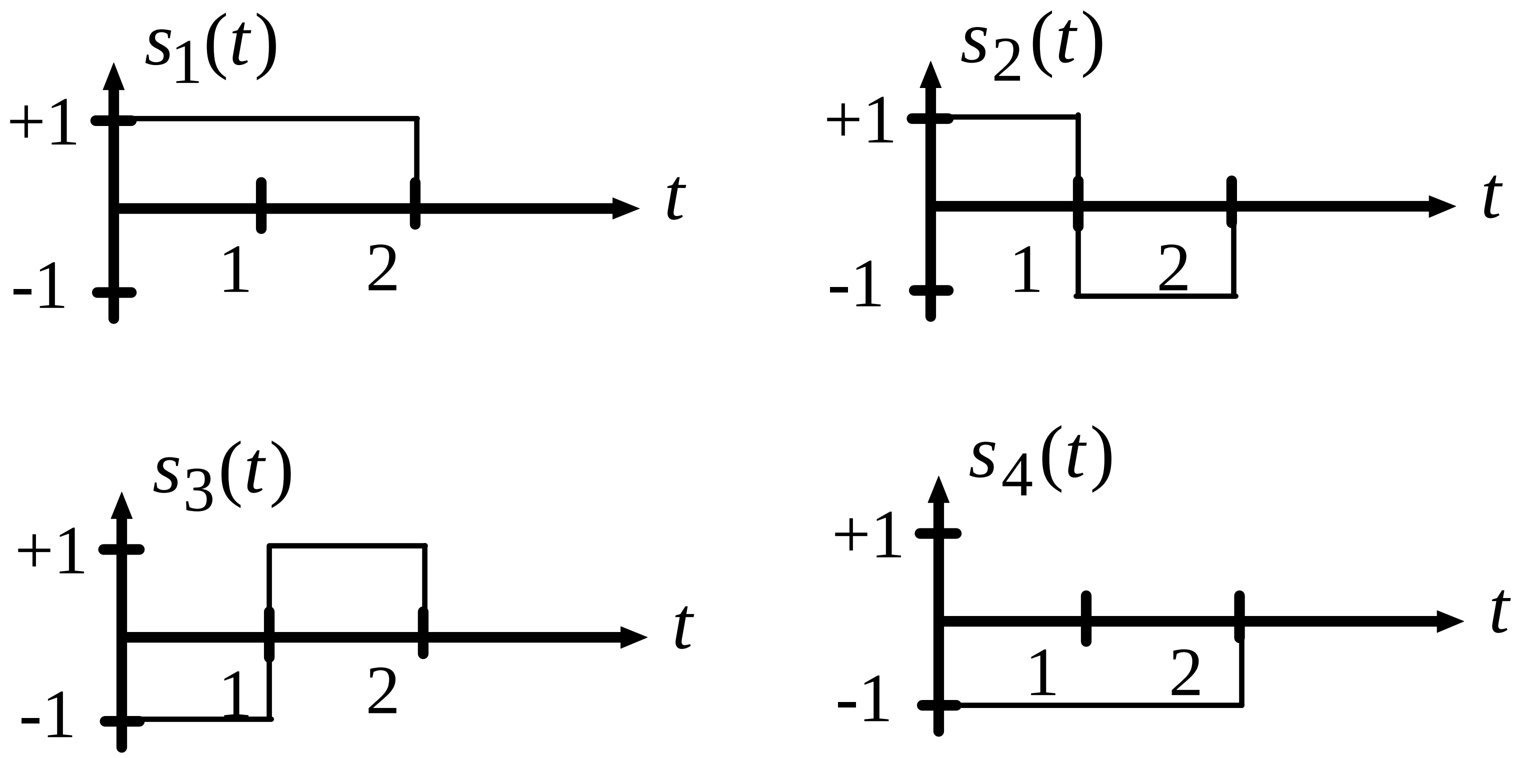
Signal Spaces

- The resulting set of basis functions can be thought of as a *signal space* by creating a space where the dimensions represent the basis functions
- For each symbol the coefficients for the set of basis functions can be represented as a vector
- The resulting vector represents a point in our signal space
- Ex: $s_1(t) = a*f_1(t) + b*f_2(t) \rightarrow [a, b]$



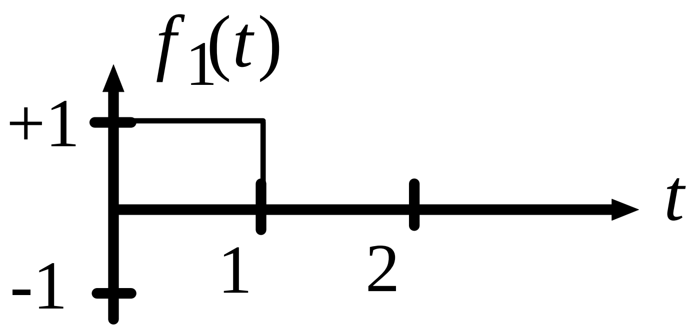
Example of Signal Space

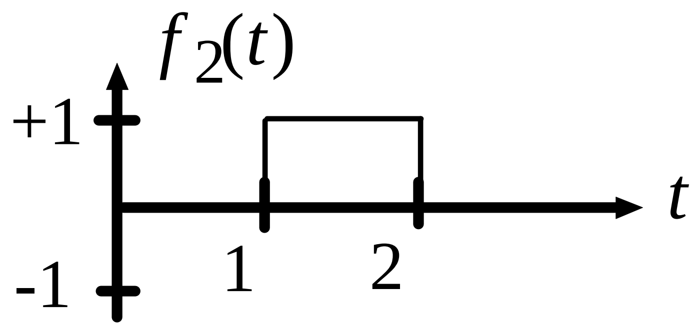
Consider the following signal signal set:



Example of Signal Space (continued)

- We can express each of the signals in terms of the following basis functions:





$$s_1(t) = 1 \cdot f_1(t) + 1 \cdot f_2(t) \qquad s_2(t) = 1 \cdot f_1(t) - 1 \cdot f_2(t)$$

$$s_3(t) = -1 \cdot f_1(t) + 1 \cdot f_2(t) \qquad s_4(t) = -1 \cdot f_1(t) - 1 \cdot f_2(t)$$

- Therefore the basis is complete

Example of Signal Space (continued)

- The basis is orthogonal:

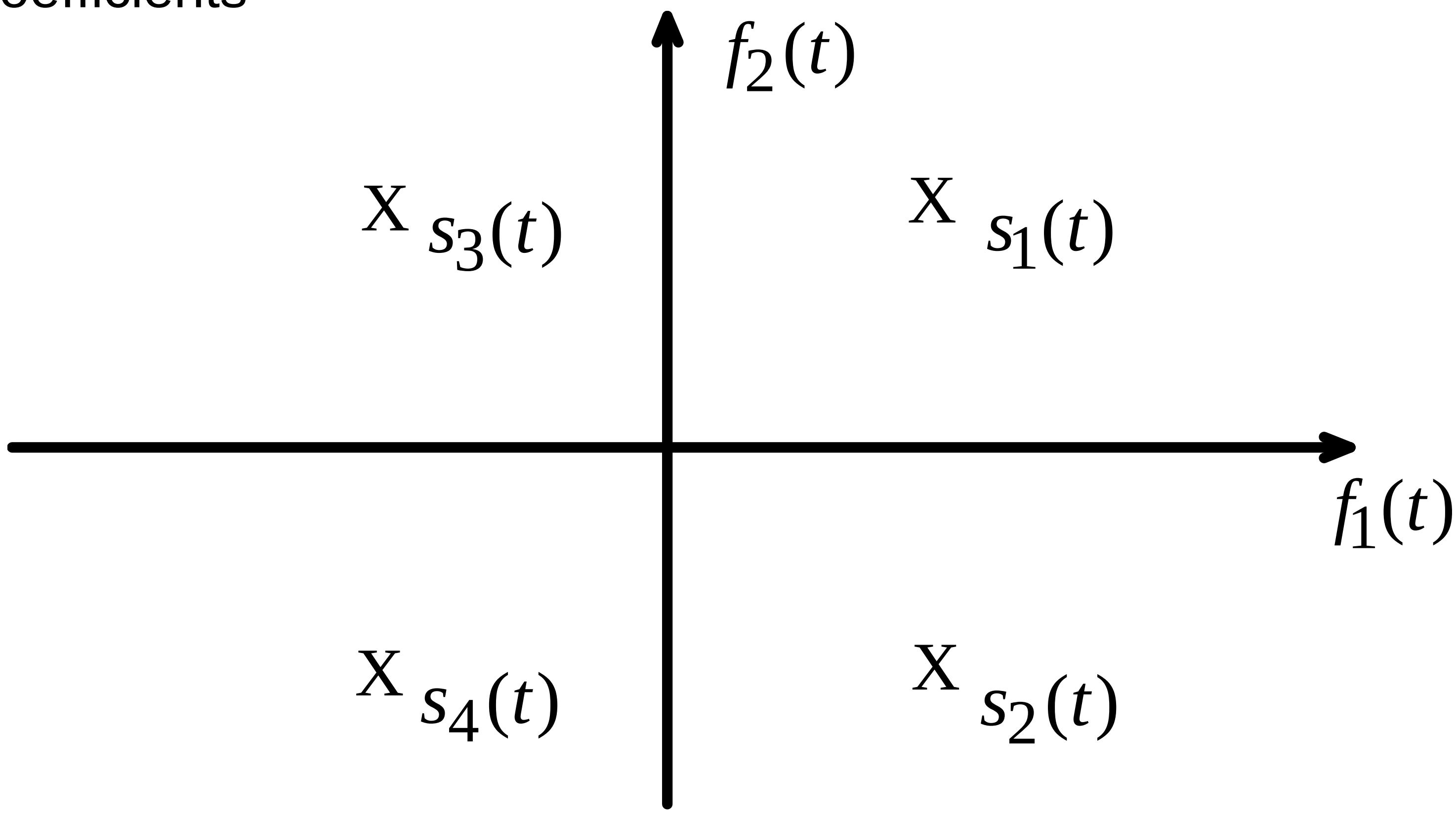
$$\int_{-\infty}^{\infty} f_1(t) f_2^*(t) dt = 0$$

- The basis is normalized:

$$\int_{-\infty}^{\infty} \left| f_1(t) \right|^2 dt = \int_{-\infty}^{\infty} \left| f_2(t) \right|^2 dt = 1$$

Signal Space Diagram

- Axes represent the basis functions. Points are placed at the coefficients



Another Example

- Suppose our signal set can be represented in I/Q form:

$$s(t) = x(t)\cos(2\pi f_c t) - y(t)\sin(2\pi f_c t)\Big|_0^T$$

where $x(t)$ and $y(t)$ are constants for $t \in [0, T]$

- Then the functions:

$$f_1(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_c t)\Bigg|_0^T,\ f_2(t) = \sqrt{\frac{2}{T}}\sin(2\pi f_c t)\Bigg|_0^T$$

form a complete orthonormal basis

Proof

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- All I/Q signals can be represented by the linear combination of these basis functions.
- These basis functions are orthogonal:

$$\int_0^T f_1(t)\, f_2^*(t)\,dt = \int_0^T \sqrt{\frac{2}{T}}\cos(2\pi f_c t)\cdot\sqrt{\frac{2}{T}}\sin(2\pi f_c t)\,dt$$

$$= \frac{2}{T}\int_0^T \frac{1}{2}\left[\sin(0)+\sin(4\pi f_c t)\right]dt$$

$$= \frac{-1}{4\pi f_c T}\left[\cos(4\pi f_c t)\right]\Big|_0^T \approx 0,\ \text{for } f_c T >> 1$$

Proof (continued)

- These basis functions are normalized:

$$\int_0^T \left| f_1(t) \right|^2 dt = \int_0^T \left| f_2(t) \right|^2 dt = \int_0^T \left(\sqrt{\frac{2}{T}} \cos(2\pi f_c t) \right)^2 dt$$

$$= \frac{2}{T} \int_0^T \frac{1}{2} \left[\cos(0) + \cos(4\pi f_c t) \right] dt \approx \frac{1}{T} [1] \Big|_0^T = 1$$

Constellation Diagrams

- Thus, constellation diagrams are simply signal space plots for modulation schemes that have only two basis functions.
- Specifically, basis functions are $\sqrt{2/T}\cos(2\pi ft)$ and $\sqrt{2/T}\sin(2\pi ft)$ over the symbol duration [0,T)
- Only good for phase modulation or amplitude modulation
- Other modulation formats require larger number of basis functions.

Energy Per Symbol

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$$E_i = \int_0^T |s_i(t)|^2 \, dt$$

$$= \int_0^T \left| \sum_k v_k f_k(t) \right|^2 dt$$

$$= \int_0^T \left(\sum_m \sum_k v_k v_m f_k(t) f_m^*(t) \right) dt$$

$$= \sum_m \sum_k v_k v_m \left[\int_0^T f_k(t) f_m^*(t) dt \right]$$

$$= \sum_k v_k^2$$

For BER comparison purposes we would like to plot signals in terms of *Energy per Symbol*

Notes on Signal Spaces

- Two entirely different signal sets can have the same geometric representation.
- The underlying geometry will determine the performance and the receiver structure for a signal set.
- In both of these cases we were fortunate enough to guess the correct basis functions.
- Is there a general method to find a complete orthonormal basis for an arbitrary signal set?
  - ***Yes: The Gram-Schmidt Procedure***
- Note that FSK will require *M* basis functions

Signal Space: *M* –PSK

- The symbols of *M*-PSK can be written as

$$s_i(t) = \cos\left(2\pi f_c t + \frac{2\pi}{M} i\right)\Bigg|_0^T, \quad i = 0, \ldots, M-1$$

- The two basis functions are

$$f_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t)\Bigg|_0^T \qquad f_2(t) = -\sqrt{\frac{2}{T}} \sin(2\pi f_c t)\Bigg|_0^T$$

$$s_i(t) = c_{1i} f_1(t) + c_{2i} f_2(t) = c_{1i}\sqrt{\frac{2}{T}} \cos\left(2\pi f_c t\right) - c_{2i}\sqrt{\frac{2}{T}} \sin\left(2\pi f_c t\right)$$

$$c_{1i} = \sqrt{\frac{T}{2}} \cos\left(\frac{2\pi}{M} i\right) \qquad c_{2i} = \sqrt{\frac{T}{2}} \sin\left(\frac{2\pi}{M} i\right)$$

Average Symbol Energy

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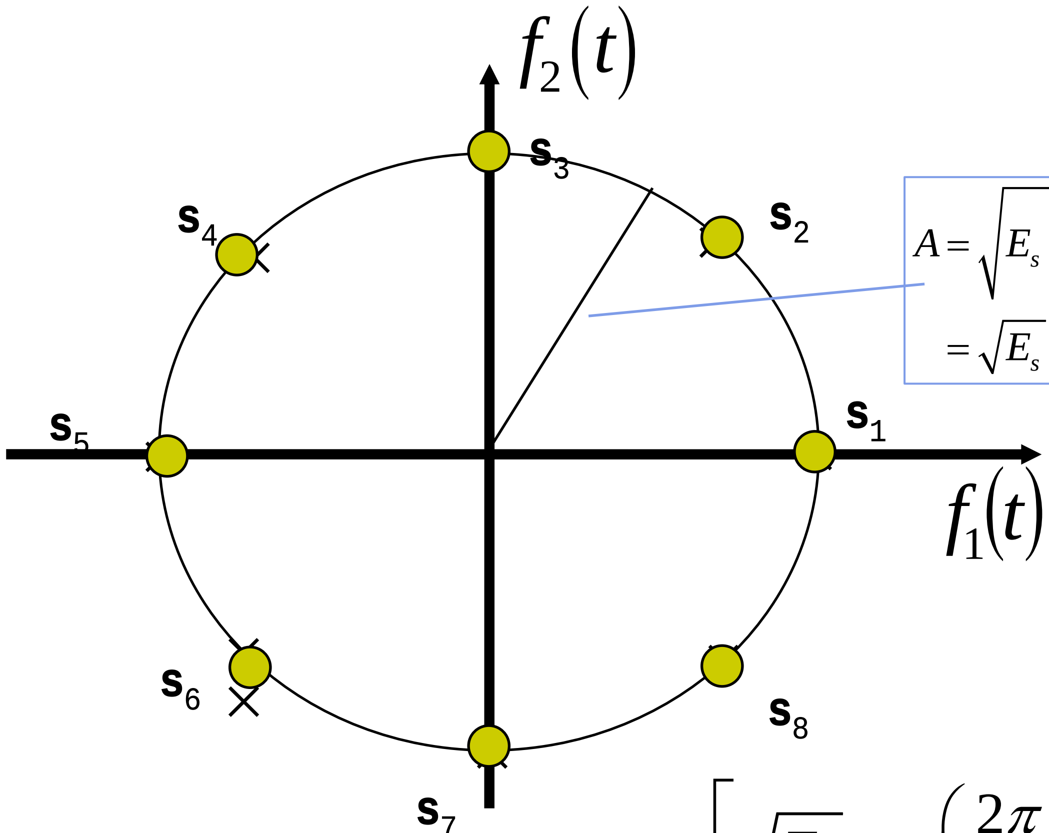
- All symbols have equal energy

$$E_s = \frac{T}{2}\cos^2\left(\frac{2\pi}{M}i\right) + \frac{T}{2}\sin^2\left(\frac{2\pi}{M}i\right)$$
$$= \frac{T}{2}$$

- Writing the symbols in terms of the energy per symbol:

$$\mathbf{s}_i = \left[\sqrt{E_s}\cos\left(\frac{2\pi}{M}i\right) \quad \sqrt{E_s}\sin\left(\frac{2\pi}{M}i\right)\right]$$

Ex: 8-ary PSK

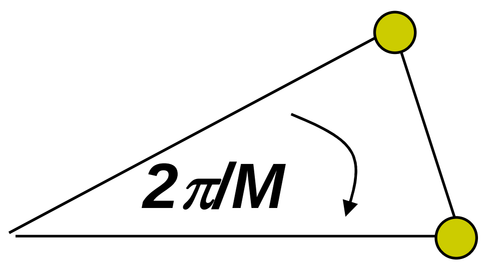


$$A = \sqrt{E_s \cos^2\left(\frac{2\pi}{M} i\right) + E_s \sin^2\left(\frac{2\pi}{M} i\right)}$$
$$= \sqrt{E_s}$$

$$\mathbf{s}_i = \left[\sqrt{E_s} \cos\left(\frac{2\pi}{M} i\right) \quad \sqrt{E_s} \sin\left(\frac{2\pi}{M} i\right) \right]$$

Distance Between Symbols

- The distance between symbols has a direct impact on the BER performance



$$d = \sqrt{A^2 + A^2 - 2A^2\cos\left(2\pi/M\right)}$$
$$= \sqrt{2A^2\left(1 - \cos\left(2\pi/M\right)\right)}$$
$$= \sqrt{4A^2\sin^2\left(\pi/M\right)}$$

- Substituting for the average energy per symbol:

$$d = \sqrt{4E_s\sin^2\left(\pi/M\right)}$$

Distance decreases dramatically as *M* increases

- Changing to energy per bit:

$$d = \sqrt{4E_b\log_2\left(M\right)\sin^2\left(\pi/M\right)}$$

Signal Space: *M*-FSK

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- Consider the signal set

$$s_i(t) = \sqrt{\frac{2}{T}} \cos\left(2\pi f_c t + 2\pi i \Delta f t\right)$$

- With baseband equivalent

$$s_i(t) = \sqrt{\frac{2}{T}} e^{j2\pi i \Delta f t}$$

- We can show that the correlation between symbols is

$$\rho_{mn} = \frac{1}{T}\int_0^T e^{j2\pi(m-n)\Delta f t} dt$$
$$= \frac{\sin\left[\pi T(m-n)\Delta f\right]}{\pi T(m-n)\Delta f} e^{j\pi T(m-n)\Delta f}$$

M- FSK (cont.)

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- If we choose Δf = 1/T, the correlation between symbols is zero thus we will need *M* basis functions (i.e., one for each symbol).

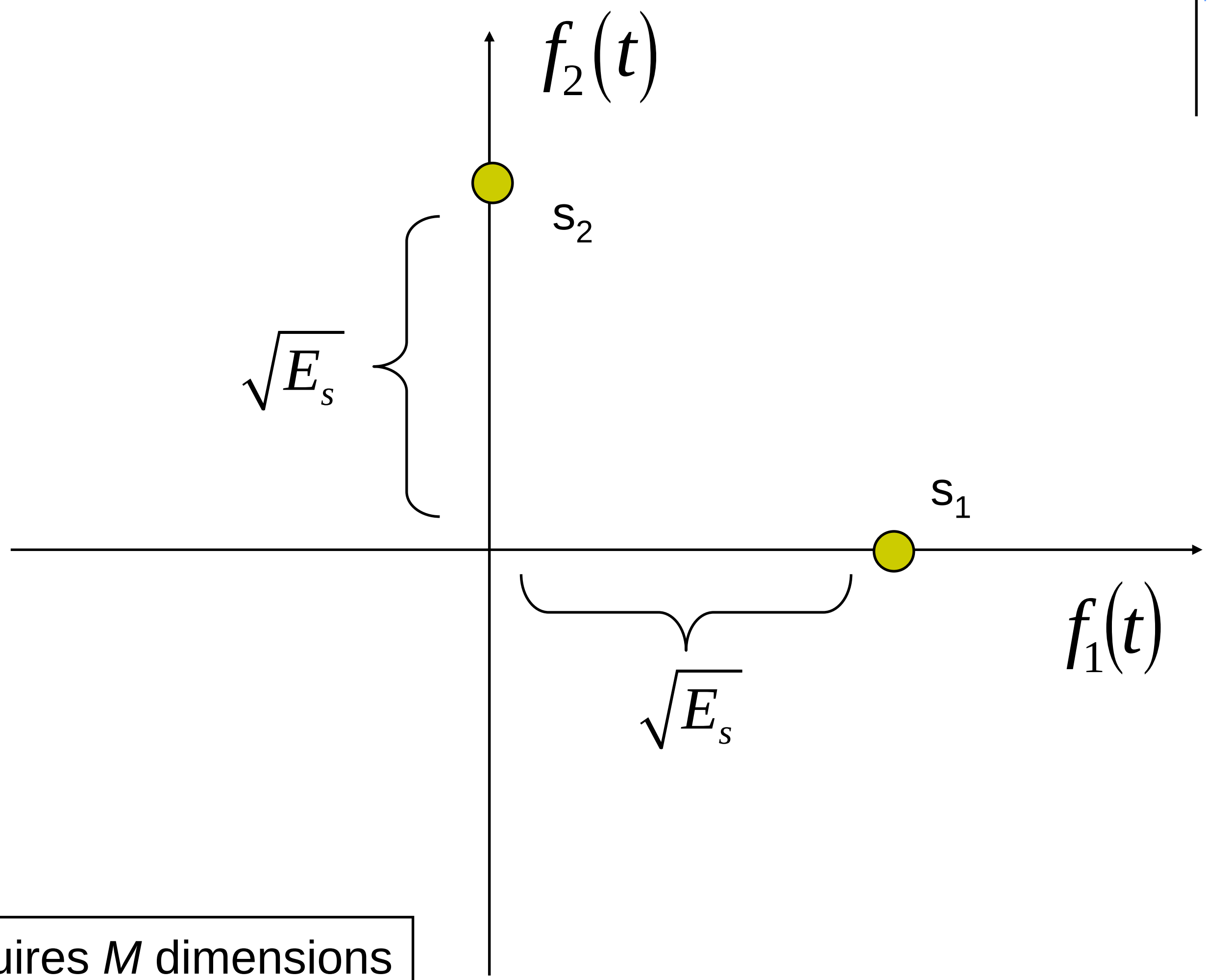
$$f_i(t) = \sqrt{\frac{2}{T}} \cos\left(2\pi f_c t + \frac{2\pi i}{T} t \right)$$

- Further, we can represent the symbols in signal space as *M*-dimensional vectors:

$$\boxed{\overline{E}_s = 1}$$

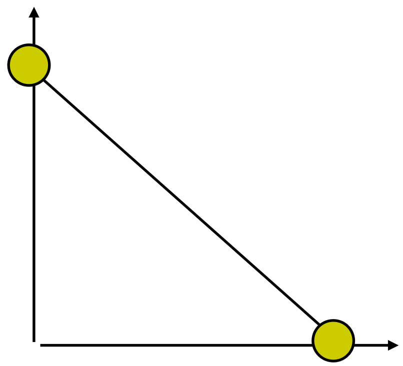
$$\mathbf{s}_1 = \begin{bmatrix} \sqrt{E} & 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$
$$\mathbf{s}_2 = \begin{bmatrix} 0 & \sqrt{E} & 0 & 0 & \dots & 0 \end{bmatrix}$$
$$\mathbf{s}_3 = \begin{bmatrix} 0 & 0 & \sqrt{E} & 0 & \dots & 0 \end{bmatrix}$$
$$\vdots$$
$$\mathbf{s}_M = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & \sqrt{E} \end{bmatrix}$$

Ex: BFSK



M-ary FSK requires *M* dimensions

Distance Between Symbols



$$d = \sqrt{2E_s}$$

- As we increase *M* the number dimensions increases and thus the distance between points does not decrease
- In fact, in terms of energy per bit:

$$d = \sqrt{2E_b \log_2\left(M\right)}$$

Distance actually increases as *M* increases (in terms of energy per bit).

Signal Space: *M* –ASK

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- The symbols of *M*-ASK can be written as

$$s_i(t) = i\cos(2\pi f_c t)\Big|_0^T, \quad i = 0,\ldots,M-1$$

- There is one basis function:

$$f_1(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_c t)\Big|_0^T$$

$$s_i(t) = c_i f_1(t) = c_i \cos(2\pi f_c t)$$

$$c_i = \sqrt{\frac{T}{2}}\, i$$

Average Symbol Energy

- All symbols have *different* energies

$$E_s = \int_0^T \left\{ \sqrt{\frac{T}{2}} i \sqrt{\frac{2}{T}} \cos\left(\omega_c t\right) \right\}^2 dt$$

$$= i^2 \frac{T}{2}$$

- Average Energy:

$$\overline{E_s} = \frac{1}{M} \sum_{i=0}^{M-1} \left(\sqrt{\frac{T}{2}} i \right)^2 = \frac{1}{M} \frac{T}{2} \sum_{i=0}^{M-1} i^2$$

Example: 4-ASK

- Symbols

$$s_0(t) = 0\Big|_0^T$$

$$s_1(t) = \cos(\omega_c t)\Big|_0^T$$

$$s_2(t) = 2\cos(\omega_c t)\Big|_0^T$$

$$s_3(t) = 3\cos(\omega_c t)\Big|_0^T$$

- Basis function

$$f_1(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_c t)\Big|_0^T$$

- Coefficients

$$c_0 = 0 \qquad c_1 = \sqrt{\frac{T}{2}}$$

$$c_2 = 2\sqrt{\frac{T}{2}} \qquad c_3 = 3\sqrt{\frac{T}{2}}$$

Example (cont.)

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- Average Energy

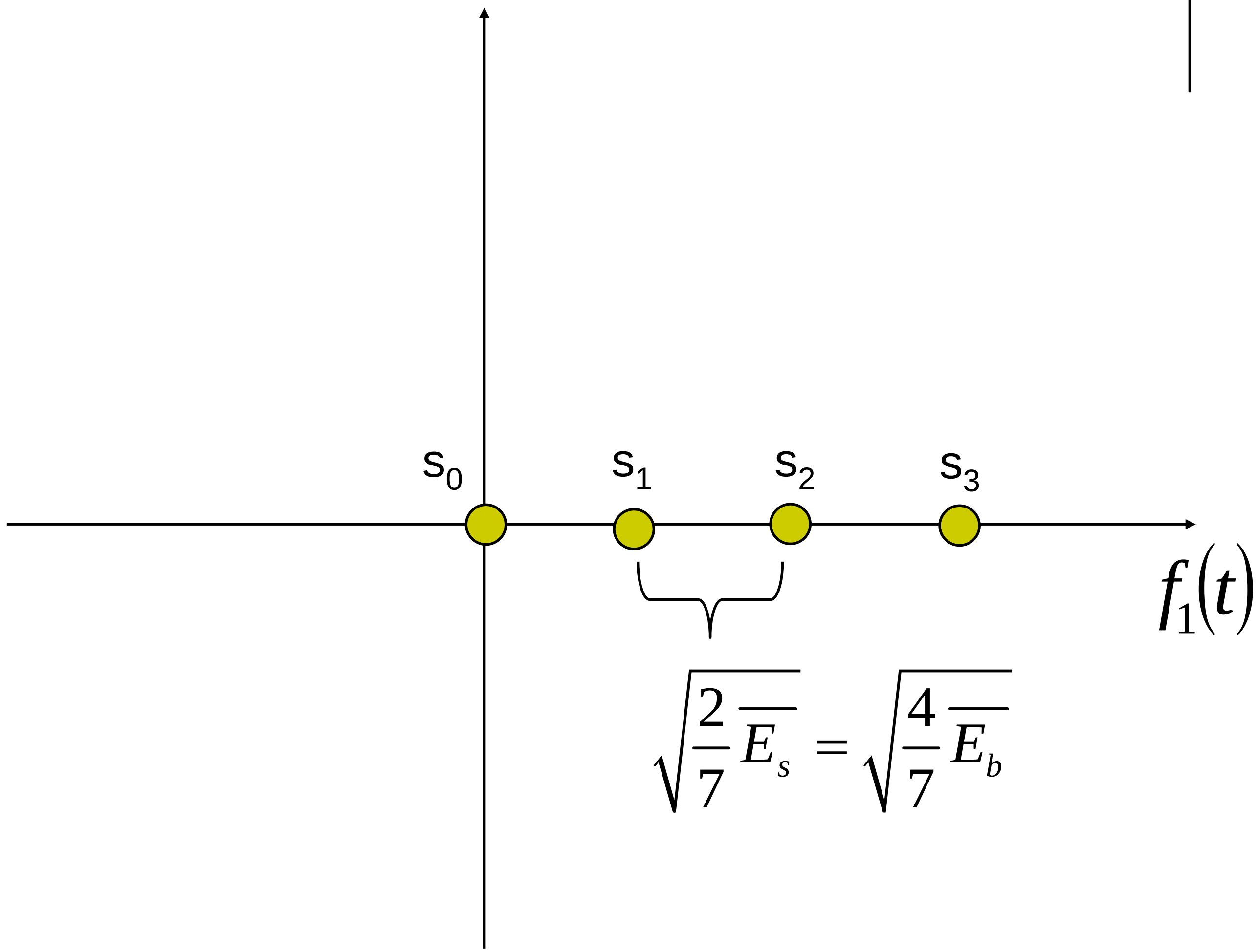
$$\overline{E_s} = \frac{1}{M}\sum_{i=0}^{3}\left(\sqrt{\frac{T}{2}}i\right)^2 = \frac{T}{2}\frac{1}{4}\{0+1+4+9\}$$

$$= \frac{7T}{4}$$

$$c_0 = 0 \qquad c_1 = \sqrt{\frac{2}{7}\overline{E_s}}$$

$$c_2 = \sqrt{\frac{8}{7}\overline{E_s}} \qquad c_3 = \sqrt{\frac{18}{7}\overline{E_s}}$$

Ex: 4-ASK



s_0
s_1
s_2
s_3
$f_1(t)$
$\sqrt{\frac{2}{7}\overline{E_s}} = \sqrt{\frac{4}{7}\overline{E_b}}$

Signal Space: 16-*QAM*

- The symbols of *QAM* can be written as

$$s_i(t) = x(t)\cos(\omega_c t) - y(t)\sin(\omega_c t)\Big|_0^T$$

$$x(t) \in \{-3,-1,1,3\},\ y(t) \in \{-3,-1,1,3\}$$

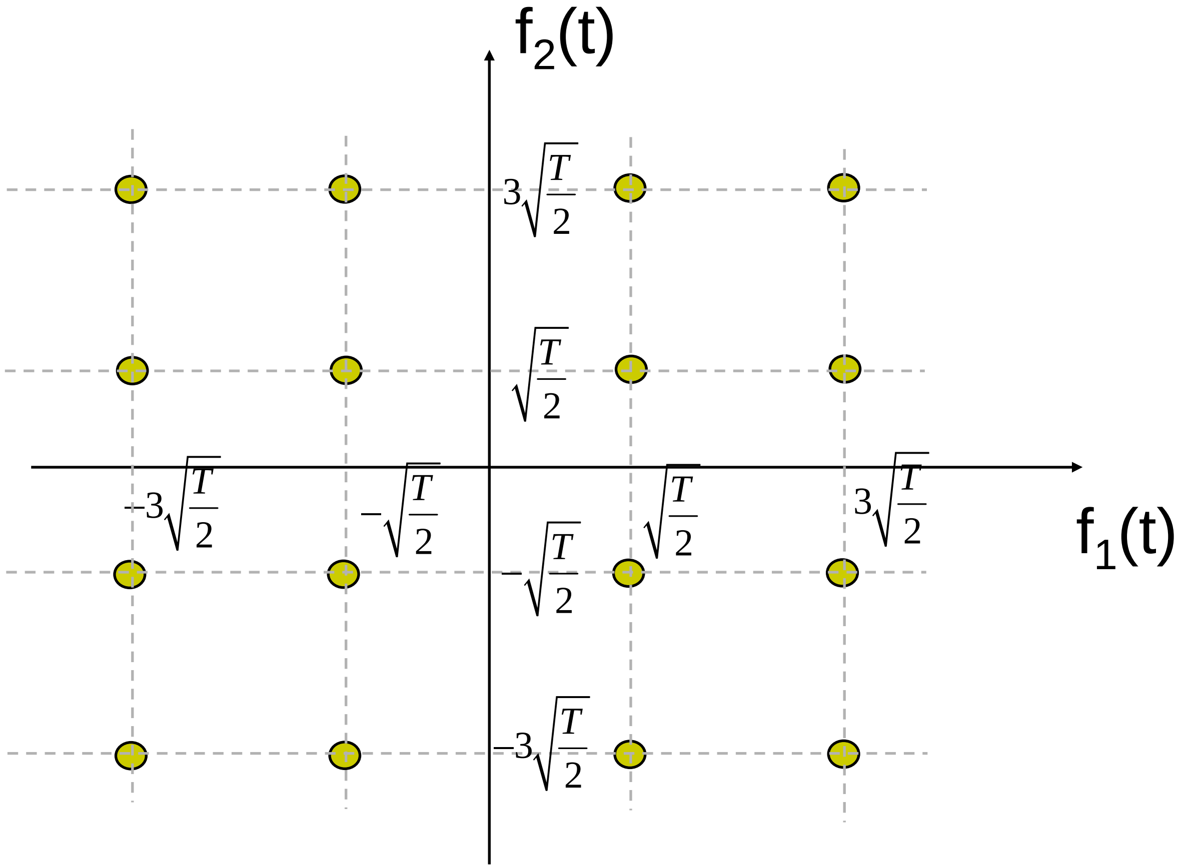
- The two basis functions are

$$f_1(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_c t)\Big|_0^T \qquad f_2(t) = -\sqrt{\frac{2}{T}}\sin(2\pi f_c t)\Big|_0^T$$

$$\begin{aligned} s_i(t) &= c_{1i} f_1(t) + c_{2i} f_2(t) \\ &= c_{1i}\cos(2\pi f_c t) - c_{2i}\sin(2\pi f_c t) \end{aligned}$$

$$\boxed{\begin{aligned} c_{1i} &\in \left\{-3\sqrt{\frac{T}{2}}, -\sqrt{\frac{T}{2}}, \sqrt{\frac{T}{2}}, 3\sqrt{\frac{T}{2}}\right\} \\ c_{2i} &\in \left\{-3\sqrt{\frac{T}{2}}, -\sqrt{\frac{T}{2}}, \sqrt{\frac{T}{2}}, 3\sqrt{\frac{T}{2}}\right\} \end{aligned}}$$

Example : 16-QAM



Average Symbol Energy

- Symbol energies

$$E_s = T$$ **Four symbols**

$$E_s = 10\frac{T}{2} = 5T$$ **Eight symbols**

$$E_s = 18\frac{T}{2} = 9T$$ **Four Symbols**

- Thus, the average symbol energy

$$\overline{E_s} = \frac{1}{16}\left\{4*T + 8*5T + 4*9T\right\}$$
$$= 5T$$

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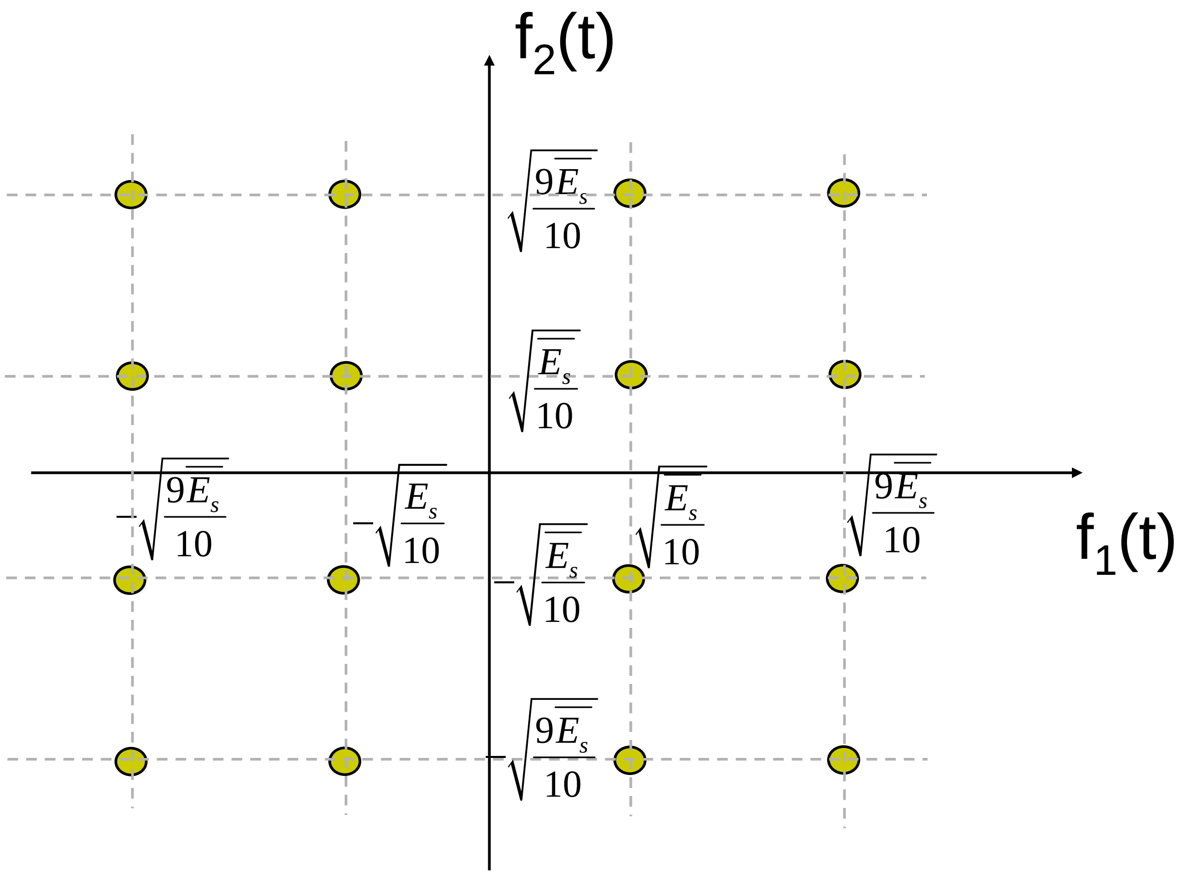
Example : 16-QAM



Pulse Shaping

- Q: How is pulse shaping handled with the signal space approach?
- A: By incorporating the pulse shape into the basis function.
  - This was implicitly done previously
  - Example: MPSK

$$f_1(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_c t)\Big|_0^T \qquad f_2(t) = -\sqrt{\frac{2}{T}}\sin(2\pi f_c t)\Big|_0^T$$

  - Can be written as

$$f_1(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_c t)\Pi\left(\frac{t-T/2}{T}\right) \qquad f_2(t) = -\sqrt{\frac{2}{T}}\sin(2\pi f_c t)\Pi\left(\frac{t-T/2}{T}\right)$$

Pulse Shaping (cont.)

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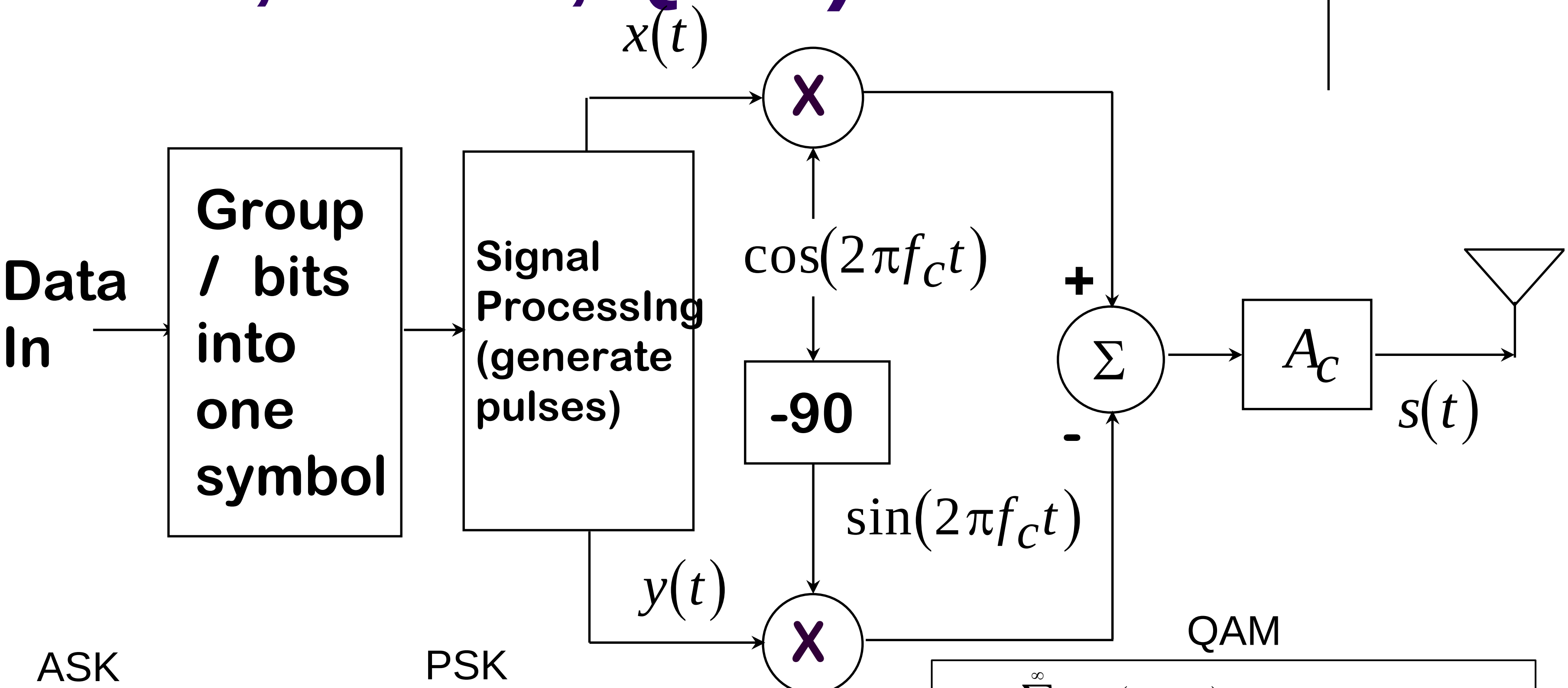
- For arbitrary pulse shaping with M-PSK

$$f_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t)\, p(t) \qquad f_2(t) = -\sqrt{\frac{2}{T}} \sin(2\pi f_c t)\, p(t)$$

- Where *p(t)* is an arbitrary pulse shape with unit energy
- The constellation diagrams would not change
  - Thus, distance properties wouldn't change and assuming matched filtering, the performance wouldn't change

Generic Transmitter (MPSK, MASK, QAM)



ASK

$$x(t) = \sum_{n=-\infty}^{\infty} A_n p(t - nT_s)$$

$$A_n = A_1, A_2, \ldots A_M$$

$$y(t) = 0$$

PSK

$$x(t) = \sum_{n=-\infty}^{\infty} A_n p(t - nT_s)$$

$$A_n = \cos(\theta_1), \cos(\theta_2), \ldots \cos(\theta_M)$$

$$y(t) = \sum_{n=-\infty}^{\infty} B_n p(t - nT_s)$$

$$B_n = \sin(\theta_1), \sin(\theta_2), \ldots \sin(\theta_M)$$

QAM

$$x(t) = \sum_{n=-\infty}^{\infty} A_n p(t - nT_s)$$

$$A_n = \frac{1}{\sqrt{E_{avg}}}\left\{-\sqrt{M}+1, -\sqrt{M}+3, \ldots \sqrt{M}-3, \sqrt{M}-1\right\}$$

$$y(t) = \sum_{n=-\infty}^{\infty} B_n p(t - nT_s)$$

$$B_n = \frac{1}{\sqrt{E_{avg}}}\left\{-\sqrt{M}+1, -\sqrt{M}+3, \ldots \sqrt{M}-3, \sqrt{M}-1\right\}$$

Summary

- We have presented the signal space representation for all of the major digital modulation schemes
  - Distance between symbols reduces for MPSK, MASK
  - Distance between symbols increases (in terms of E_b/N_o) for MFSK
- Pulse shaping can be easily incorporated into this framework
- Approach leads to straightforward transmitter (and as we will see next time) receiver implementation