ECE4634
Digital Communications
Fall 2007

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Lecture #7: Digital Pulse Modulation : PCM

Analog and Digital Communications

Overview

- We are currently studying *baseband communication systems*
- Previously we looked at
  - Sampling continuous time waveforms
    - Nyquist sampling theorem
  - Representing time samples using *discrete baseband* communication systems where samples are used to modulate a pulse train
- Digital systems are discrete in time and have a finite number of messages
- Pulse Code Modulation is one form of *digital* baseband communications
  - Often called *analog-to-digital conversion* since it is so common
  - However, this is only *one form* of analog-to-digital conversion
- What to read – 5.4-5.6

Lecture Objectives

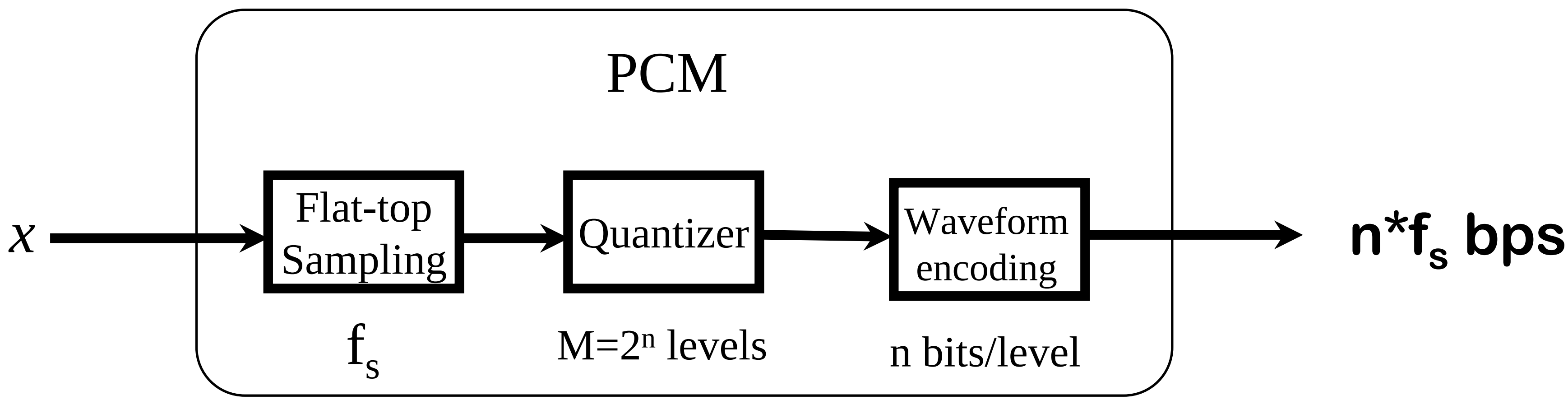
- In this lecture we will
  - Introduce the digital communication concept of Pulse Code Modulation
  - Examine (uniform scalar) quantization in detail
  - Derive the SNR of quantization
  - Demonstrate the trade-off that exists in quantization between fidelity and bit rate (i.e., bandwidth)

What is Modulation?

- Modulation is changing a parameter of a carrier in order to convey information
- Two common carriers are
  - Sinusoid (bandpass systems)
  - Pulse stream (baseband systems)
- In PCM (Pulse Code Modulation) we modulate a stream of pulses (typically we assume square pulses for convenience)

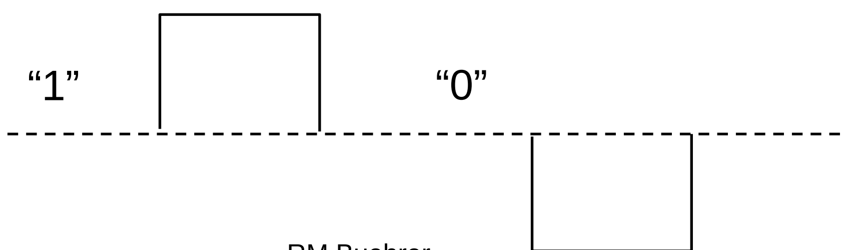
Pulse Code Modulation (PCM)

- Pulse Code Modulation refers to a system that creates a baseband signal that is generated directly from the binary quantizer output



Waveform Encoding

- Once the information is converted to bits, it must be mapped onto waveforms (i.e., modulation)
- If each bit is mapped to one of two different waveforms, we term this *binary encoding*
- If m bits are mapped to $M = 2^m$ waveforms, we term this M-ary encoding
- We will discuss specific waveforms next class
- For now assume that we use the mapping



PCM – Example

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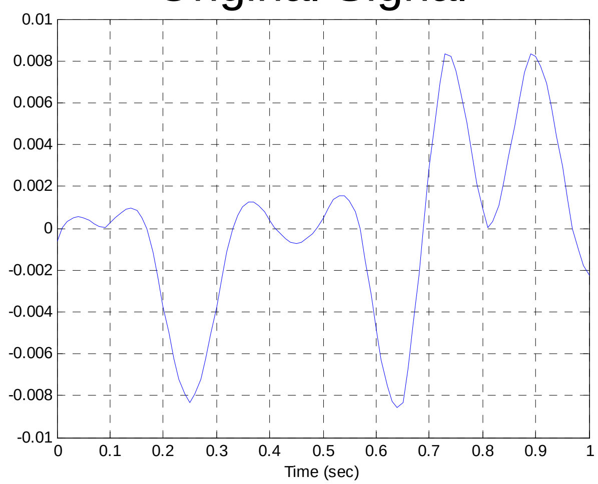
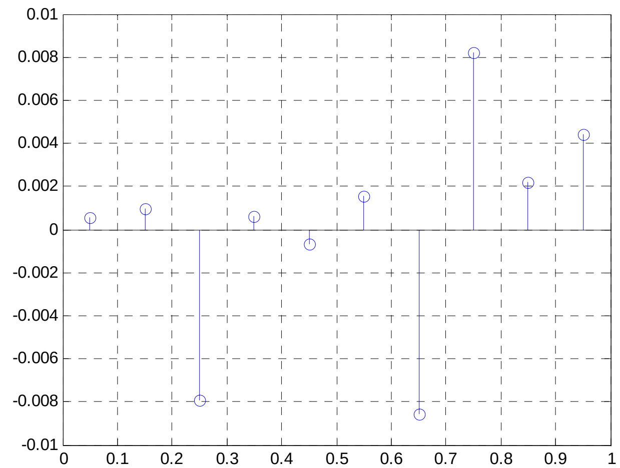
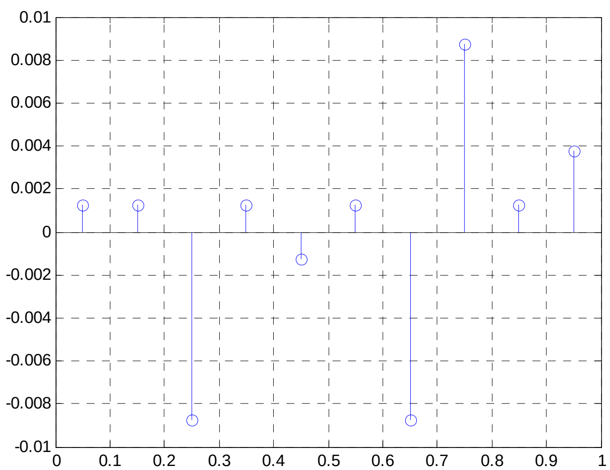
Original Signal



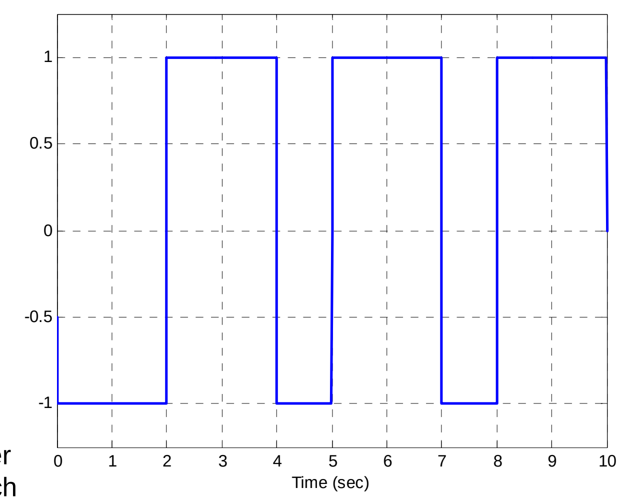
Sampled Signal



Quantized Signal



Resulting Digital Signal



Bandwidth of PCM Signals

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- Sample rate: f_s samples/second.
- Bit rate out of the quantizer:

$$f_s \log_2 M = f_s \cdot n \text{ bits / second}$$

- Bandwidth of the resulting digital signal depends on the pulses used
- Minimum theoretical bandwidth (with optimal pulse shaping which we will discuss later): $f_s \cdot n/2 \text{ Hz}$
- First null bandwidth (with rectangular pulse waveforms):

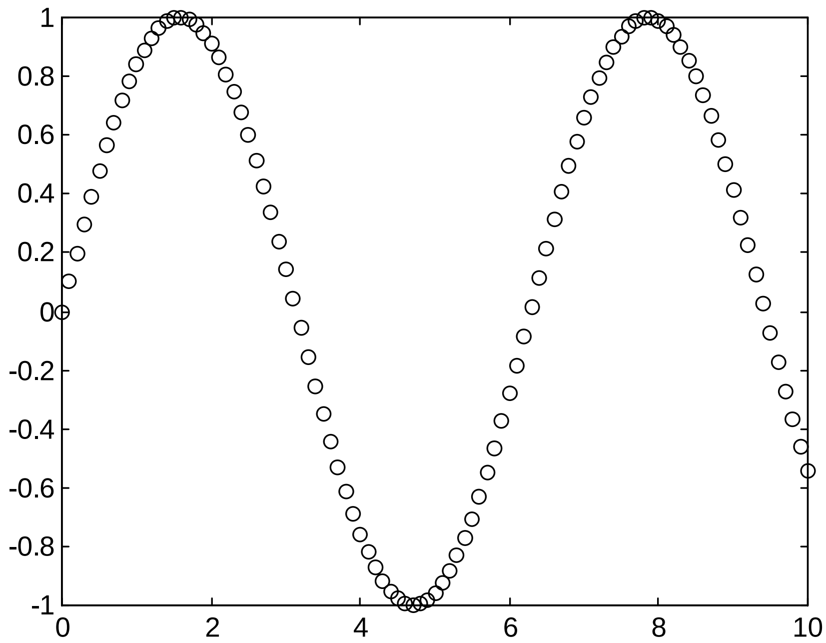
$$f_s \cdot n \text{ Hz}$$

- We'll talk more about the effects of pulse shaping in a few days

Note that the resulting bandwidth depends on the digital waveform (i.e., pulse shape) that is used.

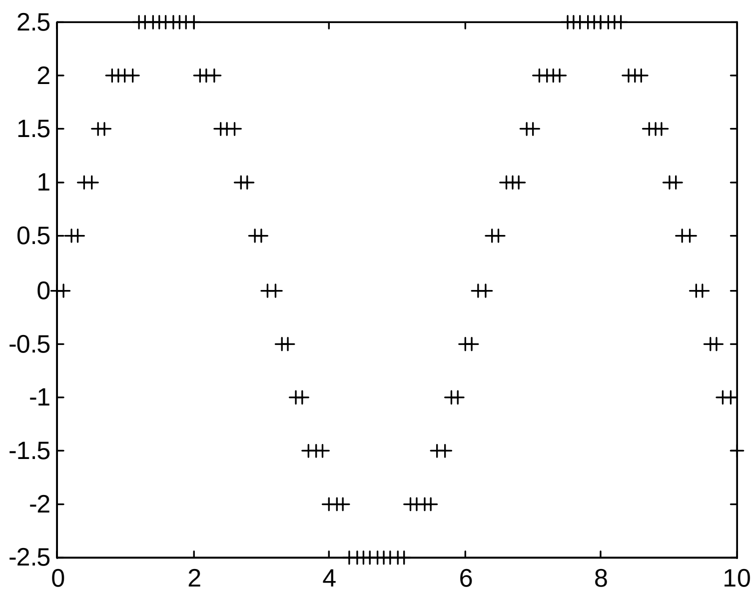
Digital Representation of Analog Signals

- Sampling analog signals makes them discrete in time:



Digital Representation of Analog Signals

- Quantization of sampled analog signals makes the samples discrete in amplitude:



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Quantization

- Continuous time signals are sampled at discrete time intervals
- Sampling may be performed without distortion provided signal is sampled at Nyquist rate
- Continuous-valued samples of data require an infinite # of bits to represent with perfect precision.
- Quantization is the process of approximating continuous-valued samples with a finite # of bits.
- Quantization always introduces some distortion.

Notation Associated with Quantization

- Let *X* be a random variable representing a sample of data.
- Then $\tilde{X} = f_Q(X)$ is the quantized value of *X.*
- A quantizer has *M* quantization levels:

$$\tilde{X} \in \{\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_M\}$$

- The *M* levels correspond to *M* quantization regions.
- The endpoints of the quantization regions are specified by $M+1$ values: $\{x_0, x_1, \ldots, x_M\}$, where $x_0 = -\infty, x_M = \infty$
- Then:

$$x_{k-1} \le x < x_k \Rightarrow \tilde{x} = f_Q(x) = \tilde{x}_k$$

Graphical Description of Quantization

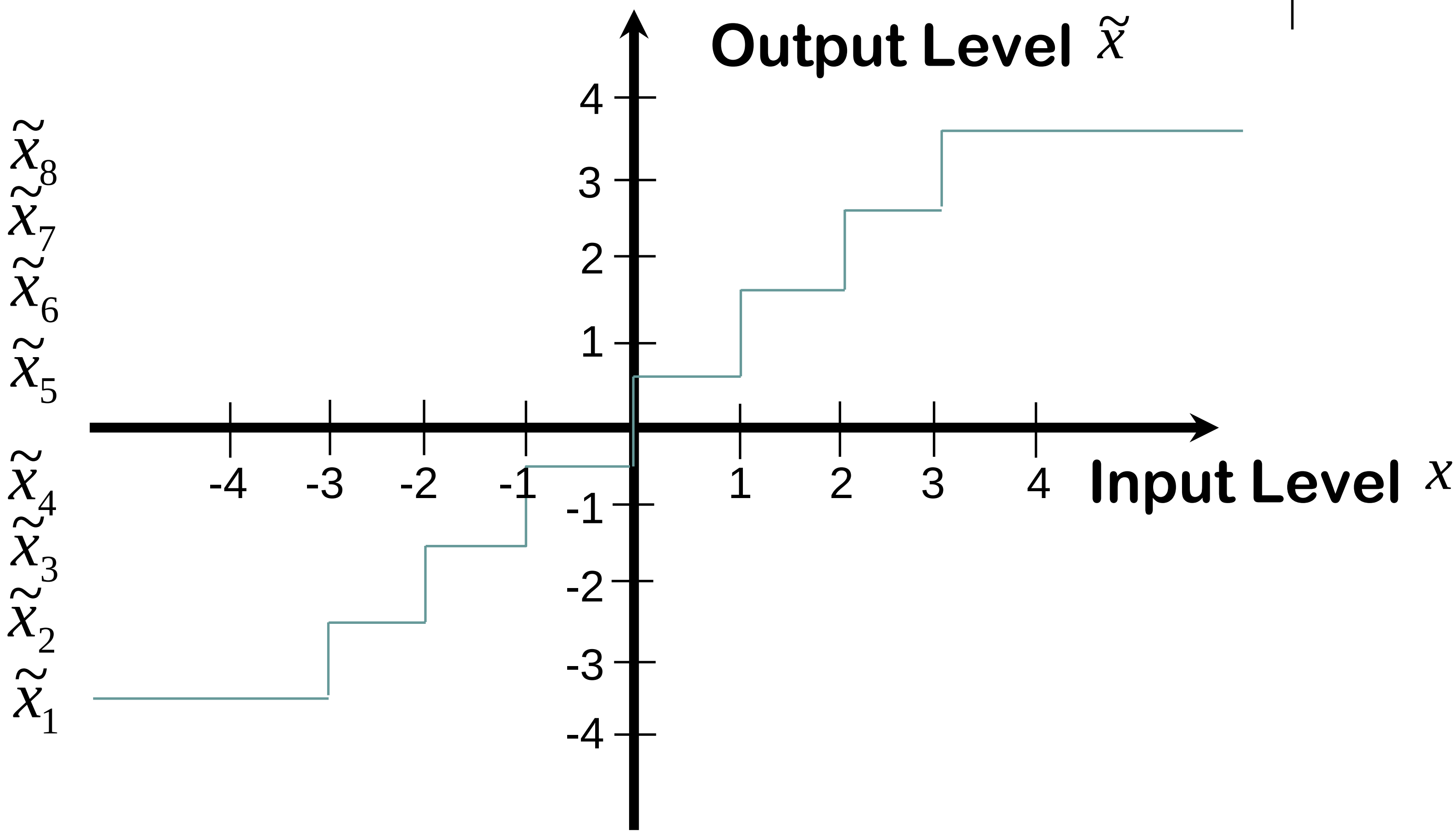

Table Representation of Quantizer

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k	x_{k-1}	x_k	$\tilde{x}_k$	Output Bits
1	$-\infty$	-3	-3.5	000
2	-3	-2	-2.5	001
3	-2	-1	-1.5	010
4	-1	0	-0.5	011
5	0	1	0.5	100
6	1	2	1.5	101
7	2	3	2.5	110
8	3	∞	3.5	111

Concise Representation of Quantizer

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- Usually, it is sufficient just to list the quantization levels.
- Example: {-3.5, -2.5, -1.5, -0.5, 0.5, 1.5, 2.5, 3.5}
- Why?
  - We assume that all points are quantized to the nearest quantization level
  - This determines where the borders of the quantization regions are
  - Any other borders would increase the error introduced by the quantizer

Practical Methods for Implementing Analog to Digital Converters

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Lower Complexity

- Counting or Ramp ADC
  - test value is incremented in equal steps until it is greater than input sample
- Serial or Successive Approximation ADC
  - uses binary search to narrow range of input sample until desired accuracy is reached
- Parallel or Flash ADC
  - input sample is compared with all possible quantization levels at once

Faster

Distortion

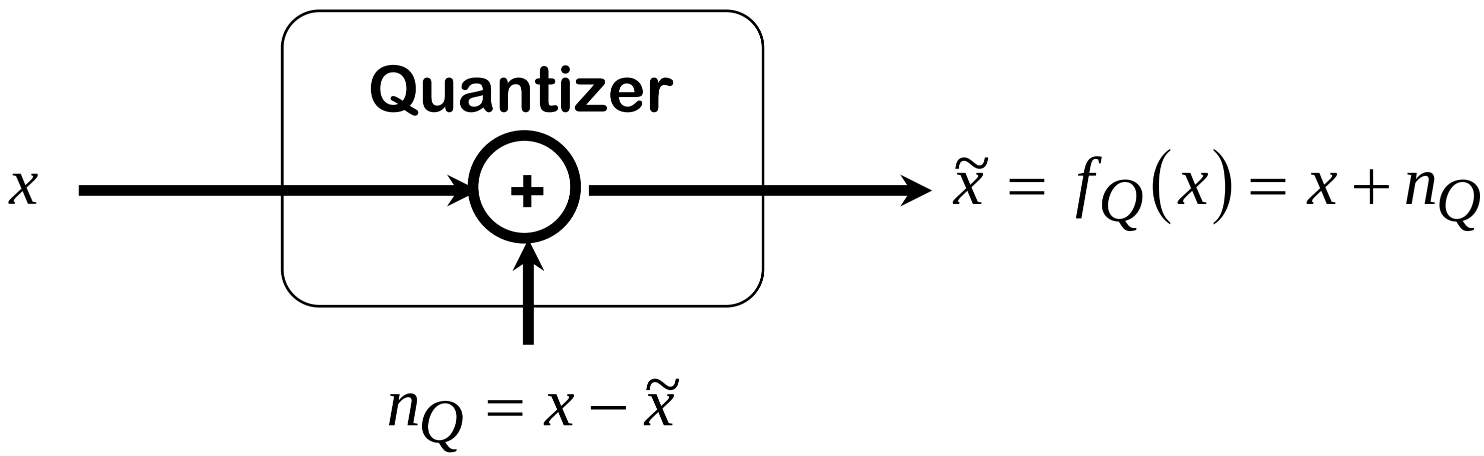
- Quantization introduces distortion into a signal.
- We want to minimize average distortion D , where

$$
\begin{aligned}
D &= E\left[\left(X - \tilde{X}\right)^2\right] \\
&= \int_{-\infty}^{\infty} \left(x - \tilde{x}\right)^2 f\left(x\right) dx \\
&= \sum_{k=1}^{M} \int_{x_{k-1}}^{x_k} \left(x - \tilde{x}_k\right)^2 f\left(x\right) dx
\end{aligned}
$$

- This measure of distortion is sometimes also called mean square error (MSE)
- MSE penalizes large errors more than small errors

Another Way of Viewing Quantization

- Quantization adds a random “noise” to the true value of the sample point
- Then $\mathrm{MSE} = E\left[n_Q^2\right]$ may be thought of as noise power
- We can define a signal-to-noise ratio (SNR) to measure performance



Signal to Noise Ratio Calculations for Quantizers

- Average SNR

$$\left(\frac{S}{N}\right)_{\text{avg}} = \frac{E\left[X^2\right]}{E\left[n_Q^2\right]} = \frac{E\left[X^2\right]}{D} = \frac{\int_{-\infty}^{\infty} x^2 f_X(x)dx}{\int_{-\infty}^{\infty} (x-\tilde{x})^2 f_X(x)dx}$$

- Sometimes we also define “peak” SNR
  - This is less important.

$$\left(\frac{S}{N}\right)_{\text{peak}} = \frac{X_{\text{peak}}^2}{E\left[n_Q^2\right]} = \frac{X_{\text{peak}}^2}{D}$$

Example 7.1: SNR Calculation

- Let: $\left\{\tilde{x}_1 = -3.5, \tilde{x}_2 = -2.5, \tilde{x}_3 = -1.5, \tilde{x}_4 = -0.5, \tilde{x}_5 = 0.5, \tilde{x}_6 = 1.5, \tilde{x}_7 = 2.5, \tilde{x}_8 = 3.5\right\}$

- Let: $f(x) = \begin{cases} 1/8, & -4 \le x \le 4 \\ 0, & \text{else} \end{cases}$

$$E\left[X^2\right] = \int_{-4}^{4} x^2 \cdot \frac{1}{8} dx = \left.\frac{x^3}{24}\right|_{-4}^{4} = \frac{128}{24} = \frac{16}{3}$$

Example 7.1 (continued)

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- Distortion:

$$D = MSE = \sum_{k=1}^{M} \int_{x_{k-1}}^{x_k} \left(x - \tilde{x}_k\right)^2 f_X(x)\, dx$$

$$= \sum_{k=1}^{8} \int_{-5+k}^{-4+k} \left(x - \left(-4.5 + k\right)\right)^2 \cdot \frac{1}{8} dx$$

- Examine the $k = 5$ term:

$$\int_0^1 (x - 0.5)^2 \cdot \frac{1}{8} dx = \frac{1}{8} \cdot \int_0^1 x^2 - x + 0.25 dx = \frac{1}{8} \cdot \left[\frac{x^3}{3} - \frac{x^2}{2} + 0.25x \right]_0^1$$

$$= \frac{1}{8} \cdot \left[\frac{1}{3} - \frac{1}{2} + \frac{1}{4} \right] = \frac{1}{8} \cdot \frac{1}{12}$$

Example 7.1 (continued)

- All 8 terms are identical. $\therefore \mathrm{MSE} = 1/12$

$$\left(\frac{S}{N}\right)_{\text{avg}} = 10\log_{10}\frac{E\left[X^2\right]}{D} = 10\log_{10}\left(\frac{16/3}{1/12}\right) = 18.1\text{dB}$$

- A standard rule of thumb is that each additional bit adds 6 dB to the SNR of uniform quantizers operating on a uniform pdf.

SNR for Uniform Quantization

- General result: $$\left(\frac{S}{N}\right)_{\text{avg}} = M^2$$
  - Assumes uniform quantizer with *M* levels
  - Assumes that input samples have uniform distribution with identical range as quantizer
  - Though some texts do not make it clear, this result applies only when these special set of conditions hold. Otherwise, we have to use integral formula
- However, a useful Rule of Thumb:
  - Each additional bit (doubling *M*) increases SNR by 6 dB

$$\left(\frac{S}{N}\right)_{avg} = 6.02n + \alpha$$

  - Where α depends on the distribution of the signal

Example 7.2: PCM Calculation

- **Problem:**
  - Suppose that an analog music signal is found to have a bandwidth of 15 kHz and that samples of the signal may be modeled as having a uniform distribution.
  - Find the minimum first-null bandwidth (assuming the use of square pulses) at which it would be possible to transmit a PCM signal while maintaining an average SNR of at least 58 dB.

Example 7.2 (continued)

- **Solution:**

$$f_s \geq 2B = 30{,}000 \text{ samples / sec}$$

$$10\log_{10} M^2 \geq 58\text{dB} \Rightarrow M \geq 794 \Rightarrow M \geq 1024$$

$$\Rightarrow n \geq 10 \text{ bits/sampl e}$$

- We will assume *n* must be an integer, thus *n* = 10
- Minimum data rate: $R = f_s n \geq 300kbps$
- First null BW: $B = f_s n = 300kHz$
- Assumes rectangular pulses are used as the waveforms

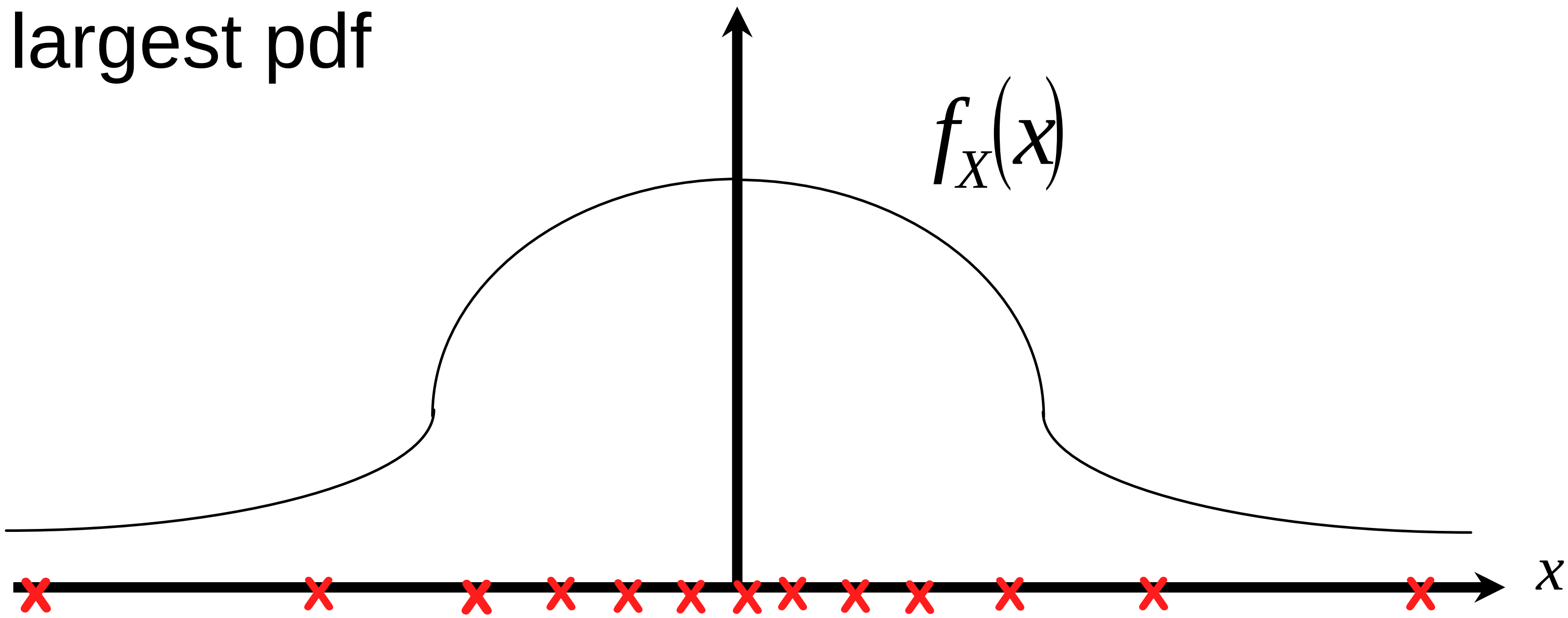
Note: Bandwidth comes into play twice here. Once for the original analog signal and once for the resulting digital signal.

Non-Uniform Quantization

- A quantizer for which
$$\tilde{x}_{k-1} - \tilde{x}_k = \Delta, \forall k \in \{1, \ldots, L-1\}$$
is called a <u>uniform quantizer</u>.
- It is sometimes better to use non-uniform spacing
- Examine the distortion measure: $D = \int_{-\infty}^{\infty} (x - \tilde{x})^2 f(x) dx$
- We wish to make $(x - \tilde{x})^2$ small when $f(x)$ is large.
- We can accept larger $(x - \tilde{x})^2$ when $f(x)$ is small.
- **Basic principle:** concentrate quantization levels in area of largest pdf.

Non-uniform Quantization

- Concentrate quantization levels in areas of largest pdf



Quantization levels = x

Difference between Uniform and Non-uniform Quantization

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- Let: $p(x) = e^{-\frac{x^2}{2}} \Big/ \sqrt{2\pi}$
- Let: $\{\tilde{x}_1 = -1.494, \tilde{x}_2 = -0.498, \tilde{x}_3 = 0.498, \tilde{x}_4 = 1.494\}$
- Numerical integration shows that:

$$D = 0.1188, E\left[X^2\right] = 1, \left(\frac{S}{N}\right)_{avg} = 10\log_{10}\left(\frac{1}{0.1188}\right) = 9.25dB$$

- Non-uniform quantization will yield better results
  - The "best" possible quantizer has

$$\left(\frac{S}{N}\right)_{avg} = 12dB$$

Drill Problem

- If time….

Summary

- Today we have examined Pulse Code Modulation which converts an analog signal to a digital signal.
- Comprised of
  - Sampling
  - Quantization
  - Pulse modulation
- Next class we will examine additional techniques for converting analog signals to baseband digital signals