ECE4634
Digital Communications
Fall 2007

Instructor: R. Michael Buehrer

Lecture #5: The Sampling Theorem

Analog and Digital Communications

Overview

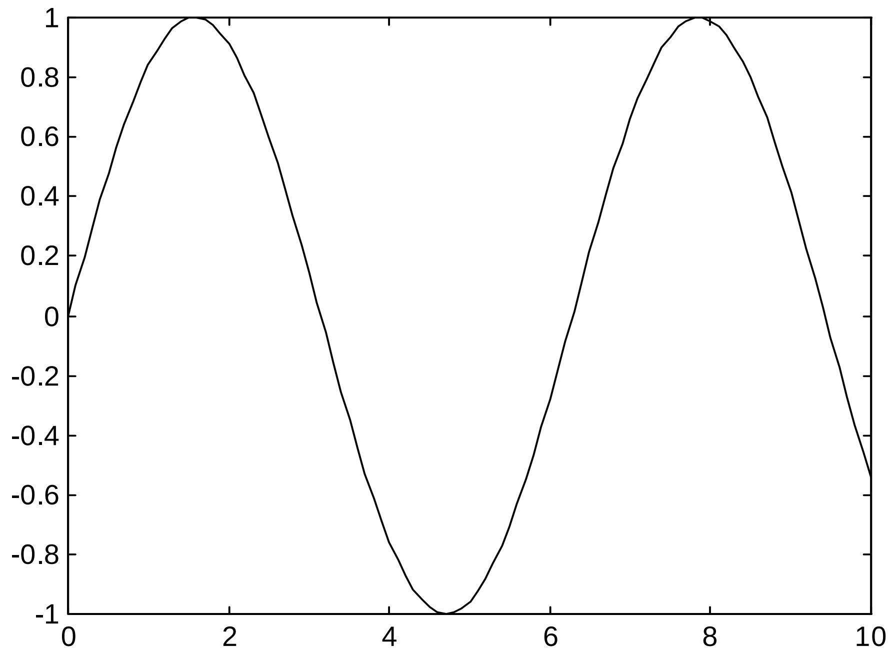
- We are studying *digital* communication systems
- Digital systems can be used to transmit either analog or digital information
- Analog information must be converted to digital format
  - This conversion includes *sampling* and some form of *quantization*
- Today we study the impact of sampling
- What to read – Section 5.1 in the text

Lecture Objectives

- The objectives of this lecture are to
  - Show that the sampling process can be done with (theoretically) no loss of information
  - Describe the conditions under which this occurs – i.e., to introduce the Sampling Theorem
  - Explain the practical significance of the sampling theorem

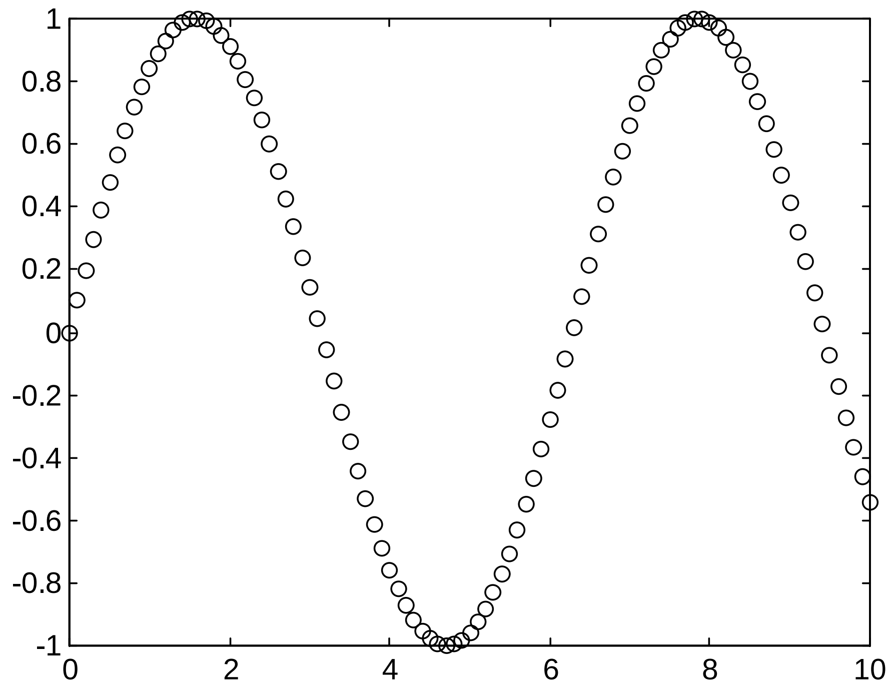
Digital Representation of Analog Signals

- Analog signals (e.g. voice,video) are continuous in time and amplitude:



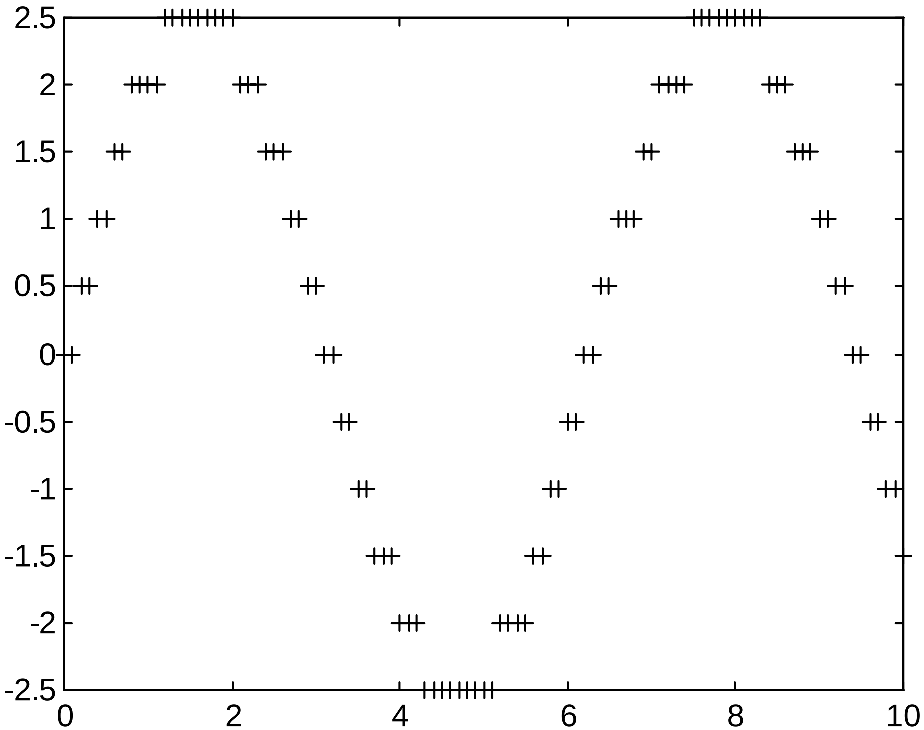
Digital Representation of Analog Signals

- Sampling analog signals makes them discrete in *time*:



Digital Representation of Analog Signals

- <u>Quantization</u> of sampled analog signals makes the samples discrete in amplitude:



•The number of discrete amplitude levels is directly related to the number of bits we are willing to use to represent each sample. Thus, we trade-off bit rate and fidelity

Digital Representation of Analog Signals

- If done properly, sampling introduces no distortion into the signal
- Quantization does introduce distortion
  - There is a tradeoff between distortion and bandwidth requirements
  - More bits per sample → less distortion
  - Fewer bits per sample → lower bandwidth requirements
- We consider sampling today.
- We will discuss quantization shortly.

The Sampling Theorem

Analog and Digital Communications

- We consider instantaneous sampling of a signal waveform ("ideal sampling" or "impulse sampling") which can be modeled as

$$w_s(t) = \underbrace{w(t)}_{\substack{\text{original}\\ \text{signal}}} \underbrace{\sum_{n=-\infty}^{\infty} \delta(t - nT_s)}_{\substack{\text{impulse}\\ \text{train}}}$$

$$= \sum_{n=-\infty}^{\infty} w(nT_s)\,\delta(t - nT_s)$$

- The train of impulse functions select sample values at regular intervals.
- How often do we have to sample to retrieve the original information? (*i.e.*, how small can T_s be?)

RM Buehrer
Virginia Tech
Fall 2007

The Sampling Theorem (continued)

Analog and Digital Communications

$$\boxed{w_s(t) = w(t)\sum_{n=-\infty}^{\infty}\delta(t - nT_s) = \sum_{n=-\infty}^{\infty} w(nT_s)\delta(t - nT_s)}$$

- **The train of impulse functions select sample values at regular intervals. Using a Fourier Series representation of the impulse train:**

$$\sum_{n=-\infty}^{\infty}\delta(t - nT_s) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} e^{jn\omega_s t}, \omega_s = \frac{2\pi}{T_s}$$

- **Rewriting, we have:**

$$w_s(t) = w(t)\sum_{n=-\infty}^{\infty}\frac{1}{T_s}e^{jn\omega_s t}$$

RM Buehrer
Virginia Tech
Fall 2007

The Sampling Theorem (continued)

Analog and Digital Communications

- Taking the Fourier Transform of signals:

$$W_s(f) = \frac{1}{T_s} W(f) * F\left\{ \sum_{n=-\infty}^{\infty} e^{jn\omega_s t} \right\}$$

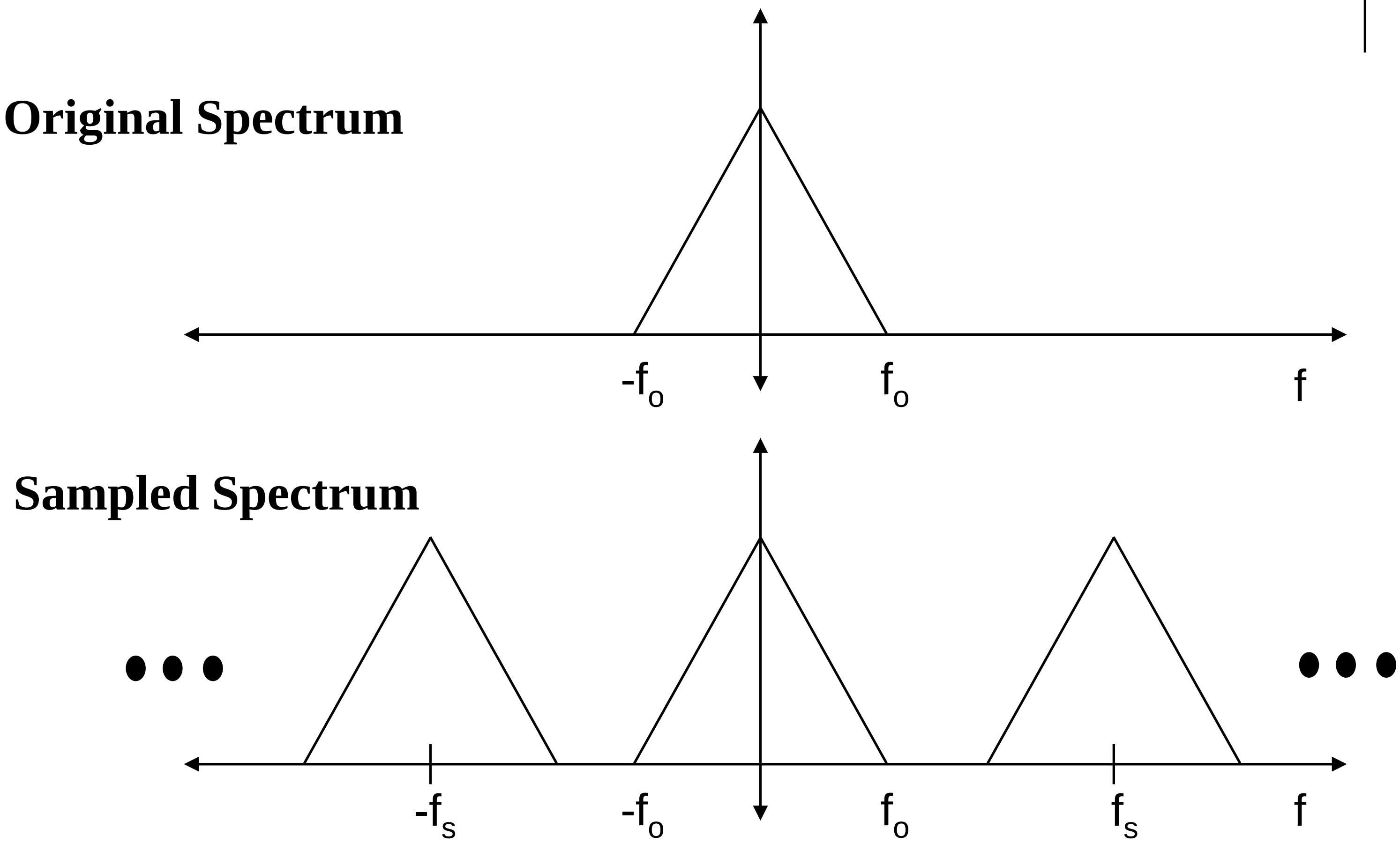
$$= \frac{1}{T_s} W(f) * \sum_{n=-\infty}^{\infty} F\left\{ e^{jn\omega_s t} \right\}$$

$$W_S(f) = \frac{1}{T_S} W(f) * \sum_{n=-\infty}^{\infty} \delta(f - nf_S),\ f_S = \frac{\omega_S}{2\pi}$$

$$\boxed{W_s(f) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} W(f - nf_s)}$$

Note: This also follows from the fact that the Fourier Transform of an impulse train is simply an impulse train.

Sampling Theorem



RM Buehrer
Virginia Tech
Fall 2007

Sampling Theorem

- Let *w(t)* be a bandlimited signal with Fourier Transform:

$$W(f) = 0, \text{ for } |f| > B$$

- ***w(t)* can be perfectly reconstructed from uniformly spaced samples, provided those samples are taken at a rate** $f_s \geq 2B$
  - *2B* is called the <u>Nyquist Rate</u>
  - If $f_s < 2B$, <u>aliasing</u> results.
  - If the signal is not strictly bandlimited, then it must be passed through lowpass filter before sampling to practically limit its bandwidth

Recovering the Signal from Sampled Waveform

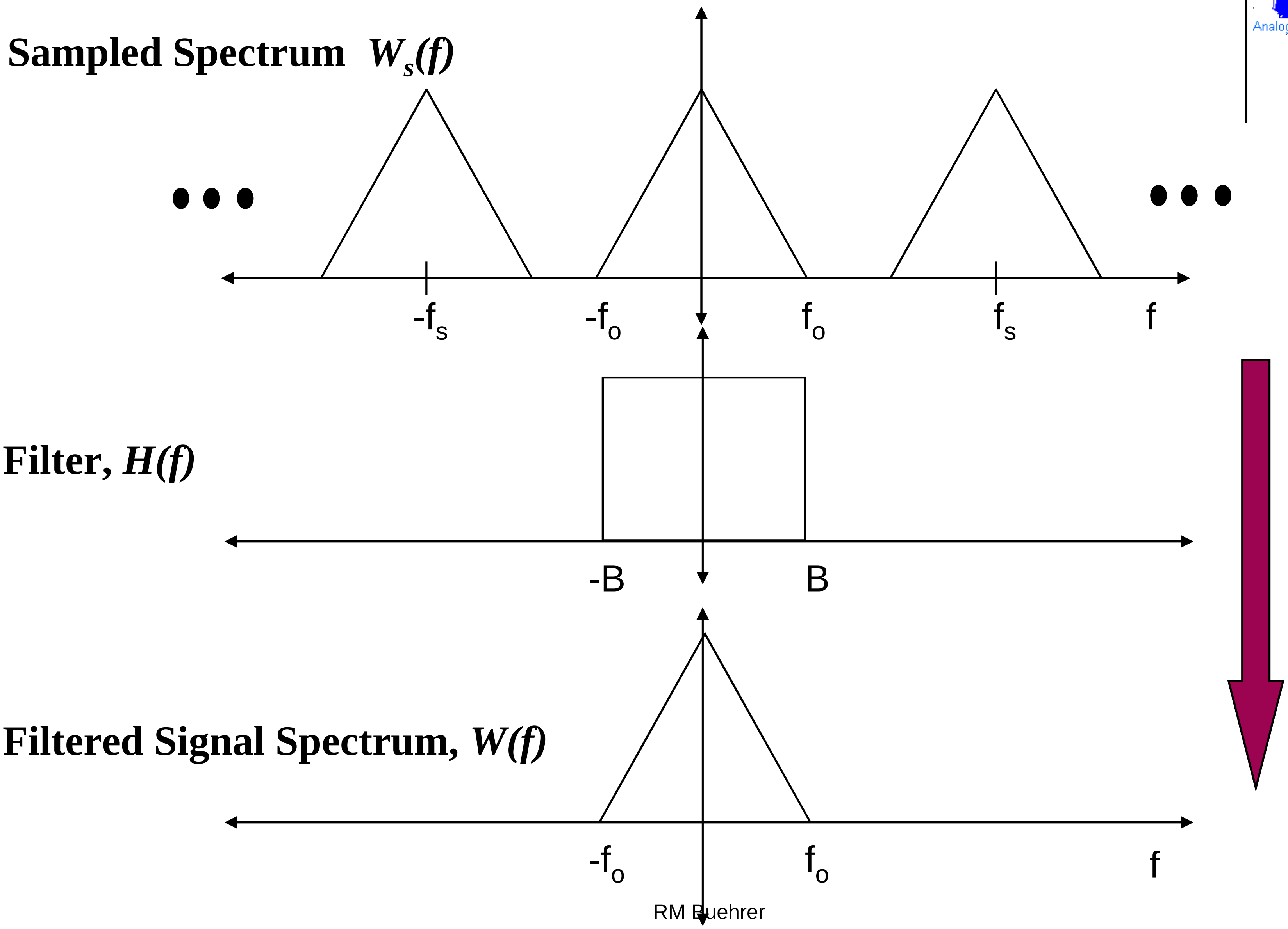
- Sampled signal: $W_s(f) = \dfrac{1}{T_s} \sum_{n=-\infty}^{\infty} W(f - nf_s)$

- Apply lowpass filter to recover original signal

$W_s(f) \longrightarrow \boxed{\textbf{Ideal Lowpass Filter w/bandwidth B}} \longrightarrow W(f)$

$$\boxed{\begin{aligned} W(f) &= W_s(f)\Pi\left(\frac{f}{2B}\right) \\ &= \left(\frac{1}{T_s}\sum_{n=-\infty}^{\infty} W(f - nf_s)\right)\Pi\left(\frac{f}{2B}\right) \\ &= \frac{1}{T_s} W(f) \end{aligned}}$$

Recovering the Original Signal



Other Versions of Sampling Theorem

Analog and Digital Communications

- We will also discuss "flat top" sampling using rectangular pulses.
  - Becomes identical to instantaneous sampling approach as pulses become short

- Random signals are band-limited if the power spectral density satisfies: $P_X(f) = 0, \text{ for } |f| > B$
  - May also be represented by samples taken at rate 2*B*

- Bandpass signals with bandwidth *B* may be represented by *complex-valued* samples taken at rate B or by *real-valued* samples taken at rate *2B*

Practical Limitations

- Band-limited Signals
  - No signal has a spectrum that goes to identically zero at some finite frequency
  - We treat some level (say 30dB below the strongest frequency component as *essentially zero.*
  - Signals are typically filtered with a low pass filter with very steep roll-off in order to ensure the signal falls below a certain value
- Ideal reconstruction
  - Using sinc functions for interpolation (i.e., using a perfect “brick-wall” frequency filter) results in *ideal recontruction*
  - Non-ideal reconstruction can be accomplished using other forms of interpolation
    - Result in non-ideal frequency filters

Example 5.1

- Consider the following time domain signal:

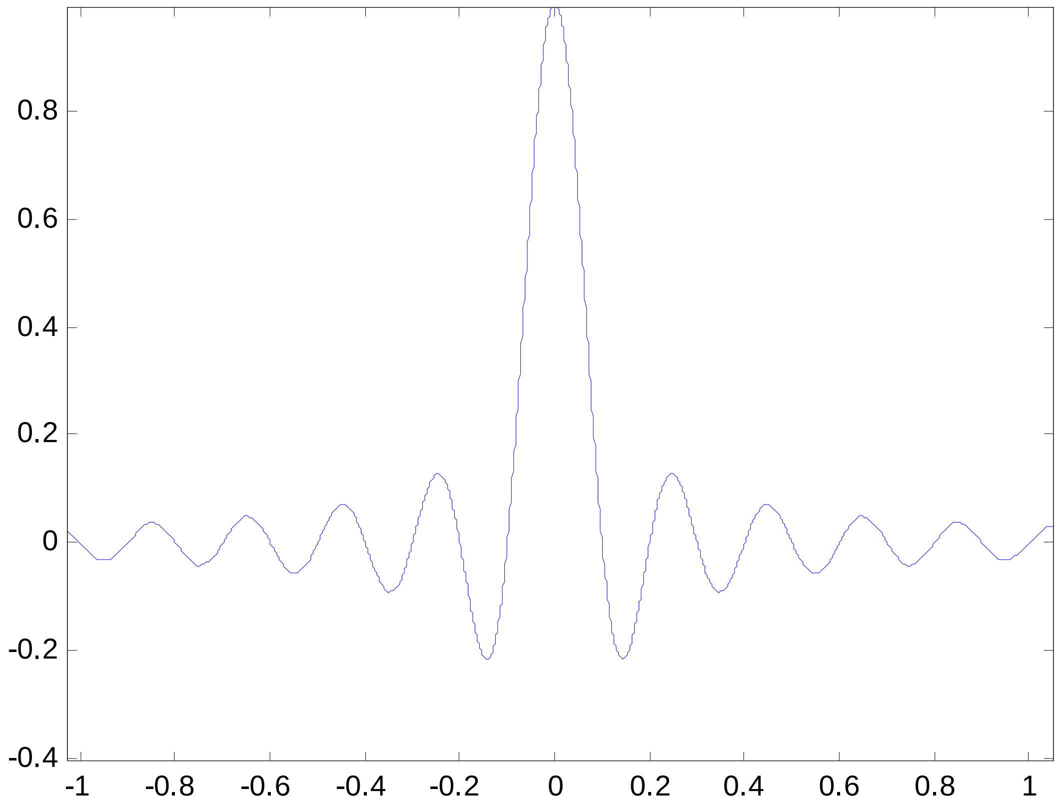
$$w(t) = \mathrm{sinc}(100000t)$$

- The Fourier Transform is :

$$W(f) = \frac{1}{100000}\Pi\left(\frac{f}{100000}\right)$$

- The bandwidth of this signal is:
  - B=50 kHz
- Therefore samples must be taken at least at rate:
  - R > 2 B = 100,000 samples/second

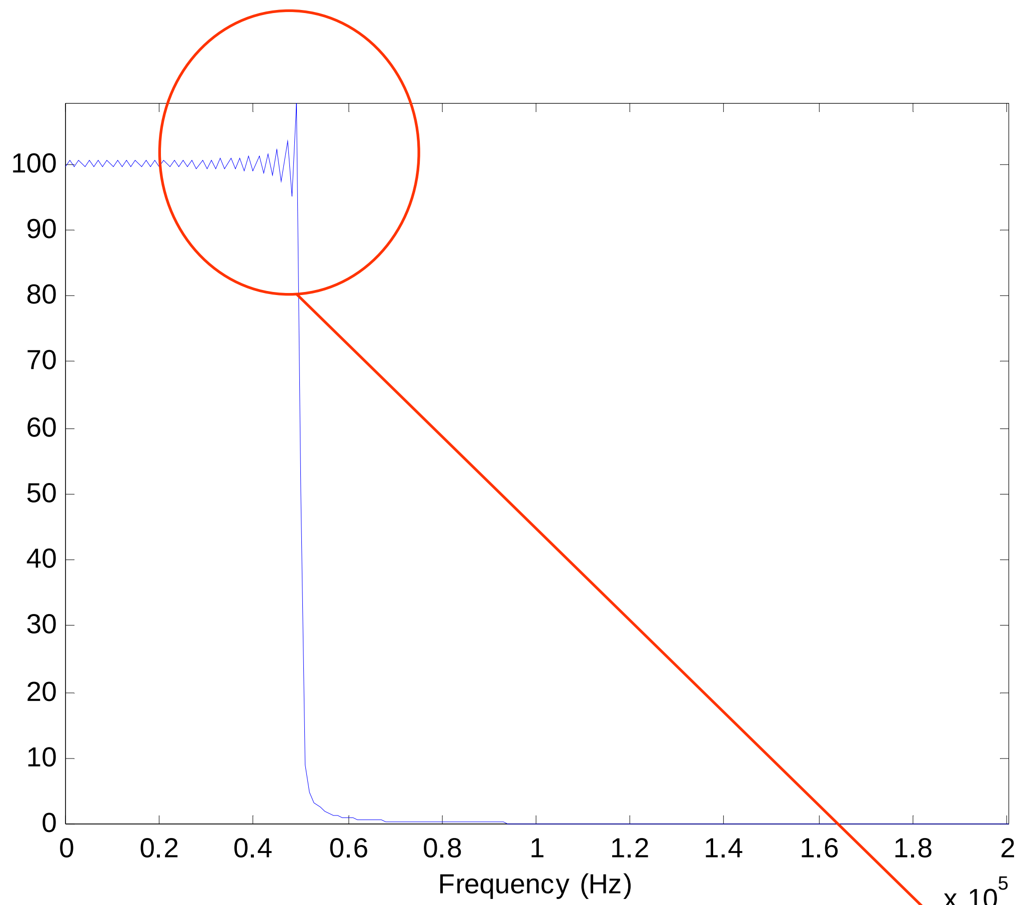
Example 5.1 (cont.)



10^{-4}

$$w(t) = \left[\mathrm{Sa}\left(100000\pi t\right)\right]$$

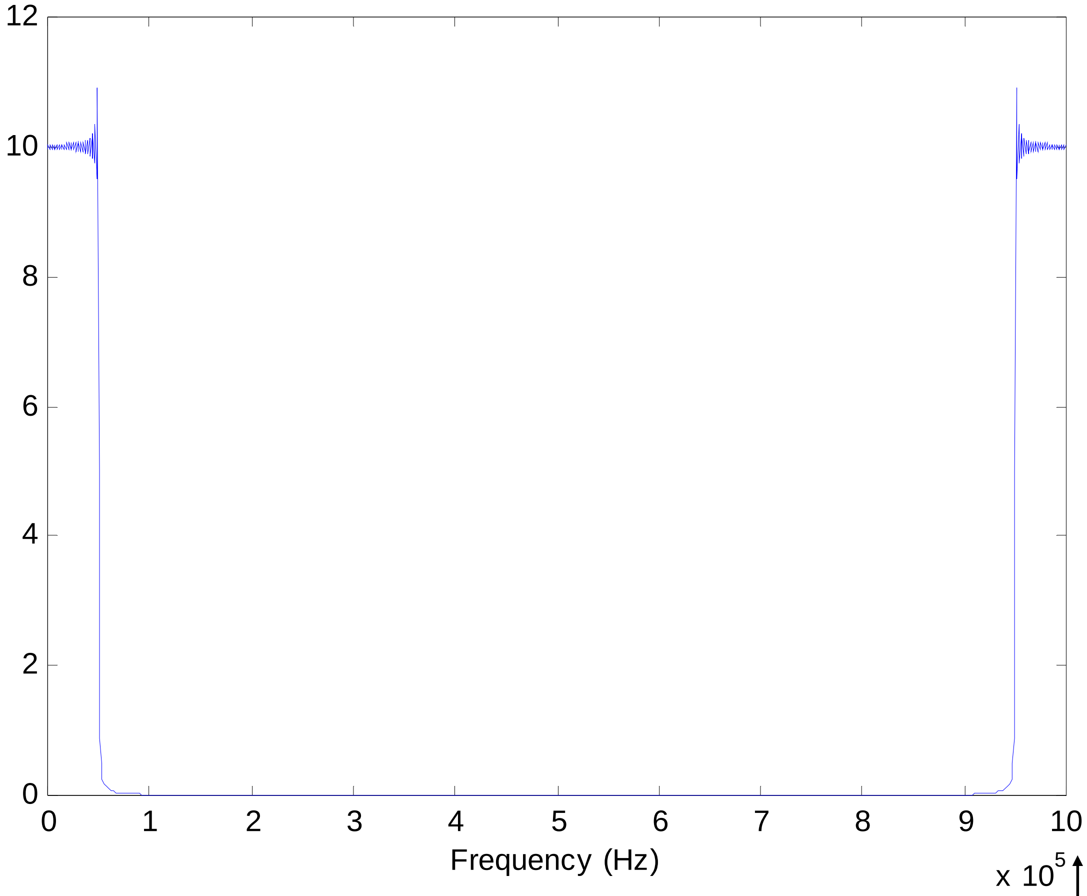
Example 5.1 (cont.)



$$W(f) = \frac{1}{100000}\Pi\left(\frac{f}{100000}\right)$$

Why isn't this a perfect square pulse?

Example 5.1 (cont.)

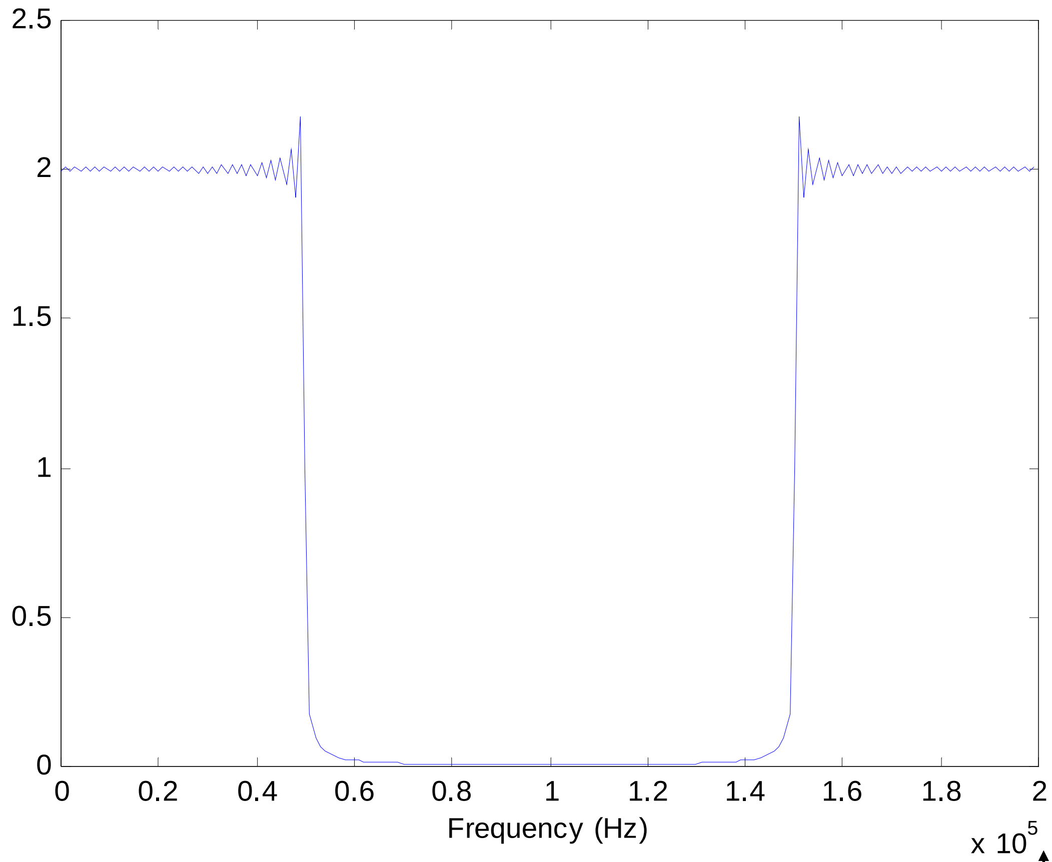


f_s

$$f_s >> 2B$$

B = 50kHz
f_s = 1MHz

Example 5.1 (cont.)

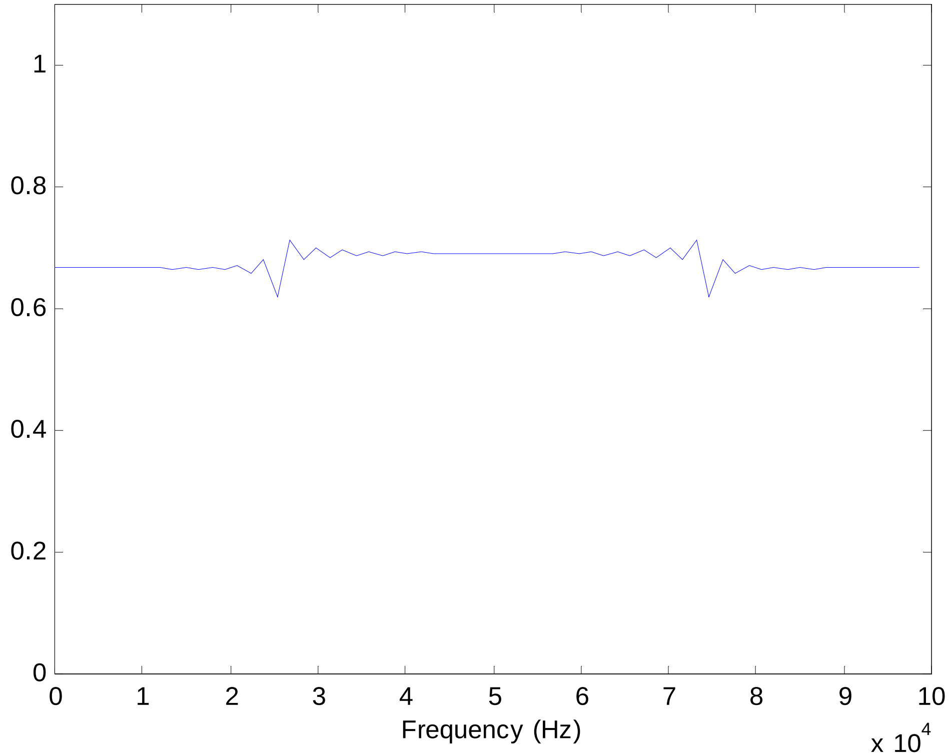


f_s

$$f_s > 2B$$

B = 50kHz
f_s = 200kHz

Example 5.1 (cont.) : Aliasing



f_s

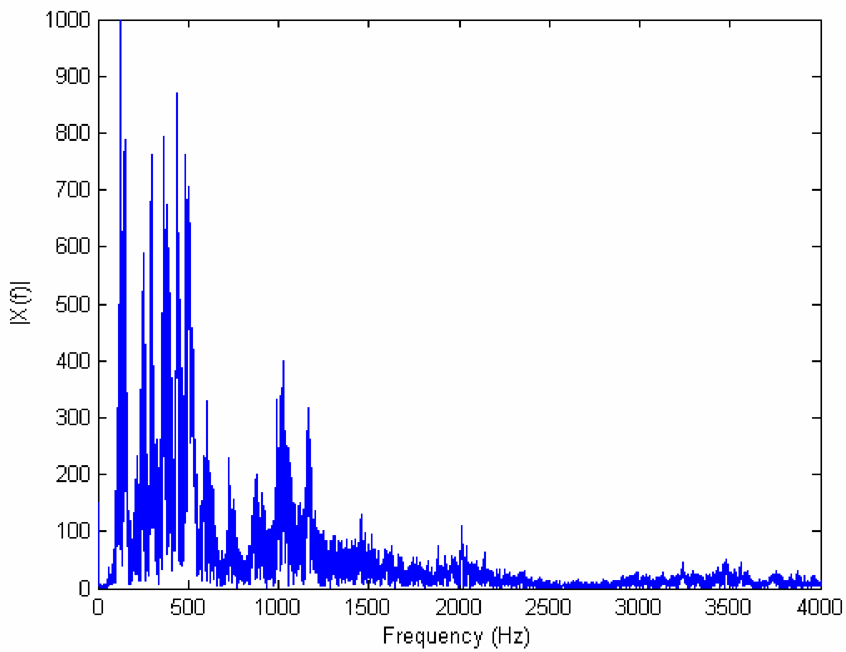
$$f_s < 2B$$

B = 50kHz
f_s = 75kHz

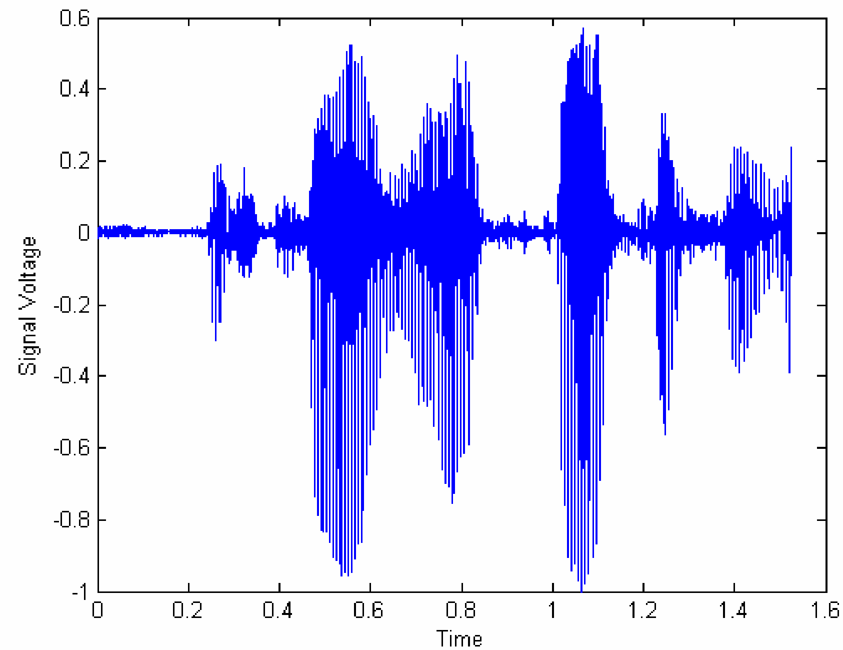
RM Buehrer
Virginia Tech
Fall 2007

Example 5.2

Original Spectrum



Time Signal

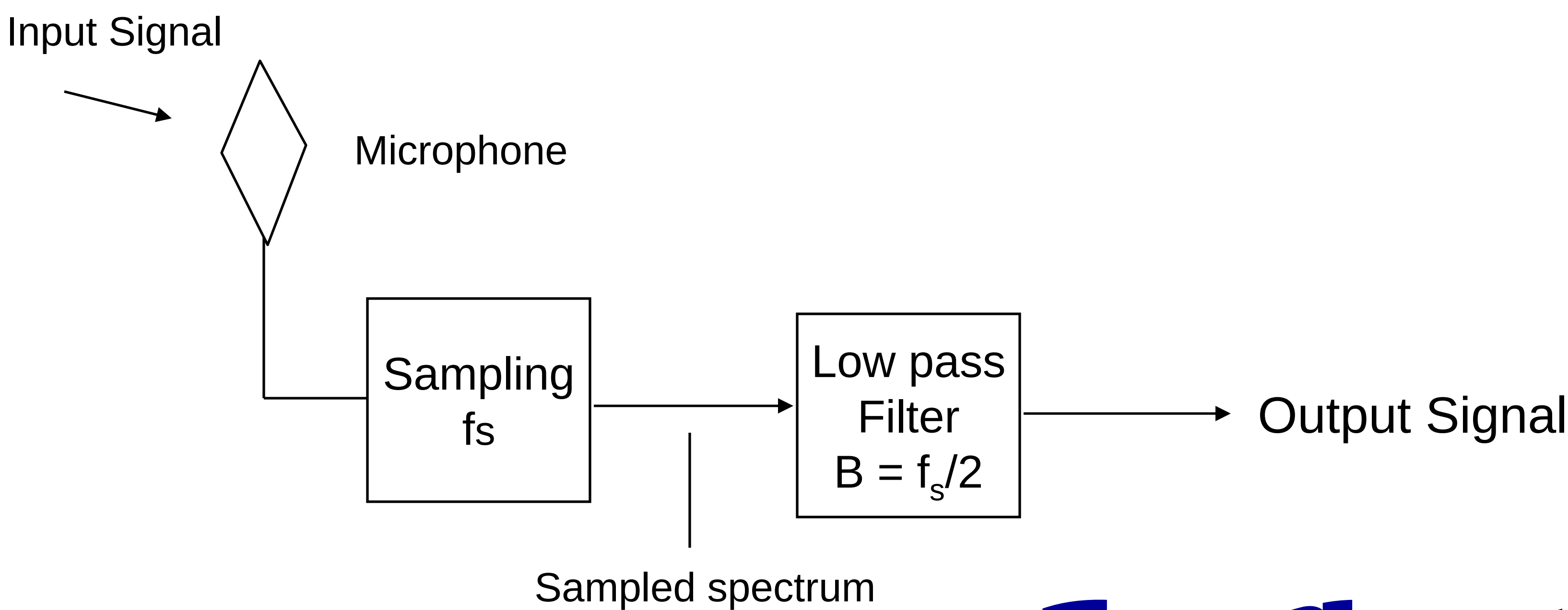


SamplingOriginal.wav

Analog and Digital Communications

Example 5.2: System

- Simple sampling and reconstruction

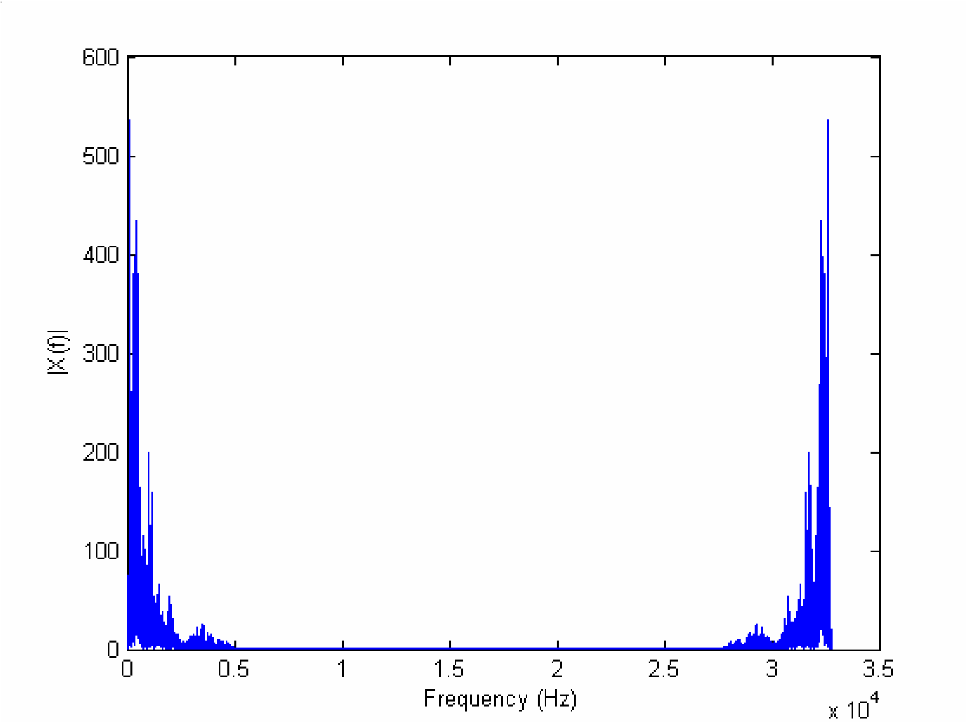


Test Signal

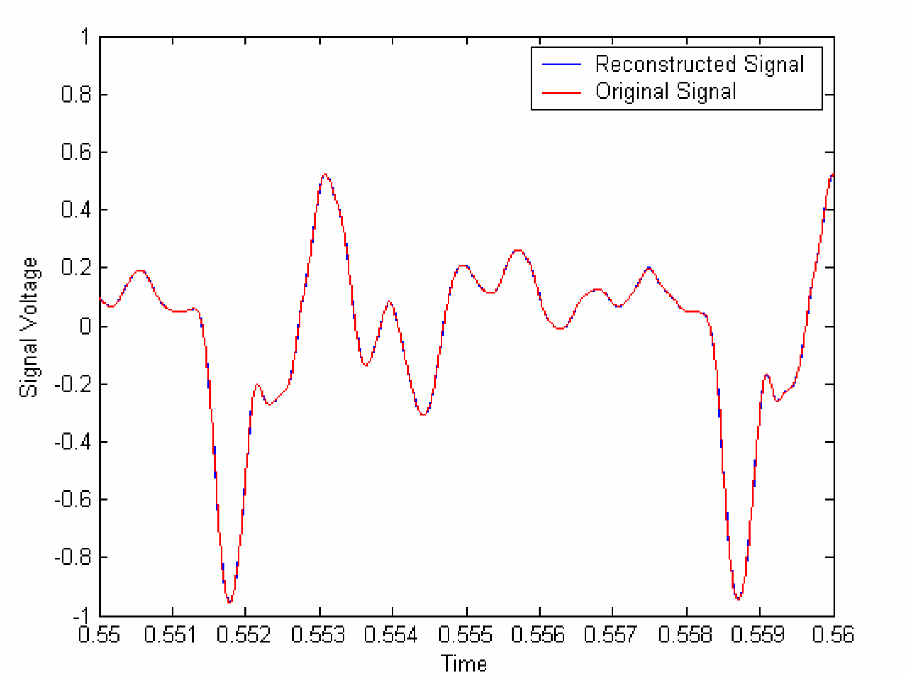
Example 5.2: f_s = 32kHz

Analog and Digital Communications

No Aliasing



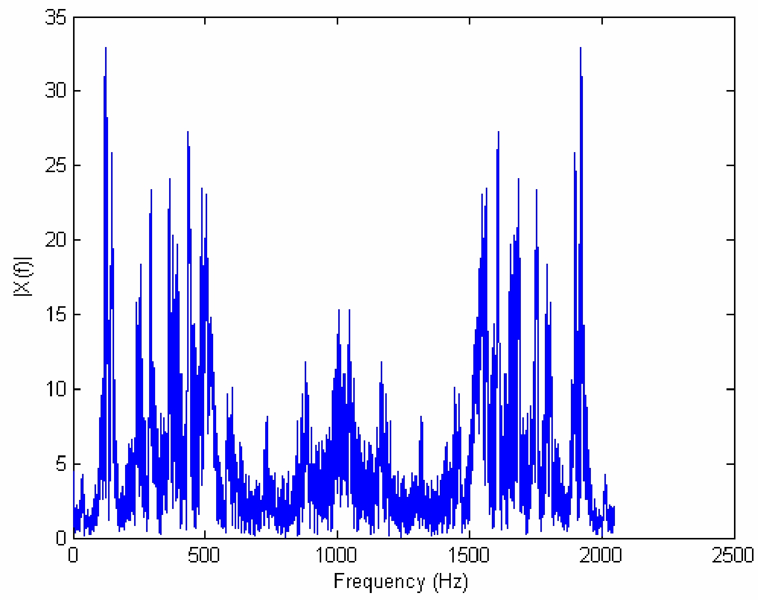
Perfect Reconstruction



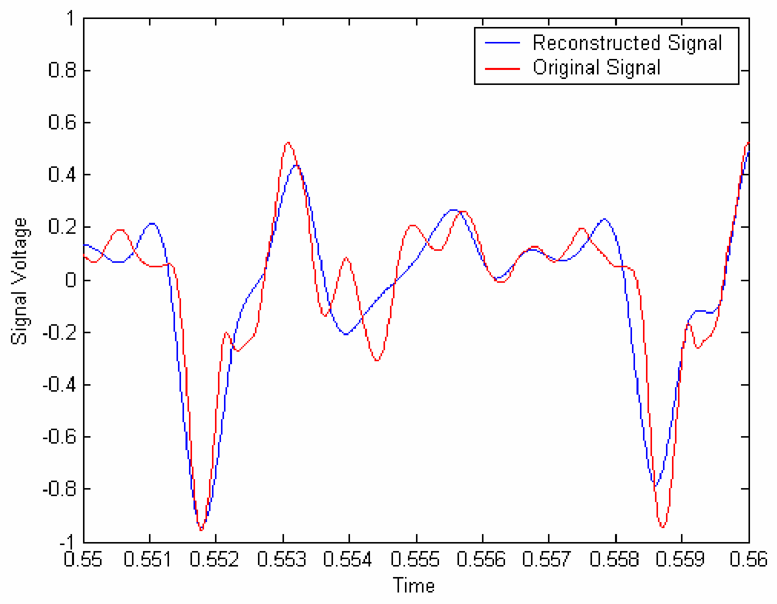
Sound Quality

Example 5.2: f_s = 2kHz

Substantial Aliasing

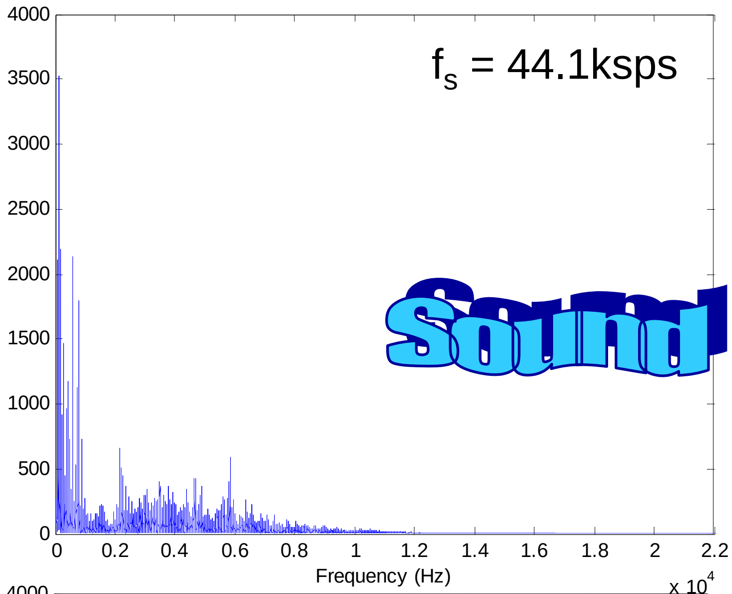


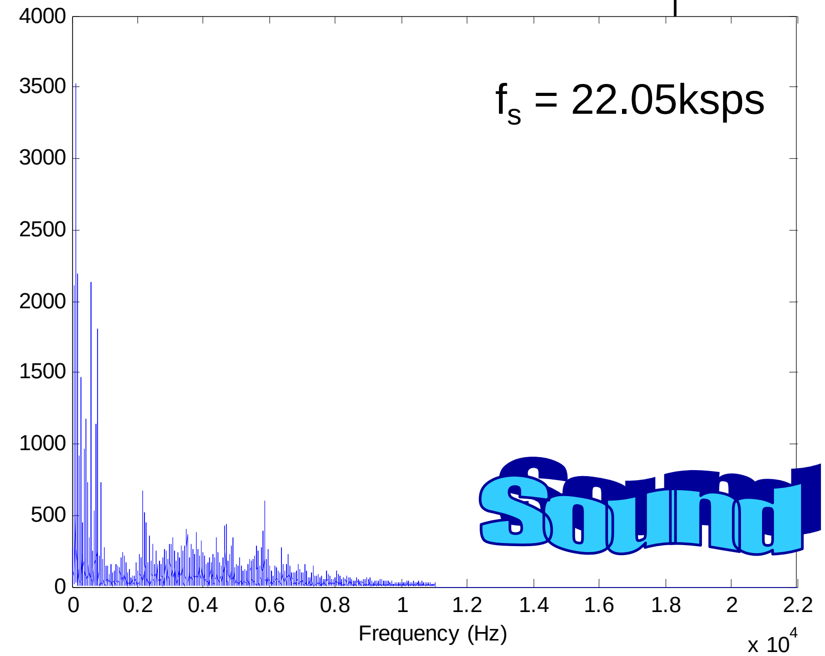
Imperfect Reconstruction

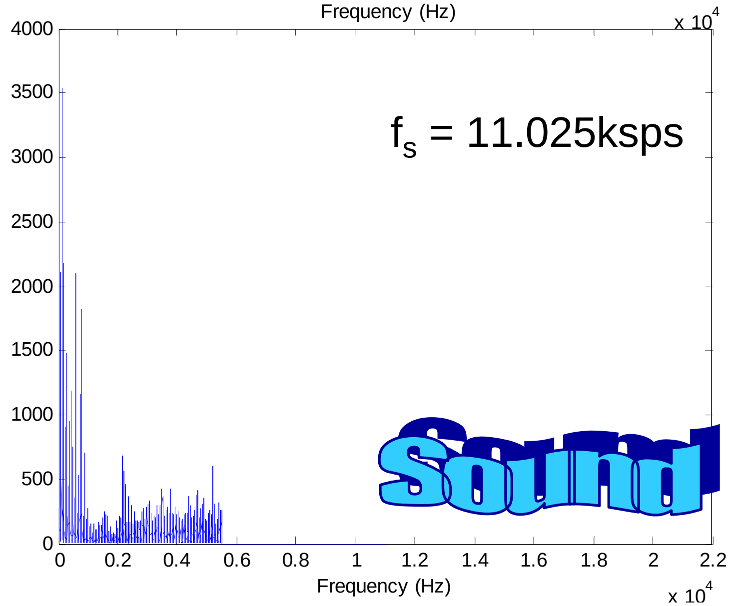


Sound Quality

Example 5.3 – Reconstructed Spectrum







Two aspects of aliasing can be discerned:

1. Loss of high frequency content
2. Distortion of lower frequency content

Practical Sampling Rates

- Speech:
  - Telephone quality speech has a bandwidth of 4 kHz
  - Most digital telephone systems sample at 8000 samples/sec
- Audio:
  - The highest frequency the human ear can hear is approximately 15 kHz
  - CDs sample at rate 44,100 samples/sec
- Video:
  - The human eye requires samples at a rate of at least 20 frames/sec to achieve smooth motion

Summary

- Today we have examined a key aspect of digital communications: Sampling
- Nyquist's Sampling Theorem tells us that sampling introduces no distortion provided that we sample at a rate equal to or greater than twice the highest frequency
- In practical scenarios we typically filter the signal before sampling in order to prevent aliasing

ECE4634
Digital Communications
Fall 2007

Appendix
Matlab Representation of Signals

Analog and Digital Communications

Representation of Signals in Matlab

Analog and Digital Communications

- Matlab uses vectors which are inherently sampled signals with limited duration (0 to *T*)
- Recall the Fourier Transform $X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft}$
- The sampled signal can be written as

$$x(\Delta t) \approx \sum_{n=0}^{N-1} x(n\Delta t)\delta(t - n\Delta t)$$

- Which has a Fourier Transform $X(f) = \sum_{n=0}^{N-1} x(n\Delta t) e^{-j2\pi fn\Delta t}$
- We are interested in discrete points thus we limit *f* to values {*0, 1/T, 2/T, ….(N-1)/T*} $f = \frac{k}{T} = \frac{k}{N\Delta t}$
- Thus we have:

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{j2\pi n\frac{k}{N}} \quad k = 0,1\ldots N-1$$

This is termed the Discrete Fourier Transform

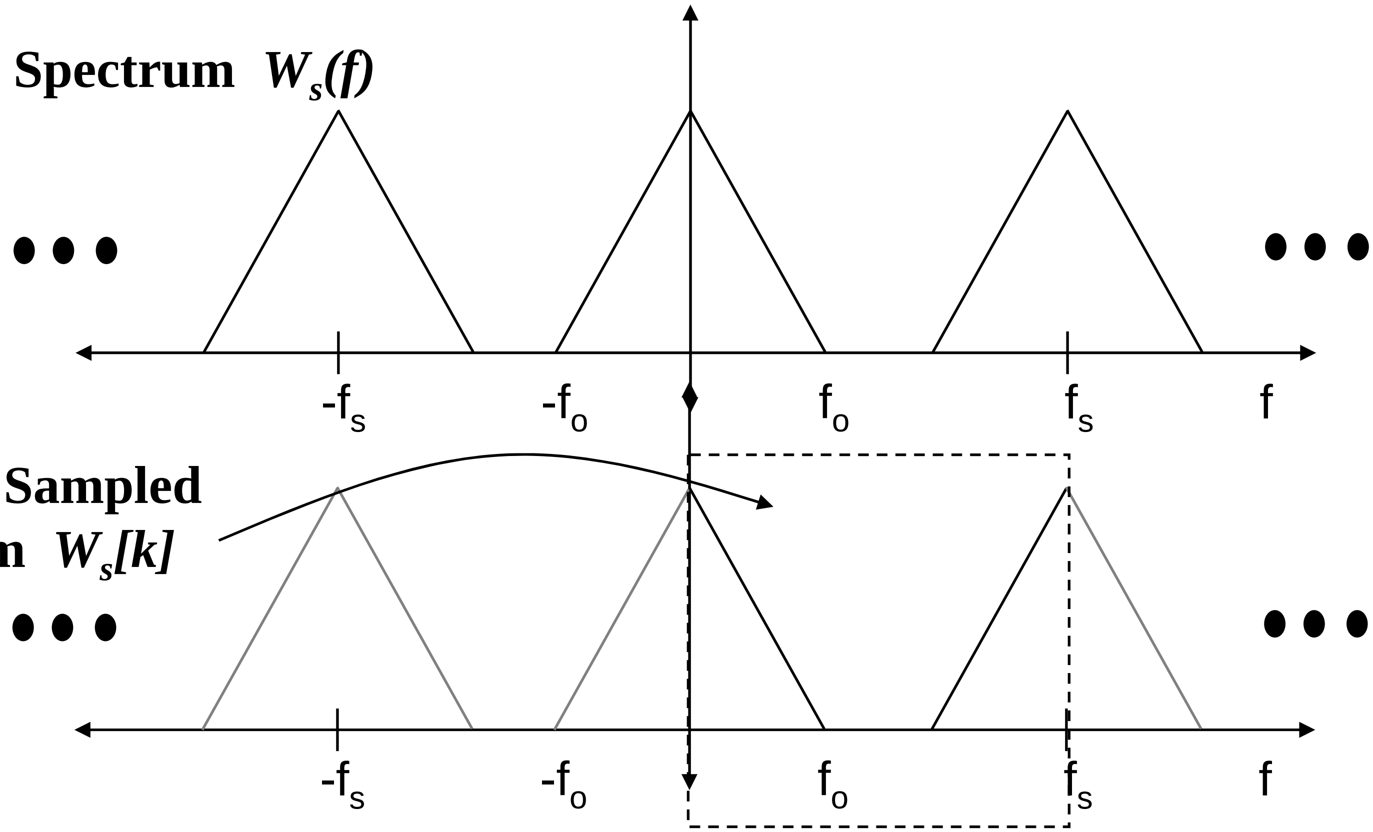
Representation of Signals in Matlab

- Sampling rate must be high enough to avoid aliasing ($f_s > 2B$)
- Use `fft` command to obtain spectrum
- Frequency domain resolution can be increased by “zero padding” - adding zeros at end of signal
- Typically we desire a plot of the *magnitude* of the spectral density of the signal [`plot(abs(fft(x)))`]
- Matlab requires that you assign the units of the scale [`f=0:fs/points:fs-fs/points`]
  - The `fft` function only returns values between 0 and f_s.

Matlab Plots

Sampled Spectrum $W_s(f)$

Discrete Sampled Spectrum $W_s[k]$



- Since the DFT only provides frequency domain points from f = 0 to $f_s/2$, the plot only shows a fraction of the actual sampled spectrum

RM Buehrer
Virginia Tech
Fall 2007

ECE4634
Digital Communications
Fall 2007

Appendix

Time domain view of the Sampling Theorem

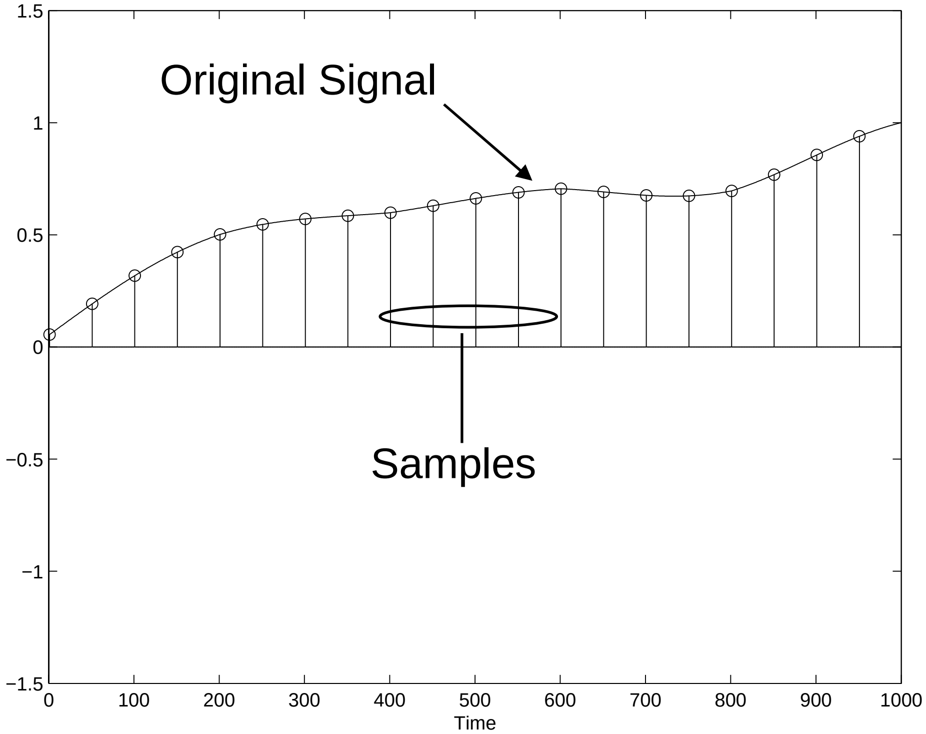
Analog and Digital Communications

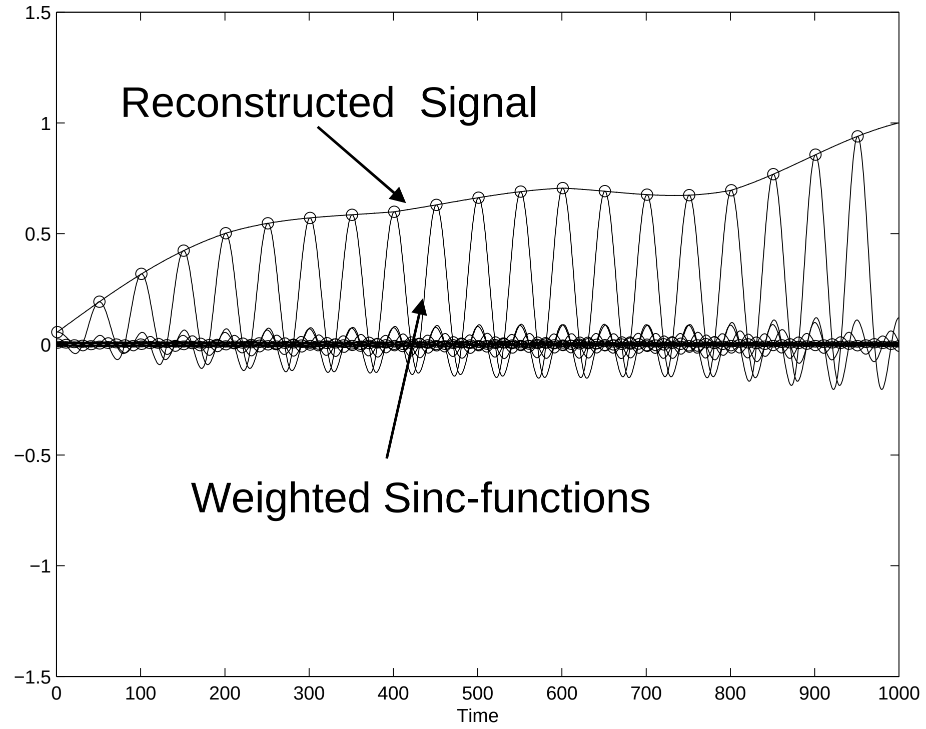
Another View of the Sampling Theorem

$$W(f) = W_s(f)\Pi\left(\frac{f}{2B}\right)$$

$$\begin{aligned}
w(t) &= w_s(t) * \Im^{-1}\left\{\Pi\left(\frac{f}{2B}\right)\right\} \\
&= w_s(t) * \mathrm{sinc}(2\mathrm{Bt}) \\
&= \left(\sum_{n=-\infty}^{\infty} w(nT_s)\delta(t - nT_s)\right) * \mathrm{sinc}(2\mathrm{Bt}) \\
&= \sum_{n=-\infty}^{\infty} w(nT_s)\mathrm{sinc}(2Bt - n2BT_s) \\
&= \sum_{n=-\infty}^{\infty} w(nT_s)\mathrm{sinc}\left(\frac{t}{T_s} - n\right)
\end{aligned}$$

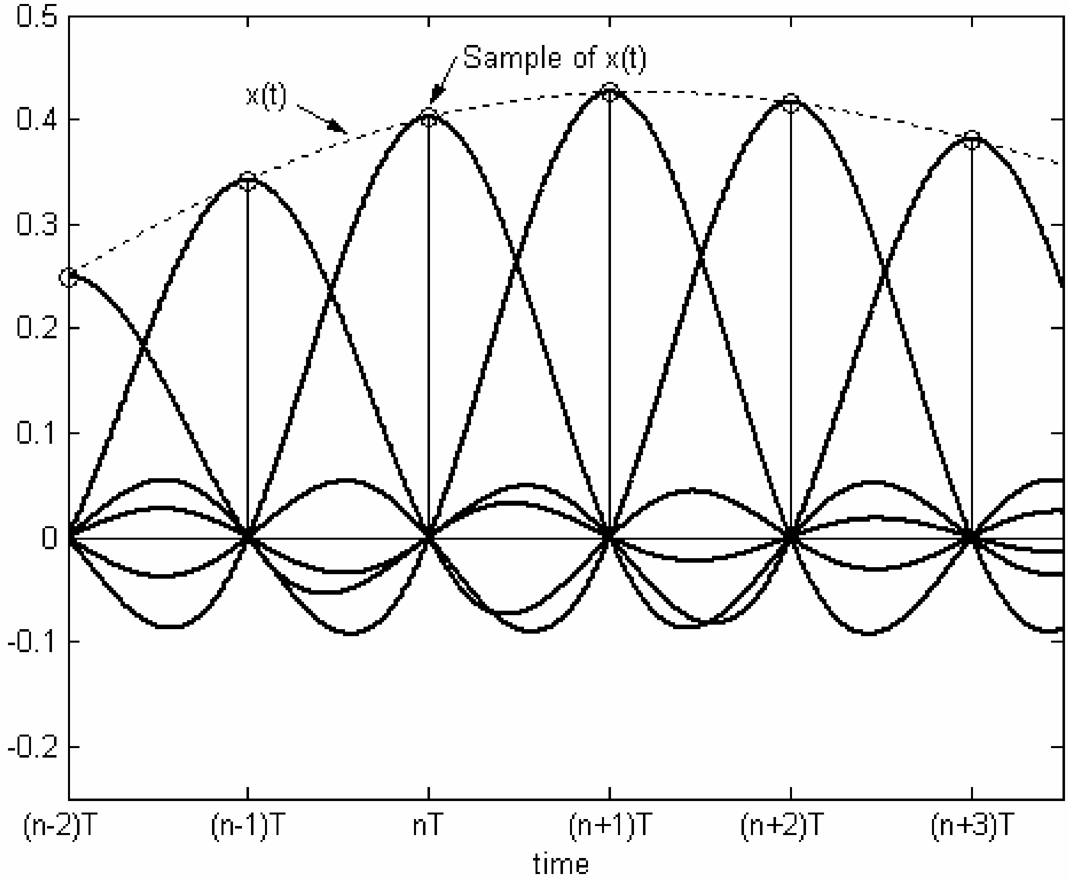
Time-Domain View of the Sampling Theorem





sin(x)/x is also referred to as the *Sampling Function*

Ideal Reconstruction



- Sinc functions provide ideal reconstruction of values between samples
- Compared to linear interpolation or some other form of interpolation, using sinc functions provides *ideal interpolation*