ECE4634
Digital Communications
Fall 2007

Instructor: R. Michael Buehrer

Lecture #11: Pulse Shaping

Analog and Digital Communications

Pulse Shaping

Analog and Digital Communications

- The spectrum of the signal is dependent on the spectrum of the pulse used, the pulse rate as well as the autocorrelation of the data.

$$P_x(f) = \frac{|F(f)|^2}{T_s} \sum_{k=-\infty}^{\infty} R(k) e^{j2\pi f k T_s}$$

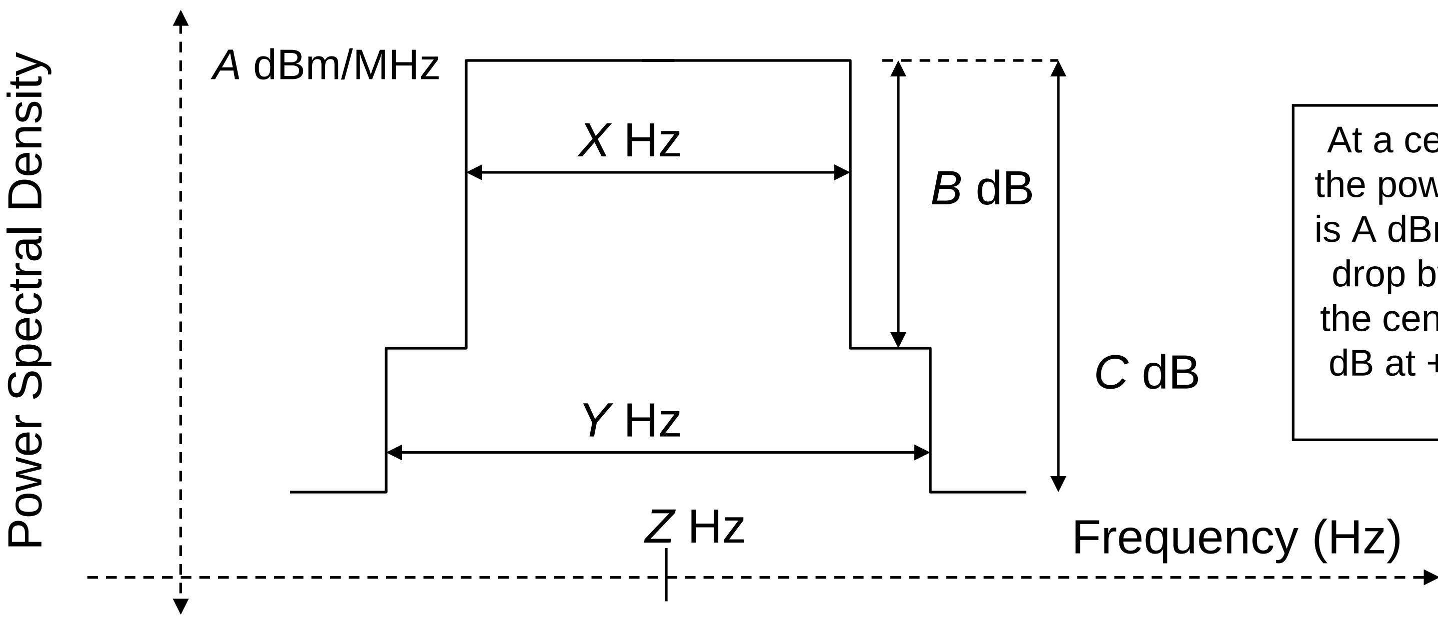
- Often we wish to control aspects of the transmission bandwidth.
- We saw last time how we can control spectral characteristics using different line codes
  - controls the autocorrelation and pulse duration
- Now we turn look at controlling spectrum by controlling the shape of the pulse used.
- This is termed pulse shaping and allows us to maximize data rate within a given bandwidth
- What to read – Sections 6.3 and 6.4

Lecture Objectives

- In this lecture we will demonstrate that
  - pulse shaping allows us to control the spectral parameters of the signal
  - there are limits to what can be done with a pulse that is limited to duration *T*
  - we can reduce bandwidth by allowing the pulse duration to be more than the time between consecutive pulses (T) but in general this leads to ISI
  - a family of pulses termed *Nyquist pulses* or *raised cosine pulses* provide minimum bandwidth with zero ISI

The Spectral Mask

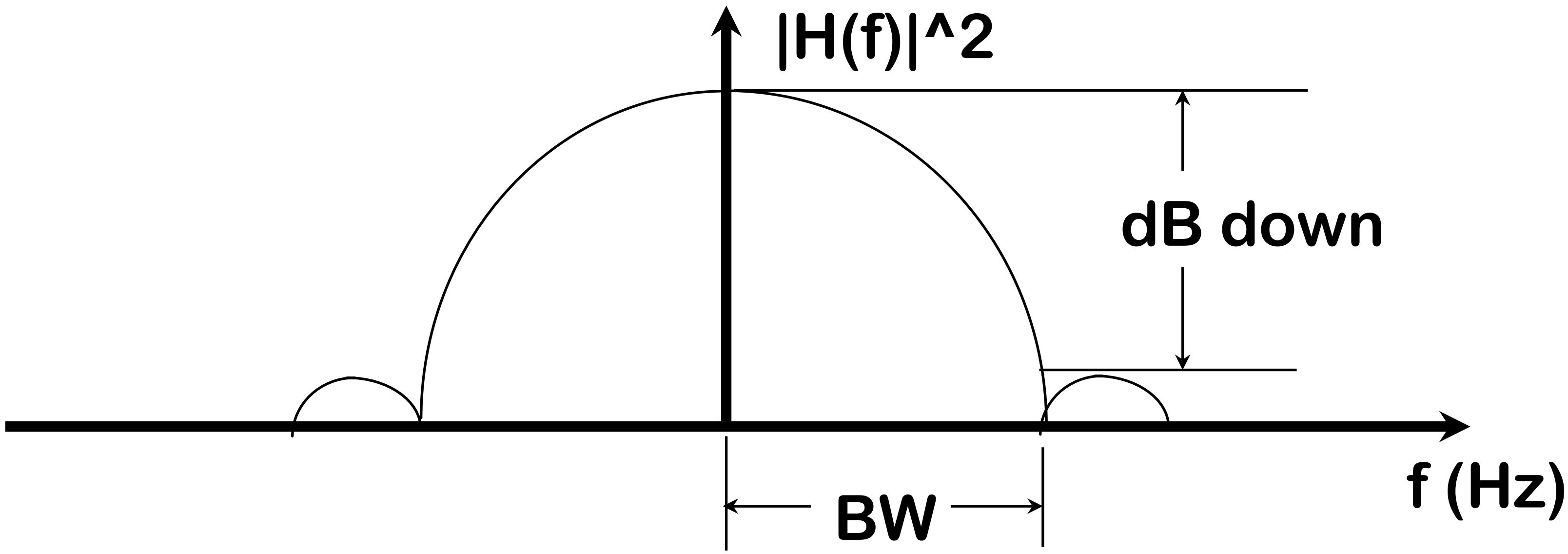
- The Federal Communications Commission (FCC) and many system designers typically specify bandwidth constraints using the *spectral mask*.
- A spectral mask describes the maximum power allowed within a certain bandwidth



At a center frequency of *Z* Hz the power spectral density limit is A dBm/MHz. The PSD must drop by *B* dB at +/- X/2 Hz of the center frequency and by *C* dB at +/- *Y/2* Hz of the center frequency.

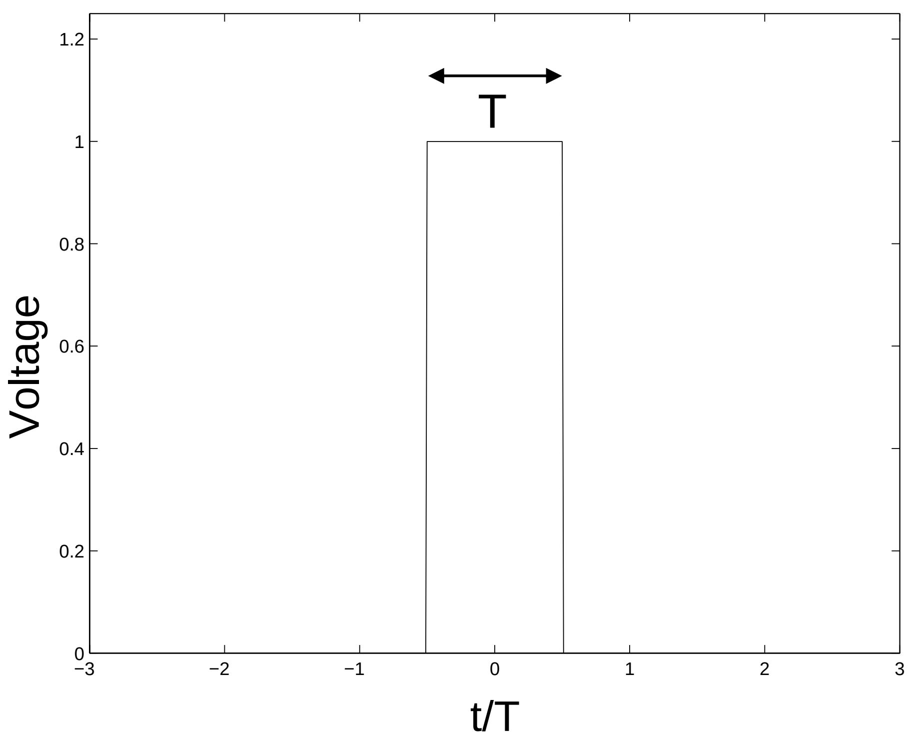
Design Criteria for Pulse Spectrum

- Two important spectral characteristics
  - First null bandwidth
  - Size of sidelobes
- These will determine how well the pulse fits within the mask at a given data rate
- Would like to “round off the corners” of pulses to avoid excessive spectral occupancy

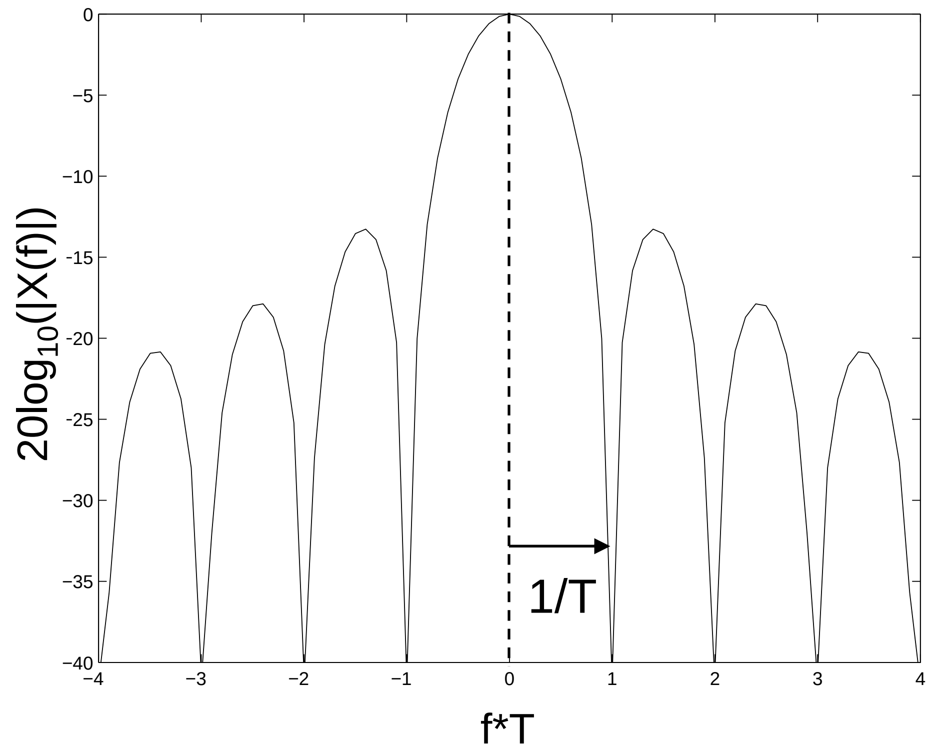


Rectangular Pulse

Time Waveform



Magnitude Spectrum



- **First Null BW: 1/T = R_s (symbol rate)**
- **First Sidelobe: 13.6 dB down**

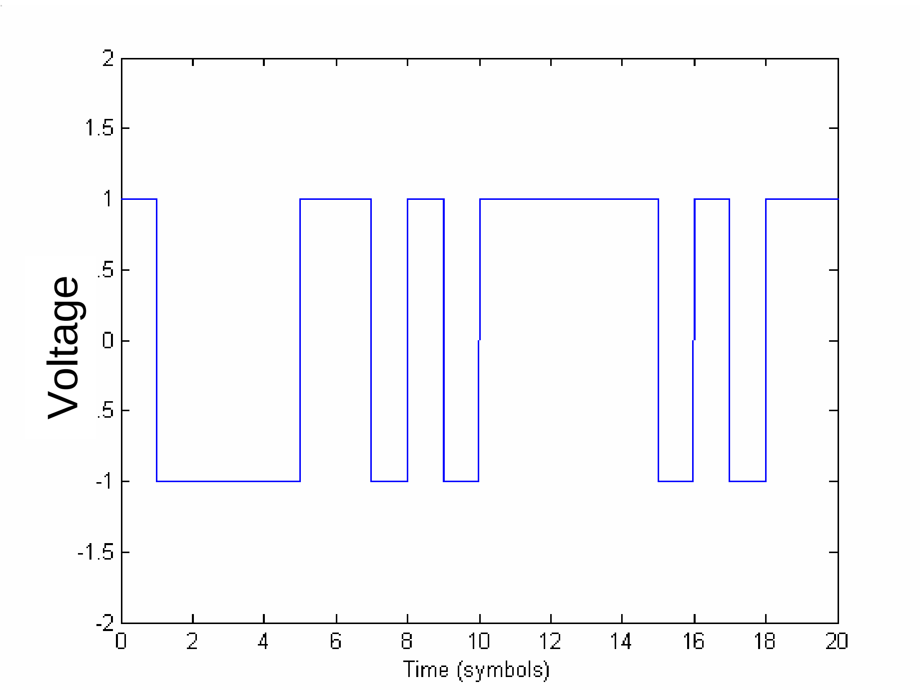
Peaks equal $\frac{2}{\pi n}$

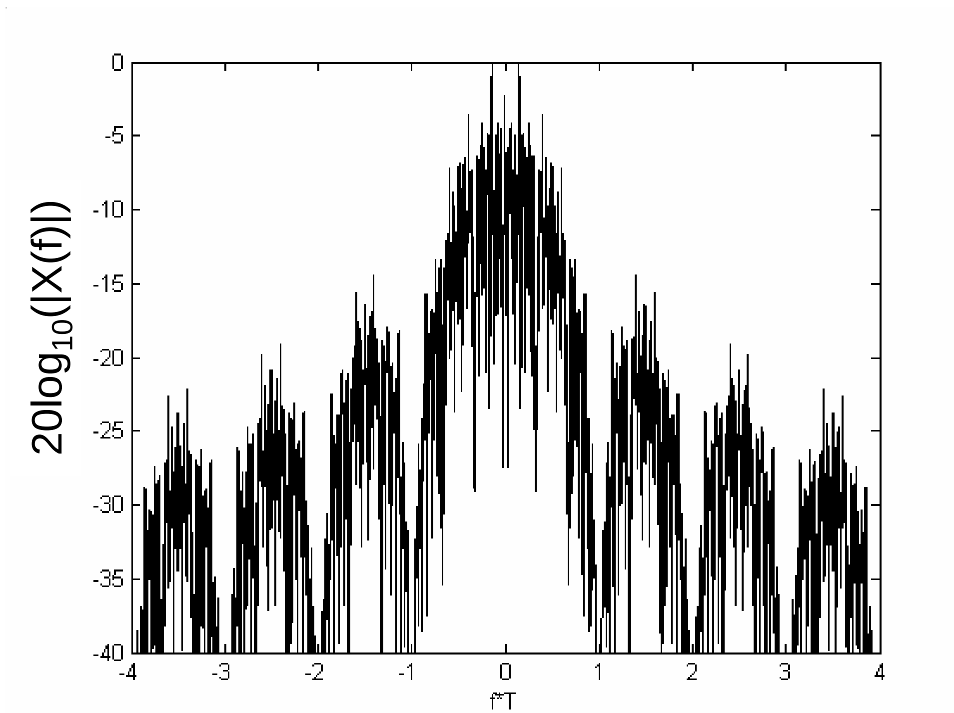
Located at $fT = \frac{2n+1}{2}$

Example Signal

Example signal

Example Spectrum





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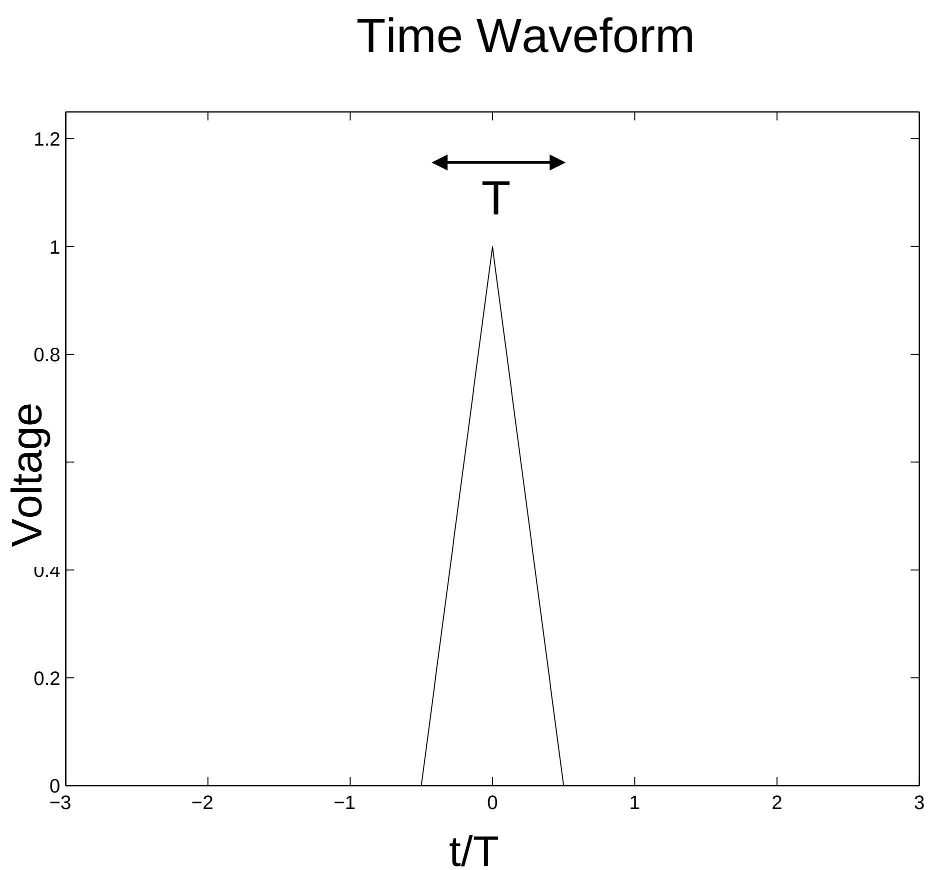
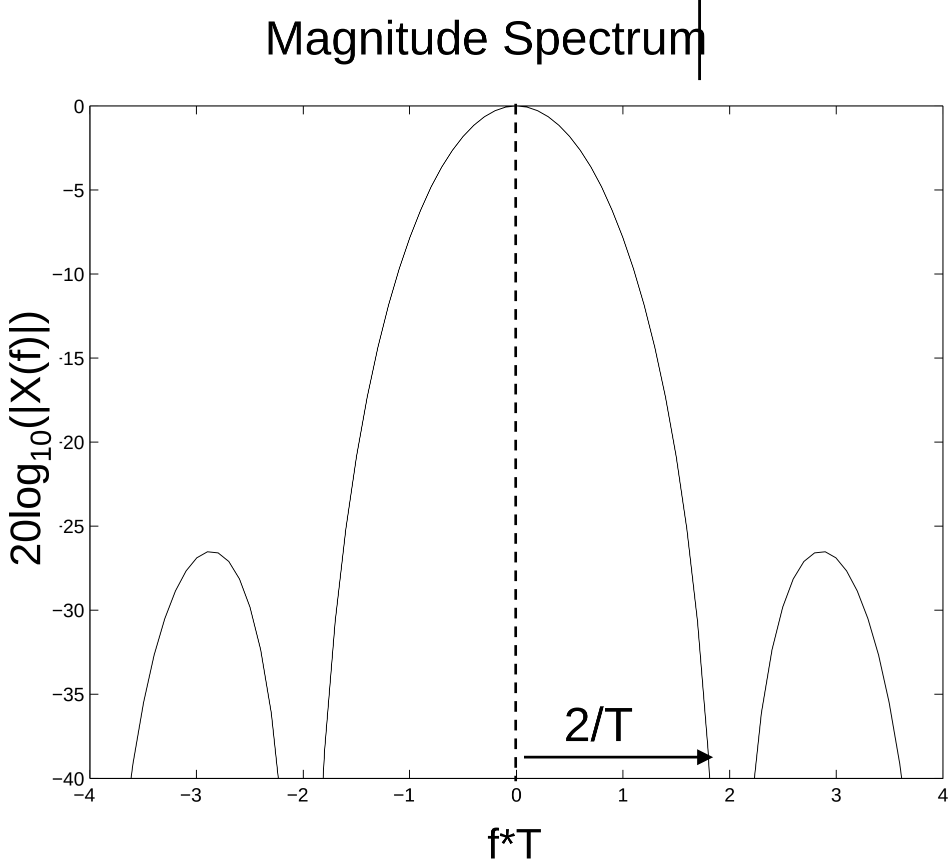
Triangular Pulse





- **First Null BW: 2/T = 2R$_s$**
- **First Sidelobe: 26 dB down**

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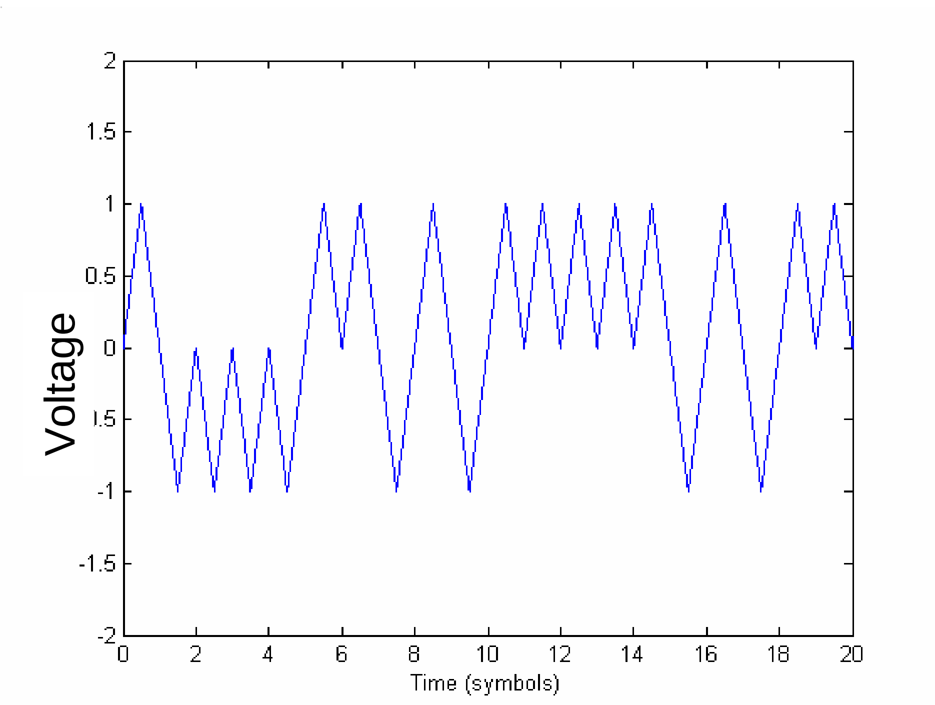
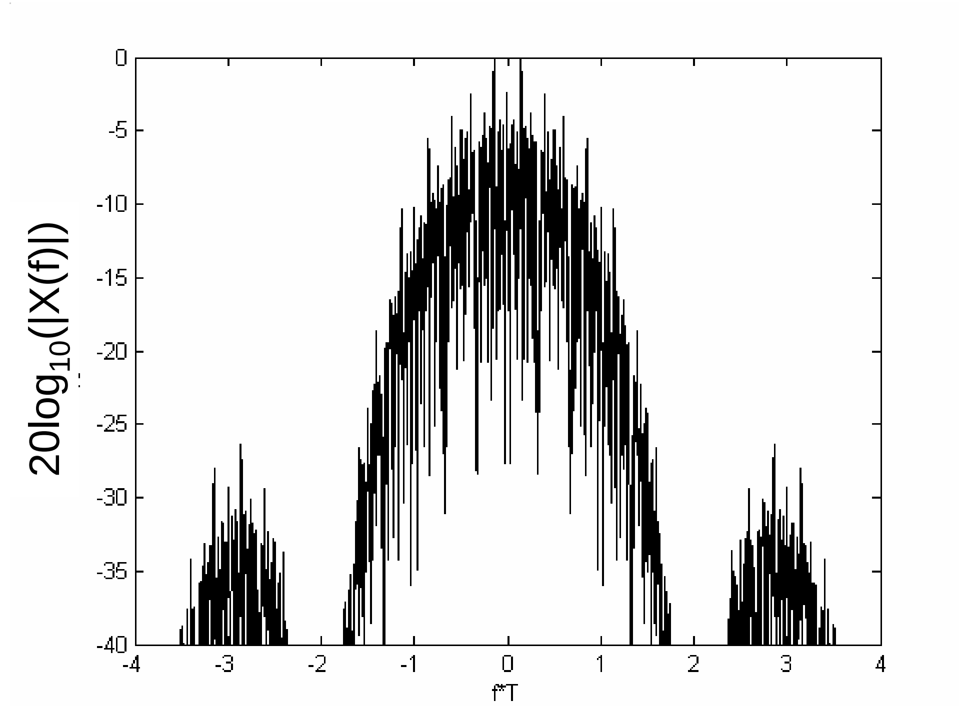
Example Signal

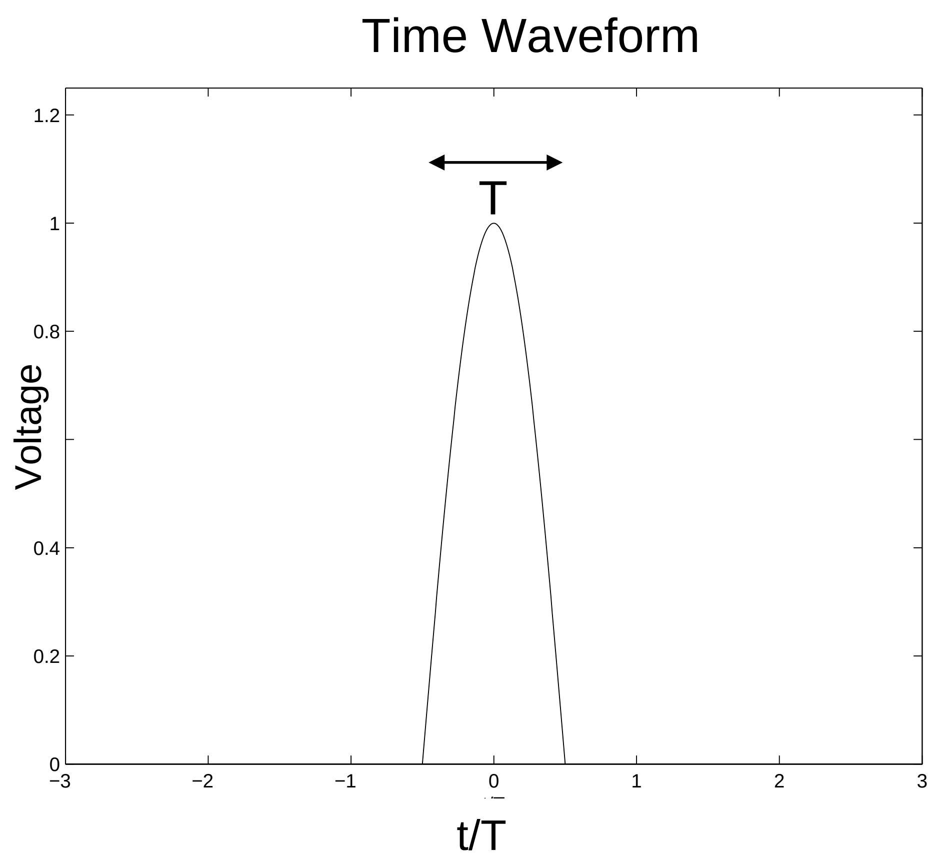
Example signal

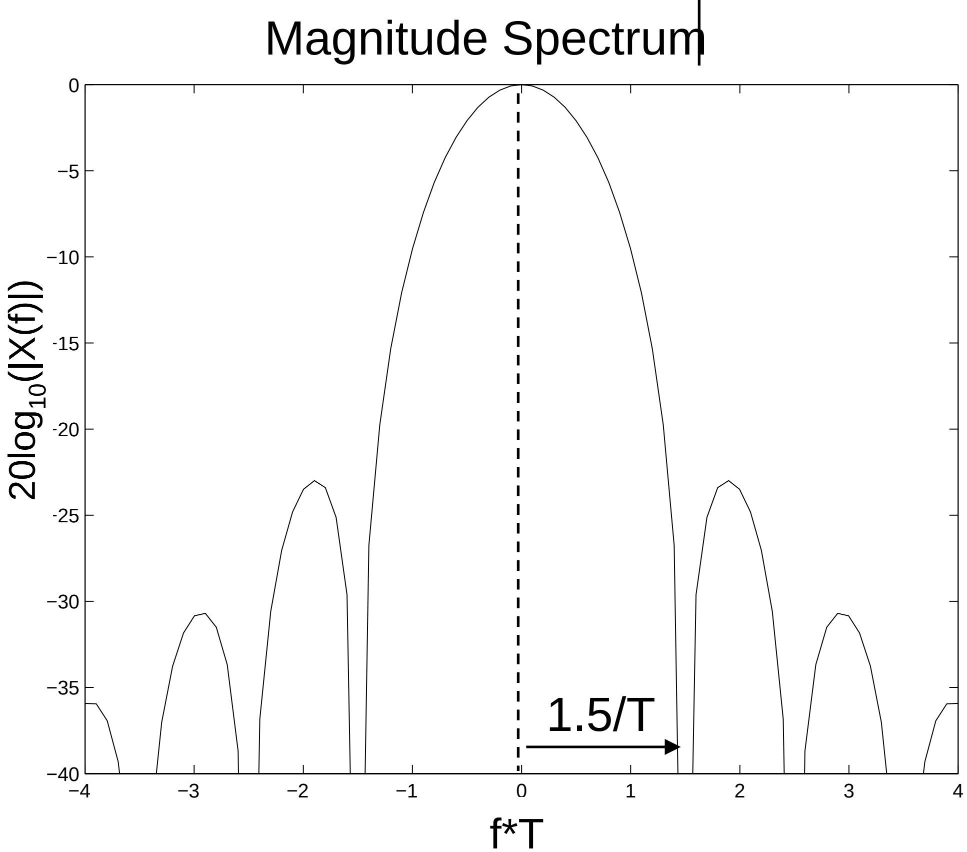


Example Spectrum



Sinusoidal Pulse Shape

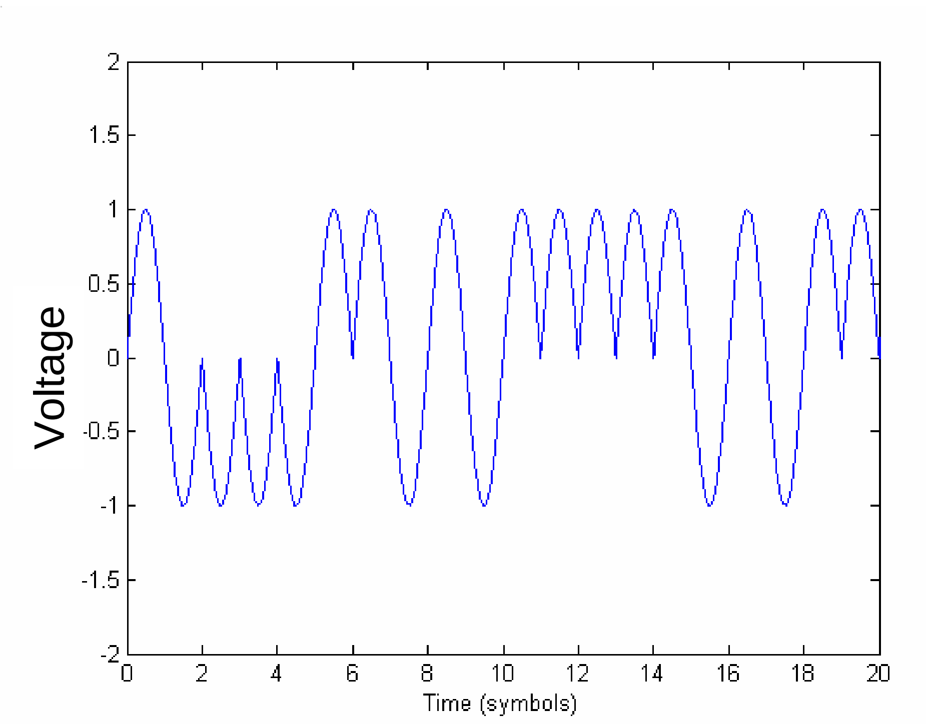




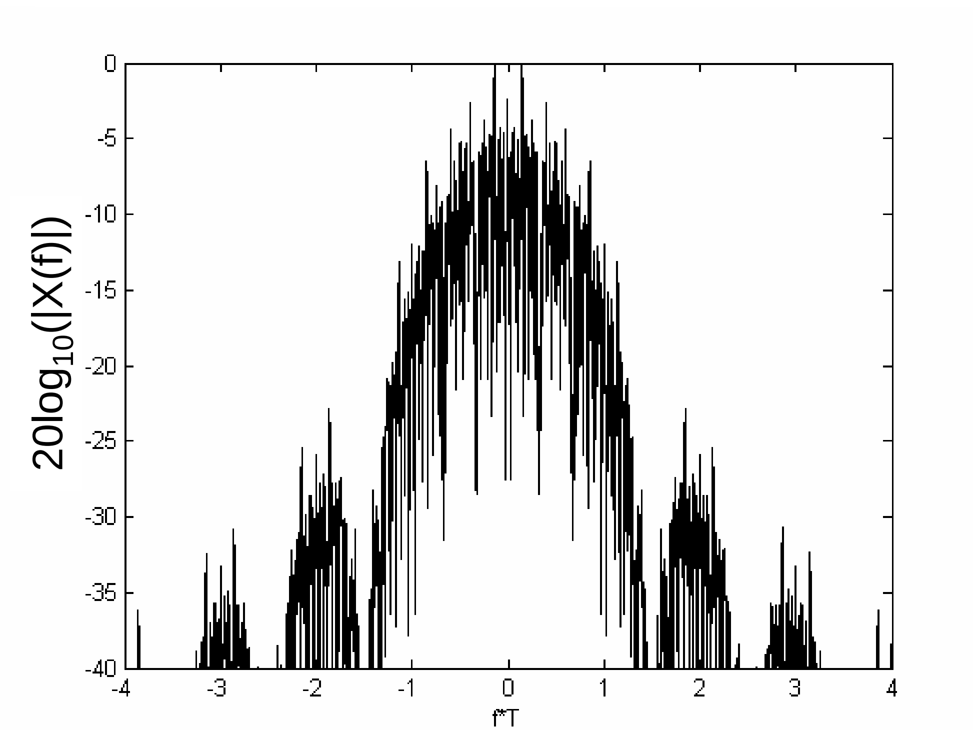
- **First Null BW: 1.5/T = 1.5R_s**
- **First Sidelobe: 22 dB down**

Example Signal

Example signal

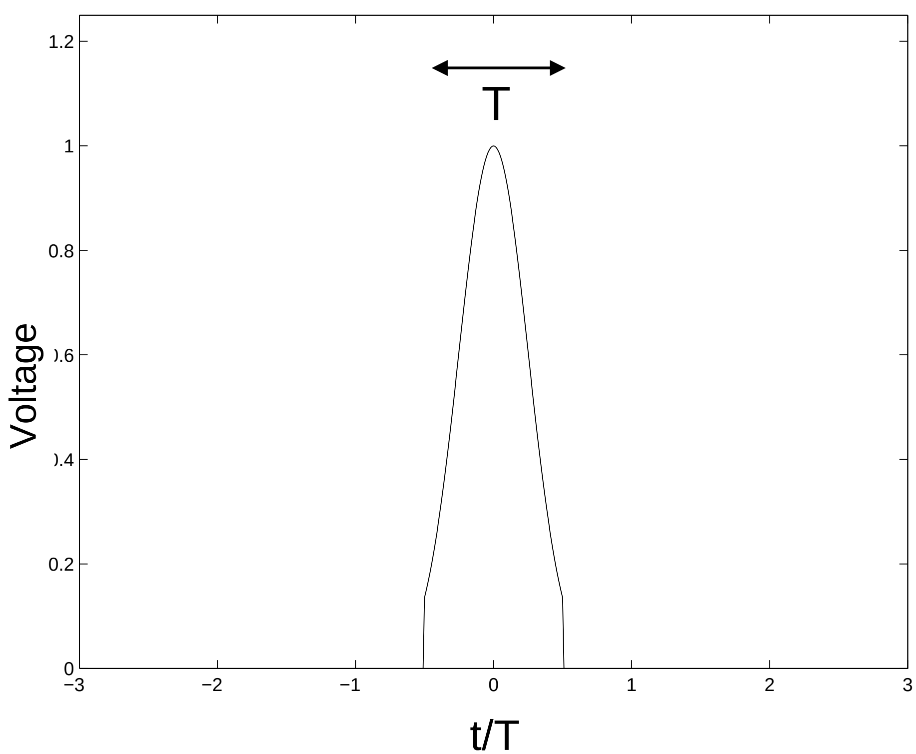


Example Spectrum

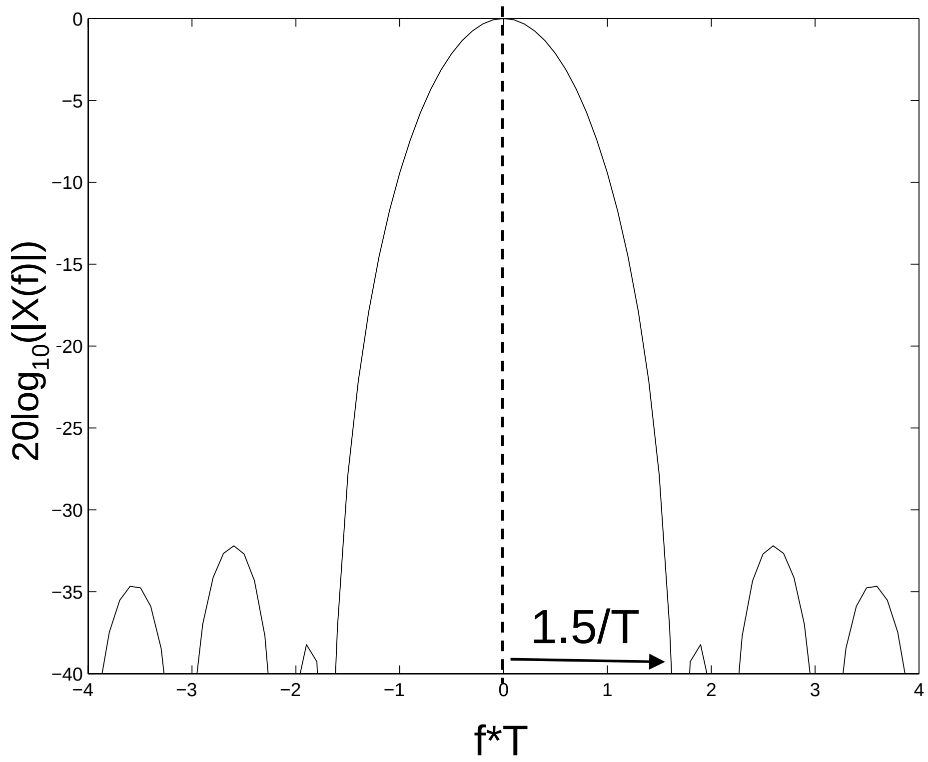


Truncated Gaussian Pulse Shape

Time Waveform



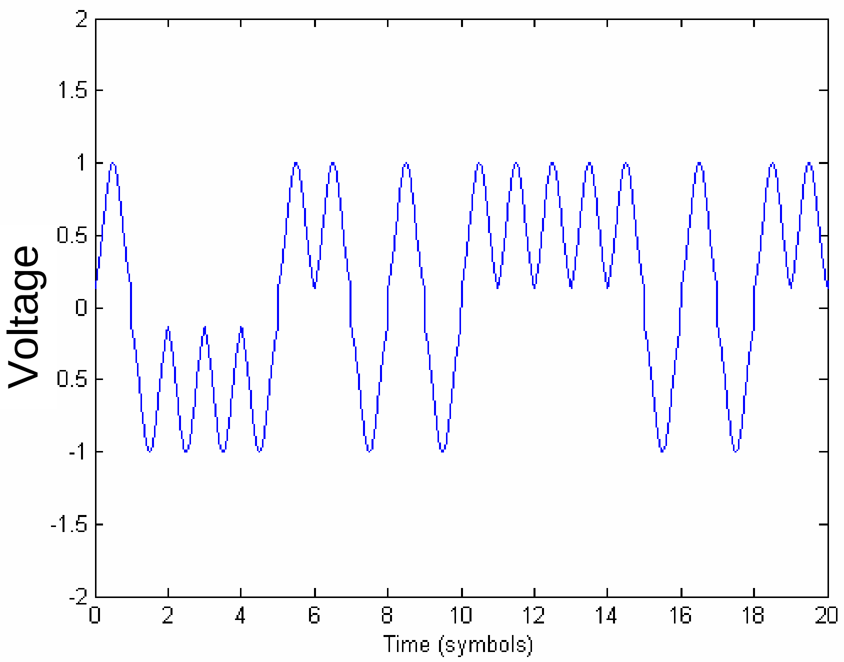
Magnitude Spectrum



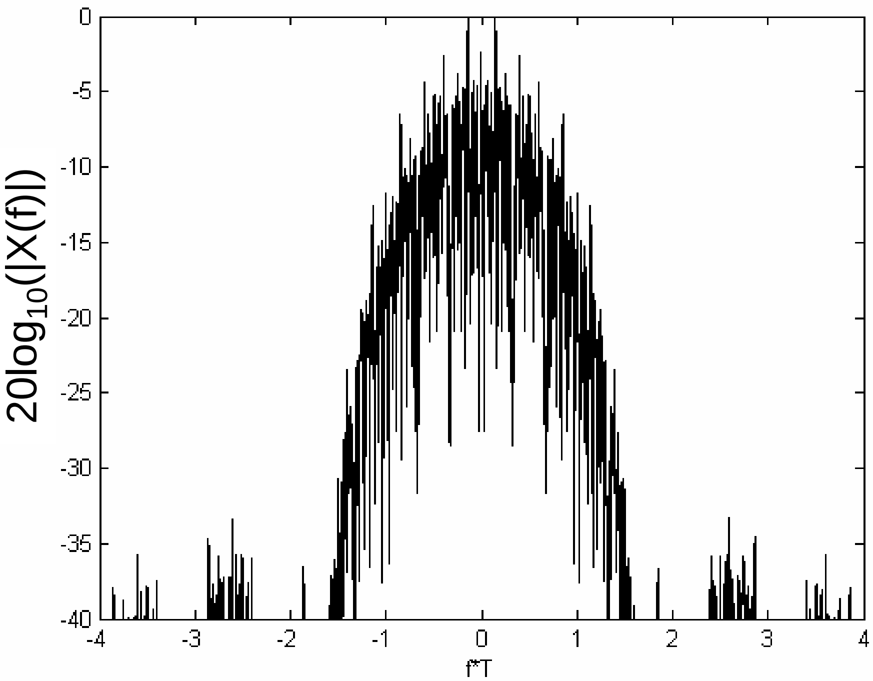
- **First Null BW: 1.5/T = 1.5Rs**
- **Largest Sidelobe: 31 dB down**

Example Signal

Example signal

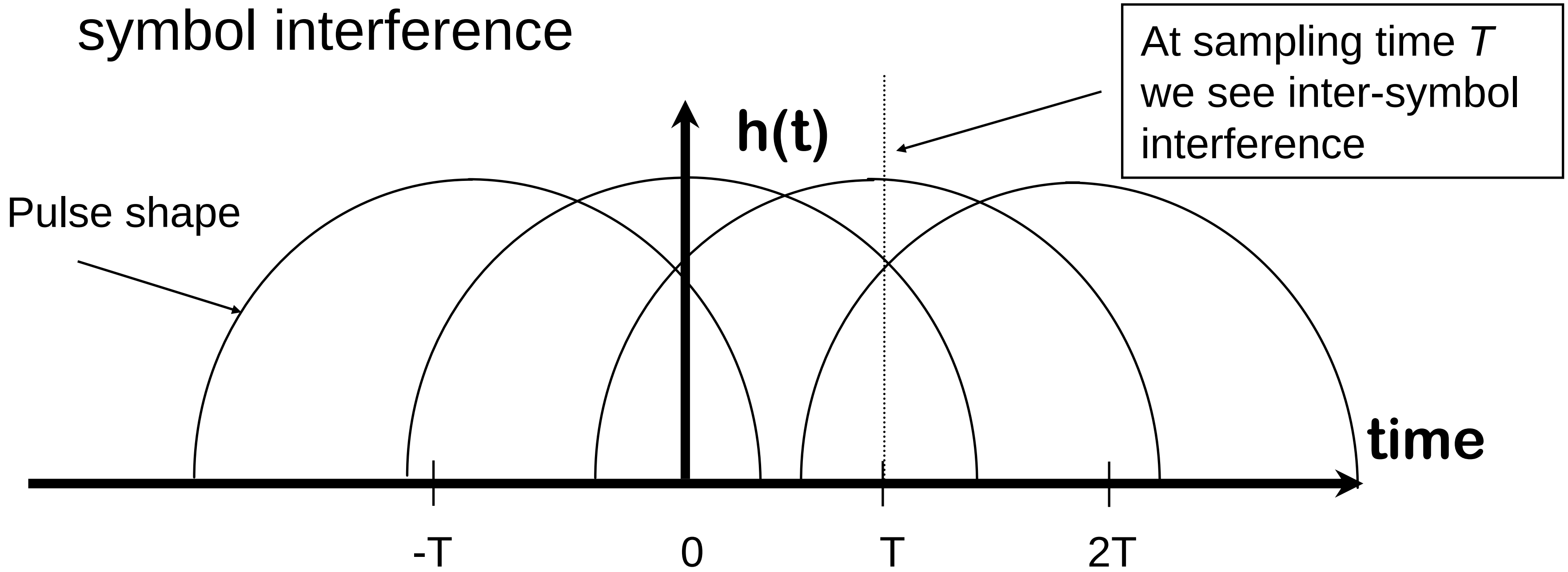


Example Spectrum



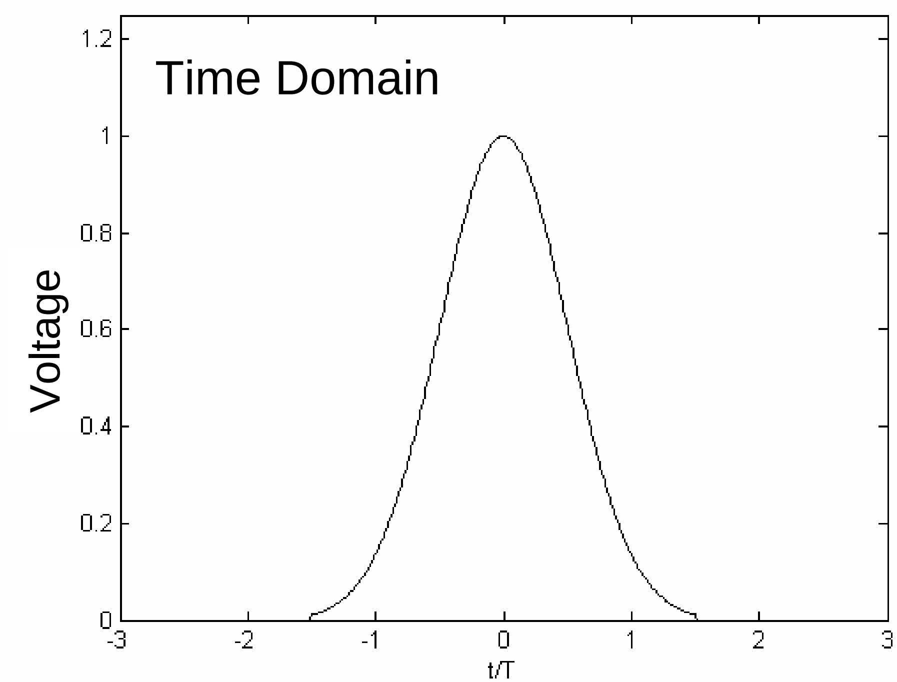
Elongating the pulse

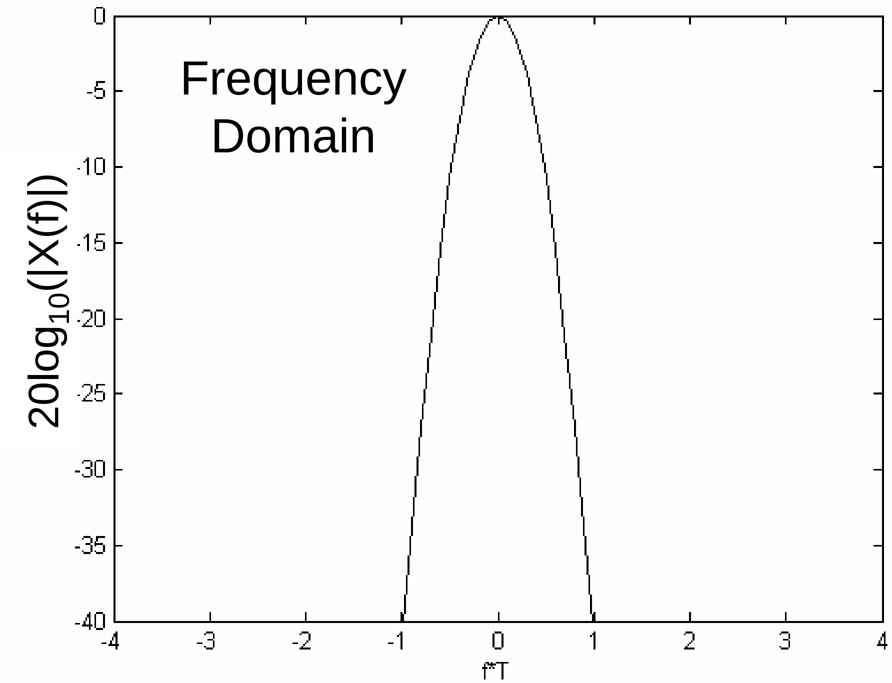
- In order to reduce the bandwidth further, we must elongate the symbols to beyond one symbol duration.
- However, if pulses overlap they may produce inter-symbol interference



Example 11.1

- We allow the pulse to go beyond one symbol duration (exponential pulse)

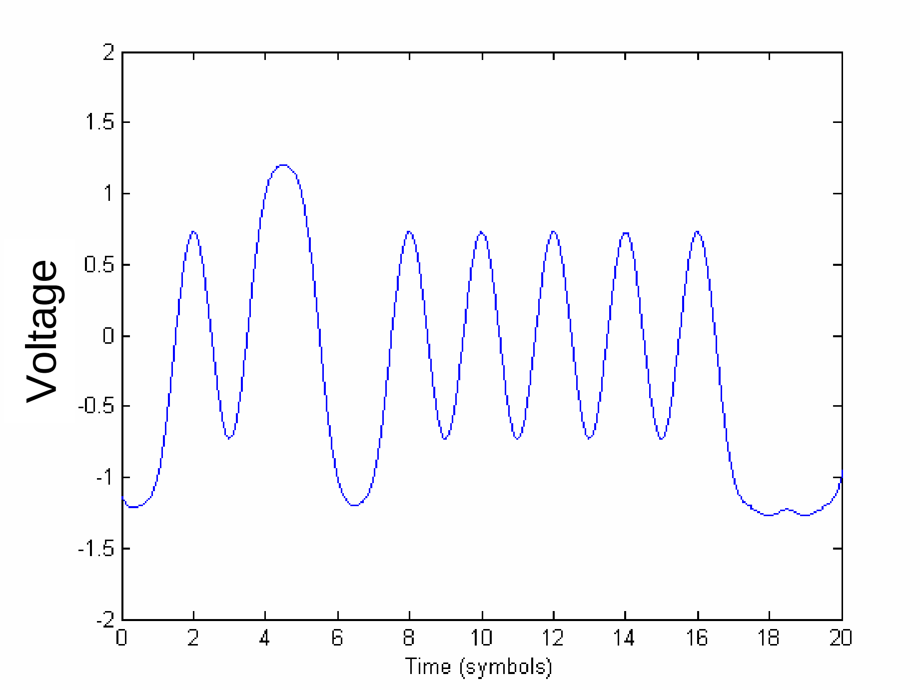


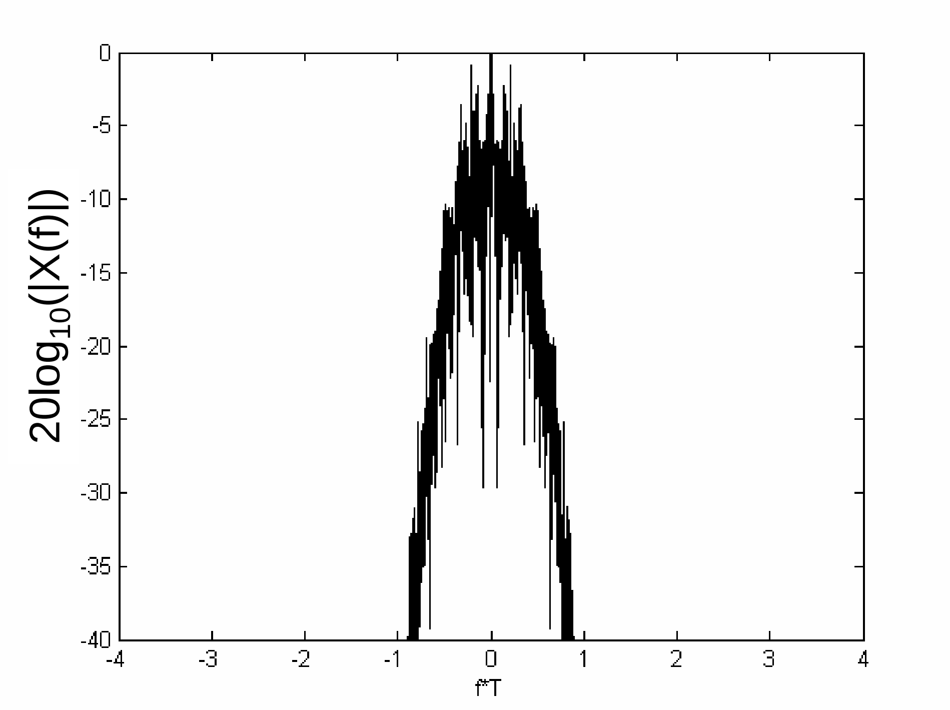


- **First Null BW: 1/T = $1R_s$**
- **First Sidelobe: > 40 dB down**

Example - continued

- Unfortunately, this leads to inter-symbol interference





Nyquist's First Criteria for Zero ISI

- Overlapping pulses will not cause inter-symbol interference if they have zero amplitude *at the time we sample the signal*.
- Mathematically we desire:

$$h\left(kT_s\right) = \begin{cases} C, & k = 0 \\ 0, & k \neq 0 \end{cases}$$

  - where *h(t)* is the pulse shape, *k* is an integer and T_s is one symbol duration
  - What is one pulse that we have examined that exhibits this property?
    - A Sinc pulse

Nyquist's First Criteria for Zero ISI

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- This requirement is equivalent to having a transfer function

$$H(f) = \begin{cases} rect\left(\dfrac{f}{2B_o}\right) + Y(f) & |f| < 2B_o \\ 0 & else \end{cases}$$

where $B_o = R_s/2$ (i.e., ½ the symbol rate) and $Y(f)$ is a real function that is even symmetric about $f=0$ and odd symmetric about $f=B_o$.

$$\boxed{\begin{array}{ll} Y(-f) = Y(f) & |f| < 2B_o \\ Y(-f + B_o) = -Y(f + B_o) & |f| < B_o \end{array}}$$

See Appendix for proof

Sinc pulses

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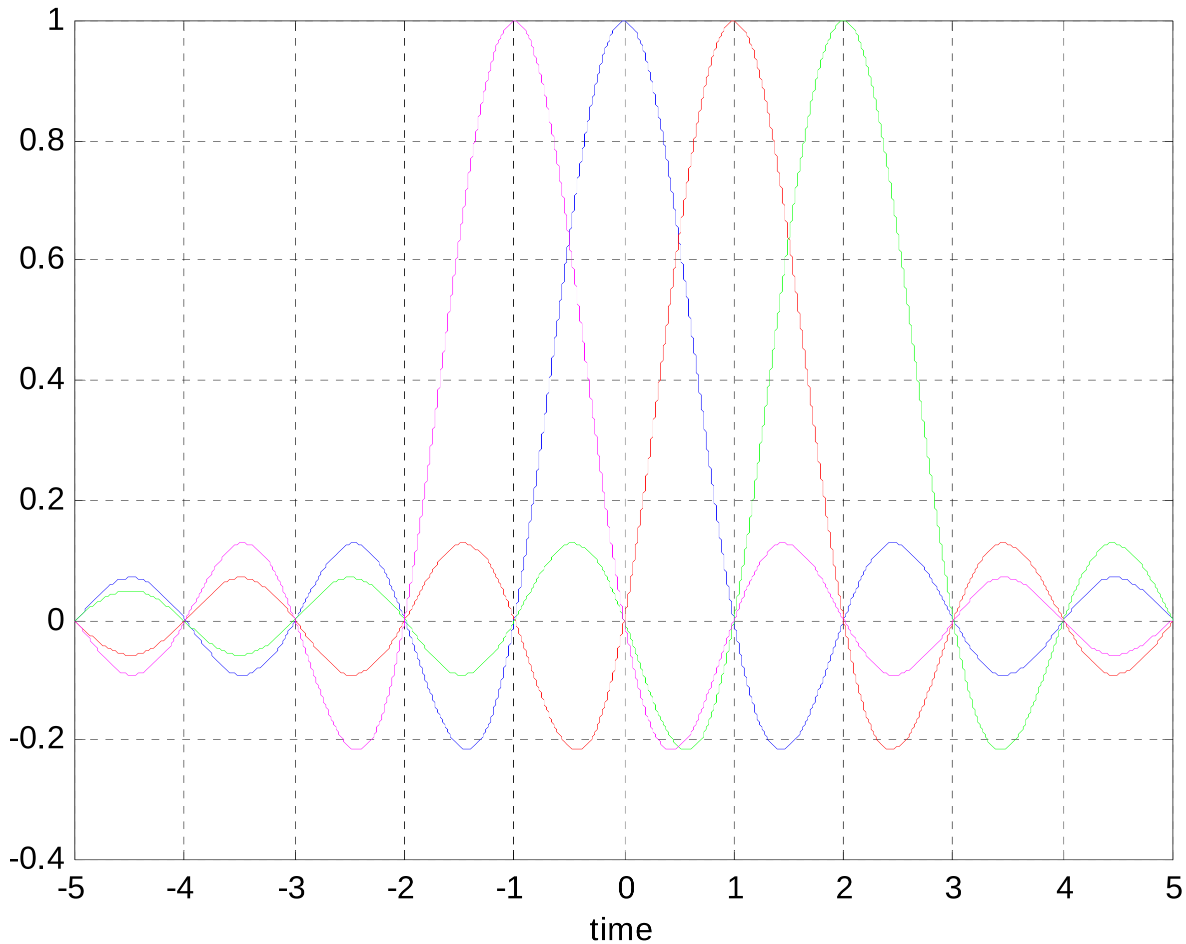


Note that at sampling times adjacent pulses equal zero

No Inter-Symbol Interference (ISI) if we are using sinc pulses.

Nyquist Filters

- Use of *sinc(t/T$_o$)* pulses does allow for zero ISI, however, it require filters in frequency that are impossible to implement ("brick wall filters") since they are non-causal.
  - We can make them causal by truncating the pulse and adding a delay
- These pulses have the minimum bandwidth possible ($B = B_o = R/2$).
- We can use (*modestly*) more practical filters and still satisfy Nyquist's zero ISI criteria if we allow the use of more bandwidth
- One set of such filters are *Raised-Cosine filters* or in the time domain *Raised-Cosine pulses*

Raised Cosine Pulse Family - Satisfies the Nyquist Criteria

- Frequency Domain:

$$H(f)=\begin{cases}\dfrac{\sqrt{E}}{2B_o} & 0\le|f|<f_1\\[2ex] \dfrac{1}{2}\dfrac{\sqrt{E}}{2B_o}\left[1+\cos\left(\dfrac{\pi\left(|f|-f_1\right)}{2\left(B_o-f_1\right)}\right)\right] & f_1\le|f|\le 2B_o-f_1\\[2ex] 0 & |f|>2B_o-f_1\end{cases}$$

  - $B = 2B_o - f_1$ is the absolute bandwidth of the filter
  - B_o and f_1 are related through *a* which is termed the roll-off factor

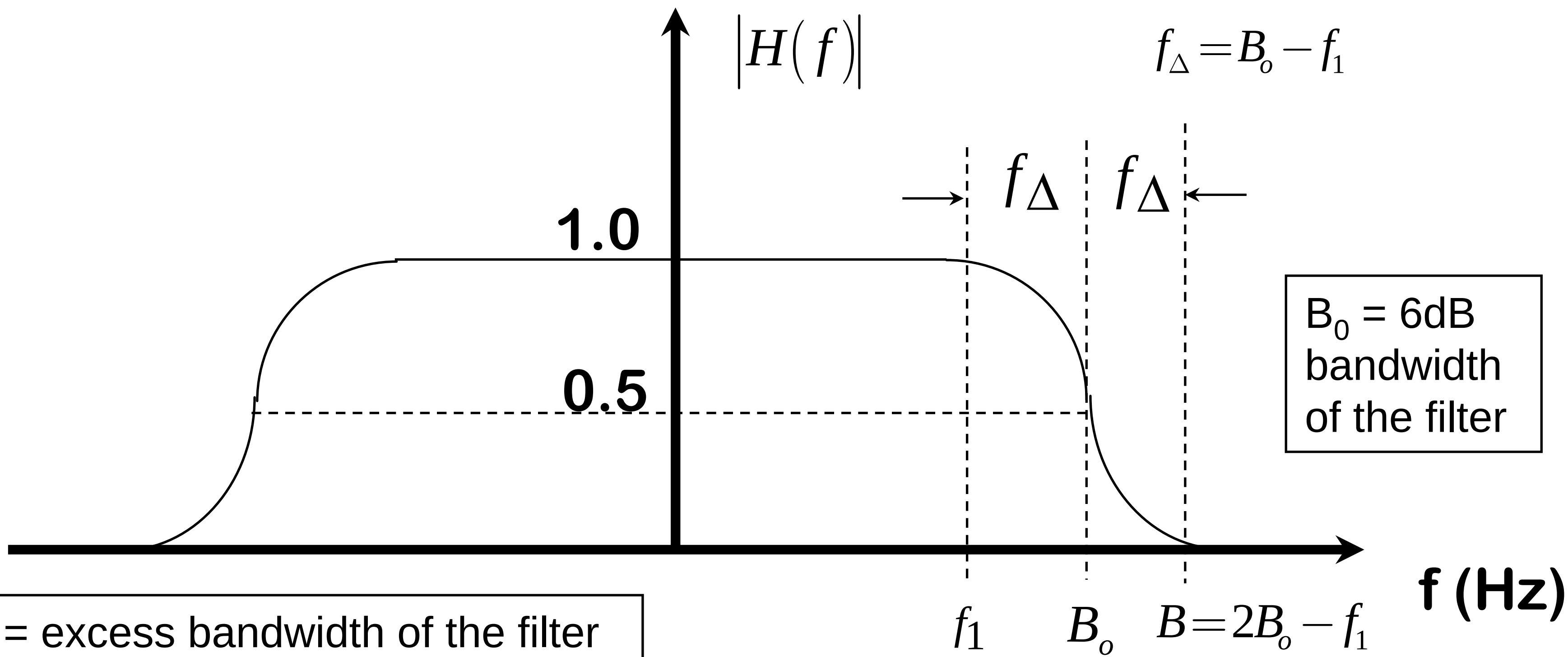
$$\alpha = 1-\frac{f_1}{B_o}$$

- Time Domain: $h(t)=F^{-1}\{H(f)\}=\sqrt{E}\left(\dfrac{\sin(2\pi B_0 t)}{2\pi B_0 t}\right)\cdot\left[\dfrac{\cos(2\pi\alpha B_o t)}{1-(4\alpha B_o t)^2}\right]$

$$=\sqrt{E}\operatorname{sinc}(2B_0 t)\cdot\left[\frac{\cos(2\pi\alpha B_o t)}{1-(4\alpha B_o t)^2}\right]$$

Spectrum of Raised Cosine Pulse

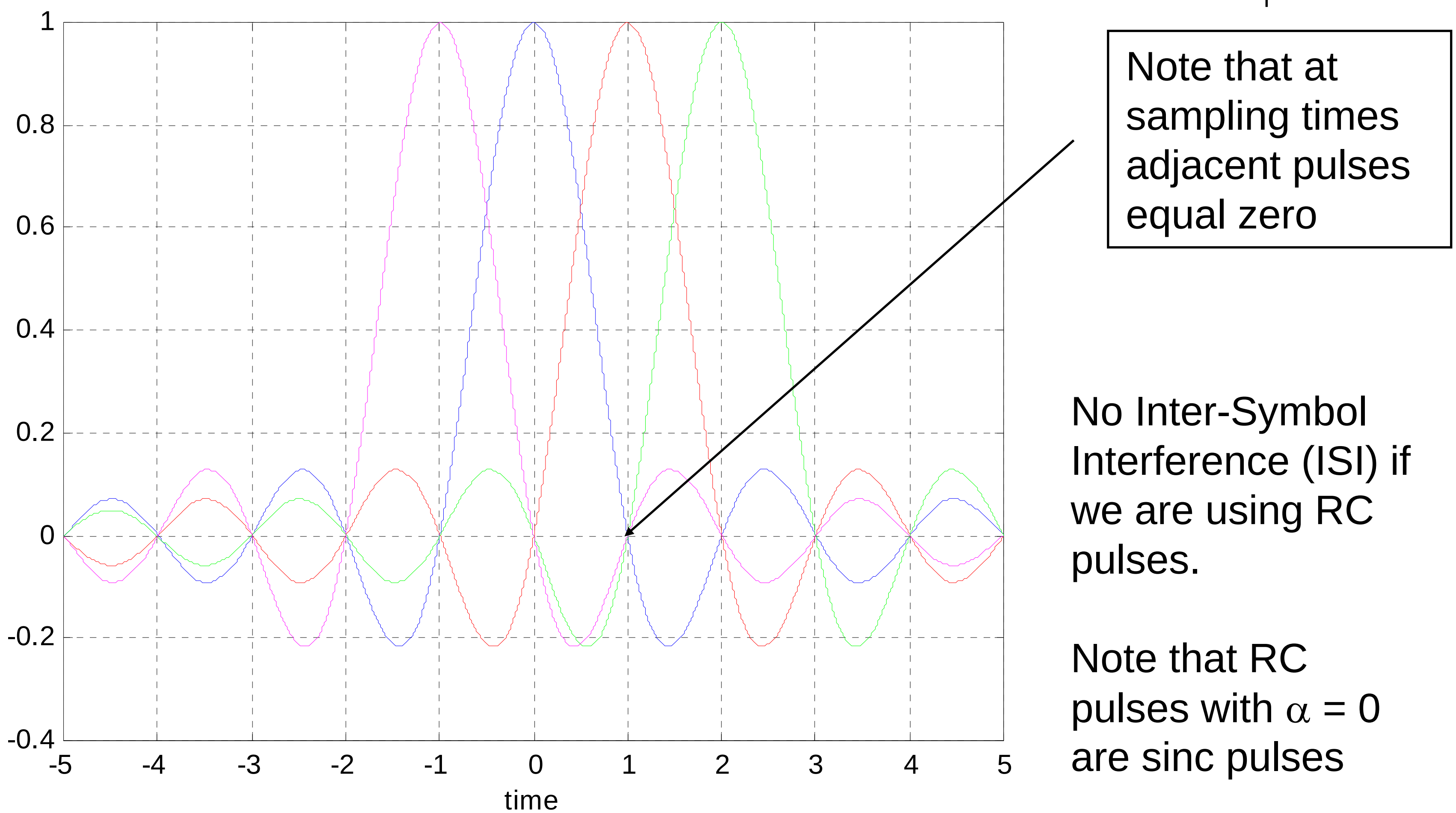
- α = 0 corresponds to *sinc( )* function and "brick wall filter"



- f_Δ = excess bandwidth of the filter since it represents the bandwidth beyond the minimum.
- As α increases, f_Δ increases

$$\alpha = 0 \longrightarrow B = B_o = R_s / 2$$
$$\alpha = 1 \longrightarrow B = 2B_o = R_s$$

Pulse Design for No ISI Raised Cosine (α=0)



No Inter-Symbol Interference (ISI) if we are using RC pulses.

Note that RC pulses with $\alpha = 0$ are sinc pulses

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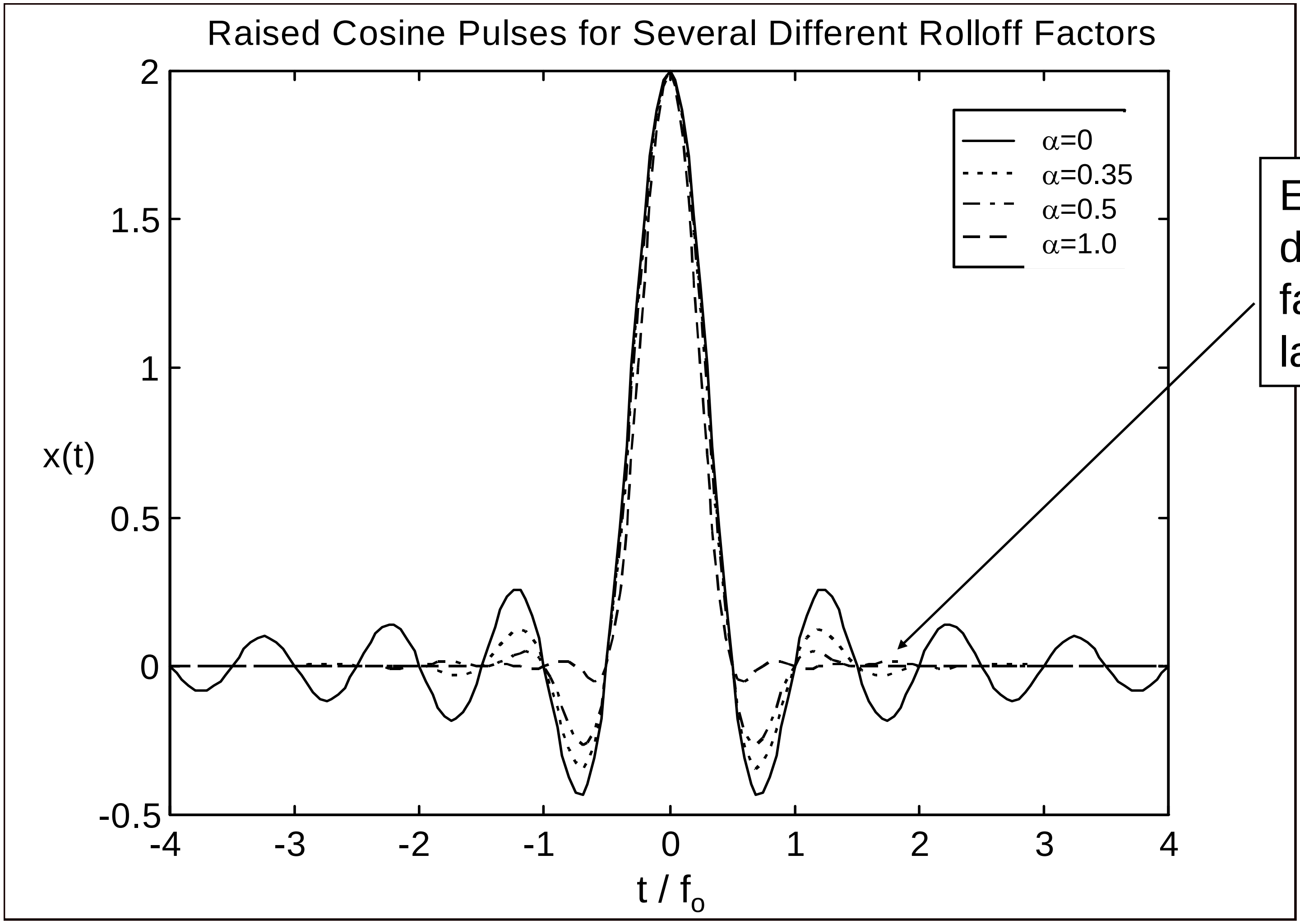
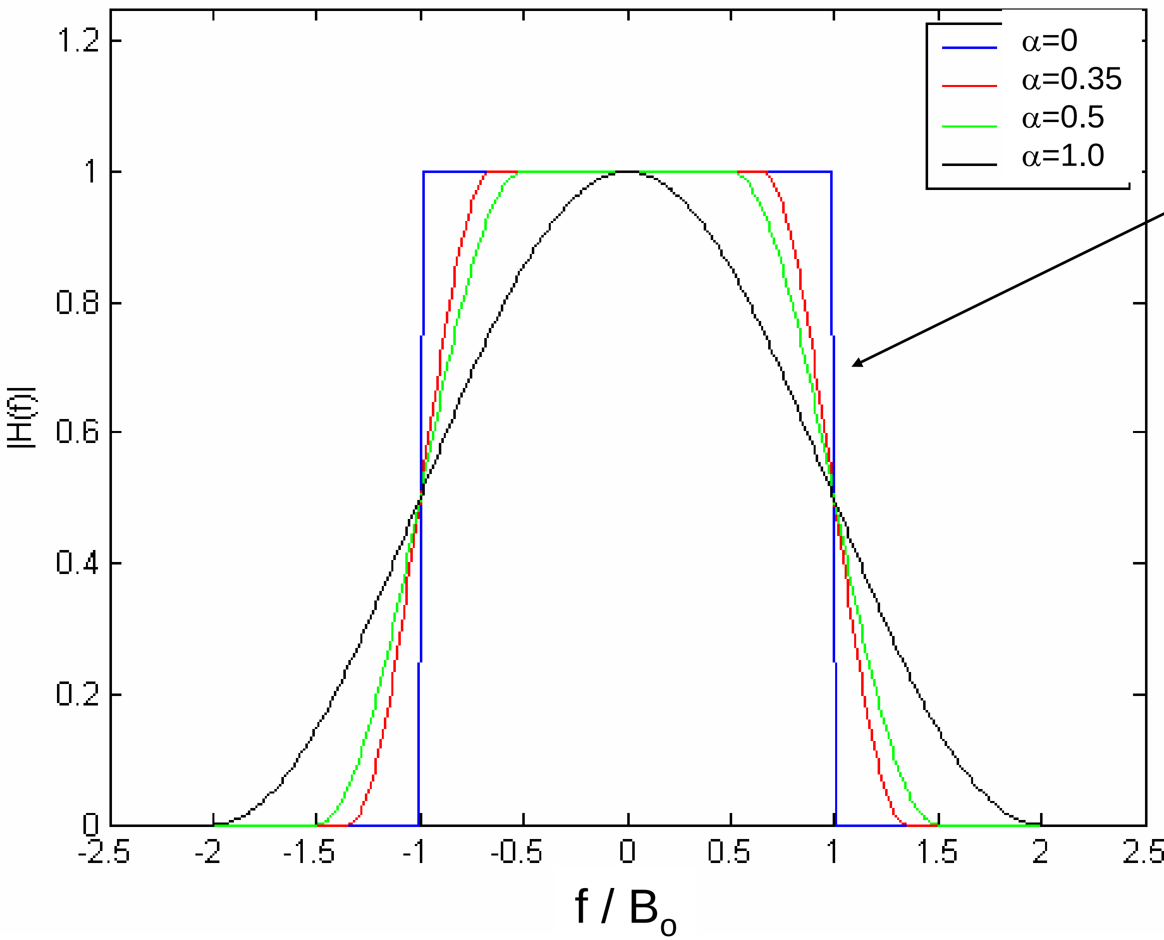
Raised Cosine Pulse - Time Domain



Envelope decays faster with larger α

$T_s = 1/2B_o$

Raised Cosine Pulse - Frequency Domain



Increasing α increases bandwidth

Note: $B_o = R_s/2$

$B_{min} = B_o = R_s/2$

$B_{max} = 2B_o = R_s$

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Bandwidth of Raised Cosine Pulses

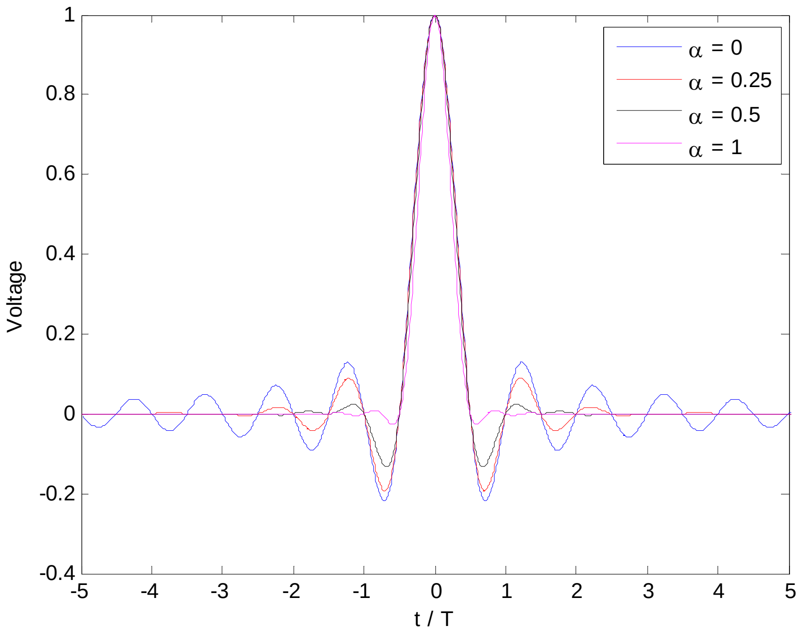
- For PCM system:

$$B = (1+\alpha) \cdot f_s \cdot n/2 \text{ Hz}$$

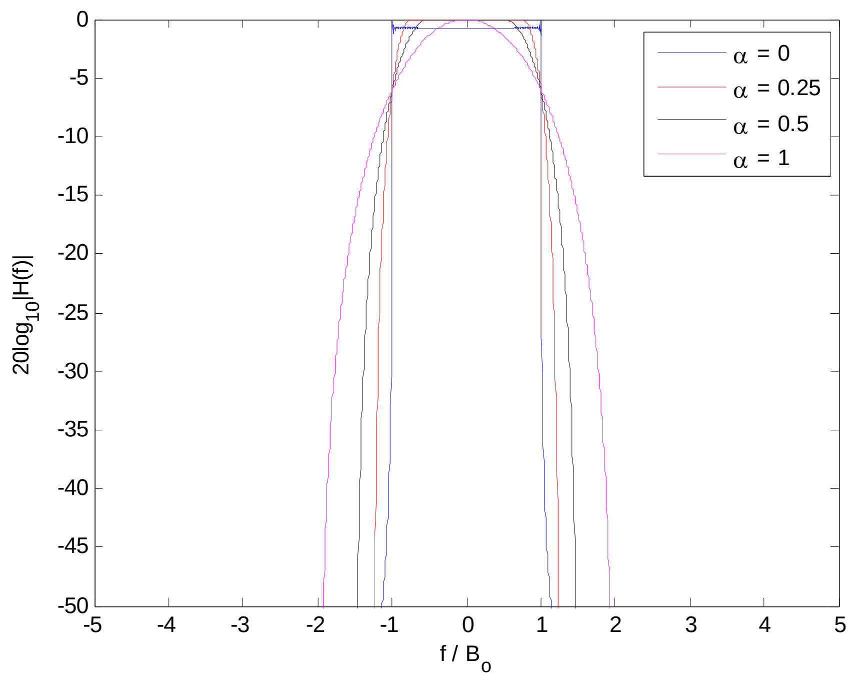
  - $0 \le \alpha \le 1$ is a parameter called "roll-off factor"
- Special cases:
  - $\alpha = 0$ is just a *sinc(.)* function
  - $\alpha = 1$ is the largest possible value
  - $\alpha = 0.35$ was used in the old U.S. Digital Cellular (IS-54/136) standard

Examples

Time Domain



Frequency Domain



In class drill

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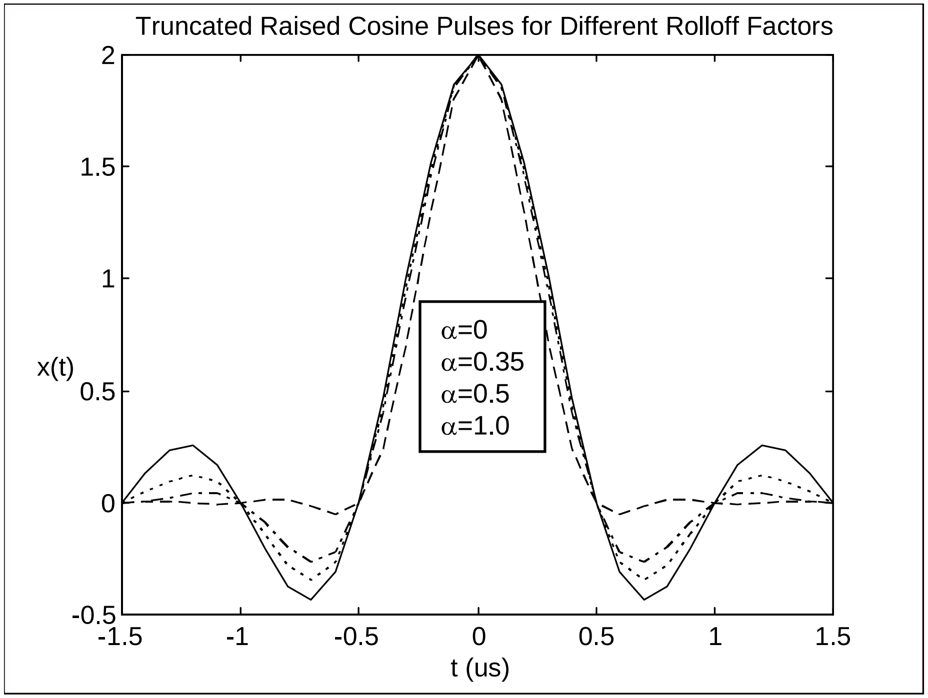
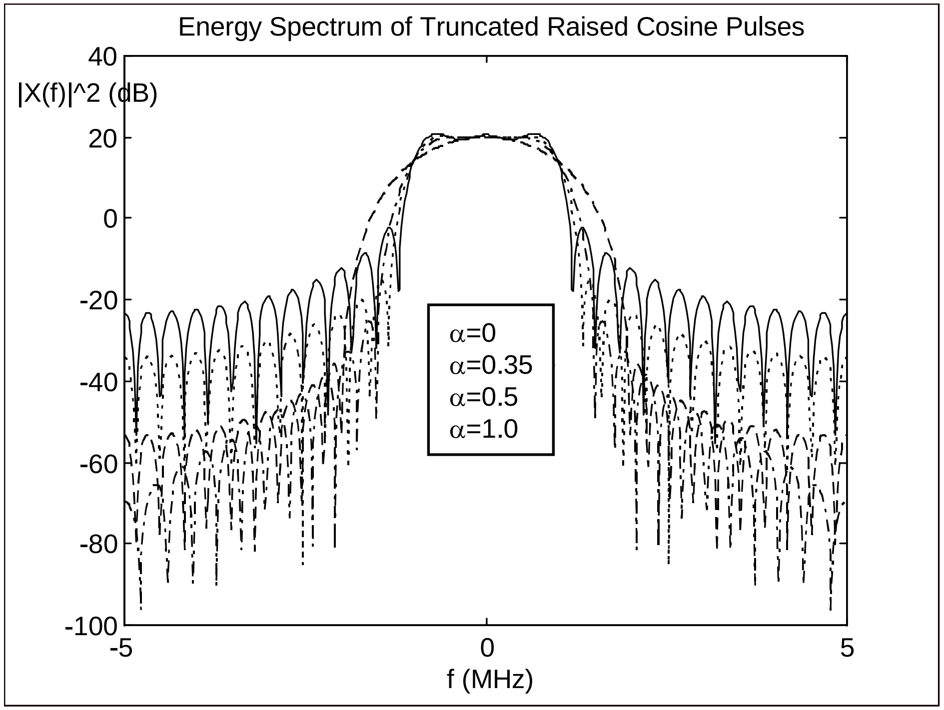
Implementation of Raised Cosine Pulse

- Can be digitally implemented with an FIR filter
- Analog filters such as Butterworth filters may approximate the tight shape of this spectrum
- Practical pulses must be truncated in time
  - Truncation leads to sidelobes - even in RC pulses
  - The larger the value of α, the less effect that a given truncation length has.
- Sometimes a “square-root” raised cosine spectrum is used when identical filters are implemented at transmitter and receiver
  - We will discuss this more when we talk about “matched filtering”

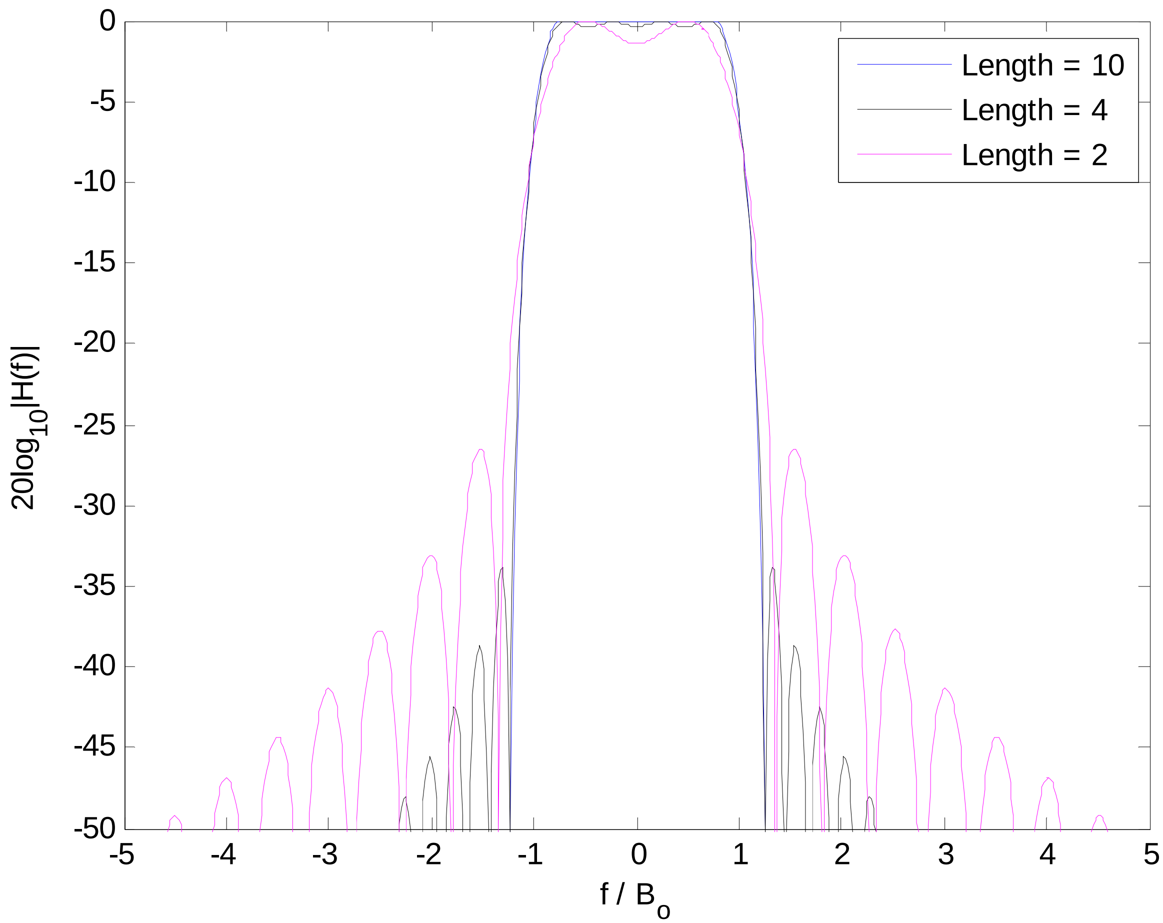
Truncated Raised Cosine Pulses

- Truncating raised cosine pulse to finite duration results in some side-lobes





Example



- $\alpha = 0.25$
- Truncated to 10, 4, and 2 symbols
- For 10-symbol long approximation, we see no side-lobes within 50dB
- As truncation length gets smaller, side-lobes rise
- Larger truncation length requires larger delay to make pulse causal.

Conclusions

- Reducing the bandwidth of the transmit signal is desirable to improve spectral efficiency (i.e., get the most bits/sec in the smallest bandwidth)
- Elongating pulses is a good way to reduce bandwidth
  - Elongating pulses indiscriminately will introduce ISI
- Intelligent pulse design can reduce bandwidth without this penalty
  - Nyquist Criterion
  - Sinc pulse and Raised Cosine pulse satisfy Nyquist Criterion
- Channel conditions can also reduce bandwidth further resulting in channel-induced ISI
  - We will discuss this more next time

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Appendix

Proof of Nyquist Filter Frequency Domain Characteristics

Analog and Digital Communications

Nyquist's First Criteria for Zero ISI

- Zero ISI in the time domain is equivalent to having a transfer function

$$H(f) = \begin{cases} rect\left(\dfrac{f}{2B_o}\right) + Y(f) & \left|f\right| < 2B_o \\ 0 & else \end{cases}$$

where $B_o = R_s/2$ (i.e., ½ the symbol rate) and $Y(f)$ is a real function that is even symmetric about $f=0$ and odd symmetric about $f=B_o$.

$$\boxed{\begin{array}{ll} Y(-f) = Y(f) & |f| < 2B_o \\ Y(-f + B_o) = -Y(f + B_o) & |f| < B_o \end{array}}$$

Proof

$$h(t) = \int_{-2B_o}^{-B_o} Y(f)e^{j\omega t}df + \int_{-B_o}^{B_o} \left(1+Y(f)\right)e^{j\omega t}df + \int_{B_o}^{2B_o} Y(f)e^{j\omega t}df$$

$$= \int_{-B_o}^{B_o} e^{j\omega t}df + \int_{-2B_o}^{2B_o} Y(f)e^{j\omega t}df$$

$$= 2B_o\left(\frac{\sin\left(2\pi B_o t\right)}{2\pi B_o t}\right) + \int_{-2B_o}^{0} Y(f)e^{j\omega t}df + \int_{0}^{2B_o} Y(f)e^{j\omega t}df$$

Via a change of variables:

$\boxed{f_1 = f + B_o}$ $\boxed{f_1 = f - B_o}$

$$h(t) = 2B_o\left(\frac{\sin\left(2\pi B_o t\right)}{2\pi B_o t}\right) + e^{-j2\pi B_o t}\int_{-B_o}^{B_o} Y(f_1 - B_o)e^{j2\pi f_1 t}df_1 + e^{-j2\pi f_o t}\int_{-B_o}^{B_o} Y(f_1 + B_o)e^{j2\pi f_1 t}df_1$$

$$= 2B_o\left(\frac{\sin\left(2\pi B_o t\right)}{2\pi B_o t}\right) + j2\sin(2\pi B_o t)\int_{-B_o}^{B_o} Y(f_1 + B_o)e^{j2\pi f_1 t}df_1$$

Due to $Y(f_1 - B_o) = -Y(f_1 + B_o)$

Proof (cont.)

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$$h(t) = 2B_o\left(\frac{\sin\left(2\pi B_o t\right)}{2\pi B_o t}\right) + j2\sin(2\pi B_o t)\int_{-B_o}^{B_o} Y(f_1 + B_o)e^{j2\pi f_1 t}df_1$$

We can see that the above equation is zero at $t=n/(2B_o)$ for n not equal to zero.

One pulse shape that satisfies this equation is

$$H(f) = \begin{cases} rect\left(\dfrac{f}{2B_o}\right) & |f| < 2B_o \\ 0 & else \end{cases}$$

$Y(f)=0$

Or $sinc(t/T_o)$ in the time domain