ECE4634
Digital Communications
Fall 2007

Instructor: R. Michael Buehrer

Lecture #3: Singularity Functions,

Energy Spectral Density,

Power Spectral Density,

Linear Systems

Analog and Digital Communications

Overview

- Information in communication systems is transferred through the use of EM waves
- At each point in the system, we observe signals. These signals can be described mathematically using both the time and the frequency domains.
- While the time domain is more familiar to most students, often the frequency domain is more intuitive for understanding certain signal characteristics
- At the receiver we observe both the desired waveform as well as undesired waveforms such as *noise* and *interference*.
- Reading
  - Sections 2.4-2.6, 2.8

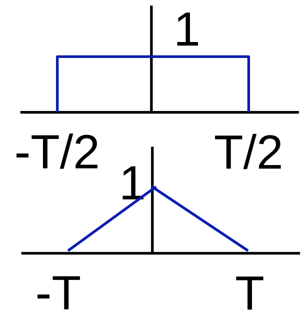
Lecture Objectives

Analog and Digital Communications

- Review of special signals with discontinuities termed *singularity functions*
  - Specifically we examine the impulse and step functions.
- Review three important frequency-domain concepts
  - Fourier Transform of periodic signal
  - Energy Spectral Density
  - Power Spectral Density
- Review simple system concept
  - Transfer Function of a linear system

Some Commonly Used Functions

- Rectangular Pulse: $rect\left(t/T\right)$
- Triangular Pulse: $tri\left(t/T\right)$



- Sinc Function: $$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$
- Unit Step Function: $$u\left(t\right) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases}$$
- Dirac Delta Function (Unit Impulse Function): $$\delta(t) = \begin{cases} 0, & t \neq 0 \\ \text{undefined}, & t = 0 \end{cases}$$ $$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

Singularity Functions

- The unit impulse, unit step, and unit ramp are part of a larger family of functions termed *singularity functions* written as $u_k(t)$ where k represents the number of times the unit impulse is differentiated
- A negative value of k represents an integral

$$u_0(t) = \delta(t)$$
$$u_{-1}(t) = u(t)$$
$$u_{-2}(t) = ramp(t)$$

Unit Step Function

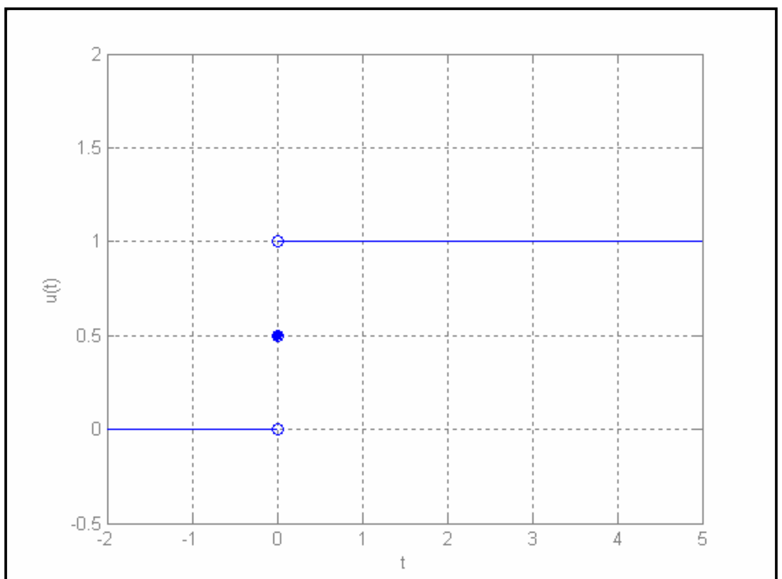
- The *unit step function* is defined as

$$u(t) = \begin{cases} 1 & t > 0 \\ \dfrac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases}$$

*-Note that the definition at t=0 is irrelevant as long as it is finite

- This function is very useful and is commonly used to represent a function or system being switched on

Since the height is “1” we call this the *unit* step function



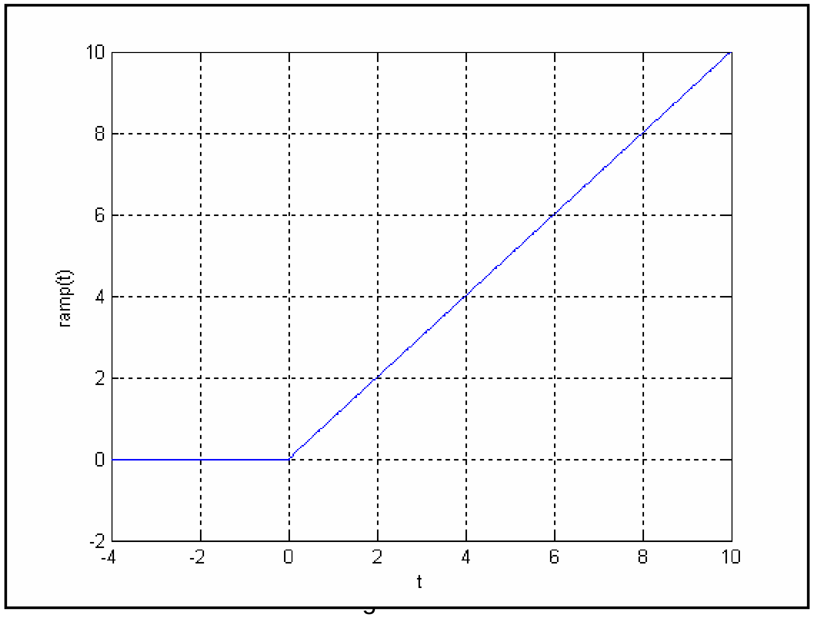
Note that there is a discontinuity at $t = 0$ which can represents a signal being switched “on”

Unit Ramp Function

- Another useful function is one which turns on at $t = 0$ and increases linearly with time
- This is termed the *unit ramp function* and is defined as

$$ramp(t) = \begin{cases} t & t > 0 \\ 0 & t \le 0 \end{cases}$$

Since the slope is "1" we call this the *unit* ramp function



Note

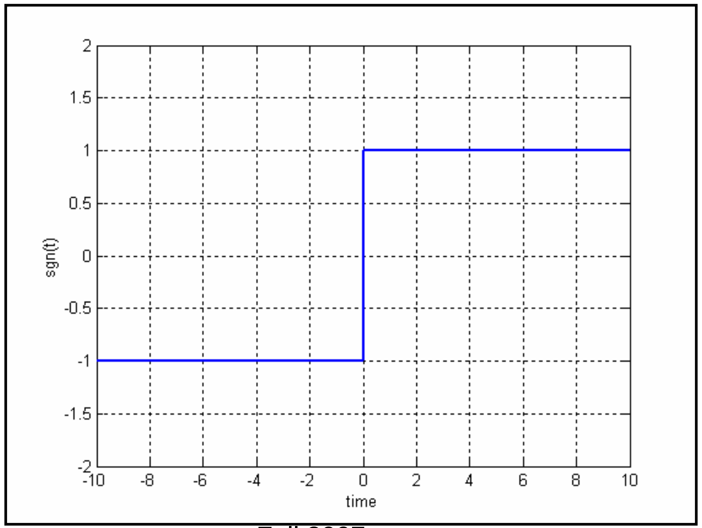
$$ramp(t) = \int_{-\infty}^{t} u(\lambda)\, d\lambda$$

Signum Function

- The signum function is related to the unit step function and is defined as

$$\operatorname{sgn}(t) = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases}$$

- This is also sometimes called the *sign* function since it essentially produces the sign of its argument



The Unit Impulse Function

- The unit impulse is defined as a function which when multiplied by another function *g(t)* (which is finite and continuous at *t=0*) and the product is integrated between limits which include *t=0*, the result is *g(0)*:

$$g(0) = \int_{-\infty}^{\infty} \delta(t) g(t) dt$$

- The impulse can thus be defined as

$$\delta(t) = 0 \quad t \neq 0 \qquad\qquad \int_{t_1}^{t_2} \delta(t) dt = \begin{cases} 1 & t_1 < 0 < t_2 \\ 0 & else \end{cases}$$

Properties of the Impulse function

Analog and Digital Communications

- The strength of an impulse is equal to the area of the impulse.
- The unit impulse has area or strength of one.
- Consider an impulse of strength *k* written as $k\,\delta(t)$:

$$\int_{-\infty}^{\infty} k\delta(\lambda)g(\lambda)\,d\lambda = k\,g(0)$$

- Equivalence property: $g(t)k\,\delta(t) = k\,g(0)\delta(t)$
- Sampling or sifting property: $\int_{-\infty}^{\infty} \delta(t-t_o)g(t)\,dt = g(t_o)$
- Replication property: $g(t)*\delta(t-t_o) = g(t-t_o)$

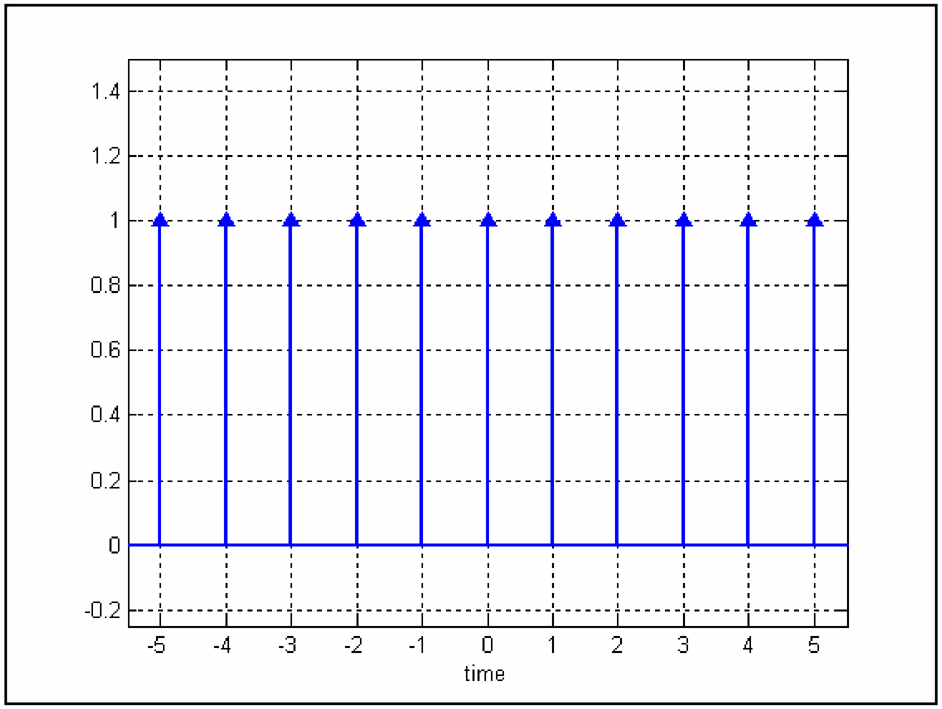
Unit Comb

- The unit comb is a sequence of uniformly spaced unit impulses (sometimes also called an *impulse train*)

$$comb(t) = \sum_{n=-\infty}^{\infty} \delta(t-n)$$

where *n* is an integer

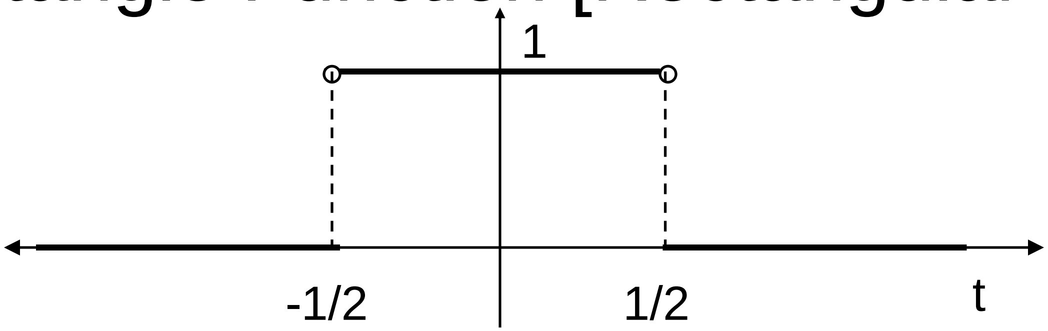
Since the strength of each impulse is “1” and the spacing of the impulses is unity, we call this the *unit* comb function



Other Functions

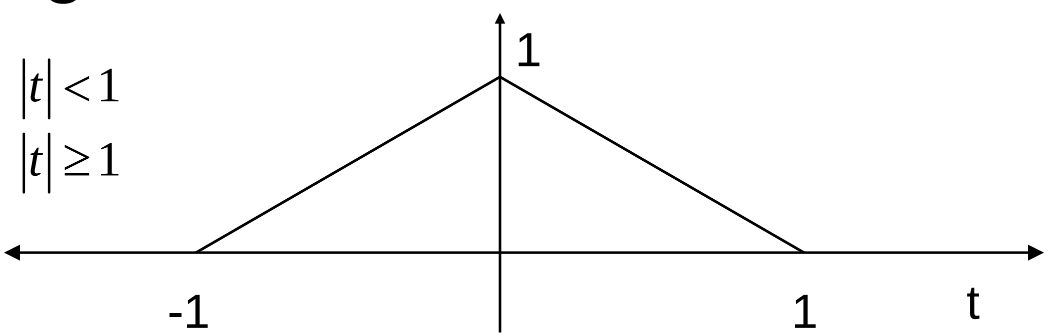
- Unit Rectangle Function [Rectangular Pulse]

$$rect(t) = \begin{cases} 1 & |t| \le \frac{1}{2} \\ 0 & |t| > \frac{1}{2} \end{cases}$$



- Unit Triangle

$$tri(t) = \begin{cases} 1-|t| & |t| < 1 \\ 0 & |t| \ge 1 \end{cases}$$



Fourier Transform of Periodic Signals

- While the Fourier Transform is applicable to energy signals, we can use it for periodic signals by using the delta function. A sinusoid has all of its energy at $f = f_o$ and the Fourier Transform is a delta function at $\pm f_o$
- Note that the Fourier Series coefficients are a valid representation of the spectrum for *periodic waveforms.* Thus, we can write the spectrum as a series of impulses each weighted by the Fourier Series coefficient $$W(f) = \sum_{n=-\infty}^{\infty} c_n \delta(f - nf_o)$$

Fourier Transform of Periodic Signals

- We can also evaluate the spectrum of a periodic signal by representing the periodic signal as the convolution of an impulse train with a single pulse.
- The FT of an impulse train is simply an impulse train. From the convolution property, we know that the spectrum is the multiplication of an impulse train with the spectrum of the pulse.
- Thus, the spectrum of a periodic signal is a sampled version of the spectrum of the signal over one period. The samples correspond to the Fourier coefficients.

Energy Spectral Density

- Parseval's Theorem:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |X(f)|^2 \, df$$

- Thus we can define Energy Spectral Density as:

$$\psi_x(f) = |X(f)|^2$$

- And

$$E = \int_{-\infty}^{\infty} \psi_x(f) df$$

- However, this only applies to *Energy* signals

Power Spectral Density

Analog and Digital Communications

- Similar to ESD, we can define Power Spectral Density (PSD) for power signals. First, recall that the power of a signal is defined as:

$$P = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x^2(t)dt = \lim_{T\to\infty}\frac{1}{T}\int_{-\infty}^{\infty} x_T^2(t)dt$$

- Where:

$$x_T(t) = x(t)\Pi(t/T)$$
$$X_T(f) = F\{x_T(t)\}$$

- From Parseval's theorem we define the PSD as:

$$S_x(f) = \lim_{T\to\infty}\left(\frac{\left|X_T(f)\right|^2}{T}\right)$$

- For random signals we will exclusively use power spectral density

Autocorrelation Function

- The *autocorrelation* *R(τ)* of a real *energy* signal is defined as:

$$R_x(\tau) = \int_{-\infty}^{\infty} x(t)x(t+\tau)dt$$

- Further, it can be shown that the autocorrelation function and ESD form a Fourier Transform pair

$$R_x(\tau) \rightleftharpoons \psi_x(f) = \left|X\left(f\right)\right|^2$$

- The *autocorrelation* *R(τ)* of a real *power* signal is defined as:

$$R_x(\tau) = \left\langle x(t)x(t+\tau)\right\rangle = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} x(t)x(t+\tau)dt$$

- It can also be shown that the autocorrelation function and PSD form a Fourier Transform pair

$$R_x(\tau) \rightleftharpoons S_x(f)$$

Quiz #1

Analog and Digital Communications

Review of LTI Systems

- A system is *linear* when superposition holds:

$$y(t) = \Im\left[a_1 x_1(t) + a_2 x_2(t)\right] = a_1 \Im\left[x_1(t)\right] + a_2 \Im\left[x_2(t)\right]$$

- A system is *time-invariant* if for a delayed input $x(t-t_o)$ the output is simply delayed by the same amount $y(t-t_o)$. In other words the *shape* of the output is the same no matter when the input is applied.

Properties of Fourier Transform: Convolution

- $$w_1(t) * w_2(t) \rightleftharpoons W_1(f) \cdot W_2(f)$$

  where the convolution operation is defined by:

$$w_1(t) * w_2(t) = \int_{-\infty}^{\infty} w_1(\lambda) w_2(t-\lambda) d\lambda$$

- A complicated operation in the time domain can be reduced to a simple operation in the frequency domain
  - Both analytical and numerical calculations can be simplified this way.

Application of Convolution Property to Linear Systems

- Consider a linear system:

$$x(t) \longrightarrow \boxed{\begin{array}{c}\textbf{Impulse Response}\\ h(t)\end{array}} \longrightarrow y(t) = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)d\lambda$$

- Output of linear system is given by:

$$y(t) = x(t) * h(t) \Leftrightarrow Y(f) = X(f)H(f)$$

  - Transform input to frequency domain
  - Apply $Y(f) = X(f)H(f)$
  - Transform answer back to time domain

Transfer Function

- Using the impulse response and the properties of the Fourier Transform we get:

$$Y(f) = H(f)X(f)$$

- Thus,

$$H(f) = \frac{Y(f)}{X(f)}$$

- *H(f)* is termed the *Transfer Function* of the system.

Summary

- Today we reviewed additional concepts that will be important in the study of communication systems including
  - Singularity functions (impulse, step, etc)
  - Energy and Power Spectral Densities
  - Linear systems: The impulse response and the transfer function
- Next week we will begin examining communication systems

Appendix

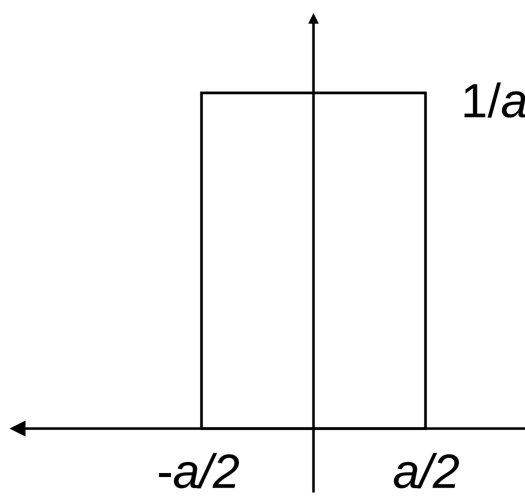
Additional information on the unit impulse

Unit Impulse Function

Analog and Digital Communications

- One of the most useful, yet strange, functions that we will use in communications is the unit impulse function, $\delta(t)$ (sometimes also called the Dirac delta function).
- To understand the unit impulse function consider a unit area pulse $\delta_a(t)$ which has width *a* and height *1/a*:

$$\delta_a(t) = \begin{cases} \dfrac{1}{a} & |t| < \dfrac{a}{2} \\ 0 & else \end{cases}$$



Area = a*1/a = 1

Unit Impulse Function (cont.)

- Now, consider a the integral of the unit pulse times a function g(t):

$$A = \int_{-\infty}^{\infty} \delta_a(t)\, g(t)\, dt$$

$$= \frac{1}{a} \int_{-a/2}^{a/2} g(t)\, dt$$

If we let the interval, *a*, get very small:

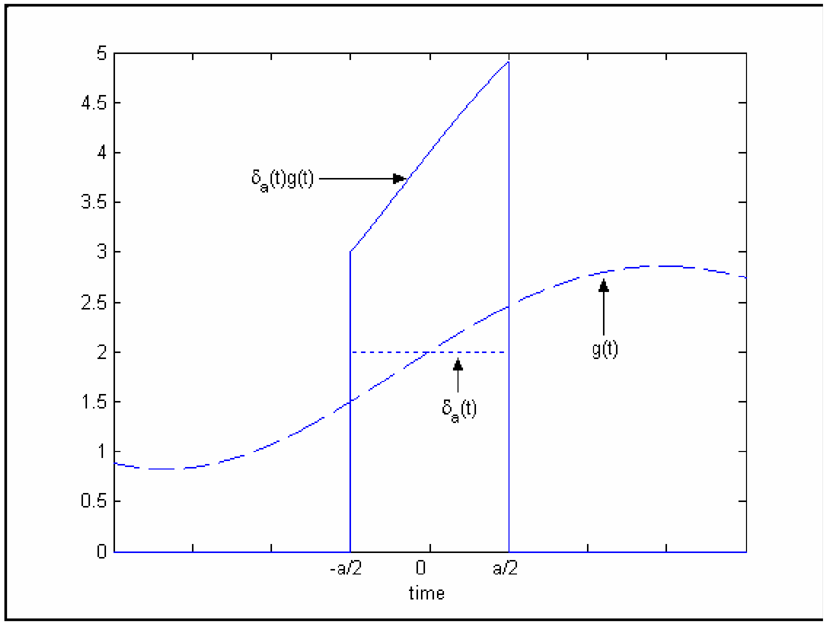
$$\lim_{a\to 0} A = \lim_{a\to 0}\left\{ \int_{-\infty}^{\infty} \delta_a(t)\, g(t)\, dt \right\}$$

$$= \lim_{a\to 0}\left\{ \frac{1}{a}\int_{-a/2}^{a/2} g(t)\, dt \right\}$$

$$= g(0) \lim_{a\to 0}\left\{ \frac{1}{a}\int_{-a/2}^{a/2} dt \right\}$$

$$= g(0) \lim_{a\to 0} \frac{1}{a} a$$

$$= g(0)$$



Thus, in the limit as *a* → 0, the function $\delta_a(t)$ has the property that it extracts the value of the function at time equal 0 when their product is integrated over any limits which include *t=0*.

Derivative of the Unit Step

- Taking the derivative of the unit step

$$\frac{d}{dt}\{u(t)\} = \frac{d}{dt}\{u(t)\}_{t\neq 0} + \lim_{\varepsilon\to 0}\left[u(t+\varepsilon)-u(t-\varepsilon)\right]\delta(t)$$

- The derivative for t< 0 is zero. The derivative for t > 0 is also zero.
- Thus we have

$$\frac{d}{dt}\{u(t)\} = 0+[1-0]\delta(t)$$

$$=\delta(t)$$

- The unit impulse function is the generalized derivative of the unit step function.
- Further

$$u(t)=\int_{-\infty}^{t}\delta(\lambda)\,d\lambda$$

Appendix

The Impulse Response

Impulse Response

- An LTI system may be characterized by its *impulse response.* If we represent the output and input of an LTI system by $y(t)$ and $x(t)$, respectively we say that the impulse response $h(t)$ is the output $y(t)$ when $x(t)=\delta(t)$ where

$$\delta(t) = \begin{cases} 0, & t \neq 0 \\ \text{undefined}, & t = 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

Impulse Response (cont.)

- The impulse response is useful because it can be used to find the output of a system when the input is *not* an impulse.
- Let us approximate a general input *x(t)* using a series of impulses

$$x(t) \approx \sum_{n=0}^{\infty} x(n\Delta t)[\delta(t - n\Delta t)]\Delta t$$

- Since the system is LTI we can approximate the output as

$$y(t) \approx \sum_{n=0}^{\infty} x(n\Delta t)\left[h(t - n\Delta t)\right]\Delta t$$

Impulse Response (cont.)

Analog and Digital Communications

- If we let Δt go to zero we get:

$$y(t) = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)d\lambda$$

- This says that the output of an LTI system is the *convolution* of the input and the impulse response. Thus we can find the output based on any input if we know the impulse response.