# Chapter 2

**2.1-1** Let us denote the signal in question by $g(t)$ and its energy by $E_g$. For parts (a) and (b)

$$E_g = \int_0^{2\pi} \sin^2 t\, dt = \frac{1}{2}\int_0^{2\pi} dt - \frac{1}{2}\int_0^{2\pi} \cos 2t\, dt = \pi + 0 = \pi$$

$$\text{(c)} \qquad E_g = \int_{2\pi}^{4\pi} \sin^2 t\, dt = \frac{1}{2}\int_{2\pi}^{4\pi} dt - \frac{1}{2}\int_{2\pi}^{4\pi} \cos 2t\, dt = \pi + 0 = \pi$$

$$\text{(d)} \qquad E_g = \int_0^{2\pi} (2\sin t)^2\, dt = 4\left[\frac{1}{2}\int_0^{2\pi} dt - \frac{1}{2}\int_0^{2\pi} \cos 2t\, dt\right] = 4[\pi + 0] = 4\pi$$

Sign change and time shift do not affect the signal energy. Doubling the signal quadruples its energy. In the same way we can show that the energy of $kg(t)$ is $k^2 E_g$.

**2.1-2** (a) $E_x = \int_0^2 (1)^2 dt = 2.$ $\qquad E_y = \int_0^1 (1)^2 dt + \int_1^2 (-1)^2 dt = 2$

$$E_{x+y} = \int_0^1 (2)^2 dt = 4. \quad E_{x-y} = \int_1^2 (2)^2 dt = 4$$

Therefore $E_{x\pm y} = E_x + E_y$.

$$\text{(b)}\, E_x = \int_0^{\pi} (1)^2 dt + \int_{\pi}^{2\pi} (-1)^2 dt = 2\pi, \qquad E_y = \int_0^{\pi/2} (1)^2 dt + \int_{\pi/2}^{\pi} (-1)^2 dt + \int_{\pi}^{3\pi/2} (1)^2 dt + \int_{3\pi/2}^{2\pi} (-1)^2 dt = 2\pi$$

$$E_{x+y} = \int_0^{\pi/2} (2)^2 dt + \int_{\pi/2}^{3\pi/2} (0)^2 dt + \int_{3\pi/2}^{2\pi} (-1)^2 dt = 4\pi$$

Similarly, we can show that $E_{x-y} = 4\pi$ Therefore $E_{x\pm y} = E_x + E_y$. We are tempted to conclude that $E_{x\pm y} = E_x + E_y$ in general. Let us see.

$$\text{(c)} \qquad E_x = \int_0^{\pi/4} (1)^2 dt + \int_{\pi/4}^{\pi} (-1)^2 dt = \pi \qquad E_y = \int_0^{\pi} (1)^2 dt = \pi$$

$$E_{x+y} = \int_0^{\pi/4} (2)^2 dt + \int_{\pi/4}^{\pi} (0)^2 dt = \pi \qquad E_{x-y} = \int_0^{\pi/4} (0)^2 dt + \int_{\pi/4}^{\pi} (-2)^2 dt = 3\pi$$

Therefore, in general $E_{x\pm y} \neq E_x + E_y$

**2.1-3**

$$P_g = \frac{1}{T_0}\int_0^{T_0} C^2 \cos^2(\omega_0 t + \theta)\, dt = \frac{C^2}{2T_0}\int_0^{T_0} [1 + \cos(2\omega_0 t + 2\theta)]\, dt$$

$$= \frac{C^2}{2T_0}\left[\int_0^{T_0} dt + \int_0^{T_0} \cos(2\omega_0 t + 2\theta)\, dt\right] = \frac{C^2}{2T_0}[T_0 + 0] = \frac{C^2}{2}$$

**2.1-4** This problem is identical to Example 2.2b, except that $\omega_1 \neq \omega_2$. In this case, the third integral in $P_g$ (see p. 19 is not zero. This integral is given by

$$I_3 = \lim_{T\to\infty} \frac{2C_1C_2}{T}\int_{-T/2}^{T/2} \cos(\omega_1 t + \theta_1)\cos(\omega_1 t + \theta_2)\, dt$$

$$= \lim_{T\to\infty} \frac{C_1C_2}{T}\left[\int_{-T/2}^{T/2} \cos(\theta_1 - \theta_2)\, dt + \int_{-T/2}^{T/2} \cos(2\omega_1 t + \theta_1 + \theta_2)\, dt\right]$$

$$= \lim_{T\to\infty} \frac{C_1C_2}{T}[T\cos(\theta_1 - \theta_2)] + 0 = C_1C_2\cos(\theta_1 - \theta_2)$$

Therefore

$$P_y = \frac{C_1^2}{2} + \frac{C_2^2}{2} + C_1 C_2 \cos(\theta_1 - \theta_2)$$

**2.1-5**

$$P_g = \frac{1}{4}\int_{-2}^{2} (t^3)^2 dt = 64/7 \qquad \textbf{(a) } P_{-g} = \frac{1}{4}\int_{-2}^{2} (-t^3)^2 dt = 64/7$$

$$\textbf{(b) } P_{2g} = \frac{1}{4}\int_{-2}^{2} (2t^3)^2 dt = 4(64/7) = 256/7 \qquad \textbf{(c) } P_{cg} = \frac{1}{4}\int_{-2}^{2} (ct^3)^2 dt = 64c^2/7$$

Sign change of a signal does not affect its power. Multiplication of a signal by a constant $c$ increases the power by a factor $c^2$.

**2.1-6**

$$\textbf{(a)} \quad P_g = \frac{1}{\pi}\int_0^{\pi} (e^{-t/2})^2\, dt = \frac{1}{\pi}\int_0^{\pi} e^{-t}\, dt = \frac{1}{\pi}[1 - e^{-\pi}]$$

$$\textbf{(b)} \quad P_g = \frac{1}{2\pi}\int_{-\pi}^{\pi} w^2(t)\, dt = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} dt = 0.5$$

$$\textbf{(c)} \quad P_g = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} w_0^2(t)\, dt = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} dt = 1$$

$$\textbf{(d)} \quad P_g = \frac{1}{4}\int_{-2}^{2} (\pm 1)^2\, dt = 1$$

$$\textbf{(e)} \quad P_g = \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{t}{2\pi}\right)^2 dt = \frac{1}{3}$$

**2.1-7**

$$P_g = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} g(t)g^*(t)\, dt = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} \sum_{k=m}^{n}\sum_{r=m}^{n} D_k D_r^* e^{j(\omega_k - \omega_r)t}\, dt$$

The integrals of the cross-product terms (when $k \neq r$) are finite because the integrands are periodic signals (made up of sinusoids). These terms, when divided by $T \to \infty$, yield zero. The remaining terms ($k = r$) yield

$$P_g = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} \sum_{k=m}^{n} |D_k|^2\, dt = \sum_{k=m}^{n} |D_k|^2$$

**2.1-8** **(a)** Power of a sinusoid of amplitude $C$ is $C^2/2$ [Eq. (2.6a)] regardless of its frequency ($\omega \neq 0$) and phase. Therefore, in this case $P = (10)^2/2 = 50$.
**(b)** Power of a sum of sinusoids is equal to the sum of the powers of the sinusoids [Eq. (2.6b)]. Therefore, in this case $P = \frac{(10)^2}{2} + \frac{(16)^2}{2} = 178$.
**(c)** $(10 + 2\sin 3t)\cos 10t = 10\cos 10t + \sin 13t - \sin 3t$. Hence from Eq. (2.6b) $P = \frac{(10)^2}{2} + \frac{1}{2} + \frac{1}{2} = 51$.
**(d)** $10\cos 5t\cos 10t = 5(\cos 5t + \cos 15t$. Hence from Eq. (2.6b) $P = \frac{(5)^2}{2} + \frac{(5)^2}{2} = 25$.
**(e)** $10\sin 5t\cos 10t = 5(\sin 15t - \sin 5t$. Hence from Eq. (2.6b) $P = \frac{(5)^2}{2} + \frac{(-5)^2}{2} = 25$.
**(f)** $e^{j\alpha t}\cos\omega_0 t = \frac{1}{2}\left[e^{j(\alpha+\omega_0)t} + e^{j(\alpha-\omega_0)t}\right]$. Using the result in Prob. 2.1-7, we obtain $P = (1/4) + (1/4) = 1/2$.

**2.2-1** For a real $a$

$$E_g = \int_{-\infty}^{\infty} (e^{-at})^2\, dt = \int_{-\infty}^{\infty} e^{-2at}\, dt = \infty$$

$$P_g = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} (e^{-at})^2\, dt = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} e^{-2at}\, dt = \infty$$

For imaginary $a$, let $a = jr$. Then

$$P_g = I1im_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} (e^{jrt})(e^{-jrt})\, dt = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} dt = 1$$

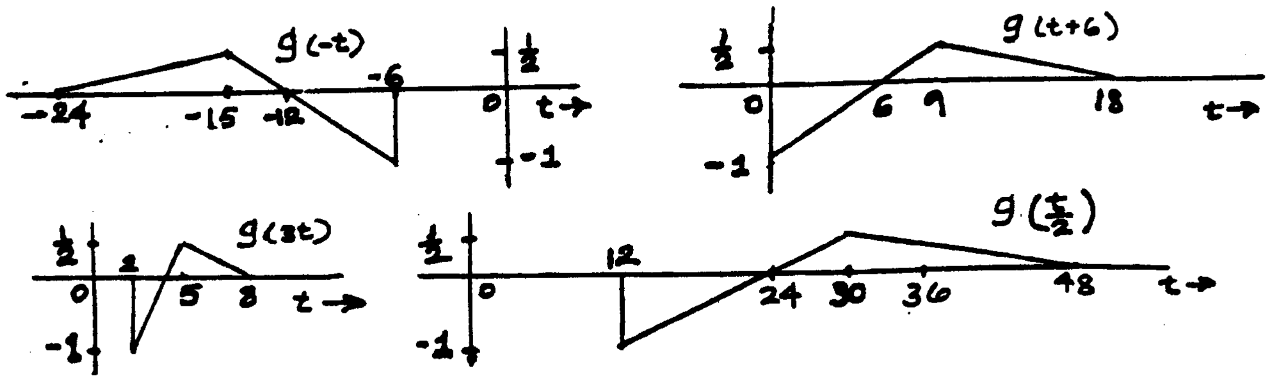



**Fig. S2.3-2**

Clearly, if $a$ is real, $e^{-at}$ is neither energy not power signal. However, if $a$ is imaginary, it is a power signal with power 1.

**2.3-1**

$$g_2(t) = g(t-1) + g_1(t-1), \quad g_3(t) = g(t-1) + g_1(t+1), \quad g_4(t) = g(t-0.5) + g_1(t+0.5)$$

The signal $g_5(t)$ can be obtained by (i) delaying $g(t)$ by 1 second (replace $t$ with $t-1$), (ii) then time-expanding by a factor 2 (replace $t$ with $t/2$), (iii) then multiply with 1.5. Thus $g_5(t) = 1.5g(\frac{t}{2} - 1)$.

**2.3-2** All the signals are shown in Fig. S2.3-2.

**2.3-3** All the signals are shown in Fig. S2.3-3

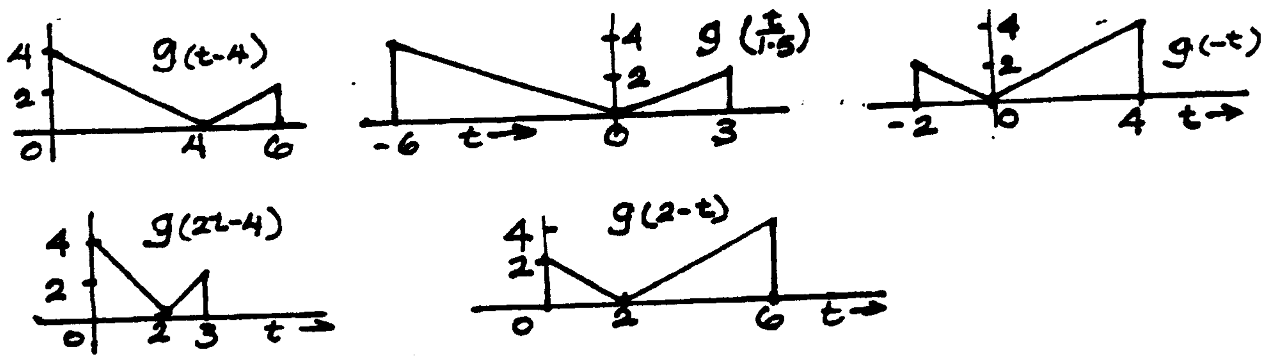



**Fig. S2.3-3**

**2.3-4**

$$E_{-g} = \int_{-\infty}^{\infty} [-g(t)]^2\,dt = \int_{-\infty}^{\infty} g^2(t)\,dt = E_g. \quad E_{g(-t)} = \int_{-\infty}^{\infty} [g(-t)]^2\,dt = \int_{-\infty}^{\infty} g^2(x)\,dx = E_g$$

$$E_{g(t-T)} = \int_{-\infty}^{\infty} [g(t-T)]^2\,dt = \int_{-\infty}^{\infty} g^2(x)\,dx = E_g. \quad E_{g(at)} = \int_{-\infty}^{\infty} [g(at)]^2\,dt = \frac{1}{a}\int_{-\infty}^{\infty} g^2(x)\,dx = E_g/a$$

$$E_{g(at-b)} = \int_{-\infty}^{\infty} [g(at-b)]^2\,dt = \frac{1}{a}\int_{-\infty}^{\infty} g^2(x)\,dx = E_g/a. \quad E_{g(t/a)} = \int_{-\infty}^{\infty} [g(t/a)]^2\,dt = a\int_{-\infty}^{\infty} g^2(x)\,dt = aE_g$$

$$E_{ag(t)} = \int_{-\infty}^{\infty} [ag(t)]^2\,dt = a^2\int_{-\infty}^{\infty} g^2(t)\,dt = a^2E_g$$

**2.4-1** Using the fact that $g(x)\delta(x) = g(0)\delta(x)$, we have

**(a)** 0 **(b)** $\frac{2}{9}\delta(\omega)$ **(c)** $\frac{1}{2}\delta(t)$ **(d)** $-\frac{1}{5}\delta(t-1)$ **(e)** $\frac{1}{2-j3}\delta(\omega+3)$ **(f)** $k\delta(\omega)$ (use L' Hôpital's rule)

**2.4-2** In these problems remember that impulse $\delta(x)$ is located at $x = 0$. Thus, an impulse $\delta(t-\tau)$ is located at $\tau = t$, and so on.

**(a)** The impulse is located at $\tau = t$ and $g(\tau)$ at $\tau = t$ is $g(t)$. Therefore

$$\int_{-\infty}^{\infty} g(\tau)\delta(t-\tau)\,d\tau = g(t)$$

**(b)** The impulse $\delta(\tau)$ is at $\tau = 0$ and $g(t-\tau)$ at $\tau = 0$ is $g(t)$. Therefore

$$\int_{-\infty}^{\infty} \delta(\tau)g(t-\tau)\,d\tau = g(t)$$

Using similar arguments, we obtain
**(c)** 1 **(d)** 0 **(e)** $e^3$ **(f)** 5 **(g)** $g(-1)$ **(h)** $-e^2$

**2.4-3** Letting $at = x$, we obtain (for $a > 0$)

$$\int_{-\infty}^{\infty} \phi(t)\delta(at)\,dt = \frac{1}{a}\int_{-\infty}^{\infty} \phi(\frac{x}{a})\delta(x)\,dx = \frac{1}{a}\phi(0)$$

Similarly for $a < 0$, we show that this integral is $-\frac{1}{a}\phi(0)$. Therefore

$$\int_{-\infty}^{\infty} \phi(t)\delta(at)\,dt = \frac{1}{|a|}\phi(0) = \frac{1}{|a|}\int_{-\infty}^{\infty} \phi(t)\delta(t)\,dt$$

Therefore

$$\delta(at) = \frac{1}{|a|}\delta(t)$$

**2.5-1** Trivial. Take the derivative of $|\mathbf{e}|^2$ with respect to $c$ and equate it to zero.

**2.5-2** **(a)** In this case $E_x = \int_0^1 dt = 1$, and

$$c = \frac{1}{E_x}\int_0^1 g(t)x(t)\,dt = \frac{1}{1}\int_0^1 t\,dt = 0.5$$

**(b)** Thus, $g(t) \approx 0.5x(t)$, and the error $e(t) = t - 0.5$ over $(0 \le t \le 1)$, and zero outside this interval. Also $E_g$ and $E_e$ (the energy of the error) are

$$E_g = \int_0^1 g^2(t)\,dt = \int_0^1 t^2\,dt = 1/3 \quad \text{and} \quad E_e = \int_0^1 (t-0.5)^2\,dt = 1/12$$

The error $(t - 0.5)$ is orthogonal to $x(t)$ because

$$\int_0^1 (t-0.5)(1)\,dt = 0$$

Note that $E_g = c^2E_x + E_e$. To explain these results in terms of vector concepts we observe from Fig. 2.15 that the error vector $\mathbf{e}$ is orthogonal to the component $c\mathbf{x}$. Because of this orthogonality, the length-square of $\mathbf{g}$ [energy of $g(t)$] is equal to the sum of the square of the lengths of $c\mathbf{x}$ and $\mathbf{e}$ [sum of the energies of $cx(t)$ and $e(t)$].

**2.5-3** In this case $E_g = \int_0^1 g^2(t)\,dt = \int_0^1 t^2\,dt = 1/3$, and

$$c = \frac{1}{E_g}\int_0^1 x(t)g(t)\,dt = 3\int_0^1 t\,dt = 1.5$$

Thus, $x(t) \approx 1.5g(t)$, and the error $e(t) = x(t) - 1.5g(t) = 1 - 1.5t$ over $(0 \le t \le 1)$, and zero outside this interval. Also $E_e$ (the energy of the error) is $E_e = \int_0^1 (1-1.5t)^2\,dt = 1/4$.

**2.5-4** **(a)** In this case $E_x = \int_0^1 \sin^2 2\pi t\,dt = 0.5$, and

$$c = \frac{1}{E_x}\int_0^1 g(t)x(t)\,dt = \frac{1}{0.5}\int_0^1 t\sin 2\pi t\,dt = -1/\pi$$

**(b)** Thus, $g(t) \approx -(1/\pi)x(t)$, and the error $e(t) = t + (1/\pi)\sin 2\pi t$ over $(0 \le t \le 1)$, and zero outside this interval. Also $E_g$ and $E_e$ (the energy of the error) are

$$E_g = \int_0^1 g^2(t)\,dt = \int_0^1 t^2\,dt = 1/3 \quad \text{and} \quad E_e = \int_0^1 [t - (1/\pi)\sin 2\pi t]^2\,dt = \frac{1}{3} - \frac{1}{2\pi^2}$$

The error $[t + (1/\pi)\sin 2\pi t]$ is orthogonal to $x(t)$ because

$$\int_0^1 \sin 2\pi t[t + (1/\pi)\sin 2\pi t]\,dt = 0$$

Note that $E_g = c^2 E_x + E_e$. To explain these results in terms of vector concepts we observe from Fig. 2.15 that the error vector **e** is orthogonal to the component $c\mathbf{x}$. Because of this orthogonality, the length of **f** [energy of $g(t)$] is equal to the sum of the square of the lengths of $c\mathbf{x}$ and **e** [sum of the energies of $cx(t)$ and $e(t)$].

**2.5-5** **(a)** If $x(t)$ and $y(t)$ are orthogonal, then we can show the energy of $x(t) \pm y(t)$ is $E_x + E_y$.

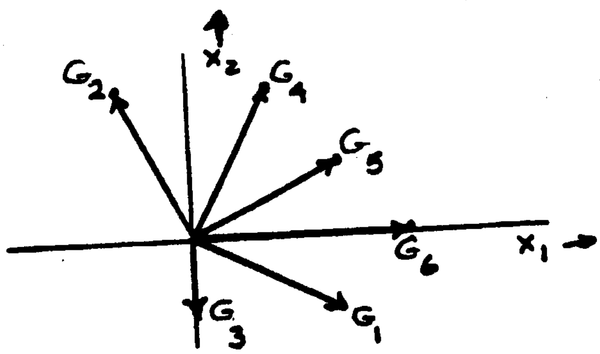
$$\int_{-\infty}^{\infty} |x(t) \pm y(t)|^2\,dt = \int_{-\infty}^{\infty} |x(t)|^2\,dt + \int_{-\infty}^{\infty} |y(t)|^2\,dt \pm \int_{-\infty}^{\infty} x(t)y^*(t)\,dt \pm \int_{-\infty}^{\infty} x^*(t)y(t)\,dt \tag{1}$$

$$= \int_{-\infty}^{\infty} |x(t)|^2\,dt + \int_{-\infty}^{\infty} |y(t)|^2\,dt \tag{2}$$

The last result follows from the fact that because of orthogonality, the two integrals of the cross products $x(t)y^*(t)$ and $x^*(t)y(t)$ are zero [see Eq. (2.40)]. Thus the energy of $x(t) + y(t)$ is equal to that of $x(t) - y(t)$ if $x(t)$ and $y(t)$ are orthogonal.
**(b)** Using similar argument, we can show that the energy of $c_1x(t) + c_2y(t)$ is equal to that of $c_1x(t) - c_2y(t)$ if $x(t)$ and $y(t)$ are orthogonal. This energy is given by $|c_1|^2E_x + |c_2|^2E_y$.
**(c)** If $z(t) = x(t) \pm y(t)$, then it follows from Eq. (1) in the above derivation that

$$E_z = E_x + E_y \pm (E_{xy} + E_{yx})$$

**2.5-6** $\mathbf{g}_1(2,-1)$, $\mathbf{g}_2(-1,2)$, $\mathbf{g}_3(0,-2)$, $\mathbf{g}_4(1,2)$, $\mathbf{g}_5(2,1)$, and $\mathbf{g}_6(3,0)$. From Fig. S2.5-6, we see that pairs $(\mathbf{g}_3, \mathbf{g}_6)$, $(\mathbf{g}_1, \mathbf{g}_4)$ and $(\mathbf{g}_2, \mathbf{g}_5)$ are orthogonal. We can verify this also analytically.




**gig. S2.5-6**

$$\mathbf{g}_3 \cdot \mathbf{g}_6 = (0 \times 3) + (-2 \times 0) = 0$$
$$\mathbf{g}_1 \cdot \mathbf{g}_4 = (2 \times 1) + (-1 \times 2) = 0$$
$$\mathbf{g}_2 \cdot \mathbf{g}_5 = (-1 \times 2) + (2 \times 1) = 0$$

We can show that the corresponding signal pairs are also orthogonal.

$$\int_{-\infty}^{\infty} g_3(t)g_6(t)\,dt = \int_{-\infty}^{\infty} [-x_2(t)][3x_1(t)]\,dt = 0$$

$$\int_{-\infty}^{\infty} g_1(t)g_4(t)\,dt = \int_{-\infty}^{\infty} [2x_1(t) - x_2(t)][x_1(t) + 2x_2(t)]\,dt = 0$$

$$\int_{-\infty}^{\infty} g_2(t)g_5(t)\,dt = \int_{-\infty}^{\infty} [-x_1(t) + 2x_2(t)][2x_1(t) + x_2(t)]\,dt = 0$$

In deriving these results, we used the fact that $\int_{-\infty}^{\infty} x_1^2 dt = \int_{-\infty}^{\infty} x_2^2(t)dt = 1$ and $\int_{-\infty}^{\infty} x_1(t)x_2(t)\,dt = 0$

**2.6-1** We shall compute $c_n$ using Eq. (2.48) for each of the 4 cases. Let us first compute the energies of all the signals.

$$E_x = \int_0^1 \sin^2 2\pi t\, dt = 0.5$$

In the same way we find $E_{g_1} = E_{g_2} = E_{g_3} = E_{g_4} = 0.5$.
Using Eq. (2.48), the correlation coefficients for four cases are found as

$$(1)\quad \frac{1}{\sqrt{(0.5)(0.5)}} \int_0^1 \sin 2\pi t \sin 4\pi t\, dt = 0 \quad (2)\quad \frac{1}{\sqrt{(0.5)(0.5)}} \int_0^1 (\sin 2\pi t)(-\sin 2\pi t)\, dt = -1$$

$$(3)\quad \frac{1}{\sqrt{(0.5)(0.5)}} \int_0^1 0.707 \sin 2\pi t\, dt = 0 \quad (4)\quad \frac{1}{\sqrt{(0.5)(0.5)}} \left[\int_0^{0.5} 0.707 \sin 2\pi t\, dt - \int_{0.5}^1 0.707 \sin 2\pi t\, dt\right] = 1.414/\pi$$

Signals $x(t)$ and $g_2(t)$ provide the maximum protection against noise.

**2.8-1** Here $T_0 = 2$, so that $\omega_0 = 2\pi/2 = \pi$, and

$$g(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi t + b_n \sin n\pi t \qquad -1 \le t \le 1$$

where

$$a_0 = \frac{1}{2}\int_{-1}^{1} t^2\, dt = \frac{1}{3}, \qquad a_n = \frac{2}{2}\int_{-1}^{1} t^2 \cos n\pi t\, dt = \frac{4(-1)^n}{\pi^2 n^2}, \qquad b_n = \frac{2}{2}\int_{-1}^{1} t^2 \sin n\pi t\, dt = 0$$

Therefore

$$g(t) = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\pi t \qquad -1 \le t \le 1$$

Figure S2.8-1 shows $g(t) = t^2$ for all $t$ and the corresponding Fourier series representing $g(t)$ over $(-1, 1)$.

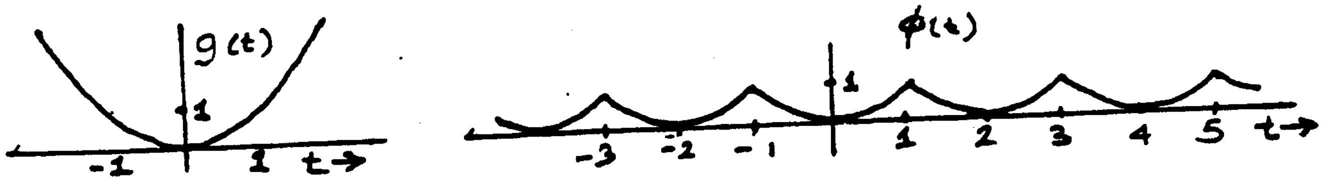



**Fig. S2.8-1**

The power of $g(t)$ is

$$P_g = \frac{1}{2}\int_{-1}^{1} t^4\, dt = \frac{1}{5}$$

Moreover, from Parseval's theorem [Eq. (2.90)]

$$P_g = C_0^2 + \sum_{1}^{\infty} \frac{C_n^2}{2} = \left(\frac{1}{3}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{4(-1)^n}{\pi^2 n^2}\right)^2 = \frac{1}{9} + \frac{8}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{1}{9} + \frac{8}{90} = \frac{1}{5}$$

(b) If the $N$-term Fourier series is denoted by $x(t)$, then

$$x(t) = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{N-1} \frac{(-1)^n}{n^2} \cos n\pi t \qquad -1 \le t \le 1$$

The power $P_x$ is required to be $99\% P_g = 0.198$. Therefore

$$P_x = \frac{1}{9} + \frac{8}{\pi^4}\sum_{n=1}^{N-1}\frac{1}{n^4} = 0.198$$

For $N = 1$, $P_x = 0.1111$; for $N = 2$, $P_x = 0.19323$. For $N = 3$, $P_x = 0.19837$, which is greater than 0.198. Thus, $N = 3$.

**2.8-2** Here $T_0 = 2\pi$, so that $\omega_0 = 2\pi/2\pi = 1$, and

$$g(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt \qquad -\pi \le t \le \pi$$

where

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} t\,dt = 0, \qquad a_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} t\cos nt\,dt = 0, \qquad b_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} t\sin nt\,dt = \frac{2(-1)^{n+1}}{n}$$

Therefore

$$g(t) = 2(-1)^{n+1}\sum_{n=1}^{\infty}\frac{1}{n}\sin nt \qquad -\pi \le t \le \pi$$

Figure S2.8-2 shows $g(t) = t$ for all $t$ and the corresponding Fourier series to represent $g(t)$ over $(-\pi, \pi)$.

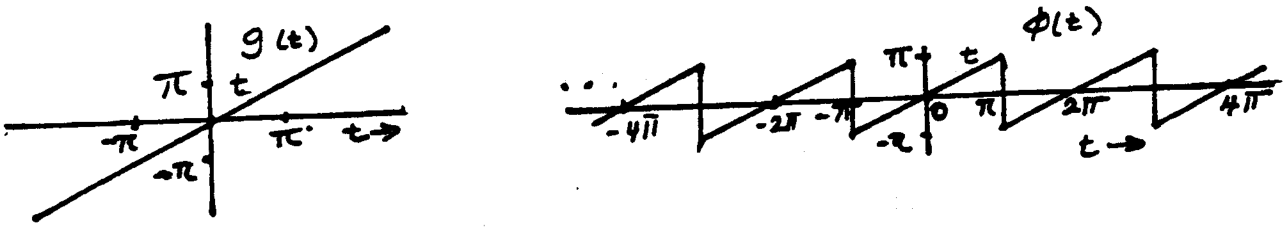



**Fig. S2.8-2**

The power of $g(t)$ is

$$P_g = \frac{1}{2\pi}\int_{-\pi}^{\pi}(t)^2\,dt = \frac{\pi^2}{3}$$

Moreover, from Parseval's theorem [Eq. (2.90)]

$$P_g = C_0^2 + \sum_{1}^{\infty}\frac{C_n^2}{2} = \frac{1}{2}\sum_{1}^{\infty}\frac{4}{n^2} = 2\sum_{1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{3}$$

(b) If the $N$-term Fourier series is denoted by $x(t)$, then

$$x(t) = 2(-1)^{n+1}\sum_{n=1}^{N}\frac{1}{n}\sin n\pi t \qquad -\pi \le t \le \pi$$

The power $P_x$ is required to be $0.90 \times \frac{\pi^2}{3} = 0.3\pi^2$. Therefore

$$P_x = \frac{1}{2}\sum_{n=1}^{N}\frac{4}{n^2} = 0.3\pi^2$$

For $N = 1$, $P_x = 2$; for $N = 2$, $P_x = 2.5$, for $N = 5$, $P_x = 2.927$, which is less than $0.3\pi^2$. For $N = 6$, $P_x = 2.9825$, which is greater than $0.3\pi^2$. Thus, $N = 6$.

**2.8-3** Recall that

$$a_0 = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g(t)\,dt \tag{1a}$$

$$a_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} g(t)\cos n\omega_0 t\,dt \tag{1b}$$

$$b_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} g(t)\sin n\omega_0 t\,dt \tag{1c}$$

Recall also that $\cos n\omega_0 t$ is an even function and $\sin n\omega_0 t$ is an odd function of $t$. If $g(t)$ is an even function of $t$, then $g(t)\cos n\omega_0 t$ is also an even function and $g(t)\sin n\omega_0 t$ is an odd function of $t$. Therefore (see hint)

$$a_0 = \frac{2}{T_0}\int_0^{T_0/2} g(t)\,dt \tag{2a}$$

$$a_n = \frac{4}{T_0}\int_0^{T_0/2} g(t)\cos n\omega_0 t\,dt \tag{2b}$$

$$b_n = 0 \tag{2c}$$

Similarly, if $g(t)$ is an odd function of $t$, then $g(t)\cos n\omega_0 t$ is an odd function of $t$ and $g(t)\sin n\omega_0 t$ is an even function of $t$. Therefore

$$a_0 = a_n = 0 \tag{3a}$$

$$b_n = \frac{4}{T_0}\int_0^{T_0/2} g(t)\sin n\omega_0 t\,dt \tag{3b}$$

Observe that, because of symmetry, the integration required to compute the coefficients need be performed over only half the period.

**2.8-4** (a) $T_0 = 4$, $\omega_0 = \frac{2\pi}{T_0} = \frac{\pi}{2}$. Because of even symmetry, all sine terms are zero.

$$g(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{2}t\right)$$

$$a_0 = 0 \text{ (by inspection)}$$

$$a_n = \frac{4}{4}\left[\int_0^1 \cos\left(\frac{n\pi}{2}t\right)dt - \int_1^2 \cos\left(\frac{n\pi}{2}t\right)dt\right] = \frac{4}{n\pi}\sin\frac{n\pi}{2}$$

Therefore, the Fourier series for $g(t)$ is

$$g(t) = \frac{4}{\pi}\left(\cos\frac{\pi t}{2} - \frac{1}{3}\cos\frac{3\pi t}{2} + \frac{1}{5}\cos\frac{5\pi t}{2} - \frac{1}{7}\cos\frac{7\pi t}{2} + \cdots\right)$$

Here $b_n = 0$, and we allow $C_n$ to take negative values. Figure S2.8-4a shows the plot of $C_n$.

(b) $T_0 = 10\pi$, $\omega_0 = \frac{2\pi}{T_0} = \frac{1}{5}$. Because of even symmetry, all the sine terms are zero.

$$g(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n}{5}t\right) + b_n \sin\left(\frac{n}{5}t\right)$$

$$a_0 = \frac{1}{5} \qquad \text{(by inspection)}$$

$$a_n = \frac{2}{10\pi}\int_{-\pi}^{\pi}\cos\left(\frac{n}{5}t\right)dt = \frac{1}{5\pi}\left(\frac{5}{n}\right)\sin\left(\frac{n}{5}t\right)\Big|_{-\pi}^{\pi} = \frac{2}{\pi n}\sin\left(\frac{n\pi}{5}\right)$$

$$b_n = \frac{2}{10\pi}\int_{-\pi}^{\pi}\sin\left(\frac{n}{5}t\right)dt = 0 \qquad \text{(integrand is an odd function of } t\text{)}$$

Here $b_n = 0$, and we allow $C_n$ to take negative values. Note that $C_n = a_n$ for $n = 0, 1, 2, 3, \cdots$. Figure S2.8-4b shows the plot of $C_n$.

(c) $T_0 = 2\pi$, $\omega_0 = 1$.

$$g(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt \quad \text{with} \quad a_0 = 0.5 \quad \text{(by inspection)}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi}\frac{t}{2\pi}\cos nt\,dt = 0, \qquad b_n = \frac{1}{\pi}\int_0^{2\pi}\frac{t}{2\pi}\sin nt\,dt = -\frac{1}{\pi n}$$

and

$$g(t) = 0.5 - \frac{1}{\pi}\left(\sin t + \frac{1}{2}\sin 2t + \frac{1}{3}\sin 3t + \frac{1}{4}\sin 4t + \cdots\right)$$
$$= 0.5 + \frac{1}{\pi}\left[\cos\left(t + \frac{\pi}{2}\right) + \frac{1}{2}\cos\left(2t + \frac{\pi}{2}\right) + \frac{1}{3}\cos\left(3t + \frac{\pi}{2}\right) + \cdots\right]$$

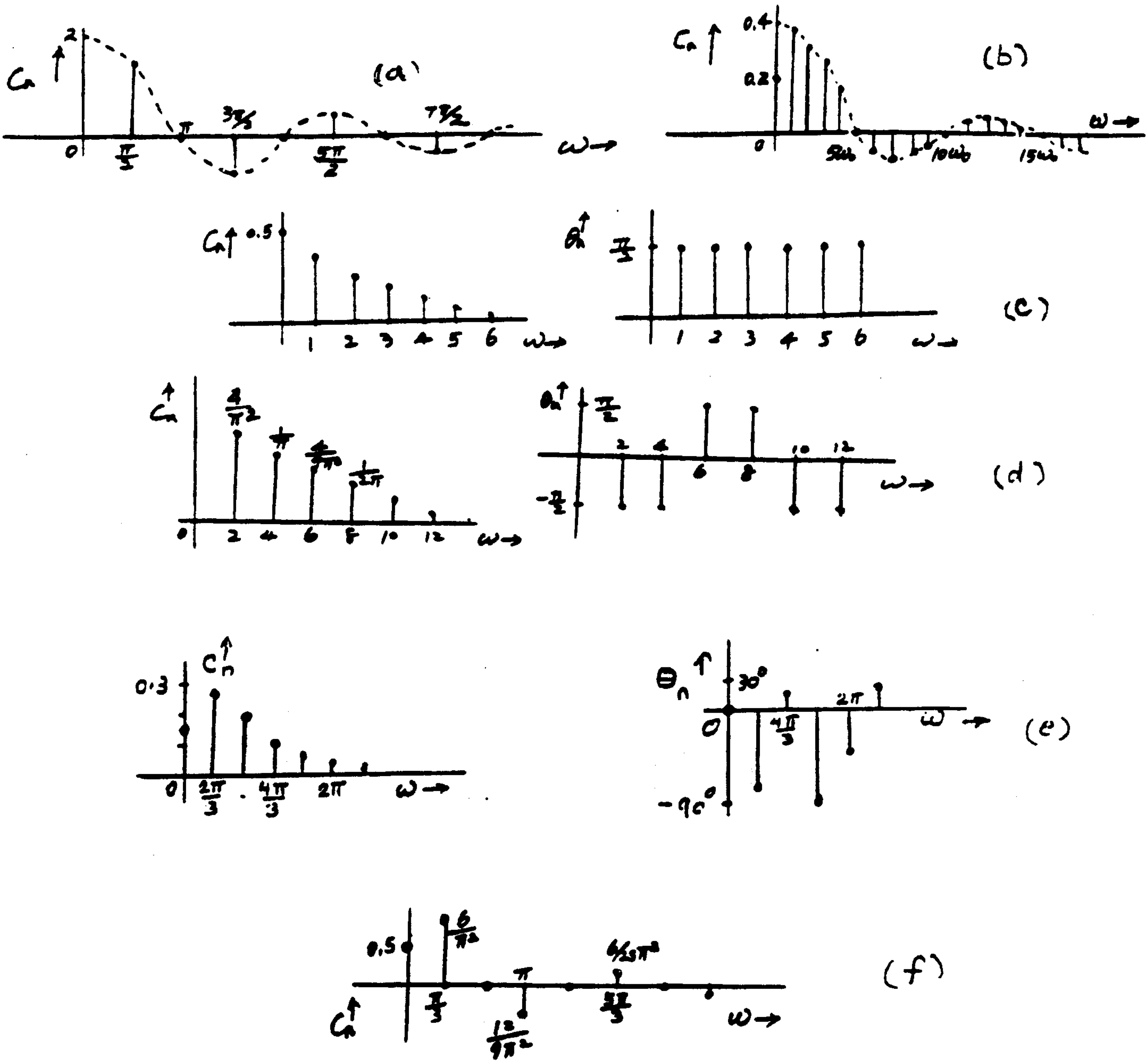



Fig. S2.8-4

The reason for vanishing of the cosines terms is that when 0.5 (the dc component) is subtracted from $g(t)$, the remaining function has odd symmetry. Hence, the Fourier series would contain dc and sine terms only. Figure S2.8-4c shows the plot of $C_n$ and $\theta_n$.

(d) $T_0 = \pi$, $\omega_0 = 2$ and $g(t) = \frac{4}{\pi}t$.

$a_0 = 0$ (by inspection).

$a_n = 0 \quad (n > 0)$ because of odd symmetry.

$$b_n = \frac{4}{\pi}\int_0^{\pi/4} \frac{4}{\pi} t \sin 2nt \, dt = \frac{2}{\pi n}\left(\frac{2}{\pi n}\sin\frac{\pi n}{2} - \cos\frac{\pi n}{2}\right)$$

$$\begin{aligned} g(t) &= \frac{4}{\pi^2}\sin 2t + \frac{1}{\pi}\sin 4t - \frac{4}{9\pi^2}\sin 6t - \frac{1}{2\pi}\sin 8t + \cdots \\ &= \frac{4}{\pi^2}\cos\left(2t - \frac{\pi}{2}\right) + \frac{1}{\pi}\cos\left(4t - \frac{\pi}{2}\right) + \frac{4}{9\pi^2}\cos\left(6t + \frac{\pi}{2}\right) + \frac{1}{\pi}\cos\left(8t + \frac{\pi}{2}\right) + \cdots \end{aligned}$$

Figure S2.8-4d shows the plot of $C_n$ and $\theta_n$.

(e) $T_0 = 3$, $\omega_0 = 2\pi/3$.

$$a_0 = \frac{1}{3}\int_0^1 t\,dt = \frac{1}{6}$$

$$a_n = \frac{2}{3}\int_0^1 t\cos\frac{2n\pi}{3}t\,dt = \frac{3}{2\pi^2 n^2}\left[\cos\frac{2\pi n}{3} + \frac{2\pi n}{3}\sin\frac{2\pi n}{3} - 1\right]$$

$$b_n = \frac{2}{3}\int_0^1 t\sin\frac{2n\pi}{3}t\,dt = \frac{3}{2\pi^2 n^2}\left[\sin\frac{2\pi n}{3} - \frac{2\pi n}{3}\cos\frac{2\pi n}{3}\right]$$

Therefore $C_0 = \frac{1}{6}$ and

$$C_n = \frac{3}{2\pi^2 n^2}\left[\sqrt{2 + \frac{4\pi^2 n^2}{9} - 2\cos\frac{2\pi n}{3} - \frac{4\pi n}{3}\sin\frac{2\pi n}{3}}\right] \quad \text{and} \quad \theta_n = \tan^{-1}\left(\frac{\frac{2\pi n}{3}\cos\frac{2\pi n}{3} - \sin\frac{2\pi n}{3}}{\cos\frac{2\pi n}{3} + \frac{2\pi n}{3}\sin\frac{2\pi n}{3} - 1}\right)$$

(f) $T_0 = 6$, $\omega_0 = \pi/3$, $a_0 = 0.5$ (by inspection). Even symmetry; $b_n = 0$.

$$\begin{aligned} a_n &= \frac{4}{6}\int_0^3 g(t)\cos\frac{n\pi}{3}\,dt \\ &= \frac{2}{3}\left[\int_0^1 \cos\frac{n\pi}{3}\,dt + \int_1^2 (2-t)\cos\frac{n\pi}{3}t\,dt\right] \\ &= \frac{6}{\pi^2 n^2}\left[\cos\frac{n\pi}{3} - \cos\frac{2n\pi}{3}\right] \end{aligned}$$

$$g(t) = 0.5 + \frac{6}{\pi^2}\left(\cos\frac{\pi}{3}t - \frac{2}{9}\cos\pi t + \frac{1}{25}\cos\frac{5\pi}{3}t + \frac{1}{49}\cos\frac{7\pi}{3}t + \cdots\right)$$

Observe that even harmonics vanish. The reason is that if the dc (0.5) is subtracted from $g(t)$, the resulting function has half-wave symmetry. (See Prob. 2.8-6). Figure S2.8-4f shows the plot of $C_n$.

**2.8-5**

An even function $g_e(t)$ and an odd function $g_o(t)$ have the property that

$$g_e(t) = g_e(-t) \qquad \text{and} \qquad g_o(t) = -g_o(-t) \tag{1}$$

Every signal $g(t)$ can be expressed as a sum of even and odd components because

$$g(t) = \underbrace{\tfrac{1}{2}[g(t) + g(-t)]}_{\text{even}} + \underbrace{\tfrac{1}{2}[g(t) - g(-t)]}_{\text{odd}}$$

From the definitions in Eq. (1), it can be seen that the first component on the right-hand side is an even function, while the second component is odd. This is readily seen from the fact that replacing $t$ by $-t$ in the first component yields the same function. The same maneuver in the second component yields the negative of that component.

To find the odd and the even components of $g(t) = u(t)$, we have

$$g(t) = g_e(t) + g_o(t)$$

where [from Eq. (1)]

$$g_e(t) = \tfrac{1}{2}[u(t) + u(-t)] = \frac{1}{2}$$

and

$$g_o(t) = \tfrac{1}{2}[u(t) - u(-t)] = \frac{1}{2}\text{sgn}(t)$$

The even and odd components of the signal $u(t)$ are shown in Fig. S2.8-5a.

Similarly, to find the odd and the even components of $g(t) = e^{-at}u(t)$, we have

$$g(t) = g_e(t) + g_o(t)$$

where

$$g_e(t) = \tfrac{1}{2}\left[e^{-at}u(t) + e^{at}u(-t)\right]$$

and

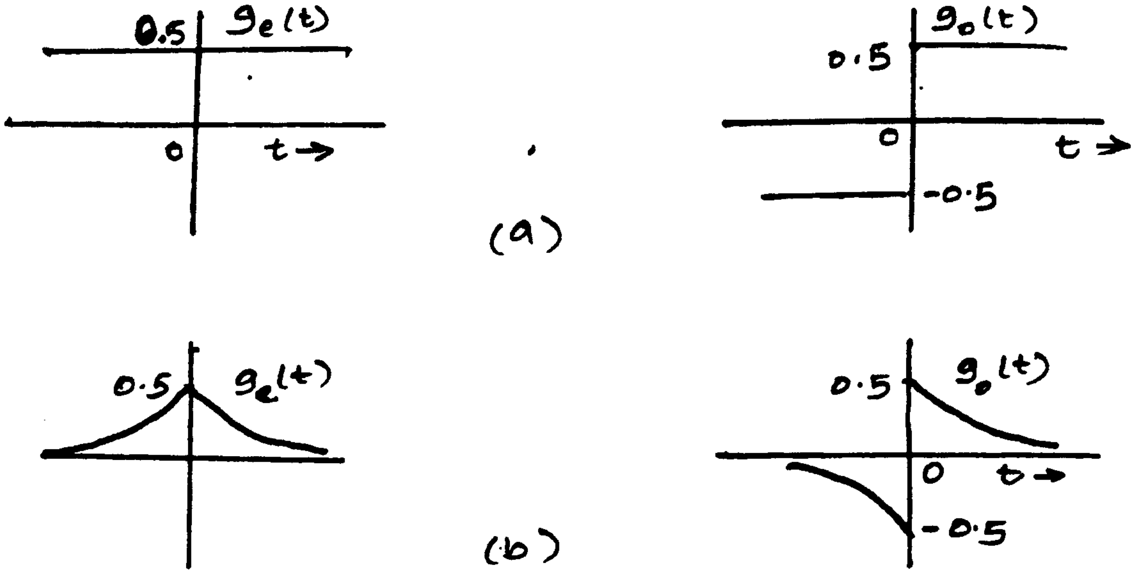



**Fig. S2.8-5**

$$g_o(t) = \tfrac{1}{2}\left[e^{-at}u(t) - e^{at}u(-t)\right]$$

The even and odd components of the signal $e^{-at}u(t)$ are shown in Fig. S2.8-5b. For $g(t) = e^{jt}$, we have

$$e^{jt} = g_e(t) + g_o(t)$$

where

$$g_e(t) = \tfrac{1}{2}\left[e^{jt} + e^{-jt}\right] = \cos t$$

and

$$g_o(t) = \tfrac{1}{2}\left[e^{jt} - e^{-jt}\right] = j\sin t$$

**2.8-6** (a) For half wave symmetry

$$g(t) = -g\left(t \pm \frac{T_0}{2}\right)$$

and

and $$a_n = \frac{2}{T_0}\int_0^{T_0} g(t)\cos n\omega_0 t\, dt = \frac{2}{T_0}\int_0^{T_0/2} g(t)\cos n\omega_0 t\, dt + \int_{T_0/2}^{T_0} g(t)\cos n\omega_0 t\, dt$$

Let $x = t - T_0/2$ in the second integral. This gives

$$\begin{aligned} a_n &= \frac{2}{T_0}\left[\int_0^{T_0/2} g(t)\cos n\omega_0 t\, dt + \int_0^{T_0/2} g\left(x + \frac{T_0}{2}\right)\cos n\omega_0\left(x + \frac{T_0}{2}\right) dx\right] \\ &= \frac{2}{T_0}\left[\int_0^{T_0/2} g(t)\cos n\omega_0 t\, dt + \int_0^{T_0/2} -g(x)[-\cos n\omega_0 x]\, dx\right] \\ &= \frac{4}{T_0}\left[\int_0^{T_0/2} g(t)\cos n\omega_0 t\, dt\right] \end{aligned}$$

In a similar way we can show that

$$b_n = \frac{4}{T_0}\int_0^{T_0/2} g(t)\sin n\omega_0 t\, dt$$

(b) (i) $T_0 = 8$, $\omega_0 = \frac{\pi}{4}$, $a_0 = 0$ (by inspection). Half wave symmetry. Hence

$$a_n = \frac{4}{8}\left[\int_0^4 g(t)\cos\frac{n\pi}{4}t\,dt\right] = \frac{1}{2}\left[\int_0^2 \frac{t}{2}\cos\frac{n\pi}{4}t\,dt\right]$$
$$= \frac{4}{n^2\pi^2}\left(\cos\frac{n\pi}{2} + \frac{n\pi}{2}\sin\frac{n\pi}{2} - 1\right) \quad (n \text{ odd})$$
$$= \frac{4}{n^2\pi^2}\left(\frac{n\pi}{2}\sin\frac{n\pi}{2} - 1\right) \quad (n \text{ odd})$$

Therefore

$$a_n = \begin{cases} \frac{4}{n^2\pi^2}\left(\frac{n\pi}{2} - 1\right) & n = 1, 5, 9, 13, \cdots \\ -\frac{4}{n^2\pi^2}\left(\frac{n\pi}{2} + 1\right) & n = 3, 7, 11, 15, \cdots \end{cases}$$

Similarly

$$b_n = \frac{1}{2}\int_0^2 \frac{t}{2}\sin\frac{n\pi}{4}t\,dt = \frac{4}{n^2\pi^2}\left(\sin\frac{n\pi}{2} - \frac{n\pi}{2}\cos\frac{n\pi}{2}\right) = \frac{4}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) \quad (n \text{ odd})$$

and

$$g(t) = \sum_{n=1,3,5,\cdots}^{\infty} a_n \cos\frac{n\pi}{4}t + b_n \sin\frac{n\pi}{4}t$$

(ii) $T_0 = 2\pi$, $\omega_0 = 1$, $a_0 = 0$ (by inspection). Half wave symmetry. Hence

$$g(t) = \sum_{n=1,3,5,\cdots}^{\infty} a_n \cos nt + b_n \sin nt$$

$$a_n = \frac{2}{\pi}\int_0^\pi e^{-t/10}\cos nt\,dt$$
$$= \frac{2}{\pi}\left[\frac{e^{-t/10}}{n^2 + 0.01}(-0.1\cos nt + n\sin nt)\right]_0^\pi \quad (n \text{ odd})$$
$$= \frac{2}{\pi}\left[\frac{e^{-\pi/10}}{n^2 + 0.01}(0.1) - \frac{1}{n^2 + 0.01}(-0.1)\right]$$
$$= \frac{2}{10\pi(n^2 + 0.01)}(e^{-\pi/10} - 1) = \frac{0.0465}{n^2 + 0.01}$$

and

$$b_n = \frac{2}{\pi}\int_0^\pi e^{-t/10}\sin nt\,dt$$
$$= \frac{2}{\pi}\left[\frac{e^{-t/10}}{n^2 + 0.01}(-0.1\sin nt - n\cos nt)\right]_0^\pi \quad (n \text{ odd})$$
$$= \frac{2n}{(n^2 + 0.01)}(e^{-\pi/10} - 1) = \frac{1.461n}{n^2 + 0.01}$$

**2.9-1 (a)**: $T_0 = 4, \omega_0 = \pi/2$. Also $D_0 = 0$ (by inspection).

$$D_n = \frac{1}{2\pi}\int_{-1}^{1} e^{-j(n\pi/2)t}\,dt - \int_1^3 e^{-j(n\pi/2)t}\,dt = \frac{2}{\pi n}\sin\frac{n\pi}{2} \qquad |n| \geq 1$$

(b) $T_0 = 10\pi$, $\omega_0 = 2\pi/10\pi = 1/5$

$$g(t) = \sum_{n=-\infty}^{\infty} D_n e^{j\frac{n}{5}t}, \qquad \text{where} \qquad D_n = \frac{1}{10\pi}\int_{\pi}^{\pi} e^{-j\frac{n}{5}t}\,dt = \frac{j}{2\pi n}\left(-2j\sin\frac{n\pi}{5}\right) = \frac{1}{\pi n}\sin\left(\frac{n\pi}{5}\right)$$

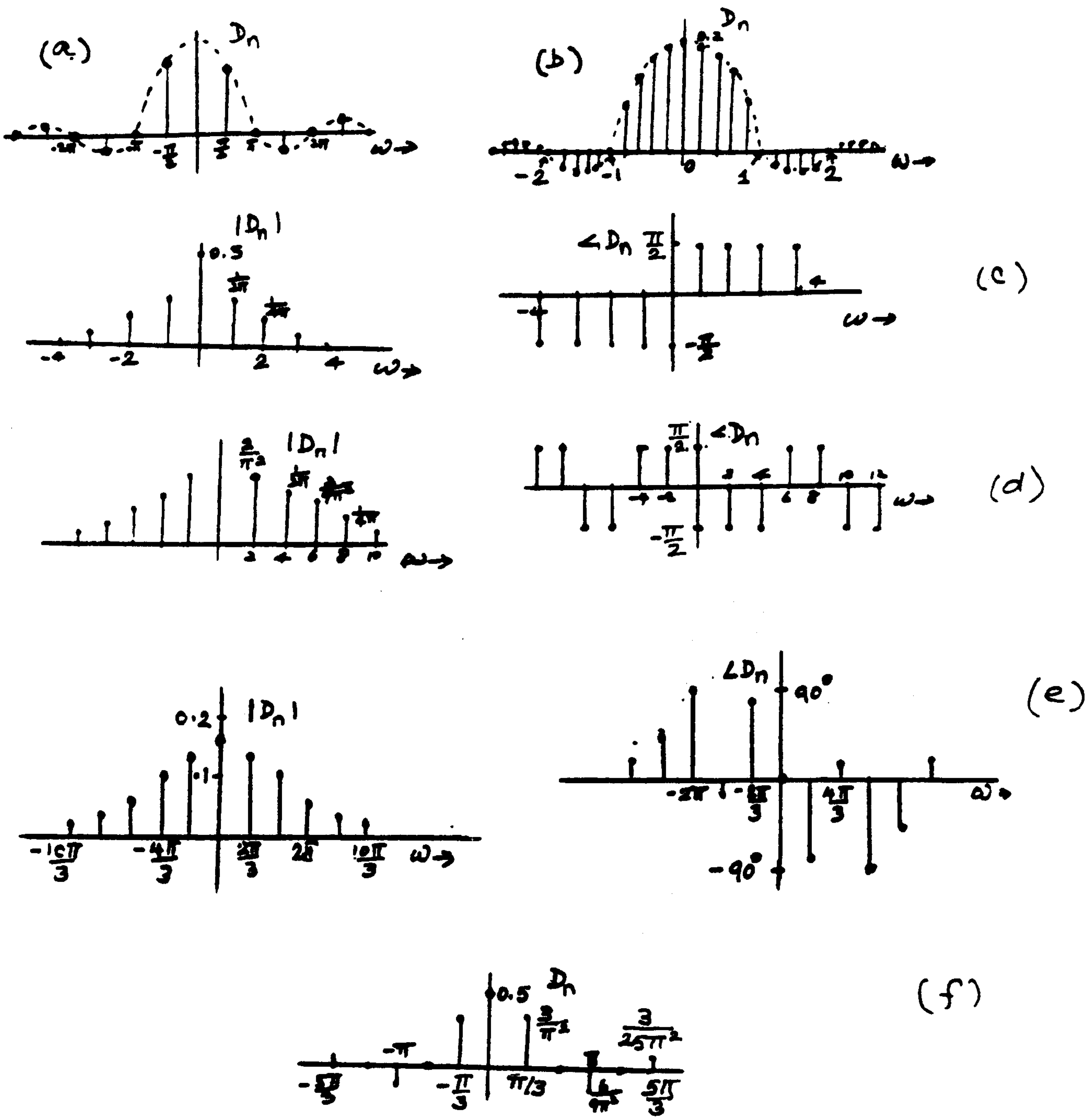



**Fig. S2.9-1**

**(c)**

$$g(t) = D_0 + \sum_{n=-\infty}^{\infty} D_n e^{jnt}, \quad \text{where, by inspection} \quad D_0 = 0.5$$

$$D_n = \frac{1}{2\pi}\int_0^{2\pi} \frac{t}{2\pi} e^{-jnt}\, dt = \frac{j}{2\pi n}, \quad \text{so that} \quad |D_n| = \frac{1}{2\pi n}, \quad \text{and} \quad \angle D_n = \begin{cases} \frac{\pi}{2} & n > 0 \\ \frac{-\pi}{2} & n < 0 \end{cases}$$

**(d)** $T_0 = \pi$, $\omega_0 = 2$ and $D_n = 0$

$$g(t) = \sum_{n=-\infty}^{\infty} D_n e^{j2nt}, \quad \text{where} \quad D_n = \frac{1}{\pi}\int_{-\pi/4}^{\pi/4} \frac{4t}{\pi} e^{-j2nt}\, dt = \frac{-j}{\pi n}\left(\frac{2}{\pi n}\sin\frac{\pi n}{2} - \cos\frac{\pi n}{2}\right)$$

(e) $T_0 = 3, \omega_0 = \frac{2\pi}{3}$.

$$g(t) = \sum_{n=-\infty}^{\infty} D_n e^{j\frac{2\pi n}{3}t}, \qquad \text{where} \qquad D_n = \frac{1}{3}\int_0^1 t\, e^{-j\frac{2\pi n}{3}t}\, dt = \frac{3}{4\pi^2 n^2}\left[e^{-j\frac{2\pi n}{3}}\left(\frac{j2\pi n}{3}+1\right) - 1\right]$$

Therefore

$$|D_n| = \frac{3}{4\pi^2 n^2}\left[\sqrt{2 + \frac{4\pi^2 n^2}{9} - 2\cos\frac{2\pi n}{3} - \frac{4\pi n}{3}\sin\frac{2\pi n}{3}}\right] \quad \text{and} \quad \angle D_n = \tan^{-1}\left(\frac{\frac{2\pi n}{3}\cos\frac{2\pi n}{3} - \sin\frac{2\pi n}{3}}{\cos\frac{2\pi n}{3} + \frac{2\pi n}{3}\sin\frac{2\pi n}{3} - 1}\right)$$

(f) $T_0 = 6, \omega_0 = \pi/3 \; D_0 = 0.5$

$$g(t) = 0.5 + \sum_{n=-\infty}^{\infty} D_n e^{\frac{j\pi n t}{3}}$$

$$D_n = \frac{1}{6}\left[\int_{-2}^{-1}(t+2)e^{-\frac{j\pi n t}{3}}\, dt + \int_{-1}^{1} e^{-\frac{j\pi n t}{3}}\, dt + \int_1^2 (-t+2)e^{-\frac{j\pi n t}{3}}\, dt\right] = \frac{3}{\pi^2 n^2}\left(\cos\frac{n\pi}{3} - \cos\frac{2\pi n}{3}\right)$$

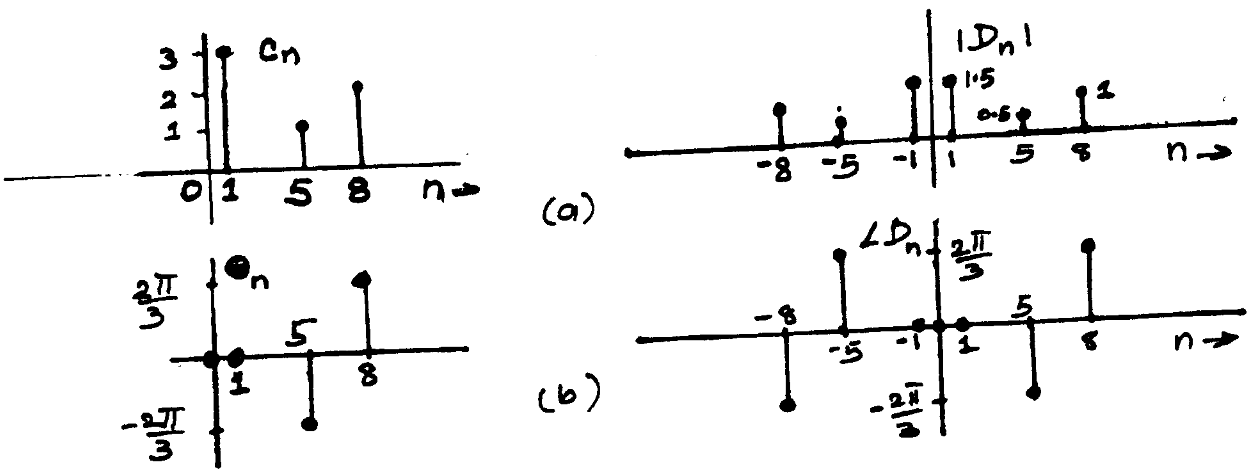



Fig. S2.9-2

**2.9-2**

$$g(t) = 3\cos t + \sin\left(5t - \frac{\pi}{6}\right) - 2\cos\left(8t - \frac{\pi}{3}\right)$$

For a compact trigonometric form, all terms must have cosine form and amplitudes must be positive. For this reason, we rewrite $g(t)$ as

$$\begin{aligned} g(t) &= 3\cos t + \cos\left(5t - \frac{\pi}{6} - \frac{\pi}{2}\right) + 2\cos\left(8t - \frac{\pi}{3} - \pi\right) \\ &= 3\cos t + \cos\left(5t - \frac{2\pi}{3}\right) + 2\cos\left(8t - \frac{4\pi}{3}\right) \end{aligned}$$

Figure S2.9-2a shows amplitude and phase spectra.

(b) By inspection of the trigonometric spectra in Fig. S2.9-2a, we plot the exponential spectra as shown in Fig. S2.9-2b. By inspection of exponential spectra in Fig. S2.9-2a, we obtain

$$\begin{aligned} g(t) &= \frac{3}{2}(e^{jt} + e^{-jt}) + \frac{1}{2}\left[e^{j(5t-\frac{2\pi}{3})} + e^{-j(5t-\frac{2\pi}{3})}\right] + \left[e^{j(8t-\frac{4\pi}{3})} + e^{-j(8t-\frac{4\pi}{3})}\right] \\ &= \frac{3}{2}e^{jt} + \left(\frac{1}{2}e^{-j\frac{2\pi}{3}}\right)e^{j5t} + \left(e^{-j\frac{4\pi}{3}}\right)e^{j8t} + \frac{3}{2}e^{-jt} + \left(\frac{1}{2}e^{j\frac{2\pi}{3}}\right)e^{-j5t} + \left(e^{j\frac{4\pi}{3}}\right)e^{-j8t} \end{aligned}$$

**2.9-3** (a)

$$g(t) = 2 + 2\cos(2t - \pi) + \cos(3t - \frac{\pi}{2})$$
$$= 2 - 2\cos 2t + \sin 3t$$

(b) The exponential spectra are shown in Fig. S2.9-3.

(c) By inspection of exponential spectra

$$g(t) = 2 + \left[e^{(2t-\pi)} + e^{-j(2t-\pi)}\right] + \frac{1}{2}\left[e^{j(3t-\frac{\pi}{2})} + e^{-j(3t-\frac{\pi}{2})}\right]$$
$$= 2 + 2\cos(2t - \pi) + \cos\left(3t - \frac{\pi}{2}\right)$$

(d) Observe that the two expressions (trigonometric and exponential Fourier series) are equivalent.


Fig. S2.9-3

**2.9-4**

$$D_n = \frac{1}{T_0}\left[\int_{-T_0/2}^{T_0/2} f(t)\cos n\omega_0 t\, dt - j\int_{-T_0/2}^{T_0/2} f(t)\sin n\omega_0 t\, dt\right]$$

If $g(t)$ is even, the second term on the right-hand side is zero because its integrand is an odd function of $t$. Hence, $D_n$ is real. In contrast, if $g(t)$ is odd, the first term on the right-hand side is zero because its integrand is an odd function of $t$. Hence, $D_n$ is imaginary.

# Chapter 3

**3.1-1**

$$G(\omega) = \int_{-\infty}^{\infty} g(t)e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} g(t)\cos\omega t\,dt - j\int_{-\infty}^{\infty} g(t)\sin\omega t\,dt$$

If $g(t)$ is an even function of $t$, $g(t)\sin\omega t$ is an odd function of $t$, and the second integral vanishes. Moreover, $g(t)\cos\omega t$ is an even function of $t$, and the first integral is twice the integral over the interval 0 to $\infty$. Thus when $g(t)$ is even

$$G(\omega) = 2\int_0^{\infty} g(t)\cos\omega t\,dt \tag{1}$$

Similar argument shows that when $g(t)$ is odd

$$G(\omega) = -2j\int_0^{\infty} g(t)\sin\omega t\,dt \tag{2}$$

If $g(t)$ is also real (in addition to being even), the integral (1) is real. Moreover from (1)

$$G(-\omega) = 2\int_0^{\infty} g(t)\cos\omega t\,dt = G(\omega)$$

Hence $G(\omega)$ is real and even function of $\omega$. Similar arguments can be used to prove the rest of the properties.

**3.1-2**

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)e^{j\omega t}\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(\omega)|e^{j\theta_g(\omega)}e^{j\omega t}\,d\omega$$

$$= \frac{1}{2\pi}\left[\int_{-\infty}^{\infty} |G(\omega)|\cos[\omega t + \theta_g(\omega)]\,d\omega + j\int_{-\infty}^{\infty} |G(\omega)|\sin[\omega t + \theta_g(\omega)]\,d\omega\right]$$

Since $|G(\omega)|$ is an even function and $\theta_g(\omega)$ is an odd function of $\omega$, the integrand in the second integral is an odd function of $\omega$, and therefore vanishes. Moreover the integrand in the first integral is an even function of $\omega$, and therefore

$$g(t) = \frac{1}{\pi}\int_0^{\infty} |G(\omega)|\cos[\omega t + \theta_g(\omega)]\,d\omega$$

For $g(t) = e^{-at}u(t)$, $G(\omega) = \frac{1}{\omega+a}$. Therefore $|G(\omega)| = 1/\sqrt{\omega^2+a^2}$ and $\theta_g(\omega) = -\tan^{-1}(\frac{\omega}{a})$. Hence

$$e^{-at} = \frac{1}{\pi}\int_0^{\infty} \frac{1}{\sqrt{\omega^2+a^2}}\cos\left[\omega t - \tan^{-1}\left(\frac{\omega}{a}\right)\right]\,d\omega$$

**3.1-3**

$$G(\omega) = \int_{-\infty}^{\infty} g(t)e^{-j\omega t}\,dt$$

Therefore

$$G^*(\omega) = \int_{-\infty}^{\infty} g^*(t)e^{j\omega t}\,dt$$

and

$$G^*(-\omega) = \int_{-\infty}^{\infty} g^*(t)e^{-j\omega t}\,dt$$

**3.1-4 (a)**

$$G(\omega) = \int_0^T e^{-at} e^{-j\omega t}\, dt = \int_0^T e^{-(j\omega + a)t}\, dt = \frac{1 - e^{-(j\omega + a)T}}{j\omega + a}$$

**(b)**

$$G(\omega) = \int_0^T e^{at} e^{-j\omega t}\, dt = \int_0^T e^{-(j\omega - a)}\, dt = \frac{1 - e^{-(j\omega - a)T}}{j\omega - a}$$

**3.1-5 (a)**

$$G(\omega) = \int_0^1 4e^{-j\omega t}\, dt + \int_1^2 2e^{-j\omega t}\, dt = \frac{4 - 2e^{-j\omega} - 2e^{-j2\omega}}{j\omega}$$

**(b)**

$$G(\omega) = \int_{-\tau}^0 -\frac{t}{\tau} e^{-j\omega t}\, dt + \int_0^\tau \frac{t}{\tau} e^{-j\omega t}\, dt = \frac{2}{\tau\omega^2}[\cos\omega\tau + \omega\tau \sin\omega\tau - 1]$$

This result could also be derived by observing that $g(t)$ is an even function. Therefore from the result in Prob. 3.1-1

$$G(\omega) = \frac{2}{\tau} \int_0^\tau t\cos\omega t\, dt = \frac{2}{\tau\omega^2}[\cos\omega\tau + \omega\tau\sin\omega\tau - 1]$$

**3.1-6 (a)**

$$g(t) = \frac{1}{2\pi} \int_{-\omega_0}^{\omega_0} \omega^2 e^{j\omega t}\, d\omega = \frac{1}{2\pi} \frac{e^{j\omega t}}{(jt)^3}[-\omega^2 t^2 - 2j\omega t + 2\Big|_{-\omega_0}^{\omega_0} = \frac{(\omega_0^2 t^2 - 2)\sin\omega_0 t + 2\omega_0 t\cos\omega_0 t}{\pi t^3}$$

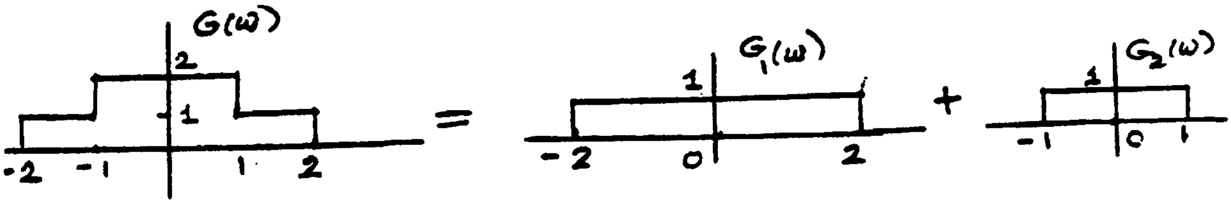



Fig. S3.1-6

(b) The derivation can be simplified by observing that $G(\omega)$ can be expressed as a sum of two gate functions $G_1(\omega)$ and $G_2(\omega)$ as shown in Fig. S3.1-6. Therefore

$$g(t) = \frac{1}{2\pi} \int_{-2}^2 [G_1(\omega) + G_2(\omega)] e^{j\omega t}\, d\omega = \frac{1}{2\pi} \left\{ \int_{-2}^2 e^{j\omega t}\, d\omega + \int_{-1}^1 e^{j\omega t}\, d\omega \right\} = \frac{\sin 2t + \sin t}{\pi t}$$

**3.1-7 (a)**

$$\begin{aligned} g(t) &= \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} \cos\omega\, e^{j\omega t}\, d\omega \\ &= \frac{e^{j\omega t}}{2\pi(1 - t^2)} \{jt\cos\omega + \sin\omega\}_{-\pi/2}^{\pi/2} \\ &= \frac{1}{\pi(1 - t^2)} \cos\left(\frac{\pi t}{2}\right) \end{aligned}$$

**(b)**

$$g(t) = \frac{1}{2\pi} \int_{-\omega_0}^{\omega_0} G(\omega) e^{j\omega t}\, d\omega = \frac{1}{2\pi} \left[ \int_{-\omega_0}^{\omega_0} G(\omega)\cos\omega t\, d\omega + j\int_{-\omega_0}^{\omega_0} G(\omega)\sin\omega t\, d\omega \right]$$

Because $G(\omega)$ is even function, the second integral on the right-hand side vanishes. Also the integrand of the first term is an even function. Therefore

$$g(t) = \frac{1}{\pi}\int_0^{\omega_0} \frac{\omega}{\omega_0}\cos t\omega \, d\omega = \frac{1}{\pi\omega_0}\left[\frac{\cos t\omega + t\omega \sin t\omega}{t^2}\right]_0^{\omega_0}$$

$$= \frac{1}{\pi\omega_0 t^2}[\cos \omega_0 t + \omega_0 t \sin \omega_0 t - 1]$$

**3.1-8** **(a)**

$$g(t) = \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} e^{-j\omega t_0} e^{j\omega t}\, d\omega = \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} e^{j\omega(t-t_0)}\, d\omega$$

$$= \frac{1}{(2\pi)j(t-t_0)} e^{j\omega(t-t_0)}\Big|_{-\omega_0}^{\omega_0} = \frac{\sin \omega_0(t-t_0)}{\pi(t-t_0)} = \frac{\omega_0}{\pi}\text{sinc}[\omega_0(t-t_0)]$$

**(b)**

$$g(t) = \frac{1}{2\pi}\left[\int_{-\omega_0}^{0} je^{j\omega t}\, d\omega + \int_0^{\omega_0} -je^{j\omega t}\, d\omega\right]$$

$$= \frac{1}{2\pi t} e^{j\omega t}\Big|_{-\omega_0}^{0} - \frac{1}{2\pi t} e^{j\omega t}\Big|_0^{\omega_0} = \frac{1-\cos \omega_0 t}{\pi t}$$

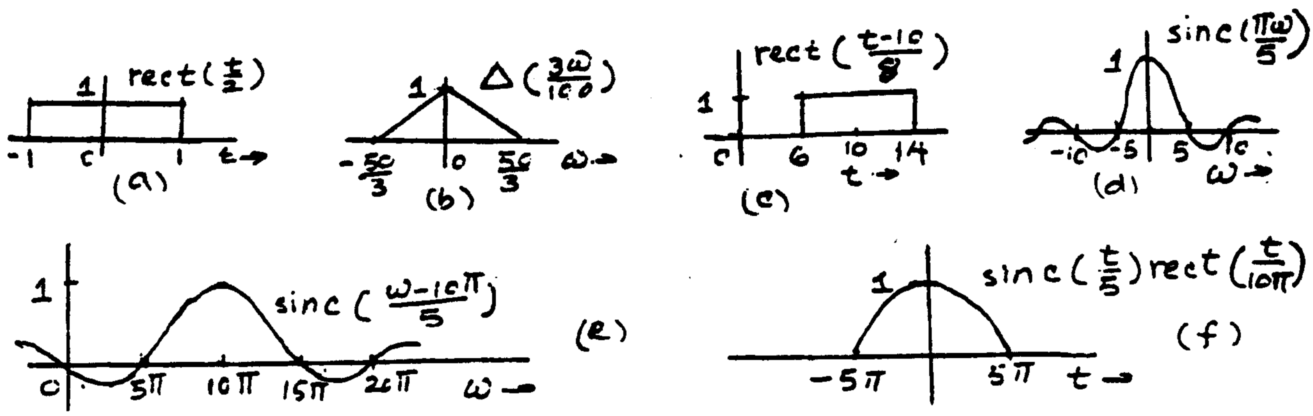



**Fig. S3.2-1**

**3.2-1** Figure S3.2-1 shows the plots of various functions. The function in part **(a)** is a gate function centered at the origin and of width 2. The function in part **(b)** can be expressed as $\Delta\left(\frac{\omega}{100/3}\right)$. This is a triangle pulse centered at the origin and of width 100/3. The function in part **(c)** is a gate function $\text{rect}(\frac{t}{8})$ delayed by 10. In other words it is a gate pulse centered at $t = 10$ and of width 8. The function in part **(d)** is a sinc pulse centered at the origin and the first zero occurring at $\frac{\pi\omega}{5} = \pi$, that is at $\omega = 5$. The function in part **(e)** is a sinc pulse $\text{sinc}(\frac{\omega}{5})$ delayed by $10\pi$. For the sinc pulse $\text{sinc}(\frac{\omega}{5})$, the first zero occurs at $\frac{\omega}{5} = \pi$, that is at $\omega = 5\pi$. Therefore the function is a sinc pulse centered at $\omega = 10\pi$ and its zeros spaced at intervals of $5\pi$ as shown in the fig. S3.2-1e. The function in part **(f)** is a product of a gate pulse (centered at the origin) of width $10\pi$ and a sinc pulse (also centered at the origin) with zeros spaced at intervals of $5\pi$. This results in the sinc pulse truncated beyond the interval $\pm 5\pi$ ($|t| \geq 5\pi$) as shown in Fig. f.

**3.2-2** The function $\text{rect}\,(t-5)$ is centered at $t = 5$, has a width of unity, and its value over this interval is unity. Hence

$$G(\omega) = \int_{4.5}^{5.5} e^{-j\omega t}\, dt = -\frac{1}{j\omega} e^{-j\omega t}\Big|_{4.5}^{5.5} = \frac{1}{j\omega}[e^{-j4.5\omega} - e^{-j5.5\omega}]$$

$$= \frac{e^{-j5\omega}}{j\omega}[e^{j\omega/2} - e^{-j\omega/2}] = \frac{e^{-j5\omega}}{j\omega}\left[2j \sin \frac{\omega}{2}\right]$$

$$= \text{sinc}\left(\frac{\omega}{2}\right) e^{-j5\omega}$$

**3.2-3**

$$g(t) = \frac{1}{2\pi}\int_{10-\pi}^{10+\pi} e^{j\omega t}\,d\omega = \left.\frac{e^{j\omega t}}{2\pi(j\omega)}\right|_{10-\pi}^{10+\pi} = \frac{1}{j2\pi\omega}\left[e^{j(10+\pi)t} - e^{j(10-\pi)t}\right]$$
$$= \frac{e^{j10t}}{j2\pi\omega}[2j\sin\pi t] = \mathrm{sinc}(\pi t)e^{j10t}$$

**3.2-4** Observe that $1 + \mathrm{sgn}(t) = 2u(t)$. Adding pairs 7 and 12 in Table 3.1 and then dividing by 2 yields the desired result.

**3.2-5** Observe that

$$\cos(\omega_0 t + \theta) = \frac{1}{2}\left[e^{j(\omega_0 t+\theta)} + e^{-j(\omega_0 t+\theta)}\right]$$
$$= \frac{1}{2}e^{j\theta}e^{j\omega_0 t} + \frac{1}{2}e^{-j\theta}e^{-j\omega_0 t}$$

Fourier transform of the above equation yields the desired result.

**3.3-1** (a)

$$\underbrace{u(t)}_{g(t)} \Longleftrightarrow \underbrace{\pi\delta(\omega) + \frac{1}{j\omega}}_{G(\omega)}$$

Application of duality property yields

$$\underbrace{\pi\delta(t) + \frac{1}{jt}}_{G(t)} \Longleftrightarrow \underbrace{2\pi u(-\omega)}_{2\pi g(-\omega)}$$

or

$$\frac{1}{2}\left[\delta(t) + \frac{1}{j\pi t}\right] \Longleftrightarrow u(-\omega)$$

Application of Eq. (3.28) yields

$$\frac{1}{2}\left[\delta(-t) - \frac{1}{j\pi t}\right] \Longleftrightarrow u(\omega)$$

But $\delta(t)$ is an even function. that is $\delta(-t) = \delta(t)$. and

$$\frac{1}{2}[\delta(t) + \frac{j}{\pi t}] \Longleftrightarrow u(\omega)$$

(b)

$$\underbrace{\cos\omega_0 t}_{g(t)} \Longleftrightarrow \underbrace{\pi[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)]}_{G(\omega)}$$

Application of duality property yields

$$\underbrace{\pi[\delta(t+\omega_0) + \delta(t-\omega_0)]}_{G(t)} \Longleftrightarrow \underbrace{2\pi\cos(-\omega_0\omega)}_{2\pi g(-\omega)} = 2\pi\cos(\omega_0\omega)$$

Setting $\omega_0 = T$ yields

$$\delta(t+T) + \delta(t-T) \Longleftrightarrow 2\cos T\omega$$

(c)

$$\underbrace{\sin\omega_0 t}_{g(t)} \Longleftrightarrow \underbrace{j\pi[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)]}_{G(\omega)}$$

Application of duality property yields

$$\underbrace{j\pi[\delta(t+\omega_0) - \delta(t-\omega_0)]}_{G(t)} \Longleftrightarrow \underbrace{2\pi\sin(-\omega_0\omega)}_{2\pi g(-\omega)} = -2\pi\sin(\omega_0\omega)$$

Setting $\omega_0 = T$ yields

$$\delta(t+T) - \delta(t-T) \Longleftrightarrow 2j \sin T\omega$$

**3.3-2** **Fig. (b)** $g_1(t) = g(-t)$ and

$$G_1(\omega) = G(-\omega) = \frac{1}{\omega^2}[e^{-j\omega} + j\omega e^{-j\omega} - 1]$$

**Fig. (c)** $g_2(t) = g(t-1) + g_1(t-1)$. Therefore

$$\begin{aligned} G_3(\omega) &= [G(\omega) + G_1(\omega)]e^{-j\omega} = [G(\omega) + G(-\omega)]e^{-j\omega} \\ &= \frac{2e^{-j\omega}}{\omega^2}(\cos\omega + \omega\sin\omega - 1) \end{aligned}$$

**Fig. (d)** $g_3(t) = g(t-1) + g_1(t+1)$

$$\begin{aligned} G_4(\omega) &= G(\omega)e^{-j\omega} + G(-\omega)e^{j\omega} \\ &= \frac{1}{\omega^2}[2 - 2\cos\omega] = \frac{4}{\omega^2}\sin^2\frac{\omega}{2} = \text{sinc}^2\left(\frac{\omega}{2}\right) \end{aligned}$$

**Fig. (e)** $g_4(t) = g(t - \frac{1}{2}) + g_1(t + \frac{1}{2})$, and

$$\begin{aligned} G_4(\omega) &= G(\omega)e^{-j\omega/2} + G_1(\omega)e^{j\omega/2} \\ &= \frac{e^{-j\omega/2}}{\omega^2}[e^{j\omega} - j\omega e^{j\omega} - 1] + \frac{e^{j\omega/2}}{\omega^2}[e^{-j\omega} + j\omega e^{-j\omega} - 1] \\ &= \frac{1}{\omega^2}[2\omega\sin\frac{\omega}{2}] = \text{sinc}\left(\frac{\omega}{2}\right) \end{aligned}$$

**Fig. (f)** $g_5(t)$ can be obtained in three steps: (i) time-expanding $g(t)$ by a factor 2 (ii) then delaying it by 2 seconds. (iii) and multiplying it by 1.5 [we may interchange the sequence for steps (i) and (ii)]. The first step (time-expansion by a factor 2) yields

$$f\left(\frac{t}{2}\right) \Longleftrightarrow 2G(2\omega) = \frac{1}{2\omega^2}(e^{j2\omega} - j2\omega e^{j2\omega} - 1)$$

Second step of time delay of 2 secs. yields

$$f\left(\frac{t-2}{2}\right) \Longleftrightarrow \frac{1}{2\omega^2}(e^{j2\omega} - j2\omega e^{j2\omega} - 1)e^{-j2\omega} = \frac{1}{2\omega^2}(1 - j2\omega - e^{-j2\omega})$$

The third step of multiplying the resulting signal by 1.5 yields

$$g_5(t) = 1.5f\left(\frac{t-2}{2}\right) \Longleftrightarrow \frac{3}{4\omega^2}(1 - j2\omega - e^{-j2\omega})$$

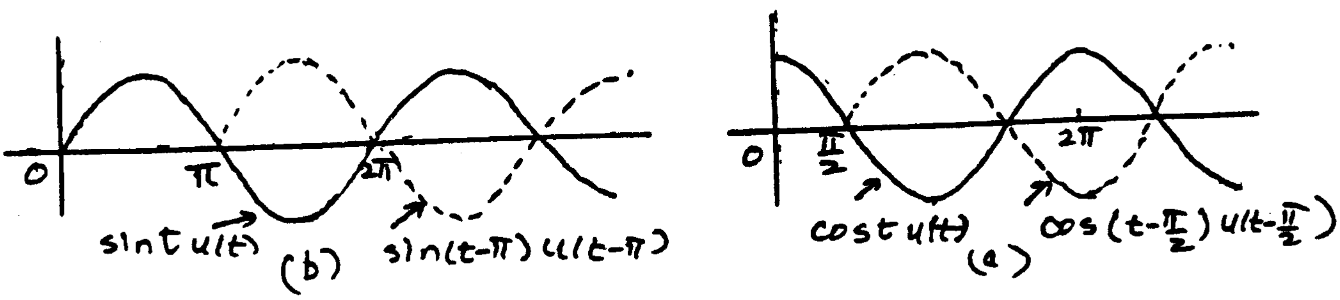



**Fig. S3.3-3**

**3.3-3** **(a)**

$$g(t) = \text{rect}\left(\frac{t+T/2}{T}\right) - \text{rect}\left(\frac{t-T/2}{T}\right)$$

$$\text{rect}\left(\frac{t}{T}\right) \Longleftrightarrow T\text{sinc}\left(\frac{\omega T}{2}\right)$$

$$\text{rect}\left(\frac{t \pm T/2}{T}\right) \Longleftrightarrow T\text{sinc}\left(\frac{\omega T}{2}\right)e^{\pm j\omega T/2}$$

and

$$G(\omega) = T\text{sinc}\left(\frac{\omega T}{2}\right)[e^{j\omega T/2} - e^{-j\omega T/2}]$$
$$= 2jT\text{sinc}\left(\frac{\omega T}{2}\right)\sin\frac{\omega T}{2}$$
$$= \frac{j4}{\omega}\sin^2\left(\frac{\omega T}{2}\right)$$

**(b)** From Fig. S3.3-3b we verify that

$$g(t) = \sin t\, u(t) + \sin(t-\pi)u(t-\pi)$$

Note that $\sin(t-\pi)u(t-\pi)$ is $\sin t\, u(t)$ delayed by $\pi$. Now, $\sin t\, u(t) \Longleftrightarrow \frac{\pi}{2j}[\delta(\omega-1) - \delta(\omega+1)] + \frac{1}{1-\omega^2}$ and

$$\sin(t-\pi)u(t-\pi) \Longleftrightarrow \{\frac{\pi}{2j}[\delta(\omega-1) - \delta(\omega+1)] + \frac{1}{1-\omega^2}\}e^{-j\pi\omega}$$

Therefore

$$G(\omega) = \{\frac{\pi}{2j}[\delta(\omega-1) - \delta(\omega+1)] + \frac{1}{1-\omega^2}\}(1 + e^{-j\pi\omega})$$

Recall that $g(x)\delta(x-x_0) = g(x_0)\delta(x-x_0)$. Therefore $\delta(\omega \pm 1)(1 + e^{-j\pi\omega}) = 0$. and

$$G(\omega) = \frac{1}{1-\omega^2}(1 + e^{-j\pi\omega})$$

**(c)** From Fig. S3.3-3c we verify that

$$g(t) = \cos t\left[u(t) - u\left(t - \frac{\pi}{2}\right)\right] = \cos t\, u(t) - \cos t\, u\left(t - \frac{\pi}{2}\right)$$

But $\sin(t - \frac{\pi}{2}) = -\cos t$. Therefore

$$g(t) = \cos t\, u(t) + \sin\left(t - \frac{\pi}{2}\right)u\left(t - \frac{\pi}{2}\right)$$
$$G(\omega) = \frac{\pi}{2}[\delta(\omega-1) + \delta(\omega+1)] + \frac{j\omega}{1-\omega^2} + \left\{\frac{\pi}{2j}[\delta(\omega-1) - \delta(\omega+1)] + \frac{1}{1-\omega^2}\right\}e^{-j\pi\omega/2}$$

Also because $g(x)\delta(x-x_0) = g(x_0)\delta(x-x_0)$.

$$\delta(\omega \pm 1)e^{-j\pi\omega/2} = \delta(\omega \pm 1)e^{\pm j\pi/2} = \pm j\delta(\omega \pm 1)$$

Therefore

$$G(\omega) = \frac{j\omega}{1-\omega^2} + \frac{e^{-j\pi\omega/2}}{1-\omega^2} = \frac{1}{1-\omega^2}[j\omega + e^{-j\pi\omega/2}]$$

**(d)**

$$g(t) = e^{-at}[u(t) - u(t-T)] = e^{-at}u(t) - e^{-at}u(t-T)$$
$$= e^{-at}u(t) - e^{-aT}e^{-a(t-T)}u(t-T)$$
$$G(\omega) = \frac{1}{j\omega + a} - \frac{e^{-aT}}{j\omega + a}e^{-j\omega T} = \frac{1}{j\omega + a}[1 - e^{-(a+j\omega)T}]$$

**3.3-4** From time-shifting property

$$g(t \pm T) \Longleftrightarrow G(\omega)e^{\pm j\omega T}$$

Therefore

$$g(t+T) + g(t-T) \Longleftrightarrow G(\omega)e^{j\omega T} + G(\omega)e^{-j\omega T} = 2G(\omega)\cos\omega T$$

We can use this result to derive transforms of signals in Fig. P3.3-4.
**(a)** Here $g(t)$ is a gate pulse as shown in Fig. S3.3-4a.

$$g(t) = \text{rect}\left(\frac{t}{2}\right) \Longleftrightarrow 2\,\text{sinc}(\omega)$$

Also $T = 3$. The signal in Fig. P3.3-4a is $g(t+3) + g(t-3)$, and

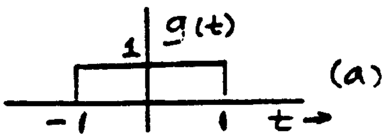
$$g(t+3) + g(t-3) \Longleftrightarrow 4\,\text{sinc}(\omega)\cos 3\omega$$

**(b)** Here $g(t)$ is a triangular pulse shown in Fig. S3.3-4b. From the Table 3.1 (pair 19)

$$g(t) = \Delta\left(\frac{t}{2}\right) \Longleftrightarrow \text{sinc}^2\left(\frac{\omega}{2}\right)$$

Also $T = 3$. The signal in Fig. P3.3-4b is $g(t+3) + g(t-3)$, and

$$g(t+3) + g(t-3) \Longleftrightarrow 2\,\text{sinc}^2\left(\frac{\omega}{2}\right)\cos 3\omega$$



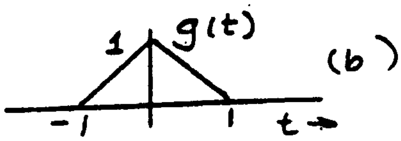




**Fig. S3.3-4**

**3.3-5** Frequency-shifting property states that

$$g(t)e^{\pm j\omega_0 t} \Longleftrightarrow G(\omega \mp \omega_0)$$

Therefore

$$g(t)\sin\omega_0 t = \frac{1}{2j}[g(t)e^{j\omega_0 t} + g(t)e^{-j\omega_0 t}] = \frac{1}{2j}[G(\omega - \omega_0) + G(\omega - \omega_0)]$$

Time-shifting property states that

$$g(t \pm T) \Longleftrightarrow G(\omega)e^{\pm j\omega T}$$

Therefore

$$g(t+T) - g(t-T) \Longleftrightarrow G(\omega)e^{j\omega T} - G(\omega)e^{-j\omega T} = 2jG(\omega)\sin\omega T$$

and

$$\frac{1}{2j}[g(t+T) - g(t-T)] \Longleftrightarrow G(\omega)\sin T\omega$$

The signal in Fig. P3.3-5 is $g(t+3) - g(t-3)$ where

$$g(t) = \text{rect}\left(\frac{t}{2}\right) \Longleftrightarrow 2\text{sinc}(\omega)$$

Therefore

$$g(t+3) - g(t-3) \Longleftrightarrow 2j[2\text{sinc}(\omega)\sin 3\omega] = 4j\,\text{sinc}(\omega)\sin 3\omega$$

**3.3-6** **Fig. (a)** The signal $g(t)$ in this case is a triangle pulse $\Delta(\frac{t}{2\pi})$ (Fig. S3.3-6) multiplied by $\cos 10t$.

$$g(t) = \Delta\left(\frac{t}{2\pi}\right)\cos 10t$$

Also from Table 3.1 (pair 19) $\Delta(\frac{t}{2\pi}) \Longleftrightarrow \pi\,\text{sinc}^2(\frac{\pi\omega}{2})$ From the modulation property (3.35), it follows that

$$g(t) = \Delta\left(\frac{t}{2\pi}\right)\cos 10t \Longleftrightarrow \frac{\pi}{2}\left\{\text{sinc}^2\left[\frac{\pi(\omega-10)}{2}\right] + \text{sinc}^2\left[\frac{\pi(\omega+10)}{2}\right]\right\}$$

The Fourier transform in this case is a real function and we need only the amplitude spectrum in this case as shown in Fig. S3.3-6a.

**Fig. (b)** The signal $g(t)$ here is the same as the signal in Fig. (a) delayed by $2\pi$. From time shifting property, its Fourier transform is the same as in part (a) multiplied by $e^{-j\omega(2\pi)}$. Therefore

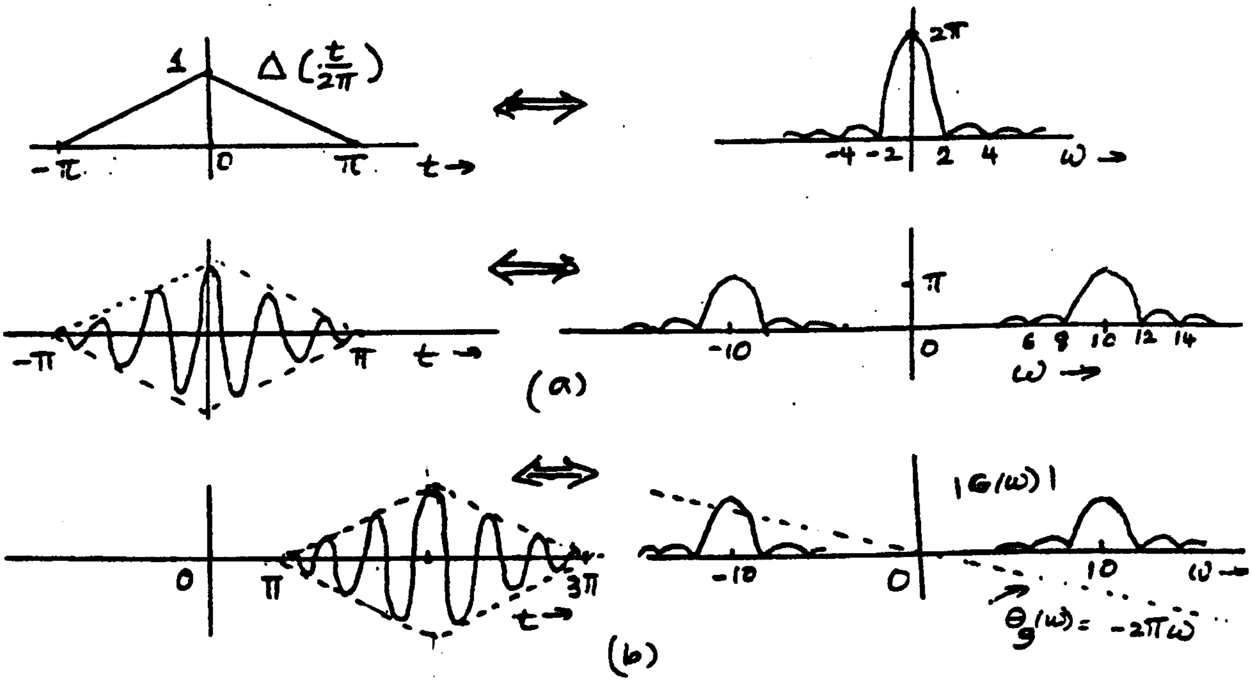



**Fig. S3.3-6**

$$G(\omega) = \frac{\pi}{2}\left\{\operatorname{sinc}^2\left[\frac{\pi(\omega-10)}{2}\right] + \operatorname{sinc}^2\left[\frac{\pi(\omega+10)}{2}\right]\right\}e^{-j2\pi\omega}$$

The Fourier transform in this case is the same as that in part (a) multiplied by $e^{-j2\pi\omega}$. This multiplying factor represents a linear phase spectrum $-2\pi\omega$. Thus we have an amplitude spectrum [same as in part (a)] as well as a linear phase spectrum $\angle G(\omega) = -2\pi\omega$ as shown in Fig. S3.3-6b. the amplitude spectrum in this case as shown in Fig. S3.3-6b.

Note: In the above solution. we first multiplied the triangle pulse $\Delta(\frac{t}{2\pi})$ by $\cos 10t$ and then delayed the result by $2\pi$. This means the signal in Fig. (b) is expressed as $\Delta(\frac{t-2\pi}{2\pi})\cos 10(t-2\pi)$.

We could have interchanged the operation in this particular case. that is, the triangle pulse $\Delta(\frac{t}{2\pi})$ is first delayed by $2\pi$ and then the result is multiplied by $\cos 10t$. In this alternate procedure. the signal in Fig. (b) is expressed as $\Delta(\frac{t-2\pi}{2\pi})\cos 10t$.

This interchange of operation is permissible here only because the sinusoid $\cos 10t$ executes integral number of cycles in the interval $2\pi$. Because of this both the expressions are equivalent since $\cos 10(t-2\pi) = \cos 10t$.

**Fig.** (c) In this case the signal is identical to that in Fig. b. except that the basic pulse is $\operatorname{rect}(\frac{t}{2\pi})$ instead of a triangle pulse $\Delta(\frac{t}{2\pi})$. Now

$$\operatorname{rect}\left(\frac{t}{2\pi}\right) \Longleftrightarrow 2\pi\operatorname{sinc}(\pi\omega)$$

Using the same argument as for part (b). we obtain

$$G(\omega) = \pi\{\operatorname{sinc}[\pi(\omega+10)] + \operatorname{sinc}[\pi(\omega-10)]\}e^{-j2\pi\omega}$$

**3.3-7** **(a)**

$$G(\omega) = \operatorname{rect}\left(\frac{\omega-4}{2}\right) + \operatorname{rect}\left(\frac{\omega+4}{2}\right)$$

Also

$$\frac{1}{\pi}\operatorname{sinc}(t) \Longleftrightarrow \operatorname{rect}\left(\frac{\omega}{2}\right)$$

Therefore

$$g(t) = \frac{2}{\pi}\operatorname{sinc}(t)\cos 4t$$

**(b)**

$$G(\omega) = \Delta\left(\frac{\omega+4}{4}\right) + \Delta\left(\frac{\omega-4}{4}\right)$$

Also

$$\frac{1}{\pi}\text{sinc}^2(t) \Longleftrightarrow \Delta\left(\frac{\omega}{4}\right)$$

Therefore

$$g(t) = \frac{2}{\pi}\text{sinc}^2(t)\cos 4t$$

**3.3-8** From the frequency convolution property, we obtain

$$g^2(t) \Longleftrightarrow \frac{1}{2\pi}G(\omega) * G(\omega)$$

The width property of convolution states that if $c_1(x) * c_2(x) = y(x)$, then the width of $y(x)$ is equal to the sum of the widths of $c_1(x)$ and $c_2(x)$. Hence, the width of $G(\omega) * G(\omega)$ is twice the width of $G(\omega)$. Repeated application of this argument shows that the bandwidth of $g^n(t)$ is $nB$ Hz ($n$ times the bandwidth of $g(t)$).

**3.3-9** (a)

$$G(\omega) = \int_{-T}^{0} e^{-j\omega t}\,dt - \int_{0}^{T} e^{-j\omega t}\,dt = -\frac{2}{j\omega}[1-\cos\omega T] = \frac{j4}{\omega}\sin^2\left(\frac{\omega T}{2}\right)$$

(b)

$$g(t) = \text{rect}\left(\frac{t+T/2}{T}\right) - \text{rect}\left(\frac{t-T/2}{T}\right)$$

$$\text{rect}\left(\frac{t}{T}\right) \Longleftrightarrow T\text{sinc}\left(\frac{\omega T}{2}\right)$$
$$\text{rect}\left(\frac{t \pm T/2}{T}\right) \Longleftrightarrow T\text{sinc}\left(\frac{\omega T}{2}\right) e^{\pm j\omega T/2}$$

and

$$\begin{aligned}
G(\omega) &= T\text{sinc}\left(\frac{\omega T}{2}\right)[e^{j\omega T/2} - e^{-j\omega T/2}] \\
&= 2jT\text{sinc}\left(\frac{\omega T}{2}\right)\sin\frac{\omega T}{2} \\
&= \frac{j4}{\omega}\sin^2\left(\frac{\omega T}{2}\right)
\end{aligned}$$

(c)

$$\frac{df}{dt} = \delta(t+T) - 2\delta(t) + \delta(t-T)$$

The Fourier transform of this equation yields

$$j\omega G(\omega) = e^{j\omega T} - 2 + e^{-j\omega T} = -2[1-\cos\omega T] = -4\sin^2\left(\frac{\omega T}{2}\right)$$

Therefore

$$G(\omega) = \frac{j4}{\omega}\sin^2\left(\frac{\omega T}{2}\right)$$

**3.3-10**

A basic demodulator is shown in Fig. S3.3-10a. The product of the modulated signal $g(t)\cos\omega_0 t$ with $2\cos\omega_0 t$ yields

$$g(t)\cos\omega_0 t \times 2\cos\omega_0 t = 2g(t)\cos^2\omega_0 t = g(t)[1+\cos 2\omega_0 t] = g(t) + g(t)\cos 2\omega_0 t$$

The product contains the desired $g(t)$ (whose spectrum is centered at $\omega = 0$) and the unwanted signal $g(t)\cos 2\omega_0 t$ with spectrum $\frac{1}{2}[G(\omega+2\omega_0]+G(\omega-2\omega_0]$. which is centered at $\pm 2\omega_0$. The two spectra are nonoverlapping because

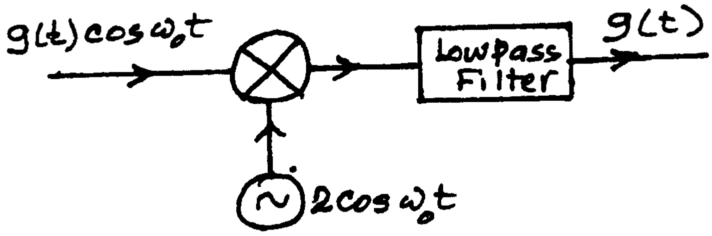



**Fig. S3.3-10**

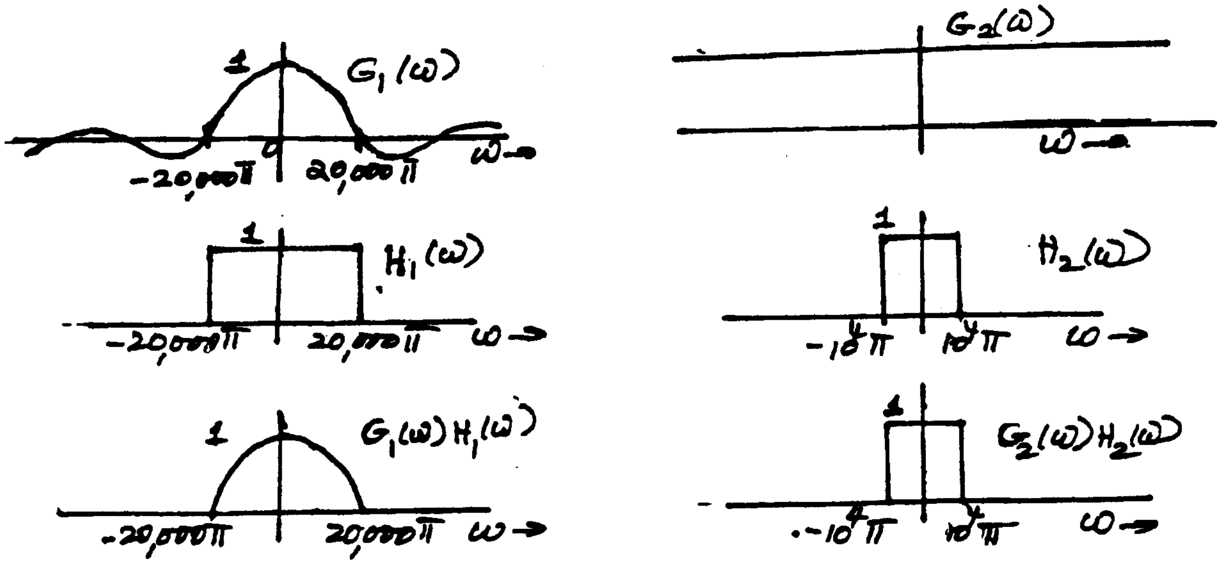



**Fig. S3.4-1**

$W < \omega_0$ (See Fig. S3.3-10b). We can suppress the unwanted signal by passing the product through a lowpass filter as shown in Fig. S3.3-10a.

**3.4-1**

$$G_1(\omega) = \text{sinc}\left(\frac{\omega}{20000}\right) \qquad \text{and} \qquad G_2(\omega) = 1$$

Figure S3.4-1 shows $G_1(\omega)$, $G_2(\omega)$, $H_1(\omega)$ and $H_2(\omega)$. Now

$$Y_1(\omega) = G_1(\omega)H_1(\omega)$$
$$Y_2(\omega) = G_2(\omega)H_2(\omega)$$

The spectra $Y_1(\omega)$ and $Y_2(\omega)$ are also shown in Fig. S3.4-1. Because $y(t) = y_1(t)y_2(t)$, the frequency convolution property yields $Y(\omega) = Y_1(\omega) * Y_2(\omega)$. From the width property of convolution, it follows that the bandwidth of $Y(\omega)$ is the sum of bandwidths of $Y_1(\omega)$ and $Y_2(\omega)$. Because the bandwidths of $Y_1(\omega)$ and $Y_2(\omega)$ are 10 kHz, 5 kHz, respectively, the bandwidth of $Y(\omega)$ is 15 kHz.

**3.5-1**

$$H(\omega) = e^{-k\omega^2} e^{-j\omega t_0}$$

Using pair 22 (Table 3.1) and time-shifting property, we get

$$h(t) = \frac{1}{\sqrt{4\pi k}} e^{-(t-t_0)^2/4k}$$

This is noncausal. Hence the filter is unrealizable. Also

$$\int_{-\infty}^{\infty} \frac{|\ln|H(\omega)||}{\omega^2+1}\,d\omega = \int_{-\infty}^{\infty} \frac{k\omega^2}{\omega^2+1}\,d\omega = \infty$$

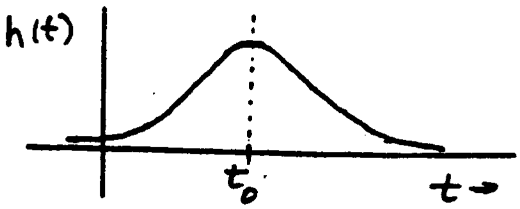



**Figure S3.5-1**

Hence the filter is noncausal and therefore unrealizable. Since $h(t)$ is a Gaussian function delayed by $t_0$. it looks as shown in the adjacent figure. Choosing $t_0 = 3\sqrt{2k}$, $h(0) = e^{-4.5} = 0.011$ or 1.1% of its peak value. Hence $t_0 = 3\sqrt{2k}$ is a reasonable choice to make the filter approximately realizable.

**3.5-2**

$$H(\omega) = \frac{2 \times 10^5}{\omega^2 + 10^{10}} e^{-j\omega t_0}$$

From pair 3, Table 3.1 and time-shifting property, we get

$$h(t) = e^{-10^5|t-t_0|}$$

The impulse response is noncausal, and the filter is unrealizable.

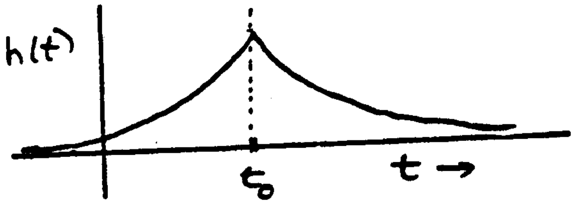



**Figure S3.5-2**

The exponential delays to 1.8% at 4 times constants. Hence $t_0 = 4/a = 4 \times 10^{-5} = 40\mu s$ is a reasonable choice to make this filter approximately realizable.

**3.5-3** From the results in Example 3.16

$$|H(\omega)| = \frac{a}{\sqrt{\omega^2 + a^2}} \qquad a = \frac{1}{RC} = 10^6$$

Also $H(0) = 1$. Hence if $\omega_1$ is the frequency where the amplitude response drops to 0.95, then

$$|H(\omega_1)| = \frac{10^6}{\sqrt{\omega_1^2 + 10^{12}}} = 0.95 \Longrightarrow \omega_1 = 328,684$$

Moreover, the time delay is given by (see Example 3.16)

$$t_d(\omega) = \frac{a}{\omega^2 + a^2} \Longrightarrow t_d(0) = \frac{1}{a} = 10^{-6}$$

If $\omega_2$ is the frequency where the time delay drops to 0.98% of its value at $\omega = 0$, then

$$t_d(\omega_2) = \frac{10^6}{\omega_2^2 + 10^{12}} = 0.98 \times 10^{-6} \Longrightarrow \omega_2 = 142,857$$

We select the smaller of $\omega_1$ and $\omega_2$, that is $\omega = 142,857$, where both the specifications are satisfied. This yields a frequency of 22,736.4 Hz.

**3.5-4** There is a typo in this example. The time delay tolerance should be 4% instead of 1%. The band of $\Delta\omega = 2000$ centered at $\omega = 10^5$ represents the frequency range from $0.99 \times 10^5$ to $1.01 \times 10^5$. Let us consider the gains and the time delays at the band edges. From Example 3.16

$$|H(\omega)| = \frac{a}{\sqrt{\omega^2 + a^2}} \qquad t_d(\omega) = \frac{a}{\omega^2 + a^2} \qquad a = 10^3$$

At the edges of the band

$$|H(0.99 \times 10^5)| = \frac{10^3}{\sqrt{(0.99\times 10^5)^2 + 10^6}} = 10.1 \times 10^{-3}, \quad \text{and} \quad |H(1.01 \times 10^5)| = \frac{10^3}{\sqrt{(1.01\times 10^5)^2+10^6}} = 9.901 \times 10^{-3}$$

The gain variation over the band is only 1.99%. Similarly, we find the time delays at the band edges as

$$t_d(0.99 \times 10^5) = \frac{10^3}{(0.99\times 10^5)^2+10^6} = \frac{1}{(99)^2 10^3}. \qquad \text{and} \qquad t_d(1.01 \times 10^5) = \frac{10^3}{(1.01\times 10^5)^2+10^6} = \frac{1}{(101)^2 \times 10^3}$$

The time delay variation over the band is 4%. Hence, the transmission may be considered distortionless. The signal is transmitted with a gain and time delay at the center of the band, that is at $\omega = 10^5$. We also find $|H(10^5) \approx 0.01$ and $t_d(10^5) \approx 1-^{-7}$. Hence, if $g(t)$ is the input, the corresponding output is

$$y(t) = 0.01\, g(t - 10^{-7})$$

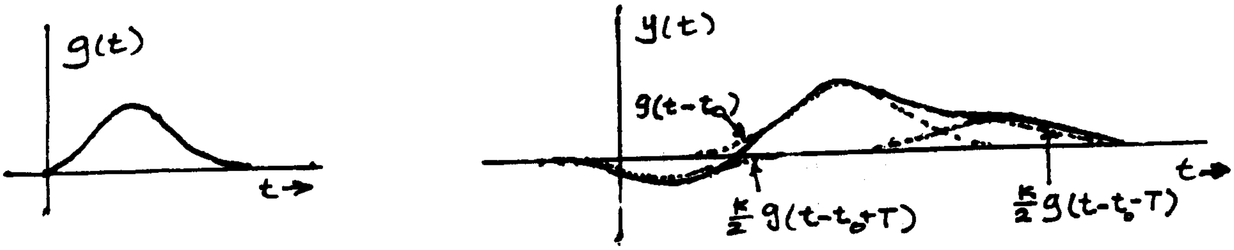



**Fig. S3.6-1**

**3.6-1**

$$\begin{aligned} Y(\omega) &= G(\omega)\, \text{rect}\left(\frac{\omega}{4\pi B}\right) e^{-j(\omega t_0 + k \sin \omega T)} \\ &\approx G(\omega)\, \text{rect}\left(\frac{\omega}{4\pi B}\right) [1 - jk \sin \omega T] e^{-j\omega t_0 t} \end{aligned}$$

This follows from the fact that $e^x \approx 1 + x$ when $x \ll 1$. Moreover, $G(\omega)\text{rect}\left(\frac{\omega}{4\pi B}\right) = G(\omega)$ because $G(\omega)$ is bandlimited to $B$ Hz. Hence

$$Y(\omega) = G(\omega) e^{-j\omega t_0} - jkG(\omega) \sin \omega T\, e^{-j\omega t_0}$$

Moreover, we can show that (see Prob. 3.3-5)

$$\frac{1}{2j}[g(t+T) - g(t-T)] \Longleftrightarrow G(\omega) \sin \omega T$$

Hence

$$y(t) = g(t - t_0) + \frac{k}{2}[g(t - t_0 - T) - g(t - t_0 + T)]$$

Figure S3.6-1 shows $g(t)$ and $y(t)$.

**3.6-2** Recall that the transfer function of an ideal time delay of $T$ seconds is $e^{-j\omega T}$. Hence, the transfer function of the equalizer in Fig. P3.6-2 is

$$H_{eq}(\omega) = a_0 + a_1 e^{-j\omega \Delta t} + a_2 e^{-j2\omega \Delta t} + \cdots + a_n e^{-jn\omega \Delta t}$$

Ideally, we require the equalizer to have

$$\begin{aligned} [H_{eq}(\omega)]_{\text{desired}} &= \frac{1}{1 + \alpha e^{-j\omega \Delta t}} \\ &= 1 - \alpha e^{-j\omega \Delta t} + \alpha^2 e^{-j2\omega \Delta t} - \alpha^3 e^{-j3\omega \Delta t} + \cdots + \cdots \end{aligned}$$

The equalizer in Fig. P3.6-2 approximates this expression if we select $a_0 = 1$, $a_1 = -\alpha$, $a_2 = \alpha^2, \cdots, a_n = (-1)^n \alpha^n$.

**3.7-1**

$$E_g = \int_{-\infty}^{\infty} g^2(t)\,dt = \frac{1}{2\pi\sigma^2}\int_{-\infty}^{\infty} e^{-t^2/\sigma^2}\,dt$$

Letting $\frac{t}{\sigma} = \frac{x}{\sqrt{2}}$ and consequently $dt = \frac{\sigma}{\sqrt{2}}dx$

$$E_g = \frac{1}{2\pi\sigma^2}\frac{\sigma}{\sqrt{2}}\int_{-\infty}^{\infty} e^{-x^2/2}\,dx = \frac{\sqrt{2\pi}}{2\sqrt{2}\pi\sigma} = \frac{1}{2\sigma\sqrt{\pi}}$$

Also from pair 22 (Table 3.1)

$$G(\omega) = e^{-\sigma^2\omega^2/2}$$

$$E_g = \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(\omega)|^2\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\sigma^2\omega^2}\,d\omega$$

Letting $\sigma\omega = \frac{x}{\sqrt{2}}$ and consequently $d\omega = \frac{1}{\sigma\sqrt{2}}dx$

$$E_g = \frac{1}{2\pi}\frac{1}{\sigma\sqrt{2}}\int_{-\infty}^{\infty} e^{-x^2/2}\,dx = \frac{\sqrt{2\pi}}{2\pi\sigma\sqrt{2}} = \frac{1}{2\sigma\sqrt{\pi}}$$

**3.7-2** Consider a signal

$$g(t) = \text{sinc}(kt) \quad \text{and} \quad G(\omega) = \frac{\pi}{k}\text{rect}\left(\frac{\omega}{2k}\right)$$

$$\begin{aligned} E_g = \int_{-\infty}^{\infty} \text{sinc}^2(kt)\,dt &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\pi^2}{k^2}\left[\text{rect}\left(\frac{\omega}{2k}\right)\right]^2 d\omega \\ &= \frac{\pi}{2k^2}\int_{-k}^{k} d\omega = \frac{\pi}{k} \end{aligned}$$

**3.7-3** Recall that

$$g_2(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G_2(\omega)e^{j\omega t}\,d\omega \quad \text{and} \quad \int_{-\infty}^{\infty} g_1(t)e^{j\omega t}\,dt = G_1(-\omega)$$

Therefore

$$\begin{aligned} \int_{-\infty}^{\infty} g_1(t)g_2(t)\,dt &= \frac{1}{2\pi}\int_{-\infty}^{\infty} g_1(t)\left[\int_{-\infty}^{\infty} G_2(\omega)e^{j\omega t}\,d\omega\right] dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} G_2(\omega)\left[\int_{-\infty}^{\infty} g_1(t)e^{j\omega t}\,dt\right] d\omega = \frac{1}{2\pi}\int G_1(-\omega)G_2(\omega)\,d\omega \end{aligned}$$

Interchanging the roles of $g_1(t)$ and $g_2(t)$ in the above development, we can show that

$$\int_{-\infty}^{\infty} g_1(t)g_2(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} G_1(\omega)G_2(-\omega)\,d\omega$$

**3.7-4** In the generalized Parseval's theorem in Prob. 3.7-3, if we identify $g_1(t) = \text{sinc}\,(2\pi Bt - m\pi)$ and $g_2(t) = \text{sinc}\,(2\pi Bt - n\pi)$, then

$$G_1(\omega) = \frac{1}{2B}\,\text{rect}\left(\frac{\omega}{4\pi B}\right)e^{\frac{jm\omega}{2B}}, \quad \text{and} \quad G_2(\omega) = \frac{1}{2B}\,\text{rect}\left(\frac{\omega}{4\pi B}\right)e^{\frac{jn\omega}{2B}}$$

Therefore

$$\int_{-\infty}^{\infty} g_1(t)g_2(t)\,dt = \frac{1}{2\pi}\frac{1}{(2B)^2}\int_{-\infty}^{\infty}\left[\text{rect}\left(\frac{\omega}{4\pi B}\right)\right]^2 e^{\frac{j(n-m)\omega}{2B}}\,d\omega$$

But rect $\left(\frac{\omega}{4\pi B}\right) = 1$ for $|\omega| \le 2\pi B$, and is 0 otherwise. Hence

$$\int_{-\infty}^{\infty} g_1(t)g_2(t)\,dt = \frac{1}{8\pi B^2}\int_{-2\pi B}^{2\pi B} e^{\frac{j(n-m)\omega}{2B}}\,d\omega = \begin{cases} 0 & n \neq m \\ \frac{1}{2B} & n = m \end{cases}$$

In evaluating the integral, we used the fact that $e^{\pm j2\pi k} = 1$ when $k$ is an integer.

**3.7-5** Application of duality property [Eq. (3.24)] to pair 3 (Table 3.1) yields

$$\frac{2a}{t^2+a^2} \Longleftrightarrow 2\pi e^{-a|\omega|}$$

The signal energy is given by

$$E_g = \frac{1}{\pi}\int_0^{\infty} |2\pi e^{-a\omega}|^2\,d\omega = 4\pi\int_0^{\infty} e^{-2a\omega}\,d\omega = \frac{2\pi}{a}$$

The energy contained within the band (0 to $W$ ) is

$$E_W = 4\pi\int_0^{W} e^{-2a\omega}\,d\omega = \frac{2\pi}{a}[1-e^{-2aW}]$$

If $E_W = 0.99E_g$, then

$$e^{-2aW} = 0.01 \quad \Longrightarrow W = \frac{2.3025}{a}\,\text{rad/s} = \frac{0.366}{a}\,\text{Hz}$$

**3.7-6** If $g^2(t) \Longleftrightarrow A(\omega)$, then the output $Y(\omega) = A(\omega)H(\omega)$, where $H(\omega)$ is the lowpass filter transfer function (Fig. S3.7-6). Because this filter band $\Delta f \to 0$, we may express it as an impulse function of area $4\pi\Delta f$. Thus,

$$H(\omega) \approx [4\pi\Delta f]\delta(\omega) \quad \text{and} \quad Y(\omega) \approx [4\pi A(\omega)\Delta f]\delta(\omega) = [4\pi A(0)\Delta f]\delta(\omega)$$

Here we used the property $g(\tau)\delta(\tau) = g(0)\delta(\tau)$ [Eq. (1.23a)]. This yields

$$y(t) = 2A(0)\Delta f$$

Next, because $g^2(t) \Longleftrightarrow A(\omega)$, we have

$$A(\omega) = \int_{-\infty}^{\infty} g^2(t)e^{-j\omega t}\,dt \quad \text{so that} \quad A(0) = \int_{-\infty}^{\infty} g^2(t)\,dt = E_g$$

Hence, $y(t) = 2E_g\Delta f$.

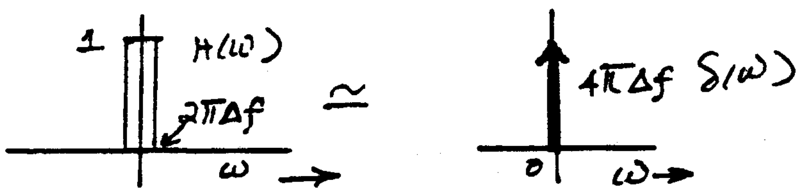



**Fig. S3.7-6**

**3.8-1** Let $g(t) = g_1(t) + g_2(t)$. Then

$$\begin{aligned}\mathcal{R}_g(\tau) &= \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}[g_1(t)+g_2(t)][g_1(t+\tau)+g_2(t+\tau)]\,dt \\ &= \mathcal{R}_{g_1}(\tau) + \mathcal{R}_{g_2}(\tau) + \mathcal{R}_{g_1g_2}(\tau) + \mathcal{R}_{g_2g_1}(\tau)\end{aligned}$$

where

$$\mathcal{R}_{xy}(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} x(t)y(t+\tau)\,dt$$

If we let $g_1(t) = C_1\cos(\omega_1 t + \theta_1)$ and $g_2(t) = C_2\cos(\omega_2 t + \theta_2)$, then

$$\mathcal{R}_{g_1g_2}(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} C_1C_2\cos(\omega_1 t + \theta_1)\cos(\omega_2 t + \omega_2\tau + \theta_2)\,dt$$

According to the argument used in Example 2.2b, the integral on the right-hand side is zero. Hence, $\mathcal{R}_{g_1g_2}(\tau) = 0$. Using the same argument, we have $\mathcal{R}_{g_2g_1}(\tau) = 0$. Therefore

$$\mathcal{R}_g(\tau) = \mathcal{R}_{g_1}(\tau) + \mathcal{R}_{g_2}(\tau) = \frac{C_1^2}{2}\cos\omega_1\tau + \frac{C_2^2}{2}\cos\omega_2\tau$$

This result can be extended to a sum of any number of sinusoids as long as the frequency of each sinusoid is distinct, hence, if

$$g(t) = \sum_{n=1}^{\infty} C_n\cos(n\omega_0 t + \theta_n) \qquad \text{then} \qquad \mathcal{R}_g(\tau) = \sum_{n=1}^{\infty}\frac{C_n^2}{2}\cos n\omega_0\tau$$

Moreover, for $g_0(t) = C_0$, $\mathcal{R}_{g_0}(\tau) = C_0^2$, and

$$\mathcal{R}_{g_0g_1}(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} C_0C_1\cos(\omega_1 t + \omega_1\tau + \theta_1)\,dt = 0$$

Thus, we can generalize the result as follows. If

$$g(t) = C_0 + \sum_{n=1}^{\infty} C_n\cos(n\omega_0 t + \theta_n) \qquad \text{then} \qquad \mathcal{R}_g(\tau) = C_0^2 + \sum_{n=1}^{\infty}\frac{C_n^2}{2}\cos n\omega_0\tau$$

and

$$S_g(\omega) = 2\pi C_0^2\delta(\omega) + \frac{\pi}{2}\sum_{n=1}^{\infty} C_n^2[\delta(\omega - n\omega_0) + \delta(\omega + n\omega_0)]$$

**3.8-2** Figure S3.8-2a shows the waveforms $x(t)$ and $x(t-\tau)$ for $\tau < T_b/2$. Let $T = NT_b$. On the average, there are $N/2$ pulses in the waveform of duration $T$. The area under the product $x(t)x(t-\tau)$ is $N/2$ times $(\frac{T_b}{2} - \tau)$ as shown in Fig. S3.8-2b. Therefore

$$\begin{aligned}\mathcal{R}_x(\tau) &= \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} x(t)x(t-\tau)\,dt \\ &= \lim_{N\to\infty}\frac{1}{NT_b}\frac{N}{2}\left(\frac{T_b}{2} - \tau\right) = \frac{1}{2}\left(\frac{1}{2} - \frac{\tau}{T_b}\right) \qquad \tau < \frac{T_b}{2} \\ &= \frac{1}{2}\left(\frac{1}{2} - \frac{|\tau|}{T_b}\right) \qquad |\tau| < \frac{T_b}{2}\end{aligned}$$

For $\frac{T_b}{2} \le |\tau| \le T_b$, there is no overlap between pulses, and $\mathcal{R}_x(\tau) = 0$. For $T_b \le |\tau| \le \frac{3T_b}{2}$, pulses again overlap. But on the average, only half pulses overlap. Hence, $\mathcal{R}_x(\tau)$ repeats every $T_b$ seconds, but only with half the magnitude, as shown in Fig. S3.8-2c. We can express $\mathcal{R}_x(\tau)$ as a sum of two components, as shown in Fig. S3.8-2d. Thus, $\mathcal{R}_x(\tau) = \mathcal{R}_1(\tau) + \mathcal{R}_2(\tau)$. The PSD is the sum of the Fourier transforms of $\mathcal{R}_1(\tau)$ and $\mathcal{R}_2(\tau)$. Hence

$$S_x(\omega) = \frac{T_b}{16}\operatorname{sinc}^2\left(\frac{\omega T_b}{4}\right) + S_2(\omega)$$

where $S_2(\omega)$ is the Fourier transform of the periodic triangle function, shown in Fig. S3.8-2d. We find the exponential Fourier series for this periodic signal to be

$$\mathcal{R}_2(\tau) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_b\tau} \qquad \omega_b = \frac{2\pi}{T_b}$$

Using Eq. (2.80), we find $D_n = \frac{1}{16}\operatorname{sinc}^2(\frac{n\pi}{2})$. Hence, according to Eq. (3.41)

$$S_2(\omega) = \frac{\pi}{8}\sum_{n=-\infty}^{\infty}\operatorname{sinc}^2\left(\frac{n\pi}{2}\right)\delta(\omega - n\omega_b) \qquad \omega_b = \frac{2\pi}{T_b}$$

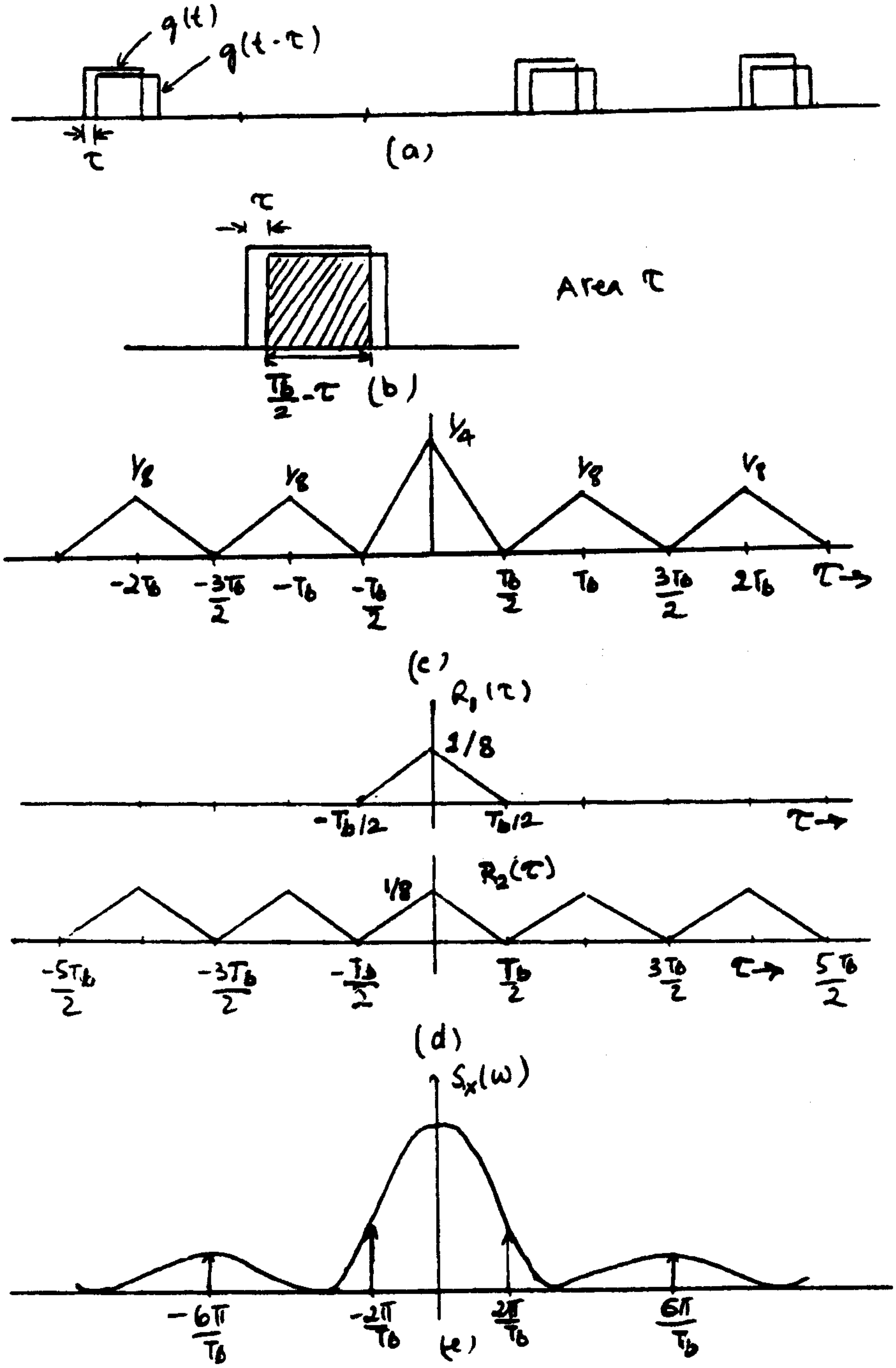



**Fig. S3.8-2**

Therefore

$$S_x(\omega) = \frac{T_b}{16}\,\text{sinc}^2\left(\frac{\omega T_b}{4}\right) + \frac{\pi}{8}\sum_{n=-\infty}^{\infty}\text{sinc}^2\left(\frac{n\pi}{2}\right)\delta(\omega - n\omega_b) \qquad \omega_b = \frac{2\pi}{T_b}$$

**3.8-3** $H(\omega) = \frac{1}{j\omega+1}$ and $|H(\omega)|^2 = \frac{1}{\omega^2+1}$.

**(a)** $\widetilde{x^2(t)} = \frac{1}{\pi}\int_0^\infty K\,d\omega = \infty$ and $\widetilde{y^2(t)} = \frac{1}{\pi}\int_0^\infty \frac{K}{\omega^2+1}\,d\omega = \frac{K}{2}$

(b) $\widetilde{x^2(t)} = \frac{1}{\pi}\int_0^1 d\omega = \frac{1}{\pi}$ and $\widetilde{y^2(t)} = \frac{1}{\pi}\int_0^1 \frac{1}{\omega^2+1}\,d\omega = \frac{1}{4}$

(c) $\widetilde{x^2(t)} = \frac{1}{\pi}\int_0^\infty \delta(\omega-1)\,d\omega = \frac{1}{\pi}$ and $\widetilde{y^2(t)} = \frac{1}{\pi}\int_0^\infty \frac{\delta(\omega-1)}{\omega^2+1}\,d\omega = \frac{1}{\pi}\int_0^\infty \frac{\delta(\omega-1)}{2}\,d\omega = \frac{1}{2\pi}$

**3.8-4** The ideal differentiator transfer function is $j\omega$. Hence, the transfer function of the entire system is

$$H(\omega) = \left(\frac{1}{j\omega+1}\right)(j\omega) = \frac{j\omega}{j\omega+1} \quad \text{and} \quad |H(\omega)|^2 = \frac{\omega^2}{\omega^2+1}$$

$$\widetilde{x^2(t)} = \frac{1}{\pi}\int_0^\infty \operatorname{rect}\left(\frac{\omega}{2}\right)d\omega = \frac{1}{\pi}\int_0^1 d\omega = \frac{1}{\pi}$$

$$\widetilde{y^2(t)} = \frac{1}{\pi}\int_0^\infty \operatorname{rect}\left(\frac{\omega}{2}\right)\frac{\omega^2}{\omega^2+1}\,d\omega = \frac{1}{\pi}\int_0^1 \frac{\omega^2}{\omega^2+1}\,d\omega = \frac{1}{\pi}\left(1-\frac{\pi}{4}\right) = 0.06831$$

# Chapter 4

**4.2-1** (i) For $m(t) = \cos 1000t$

$$\varphi_{\text{DSB-SC}}(t) = m(t)\cos 10{,}000t = \cos 1000t \cos 10{,}000t$$
$$= \frac{1}{2}[\underbrace{\cos 9000t}_{\text{LSB}} + \underbrace{\cos 11{,}000t}_{\text{USB}}]$$

(ii) For $m(t) = 2\cos 1000t + \cos 2000t$

$$\varphi_{\text{DSB-SC}}(t) = m(t)\cos 10{,}000t = [2\cos 1000t + \cos 2000t]\cos 10{,}000t$$
$$= \cos 9000t + \cos 11{,}000t + \frac{1}{2}[\cos 8000t + \cos 12{,}000t]$$
$$= \underbrace{[\cos 9000t + \frac{1}{2}\cos 8000t]}_{\text{LSB}} + \underbrace{[\cos 11{,}000t + \frac{1}{2}\cos 12{,}000t]}_{\text{USB}}$$

(iii) For $m(t) = \cos 1000t \cos 3000t$

$$\varphi_{\text{DSB-SC}}(t) = m(t)\cos 10{,}000t = \frac{1}{2}[\cos 2000t + \cos 4000t]\cos 10{,}000t$$
$$= \frac{1}{2}[\cos 8000t + \cos 12{,}000t] + \frac{1}{2}[\cos 6000t + \cos 14{,}000t]$$
$$= \frac{1}{2}\underbrace{[\cos 8000t + \cos 6000t]}_{\text{LSB}} + \frac{1}{2}\underbrace{[\cos 12{,}000t + \cos 14{,}000t]}_{\text{USB}}$$

This information is summarized in a table below. Figure S4.2-1 shows various spectra.

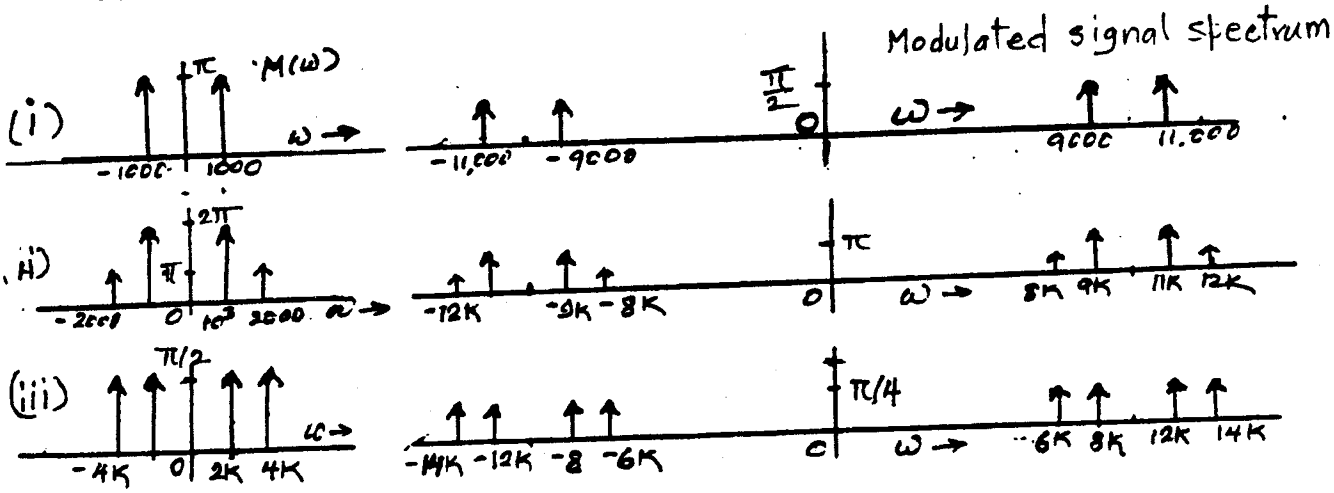



**Fig. S4.2-1**

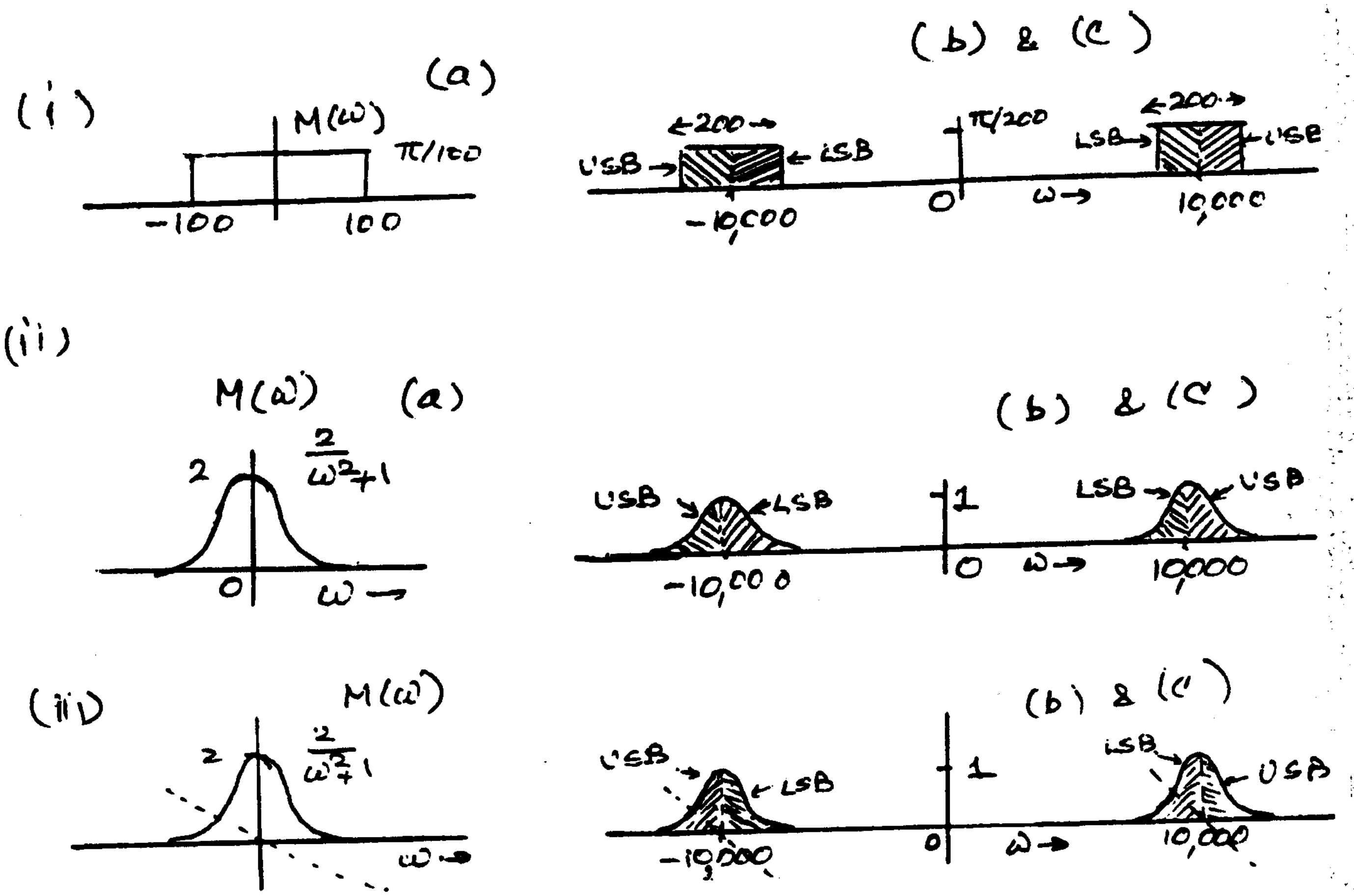



Fig. S4.2-2

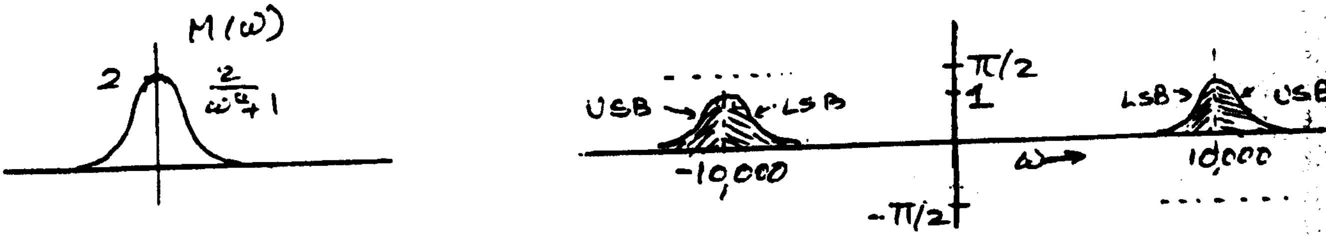



Fig. S4.2-3

| case | Baseband frequency | DSB frequency | LSB frequency | USB frequency |
|---|---|---|---|---|
| i | 1000 | 9000 and 11,000 | 9000 | 11.000 |
| ii | 1000 | 9000 and 11,000 | 9000 | 11,000 |
| | 2000 | 8000 and 12.000 | 8000 | 12,000 |
| iii | 2000 | 8000 and 12.000 | 8000 | 12,000 |
| | 4000 | 6000 and 14.000 | 6000 | 14,000 |

**4.2-2** The relevant plots are shown in Fig. S4.2-2.

**4.2-3** The relevant plots are shown in Fig. S4.2-3.

**4.2-4** (a) The signal at point $b$ is

$$\begin{aligned} y_a(t) &= m(t)\cos^3\omega_c t \\ &= m(t)\left[\frac{3}{4}\cos\omega_c t + \frac{1}{4}\cos 3\omega_c t\right] \end{aligned}$$

The term $\frac{3}{4}m(t)\cos\omega_c t$ is the desired modulated signal, whose spectrum is centered at $\pm\omega_c$. The remaining term $\frac{1}{4}m(t)\cos 3\omega_c t$ is the unwanted term, which represents the modulated signal with carrier frequency $3\omega_c$ with spectrum centered at $\pm 3\omega_c$ as shown in Fig. S4.2-4. The bandpass filter centered at $\pm\omega_c$ allows to pass the desired term $\frac{3}{4}m(t)\cos\omega_c t$, but suppresses the unwanted term $\frac{1}{4}m(t)\cos 3\omega_c t$. Hence, this system works as desired with the output $\frac{3}{4}m(t)\cos\omega_c t$.

(b) Figure S4.2-4 shows the spectra at points b and c.

(c) The minimum usable value of $\omega_c$ is $2\pi B$ in order to avoid spectral folding at dc.

(d)

$$m(t)\cos^2\omega_c t = \frac{m(t)}{2}[1+\cos 2\omega_c t]$$
$$= \frac{1}{2}m(t) + \frac{1}{2}m(t)\cos 2\omega_c t$$

The signal at point b consists of the baseband signal $\frac{1}{2}m(t)$ and a modulated signal $\frac{1}{2}m(t)\cos 2\omega_c t$, which has a carrier frequency $2\omega_c$, not the desired value $\omega_c$. Both the components will be suppressed by the filter, whose center center frequency is $\omega_c$. Hence, this system will not do the desired job.

(e) The reader may verify that the identity for $\cos n\omega_c t$ contains a term $\cos\omega_c t$ when $n$ is odd. This is not true when $n$ is even. Hence, the system works for a carrier $\cos^n\omega_c t$ only when $n$ is odd.

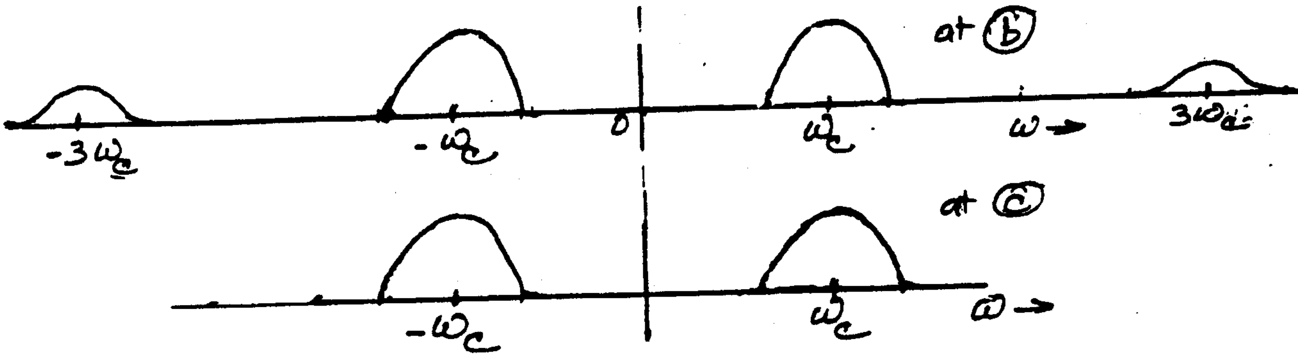



Fig. S4.2-4

**4.2-5** We use the ring modulator shown in Fig. 4.6 with the carrier frequency $f_c = 100$ kHz ($\omega_c = 200\pi \times 10^3$), and the output bandpass filter centered at $f_c = 300$ kHz. The output $v_i(t)$ is found in Eq. (4.7b) as

$$v_i(t) = \frac{4}{\pi}\left[m(t)\cos\omega_c t - \frac{1}{3}m(t)\cos 3\omega_c t + \frac{1}{5}m(t)\cos 5\omega_c t + \cdots\right]$$

The output bandpass filter suppresses all the terms except the one centered at 300 kHz (corresponding to the carrier $3\omega_c t$). Hence, the filter output is

$$y(t) = \frac{-4}{3\pi}m(t)\cos 3\omega_c t$$

This is the desired output $km(t)\cos\omega_c t$ with $k = -4/3\pi$.

**4.2-6** The resistance of each diode is $r$ ohms while conducting, and $\infty$ when off. When the carrier $A\cos\omega_c t$ is positive, the diodes conduct (during the entire positive half cycle), and when the carrier is negative the diodes are open (during the entire negative half cycle). Thus, during the positive half cycle, the voltage $\frac{R}{R+r}\phi(t)$ appears across each of the resistors $R$. During the negative half cycle, the output voltage is zero. Therefore, the diodes act as a gate in the circuit that is basically a voltage divider with a gain $2R/(R+r)$. The output is therefore

$$e_0(t) = \frac{2R}{R+r}w(t)m(t)$$

The period of $w(t)$ is $T_0 = 2\pi/\omega_c$. Hence, from Eq. (2.75)

$$w(t) = \frac{1}{2} + \frac{2}{\pi}\left[\cos \omega_c t - \frac{1}{3}\cos 3\omega_c t + \frac{1}{5}\cos 5\omega_c t + \cdots\right]$$

The output $e_o(t)$ is

$$e_o(t) = \frac{2R}{R+r} w(t) m(t) = \frac{2R}{R+r} m(t)\left[\frac{1}{2} + \frac{2}{\pi}\left(\cos \omega_c t - \frac{1}{3}\cos 3\omega_c t + \frac{1}{5}\cos 5\omega_c t + \cdots\right)\right]$$

(a) If we pass the output $e_o(t)$ through a bandband filter (centered at $\omega_c$), the filter suppresses the signal $m(t)$ and $m(t)\cos n\omega_c t$ for all $n \neq 1$, leaving only the modulated term $\frac{4R}{\pi(R+r)} m(t)\cos\omega_c t$ intact. Hence, the system acts as a modulator.

(b) The same circuit can be used as a demodulator if we use a basepass filter at the output. In this case, the input is $\phi(t) = m(t)\cos\omega_c t$ and the output is $\frac{2R}{\pi(R+r)} m(t)$.

**4.2-7** From the results in Prob. 4.2-6, the output $e_o(t) = km(t)\cos\omega_c t$, where $k = \frac{4R}{\pi(R+r)}$. In the present case, $m(t) = \sin(\omega_c t + \theta)$. Hence, the output is

$$e_o(t) = k\sin(\omega_c t + \theta)\cos\omega_c t = \frac{k}{2}[\sin\theta + \sin(2\omega_c t + \theta)]$$

The lowpass filter suppresses the sinusoid of frequency $2\omega_c$ and transmits only the dc term $\frac{k}{2}\sin\theta$.

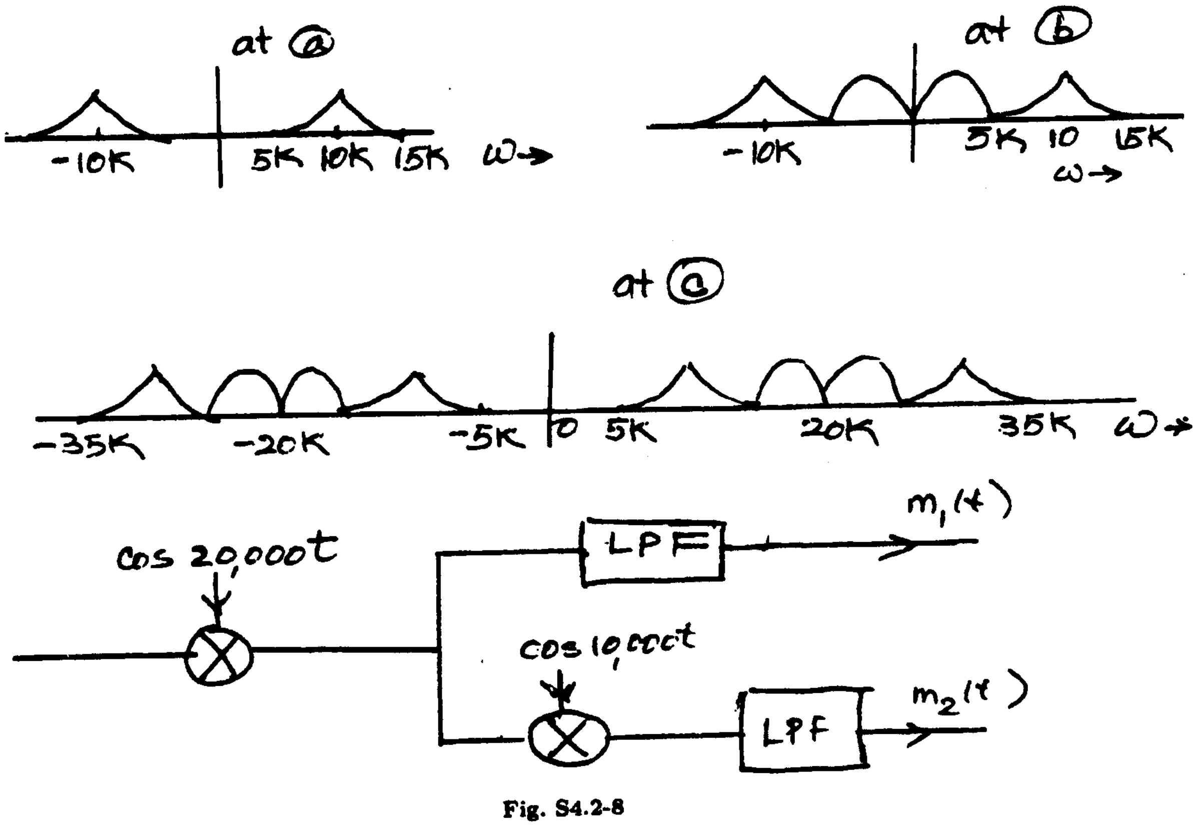



Fig. S4.2-8

**4.2-8** (a) Fig. S4.2-8 shows the signals at points a, b, and c.
(b) From the spectrum at point c, it is clear that the channel bandwidth must be at least 30,000 rad/s (from 5000 to 35,000 rad/s.).
(c) Fig. S4.2-8 shows the receiver to recover $m_1(t)$ and $m_2(t)$ from the received modulated signal.

**4.2-9** (a) S4.2-9 shows the output signal spectrum $Y(\omega)$.
(b) Observe that $Y(\omega)$ is the same as $M(\omega)$ with the frequency spectrum inverted, that is, the high frequencies are shifted to lower frequencies and vice versa. Thus, the scrambler in Fig. P4.2-9 inverts the frequency spectrum.

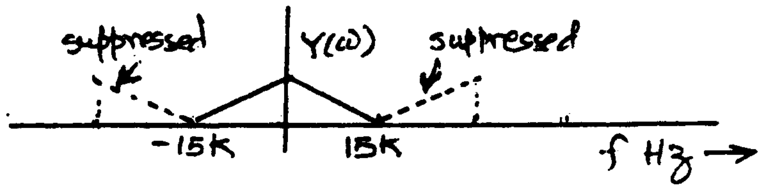



**Fig. S4.2-9**

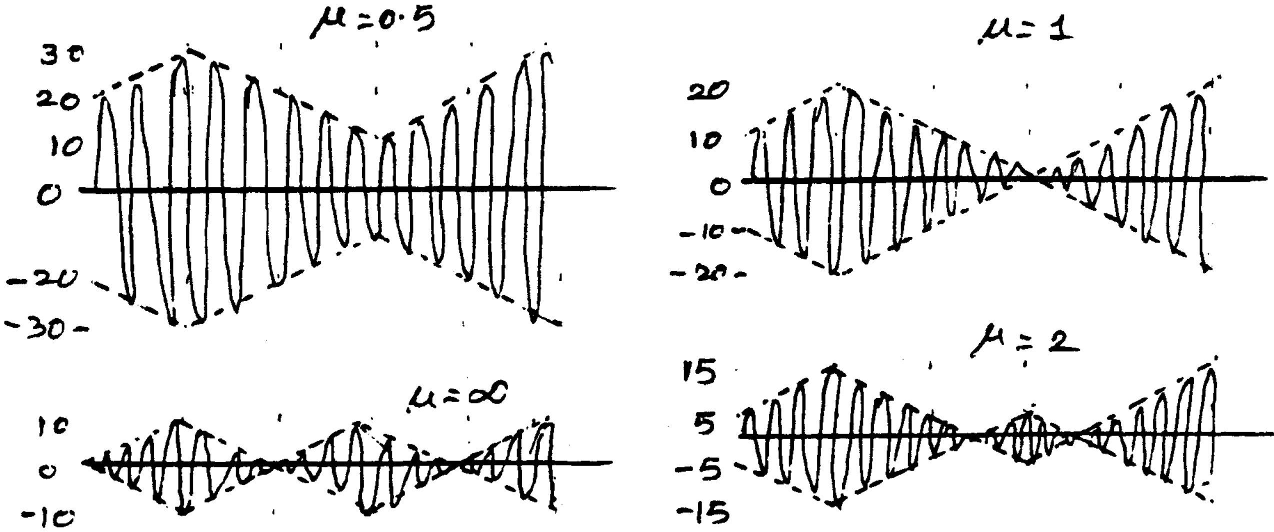



**Fig. S4.3-2**

To get back the original spectrum $M(\omega)$, we need to invert the spectrum $Y(\omega)$ once again. This can be done by passing the scrambled signal $y(t)$ through the same scrambler.

**4.2-10** We use the ring modulator shown in Fig. 4.6, except that the input is $m(t)\cos(2\pi)10^6 t$ instead of $m(t)$. The carrier frequency is 200 kHz $[\omega_c = (400\pi)10^3 t]$, and the output bandpass filter is centered at 400 kHz. The output $e_i(t)$ is found in Eq. (4.7b) as

$$e_i(t) = [m(t)\cos(2\pi)10^6 t]w_0(t) = \frac{4}{\pi}m(t)\cos(2\pi)10^6 t\left[\cos(400\pi)10^3 t - \frac{1}{3}\cos 3(400\pi)10^3 t + \frac{1}{5}\cos 5(400\pi)10^3 t + \cdots\right]$$

The product of the terms $(-1/3)\cos 3(400\pi)10^3 t$ and $(4/\pi)m(t)\cos(2\pi)10^6 t$ yields the desired term $-\frac{2}{3\pi}m(t)\cos(800\pi)10^3 t$, whose spectrum is centered at 400 kHz. It alone passes through the bandpass filter (centered at 400 kHz). All the other terms are suppressed. The desired output is

$$y(t) = -\frac{2}{3\pi}m(t)\cos(800\pi)10^3 t$$

**4.3-1** $g_a(t) = [A + m(t)]\cos\omega_c t$. Hence,

$$\begin{aligned} g_b(t) &= [A + m(t)]\cos^2\omega_c t \\ &= \frac{1}{2}[A + m(t)] + \frac{1}{2}[A + m(t)]\cos 2\omega_c t \end{aligned}$$

The first term is a lowpass signal because its spectrum is centered at $\omega = 0$. The lowpass filter allows this term to pass, but suppresses the second term, whose spectrum is centered at $\pm 2\omega_c$. Hence the output of the lowpass filter is

$$y(t) = A + m(t)$$

When this signal is passed through a dc block, the dc term $A$ is suppressed yielding the output $m(t)$. This shows that the system can demodulate AM signal regardless of the value of $A$. This is a synchronous or coherent demodulation.

**4.3-2**

$$\begin{array}{llr}
\text{(a)} & \mu = 0.5 = \dfrac{m_p}{A} = \dfrac{10}{A} & \Rightarrow A = 20 \\
\text{(b)} & \mu = 1.0 = \dfrac{m_p}{A} = \dfrac{10}{A} & \Rightarrow A = 10 \\
\text{(c)} & \mu = 2.0 = \dfrac{m_p}{A} = \dfrac{10}{A} & \Rightarrow A = 5 \\
\text{(d)} & \mu = \infty = \dfrac{m_p}{A} = \dfrac{10}{A} & \Rightarrow A = 0
\end{array}$$

This means that $\mu = \infty$ represents the DSB-SC case. Figure S4.3-2 shows various waveforms.

**4.3-3** **(a)** According to Eq. (4.10a), the carrier amplitude is $A = m_p/\mu = 10/0.8 = 12.8$. The carrier power is $P_c = A^2/2 = 78.125$.

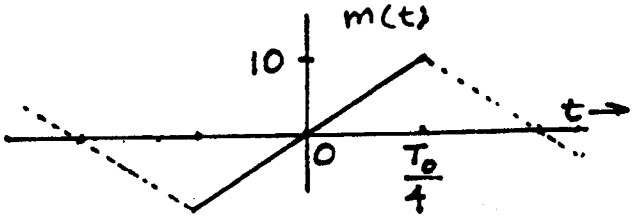



**Fig. S4.3-3**

(b) The sideband power is $\widetilde{m^2(t)}/2$. Because of symmetry of amplitude values every quarter cycle, the power of $m(t)$ may be computed by averaging the signal energy over a quarter cycle only. Over a quarter cycle $m(t)$ can be represented as $m(t) = 40t/T_0$ (see Fig. S4.3-3). Hence.

$$\widetilde{m^2(t)} = \frac{1}{T_0/4}\int_0^{T_0/4}\left[\frac{40t}{T_0}\right]^2 dt = 33.34$$

The sideband power is

$$P_s = \frac{\widetilde{m^2(t)}}{2} = 16.67$$

The efficiency is

$$\eta = \frac{P_s}{P_c + P_s} = \frac{16.67}{78.125 + 16.67} \times 100 = 19.66\%$$

**4.3-4** From Fig. S4.3-4 it is clear that the envelope of the signal $m(t)\cos\omega_c t$ is $|m(t)|$. The signal $[A + m(t)]\cos\omega_c t$ is identical to $m(t)\cos\omega_c t$ with $m(t)$ replaced by $A + m(t)$. Hence. using the previous argument, it is clear that its envelope is $|A + m(t)|$. Now, if $A + m(t) > 0$ for all $t$, then $A + m(t) = |A + m(t)|$. Therefore, the condition for demodulating AM signal using envelope detector is $A + m(t) > 0$ for all $t$.

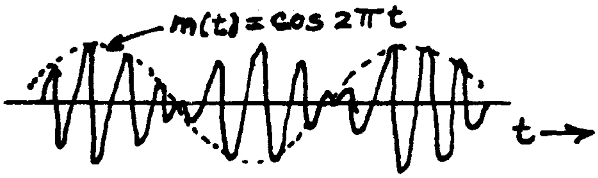



**Fig. S4.3-4**

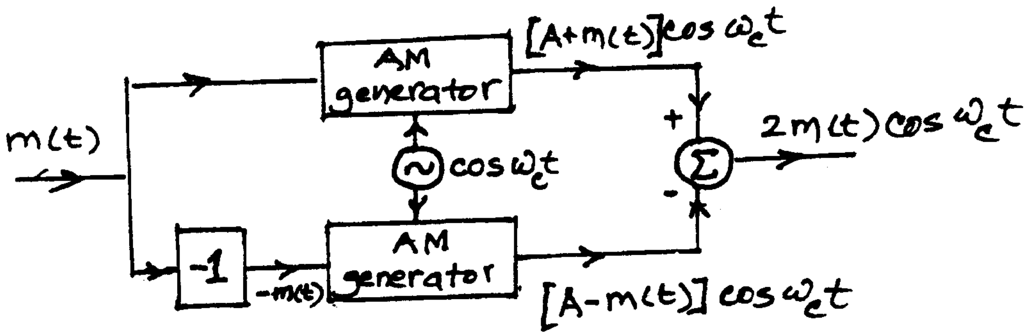



**Fig. S4.3-5**

**4.3-5** When an input to a DSB-Sc generator is $m(t)$, the corresponding output is $m(t)\cos\omega_c t$. Clearly, if the input is $A + m(t)$, the corresponding output will be $[A + m(t)]\cos\omega_c t$. This is precisely the AM signal. Thus, by adding a dc of value $A$ to the baseband signal $m(t)$, we can generate AM signal using a DSB-SC generator. The converse is generally not true. However, we can generate DSB-SC using AM generators if we use two identical AM generators in a balanced scheme shown in Fig. S4.3-5 to cancel out the carrier component.

**4.3-6** When an input to a DSB-SC demodulator is $m(t)\cos\omega_c t$, the corresponding output is $m(t)$. Clearly, if the input is $[A + m(t)]\cos\omega_c t$, the corresponding output will $A + m(t)$. By blocking the dc component $A$ from this output, we can demodulate the AM signal using a DSB-SC demodulator. The converse, unfortunately, is not true. This is because, when an input to an AM demodulator is $m(t)\cos\omega_c t$, the corresponding output is $|m(t)|$ [the envelope of $m(t)$]. Hence, unless $m(t) \geq 0$ for all $t$, it is not possible to demodulate DSB-SC signal using an AM demodulator.

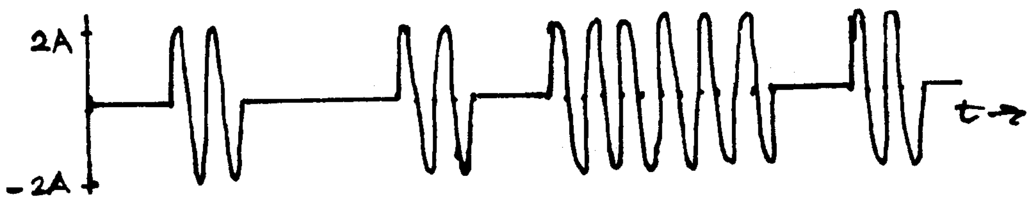



**Fig. S4.3-7**

**4.3-7** Observe that $m^2(t) = A^2$ for all $t$. Hence, the time average of $m^2(t)$ is also $A^2$. Thus

$$\widetilde{m^2(t)} = A^2 \qquad P_s = \frac{\widetilde{m^2(t)}}{2} = \frac{A^2}{2}$$

The carrier amplitude is $A = m_p/\mu = m_p = A$. Hence $P_c = A^2/2$. The total power is $P_t = P_c + P_s = A^2$. The power efficiency is

$$\eta = \frac{P_s}{P_t} \times 100 = \frac{A^2/2}{A^2} \times 100 = 0.5$$

The AM signal for $\mu = 1$ is shown in Fig. S4.3-7.

**4.3-8** The signal at point a is $[A + m(t)]\cos\omega_c t$. The signal at point b is

$$x(t) = [A + m(t)]^2\cos^2\omega_c t = \frac{A^2 + 2Am(t) + m^2(t)}{2}(1 + \cos 2\omega_c t)$$

The lowpass filter suppresses the term containing $\cos 2\omega_c t$. Hence, the signal at point c is

$$w(t) = \frac{A^2 + 2Am(t) + m^2(t)}{2} = \frac{A^2}{2}\left[1 + \frac{2m(t)}{A} + \left(\frac{m(t)}{A}\right)^2\right]$$

Usually, $m(t)/A \ll 1$ for most of the time. Only when $m(t)$ is near its peak, this condition is violated. Hence, the output at point d is

$$y(t) \approx \frac{A^2}{2} + Am(t)$$

A blocking capacitor will suppress the dc term $A^2/2$, yielding the output $Am(t)$. From the signal $w(t)$, we see that the distortion component is $m^2(t)/2$.

**4.4-1** In Fig. 4.14, when the carrier is $\cos[(\Delta\omega)t + \delta]$ or $\sin[(\Delta\omega)t + \delta]$, we have

$$\begin{aligned} x_1(t) &= 2[m_1(t)\cos\omega_c t + m_2(t)\sin\omega_c t]\cos[(\omega_c + \Delta\omega)t + \delta] \\ &= 2m_1(t)\cos\omega_c t\cos[(\omega_c + \Delta\omega)t + \delta] + 2m_2(t)\sin\omega_c t\cos[(\omega_c + \Delta\omega)t + \delta] \\ &= m_1(t)\{\cos[(\Delta\omega)t + \delta] + \cos[(2\omega_c + \Delta\omega)t + \delta]\} + m_2(t)\{\sin[(2\omega_c + \Delta\omega)t + \delta] - \sin[(\Delta\omega)t + \delta]\} \end{aligned}$$

Similarly

$$x_2(t) = m_1(t)\{\sin[(2\omega_c + \Delta\omega)t + \delta] + \sin[(\Delta\omega)t + \delta]\} + m_2(t)\{\cos[(\Delta\omega)t + \delta] - \cos[(2\omega_c + \Delta\omega)t + \delta]\}$$

After $x_1(t)$ and $x_2(t)$ are passed through lowpass filter, the outputs are

$$\begin{aligned} m_1'(t) &= m_1(t)\cos[(\Delta\omega)t + \delta] - m_2(t)\sin[(\Delta\omega)t + \delta] \\ m_2'(t) &= m_1(t)\sin[(\Delta\omega)t + \delta] + m_2(t)\cos[(\Delta\omega)t + \delta] \end{aligned}$$

**4.5-1** To generate a DSB-SC signal from $m(t)$, we multiply $m(t)$ with $\cos\omega_c t$. However, to generate the SSB signals of the same relative magnitude, it is convenient to multiply $m(t)$ with $2\cos\omega_c t$. This also avoids the nuisance of the fractions 1/2, and yields the DSB-SC spectrum $M(\omega - \omega_c) + M(\omega + \omega_c)$. We suppress the USB spectrum (above $\omega_c$ and below $-\omega_c$) to obtain the LSB spectrum. Similarly, to obtain the USB spectrum, we suppress the LSB spectrum (between $-\omega_c$ and $\omega_c$) from the DSB-SC spectrum. Figures S4.5-1 a, b and c show the three cases.

(a)From Fig. a, we can express $\varphi_{LSB}(t) = \cos 900t$ and $\varphi_{USB}(t) = \cos 1100t$.

(b)From Fig. b, we can express $\varphi_{LSB}(t) = 2\cos 700t + \cos 900t$ and $\varphi_{USB}(t) = \cos 1100t + 2\cos 1300t$.

(c)From Fig. c, we can express $\varphi_{LSB}(t) = \frac{1}{2}[\cos 400t + \cos 600t]$ and $\varphi_{USB}(t) = \frac{1}{2}[\cos 1400t + \cos 1600t]$

**4.5-2**

$$\varphi_{USB}(t) = m(t)\cos\omega_c t - m_h(t)\sin\omega_c t \qquad \text{and} \qquad \varphi_{LSB}(t) = m(t)\cos\omega_c t + m_h(t)\sin\omega_c t$$

(a) $m(t) = \cos 100t$ and $m_h(t) = \sin 100t$. Hence,

$$\varphi_{LSB}(t) = \cos 100t\cos 1000t + \sin 100t\sin 1000t = \cos(1000 - 100)t = \cos 900t$$

$$\varphi_{USB}(t) = \cos 100t\cos 1000t - \sin 100t\sin 1000t = \cos(1000 + 100)t = \cos 1100t$$

(b) $m(t) = \cos 100t + 2\cos 300t$ and $m_h(t) = \sin 100t + 2\sin 300t$. Hence,

$$\varphi_{LSB}(t) = (\cos 100t + 2\cos 300t)\cos 1000t + (\sin 100t + 2\sin 300t)\sin 1000t = \cos 900t + 2\cos 700t$$

$$\varphi_{USB}(t) = (\cos 100t + 2\cos 300t)\cos 1000t - (\sin 100t + 2\sin 300t)\sin 1000t = \cos 1100t + 2\cos 1300t$$

(c) $m(t) = \cos 100t\cos 500t = 0.5\cos 400t + 0.5\cos 600t$ and $m_h(t) = 0.5\sin 400t + 0.5\sin 600t$. Hence,

$$\varphi_{LSB}(t) = (0.5\cos 400t + 0.5\cos 600t)\cos 1000t + (0.5\sin 400t + 0.5\sin 600t)\sin 1000t = 0.5\cos 400t + 0.5\cos 600t$$

$$\varphi_{USB}(t) = (0.5\cos 400t + 0.5\cos 600t)\cos 1000t - (0.5\sin 400t + 0.5\sin 600t)\sin 1000t = 0.5\cos 1400t + 0.5\cos 1600t$$

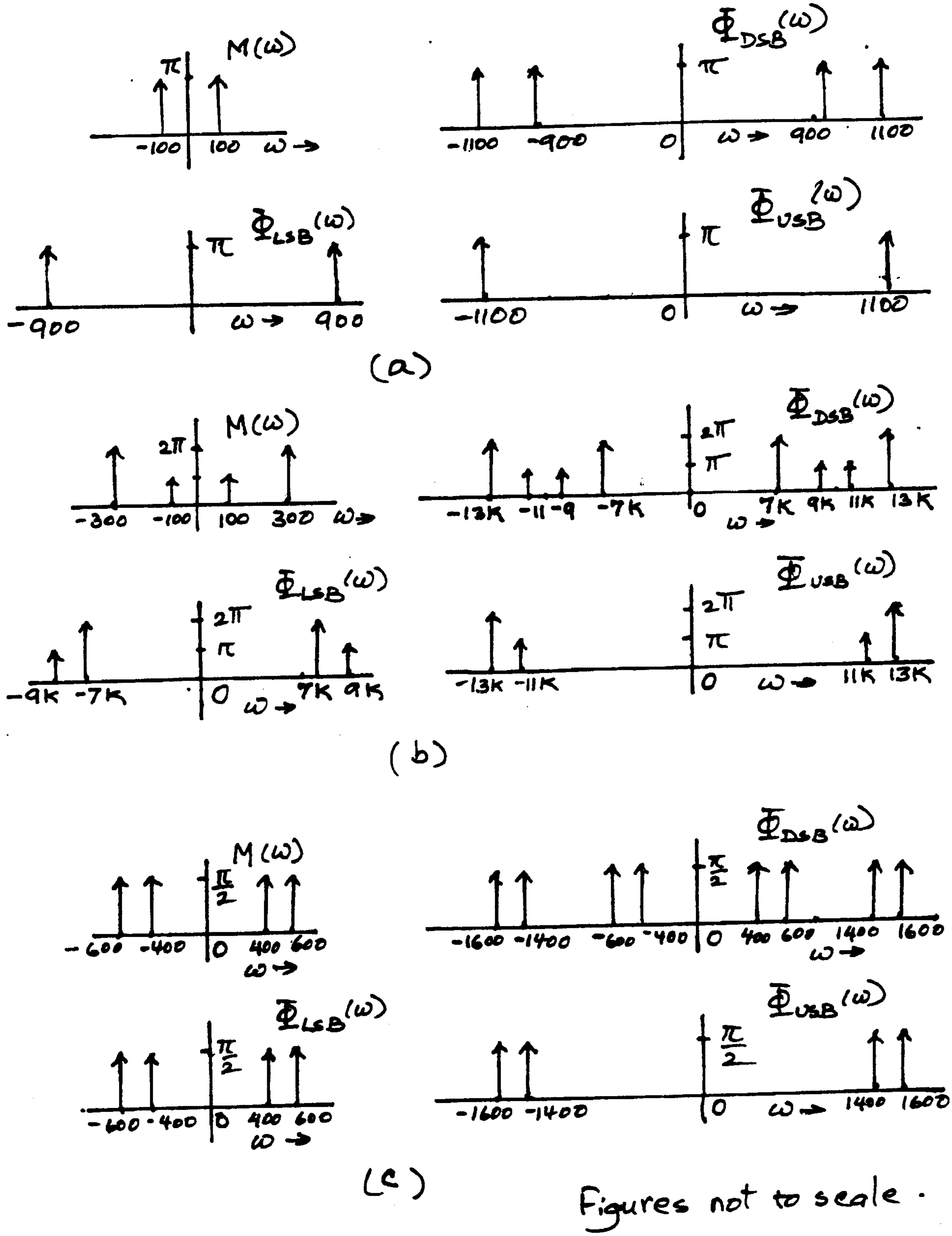

M(ω)
π
-100
100
ω →
Φ_DSB(ω)
π
-1100
-900
0
ω →
900
1100
Φ_LSB(ω)
π
-900
ω →
900
Φ_USB(ω)
π
-1100
0
ω →
1100
(a)
M(ω)
2π
-300
-100
100
300
ω →
Φ_DSB(ω)
2π
π
-13K
-11
-9
-7K
0
7K
9K
11K
13K
ω →
Φ_LSB(ω)
2π
π
-9K
-7K
0
ω →
7K
9K
Φ_USB(ω)
2π
π
-13K
-11K
0
ω →
11K
13K
(b)
M(ω)
π/2
-600
-400
0
400
600
ω →
Φ_DSB(ω)
π/2
-1600
-1400
-600
-400
0
400
600
1400
1600
ω →
Φ_LSB(ω)
π/2
-600
-400
0
400
600
ω →
Φ_USB(ω)
π/2
-1600
-1400
0
ω →
1400
1600
(c)
Figures not to scale.

Fig. S4.5-1

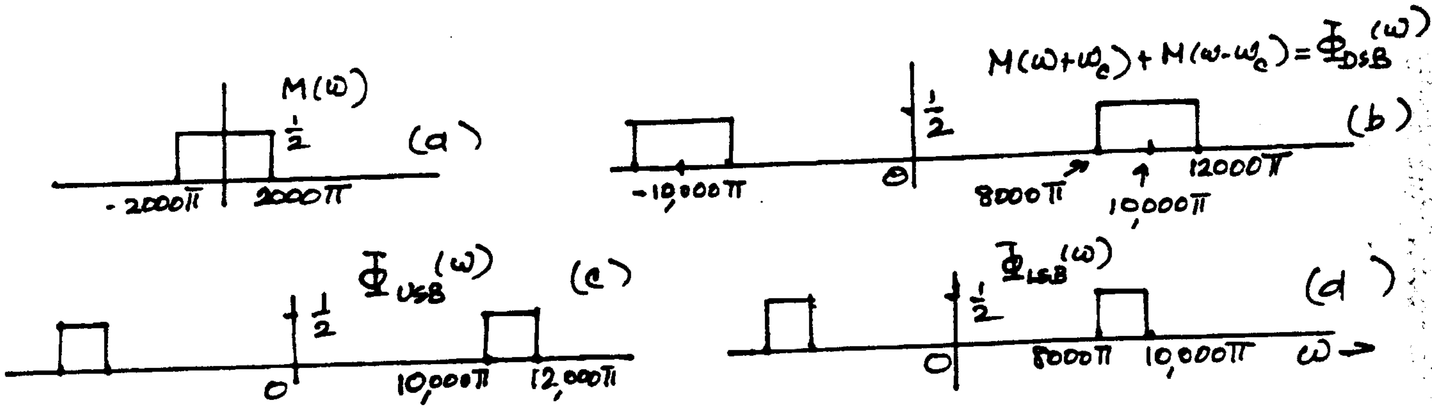



Fig. S4.5-3

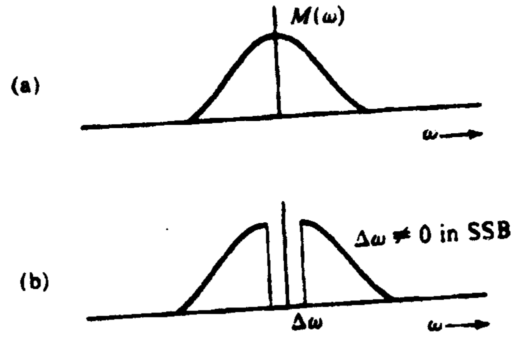



Fig. S4.5-5

**4.5-3** (a) Figure S4.5-3a shows the spectrum of $m(t)$ and Fig. S4.5-3b shows the corresponding DSB-SC spectrum $2m(t)\cos 10{,}000\pi t$.
(b) Figure S4.5-3c shows the corresponding LSB spectrum obtained by suppressing the USB spectrum.
(c) Figure S4.5-3d shows the corresponding USB spectrum obtained by suppressing the LSB spectrum.
We now find the inverse Fourier transforms of the LSB and USB spectra from Table 3.1 (pair 18) and the frequency shifting property as

$$\varphi_{LSB}(t) = 1000\,\text{sinc}\,(1000\pi t)\cos 9000\pi t$$

$$\varphi_{USB}(t) = 1000\,\text{sinc}\,(1000\pi t)\cos 11{,}000\pi t$$

**4.5-4** Because $M_h(\omega) = -jM(\omega)\,\text{sgn}\,(\omega)$, the transfer function of a Hilbert transformer is

$$H(\omega) = -j\,\text{sgn}\,(\omega)$$

If we apply $m_h(t)$ at the input of the Hilbert transformer, $Y(\omega)$, the spectrum of the output signal $y(t)$ is

$$Y(\omega) = M_h(\omega)H(\omega) = [-jM(\omega)\,\text{sgn}\,(\omega)][-j\text{sgn}\,(\omega)] = -M(\omega)$$

This shows that the Hilbert transform of $m_h(t)$ is $-m(t)$. To show that the energies of $m(t)$ and $m_h(t)$ are equal, we have

$$E_m = \int_{-\infty}^{\infty} m^2(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |M(\omega)|^2\,d\omega$$

$$E_{m_h} = \int_{-\infty}^{\infty} m_h^2(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |M_h(\omega)|^2\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |M(\omega)|^2|\text{sgn}\,(\omega)|^2\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |M(\omega)|^2\,d\omega = E_m$$

**4.5-5** The incoming SSB signal at the receiver is given by [Eq. (4.17b)]

$$\varphi_{\text{LSB}}(t) = m(t)\cos\omega_c t + m_h(t)\sin\omega_c t$$

Let the local carrier be cos $[(\omega_c + \Delta\omega)t + \delta]$. The prodeuct of the incoming signal and the local carrier is $e_d(t)$, given by

$$\begin{aligned} e_d(t) &= \varphi_{\text{LSB}}(t)\cos[(\omega_c + \Delta\omega)t + \delta] \\ &= 2[m(t)\cos\omega_c t + m_h(t)\sin\omega_c t]\cos[(\omega_c + \Delta\omega)t + \delta] \end{aligned}$$

The lowpass filter suppresses the sum frequency component centered at the frequency $(2\omega_c + \Delta\omega)$, and passes only the difference frequency component centered at the frequency $\Delta\omega$. Hence, the filter output $e_0(t)$ is given by

$$e_0(t) = m(t)\cos(\Delta\omega)t + \delta) - m_h(t)\sin(\Delta\omega)t + \delta)$$

Observe that if both $\Delta\omega$ and $\delta$ are zero, the output is given by

$$e_0(t) = m(t)$$

as expected. If only $\delta = 0$, then the output is given by

$$e_0(t) = m(t)\cos(\Delta\omega)t - m_h(t)\sin(\Delta\omega)t$$

This is an USB signal corresponding to a carrier frequency $\Delta\omega$ as shown in Fig. S4.5-5b. This spectrum is the same as the spectrum $M(\omega)$ with each frequency component shifted by a frequency $\Delta\omega$. This changes the sound of an audio signal slightly. For voice signals, the frequency shift within $\pm 20$ Hz is considered tolerable. Most US systems, however, restrict the shift to $\pm 2$ Hz.
(b) When only $\Delta\omega = 0$, the lowpass filter output is

$$e_0(t) = m(t)\cos\delta - m_h(t)\sin\delta$$

We now show that this is a phase distortion, where each frequency component of $M(\omega)$ is shifted in phase by amount $\delta$. The Fourier transform of this equation yields

$$E_0(\omega) = M(\omega)\cos\delta - M_h(\omega)\sin\delta$$

But from Eq. (4.14b)

$$M_h(\omega) = -j\text{sgn}(\omega)M(\omega) = \begin{cases} -jM(\omega) & \omega > 0 \\ M(\omega) & \omega < 0 \end{cases}$$

and

$$E_0(\omega) = \begin{cases} M(\omega)e^{j\delta} & \omega > 0 \\ M(\omega)e^{-j\delta} & \omega < 0 \end{cases}$$

It follows that the amplitude spectrum of $e_0(t)$ is $M(\omega)$, the same as that for $m(t)$. But the phase of each component is shifted by $\delta$. Phase distortion generally is not a serious problem with voice signals, because the human ear is somewhat insensitive to phase distortion. Such distortion may change the quality of speech, but the voice is still intelligible. In video signals and data transmission, however, phase distortion may be intolerable.

**4.5-6** We showed in prob. 4.5-4 that the Hilbert transform of $m_h(t)$ is $-m(t)$. Hence, if $m_h(t)$ [instead of $m(t)$] is applied at the input in Fig. 4.20, the USB output is

$$\begin{aligned} y(t) &= m_h(t)\cos\omega_c t - m(t)\sin\omega_c t \\ &= m(t)\cos\left(\omega_c t + \frac{\pi}{2}\right) + m_h(t)\sin\left(\omega_c t + \frac{\pi}{2}\right) \end{aligned}$$

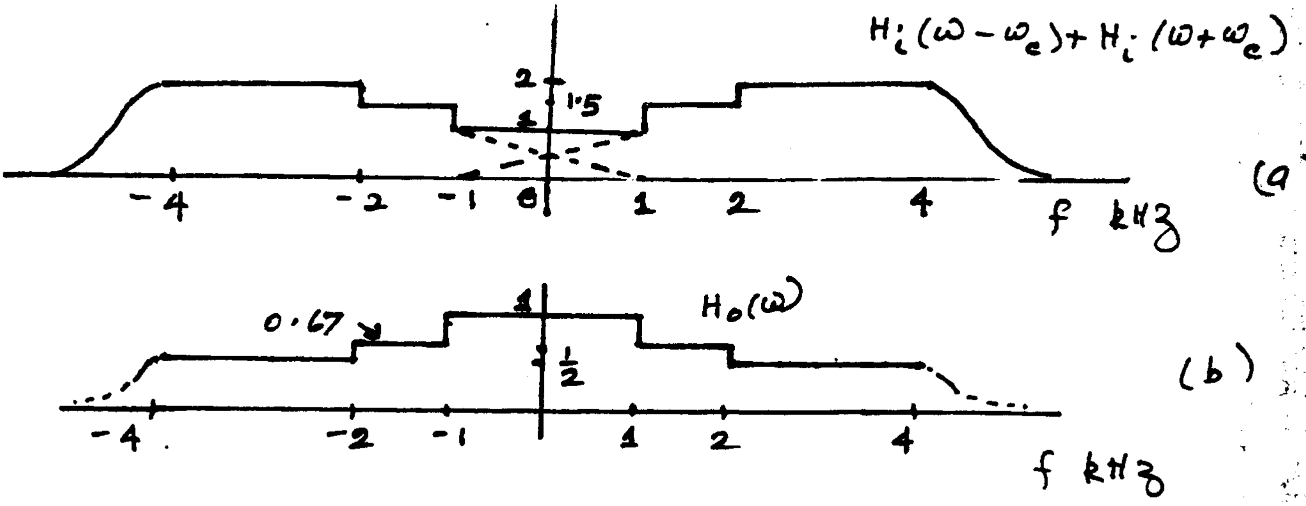



**Fig. S4.6-1**

Thus, if we apply $m_h(t)$ at the input of the Fig. 4.20, the USB output is an LSB signal corresponding to $m(t)$. The carrier also acquires a phase shift $\pi/2$. Similarly, we can show that if we apply $m_h(t)$ at the input of the Fig. 4.20, the LSB output would be an USB signal corresponding to $m(t)$ (with a carrier phase shifted by $\pi/2$).

**4.6-1** From Eq. (4.20)

$$H_o(\omega) = \frac{1}{H_i(\omega + \omega_c) + H_i(\omega - \omega_c)} \qquad |\omega| \le 2\pi B$$

Figure S4.6-1a shows $H_i(\omega - \omega_c)$ and $H_i(\omega + \omega_c)$. Figure S4.6-1b shows the reciprocal, which is $H_o(\omega)$.

**4.8-1** A station can be heard at its allocated frequency 1500 kHz as well as at its image frequency. The two frequencies are $2f_{IF}$ Hz apart. In the present case, $f_{IF} = 455$ kHz. Hence, the image frequency is $2 \times 455 = 910$ kHz apart. Therefore, the station will also be heard if the receiver is tuned to frequency $1500-910 = 590$ kHz. The reason for this is as follows. When the receiver is tuned to 590 kHz, the local oscillator frequency is $f_{LO} = 590+455 = 1045$ kHz. Now this frequency $f_{LO}$ is multiplied with the incoming signal of frequency $f_c = 1500$ kHz. The output yields the two modulated signals whose carrier frequencies are the sum and difference frequencies, which are $1500 + 1045 = 2545$ kHz and $1500 - 1045 = 455$ kHz. The sum carrier is suppressed, but the difference carrier passes through, and the station is received.

**4.8-2** The local oscillator generates frequencies in the range 1+8=9 MHz to 30+8=38 MHz. When the receiver setting is 10MHz, $f_{LO} = 10 + 8 = 18$ MHz. Now, if there is a station at $18 + 8 = 26$ MHz, it will beat (mix) with $f_{LO} = 18$ MHz to produce two signals centered at $26 + 18 = 44$ MHz and at $26 - 18 = 8$ MHz. The sum component is suppressed by the IF filter, but the difference component, which is centered at 8 MHz, passes through the IF filter.

# Chapter 5

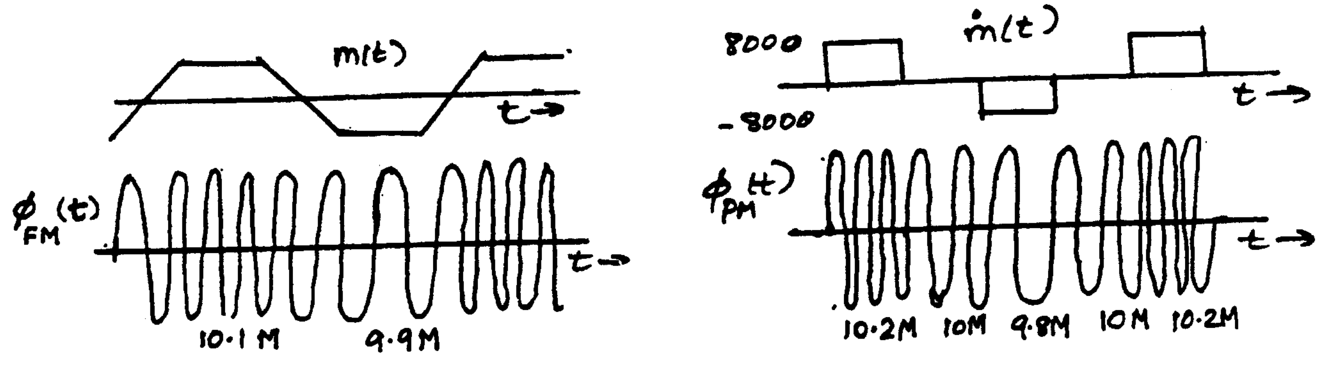



**Fig. S5.1-1**

**5.1-1** In this case $f_c = 10$ MHz. $m_p = 1$ and $m'_p = 8000$.

**For FM :**

$\Delta f = k_f m_p/2\pi = 2\pi \times 10^5/2\pi = 10^5$ Hz. Also $f_c = 10^7$. Hence, $(f_i)_{max} = 10^7 + 10^5 = 10.1$ MHz. and $(f_i)_{min} = 10^7 - 10^5 = 9.9$ MHz. The carrier frequency increases linearly from 9.9 MHz to 10.1 MHz over a quarter (rising) cycle of duration $a$ seconds. For the next $a$ seconds, when $m(t) = 1$, the carrier frequency remains at 10.1 MHz. Over the next quarter (the falling) cycle of duration $a$, the carrier frequency decreases linearly from 10.1 MHz to 9.9 MHz., and over the last quarter cycle, when $m(t) = -1$, the carrier frequency remains at 9.9 MHz. This cycles repeats periodically with the period $4a$ seconds as shown in Fig. S5.1-1a.

**For PM :**

$\Delta f = k_p m'_p/2\pi = 50\pi \times 8000/2\pi = 2 \times 10^5$ Hz. Also $f_c = 10^7$. Hence, $(f_i)_{max} = 10^7 + 2 \times 10^5 = 10.2$ MHz. and $(f_i)_{min} = 10^7 - 2 \times 10^5 = 9.8$ MHz. Figure S5.1-1b shows $\dot{m}(t)$. We conclude that the frequency remains at 10.2 MHz over the (rising) quarter cycle, where $\dot{m}(t) = 8000$. For the next $a$ seconds, $\dot{m}(t) = 0$, and the carrier frequency remains at 10 MHz. Over the next $a$ seconds, where $\dot{m}(t) = -8000$, the carrier frequency remains at 9.8 MHz. Over the last quarter cycle $\dot{m}(t) = 0$ again, and the carrier frequency remains at 10 MHz. This cycles repeats periodically with the period $4a$ seconds as shown in Fig. S5.1-1.

**5.1-2** In this case $f_c = 1$ MHz. $m_p = 1$ and $m'_p = 2000$.

**For FM :**

$\Delta f = k_f m_p/2\pi = 20,000\pi/2\pi = 10^4$ Hz. Also $f_c = 1$ MHz. Hence, $(f_i)_{max} = 10^6 + 10^4 = 1.01$ MHz. and $(f_i)_{min} = 10^6 - 10^4 = 0.99$ MHz. The carrier frequency rises linearly from 0.99 MHz to 1.01 MHz over the cycle (over the interval $-\frac{10^{-3}}{2} < t < \frac{10^{-3}}{2}$). Then instantaneously, the carrier frequency falls to 0.99 MHz and starts rising linearly to 10.01 MHz over the next cycle. The cycle repeats periodically with period $10^{-3}$ as shown in Fig. S5.1-2a.

**For PM :**

Here, because $m(t)$ has jump discontinuities, we shall use a direct approach. For convenience, we select the origin for $m(t)$ as shown in Fig. S5.1-2. Over the interval $\frac{10^{-3}}{2}$ to $\frac{10^3}{2}$, we can express the message signal as $m(t) = 2000t$. Hence,

$$\begin{aligned}
\varphi_{PM}(t) &= \cos\left[2\pi(10)^6 t + \frac{\pi}{2}m(t)\right] \\
&= \cos\left[2\pi(10)^6 t + \frac{\pi}{2}2000t\right] \\
&= \cos\left[2\pi(10)^6 t + 1000\pi t\right] = \cos\left[2\pi\left(10^6 + 500\right)t\right]
\end{aligned}$$

At the discontinuity, the amount of jump is $m_d = 2$. Hence, the phase discontinuity is $k_p m_d = \pi$. Therefore, the carrier frequency is constant throughout at $10^6 + 500$ Hz. But at the points of discontinuities, there is a

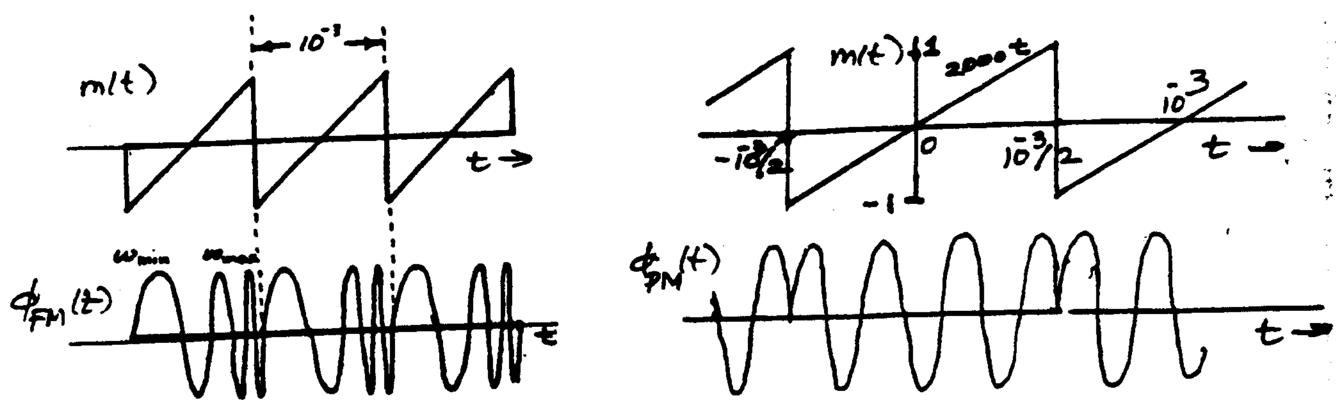



**Fig. S5.1-2**

phase discontinuity of $\pi$ radians as shown in Fig. S5.1-2b. In this case, we must maintain $k_p < \pi$ because there is a discontinuity of the amount 2. For $k_p > \pi$, the phase discontinuity will be higher than $2\pi$ giving rise to ambiguity in demodulation.

**5.1-3**

$$\text{(a)} \qquad \varphi_{PM}(t) = A\cos\left[\omega_c t + k_p m(t)\right] = 10\cos[10,000t + k_p m(t)]$$

We are given that $\varphi_{PM}(t) = 10\cos(13,000t)$ with $k_p = 1000$. Clearly, $m(t) = 3t$ over the interval $|t| \le 1$.

$$\text{(b)} \qquad \varphi_{FM}(t) = A\cos\left[\omega_c t + k_f\int^t m(\alpha)\,d\alpha\right] = 10\cos\left[10,000t + k_f\int^t m(\alpha)\,d\alpha\right]$$

$$\text{Therefore} \qquad k_f\int^t m(\alpha)\,d\alpha = 1000\int^t m(\alpha)\,d\alpha = 3000t$$

$$\text{Hence} \qquad 3t = \int^t m(\alpha)\,d\alpha \qquad \Longrightarrow m(t) = 3$$

**5.2-1** In this case $k_f = 1000\pi$ and $k_p = 1$. For

$$m(t) = 2\cos 100t + 18\cos 2000\pi t \qquad \text{and} \qquad \dot m(t) = -200\sin 100t - 36,000\pi\sin 2000\pi t$$

Therefore $m_p = 20$ and $m'_p = 36,000\pi + 200$. Also the baseband signal bandwidth $B = 2000\pi/2\pi = 1$ kHz.
**For FM :** : $\Delta f = k_f m_p/2\pi = 10,000$, and $B_{FM} = 2(\Delta f + B) = 2(20,000 + 1000) = 42$ kHz.
**For PM :** : $\Delta f = k_p m'_p/2\pi = 18,000 + \frac{100}{\pi}$ Hz, and $B_{PM} = 2(\Delta f + B) = 2(18,031.83 + 1000) = 38.06366$ kHz.

**5.2-2** $\varphi_{EM}(t) = 10\cos(\omega_c t + 0.1\sin 2000\pi t)$. Here, the baseband signal bandwidth $B = 2000\pi/2\pi = 1000$ Hz. Also,

$$\omega_i(t) = \omega_c + 200\pi\cos 2000\pi t$$

Therefore, $\Delta\omega = 200\pi$ and $\Delta f = 100$ Hz and $B_{EM} = 2(\Delta f + B) = 2(100 + 1000) = 2.2$ kHz.

**5.2-3** $\varphi_{EM}(t) = 5\cos(\omega_c t + 20\sin 1000\pi t + 10\sin 2000\pi t)$.
Here, the baseband signal bandwidth $B = 2000\pi/2\pi = 1000$ Hz. Also,

$$\omega_i(t) = \omega_c + 20,000\pi\cos 1000\pi t + 20,000\pi\cos 2000\pi t$$

Therefore, $\Delta\omega = 20,000\pi + 20,000\pi = 40,000\pi$ and $\Delta f = 20$ kHz and $B_{EM} = 2(\Delta f + B) = 2(20,000 + 1000) = 42$ kHz.

**5.2-4** The baseband signal bandwidth $B = 3 \times 1000 = 3000$ Hz.
**For FM :** $\Delta f = \frac{k_f m_p}{2\pi} = \frac{10^5 \times 1}{2\pi} = 15.951$ kHz and $B_{FM} = 2(\Delta f + B) = 37.831$ kHz.
**For PM :** $\Delta f = \frac{k_p m'_p}{2\pi} = \frac{25 \times 8000}{2\pi} = 31.831$ kHz and $B_{PM} = 2(\Delta f + B) = 66.662$ kHz.

**5.2-5** The baseband signal bandwidth $B = 5 \times 1000 = 5000$ Hz.
**For FM :** $\Delta f = \frac{k_f m_p}{2\pi} = \frac{2000\pi \times 1}{2\pi} = 1$ kHz and $B_{FM} = 2(\Delta f + B) = 2(2 + 5) = 14$ kHz.

**For PM :** To find $B_{PM}$, we observe from Fig. S5.1-2 that $\varphi_{FM}(t)$ is essentially a sequence of sinusoidal pulses of width $T = 10^{-3}$ seconds and of frequency $f_c = 1$ MHz. Such a pulse and its spectrum are depicted in Figs. 3.22c and d, respectively. The bandwidth of the pulse, as seen from Fig. 3.22d, is $4\pi/T$ rad/s or $2/T$ Hz. Hence, $B_{PM} = 2$ kHz.

**5.2-6** (a) **For FM :** $\Delta f = \frac{k_f m_p}{2\pi} = \frac{200{,}000\pi \times 1}{2\pi} = 100$ kHz and the baseband signal bandwidth $B = \frac{2000\pi}{2\pi} = 1$ kHz. Therefore

$$B_{FM} = 2(\Delta f + B) = 202 \text{ kHz}$$

**For PM :** $\Delta f = \frac{k_p m_p'}{2\pi} = \frac{10 \times 2000\pi}{2\pi} = 10$ kHz and $B_{PM} = 2(\Delta f + B) = 2(10 + 1) = 22$ kHz.

(b) $m(t) = 2 \sin 2000\pi t$, and $B = 2000\pi/2\pi = 1$ kHz. Also $m_p = 2$ and $m_p' = 4000\pi$.

**For FM :** $\Delta f = \frac{k_f m_p}{2\pi} = \frac{200{,}000\pi \times 2}{2\pi} = 200$ kHz, and

$$B_{FM} = 2(\Delta f + B) = 2(200 + 1) = 402 \text{ kHz}$$

**For PM :** $\Delta f = \frac{k_p m_p'}{2\pi} = \frac{10 \times 4000\pi}{2\pi} = 20$ kHz and $B_{PM} = 2(\Delta f + B) = 2(20 + 1) = 42$ kHz.

(c) $m(t) = \sin 4000\pi t$, and $B = 4000\pi/2\pi = 2$ kHz. Also $m_p = 1$ and $m_p' = 4000\pi$.

**For FM :** $\Delta f = \frac{k_f m_p}{2\pi} = \frac{200{,}000\pi \times 1}{2\pi} = 100$ kHz, and

$$B_{FM} = 2(\Delta f + B) = 2(100 + 2) = 204 \text{ kHz}$$

**For PM :** $\Delta f = \frac{k_p m_p'}{2\pi} = \frac{10 \times 4000\pi}{2\pi} = 20$ kHz and $B_{PM} = 2(\Delta f + B) = 2(20 + 2) = 44$ kHz.

(d) Doubling the amplitude of $m(t)$ roughly doubles the bandwidth of both FM and PM. Doubling the frequency of $m(t)$ [expanding the spectrum $M(\omega)$ by a factor 2] has hardly any effect on the FM bandwidth. However, it roughly doubles the bandwidth of PM, indicating that PM spectrum is sensitive to the shape of the baseband spectrum. FM spectrum is relatively insensitive to the nature of the spectrum $M(\omega)$.

**5.2-7** From pair 22(Table 3.1), we obtain $e^{-t^2} \Longleftrightarrow \sqrt{\pi}\, e^{-\omega^2/4}$. The spectrum $M(\omega) = \sqrt{\pi}\, e^{-\omega^2/4}$ is a Gaussian pulse, which decays rapidly. Its 3 dB bandwidth is 1.178 rad/s=0.187 Hz. This is an extremely small bandwidth compared to $\Delta f$.

Also $\dot{m}(t) = -2te^{-t^2/2}$. The spectrum of $\dot{m}(t)$ is $M'(\omega) = j\omega M(\omega) = j\sqrt{\pi}\omega\, e^{-\omega^2/4}$. This spectrum also decays rapidly away from the origin, and its bandwidth can also be assumed to be negligible compared to $\Delta f$.

**For FM :** $\Delta f = \frac{k_f m_p}{2\pi} = \frac{6000\pi \times 1}{2\pi} = 3$ kHz and $B_{FM} \approx 2\Delta f = 2 \times 3 = 6$ kHz.

**For PM :** To find $m_p'$, we set the derivative of $\dot{m}(t) = -2te^{-t^2/2}$ equal to zero. This yields

$$\ddot{m}(t) = -2e^{-t^2/2} + 4t^2 e^{-t^2/2} = 0 \quad \Longrightarrow t = \frac{1}{\sqrt{2}}$$

and $m_p' = \dot{m}(\frac{1}{\sqrt{2}}) = 0.858$, and

$\Delta f = \frac{k_p m_p'}{2\pi} = \frac{6000\pi \times 0.858}{2\pi} = 3.432$ kHz and $B_{PM} \approx 2(\Delta f) = 2(3.432) = 6.864$ kHz.

**5.3-1** The block diagram of the design is shown in Fig. S5.3-1.

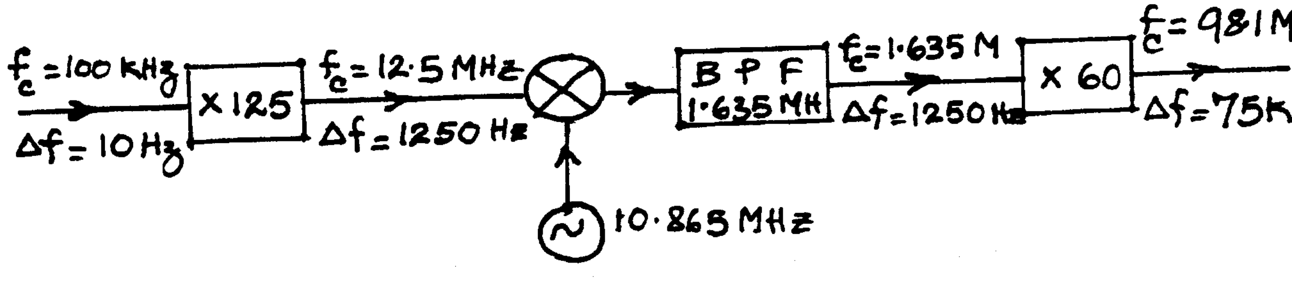



Fig. S5.3-1

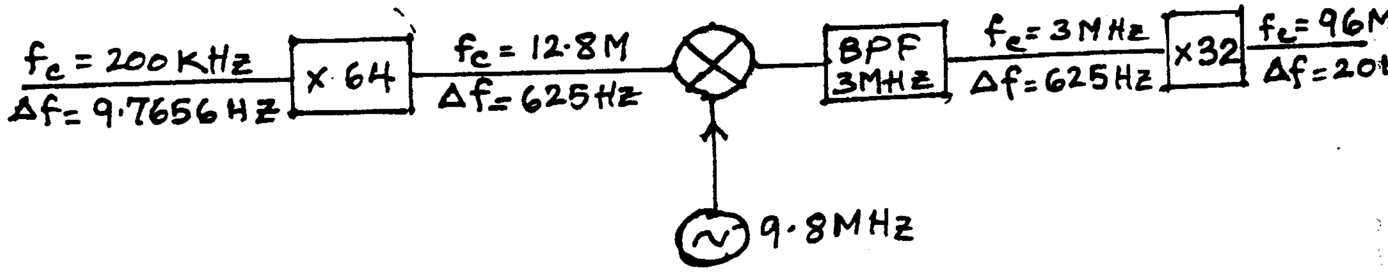



**Fig. S5.3-2**

**5.3-2** The block diagram of the design is shown in Fig. S5.3-2.

**5.4-1**

$$\text{(a)} \qquad \varphi_{PM}(t) = A\cos\left[\omega_c t + k_p m(t)\right]$$

When this $\varphi_{PM}(t)$ is passed through an ideal FM demodulator, the output is $k_p\dot{m}(t)$ This signal, when passed through an ideal integrator, yields $k_p m(t)$. Hence, FM demodulator followed by an ideal integrator acts as a PM demodulator. However, if $m(t)$ has a discontinuity, $\dot{m}(t) = \infty$ at the point(s) of discontinuity, and the system will fail.

$$\text{(b)} \qquad \varphi_{FM}(t) = A\cos\left[\omega_c t + k_f \int^t m(\alpha)\,d\alpha\right]$$

When this signal $\varphi_{FM}(t)$ is passed through an ideal PM demodulator, the output is $k_f \int^t m(\alpha)\,d\alpha$. When this signal is passed through an ideal differentiator, the output is $k_f m(t)$. Hence, PM demodulator, followed by an ideal differentiator, acts as FM demodulator regardless of whether $m(t)$ has jump discontinuities or not.

**5.4-2** Figure S5.4-2 shows the waveforms at points b, c, d, and e. The figure is self explanatory.

**5.4-3** From Eq. (5.30), the Laplace transform of the phase error $\theta_e(t)$ is given by

$$\Theta_e(s) = \frac{s}{s + AKH(s)}\Theta_i(s)$$

For $\theta_i(t) = kt^2$, $\Theta_i(s) = \frac{2k}{s^3}$, and

$$\Theta_e(s) = \frac{2k}{s^2[s + AKH(s)]}$$

The steady-state phase error [Eq. (5.33)] is

$$\lim_{t\to\infty} \theta_e(t) = \lim_{s\to 0} s\Theta_e(s) = \frac{2k}{s(s + AK)} = \infty$$

Hence, the incoming signal cannot be tracked. If

$$H(s) = \frac{s+a}{s}, \qquad \text{then} \qquad \Theta_e(s) = \frac{2k}{s^2\left[s + \frac{AK(s+a)}{s}\right]}$$

and

$$\lim_{t\to\infty} \theta_e(t) = \lim_{s\to 0} s\Theta_e(s) = \lim_{s\to 0} \frac{2k}{s^2 + AK(s+a)} = \frac{2k}{Aka}$$

Hence, the incoming signal can be tracked within a constant phase $2k/Aka$ radians. Now, if

$$H(s) = \frac{s^2 + as + b}{s^2}, \qquad \text{then} \qquad \Theta_e(s) = \frac{2k}{s^2\left[s + \frac{AK(s^2+as+b)}{s^2}\right]}$$

and

$$\lim_{t\to\infty} \theta_e(t) = \lim_{s\to 0} s\Theta_e(s) = \lim_{s\to 0} \frac{2ks}{s^3 + AK(s^2 + as + b)} = 0$$

In this case, the incoming signal can be tracked with zero phase error.

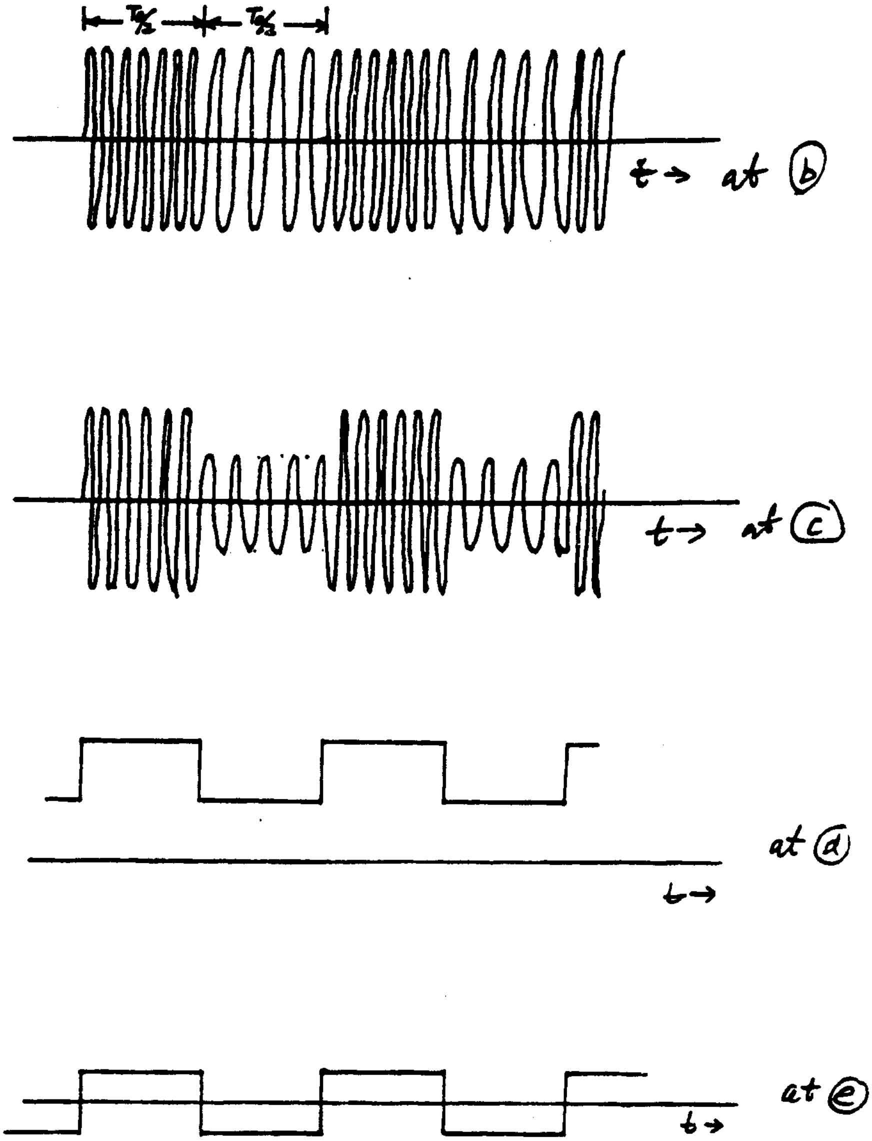



**Fig. S5.4-2**

# Chapter 6

6.1-1 The bandwidths of $g_1(t)$ and $g_2(t)$ are 100 kHz and 150 kHz, respectively. Therefore the Nyquist sampling rates for $g_1(t)$ is 200 kHz and for $g_2(t)$ is 300 kHz.
Also $g_1^2(t) \Longleftrightarrow \frac{1}{2\pi} g_1(\omega) * g_1(\omega)$, and from the width property of convolution the bandwidth of $g_1^2(t)$ is twice the bandwidth of $g_1(t)$ and that of $g_2^3(t)$ is three times the bandwidth of $g_2(t)$ (se also Prob. 4.3-10). Similarly the bandwidth of $g_1(t)g_2(t)$ is the sum of the bandwidth of $g_1(t)$ and $g_2(t)$. Therefore the Nyquist rate for $g_1^2(t)$ is 400 kHz. for $g_2^3(t)$ is 900 kHz. for $g_1(t)g_2(t)$ is 500 kHz.

6.1-2 (a)

$$\text{sinc}100\pi t) \Longleftrightarrow 0.01\text{rect}\left(\frac{\omega}{200\pi}\right)$$

The bandwidth of this signal is 100 $\pi$ rad/s or 50 Hz. The Nyquist rate is 100 Hz (samples/sec).
(b)

$$\text{sinc}^2(100\pi t) \Longleftrightarrow 0.01\Delta(\tfrac{\omega}{400\pi})$$

The bandwidth of this signal is 200 $\pi$ rad/s or 100 Hz. The Nyquist rate is 200 Hz (samples/sec).
(c)

$$\text{sinc}\,(100\pi t) + \text{sinc}\,(50\pi t) \Longleftrightarrow 0.01\text{rect}\,0.01\left(\frac{\omega}{200\pi}\right) + 0.02\text{rect}\left(\frac{\omega}{100\pi}\right)$$

The bandwidth of the first term on the right-hand side is 50 Hz and the second term is 25 Hz. Clearly the bandwidth of the composite signal is the higher of the two, that is. 100 Hz. The Nyquist rate is 200 Hz (samples/sec).
(d)

$$\text{sinc}(100\pi t) + 3\,\text{sinc}^2(60\pi t) \Longleftrightarrow 0.01\,\text{rect}(\tfrac{\omega}{200\pi}) + \tfrac{1}{20}\,\Delta(\tfrac{\omega}{240\pi})$$

The bandwidth of $\text{rect}(\frac{\omega}{200\pi})$ is 50 Hz and that of $\Delta(\frac{\omega}{240\pi})$ is 60 Hz. The bandwidth of the sum is the higher of the two, that is, 60 Hz. The Nyquist sampling rate is 120 Hz.
(e)

$$\text{sinc}(50\pi t) \Longleftrightarrow 0.02\,\text{rect}(\tfrac{\omega}{100\pi})$$
$$\text{sinc}(100\pi t) \Longleftrightarrow 0.01\,\text{rect}(\tfrac{\omega}{200\pi})$$

The two signals have bandwidths 25 Hz and 50 Hz respectively. The spectrum of the product of two signals is $1/2\pi$ times the convolution of their spectra. From width property of the convolution, the width of the convoluted signal is the sum of the widths of the signals convolved. Therefore. the bandwidth of $\text{sinc}(50\pi t)\text{sinc}(100\pi t)$ is $25 + 50 = 75$ Hz. The Nyquist rate is 150 Hz.

6.1-3 The pulse train is a periodic signal with fundamental frequency $2B$ Hz. Hence, $\omega_s = 2\pi(2B) = 4\pi B$. The period is $T_0 = 1/2B$. It is an even function of $t$. Hence, the Fourier series for the pulse train can be expressed as

$$p_{T_s}(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos n\omega_s t$$

Using Eqs. (2.72). we obtain

$$a_0 = C_0 = \frac{1}{T_0}\int_{-1/16B}^{1/16B} dt = \frac{1}{4}, \qquad a_n = C_n = \frac{2}{T_0}\int_{-1/16B}^{1/16B} \cos n\omega_s t\, dt = \frac{2}{n\pi}\sin\left(\frac{n\pi}{4}\right). \qquad b_n = 0$$

Hence.

$$\begin{aligned}\overline{g}(t) &= g(t)p_{T_s}(t)\\ &= \frac{1}{4}g(t) + \sum_{n=1}^{\infty}\frac{2}{n\pi}\sin\left(\frac{n\pi}{4}\right) g(t)\cos n\omega_s t\end{aligned}$$

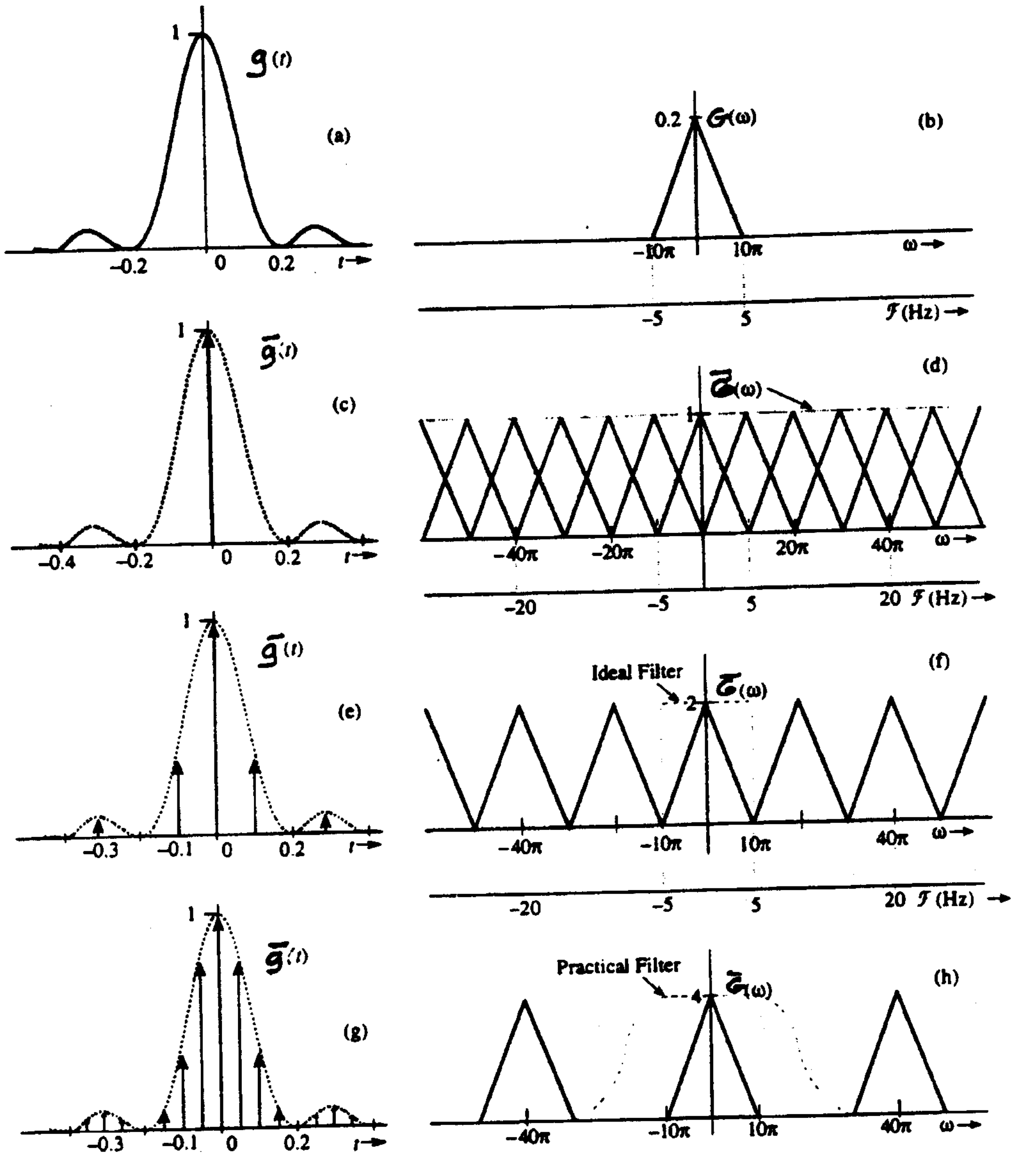



**Fig. S6.1-4**

**6.1-4** For $g(t) = \text{sinc}^2(5\pi t)$ (Fig. S6.1-4a), the spectrum is $G(\omega) = 0.2\,\Delta(\frac{\omega}{20\pi})$ (Fig. S6.1-4b). The bandwidth of this signal is 5 Hz ($10\pi$ rad/s). Consequently, the Nyquist rate is 10 Hz, that is, we must sample the signal at a rate no less than 10 samples/s. The Nyquist interval is $T = 1/2B = 0.1$ second.
Recall that the sampled signal spectrum consists of $(1/T)G(\omega) = \frac{0.2}{T}\,\Delta(\frac{\omega}{20\pi})$ repeating periodically with a period equal to the sampling frequency $f_s$ Hz. We present this information in the following Table for three sampling rates: $f_s = 5$ Hz (undersampling), 10 Hz (Nyquist rate), and 20 Hz (oversampling).

| sampling frequency $f_s$ | sampling interval $T$ | $\frac{1}{T}G(\omega)$ | comments |
|---|---|---|---|
| 5 Hz | 0.2 | $\Delta\left(\frac{\omega}{20\pi}\right)$ | Undersampling |
| 10 Hz | 0.1 | $2\Delta\left(\frac{\omega}{20\pi}\right)$ | Nyquist Rate |
| 20 Hz | 0.05 | $4\Delta\left(\frac{\omega}{20\pi}\right)$ | Oversampling |

In the first case (undersampling), the sampling rate is 5 Hz (5 samples/sec.), and the spectrum $\frac{1}{T}G(\omega)$ repeats every 5 Hz ($10\pi$ rad/sec.). The successive spectra overlap, as shown in Fig. S6.1-4d, and the spectrum $G(\omega)$ is not recoverable from $\overline{G}(\omega)$, that is, $g(t)$ cannot be reconstructed from its samples $\overline{g}(t)$ in Fig. S6.1-4c. If the sampled signal is passed through an ideal lowpass filter of bandwidth 5 Hz, the output spectrum is $\text{rect}\,(\frac{\omega}{20\pi})$.

and the output signal is $10\,\text{sinc}\,(20\pi t)$, which is not the desired signal $\text{sinc}^2(5\pi t)$. In the second case, we use the Nyquist sampling rate of 10 Hz (Fig. S6.1-4e). The spectrum $\overline{G}(\omega)$ consists of back-to-back, nonoverlapping repetitions of $\frac{1}{T}G(\omega)$ repeating every 10 Hz. Hence, $G(\omega)$ can be recovered from $\overline{G}(\omega)$ using an ideal lowpass filter of bandwidth 5 Hz (Fig. S6.1-4f). The output is $10\,\text{sinc}^2(5\pi t)$. Finally, in the last case of oversampling (sampling rate 20 Hz), the spectrum $\overline{G}(\omega)$ consists of nonoverlapping repetitions of $\frac{1}{T}G(\omega)$ (repeating every 20 Hz) with empty band between successive cycles (Fig. S6.1-4h). Hence, $G(\omega)$ can be recovered from $\overline{G}(\omega)$ using an ideal lowpass filter or even a practical lowpass filter (shown dotted in Fig. S6.1-4h). The output is $20\,\text{sinc}^2(5\pi t)$.

**6.1-5** This scheme is analyzed fully in Problem 3.4-1, where we found the bandwidths of $y_1(t)$, $y_2(t)$, and $y(t)$ to be 10 kHz, 5 kHz, and 15 kHz, respectively. Hence, the Nyquist rates for the three signals are 20 kHz, 10 kHz, and 30 kHz, respectively.

**6.1-6** (a) When the input to this filter is $\delta(t)$, the output of the summer is $\delta(t) - \delta(t-T)$. This acts as the input to the integrator. And, $h(t)$, the output of the integrator is:

$$h(t) = \int_0^t [\delta(\tau) - \delta(\tau - T)]\,d\tau = u(t) - u(t-T) = \text{rect}\left(\frac{t - \frac{T}{2}}{T}\right)$$

The impulse response $h(t)$ is shown in Fig. S6.1-6a.
(b) The transfer function of this circuit is:

$$H(\omega) = T\text{sinc}\left(\frac{\omega T}{2}\right)e^{-j\omega T/2}$$

and

$$|H(\omega)| = T\left|\text{sinc}\left(\frac{\omega T}{2}\right)\right|$$

The amplitude response of the filter is shown in Fig. S6.1-6b. Observe that the filter is a lowpass filter of bandwidth $2\pi/T$ rad/s or $1/T$ Hz.
The impulse response of the circuit is a rectangular pulse. When a sampled signal is applied at the input, each sample generates a rectangular pulse at the output, proportional to the corresponding sample value. Hence the output is a staircase approximation of the input as shown in Fig. S6.1-6c.

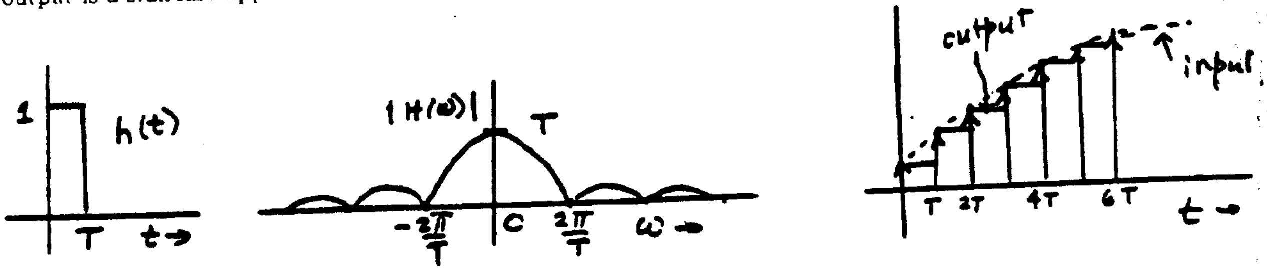



**Figure S6.1-6**

**6.1-7** (a) Figure S6.1-7a shows the signal reconstruction from its samples using the first-order hold circuit. Each sample generates a triangle of width $2T$ and centered at the sampling instant. The height of the triangle is equal to the sample value. The resulting signal consists of straight line segments joining the sample tops.
(b) The transfer function of this circuit is:

$$H(\omega) = \mathcal{F}\{h(t)\} = \mathcal{F}\left\{\Delta\left(\frac{t}{2T}\right)\right\} = T\,\text{sinc}^2\left(\frac{\omega T}{2}\right)$$

Because $H(\omega)$ is positive for all $\omega$, it also represents the amplitude response. Fig. S6.1-7b shows the impulse response $h(t) = \Delta(\frac{t}{2T})$. The corresponding amplitude response $H(\omega)$ and the ideal amplitude response (lowpass) required for signal reconstruction is shown in Fig. S6.1-7c.
(c) A minimum of $T$ secs delay is required to make $h(t)$ causal (realizable). Such a delay would cause the reconstructed signal in Fig. S6.1-7a to be delayed by $T$ secs.
(d) When the input to the first filter is $\delta(t)$, then as shown in Prob. 6.1-4, its output is a rectangular pulse $p(t) = u(t) - u(t-T)$ shown in Fig. S6.1-4a. This pulse $p(t)$ is applied to the input of the second identical filter. The output of the summer of the second filter is $p(t) - p(t-T) = u(t) - 2u(t-T) + u(t-2T)$, which is applied to the integrator. The output $h(t)$ of the integrator is the area under $p(t) - p(t-T)$, which, as

$$h(t) = \int_0^t [u(\tau) - 2u(\tau - T) + u(\tau - 2T)]\,d\tau = tu(t) - 2(t-T)u(t-T) + (t-2T)u(t-2T) = \Delta\left(\frac{t-T}{T}\right)$$

shown in Fig. S6.1-7b.

**6.1-8** Assume a signal $g(t)$ that is simultaneously timelimited and bandlimited. Let $g(\omega) = 0$ for $|\omega| > 2\pi B$. Therefore $g(\omega)\text{rect}(\frac{\omega}{4\pi B'}) = g(\omega)$ for $B' > B$. Therefore from the time-convolution property (3.43)

$$\begin{aligned} g(t) &= g(t) * [2B'\text{sinc}(2\pi B' t)] \\ &= 2B' g(t) * \text{sinc}(2\pi B' t) \end{aligned}$$

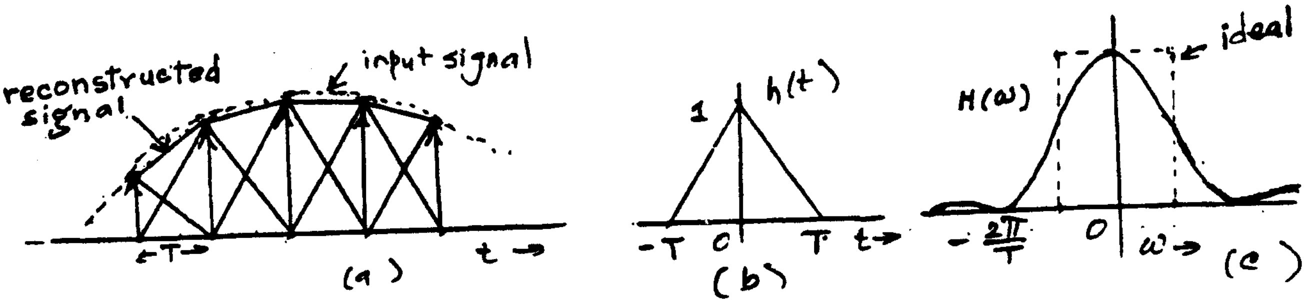



**Figure S6.1-7**

Because $g(t)$ is timelimited. $g(t) = 0$ for $|t| > T$. But $g(t)$ is equal to convolution of $g(t)$ with $\text{sinc}(2\pi B't)$ which is not timelimited. It is impossible to obtain a time-limited signal from the convolution of a time-limited signal with a non-timelimited signal.

**6.2-1** (a) Since $128 = 2^7$. we need 7 bits/character.
(b)For 100,000 characters/second . we need 700 kbits/second.
(a) 8 bits/character and 800 kbits/second.

**6.2-2**
(a) The bandwidth is 15 kHz. The Nyquist rate is 30 kHz.
(b) $65536 = 2^{16}$. so that 16 binary digits are needed to encode each sample.
(c) $30000 \times 16 = 480000$ bits/s.
(d) $44100 \times 16 = 705600$ bits/s.

**6.2-3**
(a) The Nyquist rate is $2 \times 4.5 \times 10^6 = 9$ MHz. The actual sampling rate $= 1.2 \times 9 = 10.8$ MHz.
(b) $1024 = 2^{10}$. so that 10 bits or binary pulses are needed to encode each sample.
(c) $10.8 \times 10^6 \times 10 = 108 \times 10^6$ or 108 Mbits/s.

**6.2-4** If $m_p$ is the peak sample amplitude. then

$$\text{quantization error} \leq \frac{(0.2)(m_p)}{100} = \frac{m_p}{500}$$

Because the maximum quantization error is $\frac{\Delta v}{2} = \frac{2m_p}{2L} = \frac{m_p}{L}$, it follows that

$$\frac{m_p}{L} = \frac{m_p}{500} \implies L = 500$$

Because $L$ should be a power of 2. we choose $L = 512 = 2^9$. This requires a 9-bit binary code per sample The Nyquist rate is $2 \times 1000 = 2000$ Hz. 20% above this rate is $2000 \times 1.2 = 2400$ Hz. Thus. each signal has 2400 samples/second. and each sample is encoded by 9 bits. Therefore. each signal uses $9 \times 2400 = 21.6$ kbits/second. Five such signals are multiplexed. hence. we need a total of $5 \times 21.6 = 108$ kBits/second data bits. Framing and synchronization requires additional 0.5% bits. that is. $108,000 \times 0.005 = 540$ bits. yielding a total of 108540 bits/second. The minimum transmission bandwidth is $\frac{108.54}{2} = 54.27$ kHz.

**6.2-5** Nyquist rate for each signal is 200 Hz.
The sampling rate $f_s = 2 \times$ Nyquist rate $= 400$ Hz
Total number of samples for 10 signals $= 400 \times 10 = 4000$ samples/second.
Quantization error $\leq \frac{0.25m_p}{100} = \frac{m_p}{400}$
Moreover. quantization error $= \frac{\Delta v}{2} = \frac{2m_p}{2L} = \frac{m_p}{L} = \frac{m_p}{400} \implies L = 400$
Because $L$ is a power of 2. we select $L = 512 = 2^9$. that is, 9 bits/sample.
Therefore, the minimum bit rate $= 9 \times 4000 = 36$ kbits/second.
The minimum cable bandwidth is 36/2=18 kHz.

**6.2-6** For a sinusoid. $\frac{\widetilde{m^2(t)}}{m_p^2} = 0.5$. The SNR = 47 dB =50119. From Eq. (6.16)

$$\frac{S_0}{N_0} = 3L^2\frac{\widetilde{m^2(t)}}{m_p^2} = 3L^2(0.5) = 50119 \implies L = 182.8$$

Because $L$ is a power of 2. we select $L = 256 = 2^8$. The SNR for this value of $L$ is

$$\frac{S_0}{N_0} = 3L^2\frac{\widetilde{m^2(t)}}{m_p^2} = 3(256)^2(0.5) = 98304 = 49.43 \text{ dB}$$

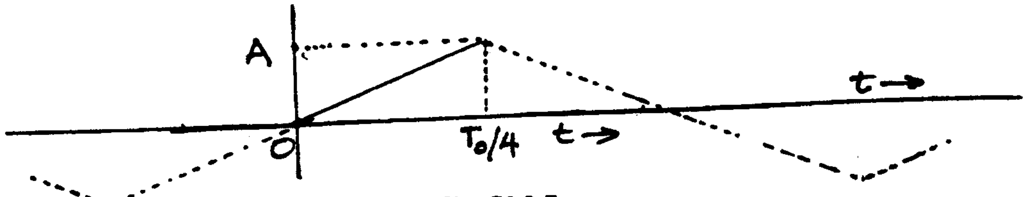



**Fig. S6.2-7**

**6.2-7** For this periodic $m(t)$, each quarter cycle takes on the same set of amplitude values. Hence, each quarter cycle contributes identical energy. Consequently, we can compute the power for this signal by averaging its energy over a quarter cycle. The equation of the first quarter cycle as shown in Fig. S6.2-7 is $m(t) = 4A/T_0$, where $A$ is the peak amplitude and $T_0$ is the period of $m(t)$. The power or the mean squared value (energy averaged over a quarter cycle) is

$$\widetilde{m^2(t)} = \frac{1}{T_0/4}\int_0^{T_0/4}\left(\frac{4A}{T_0}\right)^2 dt = \frac{A^2}{3}$$

Hence, $\frac{\widetilde{m^2(t)}}{m_p^2} = \frac{A^2/3}{A^2} = \frac{1}{3}$.
The rest of the solution is identical to that of Prob. 6.2-6. From Eq. (6.16), SNR of 47 dB is a ratio of 50119, is

$$\frac{S_0}{N_0} = 3L^2\frac{\widetilde{m^2(t)}}{m_p^2} = 3L^2(1/3) = 50119 \quad \Longrightarrow L = 223.87$$

Because $L$ is a power of 2, we select $L = 256 = 2^8$. The SNR for this value of $L$ is

$$\frac{S_0}{N_0} = 3L^2\frac{\widetilde{m^2(t)}}{m_p^2} = 3(256)^2(1/3) = 65536 = 48.16 \text{ dB}$$

**6.2-8** Here $\mu = 100$ and the SNR = 45 dB= 31,622.77. From Eq. (6.18)

$$\frac{S_0}{N_0} = \frac{3L^2}{(\ln 101)^2} = 31,622.77 \quad \Longrightarrow L = 473.83$$

Because $L$ is a power of 2, we select $L = 512 = 2^9$. The SNR for this value of $L$ is

$$\frac{S_0}{N_0} = \frac{3(512)^2}{(\ln 101)^2} = 36922.84 = 45.67 \text{ dB}$$

**6.2-9** (a) Nyquist rate = $2 \times 10^6$ Hz. The actual sampling rate is $1.5 \times (2 \times 10^6) = 3 \times 10^6$ Hz. Moreover, $L = 256$ and $\mu = 255$. From Eq. (6.18)

$$\frac{S_0}{N_0} = \frac{3L^2}{[\ln(\mu+1)]^2} = \frac{3(256)^2}{(\ln 256)^2} = 6394 = 38.06 \text{ dB}$$

(b) If we reduce the sampling rate and increase the value of $L$ so that the same number of bits/second is maintained, we can improve the SNR (because of increased $L$) with the same bandwidth. In part (a), the sampling rate is $3 \times 10^6$ Hz and each sample is encoded by 8 bits ($L = 256$). Hence, the transmission rate is $8 \times 3 \times 10^6 = 24$ Mbits/second.
If we reduce the sampling rate to $2.4 \times 10^6$ (20% above the Nyquist rate), then for the same transmission rate (24 Mbits/s), we can have $(24 \times 10^6)/(2.4 \times 10^6) = 10$ bits/sample. This results in $L = 2^{10} = 1024$. Hence, the new SNR is

$$\frac{S_0}{N_0} = \frac{3L^2}{[\ln(\mu+1)]^2} = \frac{3(1024)^2}{(\ln 256)^2} = 102300 = 50.1 \text{ dB}$$

Clearly, the SNR is increased by more than 10 dB.

**6.2-10** Equation (6.23) shows that increasing $n$ by one bit increases the SNR by 6 dB. Hence, an increase in the SNR by 12 dB (from 30 to 42) can be accomplished by increasing $n$ from 10 to 12, that is increasing by 20%.

**6.4-1** (a) From Eq. (6.33)

$$A_{max} = \frac{\sigma f_s}{\omega} \qquad \text{so that} \qquad 1 = \frac{(\sigma)(64,000)}{2\pi \times 800} \Longrightarrow \sigma = 0.0785$$

(b) $$N_0 = \frac{\sigma^2 B}{3f_s} = \frac{(0.0785)^2(3500)}{(3)(64000)} = 1.12 \times 10^{-4}$$

(c) Here $S_0 = \frac{A^2}{2} = 0.5$, and

$$\frac{S_0}{N_0} = \frac{0.5}{1.12 \times 10^{-4}} = 4.46 \times 10^3$$

(d) For uniform distribution

$$S_0 = \widetilde{m^2(t)} = \frac{m_p^2}{3} = \frac{1}{3} \qquad \text{so that} \qquad \frac{S_0}{N_0} = \frac{0.333}{1.12 \times 10^{-4}} = 2.94 \times 10^3$$

(e) For on-off signaling with a bit rate 64 kHz, we need a bandwidth of 128 kHz. For a bipolar case, we need a bandwidth of 64 kHz.

# Chapter 7

**7.2-1** For full width rect pulse $p(t) = \text{rect}\left(\frac{t}{T_b}\right)$

$$P(\omega) = T_b \operatorname{sinc}\left(\frac{\omega T_b}{2}\right)$$

For polar signaling [see Eq. (7.12)]

$$S_y(\omega) = \frac{|P(\omega)|^2}{T_b} = T_b \operatorname{sinc}^2\left(\frac{\omega T_b}{2}\right)$$

For on-off case [see Eq. (7.18b)]

$$S_y(\omega) = \frac{|P(\omega)|^2}{4T_b}\left[1 + \frac{2\pi}{T_b}\sum_{n=-\infty}^{\infty}\delta\left(\omega - \frac{2\pi n}{T_b}\right)\right]$$

$$= \frac{T_b}{4}\operatorname{sinc}^2\left(\frac{\omega T_b}{2}\right)\left[1 + \frac{2\pi}{T_b}\sum_{n=-\infty}^{\infty}\delta\left(\omega - \frac{2\pi n}{T_b}\right)\right]$$

But $\operatorname{sinc}^2\left(\frac{\omega T_b}{2}\right) = 0$ for $\omega = \frac{2\pi n}{T_b}$ for all $n \neq 0$, and $= 1$ for $n = 0$. Hence,

$$S_y(\omega) = \frac{T_b}{4}\operatorname{sinc}^2\left(\frac{\omega T_b}{2}\right) + \frac{\pi}{2}\delta(\omega)$$

For bipolar case [Eq. (7.20b)]

$$S_y(\omega) = \frac{|P(\omega)|^2}{T_b}\sin^2\left(\frac{\omega T_b}{2}\right)$$

$$= T_b \operatorname{sinc}^2\left(\frac{\omega T_b}{2}\right)\sin^2\left(\frac{\omega T_b}{2}\right)$$

The PSDs of the three cases are shown in Fig. S7.2-1. From these spectra, we find the bandwidths for all three cases to be $R_b$ Hz.

The bandwidths for the three cases, when half-width pulses are used, are as follows:

Polar and on-off: $2R_b$ Hz; bipolar: $R_b$ Hz.

Clearly, for polar and on-off cases the bandwidth is halved when full-width pulses are used. However, for the bipolar case, the bandwidth remains unchanged. The pulse shape has only a minor influence in the bipolar case because the term $\sin^2\left(\frac{\omega T_b}{2}\right)$ in $S_y(\omega)$ determines its bandwidth.

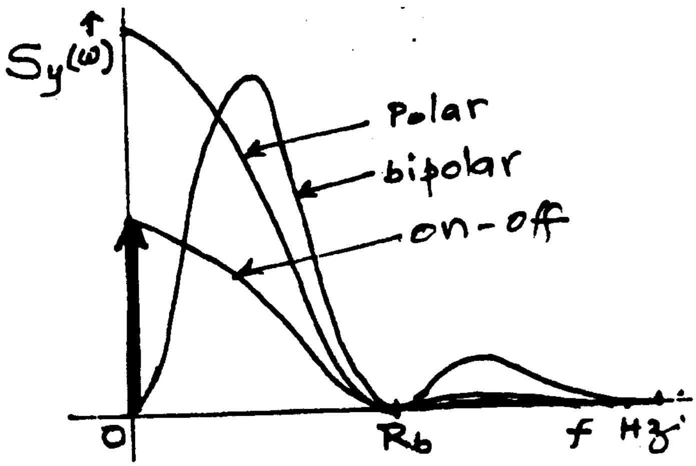



Fig. S7.2-1

7.2-2

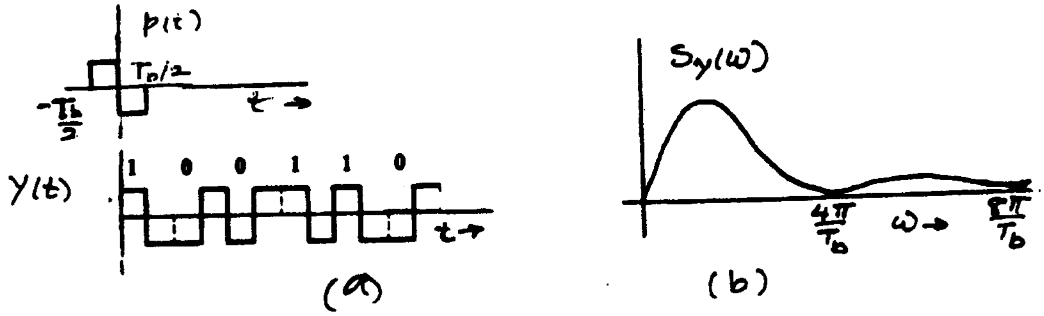



**Fig. S7.2-2**

$$P(t) = \text{rect}\left(\frac{t + \frac{T_b}{4}}{\frac{T_b}{2}}\right) - \text{rect}\left(\frac{t - \frac{T_b}{4}}{\frac{T_b}{2}}\right)$$

and

$$P(\omega) = \frac{T_b}{2}\text{sinc}\left(\frac{\omega T_b}{4}\right)e^{j\omega T_b/4} + \frac{T_b}{2}\text{sinc}\left(\frac{\omega T_b}{4}\right)e^{-j\omega T_b/4}$$
$$= jT_b \;\text{sinc}\left(\frac{\omega T_b}{4}\right)\;\sin\left(\frac{\omega T_b}{4}\right)$$

$$S_y(\omega) = \frac{|P(\omega)|^2}{T_b} = T_b\;\text{sinc}^2\left(\frac{\omega T_b}{4}\right)\;\sin^2\left(\frac{\omega T_b}{4}\right)$$

From Fig. S7.2-2, it is clear that the bandwidth is $\frac{4\pi}{T_b}$ rad/s or $2R_b$ Hz.

7.2-3 For differential code (Fig. 7.17)

$$R_0 = \lim_{N\to\infty} \frac{1}{N}\left[\frac{N}{2}(1)^2 + \frac{N}{2}(-1)^2\right] = 1$$

To compute $R_1$, we observe that there are four possible 2-bit sequences 11, 00, 01, and 10, which are equally likely. The product $a_k a_{k+1}$ for the first two combinations is 1 and is $-1$ for the last two combinations. Hence,

$$R_1 = \lim_{N\to\infty} \frac{1}{N}\left[\frac{N}{2}(1) + \frac{N}{2}(-1)\right] = 0$$

Similarly, we can show that $R_n = 0 \qquad n > 1$ Hence,

$$S_y(\omega) = \frac{|P(\omega)|^2}{T_b} = \left(\frac{T_b}{4}\right)\text{sinc}^2\left(\frac{\omega T_b}{4}\right)$$

7.2-4 (a) Fig. S7.2-4 shows the duobinary pulse train $y(t)$ for the sequence 1110001101001010.

(b) To compute $R_0$, we observe that on the average, half the pulses have $a_k = 0$ and the remaining half have $a_k = 1$ or $-1$. Hence,

$$R_0 = \lim_{N\to\infty} \frac{1}{N}\left[\frac{N}{2}(\pm 1)^2 + \frac{N}{2}(0)\right] = \frac{1}{2}$$

To determine $R_1$, we need to compute $a_k a_{k+1}$. There are four possible equally likely sequences of two bits: 11, 10, 01, 00. Since bit 0 is encoded by no pulse ($a_k = 0$), the product of $a_k a_{k+1} = 0$ for the last three of these sequences. This means on the average $\frac{3N}{4}$ combinations have $a_k a_{k+1} = 0$ and only $\frac{N}{4}$ combinations

have nonzero $a_k a_{k+1}$. Because of the duobinary rule, the bit sequence **11** can only be encoded by two consecutive pulses of the same polarity (both positive or both negative).
This means $a_k$ and $a_{k+1}$ are 1 and 1 or $-1$ and $-1$ respectively. In either case $a_k a_{k+1} = 1$. Thus, these $\frac{N}{4}$ combinations have $a_k a_{k+1} = 1$. Therefore,

$$R_1 = \lim_{N\to\infty} \frac{1}{N}\left[\frac{N}{4}(1) + \frac{3N}{4}(0)\right] = \frac{1}{4}$$

To compute $R_2$ in a similar way, we need to observe the product $a_k a_{k+2}$. For this we need to observe all possible combinations of three bits in sequence. There are eight equally likely combinations: **111, 101, 110, 100, 011, 010, 001,** and **000**. The last six combinations have either the first and/or the last bit **0**. Hence, $a_k a_{k+2} = 0$ for all these six combinations. The first two combinations are the only ones which yield nonzero $a_k a_{k+2}$. Using the duobinary rule, the first combination is encoded by three pulses of the same polarity (all positive or negative). Thus $a_k$ *and* $a_{k+2}$ are 1 and 1 or -1 and -1, respectively, yielding $a_k a_{k+2} = 1$. Similarly, because of the duobinary rule, the first and the third pulses in the second bit combination **101** are of opposite polarity yielding $a_k a_{k+2} = -1$. Thus on the average, $a_k a_{k+2} = 1$ for $\frac{N}{8}$ terms, $-1$ for $\frac{N}{8}$ terms, and 0 for $\frac{3N}{4}$ terms. Hence,

$$R_2 = \lim_{N\to\infty} \frac{1}{N}\left[\frac{N}{8}(1) + \frac{N}{8}(-1) + \frac{3N}{4}(0)\right] = 0$$

In a similar way we can show that $R_n = 0 \quad n > 1$, and from Eq. (7.10c), we obtain

$$S_y(\omega) = \frac{|P(\omega)|^2}{2T_b}(1 + \cos\omega T_b) = \frac{|P(\omega)|^2}{T_b}\cos^2\left(\frac{\omega T_b}{2}\right)$$

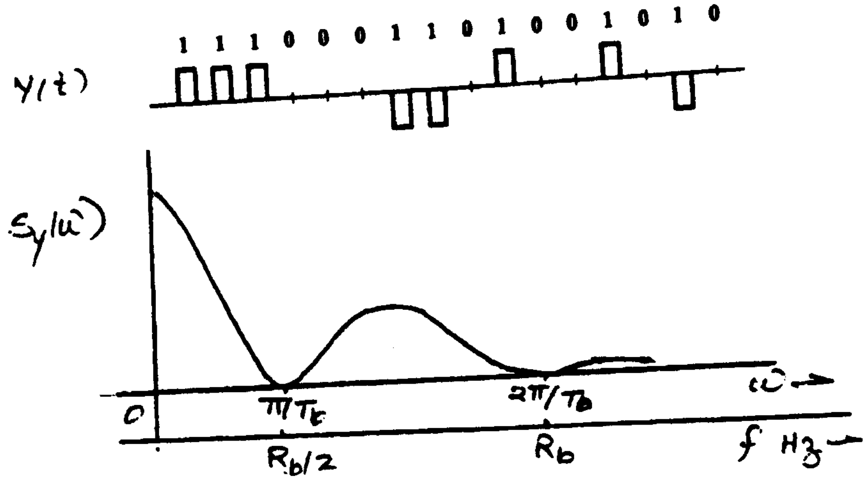



**Fig. S7.2-4**

For half-width pulse $P(t) = \text{rect}(2t/T_b)$.

$$S_y(\omega) = \frac{T_b}{4}\text{sinc}^2\left(\frac{\omega T_b}{4}\right)\cos^2\left(\frac{\omega T_b}{2}\right)$$

From Fig. S7.2-4 we observe that the bandwidth is approximately $R_b/2$ Hz.

**7.3-1** From Eq. (7.32)

$$4000 = \frac{(1+r)6000}{2} \Rightarrow r = \frac{1}{3}$$

7.3-2 Quantization error $\frac{\Delta V}{2} = \frac{m_p}{L} \le 0.01 m_p \Rightarrow L \ge 100$

(a) Because $L$ is a power of 2, we select $L = 128 = 2^7$

(b) This requires 7 bit code per sample. Nyquist rate $= 2 \times 2000 = 4$ kHz for each signal. The sampling rate $f_s = 1.25 \times 4000 = 5$ kHZ.

Eight signals require $8 \times 5000 = 40{,}000$ samples/sec.

Bit rate $= 40{,}000 \times 7 = 280$ kbits/s. From Eq. (7.32)

$$B_T = \frac{(1+r)R_b}{2} = \frac{1.2 \times 280 \times 10^3}{2} = 168 \text{ kHz.}$$

7.3-3 (a) $B_T = 2R_b \Rightarrow R_b = 1.5$ kbits/s.

(b) $B_T = R_b \Rightarrow R_b = 3$ kbits/s.

(c) $B_T = \frac{1+r}{2} R_b$. Hence, $3000 = \frac{1.25}{2} R_b \Rightarrow R_b = 4.8$ kbits/s.

(d) $B_T = R_b \Rightarrow R_b = 3$ kbits/s.

(e) $B_T = R_b \Rightarrow R_b = 3$ kbits/s.

7.3-4 (a) Comparison of $P(\omega)$ with that in Fig. 7.12 shows that this $P(\omega)$ does satisfy the Nyquist criterion with $\omega_b = 2\pi \times 10^6$ and $r = 1$. The excess bandwidth $\omega_x = \pi \times 10^6$.

(b) From Table 3.1, we find

$$p(t) = \text{sinc}^2\left(\pi \times 10^6 t\right)$$

From part (a), we have $\omega_b = 2\pi \times 10^6$ and $R_b = 10^6$. Hence, $T_b = 10^{-6}$. Observe that

$$p(n\,T_b) = 1 \qquad n = 0$$
$$= 0 \qquad n \ne 0$$

Hence $P(t)$ satisfies Eq. (7.36).

(c) the pulse transmission rate is $\frac{1}{T_b} = R_b = 10^6$ bits/s.

7.3-5 In this case $\frac{R_b}{2} = 1$ MHz. Hence, we can transmit data at a rate $R_b = 2$ MHz.

Also, $B_T = 1.2$ MHz. Hence, from Eq. (7.32)

$$1.2 \times 10^6 = \frac{1+r}{2}\left(2 \times 10^6\right) \Rightarrow r = 0.2$$

7.3-6 $f_2 = 700$ kHz. Also, $\frac{R_b}{2} = 500$ kHz and $f_x = 700 - 500 = 200$ kHz.

Hence, $r = \frac{f_x}{R_b/2} = 0.4$ and $f_1 = \frac{R_b}{2} - f_x = 500 - 200 = 300$ kHz.

7.3-7 To obtain the inverse transform of $P(\omega)$, we derive the dual of Eq. (3.35) as follows:

$$g(t-T) \Leftrightarrow G(\omega)e^{-jT\omega} \text{ and } g(t+T) \Leftrightarrow G(\omega)e^{jT\omega}$$

Hence,

$$g(t+T) + g(t-T) \Leftrightarrow 2G(\omega)\cos T\omega \tag{1}$$

Now, $P(\omega)$ in Eq. (7.34a) can be expressed as

$$P(\omega) = \frac{1}{2}\text{rect}\left(\frac{\omega}{4\pi R_b}\right) + \frac{1}{2}\text{rect}\left(\frac{\omega}{4\pi R_b}\right)\cos\left(\frac{\omega}{2R_b}\right) \tag{2}$$

Using Pair 17 (Table 3.1) and Eq. (1) above, we obtain

$$P(t) = R_b \operatorname{sinc}(2\pi R_b t) + \frac{R_b}{2}\operatorname{sinc}\left[2\pi R_b\left(t+\frac{1}{2R_b}\right)\right] + \frac{R_b}{2}\operatorname{sinc}\left[2\pi R_b\left(t-\frac{1}{2R_b}\right)\right]$$

$$= R_b\left[\operatorname{sinc}(2\pi R_b t) + \frac{1}{2}\operatorname{sinc}(2\pi R_b t + \pi) + \frac{1}{2}\operatorname{sinc}(2\pi R_b t - \pi)\right]$$

$$= R_b\left[\frac{\sin(2\pi R_b t)}{2\pi R_b t} + \frac{1}{2}\frac{\sin(2\pi R_b t+\pi)}{2\pi R_b t+\pi} + \frac{1}{2}\frac{\sin(2\pi R_b t-\pi)}{2\pi R_b t-\pi}\right]$$

$$= R_b\left[\frac{\sin(2\pi R_b t)}{2\pi R_b t} - \frac{1}{2}\frac{\sin(2\pi R_b t)}{2\pi R_b t+\pi} - \frac{1}{2}\frac{\sin(2\pi R_b t)}{2\pi R_b t-\pi}\right]$$

$$= R_b \sin(2\pi R_b t)\left[\frac{1}{2\pi R_b t} - \frac{1/2}{(2\pi R_b t+\pi)} - \frac{1/2}{(2\pi R_b t-\pi)}\right]$$

$$= R_b \sin(2\pi R_b t)\left[\frac{1}{2\pi R_b t\left(1-4R_b^2 t^2\right)}\right]$$

$$= \frac{2R_b \cos\pi R_b t \sin\pi R_b t}{2\pi R_b t\left(1-4R_b^2 t^2\right)} = \frac{R_b \cos\pi R_b t}{1-4R_b^2 t^2}\operatorname{sinc}(\pi R_b t)$$

7.3-8

$$P(\omega) = \frac{2}{R_b}\cos\left(\frac{\omega}{2R_b}\right)\operatorname{rect}\left(\frac{\omega}{2\pi R_b}\right)e^{-j\omega/2R_b}$$

$$= \frac{1}{R_b}\operatorname{rect}\left(\frac{\omega}{2\pi R_b}\right)\left[e^{j\omega/2R_b} + e^{-j\omega/2R_b}\right]e^{-j\omega/2R_b}$$

$$= \frac{1}{R_b}\operatorname{rect}\left(\frac{\omega}{2\pi R_b}\right) + \frac{1}{R_b}\operatorname{rect}\left(\frac{\omega}{2\pi R_b}\right)e^{-j\omega/2R_b}$$

Hence,

$$p(t) = \operatorname{sinc}(\pi R_b t) + \operatorname{sinc}\left[\pi R_b\left(t-\frac{1}{R_b}\right)\right]$$

$$= \frac{\sin\pi R_b t}{\pi R_b t} + \frac{\sin(\pi R_b t-\pi)}{\pi R_b t-\pi}$$

$$= \frac{\sin\pi R_b t}{\pi R_b t} - \frac{\sin\pi R_b t}{\pi R_b t-\pi} = \frac{\sin\pi R_b t}{\pi R_b t(1-R_b t)}$$

7.3-9 The Nyquist interval is $T_s = \frac{1}{R_b} = T_b$. The Nyquist samples are $p(\pm nT_b)$ for $n = 0, 1, 2, \ldots\ldots$

From Eq. (7.16), it follows that

$$p(0) = p(T_b) = 1 \text{ and } p(\pm nT_0) = 0 \text{ for all other } n.$$

Hence, from Eq. (6.10) with $T_s = T_b$, and $B = \frac{R_b}{2} = \frac{1}{2T_b}$,

$$p(t) = \operatorname{sinc}\pi R_b t + \operatorname{sinc}\left[\pi R_b\left(t-\frac{1}{R_b}\right)\right]$$

$$= \frac{\sin\ \pi R_b t}{\pi R_b} - \frac{\sin\ \pi R_b t}{\pi R_b t-\pi} = \frac{\sin\ \pi R_b t}{\pi R_b t(1-R_b t)}$$

The Fourier transform of Eq. (1) above yields

$$P(\omega)=\frac{1}{R_b}\operatorname{rect}\left(\frac{\omega}{2\pi R_b}\right)+\frac{1}{R_b}\operatorname{rect}\left(\frac{\omega}{2\pi R_b}\right)e^{-j\omega/R_b}$$
$$=\frac{1}{R_b}\operatorname{rect}\left(\frac{\omega}{2\pi R_b}\right)\left[e^{j\omega/2R_b}+e^{-j\omega/2R_b}\right]e^{-j\omega/2R_b}$$
$$=\frac{2}{R_b}\cos\left(\frac{\omega}{2R_b}\right)\operatorname{rect}\left(\frac{\omega}{2\pi R_b}\right)e^{-j\omega/2R_b}$$

7.3-10 (a) No error because the sample values of the same polarities are separated by even number of zeros and the sample values of opposite polarities are separated by odd number of zeros.
(b) The first sample value is 1 because there is no pulse before this digit. Hence the first digit is 1. The detected sequence is

**110001001101101001**

7.3-10 The first sample value is 1, indicating that the transmissions starts with a positive pulse, that is, first digit 1. The duobinary rule is violated over the digits shown by underbracket.

$$1\;2\;0\;0\;0\;-2\;0\;0\;-2\;0\;\underbrace{2\;0\;0\;-2}\;0\;2\;2\;0\;-2$$

Following are possible correct sample values in place of the 4 underbracket values: 2 2 0 –2, or 2 0 –2 –2, or 0 0 0 –2, or 2 0 0 0. These sample values represent the following 4 digit sequence: **1100**, or **1000**, or **0100**, or **1010**. Hence the 4 possible correct digit sequences are

$$\mathbf{1101001001x_1x_2x_3x_4 11100}$$

where $x_1x_2x_3x_4$ is any of the four possible sequences **1100, 1000, 0100**, or **1010**.

7.4-1 $S = 101010100000111$
From example 7.2
$$T=\left(1\oplus D^3\oplus D^5\oplus D^6\oplus D^9\oplus D^{10}\oplus D^{11}\oplus D^{12}\oplus D^{13}\oplus D^{15}\oplus\ldots\right)S$$
$$R=\left(1\oplus D^3\oplus D^5\right)T$$
$T = 101110001101001$
$R = 101010100000111 = S$

7.4-2 $S = 101010100000111$
$$T=\left(1\oplus D^2\oplus D^4\oplus D^6\oplus D^8\oplus D^{10}\oplus D^{12}\oplus D^{14}\oplus\ldots\right)S$$
$R=\left(1\oplus D^2\right)T$ (see Fig. S7.4-2)
$T = 100010000000110$
$R = 101010100000111 = S$

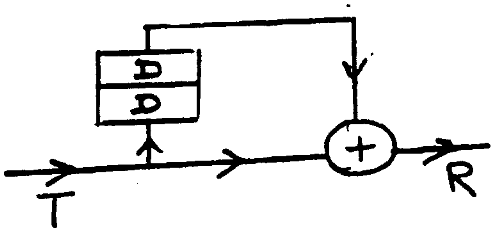


Fig. S7.4-2

7.4-3 $S = 101010100000111$
$$T=\left(1\oplus D\oplus D^2\oplus D^4\oplus D^7\oplus D^8\oplus D^9\oplus D^{11}\oplus D^{14}\right)S$$
$R=\left(1\oplus D\oplus D^3\right)T$ (see Fig. S7.4-3)
$T = 110111101001011$
$R = 101010100000111 = S$

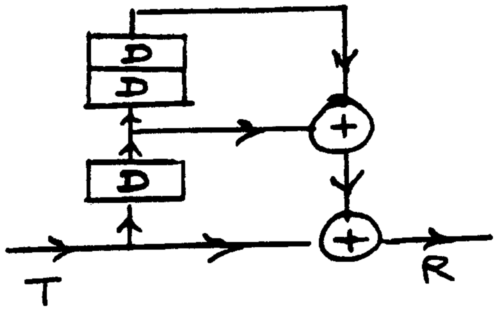



Fig. S7.4-3

**7.5-1** From Eq. (7.45), we obtain

$$\begin{bmatrix} c_{-1} \\ c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 1 & 0.3 & -0.07 \\ 0.1 & 1 & 0.3 \\ -0.002 & 0.1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.328 \\ 1.07 \\ -0.113 \end{bmatrix}$$

**7.6-1** (a) $\dfrac{A_p}{\sigma_n} = 5$

(i) For polar case $P_e = Q(5) = 2.87 \times 10^{-7}$

(ii) For on-off case $P_e = Q(5/2) = 0.00621$

(iii) For bipolar case $P_e = 1.5Q(5/2) = 0.009315$

In the following discussion, we assume $A_p = A$, the pulse amplitude.

(b) Energy of each pulse is $E_p = A^2 T_b / 2$ and there are $R_b$ pulses/second for polar case and $\dfrac{R_b}{2}$ pulses/second for on-off and bipolar case. Hence, the received powers are

$$P_{\text{polar}} = \frac{A^2 T_b}{2} R_b = \frac{A^2}{2} = \frac{(0.0015)^2}{2} = 1.125 \times 10^{-6}$$

$$P_{\text{on-off}} = \frac{A^2 T_b}{2} \times \frac{R_b}{2} = \frac{A^2}{4} = 0.5625 \times 10^{-6}$$

$$P_{\text{bipolar}} = \frac{A^2 T_b}{2} \times \frac{R_b}{2} = \frac{A^2}{4} = 0.5625 \times 10^{-6}$$

(c) For on-off case:

We require $P(\epsilon) = 2.87 \times 10^{-7} = Q(A_p / 2\sigma_n)$. Hence,

$$A_p / 2\sigma_n = 5 \text{ and } A_p = 10\sigma_n = 0.003$$

$$P_{\text{on-off}} = \frac{A^2}{4} = \frac{(0.003)^2}{4} = 2.25 \times 10^{-6}$$

For bipolar case:

$$P(\epsilon) = 2.87 \times 10^{-7} = 1.5Q(A_p / 2\sigma_n) \Rightarrow \frac{A_p}{\sigma_n} = 5.075$$

Hence

$$A = A_p = 5.075 \times 2\sigma_n = 0.003045$$

and

$$P_{\text{bipolar}} = \frac{A^2}{4} = 2.31 \times 10^{-6}$$

**7.6-2** For on-off case:

$$P_e = 10^{-6} \le Q\left(\frac{A}{2\sigma_n}\right) \Rightarrow \frac{A_p}{2\sigma_n} \ge 4.75$$

$$\sigma_n = 10^{-3} \Rightarrow A_p \geq (4.75)(2\times 10^{-3}) = 9.5\times 10^{-3}$$

For on-off case, half the pulses are zero, and for half-width rectangular pulses, the transmitted power is:

$$S_i = \frac{1}{2}\left(\frac{A_p^{\,2}}{2}\right) = \frac{A_p^{\,2}}{4} = \frac{\left(9.5\times 10^{-3}\right)}{4} = 22.56\times 10^{-6}\text{ watts.}$$

There is an attenuation of 30 dB, or equivalently, a ratio of 1000 during transmission. Therefore

$$S_T = 1000 S_i = 22.56\times 10^{-3}\text{ watts}$$

7.6-3 For polar case:

$$P_e = 10^{-6} = Q\left(\frac{A_p}{\sigma_n}\right) \Rightarrow \frac{A_p}{\sigma_n} = 4.75 \Rightarrow A_p = 4.75\times 10^{-3}$$

For polar case with half-width rectangular pulse:

$$S_i = \left(\frac{A_p^{\,2}}{2}\right) = \frac{1}{2}\left(4.5\times 10^{-3}\right)^2 = 11.28\times 10^{-6}\text{ watts}$$

$$S_T = (1000)\left(11.28\times 10^{-6} W\right) = 11.28\times 10^{-3}\text{ watts}$$

For bipolar case:

$$P_e = 10^{-6} = 1.5Q\left(\frac{A_p}{2\sigma_n}\right) \Rightarrow \frac{A_p}{2\sigma_n} = 4.835 \text{ and } A_p = 4.835\times 2\times 10^{-3} = 9.67\times 10^{-3}$$

For bipolar (or duobinary), half the pulses are zero and the receive power $S_i$ for half-width rectangular pulses is

$$S_i = \frac{A_p^{\,2}}{4} = \frac{1}{4}\left(9.67\times 10^{-3}\right)^2 = 23.38\times 10^{-6}\text{ watts}$$

$$S_T = (1000)S_i = 23.38\times 10^{-3}\text{ watts}$$

7.7-2 Sampling rate $= 2\times 4000\times 1.25 = 10{,}000$ Hz.

Quantization error $= \frac{m_p}{L} = 0.001 m_p \Rightarrow L = 1000$

Because $L$ is a power of 2, we select $L = 1024 = 2^{10}$. Hence, $n = 10$ bits/sample.

(a) Each 4-ary pulse conveys $\log_2 4 = 2$ bits of information. Hence, we need $\frac{10}{2} = 5$ 4-ary pulses/sample, and a total of $5\times 10{,}000 = 50{,}000$ 4-ary pulses/second. Therefore, the minimum transmission bandwidth is

$$\frac{50{,}000}{2} = 25\text{ kHz.}$$

(c)
$$B_T = \frac{R_b(1+r)}{2} = \frac{50{,}000(1.25)}{2} = 31.25\text{ kHz.}$$

7.7-3 (a) Each 8-ary pulse carries $\log_2 8 = 3$ bits of information. Hence, the bandwidth is reduced by a factor of 3.

(b) The amplitudes of the 8 pulses used in this 8-ary scheme are $\pm A/2$, $\pm 3A/2$, $\pm 5A/2$, and $\pm 7A/2$. Consider binary case using pulses $\pm A/2$. Let the energy of each of these pulses (of amplitude $\pm A/2$) be $E_b$. The power of this binary case is

$$P_{\text{binary}} = E_b R_b$$

Because the pulse energy is proportional to the square of the amplitude, the energy of a pulse $\pm\frac{kA}{2}$ is $k^2 E_b$. Hence, the average energy of the 8 pulses in the 8-ary case above is

$$E_{av} = \frac{E_b\left[2(\pm1)^2 + 2\left(\pm\frac{3}{2}\right)^2 + 2\left(\pm\frac{5}{2}\right)^2 + 2\left(\pm\frac{7}{2}\right)^2\right]}{8} = 21E_b$$

Hence,

$$P_{8\text{-ary}} = E_{av} \times \text{pulse rate} = 21E_b \times \frac{R_b}{3} = 7E_bR_b\,.$$

Therefore,

$$P_{8\text{-ary}} = 7P_{\text{binary}}$$

7.7-1 (a) $M = 16$. Each 16-ary pulse conveys the information of $\log_2 16 = 4$ bits. Hence, we need $\frac{12{,}000}{4} = 3000$ 16-ary pulses/second.

Minimum transmission bandwidth $= \frac{3000}{2} = 1500$ Hz.

(b) From Eq. (7.32), we have $R_b = \frac{2}{1+r}B_T$. Hence,

$$3000 = \frac{2}{1.2}B_T \Rightarrow B_T = 1800 \text{ Hz}.$$

7.7-4 (a) For polar signaling, $R_b$ bits/second requires a bandwidth of $R_b$ Hz. The half-width rectangular pulse of amplitude $\frac{A}{2}$ has energy

$$E_b = \left(\frac{A}{2}\right)^2 \frac{T_b}{2} = \frac{A^2T_b}{8}$$

The power $P$ is given by $P = E_bR_b = \frac{A^2T_b}{8}R_b = \frac{A^2}{8}$

(b) The energy of a pulse $\pm\frac{kA}{2}$ is $k^2E_b$. Hence the average energy of the M-ary pulse is

$$\begin{aligned} E_M &= \frac{1}{M}\left[2E_b + 2(\pm3)^2 + 2(\pm5)^2 + \ldots.. + 2[\pm(M-1)]^2 E_b\right] \\ &= \frac{2E_b}{M}\sum_{k=0}^{\frac{M-2}{2}}(2k+1)^2 \\ &= \frac{M^2-1}{3}E_b \end{aligned}$$

Each M-ary pulse conveys the information of $\log_2 M$ bits. Hence we require only $\frac{R_b}{\log_2 M}$ M-ary pulses/second. The power $P_M$ is given by

$$P_M = \frac{E_MR_b}{\log_2 M} = \frac{(M^2-1)R_b}{3\log_2 M}E_b = \frac{(M^2-1)A^2}{24\log_2 M} \cong \frac{M^2A^2}{24\log_2 M}$$

7.7-5 Each sample requires 8 bits $(256 = 2^8)$. Hence: $24{,}000 \times 8 = 192{,}000$ bits/sec.
$B_T = 30$ kHz

$$R = \frac{2}{1+r}B_T = \frac{2}{1.2}(30{,}000) = 50{,}000 \text{ M-ary pulses/sec.}$$

We have available up to 50,000 M-ary pulses/second to transmit 192,000 bits/sec. Hence, each pulse must transmit at least $\left(\frac{192{,}000}{50{,}000}\right) = 3.84$ bits.

⇒ choose 4 bits/pulse
⇒ $M = 16$ is the smallest acceptable value

**7.8-1** (a) Baseband polar signal at a rate of 1Mbits/sec and using full width pulses has $BW = 1\text{MHz}$. PSK doubles the $BW$ to 2MHz.

(b) FSK can be viewed as a sum of 2 ASK signals. Each ASK signal $BW = 2$ MHz. The first ASK signal occupies a band $f_{c0} \pm 1$ MHz, and the second ASK signal occupies a band $f_{c1} \pm 1$ MHz. Hence, the bandwidth is 2 MHz + 100 kHz = 2.1 MHz.

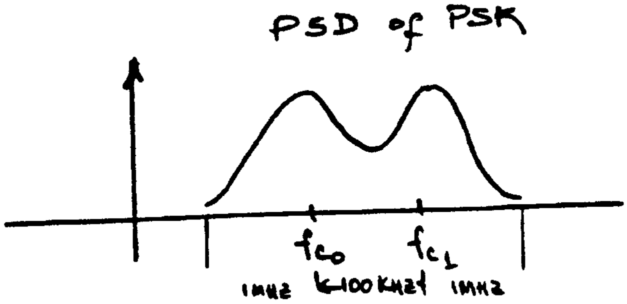



Fig.S7.8-1

**7.8-2** (a) A baseband polar signal at a rate 1 Mbits/sec using Nyquist criterion pulses at $r = 0.2$ has a

$$BW = \frac{(1+r)}{2} R_b = \frac{1.2}{2} \times 10^6 = 6.0 \times 10^5 \text{ Hz}.$$

PSK doubles $BW$ to 1.2 MHz.

(b) Similar to Prob. 7.8-1.

$$BW_{\text{FSK}} = 0.6 \text{ MHz} + 0.6 \text{ MHz} + 100 \text{ kHz}$$
$$BW_{\text{FSK}} = 1.3 \text{ MHz}$$

**7.8-3** $\log_2 M = 2$ for $M = 4$.

We need to transmit only $0.5 \times 10^6$ 4-ary pulses/sec

(a) $BW$ is reduced by a factor of 2.

$$BW_{\text{FSK}} = 1 \text{ MHz}$$

(b) In FSK, there are four center (carrier) frequencies $f_{c1}, f_{c2}, f_{c3}$, and $f_{c4}$, each separated by 100 kHz. Since ASK signal occupies band $f_c \pm 0.5$ MHz, the total bandwidth is

$0.5 \text{ MHz} + 0.5 \text{ MHz} + 100 \text{ kHz} + 100 \text{ kHz} + 100 \text{ kHz} = 1.3 \text{ MHz}.$

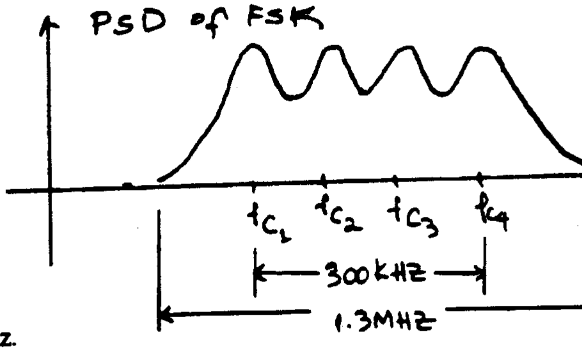



Fig. S7.8-3

**7.9-1**

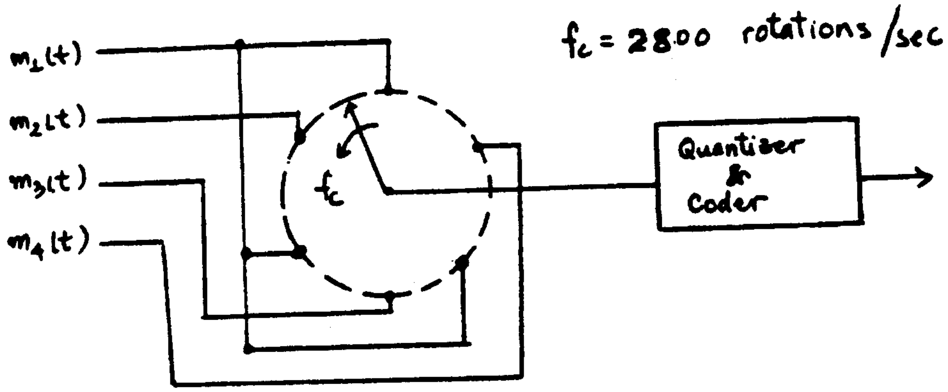



Fig. S7.9-1 (a)

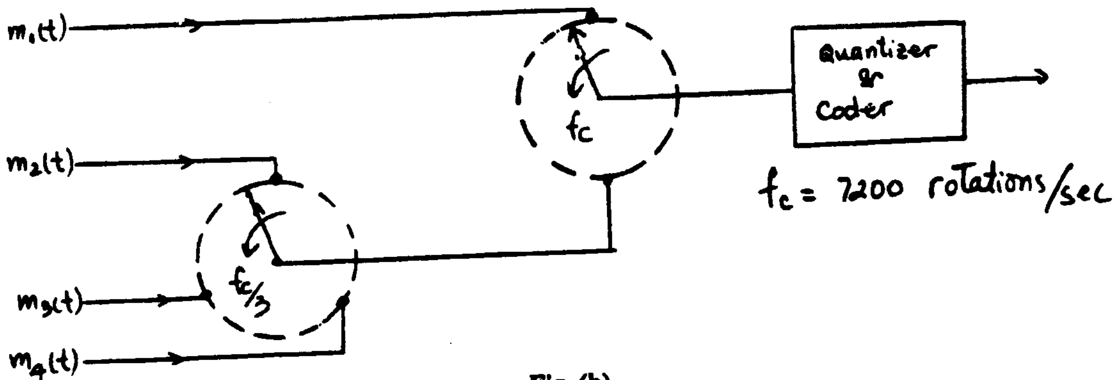



**Fig. (b)**
**Fig. S7.9-1**

Either figure (a) or (b) yields the same result.

$m_1(t)$ has 8400 samples/sec.

$m_2(t)$, $m_3(t)$, $m_4(t)$ each has 2800 samples/sec.

Hence, there are a total of 16,800 samples/sec.

7.9-2 First, we combine $m_2(t)$, $m_3(t)$, and $m_4(t)$ with a commutator speed of 700 rotations/sec. This combined signal is now multiplexed with $m_1(t)$ with a commutator speed of 2800 rotations/sec, yielding the output of 5600 samples/sec.

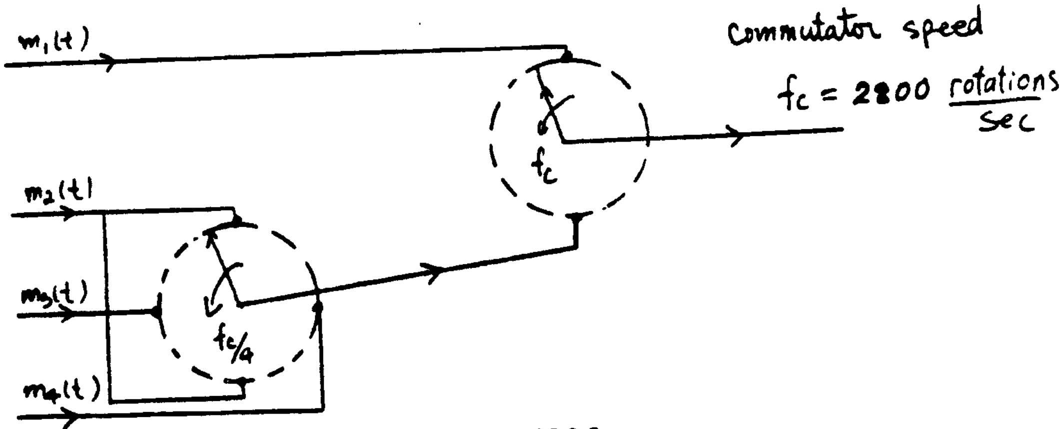
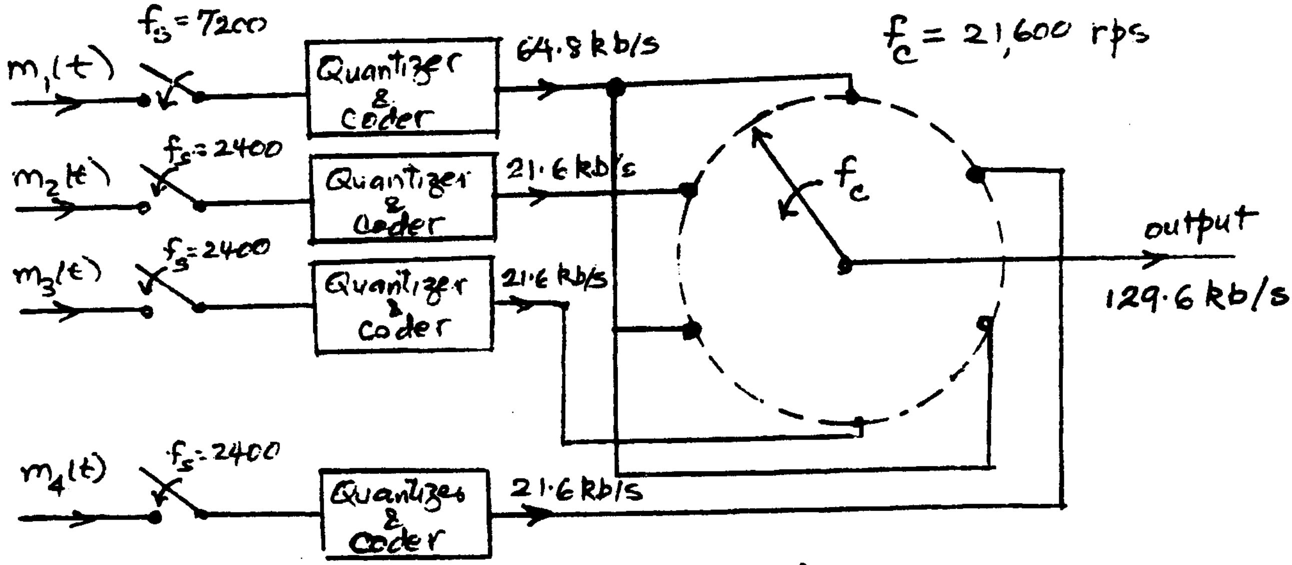


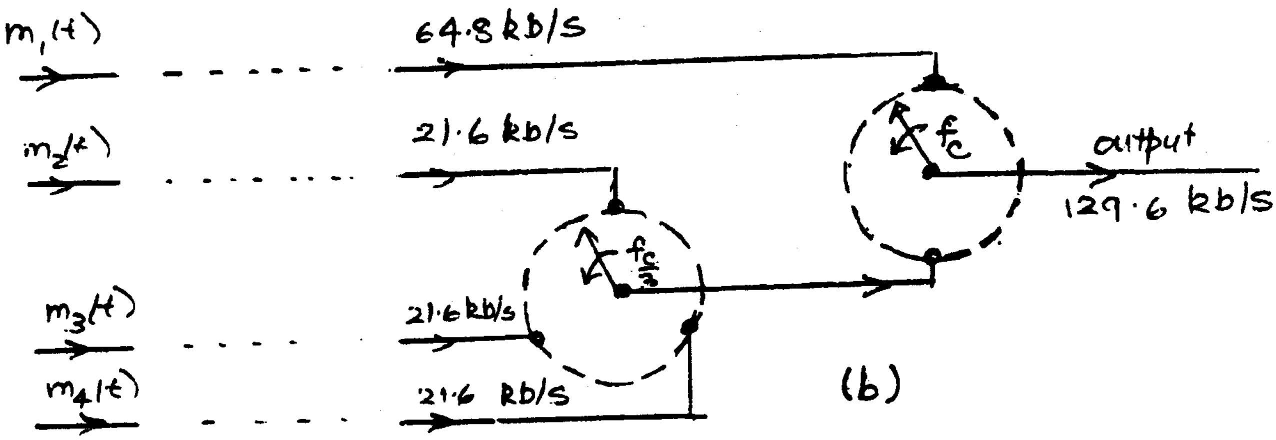

**Fig. S7.9-2**

7.9-3




(a)

Alternate arrangement




(b)

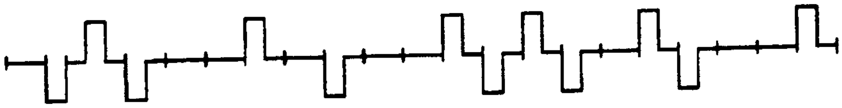
Fig S7.9-3

## Chapter 8 Exercises

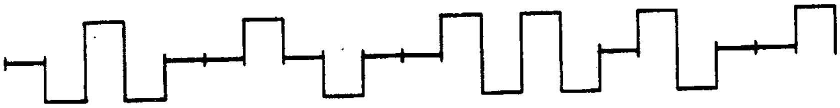
8.1-1 If a plesiochronous network operates from Cesium beam clock which is accurate to ± 3 parts in $10^{12}$, how long will it take for a DS3 signal transmitted between two networks to become out of sync if a 1/4 bit length time error results in desynchronization?
**Solution:** A DS3 bit is transmitted in $1/(44.736\cdot10^{6}) = 2.235336\cdot10^{-8}$ sec. In the worst case, both network clocks will be out of synchronization by 6 parts in $10^{12}$.

$$2.235336\cdot10^{-8}/(6\cdot10^{-12}) = 3922.27 \text{ sec/bit or } 980.57 \text{ sec/ } \tfrac{1}{4} \text{ bit}$$

8.1-2 For the bit stream 0111001010011110110001 draw an AMI waveform.
**Solution:**

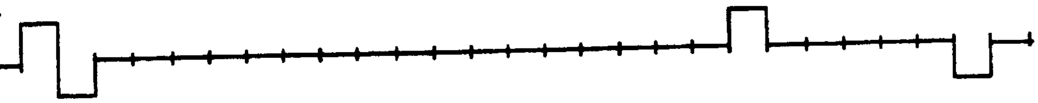


Note that typically, for illustrative purposes, the waveform is given as

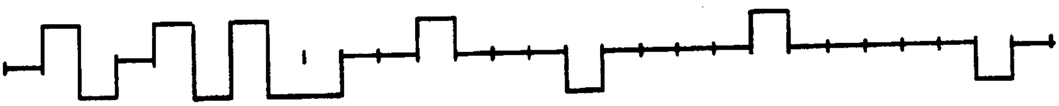


8.1-3 For the following waveforms, determine if each is a valid AMI format for a DS1 signal. If not, explain why not.

a.

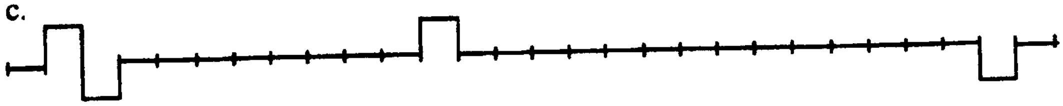


**Solution:** No. 16 0's violation



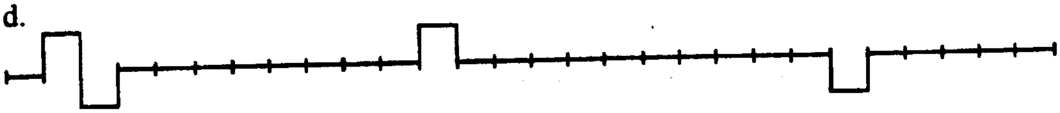
**Solution:** No. bi-polar violation

c.



**Solution:** No. 1's density violation

d.



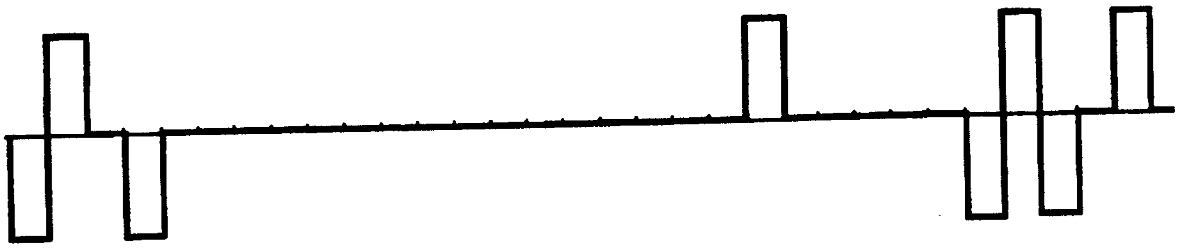
**Solution:** Yes

8.1-4 **a)** You have received the following sequence of **ESF** framing pattern sequence bits

**...00110010110010110...**

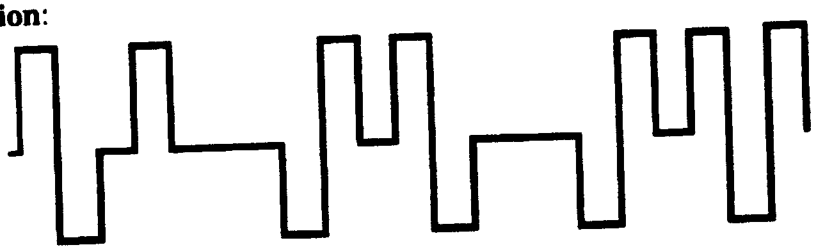
Is this a legitimate framing bit sequence in order to maintain synchronization between the T1 transmitter and receiver?
**Yes_____No_____**
If yes, why? If no, why not?
**Solution**: No. The bit sequence 0011 cannot be in an ESF framing bit sequence.

**b)** The following T1 AMI signal is received:



Is this an acceptable T1 signal?
**Yes_____No______**
**a.** If yes, explain.
**b.** If no, explain why not (what, if any, DS1 standards are violated) and draw on the figure the AMI waveform which would be transmitted by the DSU?
**Solution**: No. 16 0's violation. The 16 0's will be replaced by a pattern of 1's by the DSU.

8.1-5 The signal **11010000000000000001** is received by the DSU in a T1 data stream which uses a **B8ZS** format. Draw the output of the DSU for this signal? The first **1** is already drawn. Show the bit stream which is substituted by the DSU.
**Solution**:



8.1-6 T-1 synchronization at two distant locations is controlled by separate crystal controlled oscillators which differ in frequency by 125 parts per million. If the terminal equipment doesn't maintain sync in how many **complete D4** superframes will the faster oscillator have generated (at most) one more time slot (8-bit) than the slower oscillator ? Circle the correct answer.

a) 5
b) 10
c) 15
d) 20
e) None of the above - if "none", what is the number of D4 superframes before an extra time slot is generated?________

**Solution**: e) The faster oscillator will generate $125 \cdot 10^{-6} 1.544 \cdot 10^{6} = 193$ bits per second more than the slower oscillator. This is one frame/sec = 24.125 time slots. Hence, a time slot difference will be generated in 1/24.125 = 0.04164498 frames or 0.0034704 superframes.

8.1-7 Two plesiochronous digital networks, A and B, utilize Cesium beam clocks accurate to 3 parts in $10^{13}$. The networks are operated by independent long distance companies and are synchronized to each other by means of a UTC signal.

**a.** If a company leases a T1 line with D4 framing which is terminated at one end in network A and at the other end in network B, how often must the networks be resync'd to each other to avoid a framing bit error in the customers T1 signal in the worst case? {You may assume a framing bit error occurs when the two networks are out of sync by ≥ 1/2 of a T1 "bit time".}

**Solution:** A T1 bit time is $1/(1.544 \cdot 10^{6}) = 6.47668 \cdot 10^{-7}$ sec/bit. In the worst case, the two clocks would be off by $2 \cdot 3 = 6$ parts in $10^{13}$ or $6 \cdot 10^{-13}$ errored bits per bit transmitted. Hence, $6.47668 \cdot 10^{-7}$ sec/bit / $6 \cdot 10^{-13}$ errored bits per bit = $1.07945 \cdot 10^{6}$ seconds per errored bit or $5.39723 \cdot 10^{6}$ seconds per errored half-bit.

**b.** UTC operates via GPS satellites which are approximately 23,000 miles above the Earth. How long, in terms of T1 bits, will a correction signal take to be transmitted to the network switches?

**Solution**: The speed of light is approximately 186000 miles/sec.
23000x2 = 46000miles up and down. 46000/186000 = 0.247 sec
0.247x1544000 ≅ 381850 bits

# Chapter 10

**10.1-1** (a) $P(\text{red card}) = \frac{13+13}{52} = \frac{1}{2}$

(b) $P(\text{black queen}) = \frac{1+1}{52} = \frac{1}{26}$

(c) $P(\text{picture card}) = \frac{12}{52} = \frac{3}{13}$

(d) $P(7) = \frac{4}{52} = \frac{1}{13}$

(e) $P(n \le 5) = \frac{20}{52} = \frac{5}{13}$

**10.1-2** (a) $S = 4$ occurs as $(1,1,2),(1,2,1),(2,1,1)$. There are a total of $6 \times 6 \times 6 = 216$ outcomes.

Hence, $P(S=4) = \frac{3}{216} = \frac{1}{72}$

(b) $S = 9$ occurs as (1,2,6), (1,3,5), (1,4,4), (1,5,3), (1,6,2), (2,1,6), (2,2,5), (2,3,4), (2,4,3), (2,5,2), (2,6,1), (3,1,5), (3,2,4), (3,3,3), (3,4,2), (3,5,1), (4,1,4), (4,2,3), (4,3,2), (4,4,1), (5,1,3), (5,2,2), (5,3,1), (6,1,2), (6,2,1)

$$P(S=9) = \frac{25}{216}$$

(c) $S = 15$ occurs as (3,6,6), (4,5,6), (4,6,5), (5,4,6), (5,5,5), (5,6,4), (6,3,6), (6,4,5), (6,5,4), (6,6,3)

$$P(S=10) = \frac{10}{216}$$

**10.1-3** Note: There is a typo in this problem. The probability that the number $i$ appears should be $ki$ not $k_i$.

$$1 = \sum_{i=1}^{6} ki = k + 2k + 3k + 4k + 5k + 6k = 21k \Rightarrow k = \frac{1}{21}$$

$$P(i) = \frac{i}{21} \quad (i = 1, 2, 3, 4, 5, 6)$$

**10.1-4** We can draw 2 items out of 5 in 20 ways as follows: $0_10_2$, $0_10_3$, $0_1P_1$, $0_1P_2$, $0_20_1$, $0_20_3$, $0_2P_1$, $0_2P_2$, $0_30_1$, $0_30_2$, $0_3P_1$, $0_3P_2$, $P_10_1$, $P_10_2$, $P_10_3$, $P_1P_2$, $P_20_1$, $P_20_2$, $P_20_3$, $P_2P_1$. All these outcomes are equally likely with probability 1/20.

(i) This event $E_1 = 0_1P_1 \cup 0_1P_2 \cup 0_2P_1 \cup 0_2P_2 \cup 0_3P_1 \cup 0_3P_2 \cup P_10_1 \cup P_10_2 \cup P_10_3 \cup P_20_1 \cup P_20_2 \cup P_20_3$

Hence, $P(E_1) = \frac{12}{20} = \frac{3}{5}$

(ii) This event $E_2 = P_1P_2 \cup P_2P_1$

Hence, $P(E_2) = \frac{2}{20} = \frac{1}{10}$

(iii) This event $E_3 = 0_10_2 \cup 0_10_3 \cup 0_20_1 \cup 0_20_3 \cup 0_30_1 \cup 0_30_2$

Hence, $P(E_3) = \frac{6}{20} = \frac{3}{10}$

(iv) This event $E_4 = E_2 \cup E_3$ and both $E_2$ & $E_3$ are disjoint.

Hence, $P(E_4) = P(E_2) + P(E_3) = \frac{4}{10} = \frac{2}{5}$

**10.1-5** Let $x_{0_1}$ be the event that the first chip is oscillator and $x_{P_1}$ be the event that the first chip is PLL. Also, let $x_{0_2}$ and $x_{P_2}$ represent events that the second chip drawn is an oscillator and a PLL, respectively. Then

$$P(1 \text{ osc and } 1 \text{ PLL}) = P(x_{0_1}, x_{P_2}) + P(x_{P_1}, x_{0_2})$$

$$= P(x_{0_1})P(x_{P_2}|x_{0_1}) + P(x_{P_1})P(x_{0_2}|x_{P_1})$$

$$\left(\frac{3}{5}\times\frac{2}{4}\right)+\left(\frac{2}{5}\times\frac{3}{4}\right)=\frac{3}{5}$$

**10.1-6** Using the notation in the solution of Prob. 10.1-5, we find:

**(a)** $P(x_{0_2}|x_{P_1}) = \frac{3}{4}$

**(b)** $P(x_{0_2}|x_{0_1}) = \frac{2}{4}$

**10.1-7** **(a)** We can have $\binom{10}{2}$ ways of getting two 1's and eight 0's in 10 digits

$$\binom{10}{2} = \frac{10!}{2!8!} = 45$$

$$P(\text{two 1's and eight 0's}) = 45(0.5)^2(0.5)^8 = 45(0.5)^{10} = \frac{45}{2^{10}} = \frac{45}{1024}$$

**(b)** $P(\text{at least four 0's}) = \ 1-[P(\text{exactly one 0})]+[P(\text{exactly two 0's})]+[P(\text{exactly three 0's})]$

$$P(\text{one 0}) = \binom{10}{1}(0.5)^{10} = \frac{10}{1024} = \frac{5}{512}.$$

$$P(\text{two 0's}) = \binom{10}{2}(0.5)^{10} = \frac{45}{1024}$$

$$P(\text{three 0's}) = \binom{10}{3}(0.5)^{10} = \frac{120}{1024}$$

$$P(\text{at least four 0's}) = 1-\left(\frac{5}{512}+\frac{45}{1024}+\frac{120}{1024}\right)=\frac{849}{1024}$$

**10.1-8** **(a)** Total ways of drawing 6 balls out of 49 are

$$\binom{49}{6} = \frac{49!}{6!43!} = 13{,}983{,}816$$

Hence, $$\text{Prob(matching all 6 numbers)} = \frac{1}{13983816}$$

**(b)** To match exactly 5 number means we pick 5 of the chosen 6 numbers and the last number can be picked from the remaining 43 numbers. We can choose 5 numbers of our 6 in $\binom{6}{5} = 6$ ways and can choose one number out of 43 in $\binom{43}{1} = 43$ ways. Hence, we have $43\times 6$ combinations in which exactly 5 numbers match. Thus,

$$P(\text{matching exactly 5 numbers}) = \frac{43\times 6}{13983816} = 1.845\times 10^{-5}$$

**(c)** To match exactly 4 numbers means we pick 4 out of the chosen 6 number in $\binom{6}{4} = 15$ ways and choose 2 out of the remaining 43 numbers in $\binom{43}{2} = 903$ ways. Thus there are $15\times 903$ ways of picking exactly 4 numbers out of 6 and

$$P(\text{matching exactly 4 numbers}) = \frac{15\times 903}{13983816} = 9.686\times 10^{-4}$$

(d) Similarly, we can pick three numbers exactly in $\binom{6}{3}\binom{43}{3} = 20 \times 12341 = 246820$ ways. Hence,

$$P(\text{matching exactly 3 numbers}) = \frac{246820}{13983816} = 0.01765$$

10.1-9 (a) Let $f$ represent the system failure. Then

$$P(\bar{f}) = (1-0.01)^{10} = 0.90438$$
$$P(f) = 1 - P(\bar{f}) = 0.0956$$

(b) $P(\bar{f}) = 0.99$ and $P(f) = 0.01$

If the probability of failure of a subsystem $s_i$ is $p$, then

$$P(\bar{f}) = (1-p)^{10} \text{ or } 0.99 = (1-p)^{10} \Rightarrow P = 0.0010045$$

10.1-10 If $f$ represents the system failure and $f_u$ and $f_L$ represent the failure of the upper and the lower paths, respectively, in the system, then:

(a)
$$P(f) = P(f_u f_L) = P(f_u)P(f_L) = [P(f_u)]^2$$
$$P(f_u) = 1 - P(\bar{f}_u) = 1-(1-0.01)^{10} = 0.0956$$

and

$$P(f) = (0.0956)^2 = 0.009143$$

Reliability is $P(\bar{f}) = 1 - P(f) = 0.9908$

(b)
$$P(\bar{f}) = 0.999$$
$$P(f) = 1 - 0.999 = 0.001$$
$$P(f_u) = \sqrt{0.001} = 0.0316$$
$$P(\bar{f}_u) = (1-P)^{10} = 1 - 0.0316 \Rightarrow P = 0.003206$$

10.1-11 Let $P$ be the probability of failure of a subsystem $(s_1 \text{ or } s_2)$.

For the system in Fig. a:
The system fails if the upper and lower branches fail simultaneously. The probability of any branch not failing is

$(1-P)(1-P) = (1-P)^2$. Hence, the probability of any branch failing is $1-(1-P)^2$.

Clearly, $P_f$, the probability of the system failure is $P_f = \left[1-(1-P)^2\right]\left[1-(1-P)^2\right] \cong 4P^2 \quad P << 1$

For the system in Fig. b:
We may consider this system as a cascade of two subsystems $x_1$ and $x_2$, where $x_1$ is the parallel combination of $s_1$ and $s_1$ and $x_2$ is the parallel combination of $s_2$ and $s_2$. Let $P_f(x_i)$ be the probability of failure of $x_i$. Then

$$P_f(x_1) = P_f(x_2) = P^2$$

The system functions if neither $x_1$ nor $x_2$ fails. Hence, the probability of system not failing is $(1-P^2)(1-P^2)$. Therefore, the probability of system failing is

$$P_f = 1 - (1-P^2)(1-P^2) = 2P^2 - P^4 \cong 2P^2 \quad P << 1$$

Hence the system in Fig. a has twice the probability of failure of the system in Fig. b.

**10.1-12** There are $\binom{52}{5} = 2598960$ ways of getting 5 cards out of 52 cards. Number of ways of drawing 5 cards of the same suit (of 13 cards) is $\binom{13}{5} = 1287$. There are 4 suits. Hence there are $4 \times 1287$ ways of getting a flush. Therefore,

$$P(\text{flush}) = \frac{4 \times 1287}{2598960} = 1.9808 \times 10^{-3}$$

**10.1-13** Sum of 4 can be obtained as (1,3), (2,2) and (3,1). The two dice outcomes are independent. Let $x_1$ be the outcome of the regular die and $x_2$ be the outcome of irregular die.

$$P_{x_1x_2}(1,3) = P_{x_1}(1)P_{x_2}(3) = \frac{1}{6} \times \frac{1}{3} = \frac{1}{18}$$

$$P_{x_1x_2}(2,2) = P_{x_1}(2)P_{x_2}(2) = \frac{1}{6} \times 0 = 0$$

$$P_{x_1x_2}(3,1) = P_{x_1}(3)P_{x_2}(1) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$$

Therefore $P_s(4) = \frac{1}{18} + \frac{1}{36} = \frac{1}{12}$

Similarly,

$$P_s(5) = P_{x_1x_2}(1,4) + P_{x_1x_2}(2,3) + P_{x_1x_2}(3,2) + P_{x_1x_2}(4,1)$$
$$= \frac{1}{6} \times 0 + \frac{1}{6} \times \frac{1}{3} + \frac{1}{6} \times 0 + \frac{1}{6} \times \frac{1}{6} = \frac{1}{12}$$

**10.1-14** $B = AB \cup A^c B$

$$P(B) = P(A)P(B|A) + P(A^c)P(B|A^c)$$
$$= \left(\frac{1}{26}\right)\left(\frac{1}{51}\right) + \left(\frac{50}{52}\right)\left(\frac{2}{51}\right) = \frac{1}{26}$$
$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{\left(\frac{1}{26}\right)\left(\frac{1}{51}\right)}{\frac{1}{26}} = \frac{1}{51}$$

**10.1-15** (a) Two 1's and three 0's in a sequence of 5 digits can occur in $\binom{5}{2} = 10$ ways. The probability one such sequence is

$$P = (0.8)^2(0.2)^3 = 0.00512$$

Since the event can occur in 10 ways, its probability is

$$10 \times 0.00512 = 0.0512$$

(b) Three 1's occur with probability $\binom{5}{3}(0.8)^3(0.2)^2 = 0.2048$

Four 1's occur with probability $\binom{5}{4}(0.8)^4(0.2)^1 = 0.4096$

Five 1's occur with probability $\binom{5}{5}(0.8)^5(0.2)^0 = 0.3277$

Hence, the probability of at least three 1's occuring is

$$P = 0.2048 + 0.4096 + 0.3277 = 0.9421$$

**10.1-16** Prob(no more than 3 error) $= P(\text{no error}) + P(1\text{ error}) + P(2\text{ error}) + P(3\text{ error})$

$$= (1-P_e)^{100} + \binom{100}{1}P_e(1-P_e)^{99} + \binom{100}{2}P_e^2(1-P_e)^{98} + \binom{100}{3}P_e^3(1-P_e)^{97}$$
$$\cong (1-100P_e) + 100P_e(1-99P_e) + 4950P_e^2(1-98P_e) + 161700P_e^3(1-97P_e)$$

10.1-17 Error can occur in 10 ways. Consider case of error over the first link

$P_c$(correct detection over every link) $= (1-P_1)(1-P_2)\ldots(1-P_{10})$

$$P_E = 1 - P_c = 1-(1-P_1)(1-P_2)\ldots(1-P_{10})$$
$$= 1-\left[1-(P_1+P_2+\ldots+P_{10}) + \text{higher order terms}\right]$$
$$\cong P_1+P_2+\ldots+P_{10} \qquad P_i \ll 1$$

10.1-18 $P(\varepsilon) = \sum_{j=3}^{5}\binom{5}{j}P_e^{\,j}(1-P_e)^{5-j} = 10P_e^{\,3}(1-P_e)^2 + 5P_e^{\,4}(1-P_e) + P_e^{\,5}$

$$\cong 10P_e^{\,3}(1-P_e)^2, \qquad P_e \ll 1$$

10.1-19 (a) $P(\text{success in 1 trial}) = \frac{1}{10} = 0.1$

(b) $P(\text{success in 5 trials}) = 1 - P(\text{failure in all 5 trials})$

$$= 1 - P_{f_1}P_{f_2}P_{f_3}P_{f_4}P_{f_5}$$

$P_{f_1}$ = Prob(failure in 1st trial) = 9 / 10

$P_{f_2}$ = Prob(failure in 2nd trial) = 8 / 9

Similarly, $P_{f_3} = 7/8$, $P_{f_4} = 6/7$, and $P_{f_5} = 5/6$

Hence, $P(\text{success in 5 trials}) = 1-\left(\frac{9}{10}\right)\left(\frac{8}{9}\right)\left(\frac{7}{8}\right)\left(\frac{6}{7}\right)\left(\frac{5}{6}\right) = 1-\frac{5}{10} = 0.5$

10.1-20 Let $x$ be the event of drawing the short straw and the $P_i(x)$ denote the event that $i$th person in the sequence draws the short straw.

Now, $P_1(x) = 0.1$

$P_2(x)$ = Prob(1st person does not draw the short straw) × Prob(2nd person draws the short straw)

$$= \left[1-P_1(x)\right]\frac{1}{9} = \left(\frac{9}{10}\right)\left(\frac{1}{9}\right) = 0.1$$

Similarly,

$P_3(x)$ = Prob(neither 1st nor 2nd person draws the short straw) × Prob(3rd person draws the short straw)

$$= \left[1-P_1(x)-P_2(x)\right]\frac{1}{8} = \left(\frac{8}{10}\right)\left(\frac{1}{8}\right) = 0.1$$

Similarly, $P_4(x) = P_5(x) = \ldots = P_{10}(x) = 0.1$

10.1-21 All digits are generated independently

(a) $P(\text{all 10 digits are 0}) = (0.3)^{10}$

(b) There are $\binom{10}{2}$ ways of arranging eight 1's and two 0's. Hence,

$$P(\text{eight 1's and two 0's}) = \binom{10}{2}(0.7)^8(0.3)^2$$

(c) $P$(at least five 0's) = $P$(five 0's)+$P$(six 0's)+…+$P$(ten 0's)

$$= \binom{10}{5}(0.7)^5(0.3)^5 + \binom{10}{6}(0.7)^4(0.3)^6 + \binom{10}{7}(0.7)^3(0.3)^7 + \binom{10}{8}(0.7)^2(0.3)^8 + \binom{10}{9}(0.7)(0.3)^9 + (0.3)^{10}$$

10.2-1 $P_y(0) = P_{xy}(1,0) + P_{xy}(0,0) = P_x(1)P_{y|x}(0|1) + P_x(0)P_{y|x}(0|0)$

$$= 0.6\times 0.1 + 0.4\left[1-P_{y|x}(1|0)\right] = 0.06+0.32 = 0.38$$
$$P_y(1) = 1-P_y(0) = 0.62$$

**10.2-2** (a) $P_{x|y}(1|1) = \dfrac{P_{y|x}(1,1)P_x(1)}{P_y(1)} = \dfrac{(1-P_e)Q}{(1-Q)P_e + (1-P_e)Q}$

(b) $P_{x|y}(0|1) = 1 - P_{x|y}(1|1)$

(note that $P_y(1)$ and $P_y(0)$ are derived in Example 10.10) .

**10.2-3** (a) $P(x \ge 1) = \int_1^{\infty} \frac{1}{2} x e^{-x} dx = \frac{1}{e}$

(b) $\text{Prob}(-1 < x \le 2) = \int_{-1}^{0} -\frac{1}{2} x e^{x} dx + \int_{0}^{2} \frac{1}{2} x e^{-x} dx = 1 - \frac{1}{e} - \frac{3}{2e^2}$

(c) $\text{Prob}(x \le -2) = \int_{-\infty}^{-2} -\frac{1}{2} x e^{x} dx = \frac{3}{2e^2}$

**10.2-4**

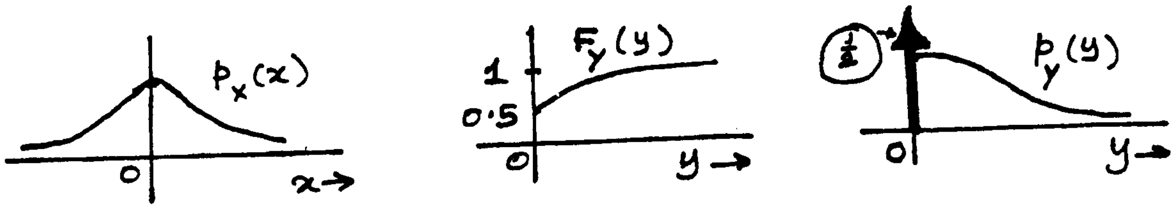



**Fig. S10.2-4**

Since this is a half-wave rectifier, $y$ assumes only positive values. So $P(y < 0) = 0$.

Hence, $F_y(y) = 0$ (for $y < 0$) and $P(y < 0^+) = \frac{1}{2}$. Hence, $F_y(0^+) = \frac{1}{2}$

**10.2-5** x is a gaussian r.v. with mean 4 and $\sigma_x = 3$

Hence,

(a) $P(x \ge 4) = Q\left(\frac{4-4}{3}\right) = Q(0) = 0.5$

(b) $P(x \ge 0) = Q\left(\frac{0-4}{3}\right) = 1 - Q\left(\frac{4}{3}\right) = 1 - 0.09176 = 0.9083$

(c) $P(x \ge -2) = Q\left(\frac{-2-4}{3}\right) = 1 - Q(2) = 1 - 0.02275 = 0.9773$

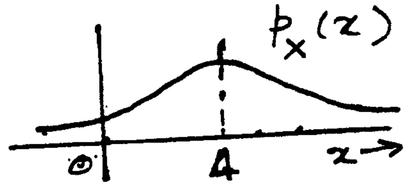



**Fig. S10.2-5**

**10.2-6** (a) From the sketch it is obvious that x is not gaussian. However, it is a unilateral (rectified) version of Gaussian PDF. Hence, we can use the expression of Gaussian r.v. with a multiplier of 2.
For a gaussian r.v.

$$p_y(y) = \frac{1}{4\sqrt{2\pi}} e^{-x^2/32} \quad \text{with } \sigma_y = 4$$

(b) Hence, (i) $P(x \ge 1) = 2P(y \ge 1) = 2Q\left(\frac{1}{4}\right) = 0.8026$

(ii) $P(1 < x \le 2) = 2P(1 < y \le 2) = 2\left[Q\left(\frac{1}{4}\right) - Q\left(\frac{2}{4}\right)\right] = 0.1856$

(c) If we take a Gaussian random variable y

$$p_y(y) = \frac{1}{4\sqrt{2\pi}} e^{-y^2/32}$$

and rectify y (all negative of y multipled by –1), the resulting variable is the desired random variable x.

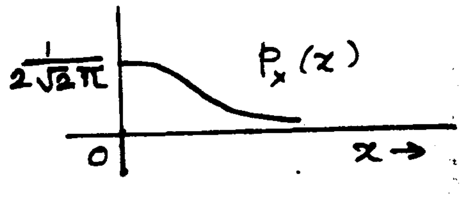



**Fig. S10.2-6**

**10.2-7** The volume V under $p_{xy}(x,y)$ must be unity.

$$V=\frac{1}{2}(1\times 1)A=\frac{A}{2}=1, \quad A=2$$

$$p_x(x)=\int_y p_{xy}(x,y)dy$$

But $y=-x+1$ and the limits on $y$ are 0 to $1-x$. Therefore,

$$p_x(x)=\int_0^{1-x} 2dy=2y\Big|_0^{1-x}=\begin{cases} 2(1-x) & 0\le x\le 4 \\ 0 & \text{otherwise}\end{cases}$$

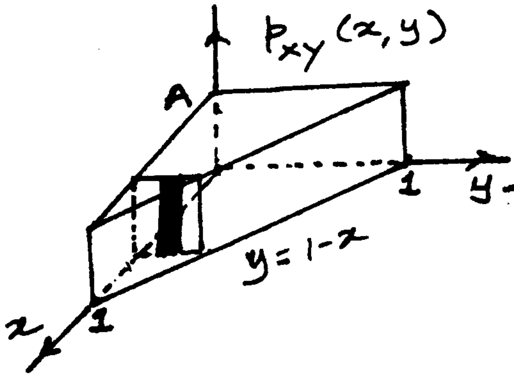



Fig. S10.2-7

Similarly, $\quad p_y(y)=\begin{cases} 2(1-y) & 0\le y\le 1 \\ 0 & \text{otherwise}\end{cases}$

$$p_{x|y}(x|y)=\frac{p_{xy}(x,y)}{p_y(y)}=\frac{2}{2(1-y)}=\begin{cases} 1/1-y & 0\le y\le 1 \\ 0 & \text{otherwise}\end{cases}$$

Similarly, $p_{y|x}(y|x)=\begin{cases} 1/1-x & 0\le x\le 1 \\ 0 & \text{otherwise}\end{cases}$

Clearly x and y are not independent.

**10.2-8**
$$p_{xy}(x,y)=xy\,e^{-(x^2+y^2)/2}u(x)u(y)$$

**(a)**
$$p_x(x)=\int_0^{\infty} xy\,e^{-(x^2+y^2)/2}u(x)dy=x\,e^{-x^2/2}u(x)$$

Similarly,
$$P_y(y)=ye^{-y^2/2}u(y)$$

$$p_x(x|y=y)=\frac{p_{xy}(x,y)}{p_y(y)}=xe^{-x^2/2}u(x)$$

and
$$p_y(y|x=x)=\frac{p_{xy}(x,y)}{p_x(x)}=ye^{-y^2/2}u(y)$$

**(b)** From results in (a), it is obvious that x and y are independent.

**10.2-9**
$$p_x(x)=\int_{-\infty}^{\infty}p_{xy}(x,y)dy=\frac{1}{2\pi\sqrt{M}}\int_{-\infty}^{\infty}e^{-(ax^2+by^2-2cxy)/2M}dy$$

$$=\frac{1}{2\pi\sqrt{M}}e^{-ax^2/2M}\int_{-\infty}^{\infty}e^{-(by^2-2cxy)/2M}dy=\frac{1}{\sqrt{2\pi b}}e^{-x^2/2b}$$

Similarly we can show that $p_y(y)=\frac{1}{\sqrt{2\pi a}}e^{-y^2/2a}$

Therefore
$$p_{x|y}(x|y)=\frac{p_{xy}(x,y)}{p_x(x)}=\sqrt{\frac{a}{2\pi M}}e^{-a\left(x-\frac{c}{a}y\right)^2/2M}$$

$$p_{y|x}(y|x)=\frac{p_{xy}(x,y)}{p_y(y)}=\sqrt{\frac{b}{2\pi M}}e^{-b\left(x-\frac{c}{b}y\right)^2/2M}$$

10.2-10 $K\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\left(x^2+xy+y^2\right)}dxdy = K\int_{-\infty}^{\infty} e^{-x^2}\left[\int_{-\infty}^{\infty} e^{-y^2-xy}dy\right]dx = 1$

But $\int_{-\infty}^{\infty} e^{-y^2-xy}dy = \sqrt{\pi}e^{x^2/4}$ and, $K\sqrt{\pi}\int_{-\infty}^{\infty} e^{-3x^2/4}dx = K\sqrt{\pi}\left(\sqrt{\frac{4\pi}{3}}\right) = 1$

Hence, $K = \frac{1}{\pi}\sqrt{\frac{3}{4}}$

$$p_x(x) = K\int_{-\infty}^{\infty} e^{-\left(x^2+xy+y^2\right)}dy = Ke^{-x^2}\int_{-\infty}^{\infty} e^{-y^2-xy}dy = K\sqrt{\pi}e^{-3x^2/4} = \sqrt{\frac{3}{4\pi}}e^{-3x^2/4}$$

Because of symmetry of $p_{xy}(x,y)$ with respect to $x$ and $y$. $p_y(y) = \sqrt{\frac{3}{4\pi}}e^{-3y^2/4}$

$$p_{x|y}(x|y) = \frac{p_{xy}(x,y)}{p_y(y)} = \frac{1}{\sqrt{\pi}}e^{-\left(x^2+xy+\frac{y^2}{4}\right)}$$

and

$$p_{y|x}(x|y) = \frac{p_{xy}(x,y)}{p_x(x)} = \frac{1}{\sqrt{\pi}}e^{-\left(\frac{x^2}{4}+xy+y^2\right)}$$

Since $p_{xy}(x,y) \neq p_x(x)p_y(y)$, x and y are not independent.

10.2-11 $P_e = P(\epsilon|1)P_x(1) + P(\epsilon|0)P_x(0)$

If the optimum threshold is $a$, then

$$P(\epsilon|1) = 1 - Q\left(\frac{a-A_p}{\sigma_n}\right) = Q\left(\frac{A_p-a}{\sigma_n}\right)$$

$$P(\epsilon|0) = Q\left(\frac{a+A_p}{\sigma_n}\right)$$

$$P_e = Q\left(\frac{A_p-a}{\sigma_n}\right)P_x(1) + Q\left(\frac{a+A_p}{\sigma_n}\right)P_x(0)$$

$$\frac{dP_e}{da} = \frac{1}{2\pi\sigma_n}\left[e^{-\left(A_p-a\right)^2/2\sigma_n^2}P_x(1) - e^{-\left(A_p+a\right)^2/2\sigma_n^2}P_x(0)\right] = 0$$

Hence, $e^{-\left(A_p-a\right)^2/2\sigma_n^2}P_x(1) = e^{-\left(A_p+a\right)^2/2\sigma_n}P_x(0)$

And $a = \frac{\sigma_n^2}{2A_p}\ln\left[\frac{P_x(0)}{P_x(1)}\right]$

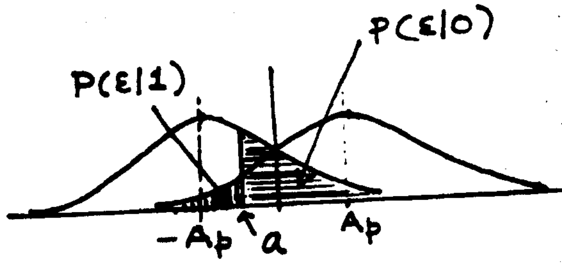



Fig. S10.2-11

10.3-1 $\bar{x} = 2$, $\sigma_x = 3$, $p_x(x) = \frac{1}{3\sqrt{2\pi}}e^{-(x-2)^2/18}$

**10.3-2** $p_x(x) = \frac{1}{2}|x|e^{-|x|}$

Because of even symmetry of $p_x(x)$, $\bar{x} = 0$ and

$$\bar{x}^2 = 2\int_0^\infty x^2 p_x(x)dx = 2\int_0^\infty x^2 \frac{1}{2}xe^{-x}dx$$

$$= \int_0^\infty x^3 e^{-x}dx = 3! = 6$$

$$\overline{x^2} = \sigma_x{}^2 + \bar{x}^2 = \sigma_x^2 = 3! = 6$$

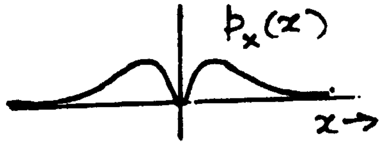



Fig. S10.3-2

**10.3-3** $p_y(y) = \frac{1}{2}\delta(y) + \frac{1}{\sigma\sqrt{2\pi}}e^{-y^2/2\sigma^2}u(y)$. Therefore

$$\bar{y} = \int_{-\infty}^\infty y p_y(y)dy = \int_{-\infty}^\infty \frac{1}{2}y\delta(y)dy + \int_{-\infty}^\infty \frac{1}{\sigma\sqrt{2\pi}}ye^{-y^2/2\sigma^2}dy$$

$$= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^\infty ye^{-y^2/2\sigma^2}dy = 0.399\sigma$$

$$\overline{y^2} = \int_{-\infty}^\infty y^2 p_y(y)dy = \int_{-\infty}^\infty \frac{1}{4}y^2\delta(y)dy + \int_{-\infty}^\infty \frac{1}{\sigma\sqrt{2\pi}}e^{-y^2/2\sigma^2}dy$$

$$= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^\infty y^2 e^{-y^2/2\sigma^2}dy = \frac{\sigma^2}{2}$$

$$\sigma_y{}^2 = \overline{y^2} - (\bar{y})^2 = \frac{\sigma^2}{2} - (0.399\sigma)^2 = 0.3408\sigma^2$$

**10.3-4** $\sigma_x{}^2 = \int_0^\infty x^2 p_x(x)dx = 2\int_0^\infty x^2 \frac{1}{4\sqrt{2\pi}}e^{-x^2/32}dx = 2\times\frac{4^2}{2}$,

Because $\bar{x} = \frac{1}{2\sqrt{2\pi}}\int_0^\infty xe^{-x^2/32}dx = \frac{8}{\sqrt{2\pi}}$ $\qquad$ $\overline{x^2} = \sigma_x{}^2 - (\bar{x})^2 = 2\frac{4^2}{2} - \left(\frac{8}{\sqrt{2\pi}}\right)^2 = \frac{16\pi - 32}{\pi}$

**10.3-5** The area of the triangle must be 1. Hence $K = \frac{1}{2}$ and $p_x(x) = \frac{1}{8}(x+1) \quad -1 \le x \le 3$

$$\bar{x} = \int_{-1}^3 xp_x(x)dx = \frac{1}{8}\int_0^4 y(y-1)dy = \frac{1}{8}\left(\frac{y^3}{3} - \frac{y^2}{2}\right)_0^4 = \frac{1}{8}\left(\frac{64}{3} - \frac{16}{2}\right) = \frac{5}{3}$$

$$\overline{x^2} = \frac{1}{8}\int_{-1}^3 x^2(x+1)dx = \frac{1}{8}\left(\frac{x^4}{4} + \frac{x^3}{3}\right)_{-1}^3$$

$$= \frac{1}{8}\left[\frac{81}{4} + \frac{27}{3} - \frac{1}{4} + \frac{1}{3}\right] = \frac{11}{3}$$

$$\sigma_x{}^2 = \bar{x}^2 - (\bar{x})^2 = \frac{11}{3} - \frac{25}{9} = \frac{8}{9}$$

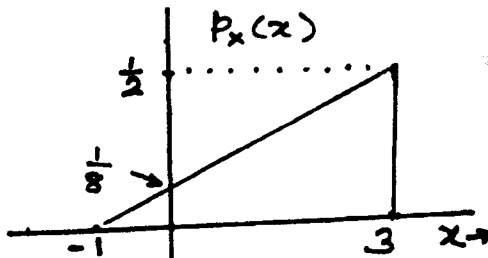



Fig. S10.3-5

**10.3-6** $\bar{x} = \sum_{i=2}^{12} x_i P_x(x_i) = \frac{1}{36}(2) + \frac{2}{36}(3) + \frac{3}{36}(4) + \frac{4}{36}(5) + \frac{5}{36}(6) + \frac{6}{36}(7) +$

$$\frac{5}{36}(8) + \frac{4}{36}(9) + \frac{3}{36}(10) + \frac{2}{36}(11) + \frac{1}{36}(12) = \frac{256}{36} = 7$$

$$\overline{x^2} = \sum_{i=2}^{12} x_i{}^2 P_x(x_i) = \frac{1}{36}(4) + \frac{2}{36}(9) + \frac{3}{36}(16) + \frac{4}{36}(25) + \frac{5}{36}(36) + \frac{6}{36}(49) +$$

$$\frac{5}{36}(64) + \frac{4}{36}(81) + \frac{3}{36}(100) + \frac{2}{36}(121) + \frac{1}{36}(144) = 54.83$$

10.3-7 $\overline{\mathrm{x}^n} = \frac{1}{\sigma_x\sqrt{2\pi}}\int_{-\infty}^{\infty} x^n e^{-x^2/2\sigma_x^2}dx$. For $n$ odd, the integrand is an odd function of $x$. Therefore $\overline{\mathrm{x}^n} = 0$.

For $n$ even, we find from tables

$$\overline{\mathrm{x}^n} = \begin{cases} (1)(3)(5)...(n-1)\sigma_x^2 & n \quad \text{even} \\ 0 & n \quad \text{odd} \end{cases}$$

10.3-8 Let $\mathrm{x}_i$ be the outcomes of the $i$th die. Then,

$$\bar{\mathrm{x}}_i = \frac{1+2+3+4+5+6}{6} = \frac{7}{2} \qquad i = 1, 2, ..., 10$$

$$\overline{\mathrm{x}_i^2} = \frac{1^2+2^2+3^2+4^2+5^2+6^2}{6} = \frac{91}{6}$$

$$\sigma_{\mathrm{x}_i}^2 = \overline{\mathrm{x}_i^2} - (\bar{\mathrm{x}}_i)^2 = \frac{35}{12}$$

If x is a RV representing the sum, then

$$\bar{\mathrm{x}} = \bar{\mathrm{x}}_1 + \bar{\mathrm{x}}_2 + ... + \bar{\mathrm{x}}_{10} = 10\left(\frac{7}{2}\right) = 35$$

$$\sigma_{\mathrm{x}}^2 = \sigma_{\mathrm{x}_1}^2 + \sigma_{\mathrm{x}_2}^2 + ... + \sigma_{\mathrm{x}_{10}}^2 = 10\left(\frac{35}{12}\right) = \frac{175}{6}$$

$$\overline{\mathrm{x}^2} = \sigma_{\mathrm{x}}^2 + \bar{\mathrm{x}}^2 = \frac{175}{6} + (35)^2 = 1254.167$$

10.4-1 $p_{\mathrm{x}}(x) = \frac{1}{2}\delta(x) + \frac{1}{2}\delta(x-3)$

$$p_{\mathrm{n}}(n) = \frac{1}{2\sqrt{2\pi}}e^{-n^2/8}$$

$$\mathrm{y} = \mathrm{x} + \mathrm{n}$$

$$p_{\mathrm{y}}(y) = p_{\mathrm{x}}(x) * p_{\mathrm{n}}(n) = \left[\frac{1}{2}\delta(x) + \frac{1}{2}\delta(x-3)\right] * \frac{1}{2\sqrt{2\pi}}e^{-n^2/8}$$

$$= \frac{1}{2}\int_{-\infty}^{\infty}\delta(x)\left[\frac{1}{2\sqrt{2\pi}}e^{-(y-x)^2/8}\right]dx + \frac{1}{2}\int_{-\infty}^{\infty}\delta(x-3)\left[\frac{1}{2\sqrt{2\pi}}e^{-(y-x)^2/8}\right]dx$$

$$= \frac{1}{4\sqrt{2\pi}}e^{-y^2/8} + \frac{1}{4\sqrt{2\pi}}e^{-(y-3)^2/8}$$

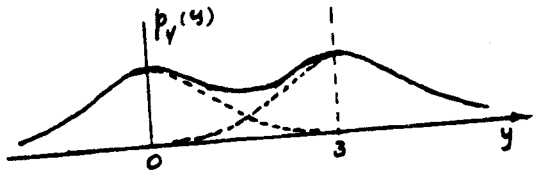



Fig. S10.4-1

10.4-2

$$p_{\mathrm{x}}(x) = 0.4\delta(x) + 0.6\delta(x-3)$$

$$p_{\mathrm{n}}(n) = \frac{1}{2\sqrt{2\pi}}e^{-n^2/8}$$

and

$$p_{\mathrm{y}}(y) = \frac{1}{5\sqrt{2\pi}}e^{-y^2/8} + \frac{3}{10\sqrt{2\pi}}e^{-(y-3)^2/8}$$

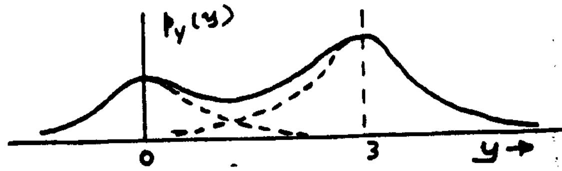



Fig. S10.4-2

10.4-3 $p_{\mathrm{x}}(x) = Q\delta(x-1) + (1-Q)\delta(x+1)$, $p_{\mathrm{n}}(n) = P\delta(n-1) + (1-Q)\delta(n+1)$

$$p_{\mathrm{y}}(y) = [Q\delta(y-1) + (1-Q)\delta(y+1)] * [P\delta(y-1) + (1-P)\delta(y+1)]$$

$$= (P+Q-2PQ)\delta(y) + PQ\delta(y-2) + (1-P)(1-Q)\delta(y+2)$$

**10.4-4**
$$p_z(z) = p_x(x) * p_y(y)$$

Taking Fourier transform of both sides, we have

$$P_z(\omega) = P_x(\omega) * P_y(\omega) \qquad \begin{cases} P_x(\omega) = e^{-\sigma_x^2\omega^2} e^{-j\omega\bar{x}} \\ P_y(\omega) = e^{-\sigma_y^2\omega^2} e^{-j\omega\bar{y}} \end{cases}$$

$$= e^{-\left(\sigma_x^2+\sigma_y^2\right)\omega^2} e^{-j\omega(\bar{x}+\bar{y})}$$

Taking inverse Fourier transform we get

$$p_z(z) = \frac{1}{\sqrt{2\pi\left(\sigma_x^2+\sigma_y^2\right)}} e^{-\left[z-(\bar{x}+\bar{y})\right]^2/2\left(\sigma_x^2+\sigma_y^2\right)}$$

It is clear that $\bar{z} = \bar{x} + \bar{y}$ and $\sigma_z^2 = \sigma_x^2 + \sigma_y^2$

**10.5-1** For any real $a$, $\overline{\left[a(x-\bar{x})-(y-\bar{y})\right]^2} \ge 0$, or $a^2\sigma_x^2 + \sigma_y^2 - 2a\sigma_{xy} \ge 0$. Hence, the discriminant of this quadratic in $a$ must be nonpositive, that is:

$$4\sigma_{xy}^2 - 4\sigma_x^2\sigma_y^2 \le 0, \text{ that is, } \left|\frac{\sigma_{xy}}{\sigma_x\sigma_y}\right| \le 1 \text{ or } |\rho| \le 1$$

**10.5-2** When $y = K_1x + K_2$ Hence, $\bar{y} = K_1\bar{x} + K_2$

$\sigma_y^2 = K_1^2\sigma_x^2$ and $\sigma_{xy} = \overline{(x-\bar{x})(y-\bar{y})} = \overline{(x-\bar{x})(K_1x + K_2 - K_1\bar{x} - K)} = K_1\sigma_x^2$. Hence,

$\rho_{xy} = \dfrac{\sigma_{xy}}{\sigma_x\sigma_y} = K_1\sigma_x^2 / K_1\sigma_x^2 = 1$ if $K_1$ is positive. If $K_1$ is negative, $\sigma_{xy} = K_1\sigma_x^2$ is negative.

But $\sigma_x$ and $\sigma_y$ are both positive. Hence, $\rho_{xy} = -1$

**10.5-3** $\bar{x} = \int_0^{2\pi} \cos\theta\, p(\theta) d\theta = \dfrac{1}{2\pi}\int_0^{2\pi} \cos\theta\, d\theta = 0$ Similarly, $\bar{y} = 0$

$$\sigma_{xy} = \overline{xy} = \overline{\cos\theta\sin\theta} = \frac{1}{2}\overline{\sin 2\theta} = \frac{1}{2}\int_0^{2\pi} \sin 2\theta\, p(\theta) d\theta = \frac{1}{4\pi}\int_0^{2\pi} \sin 2\theta\, d\theta = 0$$

Hence, $\sigma_{xy} = \bar{x}\,\bar{y} = 0$ and x, y are uncorrelated. But $x^2 + y^2 = 1$.

Hence, x and y are not independent.

**10.6-1** In this case

$R_{11} = R_{22} = R_{33} = \overline{m_k^2} = P_m$

$R_{12} = R_{21} = R_{23} = R_{32} = R_{01} = 0.825P_m$

$R_{13} = R_{31} = R_{02} = 0.562P_m$

$R_{03} = 0.308P_m$

Substituting these values in Eq. (10.86) yields: $a_1 = 1.1025$, $a_2 = -0.2883$, $a_3 = -0.0779$

From Eq. (10.87), we obtain

$\overline{\varepsilon^2} = \left[1 - (0.825a_1 + 0.562a_2 + 0.308a_3)\right]P_m = 0.2753P_m$

Hence, the SNR improvement is

$$10\log\left(\frac{P_m}{0.2753P_m}\right) = 5.63 \text{ dB.}$$

# Chapter 11

**11.1-1** This is clearly a non-stationary process. For example, amplitudes of all sample functions are zero at same instants (one is shown with a dotted line). Hence, the statistics clearly depend on $t$.

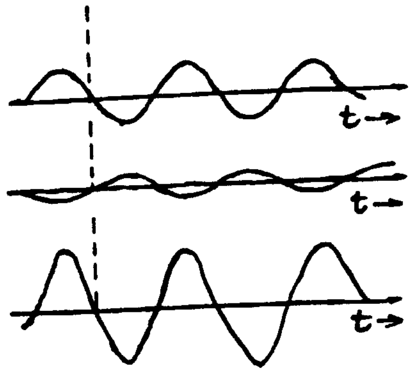



**Fig. S11.1-1**

**11.1-2** Ensemble statistics varies with $t$. This can be seen by finding

$\overline{x(t)} = \overline{A\cos(\omega t + \theta)} = A\int_0^{100} \cos(\omega t + \theta) p(\omega) d\omega$

$= \dfrac{A}{100t}\int_0^{100} \cos(\omega t + \theta) d\omega$. This is a function of $t$.

Hence, the process is non-stationary.

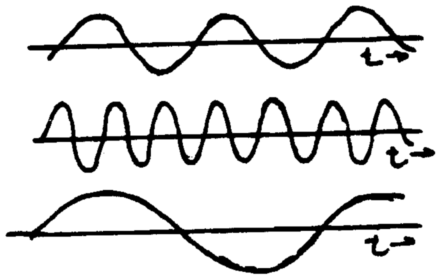



**Fig. S11.1-2**

**11.1-3** This is clearly a non-stationary process since its statistics depend on $t$. For example, at $t = 0$, the amplitudes of all sample functions is $b$. This is not the case at other values of $t$.

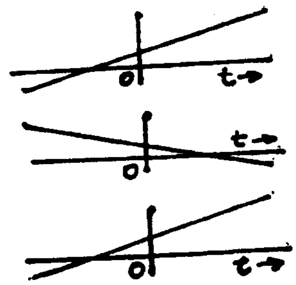



**Fig. S11.1-3**

**11.1-4** $x(t) = \mathrm{a}\cos(\omega t + \theta)$

$$\overline{x(t)} = \overline{\mathrm{a}\cos(\omega t + \theta)} = \bar{\mathrm{a}}\cos(\omega t + \theta) = \cos(\omega t + \theta)\int_{-A}^{A} a p_{\mathrm{a}}(a) da$$

$$= \left[\cos(\omega t + \theta)/2A\right]\int_{-A}^{A} a da = 0$$

$$R_x(t_1, t_2) = \overline{\mathrm{a}^2\cos(\omega t_1 + \theta)\cos(\omega t_2 + \theta)} = \cos(\omega t_1 + \theta)\cos(\omega t_2 + \theta)\overline{\mathrm{a}^2}$$

$$= \cos(\omega t_1 + \theta)\cos(\omega t_2 + \theta)\int_{-A}^{A} \frac{a^2}{2A} da$$

$$= \frac{A^2}{3}\cos(\omega t_1 + \theta)\cos(\omega t_2 + \theta)$$

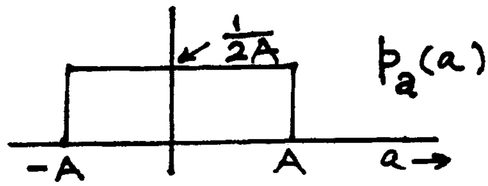



**Fig. S11.1-4**

**11.1-5** $\overline{x(t)} = \overline{a\cos(\omega t+\theta)} = \int_0^{100} \cos(\omega t+\theta)p(\omega)\,d\omega$

$$= \frac{a}{100}\int_0^{100}\cos(\omega t+\theta)\,d\omega = \frac{a}{100t}\sin(\omega t+\theta)\Big|_0^{100}$$

$$= \frac{a}{100t}\left[\sin(100t+\theta)-\sin\theta\right]$$

Using this result, we obtain

$$R_x(t_1,t_2) = a^2\overline{\cos(\omega t_1+\theta)\cos(\omega t_2+\theta)} = \frac{a^2}{2}\overline{\cos[\omega(t_1+t_2)+2\theta]} + \overline{\cos\omega(t_1-t_2)}$$

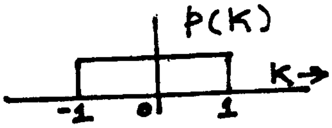
$$= \frac{a^2}{200(t_1+t_2)}\left[\sin[100(t_1+t_2)+2\theta]-\sin 2\theta\right] + \frac{a^2}{200(t_1-t_2)}\left[\sin 100(t_1-t_2)\right]$$

**11.1-6** $\overline{x(t)} = \overline{at+b} = \bar{a}t+b$. But $\bar{a}=0$ Hence, $\overline{x(t)} = b$

Also, $\bar{a}=0$, $\overline{a^2} = \int_{-2}^{2} a^2 p(a)\,da = \frac{1}{4}\frac{a^3}{3}\Big|_{-2}^{2} = \frac{4}{3}$

$$R_x(t_1,t_2) = \overline{(at_1+b)(at_2+b)} = \overline{a^2 t_1t_2 + a(t_1b+t_2b)+b^2}$$

$$= \overline{a^2}t_1t_2 + \bar{a}b(t_1+t_2) + \overline{b^2} = \frac{4}{3}t_1t_2 + b^2$$




**11.1-7** (b) $\overline{x(t)} = \overline{K} = 0$

(c)

$R_x(t_1,t_2) = \overline{KK} = \overline{K^2} = \int_{-1}^{1} K^2 p(K)dK = \frac{1}{2}\int_{-1}^{1}K^2 dK = \frac{1}{3}$

(d) The process is W.S.S. Since $\overline{x(t)}=0$ and $R_x(t_1,t_2) = \frac{1}{3}$

(e) The process is not ergodic since the time mean of each sample function is different from that of the other and it is not equal to the ensemble mean $(\bar{x}=0)$

(f) $\overline{x^2} = R_x 0 = \frac{1}{3}$

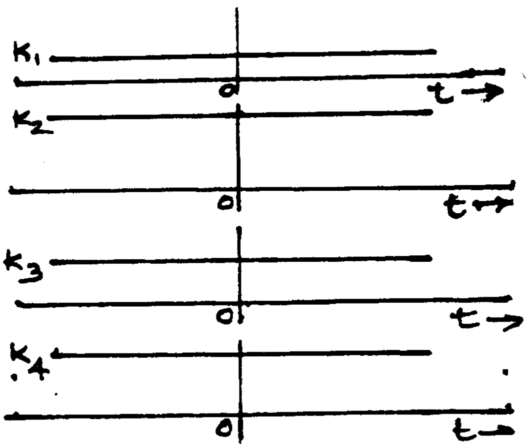
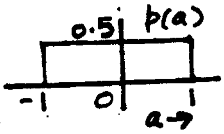
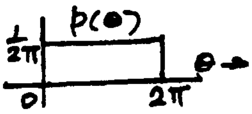



**Fig. S11.1-7**




**11.1-8** $x(t) = a\cos(\omega_c t+\theta)$

$\bar{a} = 0 \qquad \overline{a^2} = \frac{1}{3}$

(b) $\overline{x(t)} = \overline{a\cos(\omega_c t+\theta)} = \bar{a}\,\overline{\cos(\omega_c t+\theta)} = 0$

(c) $R_x(t_1,t_2) = \overline{a^2}\;\overline{\cos(\omega_c t_1+\theta)\cos(\omega_c t_2+\theta)}$

$$= \frac{1}{3}\left\{\overline{\cos\omega_c(t_1-t_2)} + \overline{\cos[\omega_c(t_1+t_2)+2\theta]}\right\}$$

$$= \frac{1}{3}\cos\omega_c(t_1-t_2) + \frac{1}{2\pi}\int_0^{2\pi}\cos[\omega_c(t_1-t_2)+2\theta]d\theta$$

$$= \frac{1}{3}\cos\omega_c(t_1-t_2)$$

(d) The process is W.S.S.

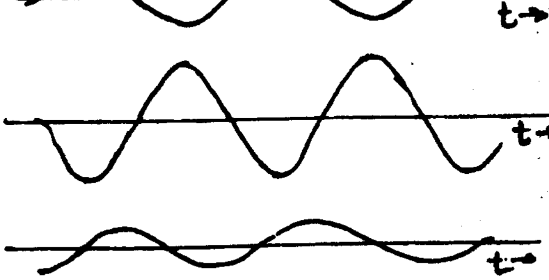



**Fig. S11.1-8**

(e) The process is not ergodic. Time means of each sample function is different and is not equal to the ensemble mean.

(f) $\overline{x^2} = R_x(0) = \dfrac{1}{3}$

**11.2-1** (a), (d), and (e) are valid PSDs. Others are not valid PSDs. PSD is always a real, non-negative and even function of $\omega$. Processes in (b), (c), (f), and (g) violate these conditions.

**11.2-2** (a) Let $x(t) = x_1$ and $x(t+\tau) = x_2$ Then,

$\overline{(x_1 \pm x_2)^2} = \overline{x_1^2} + \overline{x_2^2} + 2\overline{x_1 x_2} \geq 0,\quad \overline{x_1^2} + \overline{x_2^2} \geq \pm\, 2\overline{x_1 x_2}$

But, $\overline{x_1 x_2} = R_x(\tau)$ and $\overline{x_1^2} = \overline{x_2^2} = R_x(0)$ Hence, $R_x(0) \geq |R_x(\tau)|$

(b) $R_x(\tau) = \overline{x(t)x(t+\tau)},\quad \lim\limits_{\tau\to\infty} R_x(\tau) = \lim\limits_{\tau\to\infty} \overline{x(t)x(t+\tau)}$

As $\tau \to \infty$, $x(t)$ and $x(t+\tau)$ become independent, so $\lim\limits_{\tau\to\infty} R_x(\tau) = \overline{x(t)x(t+\tau)} = (\overline{x})(\overline{x}) = \overline{x}^2$

**11.2-3** $R_x(\tau) = 0$ for $\tau = \pm\dfrac{n}{2B}$ and its Fourier transform $S_x(\omega)$ is bandlimited to $B$ Hz. Hence, $R_x(\tau)$ is a waveform bandlimited to $B$ Hz and according to Eq. 6.10b

$R_x(\tau) = \sum\limits_{n=-\infty}^{\infty} R_x\left(\dfrac{n}{2B}\right) \operatorname{sinc}(2\pi B\tau - n)$. Since $R_x\left(\dfrac{n}{2B}\right) = 0$ for all $n$ except $n = 0$.

$R_x(\tau) = R_x(0) \operatorname{sinc}(2\pi B\tau)$ and $S_x(\omega) = \dfrac{R_x(0)}{2B} \operatorname{rect}\left(\dfrac{\omega}{4\pi B}\right)$. Hence, $x(t)$ is a white process bandlimited to $B$ Hz.

**11.2-4** $R_x(\tau) = P_{x_1x_2}(1, 1) + P_{x_1x_2}(-1,-1) - P_{x_1x_2}(-1, 1) - P_{x_1x_2}(1,-1)$

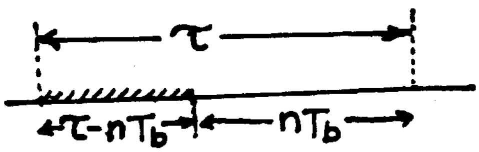
But because of symmetry of 1 and 0,

$P_{x_1x_2}(1, 1) = P_{x_1x_2}(-1,-1)$ and $P_{x_1x_2}(-1, 1) = P_{x_1x_2}(1,-1)$

and $R_x(\tau) = 2\left[P_{x_1x_2}(1, 1) - P_{x_1x_2}(1,-1)\right]$

$= 2P_{x_1}(1)\left[P_{x_2|x_1}(1|1) - P_{x_2|x_1}(-1|1)\right]$

$= 2P_{x_1}(1)\left[P_{x_2|x_1}(1|1) - \left(1 - P_{x_2|x_1}(1|1)\right)\right] = 2P_{x_2|x_1}(1|1) - 1$




Consider the case $nT_b < |\tau| < (n+1)T_b$. In this case, there are at least $n$ nodes and a possibility of $(n+1)$ nodes

$$\text{Prob}[(n+1)\text{nodes}] = \frac{\tau - nT_b}{T_b} = \frac{\tau}{T_b} - n$$

$$\text{Prob}(n \text{ nodes}) = 1 - \text{Prob}[(n+1)\text{nodes}] = (n+1) - \frac{\tau}{T_b}$$

The event $(x_2 = 1|x_1 = 1)$ can occur if there are N nodes and no state change at any node or state change at only 2 nodes or state change at only 4 nodes, etc.

Hence, $P_{x_2|x_1}(1|1) = \text{Prob}[(n+1)\text{nodes}]$ Prob(state change at even number of nodes) +
Prob($n$ nodes) Prob(State changes at eveen number of nodes)

The number of ways in which changes at $K$ nodes out of $N$ nodes occur is $\binom{N}{K}$. Hence,

$$P_{x_2|x_1}(1|1) = \left[\binom{n+1}{0}(0.6)^0(0.4)^{n+1} + \binom{n+1}{2}(0.6)^2(0.4)^{n-1} + \ldots\right]\left(\frac{|\tau|}{T_b} - n\right) +$$

$$\left[\binom{n}{0}(0.6)^0(0.4)^n+\binom{n}{2}(0.6)^2(0.4)^{n-2}+\ldots..\right]\left(n+1-\frac{|\tau|}{T_b}\right)$$

and $R_x(\tau)=2P_{x_1|x_2}(1|1)-1$ This yields

$$R_x(\tau)=1-1.2\frac{|\tau|}{T_b} \qquad |\tau|<T_b \qquad (n=0)$$

$$=-0.44+0.24\frac{|\tau|}{T_b} \qquad T_b<|\tau|<2T_b \qquad (n=1)$$

$$=0.136-0.048\frac{|\tau|}{T_b} \qquad 2T_b<|\tau|<3T_b \qquad (n=2)$$

and so on.

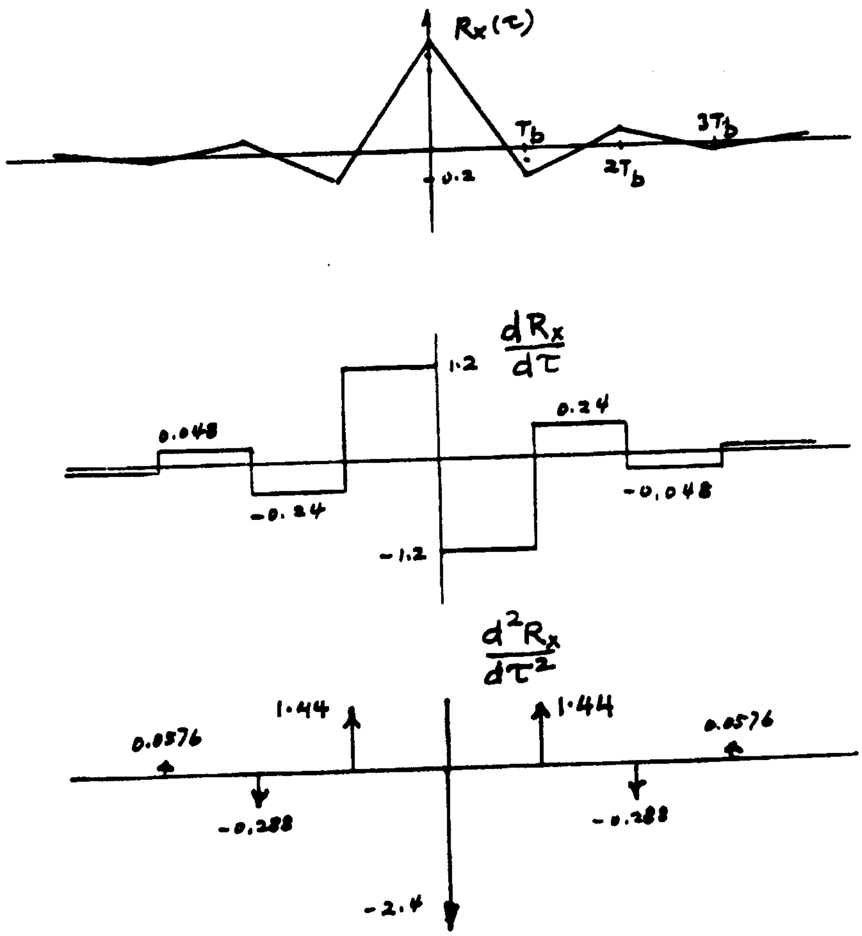



Fig. S11.2-4

The PSD can be found by differentiating $R_x(\tau)$ twice. The second derivative $d^2R_x/d\tau^2$ is a sequence of impulses as shown in Fig. S11.2-4. From the time-differentiation property,

$\frac{d^2}{d\tau^2}R_x(\tau) \leftrightarrow (j\omega)^2 S_x(\omega) = -\omega^2 S_x(\omega)$. Hence, recalling that $\delta(\tau - T) \Leftrightarrow e^{-j\omega T}$, we have

$$-\omega^2 S_x(\omega) = \frac{1}{T_b}\left[-2.4 + 1.44\left(e^{j\omega T_b} + e^{-j\omega T_b}\right) - 0.288\left(e^{j2\omega T_b} + e^{-j2\omega T_b}\right) + \ldots\right]$$
$$= \frac{1}{T_b}\left[-2.4 + 2.88\cos\omega T_b - 0.576\cos 2\omega T_b + 0.1152\cos 3\omega T_b + \ldots\right]$$

and

$$S_x(\omega) = \frac{1}{T_b\omega^2}\left[2.4 - 2.88\left(\cos\omega T_b - \frac{1}{5}\cos 2\omega T_b + \frac{1}{25}\cos 3\omega T_b - \frac{1}{125}\cos 4\omega T_b + \ldots\right)\right]$$

11.2-5 Because $S_m(\omega)$ is a white process bandlimited to $B$, $R_m(\tau) = R_m(0)\ \text{sinc}\ (2B\tau)$ and

$$R_m\left(\frac{n}{2B}\right) = 0, \quad n = \pm 1,\ \pm 2,\ \pm 3\ldots$$

This shows that $\overline{x(t)x\left(t + \frac{n}{2B}\right)} = R_m\left(\frac{n}{2B}\right) = 0$

Thus, all Nyquist sample are uncorrelated. Now, from Eq. 11.29,

$$S_y(\omega) = \frac{|P(\omega)|^2}{T_b}\left[R_0 + \sum_{n=1}^{\infty} R_m \cos n\omega_0 T_b\right]$$

$R_n = \overline{a_k a_{k+n}} = 0 \qquad n \ge 1$ and where $a_k$ is the *kth* Nyquist sample.

$R_0 = \overline{a_k^2} = \overline{x^2} = R_m(0)$. Hence,

$$S_y(\omega) = \frac{|P(\omega)|^2}{T_b} R_m(0) = 2BR_m(0)\left|P(\omega)^2\right| \qquad \text{since } T_b = \frac{1}{2B}$$

11.2-6 For duobinary

$$P_{a_k}(1) = P_{a_k}(-1) = 0.25 \text{ and } P_{a_k}(0) = 0.5$$
$$\overline{a_k} = (1)\frac{1}{4} + (-1)\frac{1}{4} + 0\left(\frac{1}{2}\right) = 0$$
$$R_0 = \overline{a_k^2} = (1)^2\frac{1}{4} + (-1)^2\frac{1}{4} + 0^2\left(\frac{1}{2}\right) = \frac{1}{2}$$
$$R_1 = \overline{a_k a_{k+1}} = \sum_{a_k}\sum_{a_{k+1}} a_k a_{k+1} P_{a_k a_{k+1}}(a_k a_{k+1})$$

Because $a_k$ and $a_{k+1}$ each can take 3 values (0, 1, -1), the double sum on the right-hand side of the above equation has 9 terms out of which only 4 are nonzero. Thus,

$$R_1 = (1)(1)P_{a_k a_{k+1}}(1,1) + (-1)(-1)P_{a_k a_{k+1}}(-1)(-1) + (1)(-1)P_{a_k a_{k+1}}(1)(-1) - (-1)(1)P_{a_k a_{k+1}}(-1)(1)$$

Because of duobinary rule, the neighboring pulses must have the same polarities. Hence,

$$P_{a_k a_{k+1}}(1,1) = P_{a_k}(1)P_{a_{k+1}|a_k}(1|1) = \frac{1}{4}\left(\frac{1}{2}\right) = \frac{1}{8}$$

Similarly, $P_{a_k a_{k+1}}(-1,-1) = \frac{1}{8}$ Hence, $R_1 = \frac{1}{4}$

Also $\qquad R_2 = \overline{a_k a_{k+2}}$

In this case, we have the same four terms as before, but $a_k$ and $a_{k+2}$ are the pulse strengths separated by one time slot. Hence, by duobinary rule,

$$P_{a_k a_{k+2}}(1,1) = P_{a_k}(1)P_{a_{k+2}|a_k}(1|1) = \frac{1}{4}\left(\frac{1}{4}\right) = \frac{1}{16}$$

Similarly, $P_{a_k a_{k+2}}(-1,-1) = \frac{1}{16}$

In a similar way, we can show that $P_{a_k a_{k+2}}(1,-1) = P_{a_k a_{k+2}}(-1,1) = \frac{1}{16}$

Hence, $R_2 = 0$

Using a similar procedure, we can show that $R_n = 0$ for $n \geq 2$. Thus, from Eq. (11.29) and noting that $R_n$ is an even function of $n$, we obtain $S_y(\omega) = \frac{|P(\omega)|^2}{T_b}\left[\frac{1}{2}(1+\cos\omega T_b)\right] = \frac{|P(\omega)|^2}{T_b}\cos^2\left(\frac{\omega T_b}{2}\right)$

For half-width rectangular pulse $P(\omega) = \frac{T_b}{2}\operatorname{sinc}\left(\frac{\omega T_b}{4}\right)$ and $S_y(\omega) = \frac{T_b}{4}\operatorname{sinc}^2\left(\frac{\omega T_b}{4}\right)\cos^2\left(\frac{\omega T_b}{2}\right)$

**11.2-7** $\overline{a_k} = (1)Q + (-1)(1-Q) = 2Q-1$

$R_0 = \overline{a_k^2} = (1)^2 Q + (-1)^2(1-Q) = 1$

Because all digits are independent,

$R_n = \overline{a_k a_{k+1}} = \bar{a}_k \bar{a}_{k+1} = (2Q-1)^2$ Hence,

$$S_y(\omega) = \frac{|P(\omega)|^2}{T_b}\left[1 + 2(2Q-1)^2\left(\sum_{n=1}^{\infty}\cos n\omega T_b\right)\right]$$

**11.2-8** Approximate impulses by rectangular pulses each of height $h$ and width $\varepsilon$ such that $h\varepsilon = 1$ and $\varepsilon \to 0$ (Fig. S11.2-8a)

$R_x(\tau) = \sum_{x_1}\sum_{x_2} x_1 x_2 P_{x_1 x_2}(x_1 x_2)$

Since $x_1$ and $x_2$ can take only two values $h$ and 0, there will only be 4 terms in the summation, out of which only one is nonzero (corresponding to $x_1 = h,\ x_2 = h$). Hence,

$R_x(\tau) = h^2 P_{x_1 x_2}(h,h) = h^2 P_{x_1}(h) P_{x_2|x_1}(h|h)$

Since there are $\alpha$ pulses/second, pulses occupy $\alpha\varepsilon$ fraction of time. Hence,

$P_{x_1}(h) = \alpha\varepsilon$ and $R_x(\tau) = h^2 \alpha\varepsilon P_{x_2|x_1}(h|h) = \alpha h P_{x_2|x_1}(h|h)$.

Now, consider the range $|\tau| < \varepsilon$. $P_{x_2|x_1}(h|h)$ is the Prob$(x_2) = h$, given that $x_1 = h$. This means $x_1$ lies on one of the impulses. Mark off an interval of $\tau$ from the edge of this impulse (see fig. S11.2-8b). If $x_1$ lies in the hatched interval, $x_2$ falls on the same pulse.

Hence,

$P_{x_2|x_1}(h|h) = \text{Prob}(x_1 \text{ lie in the hatched region}) = \frac{\varepsilon - \tau}{\varepsilon} = 1 - \frac{\tau}{\varepsilon}$

and $R_x(\tau) = \alpha h\left(1 - \frac{\tau}{\varepsilon}\right)$

Since $R_x(\tau)$ is an even function of $\tau$, $R_x(\tau) = \alpha h\left(1 - \frac{|\tau|}{\varepsilon}\right)$

In the limit as $\varepsilon \to 0$, $R_x(\tau)$ becomes an impulse of strength $\alpha$.

$R_x(\tau) = \alpha\delta(\tau) \quad |\tau| = 0$.

When $\tau > \varepsilon$, $x_1$ and $x_2$ become independent. Hence,

$$P_{x_2|x_1}(h|h) = P_{x_2}(h) = \alpha\varepsilon$$

$$R_x(\tau) = \alpha^2 h\varepsilon = \alpha^2 \qquad |\tau| > 0$$

Hence, $$R_x(\tau) = \alpha\delta(\tau) + \alpha^2$$

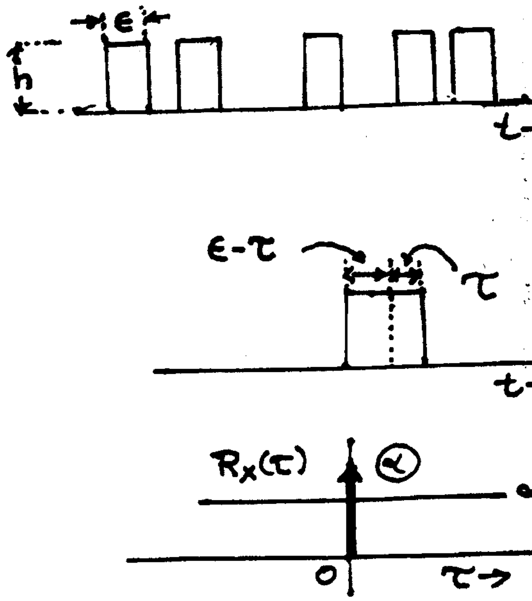



Fig. S11.2-8

11.2-9 In this case the autocorrelation function at $\tau = 0$ remain same as in Prob 11.2-8. But for $\tau > 0$ whenever $x(t)$, $x(t+\tau)$ are both nonzero, the product $x(t)x(t+\tau)$ is equally likely to be $h^2$ and $-h^2$. Hence, $R_x(\tau) = 0, \quad \tau \neq 0$ and $R_x(\tau) = \alpha\delta(\tau)$

11.2-10 The process in this problem represents the model for the thermal noise in conductors. A typical sample function of this process is shown in Fig. S11.2-10. The signal $x(t)$ changes abruptly in amplitude at random instants. The average number of changes or shifts in amplitudes are $\beta$ per second, and the number of changes are Poisson-distributed. The amplitude after a shift is independent of the amplitude prior to the shift. The first-order probability density of the process is $p(x;t)$. It can be shown that this process is stationary of order 2. Hence, $p(x;t)$ can be expressed as $p(x)$. We have

$$R_x(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 p_{x_1 x_2}(x_1, x_2)dx_1 dx_2$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 p_{x_1}(x_1) p_{x_2}(x_2|x_1 = x_1)dx_1 dx_2 \tag{1}$$

To calculate $p_{x_2}(x_2|x_1 = x_1)$, we observe that in $\tau$ seconds (interval between $x_1$ and $x_2$), there are two mutually exclusive possibilities; either there may be no amplitude shift $(x_2 = x_1)$, or there may be an amplitude shift $(x_2 \neq x_1)$. We can therefore express $p_{x_2}(x_2|x_1 = x_1)$ as

$$p_{x_2}(x_2|x_1 = x_1) = p_{x_2}(x_2|x_1 = x_1, \text{ no amplitude shift})P(\text{no amplitude shift}) +$$
$$p_{x_2}(x_2|x_1 = x_1, \text{ amplitude shift})P(\text{amplitude shift})$$

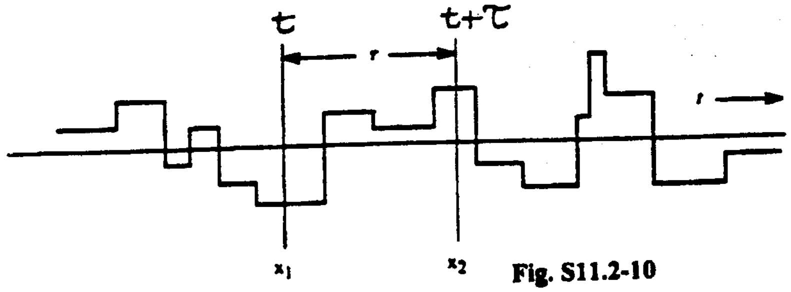



**Fig. S11.2-10**

The number of amplitude shifts are given to have Poisson distribution. The probability of $k$ shifts in $\tau$ seconds is given by

$$p_k(\tau) = \frac{(\beta\tau)^k}{k!}e^{-\beta\tau}$$

where there are on the average $\beta$ shifts per second. The probability of no shifts is obviously $p_0(\tau)$, where

$p_0(\tau) = e^{-\beta\tau}$

The probability of amplitude shift $= 1 - p_0(\tau) = 1 - e^{-\beta\tau}$. Hence

$$p_{x_2}(x_2|x_1 = x_1) = e^{-\beta\tau} p_{x_2}(x_2|x_1 = x_1, \text{ no amplitude shift}) + \left(1 - e^{-\beta\tau}\right) p_{x_2}(x_2|x_1 = x_1, \text{ amplitude shift}) \tag{2}$$

when there is no shift, $x_2 = x_1$ and the probability density of $x_2$ is concentrated at the single value $x_1$. This is obviously an impulse located at $x_2 = x_1$. Thus,

$$p_{x_2}(x_2|x_1 = x_1, \text{ no amplitude shift}) = \delta(x_2 - x_1) \tag{3}$$

whenever there are one or more shifts involved, in general, $x_2 \neq x_1$. Moreover, we are given that the amplitudes before and after a shift are independent. Hence,

$$p_{x_2}(x_2|x_1 = x_1, \text{ amplitude shift}) = p_{x_2}(x_2) = p(x) \tag{4}$$

where $p_{x_2}(x_2)$ is the first-order probability density of the process. This is obviously $p(x)$. Substituting Eqs. (3) and (4) in Eq. (2), we get

$$\begin{aligned} p_{x_2}(x_2|x_1 = x_1) &= e^{-\beta\tau}\delta(x_2 - x_1) + (1 - e^{-\beta\tau})p_{x_2}(x_2) \\ &= e^{-\beta\tau}\left[\delta(x_2 - x_1) + (e^{\beta\tau} - 1)p_{x_2}(x_2)\right] \end{aligned}$$

Substituting this equation in Eq. (1), we get

$$\begin{aligned} R_x(\tau) &= e^{-\beta\tau}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1x_2 p_{x_1}(x_1)\left[\delta(x_2 - x_1) + (e^{\beta\tau} - 1)p_{x_2}(x_2)\right]dx_1dx_2 \\ &= e^{-\beta\tau}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1x_2 p_{x_1}(x_1)\delta(x_2 - x_1)dx_1dx_2 + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1x_2(e^{\beta\tau} - 1)p_{x_1}(x_1)p_{x_2}(x_2)dx_1dx_2\right] \\ &= e^{-\beta\tau}\left[\int_{-\infty}^{\infty} x_1^2 p_{x_1}(x_1)dx_1 + (e^{\beta\tau} - 1)\int_{-\infty}^{\infty} x_1 p_{x_1}(x_1)dx_1\int_{-\infty}^{\infty} x_2 p_{x_2}(x_2)dx_2\right] \\ &= e^{-\beta\tau}\left[\overline{x^2} + (e^{\beta\tau} - 1)\bar{x}^2\right] \end{aligned} \tag{5}$$

where $\bar{x}$ and $\overline{x^2}$ are the mean and the mean-square value of the process. For a thermal noise $\bar{x} = 0$ and Eq. (5) becomes

$$R_x(\tau) = \overline{x^2}e^{-\beta\tau} \qquad \tau > 0$$

Since autocorrelation is an even function of $\tau$, we have

$$R_x(\tau) = \overline{x^2}e^{-\beta|\tau|}$$

and

$$S_x(\omega) = \frac{2\beta\overline{x^2}}{\beta^2 + \omega^2}$$

**11.3-1** For any real number $a$, $\overline{(ax - y)^2} \geq 0$

$$a^2\overline{x^2} + \overline{y^2} - 2a\overline{xy} \geq 0$$

Therefore the discriminant of the quadratic in $a$ must be non-positive. Hence,

$$\left(2\overline{xy}\right)^2 < 4\overline{x^2}\cdot\overline{y^2} \text{ or } \left(\overline{xy}\right)^2 < \overline{x^2}\ \overline{y^2}$$

Now, identify x with $x(t_1)$ and y with $y(t_2)$, and the result follows.

**11.3-2**

$$\begin{aligned} R_u(\tau) &= \overline{u(t)u(t+\tau)} = \overline{[x(t) + y(t)][x(t+\tau) + y(t+\tau)]} \\ &= R_x(\tau) + R_y(\tau) + R_{xy}(\tau) + R_{yx}(\tau) = R_x(\tau) + R_y(\tau) \end{aligned}$$

since $x(t)$ and $y(t)$ are independent.

$$\begin{aligned} R_v(\tau) &= \overline{[2x(t) + 3y(t)][2x(t+\tau) + 3y(t+\tau)]} \\ &= 4R_x(\tau) + 9R_y(\tau) \qquad \text{since } R_{xy}(\tau) = R_{yx}(\tau) = 0 \end{aligned}$$

$$R_{uv}(\tau) = \overline{[x(t) + y(t)][2x(t+\tau) + 3y(t+\tau)]} = 2R_x(\tau) + 3R_y(\tau)$$

$$R_{uv}(\tau) = R_{uv}(-\tau) = 2R_x(\tau) + 3R_y(\tau)$$

**11.3-3** $R_{xy}(\tau) = AB\overline{\cos(\omega_0 t + \phi)\cos[n\omega_0(t+\tau) + n\phi]}$

$$= \frac{AB}{2}\left\{\overline{\cos[\omega_0 t + n\omega_0(t+\tau) + (n+1)\phi]} + \overline{\cos[n\omega_0(t+\tau) - \omega_0 t + (n-1)\phi]}\right\}$$

$$\overline{\cos[\omega_0 t + n\omega_0(t+\tau) + (n+1)\phi]} = \frac{1}{2\pi}\int_0^{2\pi} \cos[\omega_0 t + n\omega_0(t+\tau) + (n+1)\phi]d\phi = 0$$

Similarly, $\overline{\cos[n\omega_0(t+\tau) - \omega_0 t + (n-1)\phi]} = 0$ and $R_{xy}(\tau) = 0$

11.3-4
$$x(t) = C_o + \sum_{n=1}^{\infty} C_n \cos n\omega_0(t - \mathrm{b}) + \theta_n$$
$$= C_o + \sum_{n=1}^{\infty} C_n(n\omega_0 t - n\omega_0 \mathrm{b} + \theta_n)$$

Since b is a r.v. uniformly distributed in the range $(0, T_b)$, $\omega_0 \mathrm{b} = \dfrac{2\pi \mathrm{b}}{T_b}$ is a r.v. uniformly distributed in the range $(0, 2\pi)$.

Using the argument in problem 11.3-3, we observe that all harmonics are incoherent. Hence the autocorrelation function of $R_x(\tau)$ is the sum of autocorrelation function of each term. Hence follows the result.

11.4-1 (a) $S_1(\omega) = 2KTR_1$ and $S_2(\omega) = 2KTR_2$

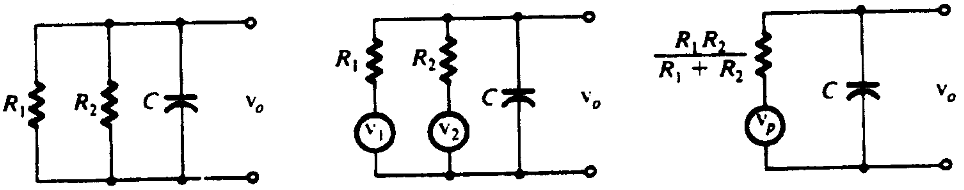
Since the two sources are incoherent, the principle of superposition applies to the PSD.

If $S_{o_1}(\omega)$ and $S_{o_2}(\omega)$ are the PSD's at the output terminals due to $S_1(\omega)$ and $S_2(\omega)$ respectively, then

$$S_{o_1}(\omega) = |H_1(\omega)|^2 S_1(\omega) \text{ and } S_{o2}(\omega) = |H_2(\omega)|^2 S_2(\omega)$$

where

$$H_1(\omega) = \frac{\dfrac{R_2 / j\omega C}{R_2 + 1/j\omega C}}{R_1 + \dfrac{R_2 / j\omega C}{R_2 + 1/j\omega C}} = \frac{\dfrac{R_2}{j\omega R_2 C + 1}}{R_1 + \dfrac{R_2}{j\omega R_2 C + 1}} = \frac{R_2}{R_1(j\omega R_2 C + 1) + R_2} = \frac{R_2}{j\omega R_1 R_2 C + R_1 + R_2}$$




(a) (b) (c)

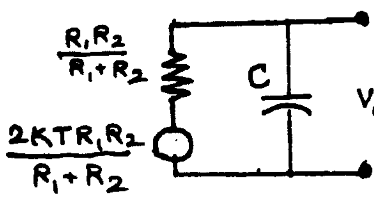
Fig. S11.4-1

Similarly,

$$H_2(\omega) = \frac{R_1}{R_2(j\omega R_1 C + 1) + R_1} = \frac{R}{j\omega R_1 R_2 C + R_1 + R_2}$$

$$S_{o_1}(\omega) = \frac{2KTR_1 R_2^2}{\omega^2 R_1^2 R_2^2 C^2 + (R_1 + R_2)^2} \quad \text{and} \quad S_{o_2}(\omega) = \frac{2KTR_2 R_1^2}{\omega^2 R_1^2 R_2^2 C^2 + (R_1 + R_2)^2}$$

$$S_{vo}(\omega) = S_{o_1}(\omega) + S_{o_2}(\omega) = \frac{2KTR_1 R_2(R_1 + R_2)}{\omega^2 R_1^2 R_2^2 C^2 + (R_1 + R_2)^2}$$




(b) $$H(\omega) = \frac{1/j\omega C}{\dfrac{1}{j\omega C} + \dfrac{R_1 R_2}{R_1 + R_2}} = \frac{R_1 + R_2}{j\omega C R_1 R_2 + (R_1 + R_2)}$$

$$S_{vo} = |H(\omega)^2| \frac{2KTR_1 R_2}{R_1 + R_2}$$

$$= \frac{(R_1 + R_2)^2}{\omega^2 C^2 R_1^2 R_2^2 + (R_1 + R_2)^2} \cdot \frac{2KTR_1 R_2}{R_1 + R_2} = \frac{2KTR_1 R_2 (R_1 + R_2)}{\omega^2 R_1^2 R_2^2 C^2 + (R_1 + R_2)^2}$$

which is the same as that found in part (a).

11.4-2 $y(t)=\int_{-\infty}^{\infty} h(\alpha)x(t-\alpha)d\alpha$

$R_{xy}(\tau)=\overline{x(t)y(t+\tau)}=\overline{x(t)\int_{-\infty}^{\infty} h(\alpha)x(t+\tau-\alpha)d\alpha}$

$=\int_{-\infty}^{\infty} h(\alpha)\overline{x(t)x(t+\tau-\alpha)}d\alpha=\int_{-\infty}^{\infty} h(\alpha)R_x(\tau-\alpha)d\alpha=h(\tau)*R_x(\tau)$ and $S_{xy}(\omega)=H(\omega)S_x(\omega)$

In Fig. 11.13, $H(\omega)=\dfrac{\frac{1}{j\omega C}}{R+\frac{1}{j\omega C}}=\dfrac{1}{j\omega RC+1}$

and $S_{nv_o}(\omega)=2KTR/(j\omega RC+1)$ and $R_{nv_o}(\tau)=2KTR\,e^{-\tau/RC}u(\tau)$

11.4-3 (a) We have found $R_x(\tau)$ of impulse noise in Prob. 11.2-8

$R_x(\tau)=\alpha\delta(\tau)+\alpha^2$, and $S_x(\omega)=\alpha+2\pi\alpha^2\delta(\omega)$

Hence,

$S_y(\omega)=|H(\omega)|^2\left[\alpha+2\pi\alpha^2\delta(\omega)\right]=2\pi\alpha^2|H(0)|^2\delta(\omega)+\alpha|H(\omega)|^2$

and $R_y(\tau)=\mathfrak{F}^{-1}\left[S_y(\omega)\right]=\alpha^2|H(0)|^2+\alpha h(\tau)*h(-\tau)$

(b) $h(t)=\dfrac{q}{\tau}\cdot e^{-t/\tau}u(t)$. $H(\omega)=\dfrac{q}{\tau}\cdot\dfrac{1}{j\omega+\frac{1}{\tau}}$

$|H(\omega)|^2=\dfrac{q^2}{1+\omega^2\tau^2}$, and $R_x(t)=\alpha^2q^2+\alpha\mathfrak{F}^{-1}\left[\dfrac{q^2}{1+\omega^2\tau^2}\right]=\alpha^2q^2+\dfrac{\alpha q^2}{2\tau}\cdot e^{-|t|/\tau}$

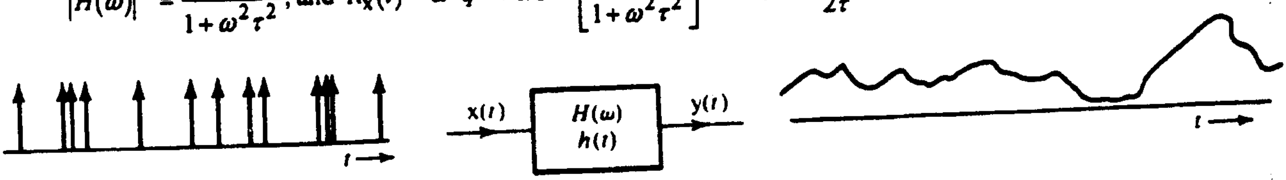



**Fig. S11.4-3**

11.5-1 $n(t)=n_c(t)\cos\omega_c t+n_s(t)\sin\omega_c t$

The PSD of $n_c(t)$ and $n_s(t)$ are identical. They are shown in Fig. S11.5-1. Also, $\overline{n^2}$ is the area under $S_n(\omega)$, and is given by $\overline{n^2}=2\left[\dfrac{\mathcal{N}}{2}\times10^4+\dfrac{10^4}{2}\left(\dfrac{\mathcal{N}}{2}\cdot\dfrac{1}{2}\right)\right]=1.25\times10^4\mathcal{N}$

$\overline{n_c^2}\left(\text{or } \overline{n_s^2}\right)$ is the area under $S_{n_c}(\omega)$, and is given by $\overline{n_c^2}=\overline{n_s^2}=2\left[5000\mathcal{N}+\dfrac{\mathcal{N}}{2}\cdot\dfrac{1}{2}\times5000\right]=1.25\times10^4\mathcal{N}$

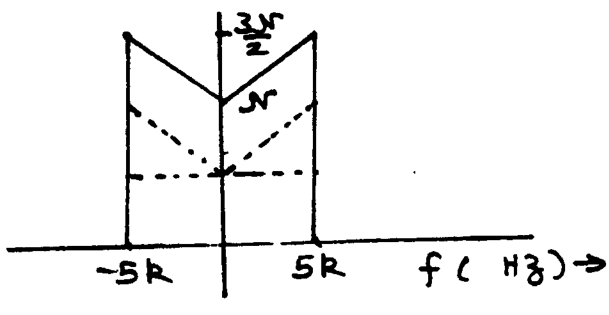



**Fig. S11.5-1**

**11.5-2** We follow a procedure similar to that of the solution of Prob. 11.5-1 except that the center frequencies are different. For the 3 center frequencies $S_{n_c}(\omega)$ [or $S_{n_s}(\omega)$] are shown in Fig. S11.5-2. In all the three cases, the area under $S_{n_c}(\omega)$ is the same, viz., $1.25\times10^4\mathcal{N}$. Thus in all 3 cases

$\overline{n_c^2} = \overline{n_s^2} = 1.25\times10^4\mathcal{N}$

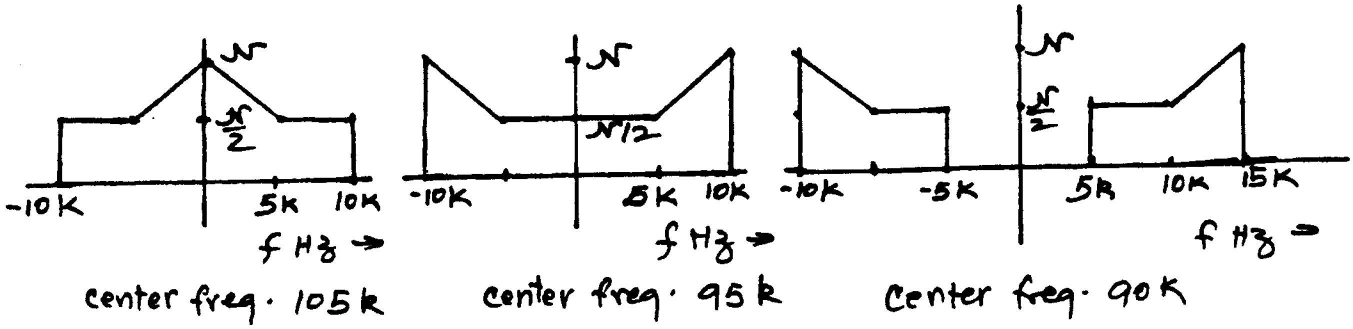



Fig. S11.5-2

**11.5-3** $\overline{n^2(t)} = \overline{n_c^2(t)} = \overline{n_s^2(t)} = 2\left[\frac{1}{2}\times10^{-3}\times100\times10^3\right] = 100$

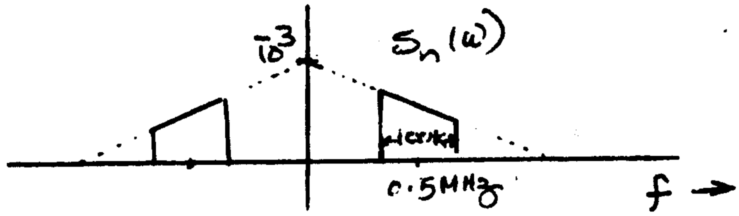



Fig. S11.5-3

**11.5-4** (a) $H_{op}(\omega) = \dfrac{S_m(\omega)}{S_m(\omega)+S_n(\omega)} = \dfrac{\frac{6}{9+\omega^2}}{\frac{6}{9+\omega^2}+6} = \dfrac{6}{6\omega^2+60} = \dfrac{1}{\omega^2+10}$

(b) $h_{op}(t) = \dfrac{1}{2\sqrt{10}}\cdot e^{-\sqrt{10}|t|}$

(c) The time constant is $\dfrac{1}{\sqrt{10}}$. Hence, a reasonable value of time-delay required to make this filter realizable is $\dfrac{3}{\sqrt{10}} = 0.949$ sec.

(d) Noise power at the output of the filter is

$$N_o = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{S_m(\omega)S_n(\omega)}{S_m(\omega)+S_n(\omega)}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{6}{\omega^2+10}d\omega = \frac{6}{2\pi\sqrt{10}}\tan^{-1}\frac{\omega}{\sqrt{10}}\Bigg|_{-\infty}^{\infty} = \frac{3}{\sqrt{10}}$$

The signal power at the output and the input are identical

$$S_i = S_o = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{6}{9+\omega^2}d\omega = 1$$

$$\text{SNR} = \frac{S_0}{N_o} = \frac{\sqrt{10}}{3} = 1.054$$

**11.5-5** (a)

$$H_{op}(\omega) = \frac{S_m(\omega)}{S_m(\omega)+S_n(\omega)} = \frac{\dfrac{4}{\omega^2+4}}{\dfrac{4}{\omega^2+4}+\dfrac{32}{\omega^2+64}}$$

$$= \frac{\omega^2+64}{9\omega^2+96} = \frac{1}{9}\left[1+\frac{53.33}{\omega^2+10.67}\right]$$

(b)

$$h_{op}(t) = \frac{1}{9}\delta(t) + 8.163e^{-3.266|t|}$$

(c) The time constant of the filter is 0.306 sec.
A reasonable value of time-delay required to make this filter realizable is $3 \times 0.306 = 0.918$ sec.

(d) Noise power at the output of the filter is

$$N_o = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{S_m(\omega)S_n(\omega)}{S_m(\omega)+S_n(\omega)}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{32}{9(\omega^2+10.67)}d\omega = 0.544$$

The signal power is

$$S_i = S_o = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{4}{4+\omega^2}d\omega = 1$$

$$\frac{S_0}{N_o} = \frac{1}{0.544} = 1.838$$

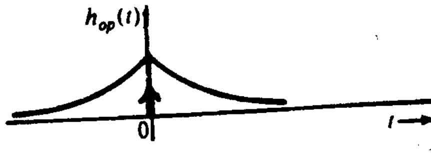



Fig. S11.5-5

# Chapter 12

**12.1-1** $\dfrac{S_o}{N_o} = \gamma = \dfrac{S_i}{\mathcal{N}B}$, $\quad \mathcal{N} = 2 \times S_n(\omega) = 2 \times 10^{-8}$, $B = \dfrac{\alpha}{2\pi} = 4000$ Hz.

$$\gamma = 1000 = \frac{S_i}{2 \times 10^{-8} \times 4000} \Rightarrow S_i = 0.08$$

Also, $H_c(\omega) = 10^{-3}$. Hence, $S_T = \dfrac{S_i}{|H_c(\omega)|^2} = 8 \times 10^4$

$$S_T = \frac{1}{2\pi}\beta[2 \times 8000\pi] = 8 \times 10^4 \Rightarrow \beta = 10$$

**12.1-2**

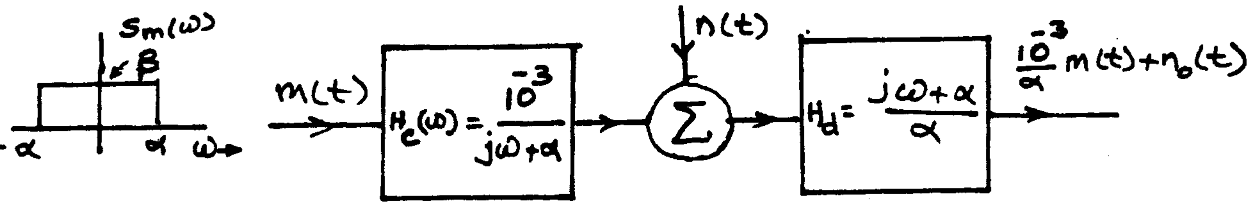



**Fig. S12.1-2**

$$S_{n_o}(\omega) = S_n(\omega)|H_d(\omega)|^2 = 10^{-10}\left(\frac{\omega^2+\alpha^2}{\alpha^2}\right) \qquad \alpha = 8000\pi$$

$$N_o = \frac{1}{\pi}\int_0^\alpha 10^{-10}\left(\frac{\omega^2+\alpha^2}{\alpha^2}\right)d\omega = \frac{10^{-10}}{\alpha^2\pi}\left(\frac{\omega^3}{3}+\alpha^2\omega\right)\Bigg|_0^\alpha = \frac{32}{3} \times 10^{-7}$$

$$35 \text{ dB} = 3162 = \frac{S_o}{N_o} = \frac{S_o}{\frac{32}{3} \times 10^{-7}} \Rightarrow S_o = 3.37 \times 10^{-3}$$

But $s_o(t) = \dfrac{10^{-3}}{\alpha}m(t)$. Hence,

$$S_o = \frac{10^{-6}}{\alpha^2}\overline{m^2(t)} = 3.37 \times 10^{-3} \Rightarrow \overline{m^2(t)} = 215.7 \times 10^9$$

Also, $\overline{m^2} = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \beta d\omega = \dfrac{\beta\alpha}{\pi} = 8000\beta = 215.7 \times 10^9$

Hence, $\beta = 26.96 \times 10^6$ and $S_m(\omega) = 26.96 \times 10^6 \operatorname{rect}\left(\dfrac{\omega}{2\alpha}\right)$

$$S_i = \frac{1}{\pi}\int_0^\alpha S_m(\omega)|H_c(\omega)|^2 d\omega = \frac{1}{\pi}\int_0^\alpha 26.96 \times 10^6\left(\frac{10^6}{\omega^2+\alpha^2}\right)d\omega$$

$$= \frac{26.96}{\alpha\pi}\tan^{-1}\frac{\omega}{\alpha}\Bigg|_0^\alpha = \frac{26.96}{4\alpha} = 2.68 \times 10^{-4}$$

$$S_T = \frac{1}{\pi}\int_0^\alpha S_m(\omega)d\omega = \frac{1}{\pi}\int_0^\alpha 26.96 \times 10^6 d\omega = \frac{26.96 \times 10^6\alpha}{\pi} = 68.65 \times 10^9$$

**12.2-1** (a) $30 \text{ dB} = 1000 = \dfrac{S_o}{N_o} = \gamma = \dfrac{S_i}{\mathcal{N}B} = \dfrac{S_i}{10^{-10} \times 4000} \Rightarrow S_i = 4 \times 10^{-4}$

(b) From Eq. (12.7), $N_o = \mathcal{N}B = 10^{-10}(4000) = 4 \times 10^{-7}$

(c) $S_i = |H_c(\omega)|^2 S_T$ and $10^{-8}S_T = 4 \times 10^{-4} \Rightarrow S_T = 4 \times 10^4$

12.2-2 (a) $\frac{S_o}{N_o} = 1000 = \frac{S_i}{\mathcal{N}B} = \frac{S_i}{10^{-10} \times 4000} \Rightarrow S_i = 4 \times 10^{-4}$

(b) $N_o = \mathcal{N}B = 10^{-10} \times 8000 = 4 \times 10^{-7}$

(c) $S_i = |H_c(\omega)|^2 S_T = 10^{-8} S_T = 4 \times 10^{-4} \Rightarrow S_T = 4 \times 10^4$

12.2-3 Let the signals $m_1(t)$ and $m_2(t)$ be transmitted over the same band by carriers of the same frequency $(\omega_c)$, but in phase quadrature. The two transmitted signals are $\sqrt{2}[m_1(t)\cos\omega_c t + m_2(t)\sin\omega_c t]$

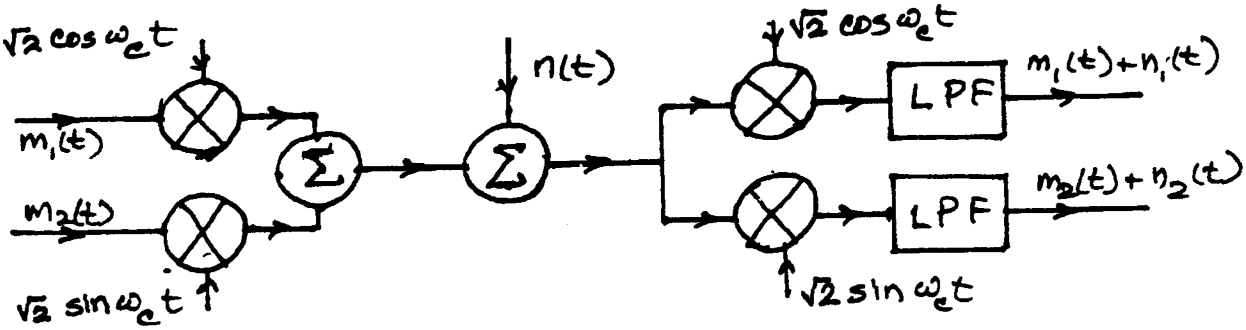



**Fig. S12.2-3**

The bandpass noise over the channel is $n_c(t)\cos\omega_c t + n_s(t)\sin\omega_c t$. Hence, the received signal is

$$\left[\sqrt{2}m_1(t) + n_c(t)\right]\cos\omega_c t + \left[\sqrt{2}m_2(t) + n_s(t)\right]\sin\omega_c t$$

Eliminating the high frequency terms, we get the output of the upper lowpass filter as $m_1(t) + \frac{1}{\sqrt{2}}n_c(t)$

Similarly, the output of the lower demodulator is $m_2(t) + \frac{1}{\sqrt{2}}n_s(t)$

These are similar to the outputs obtained for DSB-SC on page 535. Hence, we have $\frac{S_o}{N_o} = \gamma$ for both QAM channels.

12.2-4 (a) $\mu = \frac{-[m(t)]\text{min}}{A} = \frac{m_p}{A}$ Hence, $m_p = \mu A$

(b) $\frac{S_o}{N_o} = \frac{\overline{m^2}}{A^2 + \overline{m^2}} \cdot \gamma = \frac{\overline{m^2}}{\frac{m_p^2}{\mu^2} + \overline{m^2}} \cdot \gamma = \frac{\mu^2}{\kappa^2 + \mu^2}\gamma$ where $\kappa^2 = \frac{m_p^2}{\overline{m^2}}$

(c) For tone modulation $\kappa^2 = \frac{m_p^2}{m_p^2/2} = 2$ and for $\mu = 1$, $\frac{S_o}{N_o} = \frac{1}{2+1}\gamma = \frac{\gamma}{3}$

(d) Ratio $\frac{S_T}{S_T'} = \frac{A^2 + \overline{m^2}}{\overline{m^2}} = \frac{m_p^2 + \overline{m^2}}{\overline{m^2}} = \frac{m_p^2}{\overline{m^2}} + 1 \approx \kappa$ if $\kappa^2 >> 1$

12.2-5 (a) From Prob. 12.2-4, $\frac{S_o}{N_o} = \frac{\mu^2}{\kappa^2+\mu^2}\gamma$. For $3\sigma$ − loading, $m_p = 3\sigma_m$ and $\kappa^2 = \frac{m_p^2}{\overline{m^2}} = \frac{(3\sigma_m)^2}{\sigma_m^2} = 9$

and when $\mu = 1$, $\frac{S_o}{N_o} = \frac{1}{9+1}\gamma = \frac{\gamma}{10}$

(b) When $\mu = 0.5$, $\frac{S_o}{N_o} = \frac{(0.5)^2}{9+(0.5)^2}\gamma \approx \frac{\gamma}{36}$

12.2-6 For tone modulation, let $m(t) = \mu A\cos\omega_m t$. For BSB-SC,

$$\phi_{DSB}(t) = \sqrt{2}\mu A\cos\omega_m t\cdot\cos\omega_c t$$
$$= \frac{\mu A}{\sqrt{2}}\left[\cos(\omega_c+\omega_m)t + \cos(\omega_c-\omega_m)t\right]$$

$S_i = \frac{\mu^2A^2}{4} + \frac{\mu^2A^2}{4} = \frac{\mu^2A^2}{2}$ and $m_p = \frac{\mu A}{\sqrt{2}} + \frac{\mu A}{\sqrt{2}} = \sqrt{2}\mu A$ Hence, the peak power

$S_p = \left(\sqrt{2}\mu A\right)^2 = 2\mu^2A^2$ and $\frac{S_o}{N_o} = \gamma = \frac{S_i}{\mathcal{N}B} = \frac{S_p}{4\mathcal{N}B}$ where $S_i = \frac{1}{4}S_p$

For SSB-SC

$$\phi_{SSB}(t) = m(t)\cos\omega_c t + m_h(t)\sin\omega_c t$$
$$= \mu A\cos\omega_m t\cos\omega_c t + \mu A\sin\omega_m t\sin\omega_c t = \mu A\cos(\omega_c-\omega_m)t$$

$S_i = \frac{\mu^2A^2}{2}$ and $m_p = \mu A$. Hence, $S_p = \mu^2A^2$ and $\frac{S_o}{N_o} = \gamma = \frac{S_i}{\mathcal{N}B} = \frac{\mu^2A^2}{2\mathcal{N}B} = \frac{S_p}{2\mathcal{N}B}$

For AM

$$\phi_{AM}(t) = A(1+\mu\cos\omega_m t)\cos\omega_c t$$
$$S_i = \frac{A^2}{2} + \frac{\overline{m^2}}{2} = \frac{A^2}{2} + \frac{\mu^2A^2}{2}$$
$$m_p = A(1+\mu) \text{ and } S_p = A^2(1+\mu)^2.$$

Hence,

$$S_i = \frac{S_p(2+\mu^2)}{4(1+\mu)^2} \text{ and } \frac{S_o}{N_o} = \frac{\overline{m^2}}{A^2+\overline{m^2}}\gamma = \frac{\mu^2A^2/2}{A^2+(\mu^2A^2/2)}\cdot\frac{S_i}{\mathcal{N}B} = \left(\frac{\mu^2}{2+\mu^2}\right)\left[\frac{S_p(2+\mu^2)}{4(1+\mu)^2\mathcal{N}B}\right]$$

Under best condition, ie., for $\mu = 1$, $\frac{S_o}{N_o} = \frac{S_p}{10\mathcal{N}B}$

Hence, for a given peak power (given $S_p$) DSB-SC has 6dB superiority, and SSB-SC has 9dB superiority over AM. These results are derived for tone modulation and for $\mu = 1$ (the case most favorable for AM).

12.2-7 For $4\sigma$ loading, $m_p = 4\sigma_m$ and the carrier amplitude $A = mp = 4\sigma_m$ (for $\mu = 1$). For Gaussian $m(t)$,

$\overline{m^2} \cong \sigma_m^2$ (assuming $\overline{m} = 0$)

$$\text{Prob}(E \ge A) = \int_A^\infty \frac{E_n}{\sigma_n^2}e^{-E_n^2/2\sigma_n^2}dE_n = e^{-A^2/2\sigma_n^2} = 0.01$$

Hence, $\frac{A^2}{2\sigma_n^2} = 8\frac{\sigma_m^2}{\sigma_n^2} = 4.605$ and $S_i = \frac{A^2+\overline{m^2}}{2} = \frac{16\sigma_m^2+\sigma_m^2}{2} = \frac{17}{2}\cdot\sigma_m^2$

Therefore, $\gamma_{\text{Thresh}} = \frac{S_i}{\mathcal{N}B} = \frac{17}{2}\cdot\frac{\sigma_m^2}{\mathcal{N}B} = \frac{17}{8}\left(\frac{4\sigma_m^2}{\mathcal{N}B}\right) = \frac{17}{8}(4.605) = 9.79\text{dB}$

12.3-1 $\frac{S_o}{N_o} = 28\text{dB} = 631$. Hence,

$$\frac{S_o}{N_o} = 631 = 3\beta^2\gamma\frac{\overline{m^2(t)}}{m_p^2}$$
$$= 3(2)^2\gamma\frac{\sigma_m^2}{(3\sigma_m)^2}$$

Therefore, $\gamma = \frac{631 \times 9}{12} = 473.25$

(a) Also, $\gamma = \frac{S_i}{\mathcal{N}B} \Rightarrow S_i = \gamma\mathcal{N}B = 473.25 \times 2 \times 10^{-10} \times 15000 = 1.4197 \times 10^{-3}$

(b) $\beta = \frac{\Delta\omega}{2\pi B} = \frac{k_f m_p}{2\pi B} \Rightarrow 2 = \frac{k_f(3\sigma_m)}{30{,}000\pi} \Rightarrow k_f\sigma_m = 20{,}000\pi$

$S_o = \alpha^2 k_f^2 \overline{m^2(t)} = \alpha^2 k_f^2 \sigma_m^2 = \left(10^{-4}\right)^2 (20{,}000\pi)^2 = 4\pi^2$

(c) $N_o = \frac{S_o}{631} = 0.0199$

12.3-2 $m_p = B,\ m_p' = \frac{4B}{T_o}$ and bandwidth $= \frac{3}{T_o}$. Hence,

$$\frac{(S_o/N_o)_{PM}}{(S_o/N_o)_{FM}} = \frac{(2\pi \times 3/T_o)^2 B^2}{3(4B/T_o)^2} = \frac{3\pi^2}{4}$$

12.3-3 $m(t) = \cos^3\omega_o t$ and $m_p = 1$

$\dot{m}(t) = -3\omega_o\cos^2\omega_o t\sin\omega_o t$ and $\ddot{m}(t) = -3\omega_o\left[\omega_o\cos^2\omega_o t\cos\omega_o t - 2\omega_o\cos\omega_o t\sin^2\omega_o t\right]$

For a maximum

$\ddot{m}(t) = 0$. This yields $\cos^2\omega_o t = 2\sin^2\omega_0 t$

or $1 - \sin^2\omega_o t = 2\sin^2\omega_o t \Rightarrow \sin\omega_o t = \frac{1}{\sqrt{3}},\quad \cos\omega_o t = \sqrt{\frac{2}{3}}$

and

$$m_p' = \left|-3\omega_o\cos^2\omega_o t\sin\omega_o t\right| = 3\omega_o\left(\frac{2}{3}\right)\left(\frac{1}{\sqrt{3}}\right) = \frac{2}{\sqrt{3}}\omega_o$$

$$\frac{(S_o/N_o)_{PM}}{(S_o/N_o)_{FM}} = \frac{(3\omega_o)^2 m_p^2}{3m_p'^2} = \frac{9\omega_o^2}{3\left(\frac{4}{3}\omega_o^2\right)} = 2.25$$

12.3-4 $m(t) = a_1\cos\omega_1 t + a_2\cos\omega_2 t,\qquad m_p = a_1 + a_2$

$\dot{m}(t) = -(a_1\omega_1\sin\omega_1 t + a_2\omega_2\sin\omega_2 t),\qquad m_p' = a_1\omega_1 + a_2\omega_2$

$$\frac{(S_o/N_o)_{PM}}{(S_o/N_o)_{FM}} = \frac{(2\pi B)^2 m_p^2}{3m_p'^2} = \frac{\omega_2^2(a_1+a_2)^2}{3(a_1\omega_1 + a_2\omega_2)^2}$$

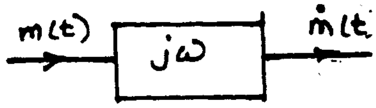
$$= \frac{\omega_2^2 a_2^2\left(1 + \frac{a_1}{a_2}\right)^2}{3\omega_2^2 a_2^2\left(1 + \frac{a_1\omega_1}{a_2\omega_2}\right)^2}$$

$$= \frac{(1+x)^2}{3(1+xy)^2}$$

**12.3-5** Error in this problem. There should be $4\pi^2$ in the denominator (see below).

$S_{\dot m}(\omega) = \omega^2 S_m(\omega)$. Hence,

$$\int_{-\infty}^{\infty} [\dot m(t)]^2 dt = \int_{-\infty}^{\infty} S_{\dot m}(2\pi f) df = \int_{-\infty}^{\infty} 4\pi^2 f^2 S_m(2\pi f) df$$




From Eq. (12.42a)

$$\overline{B_m^2} = \frac{\int f^2 S_m(2\pi f) df}{\int S_m(2\pi f) df} = \frac{1}{4\pi^2} \frac{\int_{-\infty}^{\infty} [\dot m(t)]^2 dt}{\int_{-\infty}^{\infty} m^2(t) dt}$$

These results are true for a waveform $m(t)$

**12.3-6**

$$\overline{B_m^2} = \frac{\int_{-\infty}^{\infty} \frac{f^2}{1+(f/f_o)^{2k}} df}{\int_{-\infty}^{\infty} \frac{1}{1+(f/f_o)^{2k}} df} = \frac{f_o^3 \int_{-\infty}^{\infty} \frac{x^2}{1+x^{2k}} dx}{f_o \int_{-\infty}^{\infty} \frac{1}{1+x^{2k}} dx}$$

$$= \frac{f_o^2 \left( \frac{\pi}{2k \sin\left(\frac{3\pi}{2k}\right)} \right)}{\frac{\pi}{2k \sin\left(\frac{\pi}{2k}\right)}} = f_o^2 \frac{\sin\left(\frac{\pi}{2k}\right)}{\sin\left(\frac{3\pi}{2k}\right)}$$

The definite integrals are found from integral tables.

As $k \to \infty$, $\overline{B_m{}^2} = f_o^2 \dfrac{\sin(\pi/2k)}{\sin(3\pi/2k)} \to f_o^2 \dfrac{(\pi/2k)}{(3\pi/2k)} = \dfrac{1}{3} f_o^2$

**12.3-7** $S_m(\omega) = \dfrac{|\omega|}{\sigma^2} e^{-\omega^2/2\sigma^2}$, $\overline{m^2} = 2\int_0^{\infty} \dfrac{\omega}{\sigma^2} e^{-\omega^2/2\sigma^2} d\omega = 2$

Hence, the normalized PSD's is $\dfrac{|\omega|}{2\sigma^2} e^{-\omega^2/2\sigma^2}$

If $W = 2\pi B$, then $\overline{W^2} = \overline{(2\pi B)^2} = 2\int_0^{\infty} \dfrac{\omega^3}{2\sigma^2} e^{-\omega^2/2\sigma^2} d\omega = 2\sigma^2$. If $p(W)$ is the power within the band $-W$ to $W$.

$$p(W) = 2\int_0^{W} \frac{\omega}{2\sigma^2} e^{-\omega^2/2\sigma^2} d\omega = 2\left[1 - e^{-W^2/2\sigma^2}\right],$$

$p(\infty) = 2$, and $\dfrac{p(W)}{p(\infty)} = 1 - e^{-W^2/2\sigma^2} = 0.99 \Rightarrow W = 3.03\sigma$, $B = 0.482\sigma$

$\dfrac{p(W)}{p(\infty)} = 0.95 \Rightarrow W = 2.45\sigma$, $B = 0.395\sigma$

$\dfrac{p(W)}{p(\infty)} = 0.9 \Rightarrow W = 2.15\sigma$, $B = 0.342\sigma$

$x = 0.99$ $\qquad \dfrac{W^2}{3} = 3.06\sigma^2 > \overline{W^2} \Rightarrow$ PM superior

$x = 0.95 \qquad \dfrac{W^2}{3} = 2.00\sigma^2 = \overline{W^2} \Rightarrow$ PM and FM equal

$x = 0.9 \qquad \dfrac{W^2}{3} = 1.54\sigma^2 < \overline{W^2} \Rightarrow$ FM superior

**12.3-8** $m(t) = a_1 \cos\omega_1 t + a_2 \cos\omega_2 t$, and

$$S_m(f) = \frac{a_1^2}{2}[\delta(f - f_1) + \delta(f + f_1)] + \frac{a_2^2}{2}[\delta(f - f_2) + \delta(f + f_2)]$$

$$\overline{m^2} = \int_{-\infty}^{\infty} S_m(f)df = \left(a_1^2 + a_2^2\right)/2$$

$$\overline{B_m^2} = \left(\int_{-\infty}^{\infty} f^2 S_m(f)df \Big/ \overline{m^2}\right) = \frac{2}{a_1^2 + a_2^2}\left[\frac{a_1^2 f_1^2}{2} + \frac{a_2^2 f_2^2}{2}\right] = \frac{a_1^2 f_1^2 + a_2^2 f_2^2}{a_1^2 + a_2^2},$$

Since $B = f_2$, PM is superior to FM if $f_2^2 > \dfrac{3\left(a_1^2 f_1^2 + a_2^2 f_2^2\right)}{a_1^2 + a_2^2}$,

that is, if $\dfrac{(a_1/a_2)^2(f_1/f_2)^2 + 1}{(a_1/a_2)^2 + 1} < \dfrac{1}{3}$ or if $1 + x^2 y^2 < \dfrac{1 + x^2}{3}$

**12.3-9** (a) $\dfrac{S_o}{N_o} = 3\beta^2\gamma \dfrac{\overline{m^2}}{m_p^2} = \dfrac{1}{3}\beta^2\gamma$. Since $m_p = 3\sigma$, $\overline{m^2} = \sigma^2$ and 23.4 dB = 218.8, $218.8 = \dfrac{1}{3}\beta^2\gamma = \dfrac{1}{3}(2)^2\gamma$

$\gamma = \dfrac{218.8 \times 3}{4} = 164.1$. Also, $\gamma_{\text{Thresh}} = 20(\beta + 1)$

So $\beta_{\text{Thresh}} = \dfrac{164.1}{20} - 1 = 7.21$

$\dfrac{S_o}{N_o} = \dfrac{1}{3}\beta^2\gamma = \dfrac{1}{3}(7.21)^2(164.1) = 2844 = 34.53$ dB $\qquad$ (40 dB = 10,000)

(b) $\dfrac{S_o}{N_o} = \dfrac{1}{3}\beta^2\gamma = \dfrac{1}{3}\left(\dfrac{\gamma_{Th} - 20}{20}\right)^2 \gamma_{Th} = 10{,}000$ or $(\gamma_{Th} - 20)^2 = \dfrac{12 \times 10^7}{\gamma_{Th}} \quad \Rightarrow \quad \gamma = 242.5$

Required increase in $\gamma = \dfrac{242.5}{164} = 1.479 = 1.7$ dB

**12.3-10** From Eq. (12.40) $\beta^2 = \dfrac{1}{3}\left[\dfrac{1}{1 + \left(\overline{m^2}/m_p^2\right)}\right]$

(1) Tone modulation $\qquad \beta^2 = \dfrac{1}{3}\left(\dfrac{1}{1 + 0.5}\right) \Rightarrow \beta = 0.47$

(2) Gaussian with $3\sigma$ – loading $\qquad \beta^2 = \dfrac{1}{3}\left(\dfrac{1}{1 + 1/9}\right) \Rightarrow \beta = 0.547$

(3) Gaussian with $4\sigma$ – loading $\qquad \beta^2 = \dfrac{1}{3}\left(\dfrac{1}{1 + 1/16}\right) \Rightarrow \beta = 0.56$

where For tone modulation, $\qquad \dfrac{\overline{m^2}}{m_p^2} = 0.5$.

For Gaussian modulation with $3\sigma$ – loading, $\dfrac{\overline{m^2}}{m_p^2} = \dfrac{\sigma^2}{(3\sigma)^2} = \dfrac{1}{9}$.

For Gaussian modulation with $4\sigma$ – loading, $\dfrac{\overline{m^2}}{m_p^2} = \dfrac{\sigma^2}{(4\sigma)^2} = \dfrac{1}{16}$

**12.3-11** Let us first analyze the L+R channel. In this case, the demodulator output signal, when passed through the 0-15 kHz (lowpass) filter, is given by $(L+R)' + n_0(t)$, where $S_{n_0}(\omega) = \dfrac{\mathcal{N}\omega^2}{A^2}$ [see Eq. (12.33)].

When this signal is passed through the de-emphasis filter $H_d(\omega) = \dfrac{\omega_1}{j\omega + \omega_1}$, the signal is restored to (L+R) and the output noise power $N_o'$ is given by

$$N_o' = \frac{1}{\pi}\int_0^W |H_d(\omega)|^2 S_{n_0}(\omega)d\omega = \frac{\mathcal{N}}{\pi A^2}\int_0^W \frac{\omega_1^2\omega^2}{\omega^2+\omega_1^2}d\omega = \frac{\mathcal{N}\omega_1^2}{\pi A^2}\left[W - \omega_1 \tan^{-1}\frac{W}{\omega_1}\right]$$

Let us now consider the (L-R) channel.
Let $\omega_c = 2\pi \times 38{,}000$ and $\omega_1 = 2\pi \times 2100$.
The received signal is FM demodulated (Fig. 5.19c). The PSD of the noise at the output of the FM demodulator is $S_{n_0}(\omega) = \mathcal{N}\omega^2 / A^2$ [see Eq. (12.33)] The output of the FM demodulator is separated into $(L+R)'$ over 0-15 kHz and $(L-R)' \cos\omega_c t$ over the band $38 \pm 15$ or 23 kHz to 53 kHz. Let us consider the signal over this passband, where the noise can be expressed as $n_c(t)\cos\omega_c t + n_s(t)\sin\omega_c t$. The signal is $(L-R)' \cos\omega_c t$. Hence, the received signal is $\left[(L-R)' + n_c(t)\right]\cos\omega_c + n_s(t)\sin\omega_c t$. This signal is multiplied by $2\cos\omega_c t$ and then lowpass-filtered to yield the output $(L-R)' + n_c(t)$. But

$$S_{n_c}(\omega) = S_n(\omega+\omega_c) + S_n(\omega-\omega_c) = \frac{\mathcal{N}}{A^2}\left[(\omega+\omega_c)^2 + (\omega-\omega_c)^2\right]$$

When this signal is passed through de-emphasis filter $H_d(\omega) = \dfrac{\omega_1}{j\omega+\omega_1}$, the signal is restored to (L-R) and the output noise power $N_o''$ is given by

$$N_o'' = \frac{1}{\pi}\int_0^W |H_d(\omega)|^2 S_{n_c}(\omega)d\omega = \frac{\mathcal{N}}{\pi A^2}\int_0^W \left[(\omega+\omega_c)^2 + (\omega-\omega_c)^2\right]\frac{\omega_1^2}{\omega^2+\omega_1^2}d\omega$$

$$= \frac{2\mathcal{N}\omega_1^2}{\pi A^2}\left[W + \frac{\omega_c^2 - \omega_1^2}{\omega_1}\tan^{-1}\frac{W}{\omega_1}\right] \qquad W = 2\pi \times 15{,}000$$

Hence, the (L-R) channel is noisier than (L+R) channel by factor $\dfrac{N_o''}{N_o'}$ given by

$$\frac{N_o''}{N_o'} = \frac{2\left(W + \dfrac{\omega_c^2+\omega_1^2}{\omega_1}\tan^{-1}\dfrac{W}{\omega_1}\right)}{W - \omega_1\tan^{-1}\left(\dfrac{W}{\omega_1}\right)} = \frac{2\left(B + \dfrac{f_c^2+f_1^2}{f_1}\tan^{-1}\dfrac{B}{f_1}\right)}{B - f_1\tan^{-1}\left(\dfrac{B}{f_1}\right)}$$

Substituting $B = 15{,}000$, $f_c = 38{,}000$, $f_1 = 2100$ in this equation yields:

$$\frac{N_o''}{N_o'} = 166.16 = 22.2 \text{ dB.}$$

**12.4-1** $L = M^n \Rightarrow n = \log_M L$

$$\frac{S_o}{N_o} = 3L^2 \frac{\overline{m^2(t)}}{m_p^2}$$

$$= 3M^{2n}\left(\frac{\overline{m^2}}{m_p^{\,2}}\right)$$

**12.4-2** $\frac{S_o}{N_o} = 55 \text{ dB} = 316200$

For uniform distribution

$$\overline{m^2} = \frac{1}{2m_p}\int_{-m_p}^{m_p} m^2\, dm$$

$$= \frac{1}{3} m_p$$

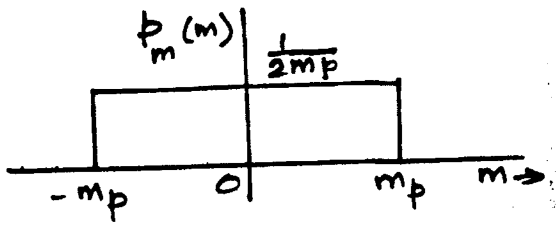



Fig. S12.4-2

(a) $316200 = 3(2)^{2n}\left(\frac{\overline{m^2}}{m_p^{\,2}}\right)$

$$= 3(2)^{2n}\left(\frac{1}{3}\right) = 2^{2n}$$

$2n = 18.27$

Since $n$ must be an integer, choose $n = 10$ and $L = 1024$

(b) $\frac{S_o}{N_o} = 3(2)^{20}\frac{1}{3} = 1.048576 \times 10^6 = 60.17$ dB.

$B_{PCM} = 2nB = 90$ MHz (assuming bipolar signaling)

(c) To increase the SNR by 6 dB, increase $n$ by 1, that is $n = 11$. Then the new bandwidth of transmission is $22 \times 4.5 = 99$ MHz.

**12.4-3** $S_i = 2BnE_p, \quad E_p = 2 \times 10^{-5}, \quad B = 4000, \quad n = 8$

$$S_i = 2 \times 4000 \times 8 \times 2 \times 10^{-5} = 1.28$$

$$\gamma = \frac{S_i}{\mathcal{N}B} = \frac{1.28}{2 \times 6.25 \times 10^{-7} \times 4000} = 2.56 \times 10^2$$

$$Q\left(\sqrt{\frac{\gamma}{n}}\right) = Q\sqrt{32} = 7.569 \times 10^{-9}$$

$B_m = nB = 8 \times 8000 = 64$ kHz (assuming bipolar line code)

(a) $$\frac{S_o}{N_o} = \frac{3(2)^{2n}}{1 + 4(2^{2n} - 1)Q\left(\sqrt{\frac{\gamma}{n}}\right)}\left(\frac{\overline{m^2}}{m_p^{\,2}}\right),$$

where $\sqrt{\frac{\gamma}{n}} = \sqrt{\frac{2E_p}{\mathcal{N}}} = \sqrt{\frac{2 \times 2 \times 10^{-5}}{2 \times 6.25 \times 10^{-7}}} = \sqrt{\frac{256}{8}} = \sqrt{32}$

So, $$\frac{S_o}{N_o} = \frac{3(2)^{16}}{1 + 4(2^{16} - 1)Q(\sqrt{32})}\left(\frac{1}{9}\right) \cong 21845 = 43.4 \text{ dB}.$$

(b) If power is reduced by 10 dB, then $\gamma = 25.6$, $\quad Q(\sqrt{3.2}) = Q(1.79) = 0.0367$ and

$$\frac{S_o}{N_o} = \frac{3(2)^{16}}{1 + 4(2^{16} - 1)Q(\sqrt{3.2})}\left(\frac{1}{9}\right) \cong 2.27 = 3.56 \text{ dB}.$$

The table below gives SNR for various values of $n$ under the reduced power.

(d)

| $n$ | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| So/No | 7.22 dB | 11.70 dB | 10.97 dB | 8.36 dB | 6.35 dB |

Hence, $n = 3$ yields the optimum SNR. The bandwidth in this case is $B_m = 3 \times 8000 = 24$ kHz.

12.4-4 $1 - P_E = P$ (correct detection over all K links) + smaller order terms

$$\cong (1-P_e)^{K-1}(1-P_e') \cong [1-(K-1)P_e][1-P_e'] \cong 1 - P_e' - (K-1)P_e$$

So $P_E = P_e' + (K-1)P_e$

(b) $\gamma = 25 \text{ dB} = 316.2, \quad \gamma = 23 \text{ dB} = 199.5$

$$P_e = Q\left(\sqrt{316.2/8}\right) = Q(6.287) = 1.6 \times 10^{-10}$$

$$P_e' = Q\left(\sqrt{199.5/8}\right) = Q(4.994) = 3 \times 10^{-7}$$

$$P_E = 99 \times 1.6 \times 10^{-10} + 3 \times 10^{-7} = 3.16 \times 10^{-7} \cong P_e'$$

12.4-5 $\overline{|\mathrm{m}|} = \int_0^A m p_{\mathrm{m}}(m)dm = \int_0^A m \frac{1}{A} dm = \frac{A}{2}$

$$\overline{\mathrm{m}^2} = \int_{-\infty}^{\infty} m^2 p_{\mathrm{m}}(m)dm = \int_{-A}^{A} m^2 \frac{1}{2A} dm = \frac{A^2}{3}$$

$$\sigma_{\mathrm{m}}^2 = \overline{\mathrm{m}^2} - (\overline{\mathrm{m}})^2 = \frac{A^2}{3}$$

$$\frac{S_o}{N_o} = \frac{3L^2}{[\ln(1+\mu)]^2} \frac{\sigma_{\mathrm{m}}^2/m_p^2}{(\sigma_{\mathrm{m}}^2/m_p^2) + (2\overline{|\mathrm{m}|}/\mu m_p) + (1/\mu^2)}$$

$$= \frac{3(2^8)^2}{(\ln 256)^2} \frac{\sigma m^2/m_p^2}{\frac{\sigma_m^2}{m_p^2} + \frac{A}{255 m_p} + \frac{1}{(255)^2}} = 6383 \frac{\frac{\sigma_m^2}{m_p^2}}{\frac{\sigma_m^2}{m_p^2} + 0.0068\frac{\sigma_{\mathrm{m}}}{\mathrm{m_p}} + 1.53 \times 10^{-5}}$$

12.5-1 As noted on Pg. (570), the optimum filters for DSB-SC and SSB-SC can be obtained from Eqs. (12.83a) and (12.83b), provided we substitute $\frac{1}{2}[S_{\mathrm{m}}(\omega+\omega_c) + S_{\mathrm{m}}(\omega-\omega_c)]$ for $S_m(\omega)$ in these equations. Let

$$\hat{S}_{\mathrm{m}}(\omega) = \frac{1}{2}[S_{\mathrm{m}}(\omega+\omega_c) + S_{\mathrm{m}}(\omega-\omega_c)]$$

$$= \frac{1}{2}\left[\frac{\alpha^2}{(\omega+\omega_c)^2+\alpha^2} + \frac{\alpha^2}{(\omega-\omega_c)^2+\alpha^2}\right]$$

$$= \frac{\alpha^2(\omega^2+\omega_c^2+\alpha^2)}{(\omega^2+\omega_c^2+\alpha^2)^2 - 4\omega^2\omega_c^2} \qquad \begin{matrix}\alpha = 3000\pi \\ \omega_c = 2\pi\times 10^5\end{matrix} \qquad (1)$$

We shall also require the power of $\hat{S}_{\mathrm{m}}(\omega)$.

$$I = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{S}_{\mathrm{m}}(\omega)d\omega$$

We can simplify the evaluation of this integral by recognizing that the power of the modulated signal $\mathrm{m}(t)\cos\omega_c t$ is half the power of $\mathrm{m}(t)$. Hence,

$$I = \frac{1}{2}\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{\mathrm{m}}(\omega)d\omega = \frac{1}{2\pi}\int_0^{\infty} \frac{\alpha^2}{\omega^2+\alpha^2}d\omega = \frac{\alpha}{2\pi}\tan^{-1}\frac{\omega}{\alpha}\bigg|_0^{\infty} = \frac{\alpha}{4} \qquad (2)$$

We shall use the PDE system shown in Fig. 12.19

**(a)** For this system

$$|H_p(\omega)|^2 = \frac{S_T\sqrt{S_n(\omega)/\hat{S}_m(\omega)}}{|H_c(\omega)|\int_{-\infty}^{\infty}\frac{\sqrt{\hat{S}_m(\omega)S_n(\omega)}}{|H_c(\omega)|}df} \tag{3}$$

Because $H_c(\omega)$ and $S_n(\omega)$ are constants, we have

$$|H_p(\omega)|^2 = \frac{S_T\sqrt{1/\hat{S}_m(\omega)}}{\int_{-\infty}^{\infty}\sqrt{\hat{S}_m(\omega)}df} = \frac{10^3/\sqrt{\hat{S}_m(\omega)}}{\frac{1}{\pi}\int_0^{\infty}\sqrt{\hat{S}_m(\omega)}d\omega}$$

where $\hat{S}_m(\omega)$ is found in Eq. (1). Also from Eq. (12.83b)

$$|H_d(\omega)|^2 = \frac{G^2}{S_T}\frac{\int_{-\infty}^{\infty}\sqrt{\hat{S}_m(\omega)}df}{\sqrt{1/\hat{S}_m(\omega)}} = \frac{10^4\int_0^{\infty}\sqrt{\hat{S}_m(\omega)}d\omega}{10^3\pi/\sqrt{\hat{S}_m(\omega)}} \tag{4}$$

**(b)** The output signal is $Gm(t)$. Hence, $S_o = G^2\overline{m^2(t)}$

We have already found the power of $m(t)$ to be $2(\alpha/4) = 2$. Hence

$$S_o = \frac{G^2\alpha}{2} = \frac{(10^{-2})^2(3000\pi)}{2} = \frac{3\pi}{20}$$

To find the output noise power $N_o$, we observe that the noise signal with PSD $S_n(\omega) = 2\times10^{-9}$ passes through the dc-emphasis filter $H_d(\omega)$ in Eq. (4) above. Hence, $\hat{S}_n(\omega)$ the noise PSD at the output of $H_d(\omega)$ is

$$\hat{S}_n(\omega) = S_n(\omega)|H_d(\omega)|^2 = \frac{2\times10^{-16}\int_0^{\infty}\sqrt{\hat{S}_m(\omega)}d\omega}{\pi/\sqrt{\hat{S}_m(\omega)}}$$

Also, the output noise power is $n_c(t)/\sqrt{2}$ and $N_o = \frac{\overline{n_c^2}}{2}$ [see Eq. (12.6b)], where $\overline{n_c^2} = \overline{n^2} = \frac{1}{\pi}\int_0^{\infty}\hat{S}_n(\omega)d\omega$

and

$$\frac{S_o}{N_o} = \frac{3\pi/20}{\frac{1}{\pi}\int_0^{\infty}\hat{S}_n(\omega)d\omega} = \frac{3\pi^2}{10\int_0^{\infty}\hat{S}_n(\omega)d\omega}$$

**12.5-2** Similar to Prob. 12.5-1

**12.5-3** The improvement ratio in FM is $\dfrac{B^3\overline{m^2}}{6\left[\int_0^B f\sqrt{S_m(\omega)}df\right]^2}$, where

$$\overline{m^2} = 2\int_0^{\infty}S_m(\omega)df = \int_0^B\beta df = 2B\beta \text{ and } \int_0^B f\sqrt{S_m(\omega)}df = \int_0^B f\sqrt{\beta}df = \sqrt{\beta}\left.\frac{f^2}{2}\right|_0^B = \sqrt{\beta}\frac{B^2}{2}$$

Hence, the improvement ratio is

$$\frac{B^3(2B\beta)}{6\left[\sqrt{\beta}\frac{B^2}{2}\right]^2} = \frac{4}{3} \cong 1.3\text{ dB.}$$

**13.1-1**

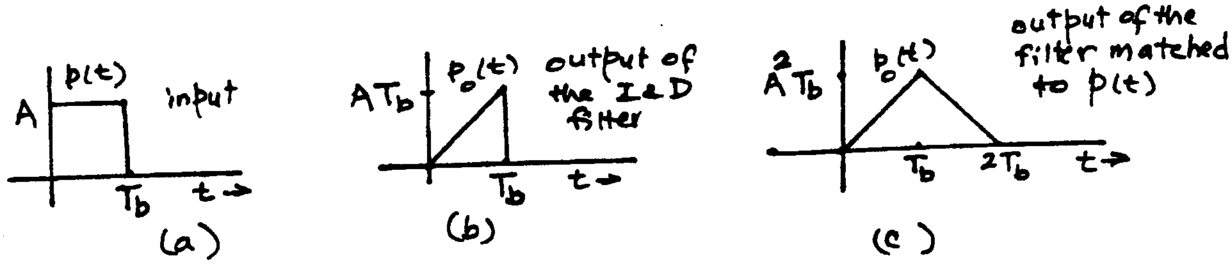



**Fig. S13.1-1**

For the integrate and dump filter (I&D), the output is the integral of $p(t)$. Hence, at $t = T_b$, $p_o(T_b) = AT_b$. If we apply $\delta(t)$ at the input of this filter, the output $h(t) = u(t) - u(t - T_b)$.
Hence,

$$H(\omega) = T_b \operatorname{sinc}\left(\frac{\omega T_b}{2}\right) e^{-j\omega T_b/2}$$

and

$$\overline{n_o^2(t)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\mathcal{N}}{2} T_b^2 \operatorname{sinc}^2\left(\frac{\omega T_b}{2}\right) d\omega$$
$$= \frac{\mathcal{N} T_b}{2}\frac{1}{\pi}\int_{-\infty}^{\infty} \operatorname{sinc}^2(x)\, dx = \frac{\mathcal{N} T_b}{2}$$

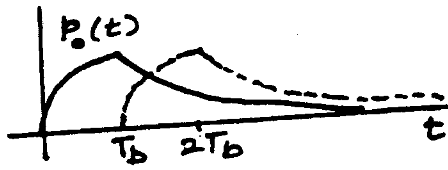
and

$$\rho^2 = \frac{p_o^2(T_b)}{\overline{n_o^2(t)}} = \frac{A^2 T_b^2}{\mathcal{N} T_b/2} = \frac{2}{\mathcal{N}} E_p$$

This is exactly the value of $\rho^2$ for the matched filter.

**13.1-2** The output $p_o(t)$ of this R-C filter is

$$p_o(t) = A\left(1 - e^{-t/RC}\right) \qquad 0 \le t \le T_b$$
$$= A\left(1 - e^{-T_b/RC}\right)e^{-(t-T_b)/RC} \qquad t > T_b.$$




The maximum value of $p_o(t)$ is $A_p$, which occurs at $T_b$:

$$A_p = p_o(T_b) = A\left(1 - e^{-T_b/RC}\right)$$
$$\sigma_n^2 = \frac{1}{2\pi}\cdot\frac{\mathcal{N}}{2}\int_{-\infty}^{\infty}\frac{d\omega}{1+\omega^2R^2C^2} = \frac{\mathcal{N}}{4RC}$$

and

$$\rho^2 = \frac{A_p^2}{\sigma_n^2} = \frac{4A^2RC\left(1-e^{-T_b/RC}\right)^2}{\mathcal{N}}.$$
$$= \frac{4A^2T_b}{\mathcal{N}}\cdot\frac{\left(1-e^{-T_b/RC}\right)^2}{T_b/RC}$$

We now maximize $\rho^2$ with respect to $RC$. Letting $x = T_b/RC$, we have

$$\rho^2 = \frac{4A^2T_b}{\mathcal{N}} \cdot \frac{\left(1-e^{-x}\right)^2}{x}$$

and

$$\frac{d\rho^2}{dx} = \frac{2xe^{-x}\left(1-e^{-x}\right)-\left(1-e^{-x}\right)^2}{x^2} = 0$$

This gives

$$2xe^{-x} = 1-e^{-x} \qquad \text{or} \qquad 1+2x = e^x$$

and

$$x \cong 1.26 \qquad \text{or} \qquad \frac{1}{RC} = \frac{1.26}{T_b}$$

Hence,

$$\rho^2_{\max} = (0.816)\frac{2A^2T_b}{\mathcal{N}}$$

Observe that for the matched filter,

$$\rho^2_{\max} = \frac{2E_p}{\mathcal{N}} = \frac{2A^2T_b}{\mathcal{N}}$$

13.2-1 $\beta^2_{\max} = \frac{2}{\mathcal{N}}\int_0^{T_b}\left[p(t)-q(t)\right]^2 dt = \frac{E_p+E_q-2E_{pq}}{\mathcal{N}/2}$

The energy of $p(t)$ is $T_b$ times the power of $p(t)$.
Hence,

$$E_p = \frac{A^2\left(1-m^2\right)}{2}T_b + \frac{A^2m^2}{2}T_b = \frac{A^2T_b}{2} = E_b$$

Similarly, $E_q = \frac{A^2T_b}{2} = E_b$

$$E_{pq} = \int_0^{T_b} p(t)q(t)dt = \int_0^{T_b}\left[-A^2\left(1-m^2\right)\cos^2\omega_c t + A^2m^2\sin^2\omega_c t\right]dt$$

$$= -\frac{A^2}{2}T_b + A^2m^2T_b$$

Hence,

$$\beta^2_{\max} = \frac{4A^2T_b\left(1-m^2\right)}{\mathcal{N}} = \frac{8E_b\left(1-m^2\right)}{\mathcal{N}}$$

and

$$P_e = Q\left(\frac{\beta_{\max}}{2}\right) = Q\left(\sqrt{\frac{2E_b\left(1-m^2\right)}{\mathcal{N}}}\right)$$

13.2-2 Let $C_1$ be the cost of error when 1 is transmitted, and $C_0$ be the cost of error when 0 is transmitted. Let the optimum threshold be $a_o$ in Fig. S13.2-2. Then:

$$C_1 = C_{10}\,p\left(\epsilon|\mathrm{m}=1\right) = C_{10}\,Q\left(\frac{A_p-a_o}{\sigma_n}\right)$$

$$C_0 = C_{01}\,p\left(\epsilon|\mathrm{m}=0\right) = C_{01}\,Q\left(\frac{A_p+a_o}{\sigma_n}\right)$$

The average cost of an error is

$$C = P_m(1)\, C_1 + P_m(0)\, C_0 \tag{1}$$

If $P_m(1) = P_m(0) = 0.5$

$$C = \frac{1}{2}(C_1 + C_0) = \frac{1}{2}\left[C_{10} Q\left(\frac{A_p - a_o}{\sigma_n}\right) + C_{01} Q\left(\frac{A_p + a_o}{\sigma_n}\right)\right]$$

For optimum threshold $dC/da_0 = 0$. Hence, to compute $dC/da_0$, we observe that

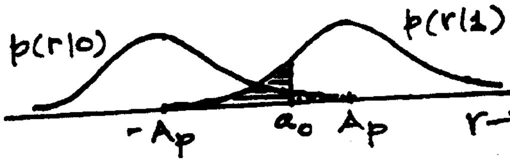



Fig. S13.2-2

$$Q(x) = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy$$

and

$$\frac{dQ}{dx} = -\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

Hence,

$$\frac{dC}{da_o} = \frac{1}{2\sigma_n \sqrt{2\pi}} \left[ C_{10}\, e^{-\frac{(A_p - a_o)^2}{2\sigma_n^2}} - C_{01}\, e^{-\frac{(A_p + a_o)^2}{2\sigma_n^2}} \right] = 0$$

Hence,

$$\frac{C_{01}}{C_{10}} = e^{\left[\frac{(A_p + a_o)^2}{2\sigma_n^2} - \frac{(A_p - a_o)^2}{2\sigma_n^2}\right]}$$

and

$$\ln\left(\frac{C_{01}}{C_{10}}\right) = \frac{2a_o A_p}{\sigma_n^2} \quad \text{and} \quad a_o = \frac{\sigma_n^2}{2A_p} \ln\left[\frac{C_{01}}{C_{10}}\right]$$

But

$$\sigma_n^2 = \frac{\mathcal{N} E_p}{2} \quad \text{and} \quad A_p = E_p.$$

Hence,

$$a_o = \frac{\mathcal{N}}{4} \ln\left[\frac{C_{01}}{C_{10}}\right]$$

**13.2-3** We follow the procedure in the solution of Prob. 13.2-2. The only difference is $P_m(1)$ and $P_m(0)$ are not 0.5. Hence,

$$C = P_m(1)\, C_1 + P_m(0)\, C_0 = P_m(1)\, C_{10}\, Q\left(\frac{A_p - a_o}{\sigma_n}\right) + P_m(0)\, C_{01}\, Q\left(\frac{A_p + a_o}{\sigma_n}\right)$$

and

$$\frac{dC}{da_o} = \frac{1}{2\sigma_n \sqrt{2\pi}} \left[ P_m(1)\, C_{10}\, e^{-\frac{(A_p - a_o)^2}{2\sigma_n^2}} - P_m(0)\, C_{01}\, e^{-\frac{(A_p + a_o)^2}{2\sigma_n^2}} \right] = 0$$

Hence,

$$\ln\left[\frac{P_m(0)\, C_{01}}{P_m(1)\, C_{10}}\right] = \frac{2a_0 A_p}{\sigma_n^2} \Rightarrow a_0 = \frac{\sigma_n^2}{2A_p} \ln\left[\frac{P_m(0)\, C_{01}}{P_m(1)\, C_{10}}\right]$$

But

$$\sigma_n^2 = \frac{\mathcal{N} E_p}{2} \text{ and } A_p = E_p$$

Hence,

$$a_0 = \frac{\mathcal{N}}{4} \ln \left[ \frac{P_m(0)\, C_{01}}{P_m(1)\, C_{10}} \right]$$

**13.5-1**

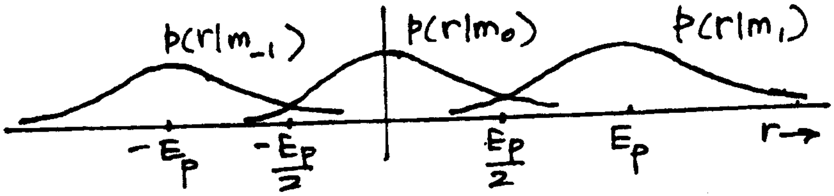



**Fig. S13.5-1**

$$\left.\begin{aligned} p(r|m_{-1}) &= \frac{1}{\sigma_n \sqrt{2\pi}} e^{-(r+E_p)^2/2\sigma_n^2} \\ P(r|m_0) &= \frac{1}{\sigma_n \sqrt{2\pi}} e^{-r^2/2\sigma_n^2} \\ P(r|m_1) &= \frac{1}{\sigma_n \sqrt{2\pi}} e^{-(r-E_p)^2/2\sigma_n^2} \end{aligned}\right\} \quad \sigma_n^2 = \frac{\mathcal{N} E_p}{2}$$

The thresholds are $\pm E_p/2$ and

$$P(\epsilon|m_0) = 2Q\left(\frac{E_p/2}{\sigma_n}\right) = 2Q\left(\sqrt{\frac{E_p}{2\mathcal{N}}}\right)$$

$$P(\epsilon|m_1) = P(\epsilon|m_{-1}) = Q\left(\frac{E_p/2}{\sigma_n}\right) = Q\left(\sqrt{\frac{E_p}{2\mathcal{N}}}\right)$$

$$P_e = \frac{1}{3}\left[2Q\left(\sqrt{\frac{E_p}{2\mathcal{N}}}\right) + Q\left(\sqrt{\frac{E_p}{2\mathcal{N}}}\right) + Q\left(\sqrt{\frac{E_p}{2\mathcal{N}}}\right)\right]$$

$$= \frac{4}{3} Q\left(\sqrt{\frac{E_p}{2\mathcal{N}}}\right)$$

**13.5-2** Here, $p(t)$ and $q(t)$ are identified with $3p(t)$ and $p(t)$, respectively. Hence,

$$H(\omega) = [3P(-\omega) - P(-\omega)]e^{-j\omega T_b} = 2P(-\omega)e^{-j\omega T_b}$$

and

$$h(t) = 2p(T_b - t)$$

$$a_o = \frac{1}{2}[E_{3p} - E_p] = \frac{1}{2}[9E_p - E_p] = 4E_p$$

But multiplication of $h(t)$ by a constant does not affect the performance. Hence we shall choose $h(t)$ to be $p(T_b - t)$ rather than $2p(T_b - t)$. This will also halve the threshold to $a_o = 2E_p$. This is shown in Fig. S13.5-2. Also,

$$E_{pq} = \int_0^{T_b} [3p(t)]p(t)dt = 3E_p$$

and

$$P_e = Q\left(\sqrt{\frac{E_{3p}+E_p-2E_{pq}}{\mathcal{N}}}\right) = Q\left(\sqrt{\frac{9E_p+E_p-6E_p}{\mathcal{N}}}\right) = Q\left(\sqrt{\frac{4E_p}{\mathcal{N}}}\right)$$

The energy/bit is $E_b = \frac{9E_p+E_p}{2} = 5Ep$ Hence,

$$P_e = Q\left(\sqrt{\frac{0.8E_b}{\mathcal{N}}}\right)$$

13.5-3 For $M = 2$, $\mathcal{N} = 2\times10^{-8}$
For 256,000 bps the baseband transmission requires a minimum bandwidth 128 kHz. But amplitude modulation doubles the bandwidth.
Hence

$$B_T = 256 \text{ kHz}$$

$$10^{-7} = Q\left(\sqrt{\frac{2E_b}{\mathcal{N}}}\right) \Rightarrow E_b = 2.7\times10^{-7}$$

$$S_i = E_bR_b = 2.7\times10^{-7}\times256{,}000 = 0.069W$$

For $M = 16$

$$B_T = \frac{256{,}000}{\log_2 16} = 64 \text{ kHz}$$

$$P_{eM} = P_b \log_2 16 = 4\times10^{-7} = \frac{2(15)}{16}Q\left(\sqrt{\frac{24E_b}{255\mathcal{N}}}\right)$$

This yields $E_b = 5.43\times10^{-6}$

$$S_i = E_bR_b = 5.43\times10^{-6}\times256{,}000 = 1.39W$$

For $M = 32$

$$B_T = \frac{256{,}000}{\log_2 32} = 51.2 \text{ kHz}$$

$$P_{eM} = P_b \log_2 32 = 5\times10^{-7} = \frac{2(31)}{32}Q\left(\sqrt{\frac{30E_b}{1023\mathcal{N}}}\right)$$

This yields $E_b = 1.719\times10^{-5}$

$$S_i = E_bR_b = 1.719\times10^{-5}\times256{,}000 = 4.4W$$

13.5-4 For $M = 2$ and $\mathcal{N} = 2\times10^{-8}$

This case is identical to MASK for $M = 2$

$$10^{-7} = Q\left(\sqrt{\frac{2E_b}{\mathcal{N}}}\right) \Rightarrow E_b = 2.7\times10^{-7}$$

$$S_i = E_bR_b = 2.7\times10^{-7}\times256{,}000 = 0.069W$$

$$B_T = \frac{256{,}000}{2}\times2 = 256 \text{ kHz}$$

For $M = 16$

$$P_{eM} = (\log_2 16)P_b = 4P_b = 4 \times 10^{-7}$$

$$4 \times 10^{-7} \cong 2Q\left(\sqrt{\frac{2\pi^2 \times 4E_b}{256\mathcal{N}}}\right) \Rightarrow E_b = 1.67 \times 10^{-6}$$

$$S_i = E_b R_b = 1.67 \times 10^{-6} \times 256{,}000 = 0.4275W$$

In MPSK, the minimum bandwidth is equal to the number of M-ary pulses/second. Hence,

$$B_T = \frac{256{,}000}{\log_2 16} = 64 \text{ kHz}$$

For $M = 32$

$$P_{eM} = (\log_2 32)P_b = 5 \times 10^{-7}$$

$$5 \times 10^{-7} \cong 2Q\left(\sqrt{\frac{2\pi^2(5E_b)}{1024\mathcal{N}}}\right) \Rightarrow E_b = 5.24 \times 10^{-6}$$

$$S_i = E_b R_b = 5.24 \times 10^{-6} \times 256{,}000 = 1.34W$$

$$B_T = \frac{256{,}000}{\log_2 32} = 51.2 \text{ kHz}$$

# Chapter 14

**14.1-1** The following signals represent 2 sets of 5 mutually orthogonal signals.

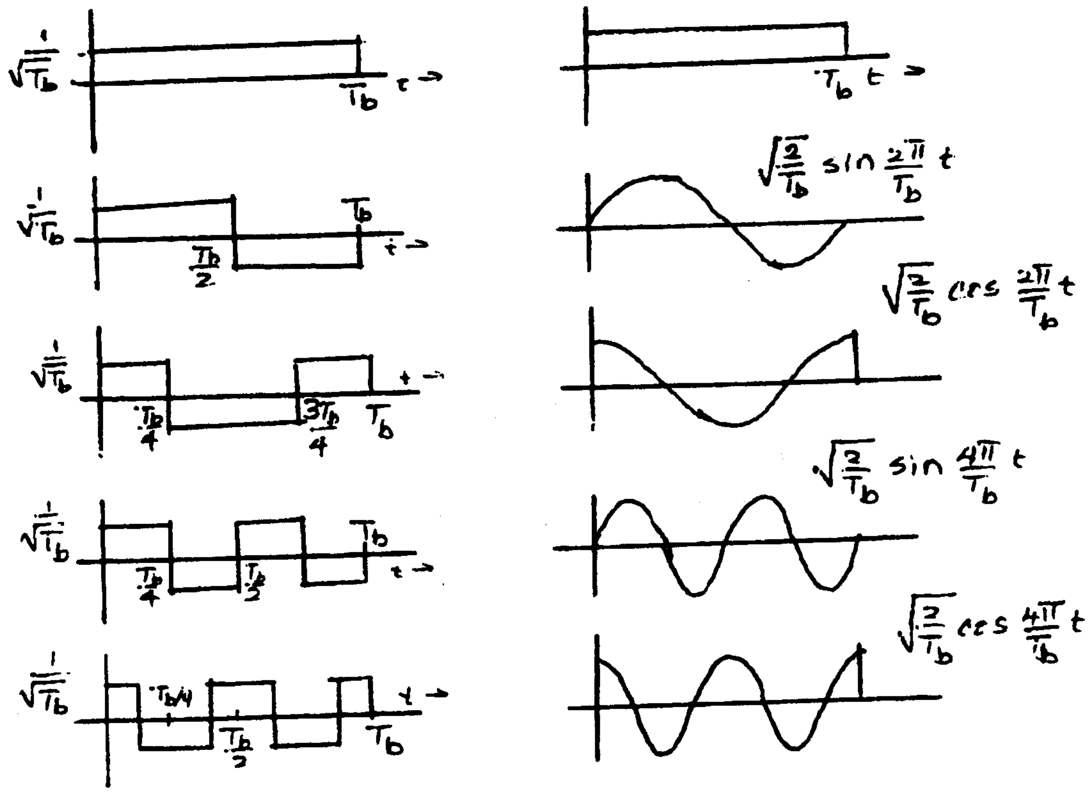



**Fig. S14.1-1**

**14.1-2**

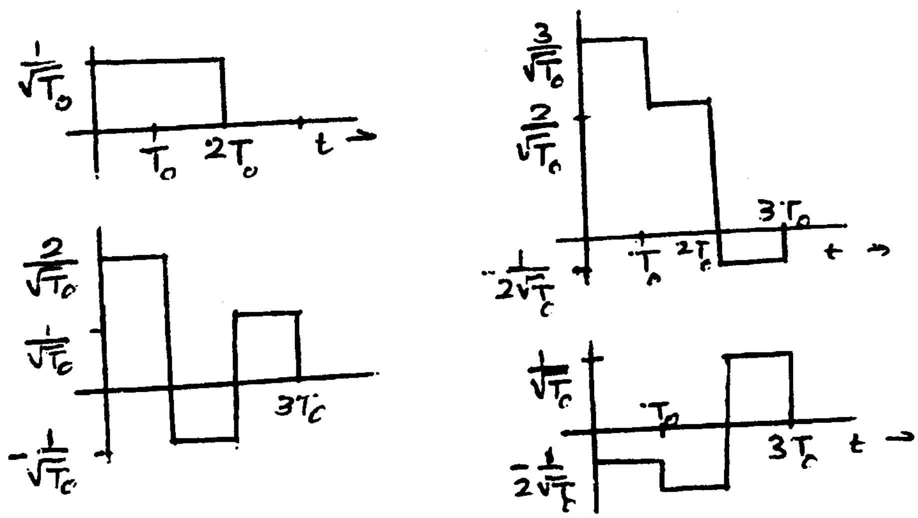
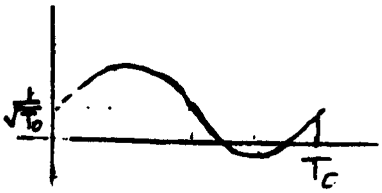
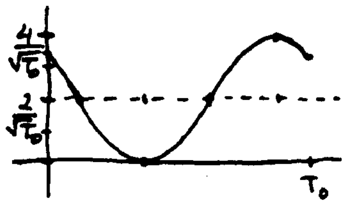



**Fig.S14.1-2**

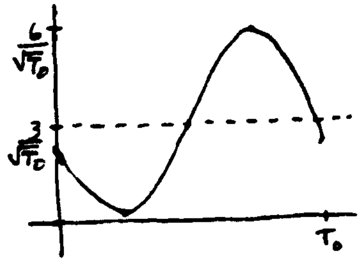
**14.1-3** 1) $(1, 1, 0)$ is $\frac{1}{\sqrt{T_0}}\left[1+\sqrt{2}\sin\omega_0 t\right] \quad \omega_0 = \frac{2\pi}{T_0}$




2) $(2, 1, 1)$ is $\frac{1}{\sqrt{T_0}}\left[2-\sqrt{2}\sin\omega_0 t+\sqrt{2}\cos\omega_0 t\right] = \frac{1}{\sqrt{T_0}}\left[2+2\cos\left(\omega_0 t+\frac{\pi}{4}\right)\right]$




3) $(3, 2, -\frac{1}{2})$ is $\frac{1}{\sqrt{T_0}}\left[3+2\sqrt{2}\sin\omega_0 t-\frac{1}{\sqrt{2}}\cos\omega_0 t\right] = \frac{1}{\sqrt{T_0}}\left[3+2.91\cos\left(\omega_0 t-104^\circ\right)\right]$




4) $(-\frac{1}{2}, -1, 1)$ is $\frac{1}{\sqrt{T_0}}\left[-\frac{1}{2}-\sqrt{2}\sin\omega_0 t+\sqrt{2}\cos\omega_0 t\right] = \frac{1}{\sqrt{T_0}}\left[-\frac{1}{2}+2\cos\left(\omega_0 t+\frac{\pi}{4}\right)\right]$

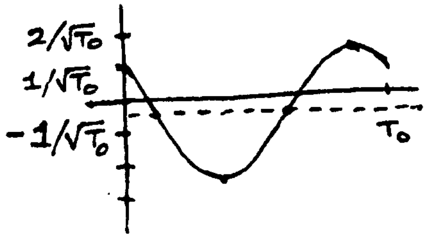



**Fig. S14.1-3**

**14.1-4**

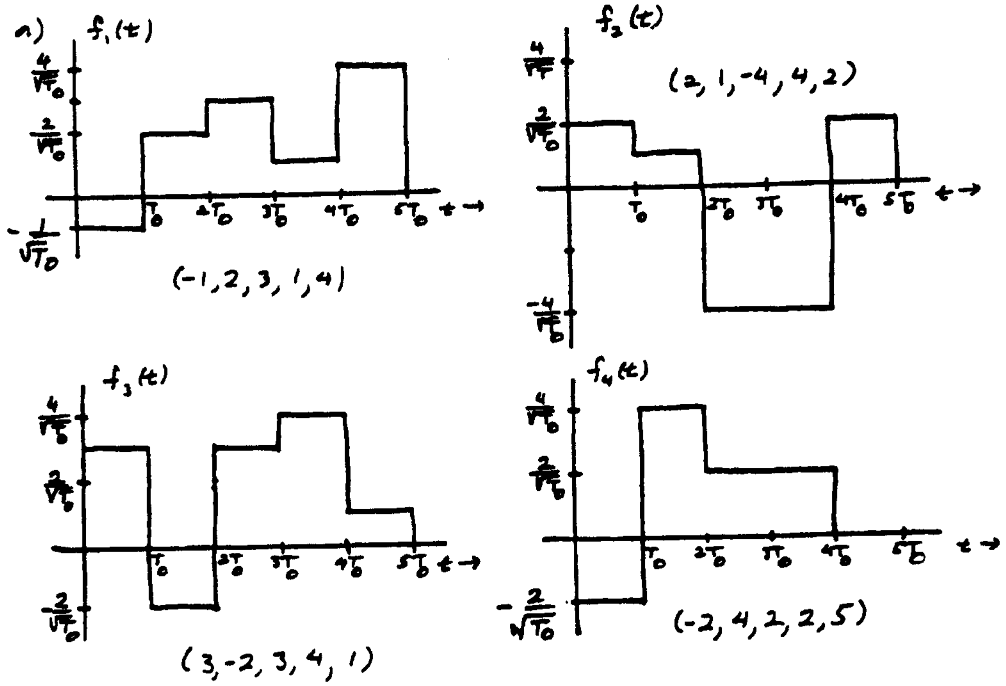



**Fig. S14.1-4**

b) The energy of each signal is:

$$E_1 = \frac{1+4+9+1+16}{T_0} T_0 = 31 \qquad E_2 = \frac{4+1+16+16+4}{T_0} T_0 = 41$$

$$E_3 = \frac{9+4+9+16+1}{T_0} T_0 = 39 \qquad E_4 = \frac{4+16+4+4+0}{T_0} T_0 = 28$$

c) $F_3 \cdot F_4 = (-6-8+6+8+0) = 0$. Hence, $f_3(t)$ and $f_4(t)$ are orthogonal.

**14.2-1** Let $\mathrm{x}(t) = \mathrm{x}_1, \quad \mathrm{x}(t+1) = \mathrm{x}_2 \quad \mathrm{x}(t+2) = \mathrm{x}_3$

We wish to determine

$$p_{\mathrm{x}_1\mathrm{x}_2\mathrm{x}_3}(x_1, x_2, x_3)$$

Since the process $\mathrm{x}(t)$ is Gaussian, $\mathrm{x}_1$, $\mathrm{x}_2$, $\mathrm{x}_3$ are jointly Gaussian with identical variance $\left(\sigma_{\mathrm{x}_1}^2 = \sigma_{\mathrm{x}_2}^2 = \sigma_{\mathrm{x}_3}^2 = R_{\mathrm{x}}(0) = 1\right)$. The covariance matrix is:

$$K = \begin{bmatrix} \sigma_{\mathrm{x}_1}^2 & \sigma_{\mathrm{x}_1\mathrm{x}_2} & \sigma_{\mathrm{x}_1\mathrm{x}_3} \\ \sigma_{\mathrm{x}_2\mathrm{x}_1} & \sigma_{\mathrm{x}_2}^2 & \sigma_{\mathrm{x}_2\mathrm{x}_3} \\ \sigma_{\mathrm{x}_3\mathrm{x}_1} & \sigma_{\mathrm{x}_3\mathrm{x}_2} & \sigma_{\mathrm{x}_3}^2 \end{bmatrix}$$

Also

$$\sigma_{\mathrm{x}_1\mathrm{x}_2} = \sigma_{\mathrm{x}_2\mathrm{x}_1} = \overline{\mathrm{x}_1\mathrm{x}_2} = \overline{\mathrm{x}(t)\,\mathrm{x}(t+1)} = R_{\mathrm{x}}(1) = \frac{1}{e}$$

$$\sigma_{\mathrm{x}_2\mathrm{x}_3} = \sigma_{\mathrm{x}_3\mathrm{x}_2} = \overline{\mathrm{x}_2\mathrm{x}_3} = \overline{\mathrm{x}(t+1)\,\mathrm{x}(t+2)} = R_{\mathrm{x}}(1) = \frac{1}{e}$$

$$\sigma_{\mathrm{x}_1\mathrm{x}_3} = \sigma_{\mathrm{x}_3\mathrm{x}_1} = \overline{\mathrm{x}_1\mathrm{x}_3} = \overline{\mathrm{x}(t)\,\mathrm{x}(t+2)} = R_{\mathrm{x}}(2) = \frac{1}{e^2}$$

so

$$K=\begin{bmatrix}1 & \frac{1}{e} & \frac{1}{e^2}\\ \frac{1}{e} & 1 & \frac{1}{e}\\ \frac{1}{e^2} & \frac{1}{e} & 1\end{bmatrix} \quad \text{and} \quad |K|=\left(1-\frac{1}{e^2}\right)^2$$

$$\Delta_{11}=\Delta_{33}=1-\frac{1}{e^2} \text{ and } \Delta_{12}=\Delta_{21}=\Delta_{23}=\Delta_{32}=\frac{1}{e^3}-\frac{1}{e}$$

$$\Delta_{13}=\Delta_{31}=0, \quad \Delta_{22}=1-\frac{1}{e^4}$$

And

$$p_{x_1x_2x_3}(x_1x_2x_3)=\frac{1}{(2\pi)^{3/2}\sqrt{|K|}}e^{-\sum\limits_i\sum\limits_j \Delta_{ij}x_ix_j}$$

**14.3-1**

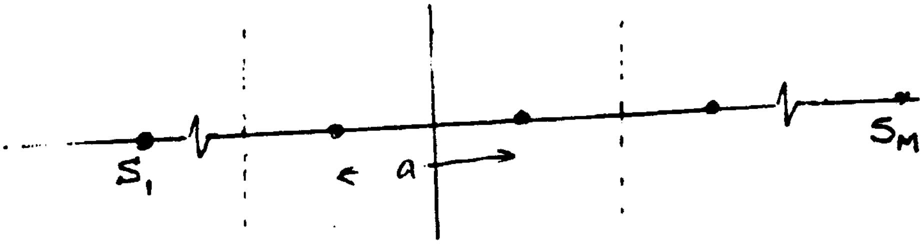



**Fig. S14.3-1**

$$P(C|m_1)=\text{Prob}\left(n_1<\frac{a}{2}\right) \text{ and } P(C|m_M)=\text{Prob}\left(n_1>\frac{-a}{2}\right)$$

$$P(C|m_2)=P(C|m_3)=\ldots\ldots=P(C|m_{M-1})=\text{Prob}\left(|n_1|<\frac{a}{2}\right)$$

Hence

$$P(C|m_1)=P(C|m_M)=\frac{1}{\sqrt{\pi\mathcal{N}}}\int_{-\infty}^{a/2}e^{-n_1^2/\mathcal{N}}\,dn_1=1-Q\left(\frac{a}{\sqrt{2\mathcal{N}}}\right)$$

and

$$P(C|m_2)=P(C|m_3)=\ldots\ldots P(C|m_{M-1})=\frac{1}{\sqrt{\pi\mathcal{N}}}\int_{-a/2}^{a/2}e^{-n_1^2/\mathcal{N}}\,dn_1=1-2Q\left(\frac{a}{\sqrt{2\mathcal{N}}}\right)$$

Hence

$$\begin{aligned}P_{eM}&=1-P(C)\\ &=1-\left[P(m_1)P(C|m_1)+P(m_2)P(C|m_2)+\ldots\ldots+P(m_M)P(C|m_M)\right]\\ &=1-\frac{1}{M}\left[M+2(M-1)Q\left(\frac{a}{\sqrt{2\mathcal{N}}}\right)\right]=\frac{2(M-1)}{M}Q\left(\frac{a}{\sqrt{2\mathcal{N}}}\right)\end{aligned}$$

The signal energies are $\left(\pm\frac{a}{2}\right)^2, \left(\pm\frac{3a}{2}\right)^2, \ldots\ldots, \left(\pm\frac{M-1}{2}a\right)^2$

Hence the average pulse energy $\overline{E}$ is

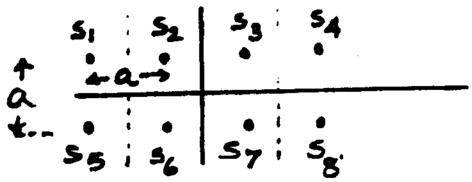
$$\overline{E} = \frac{2}{M}\left[\frac{a^2}{4} + \frac{9a^2}{4} + \frac{25a^2}{4} + \ldots\ldots + \frac{(M-1)^2}{4}a^2\right]$$

$$= \frac{a^2}{2M}\sum_{k=0}^{\frac{M-2}{2}}(2k+1)^2 = \frac{(M^2-1)a^2}{12}$$

Also
$$E_b = \frac{\overline{E}}{\log_2 M} = \frac{(M^2-1)a^2}{12\log_2 M}$$

Hence
$$P_{eM} = \frac{2(M-1)}{M} Q\left(\sqrt{\frac{6(\log_2 M)}{(M^2-1)}\frac{E_b}{\mathcal{N}}}\right)$$

Which agrees with the result in Eq. (13.52c)




14.3-2 $P(C|m_1) = P(C|m_4) = P(C|m_5) = P(C|m_8)$

$P(C|m_2) = P(C|m_3) = P(C|m_6) = P(C|m_7)$

$$P(C|m_1) = P\left(n_1 < \frac{a}{2},\ n_2 > \frac{-a}{2}\right)$$

$$= \left[1 - Q\left(\frac{a}{\sqrt{2\mathcal{N}}}\right)\right]\left[1 - Q\left(\frac{a}{\sqrt{2\mathcal{N}}}\right)\right] = \left[1 - Q\left(\frac{a}{\sqrt{2\mathcal{N}}}\right)\right]^2$$

$$P(C|m_2) = P\left(|n_1| < \frac{a}{2},\ n_2 > \frac{-a}{2}\right)$$

$$= \left[1 - 2Q\left(\frac{a}{\sqrt{2\mathcal{N}}}\right)\right]\left[1 - Q\left(\frac{a}{\sqrt{2\mathcal{N}}}\right)\right]$$

and

$$P(C) = \frac{1}{2}\left[P(C|m_1) + P(C|m_2)\right] = \frac{1}{2}\left[1 - Q\left(\frac{a}{\sqrt{2\mathcal{N}}}\right)\right]\left[2 - 3Q\left(\frac{a}{\sqrt{2\mathcal{N}}}\right)\right]$$

$$P_{eM} = 1 - P(C) = \frac{1}{2}Q\left(\frac{a}{\sqrt{2\mathcal{N}}}\right)\left[5 - 3Q\left(\frac{a}{\sqrt{2\mathcal{N}}}\right)\right]$$

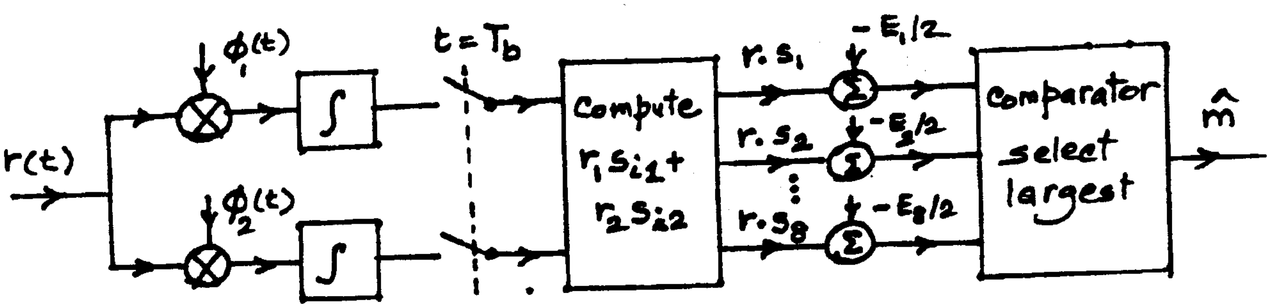



Fig. S14.3-2

The average pulse energy $\overline{E}$ is

$$\overline{E} = \frac{1}{8}\left(4\left[\left(\frac{a}{2}\right)^2 + \left(\frac{a}{2}\right)^2\right] + 4\left[\left(\frac{a}{2}\right)^2 + \left(\frac{3a}{2}\right)^2\right]\right) = \frac{3a^2}{2}$$

$$E_b = \frac{\overline{E}}{\log_2 8} = \frac{a^2}{2}$$

and

$$P_{eM} = \frac{1}{2}Q\left(\sqrt{\frac{E_b}{\mathcal{N}}}\right)\left[5 - 3Q\left(\sqrt{\frac{E_b}{\mathcal{N}}}\right)\right]$$

$$\cong 2.5Q\left(\sqrt{\frac{E_b}{\mathcal{N}}}\right) \quad \text{assuming } Q\left(\sqrt{\frac{E_b}{\mathcal{N}}}\right) << 1$$

This performance is considerably better than MASK in Prob. 14.3-1, which yields

$$P_{eM} = 1.75Q\left(\sqrt{\frac{0.286E_b}{\mathcal{N}}}\right) \text{for } M = 8$$

14.3-3 In this case, constants $a_k$'s are same for $k = 1, 2, \ldots M$. Hence, the optimum receiver is the same as that in Fig. 14.8 with terms $a_k$'s omitted.
We now compare $r \cdot s_1, r \cdot s_2, \ldots r \cdot s_M$.
Since $r \cdot s_k = \sqrt{E}r\cos\theta_k$ is the angle between $r$ and $s_k$, it is clear that we are to pick that signal $s_k$ with which $r$ has the smallest angle. In short, the detector is a phase comparator. It chooses that signal which is at the smallest angle with $r$.

14.3-4 Because of symmetry,

$$P(C|m_1) = P(C|m_2) = \ldots = P(C|m_M)$$

where $M = 2^N$

$$S_{ij} = \frac{d}{2} \text{ or } \frac{-d}{2}$$

and $\quad E_1 = E_2 = \ldots = E_M = \dfrac{Nd^2}{4} = E$

Let

$$S_1 = \left(\frac{-d}{2}, \frac{-d}{2}, \ldots, \frac{-d}{2}\right)$$

Then

$$P(C|m_1) = P\left(n_1 < \frac{d}{2}, n_2 < \frac{d}{2}, \ldots, n_N < \frac{d}{2}\right)$$

$$= \left[1 - Q\left(\frac{d}{\sqrt{2\mathcal{N}}}\right)\right]^N$$

$$= \left[1 - Q\left(\sqrt{\frac{2E}{\mathcal{N}N}}\right)\right]^N$$

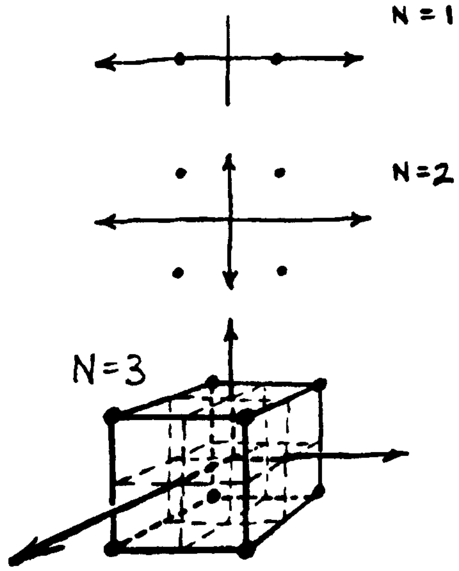



Fig. S14.3-4

and

$$P(C) = P(C|m_1)$$

$$P_{eM} = 1 - P(C) = 1 - \left[1 - 2Q\left(\sqrt{\frac{2E}{\mathcal{N}N}}\right)\right]^N$$

Here, $M = 2^N$. Hence, each symbol carries the information $\log_2 M = N$ bits.

Hence

$$E_b = E/\mathcal{N}$$

and

$$P_{eM} = 1 - \left[1 - 2Q\left(\sqrt{\frac{2E_b}{\mathcal{N}}}\right)\right]^N$$

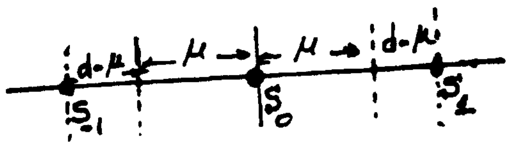



Fig. S14.3-5

14.3-5

$$\mu = \frac{\mathcal{N}}{2d}\ln\frac{P(m_0)}{P(m_1)} + \frac{d}{2}$$

$$= \frac{\mathcal{N}}{2d}\ln 2 + \frac{d}{2}$$

$$P(C|m_1) = P(C|m_{-1}) = \text{Prob.}[n_1 > -(d-\mu)] = 1 - Q\left(\frac{d-\mu}{\sqrt{\mathcal{N}/2}}\right)$$

$$P(C|m_0) = \text{Prob}\ (|n_1| < \mu) = 1 - 2Q\left(\frac{\mu}{\sqrt{\mathcal{N}/2}}\right)$$

$$P(C) = \frac{1}{2}P(C|m_0) + \frac{1}{4}P(C|m_1) + \frac{1}{4}P(C|m_{-1})$$

$$= \frac{1}{2} - \left[2 - Q\left(\frac{d-\mu}{\sqrt{\mathcal{N}/2}}\right)\right] - 2Q\left(\frac{\mu}{\sqrt{\mathcal{N}/2}}\right)$$

$$= 1 - \frac{1}{2}Q\left(\frac{d^2 - \mathcal{N}\ln 2}{2d\sqrt{\mathcal{N}/2}}\right) - Q\left(\frac{d^2 + \mathcal{N}\ln 2}{2d\sqrt{\mathcal{N}/2}}\right)$$

Also

$$\overline{E} = 0.5(0) + 0.25d^2 + 0.25d^2 = \frac{d^2}{2}$$

Hence

$$P_{eM} = 1 - P(C) = \frac{1}{2}Q\left(\frac{\frac{2\overline{E}}{\mathcal{N}} - \ln 2}{2\sqrt{\overline{E}/\mathcal{N}}}\right) + Q\left(\frac{\frac{2\overline{E}}{\mathcal{N}} + \ln 2}{2\sqrt{\overline{E}/\mathcal{N}}}\right)$$

14.3-6

$$P_{eM} = 1 - P(C)$$

$$P(C) = \frac{1}{2}\left[P(C|m_1) + P(C|m_2)\right]$$

$$P(C|m_1) = \frac{1}{\pi\mathcal{N}}\iint_{R_1} e^{-\left[(q_1-d)^2 + q_2^2\right]/\mathcal{N}}\, dq_1 dq_2$$

$$= \frac{1}{\pi\mathcal{N}}\int_0^{\frac{3d}{2}}\int_{-q_1\tan(\pi/8)}^{q_1\tan(\pi/8)} e^{-\left[(q_1-d)^2 + q_2^2/\mathcal{N}\right]}\, dq_2 dq_1$$

$$P(C|m_2) = \frac{1}{\pi\mathcal{N}} \iint_{R_2} e^{-\left[(q_1-2d)^2+q_2^2\right]/\mathcal{N}} dq_1 dq_2$$

$$= \frac{1}{\pi\mathcal{N}} \int_{\frac{3d}{2}}^{\infty} \int_{-q_1 \tan(\pi/8)}^{q_1 \tan(\pi/8)} e^{-\left[(q_1-2d)^2+q_2^2\right]/\mathcal{N}} dq_2 dq_1$$

$$\overline{E} = \frac{d^2+4d^2}{2} = \frac{5}{2}d^2$$

and

$$d = \sqrt{0.4\overline{E}} = \sqrt{0.1E_b}$$

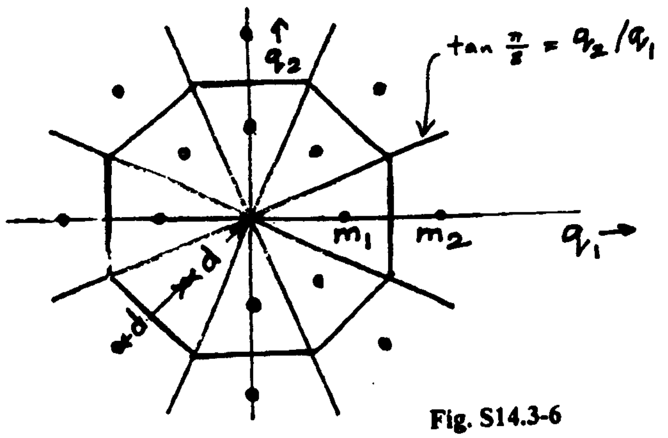
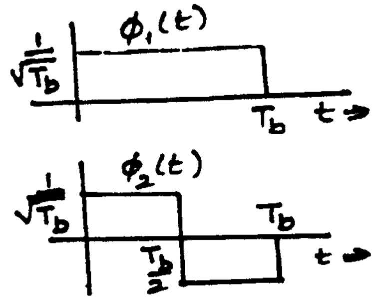



**Fig. S14.3-6**

14.3-7



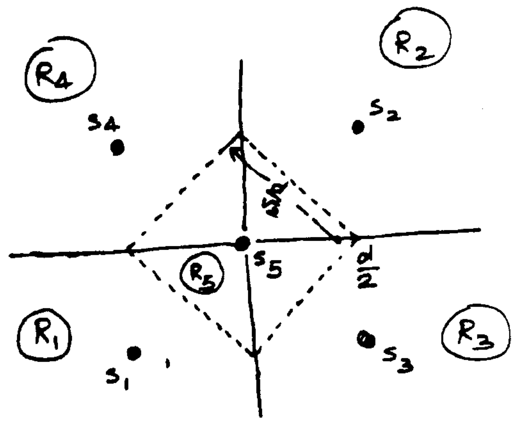




**Fig. S14.3-7a**

Note that

$$s_1 = -\frac{d}{2}\phi_1 - \frac{d}{2}\phi_2, \quad s_2 = \frac{d}{2}\phi_1 + \frac{d}{2}\phi_2$$

$$s_3 = \frac{d}{2}\phi_1 - \frac{d}{2}\phi_2, \quad s_4 = -\frac{d}{2}\phi_1 + \frac{d}{2}\phi_2$$

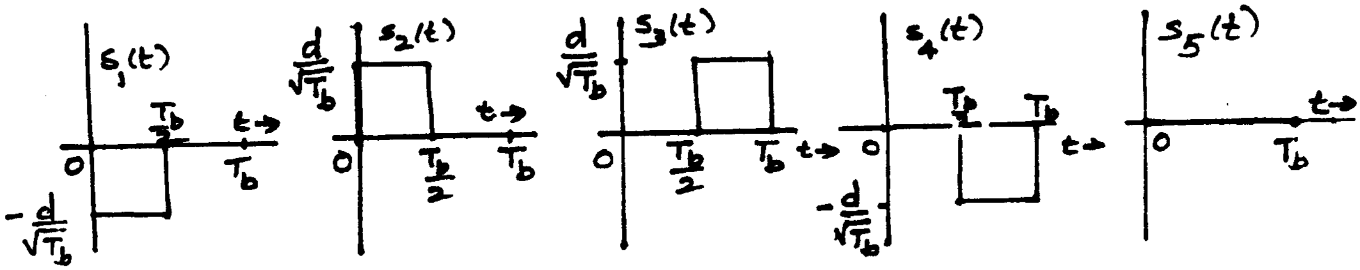



**Fig. S14.3-7b**

(c) $$P(C|m_5) = \text{Prob}\left(\text{noise originating from } s_5 \text{ remains within the square of side } \frac{d}{\sqrt{2}}\right)$$

$$= P\left(|n_1| < \frac{d}{2\sqrt{2}}, \quad |n_2| < \frac{d}{2\sqrt{2}}\right)$$

$$= \left[1 - 2Q\left(\frac{d}{2\sqrt{2}\sigma_n}\right)\right]^2 \cong 1 - 4Q\left(\frac{d}{2\sqrt{2}\sigma_n}\right), \quad \text{assuming } Q\left(\frac{d}{2\sqrt{2}\sigma_n}\right) << 1$$

and

$$P(\epsilon|m_5) \cong 4Q\left(\frac{d}{2\sqrt{2}\sigma_n}\right)$$

We also observe that $\overline{E}$, the average energy is $\overline{E} = \frac{1}{5}\left(\frac{4d^2}{2}\right) = 0.4d^2$

$$\frac{\overline{E}}{\mathcal{N}} = \frac{0.4d^2}{\mathcal{N}} = \frac{0.2d^2}{\sigma_n^2} \text{ and } \frac{d}{2\sigma_n} = \sqrt{\frac{5\overline{E}}{4\mathcal{N}}} \text{ and } \frac{d}{2\sqrt{2}\sigma_n} = \sqrt{\frac{5\overline{E}}{8\mathcal{N}}}$$

Therefore $$P(\epsilon|m_5) \cong 4Q\left(\sqrt{\frac{5\overline{E}}{8\mathcal{N}}}\right)$$

The decision region $R_2$ for $m_2$ is shown in Fig. a and again in Fig. C-1. $R_2$ can be expressed as the first quadrant (horizontally hatched area in Fig. C-1) $-A_1$. Thus

$$P(C|m_2) = \text{noise originating from } s_2 \text{ lie in } R_2$$

$$= P(\text{noise lie in 1st quadrant}) - P(\text{noise lie in } A_1)$$

$$= \left[1 - Q\left(\frac{d}{2\sigma_n}\right)\right]^2 - P(\text{noise originating from } s_2 \text{ lie in } A_1)$$

But $P(\text{noise lie in } A_1) = \frac{1}{4}[P(\text{noise lie within outer square}) - P(\text{noise lie within inner square})]$ (See Fig. C-2)

$$= \frac{1}{4}\left[P\left(|n_1|, |n_2| < \frac{d}{2}\right) - P\left(|n_1|, |n_2| < \frac{d}{2\sqrt{2}}\right)\right]$$

$$= \frac{1}{4}\left\{\left[1 - 2Q\left(\frac{d}{2\sigma_n}\right)\right]^2 - \left[1 - 2Q\left(\frac{d}{2\sqrt{2}\sigma_n}\right)\right]^2\right\}$$

$$\cong \frac{1}{4}\left[-4Q\left(\frac{d}{2\sigma_n}\right)+4Q\left(\frac{d}{2\sqrt{2}\sigma_n}\right)\right]$$

and

$$P(C|m_2)=1-2Q\left(\frac{d}{2\sigma_n}\right)+Q\left(\frac{d}{2\sigma_n}\right)-Q\left(\frac{d}{2\sqrt{2}\sigma_n}\right)$$

$$=1-Q\left(\frac{d}{2\sigma_n}\right)-Q\left(\frac{d}{2\sqrt{2}\sigma_n}\right)$$

$$=1-Q\left(\sqrt{\frac{5\bar{E}}{4\mathcal{N}}}\right)-Q\left(\sqrt{\frac{5\bar{E}}{8\mathcal{N}}}\right)$$

and

$$P(\varepsilon|m_2)=Q\left(\sqrt{\frac{5\bar{E}}{4\mathcal{N}}}\right)+Q\left(\sqrt{\frac{5\bar{E}}{8\mathcal{N}}}\right)$$

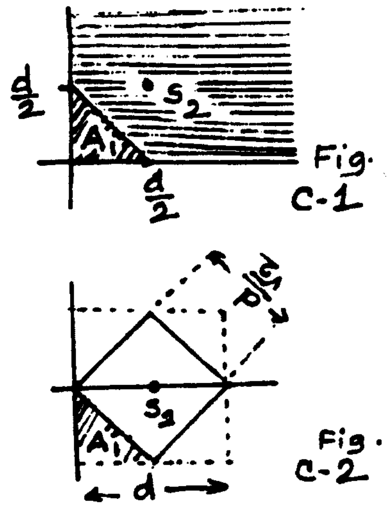



Fig. S14.3-7c

Moreover, by symmetry

$$P(\varepsilon|m_2)=P(\varepsilon|m_1)=P(\varepsilon|m_3)=P(\varepsilon|m_4)$$

Hence

$$P_{eM}=\frac{1}{5}\left[\sum_{i=1}^{5}P(\varepsilon|m_i)\right]$$

$$=0.8\left[2Q\left(\sqrt{\frac{5\bar{E}}{8\mathcal{N}}}\right)+Q\left(\sqrt{\frac{5\bar{E}}{4\mathcal{N}}}\right)\right]$$

**14.3-8**

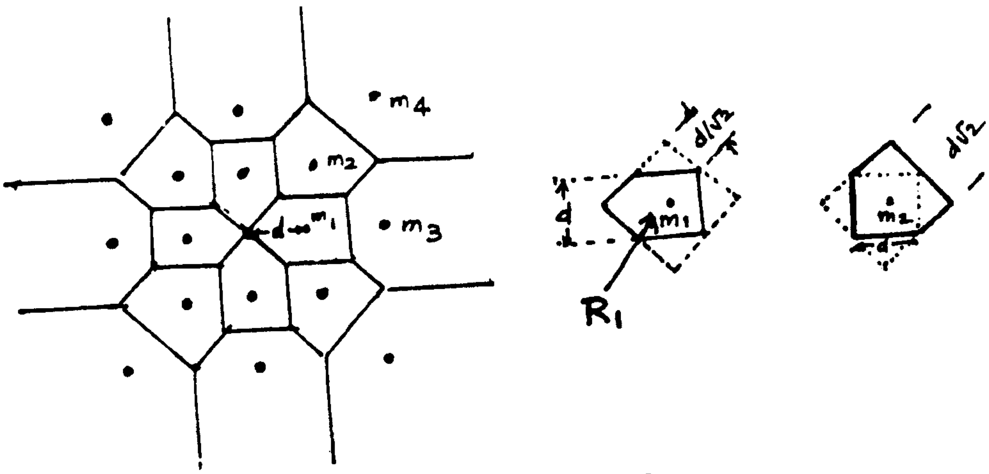



Fig. S14.3-8

$$P(C)=\frac{1}{16}\sum_{i=1}^{4}4P(C|m_i)=\frac{1}{4}\left[P(C|m_1)+P(C|m_2)+P(C|m_3)+P(C|m_4)\right]$$

The decision region $R_1$ for $m_1$ (see Figure) can be expressed as

$R_1$ = outer square of side $d\sqrt{2}-\frac{3}{4}$ (outer square $-$ inner square of side $d$)

$=\frac{1}{4}$ outer square of side $d\sqrt{2}+\frac{3}{4}$ inner square of side $d$

Now $P(C|m_1) = \text{Prob}(\text{noise originating from } m_1 \text{ lies in } R_1)$

$$= \frac{1}{4}P(\text{n lie in outer square}) + \frac{3}{4}P(\text{n lie in inner square})$$

$$= \frac{1}{4}P\left(|\text{n}| < \frac{d}{\sqrt{2}}\right) + \frac{3}{4}P\left(|\text{n}| < \frac{d}{2}\right)$$

$$= \frac{1}{4}\left[1 - 2Q\left(\frac{d}{\sqrt{\mathcal{N}}}\right)\right]^2 + \frac{3}{4}\left[1 - 2Q\left(\frac{d}{\sqrt{2\mathcal{N}}}\right)\right]^2$$

Similarly $R_2$, the decision region for $m_2$ (see figure above) can be expressed as

$$R_2 = \text{outer square of side } d\sqrt{2} - \frac{1}{2}(\text{outer square} - \text{inner square of side } d)$$

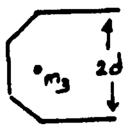
$$= \frac{1}{2}\text{outer square of side } d\sqrt{2} - \frac{1}{2}\text{inner square of side } d$$

and

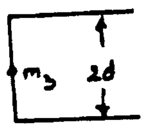
$$P(C|m_2) = \text{noise originating from } m_2 \text{ lie in } R_2$$

$$= \frac{1}{2}P\left(|\text{n}| < \frac{d}{\sqrt{2}}\right) + \frac{1}{2}P\left(|\text{n}| < \frac{d}{2}\right)$$

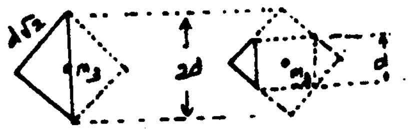
$$= \frac{1}{2}\left[1 - 2Q\left(\frac{d}{\sqrt{\mathcal{N}}}\right)\right]^2 + \frac{1}{2}\left[1 - 2Q\left(\frac{d}{\sqrt{2\mathcal{N}}}\right)\right]^2$$




$R_3$




$R_A$




$R_B$ $R_a$

The decision region $R_3$ for $m_3$ can be expressed as

$$R_3 = R_A + R_B - R_C$$

and

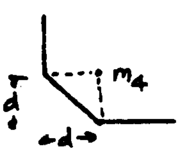
$$P(C|m_3) = \text{Prob}(\text{noise originating from } m_3 \text{ lie in } R_3)$$

$$= P(\text{noise in } R_A) + P(\text{noise in } R_B) - P(\text{noise in } R_C)$$

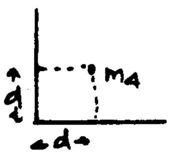
$$= P(\text{n}_1 > 0, |\text{n}_2| < d) + \frac{1}{2}P\left(|\text{n}_1|, |\text{n}_2| < \frac{d}{\sqrt{2}}\right) - \frac{1}{4}\left[P\left(|\text{n}_1|, |\text{n}_2| < \frac{d}{\sqrt{2}}\right) - P\left(|\text{n}_1|, |\text{n}_2| < \frac{d}{2}\right)\right]$$

$$= \frac{1}{2}\left[1 - 2Q\left(\sqrt{\frac{2}{\mathcal{N}}}d\right)\right] + \frac{1}{2}\left[1 - 2Q\left(\frac{d}{\sqrt{\mathcal{N}}}\right)\right]^2 - \frac{1}{4}\left\{\left[1 - 2Q\left(\frac{d}{\sqrt{\mathcal{N}}}\right)\right]^2 - \left[1 - 2Q\left(\frac{d}{\sqrt{2\mathcal{N}}}\right)\right]^2\right\}$$

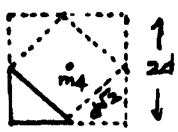
$$= \frac{1}{2}\left[1 - 2Q\left(\sqrt{\frac{2}{\mathcal{N}}}d\right)\right] + \frac{1}{4}\left[1 - 2Q\left(\frac{d}{\sqrt{\mathcal{N}}}\right)\right]^2 + \frac{1}{4}\left[1 - 2Q\left(\frac{d}{\sqrt{2\mathcal{N}}}\right)\right]^2$$




$R_4$




$R_A$




$R_B$

The decision region $R_4$ for $m_4$ can be expressed as

$$R_4 = R_A - R_B$$

and

$$P(C|m_4) = P(n_1 > -d,\ n_2 > -d) - \frac{1}{4}\left\{P(|n_1|, |n_2| < d) - P\left(|n_1|, |n_2| < \frac{d}{\sqrt{2}}\right)\right\}$$

$$= \left[1 - Q\left(d\sqrt{\frac{2}{\mathcal{N}}}\right)\right]^2 - \frac{1}{4}\left[1 - 2Q\left(d\sqrt{\frac{2}{\mathcal{N}}}\right)\right]^2 + \frac{1}{4}\left[1 - 2Q\left(\frac{d}{\sqrt{\mathcal{N}}}\right)\right]^2$$

For any practical scheme $Q(\cdot) << 1$, and we can express

$$[1 - kQ(\cdot)]^2 \cong 1 - 2kQ(\cdot)$$

Using this approximation, we have

$$P(C|m_1) \cong 1 - Q\left(\frac{d}{\sqrt{\mathcal{N}}}\right) - 3Q\left(\frac{d}{\sqrt{2\mathcal{N}}}\right)$$

$$P(C|m_2) \cong 1 - 2Q\left(\frac{d}{\sqrt{\mathcal{N}}}\right) - 2Q\left(\frac{d}{\sqrt{2\mathcal{N}}}\right)$$

$$P(C|m_3) \cong 1 - 2Q\left(d\sqrt{\frac{2}{\mathcal{N}}}\right) - Q\left(\frac{d}{\sqrt{\mathcal{N}}}\right) - Q\left(\frac{d}{\sqrt{2\mathcal{N}}}\right)$$

$$P(C|m_4) \cong 1 - Q\left(d\sqrt{\frac{2}{\mathcal{N}}}\right) - Q\left(\frac{d}{\sqrt{\mathcal{N}}}\right)$$

Hence

$$P(C) = \frac{1}{4}\left[P(C|m_1) + P(C|m_2) + P(C|m_3) + P(C|m_4)\right]$$

$$= 1 - \frac{3}{2}Q\left(\frac{d}{\sqrt{2\mathcal{N}}}\right) - \frac{5}{4}Q\left(\frac{d}{\sqrt{\mathcal{N}}}\right) - \frac{3}{4}Q\left(d\sqrt{\frac{2}{\mathcal{N}}}\right)$$

Now $E_1 = d^2$, $E_2 = 2d^2$, $E_3 = 4d^2$, and $E_4 = 8d^2$.

Therefore $\overline{E} = \frac{1}{4}\left(d^2 + 2d^2 + 4d^2 + 8d^2\right) = \frac{15}{4}d^2$

And $E_b = \dfrac{\overline{E}}{\log_2 16} = \dfrac{\overline{E}}{4}$

so that $\dfrac{E_b}{\mathcal{N}} = \dfrac{\overline{E}}{4\mathcal{N}} = \dfrac{15}{16}\dfrac{d^2}{\mathcal{N}}$

Therefore

$$P(C) = 1 - \frac{3}{2}Q\left(\sqrt{\frac{8}{15}\frac{E_b}{\mathcal{N}}}\right) - \frac{5}{4}Q\left(\sqrt{\frac{16}{15}\frac{E_b}{\mathcal{N}}}\right) - \frac{3}{4}Q\left(\sqrt{\frac{32}{15}\frac{E_b}{\mathcal{N}}}\right)$$

Moreover

$$Q\left(\sqrt{\frac{8}{15}\frac{E_b}{\mathcal{N}}}\right) >> Q\left(\sqrt{\frac{16}{15}\frac{E_b}{\mathcal{N}}}\right) >> Q\left(\sqrt{\frac{32}{15}\frac{E_b}{\mathcal{N}}}\right)$$

Hence

$$P(C) \cong 1 - \frac{3}{2}Q\left(\sqrt{\frac{8}{15}\frac{E_b}{\mathcal{N}}}\right)$$

And

$$P_{eM} = 1 - P(C) = \frac{3}{2} Q\left(\sqrt{\frac{8}{15}\frac{E_b}{\mathcal{N}}}\right)$$

Comparison of this result with that in Example 14.3[Eq.(14.57)] shows that this configuration requires approximately 1.5 times the power of the system in Example 14.3 to achieve the same performance.

**14.3-9** If $s_1$ is transmitted, we have

$$b_1 = E + a + \sqrt{E}\,n_1 \quad , \quad b_{-1} = -E + a - \sqrt{E}\,n_1$$
$$b_2 = a = \sqrt{E}\,n_2 \quad , \quad b_{-2} = a - \sqrt{E}\,n_2$$
$$\vdots \qquad\qquad\qquad \vdots$$
$$b_k = a + \sqrt{E}\,n_k \quad , \quad b_{-k} = a - \sqrt{E}\,n_k$$

and

$$P(C|m_1) = \text{Prob.}(b_1 > b_{-1},\ b_2,\ b_{-2},\ \cdots\ b_k,\ b_{-k})$$

Note that

$$b_1 > b_{-1} \text{ implies } E + a + \sqrt{E}\,n_1 > -E + a - \sqrt{E}\,n_1 \quad \text{or } n_1 > -\sqrt{E}$$
$$b_1 > b_2 \text{ implies } E + a + \sqrt{E}\,n_1 > a + \sqrt{E}\,n_2 \quad \text{or } n_2 < \sqrt{E} + n_1$$
$$b_1 > b_{-2} \text{ implies } E + a + \sqrt{E}\,n_1 > a - \sqrt{E}\,n_2 \quad \text{or } n_2 > -\left(\sqrt{E} + n_1\right)$$

Hence

$$b_1 > b_2 \text{ and } b_{-2} \text{ implies } -\left(n_1 + \sqrt{E}\right) < n_2 < \left(n_1 + \sqrt{E}\right)$$

Similarly

$$b_1 > b_k \text{ and } b_{-k} \text{ implies } -\left(n_1 + \sqrt{E}\right) < n_k < \left(n_1 + \sqrt{E}\right)$$

Hence

$$P(C|m_1) = \text{Prob.}(b_1 > b_{-1},\ b_2,\ b_{-2},\ b_3,\ b_{-3},\cdots b_k,\ b_{-k})$$
$$= \text{Prob.}\left[n_1 > -\sqrt{E},\ |n_2| < \left(n_1 + \sqrt{E}\right),\ |n_3| < \left(n_1 + \sqrt{E}\right),\ \cdots,\ |n_k| < \left(n_1 + \sqrt{E}\right)\right]$$

Since $n_1,\ n_2, \cdots n_k$ are all independent gaussian random variables each with variance $\mathcal{N}/2$,

$$P(C|m_1) = \left[P\left(n_1 > -\sqrt{E}\right) P\left(|n_2| < n_1 + \sqrt{E}\right) P\left(|n_3| < n_1 + \sqrt{E}\right) \cdots P\left(|n_k| < n_1 + \sqrt{E}\right)\right]$$
$$= \frac{1}{\sqrt{\pi\mathcal{N}}} \int_{-\sqrt{E}}^{\infty} e^{-n_1^2/\mathcal{N}} \left[\int_{-(n_1+\sqrt{E})}^{n_1+\sqrt{E}} e^{-n^2/\mathcal{N}} dn\right]^{N-1} dn_1$$
$$= \frac{1}{\sqrt{\pi\mathcal{N}}} \int_{-\sqrt{E}}^{\infty} e^{-n_1^2/\mathcal{N}} \left[1 - 2Q\left(\frac{n_1 + \sqrt{E}}{\sqrt{\mathcal{N}/2}}\right)\right]^{N-1} dn_1$$

Let $y = \dfrac{n_1 + \sqrt{E}}{\sqrt{\mathcal{N}/2}}$

$$P_{eM} = 1 - P(C|m_1) = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\sqrt{8E/\mathcal{N}}}^{\infty} e^{-\left(y - \sqrt{\frac{2E}{\mathcal{N}}}\right)^2/2} \left[1 - 2Q(y)\right]^{N-1} dy$$

Also $$E_b = \frac{E}{\log_2 2N}$$

**14.4-1** The on-off signal set and its minimum energy equivalent set are shown in Figs. (a) and (b), respectively. The minimum energy equivalent set of orthogonal signal set in Fig. (c) is also given by the set in Fig. (b). Hence, on-off (Fig. a) and orthogonal (Fig. c) have identical error probability. The set in Fig. (b) is polar with half the energy of on-off or orthogonal signals.

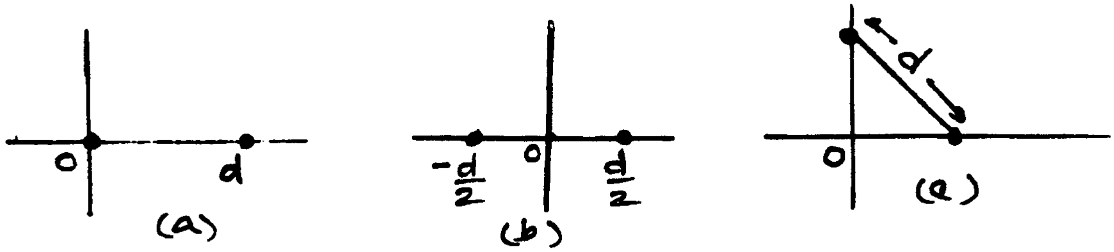



**Fig S14.4-1**

**14.4-2** Here

$$\left.\begin{aligned}\phi_1(t) &= \sqrt{\frac{2}{T_M}}\cos\omega_0 t = \sqrt{40}\cos\omega_0 t\\ \phi_2(t) &= \sqrt{40}\sin\omega_0 t\end{aligned}\right\}\ \omega_0 = \frac{2\pi}{T_M}$$

Therefore

$$\begin{array}{l|l} s_1(t) = \sqrt{20}\,\phi_2(t) & s_1 = \sqrt{\ }\ \Phi_2 \\ s_2(t) = \sqrt{5}\,\phi_1(t) & s_2 = \sqrt{5}\,\Phi_1 \\ s_3(t) = -\sqrt{5}\,\phi_1(t) & s_3 = -\sqrt{5}\,\Phi \end{array}$$

The vector $a = \frac{1}{3}\sum s_i = \frac{1}{3}\left[\sqrt{10}\,\Phi_1 - \sqrt{10}\,\Phi_1 + \sqrt{20}\,\Phi_2\right] = \frac{\sqrt{20}}{3}\Phi_2$ .

Hence the minimum energy signal set is given by

$$\hat{s}_1(t) = s_1(t) - \frac{\sqrt{20}}{3}\phi_2(t) = \sqrt{20}\,\phi_2(t) - \frac{\sqrt{20}}{3}\phi_2(t) = \frac{40\sqrt{2}}{3}\sin\omega_0 t$$

$$\hat{s}_2(t) = s_2(t) - \frac{\sqrt{20}}{3}\phi_2(t) = \sqrt{5}\,\phi_1(t) - \frac{\sqrt{20}}{3}\phi_2(t) = 10\sqrt{2}\cos\omega_0 t - \frac{20\sqrt{2}}{3}\sin\omega_0 t$$

$$\hat{s}_3(t) = s_3(t) - \frac{\sqrt{20}}{3}\phi_2(t) = -\sqrt{5}\,\phi_1(t) - \frac{\sqrt{20}}{3}\phi_2(t) = -10\sqrt{2}\cos\omega_0 t - \frac{20\sqrt{2}}{3}\sin\omega_0 t$$

The optimum receiver – a suitable form – in this case would be that shown in Fig. 14.8a or b.

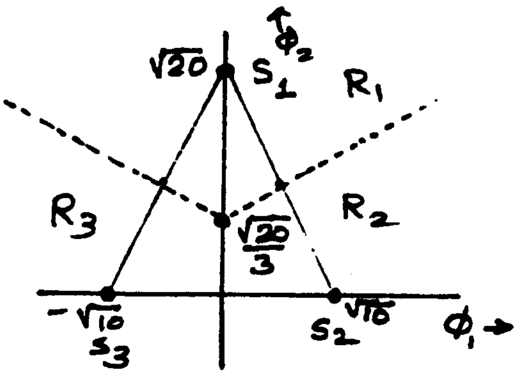



**Fig. S14.4-2**

**14.4-3** To find the minimum energy set, we have $a = \frac{1}{4}(s_1 + s_2 + s_3 + s_4) = -\phi_1 - \phi_2$

Hence the new minimum energy set is

$$\hat{s}_1 = s_1 - \phi_1 - \phi_2 = \sqrt{3}\,\phi_1 + \phi_2,\ \hat{s}_2 = -\phi_1 + \sqrt{3}\,\phi_2,\ \hat{s}_3 = -\sqrt{3}\,\phi_1 - \phi_2,\ \hat{s}_4 = \phi_1 - \sqrt{3}\,\phi_2$$

Note that all the four signals form vertices of a square because $(\hat{s}_1\ \hat{s}_2)$, $(\hat{s}_2\ \hat{s}_3)$, $(\hat{s}_3\ \hat{s}_4)$, and $(\hat{s}_4\ \hat{s}_1)$ are orthogonal. The distance between these signal pairs is always $2\sqrt{2}$. This set is shown in Fig. S14.4-3a.

The signal set is now rotated so as to yield a new set shown in Fig. S14.4-3b.

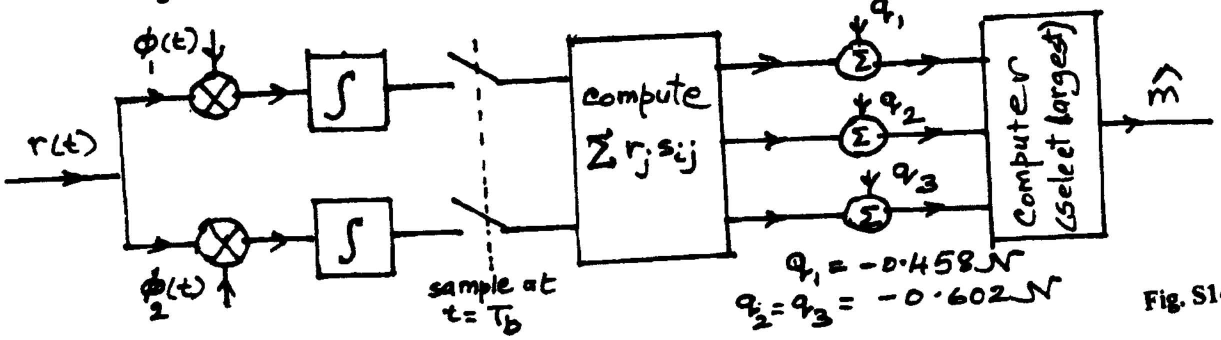



Fig. S14.4-3

Observing symmetry we obtain

$$P(C) = P(C|m_1) = P(C|m_2) = P(C|m_3) = P(C|m_4)$$

$$= P(n_1 > -\sqrt{2} \text{ and } n_2 > -\sqrt{2})$$

$$= \left[1 - Q\left(\frac{\sqrt{2}}{\sqrt{\mathcal{N}/2}}\right)\right]^2$$

$$= \left[1 - Q\left(\frac{2}{\sqrt{\mathcal{N}}}\right)\right]^2 = \left[1 - Q\left(\frac{2}{\sqrt{0.4}}\right)\right]^2 = [1 - Q(3.16)]^2$$

$$P_{eM} = 1 - P(C) = 1 - [1 - Q(3.16)]^2 \cong 1.58 \times 10^{-3}$$

14.4-4 $\quad s_1 = \dfrac{2}{10\sqrt{10}}\phi_1 = \dfrac{1}{5\sqrt{10}}\phi_1$, $s_2 = \dfrac{1}{5\sqrt{10}}\phi_2$ and $s_3 = \dfrac{1}{10\sqrt{10}}[\phi_1 + \phi_2]$

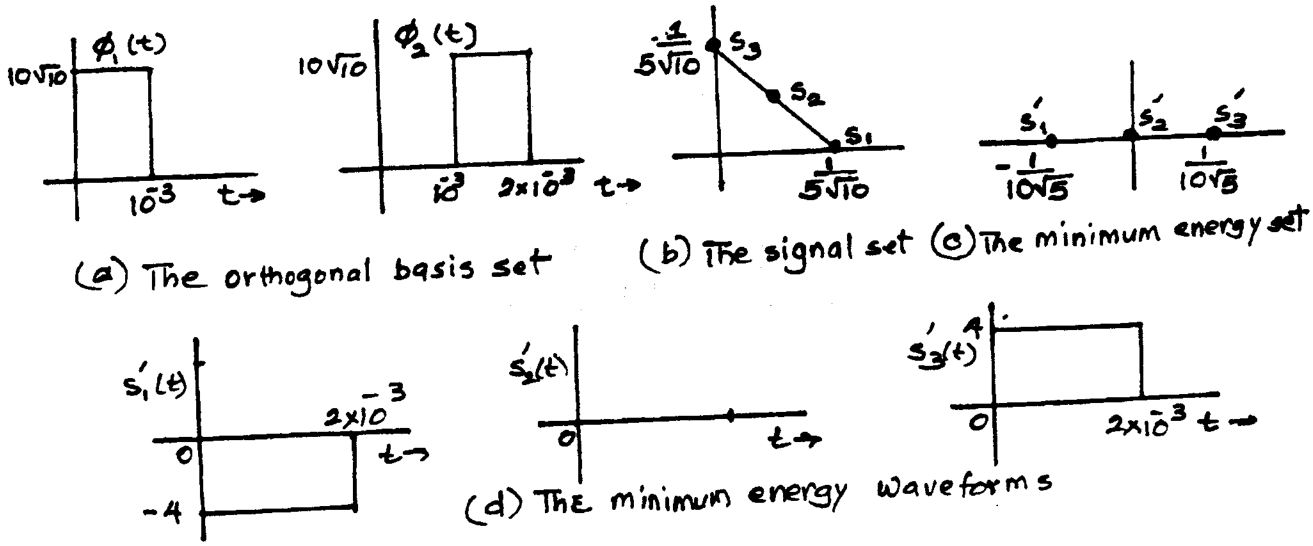



Fig. S14.4-4

$$P(C|m_3) = P(C|m_1) = P\left(n_1 < \frac{1}{20\sqrt{5}}\right) = 1 - Q\left(\frac{1/20\sqrt{5}}{\sqrt{10^{-5}}}\right) = 1 - Q(7.07)$$

$$P(C|m_2) = P\left(|n| < \frac{1}{20\sqrt{5}}\right) = 1 - 2Q(7.07)$$

and

$$P(C) = \frac{1}{3}\left[2P(C|m_1) + P(C|m_2)\right] = \frac{1}{3}\left[2 - 2Q(7.07) + 1 - 2Q(7.07)\right]$$

$$= 1 - \frac{4}{3}Q(7.07)$$

$$P_{eM} = 1 - P(C) = \frac{4}{3}Q(7.07) \cong 1.03 \times 10^{-12}$$

Also $\quad E_1 = E_3 = \left(\frac{1}{5\sqrt{10}}\right)^2 = 4 \times 10^{-3}$

$$E_2 = \left(\frac{1}{10\sqrt{10}}\right)^2 + \left(\frac{1}{10\sqrt{10}}\right)^2 = 2 \times 10^{-3} \quad \overline{E} = \frac{1}{3}(E_1 + E_2 + E_3) = \frac{1}{3} \times 10^{-2}$$

Mean energy of the minimum energy set:

$$E_{\min} = \frac{1}{3}\left(2 \times 10^{-3} + 0 + 2 \times 10^{-3}\right) = \frac{4}{3} \times 10^{-3}$$

**14.4-5** The use of Eq. (14.76) and signal rotation shows that the minimum energy set in this case is identical to that in Prob. 14.4-4. Hence the minimum energy set is as shown in Fig. S14.4-4c. this situation is identical to that in Prob. 14.3-5 with $d = \frac{1}{10\sqrt{5}}$. From the results in the solution of Prob. 14.3-5, we have

$$\overline{E} = \frac{d^2}{2} = 10^{-3}$$

Also, we are given $s_n(\omega) = \frac{\mathcal{N}}{2} = 10^{-5}$. Hence, $\mathcal{N} = 2 \times 10^{-5}$.

**(a)** From the solution of Prob. 14.3-5

$$P_{eM} = \frac{1}{2}Q(7.02) + Q(7.12) = 1.09 \times 10^{-12}$$

**(b)** and **(c)** identical to those in Prob. 14.4-4

**14.4-6** **(a)** The center of gravity of the signal set is $(\mathbf{s}_1 + \mathbf{s}_2)/2$

Hence, the minimum energy signal set is

$$\mathbf{x}_1 = \mathbf{s}_1 - \frac{(\mathbf{s}_1 + \mathbf{s}_2)}{2} = \frac{\mathbf{s}_1 - \mathbf{s}_2}{2} \quad \& \quad \mathbf{x}_2 = \mathbf{s}_2 - \frac{(\mathbf{s}_1 + \mathbf{s}_2)}{2} = \frac{\mathbf{s}_2 - \mathbf{s}_1}{2}$$

The minimum energy signals are

$$\left.\begin{aligned} x_1(t) &= 0.5 - 0.707\sin\frac{\omega_0 t}{2} \\ x_2(t) &= 0.707\sin\frac{\omega_0 t}{2} - 0.5 \end{aligned}\right\} \quad \omega_0 = 2000\pi$$

**(b)** $$E_{x_1} = \int_0^{0.001}\left(0.5 - 0.707\sin\frac{\omega_0 t}{2}\right)^2 dt = 0.4984 \times 10^{-5}$$

$E_{x_2} = E_{x_1} = 0.4984 \times 10^{-5}$. We are given $\mathcal{N} = 5 \times 10^{-6}$

$$P_b = Q\left(\sqrt{\frac{2E_{x_1}}{\mathcal{N}}}\right) = Q(4.465) = 0.41 \times 10^{-5}$$

(c) We use Gram-Schmidt orthogonalization procedure in appendix C to obtain

$$y_1(t) = s_1(t)$$

$$y_2(t) = \sqrt{2}\sin\omega_0 t - \frac{\int_0^{.001}\sqrt{2}\sin\omega_0 t\,dt}{.001} = \sqrt{2}\sin\omega_0 t - \frac{2\sqrt{2}}{\pi}$$

$$\hat{y}_1 = \frac{y_1}{|y_1|}$$

$$|y_1| = \sqrt{\int y_1^2\,dt} = \sqrt{.001} \qquad \hat{y}_1 = \frac{y_1}{\sqrt{.001}} = 31.6y_1$$

$$y_2^2(t) = 1 - \cos 2\omega_0 t - \frac{8}{\pi}\sin\omega_0 t + \frac{8}{\pi^2}$$

$$|y_2| = \sqrt{\int y_2^2\,dt} = \sqrt{.001\left(1-\frac{8}{\pi^2}\right)} \qquad \hat{y}_2 = \frac{y_2}{\sqrt{.001\left(1-\frac{8}{\pi^2}\right)}} = 72.2y_2$$

$$s_1(t) = \frac{1}{31.6}\hat{y}_1(t) = .0316\hat{y}_1(t)$$

$$s_2(t) = \frac{1}{72.7}\hat{y}_2(t) + \frac{1}{35.1}\hat{y}_1(t) = .0138\hat{y}_2(t) + .0285\hat{y}_1(t)$$

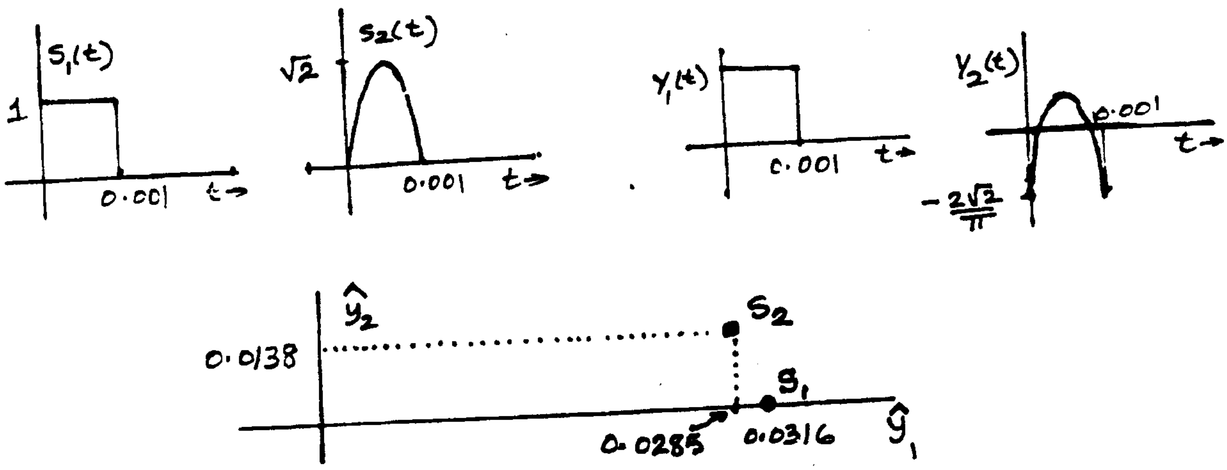



Fig. S14.4-6

# Chapter 15

**15-1.1** $P_1 = 0.4,\ P_2 = 0.3,\ P_3 = 0.2$ and $P_4 = 0.1$

$$H(m) = -(P_1 \log P_1 + P_2 \log P_2 + P_3 \log P_3 + P_4 \log P_4)$$
$$= 1.846 \text{ bits (source entropy)}$$

There are $10^6$ symbols/s. Hence, the rate of information generation is $1.846 \times 10^6$ bits/s.

**15.1-2** Information/element = $\log_2 10 = 3.32$ bits.

Information/picture frame = $3.32 \times 300{,}000 = 9.96 \times 10^5$ bits.

**15.1-3** Information/word = $\log_2 10000 = 13.3$ bits.

Information content of 1000 words = $13.3 \times 1000 = 13{,}300$ bits.

The information per picture frame was found in Problem 15.1-2 to be $9.96 \times 10^5$ bits. Obviously, it is not possible to describe a picture completely by 1000 words, in general. Hence, a picture is worth 1000 words is very much an underrating or understating the reality.

**15.1-4** **(a)** Both options are equally likely. Hence,

$$I = \log\left(\frac{1}{0.5}\right) = 1 \text{ bit}$$

**(b)** $P(2 \text{ lanterns}) = 0.1$

$I(2 \text{ lanterns}) = \log_2 10 = 3.322$ bits

**15.1-5** **(a)** All 27 symbols equiprobable and $P(x_i) = 1/27$.

$$H_1(x) = 27\left(\frac{1}{27}\log_2 27\right) = 4.755 \text{ bits / symbol}$$

**(b)** Using the probability table, we compute

$$H_w(x) = -\sum_{i=1}^{27} P(x_i) \log P(x_i) = 4.127 \text{ bits / symbol}$$

**(c)** Using Zipf's law, we compute entropy/word $H_w(x)$.

$$H_w(x) = -\sum_{r=1}^{8727} P(r) \log P(r)$$
$$= -\sum_{r=1}^{8727} \frac{0.1}{r} \log\left(\frac{0.1}{r}\right) = 9.1353 \text{ bits / word.}$$

H/letter = 11/82/5.5=2.14 bits/symbol.

Entropy obtained by Zipf's law is much closer to the real value than $H_1(x)$ or $H_2(x)$.

**15.2-1** $H(\mathrm{m}) = \sum_{i=1}^{7} P_i \log P_i = \frac{63}{32}$ bits

| Message | Probability | Code | $S_1$ | | $S_2$ | | $S_3$ | | $S_4$ | | $S_5$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $m_1$ | 1/2 | 0 | 1/2 | 0 | 1/2 | 0 | 1/2 | 0 | 1/2 | 0 | 1/2 | 0 |
| $m_2$ | 1/4 | 10 | 1/4 | 10 | 1/4 | 10 | 1/4 | 10 | 1/4 | 10 | 1/2 | 1 |
| $m_3$ | 1/8 | 110 | 1/8 | 110 | 1/8 | 110 | 1/8 | 110 | 1/4 | 11 | | |
| $m_4$ | 1/16 | 1110 | 1/16 | 1110 | 1/16 | 1110 | 1/8 | 111 | | | | |
| $m_5$ | 1/32 | 11110 | 1/32 | 11110 | 1/16 | 1111 | | | | | | |
| $m_6$ | 1/64 | 111110 | 1/32 | 11111 | | | | | | | | |
| $m_7$ | 1/64 | 111111 | | | | | | | | | | |

$$L = \sum_i P_i L_i = \frac{1}{2}(1) + \frac{1}{4}(2) + \frac{1}{8}(3) + \frac{1}{16}(4) + \frac{1}{32}(5) + \frac{1}{64}(6) + \frac{1}{64}(6)$$

$$= \frac{63}{32} \text{ binary digits}$$

Efficiency $\eta = \frac{H(\mathrm{m})}{L} \times 100 = 100\%$

Redundancy $\gamma = (100 - \eta) = 0\%$

**15.2-2** $H(\mathrm{m}) = -\sum_{i=1}^{7} P_i \log P_i = 2.289$ bits

$$= \frac{2.289}{\log_2 3} = 1.4442 \quad \text{3-ary units}$$

| Message | Probability | Code | $S_1$ | | $S_2$ | |
|---|---|---|---|---|---|---|
| $m_1$ | 1/3 | 0 | 1/3 | 0 | 1/3 | 0 |
| $m_2$ | 1/3 | 1 | 1/3 | 1 | 1/3 | 1 |
| $m_3$ | 1/9 | 20 | 1/9 | 20 | 1/3 | 2 |
| $m_4$ | 1/9 | 21 | 1/9 | 21 | | |
| $m_5$ | 1/27 | 220 | 1/9 | 22 | | |
| $m_6$ | 1/27 | 221 | | | | |
| $m_7$ | 1/27 | 222 | | | | |

$$L = \sum_{i=1}^{7} P_i L_i = \frac{1}{3}(1) + \frac{1}{3}(1) + \frac{1}{9}(2) + \frac{1}{9}(2) + 3\frac{1}{27}(3)$$

$$= \frac{13}{9} \quad \text{3-ary digits}$$

$$= 1.4442 \quad \text{3-ary digits}$$

Efficiency $\eta = \frac{H(\mathrm{m})}{L} = \frac{1.4442}{1.4442} \times 100 = 100\%$

Redundancy $\gamma = (1 - \eta)100 = 0\%$

**15.2-3** $H(\mathrm{m}) = -\sum_{i=1}^{4} P_i \log P_i = 1.69$ bits

| Message | Probability | Code | $s_1$ | | $s_2$ | |
|---|---|---|---|---|---|---|
| $m_1$ | 0.5 | 0 | 0.5 | 0 | 0.5 | 0 |
| $m_2$ | 0.3 | 1 0 | 0.3 | 1 0 | 0.5 | 1 |
| $m_3$ | 0.1 | 1 1 0 | 0.2 | 1 1 | | |
| $m_4$ | 0.1 | 1 1 1 | | | | |

$$L = \sum P_i L_i = 0.5(1) + 0.3(2) + 0.1(3) + 0.1(3) = 1.7 \text{ binary digits}$$

$$\text{Efficiency } \eta = \frac{H(\mathrm{m})}{L} \times 100 = \frac{1.69}{1.7} \times 100 = 99.2\%$$

$$\text{Redundancy } \gamma = (1-\eta)100 = 0.8\%$$

For ternary coding, we need one dummy message of probability 0. Thus,

| Message | Probability | Code | $s_1$ | |
|---|---|---|---|---|
| $m_1$ | 0.5 | 0 | 0.5 | 0 |
| $m_2$ | 0.3 | 1 | 0.3 | 1 |
| $m_3$ | 0.1 | 20 | 0.2 | 2 |
| $m_4$ | 0.1 | 21 | | |
| $m_5$ | 0 | 22 | | |

$$L = 0.5(1) + 0.3(1) + 0.1(2) + 0.1(2) = 1.2 \quad 3\text{-ary digits}$$

$$H(\mathrm{m}) = 1.69 \text{ bits } = \frac{1.69}{\log_2 3} = 1.0663 \quad 3\text{-ary units}$$

$$\text{Efficiency } \eta = \frac{H(\mathrm{m})}{L} \times 100 = \frac{1.0663}{1.2} \times 100 = 88.86\%$$

$$\text{Redundancy } \gamma = (1-\eta)100 = 11.14\%$$

**15.2-4**

| Message | Probability | Code | $s_1$ | | $s_2$ | |
|---|---|---|---|---|---|---|
| $m_1$ | 1/2 | 0 | 1/2 | 0 | 1/2 | 0 |
| $m_2$ | 1/4 | 1 | 1/4 | 1 | 1/4 | 1 |
| $m_3$ | 1/8 | 20 | 1/8 | 20 | 1/4 | 2 |
| $m_4$ | 1/16 | 21 | 1/16 | 21 | | |
| $m_5$ | 1/32 | 220 | 1/16 | 22 | | |
| $m_6$ | 1/64 | 221 | | | | |
| $m_7$ | 1/64 | 222 | | | | |

$$L = \sum P_i L_i = \frac{21}{16} \quad 3\text{-ary digits}$$

$$\text{From Problem 15.2-1, } H(\mathrm{m}) = \frac{63}{32} \text{ bits} = 1.242 \quad 3\text{-ary units}$$

$$\text{Efficiency } \eta = \frac{H(\mathrm{m})}{L} \times 100 = \frac{1.242}{1.3125} \times 100 = 94.63\%$$

$$\text{Redundancy } \gamma = (1-\eta)100 = 5.37\%$$

**15.2-5**

| Message | Probability | Code | $s_1$ | | $s_2$ | | $s_3$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $m_1$ | 1/3 | 1 | 1/3 | 1 | 1/3 | 1 | 1/3 | 1 | 1/3 | 1 | 2/3 | 0 |
| $m_2$ | 1/3 | 00 | 1/3 | 00 | 1/3 | 00 | 1/3 | 00 | 1/3 | 00 | 1/3 | 1 |
| $m_3$ | 1/9 | 011 | 1/9 | 011 | 1/9 | 011 | 2/9 | 010 | 1/3 | 01 | | |
| $m_4$ | 1/9 | 0100 | 1/9 | 0100 | 1/9 | 0100 | 1/9 | 011 | | | | |
| $m_5$ | 1/27 | 01010 | 1/27 | 01010 | 1/9 | 0101 | | | | | | |
| $m_6$ | 1/27 | 010110 | 2/27 | 01011 | | | | | | | | |
| $m_7$ | 1/27 | 010111 | | | | | | | | | | |

$$L = \sum P_i L_i = \frac{65}{27} = 2.4074 \text{ binary digits}$$

$H(\text{m}) = 2.289$ bits (See Problem 15.2 - 2).

$$\text{Efficiency } \eta = \frac{H(\text{m})}{L} \times 100 = \frac{2.289}{2.4074} \times 100 = 95.08\%$$

$$\text{Redundancy } \gamma = (1-\eta)100 = 4.92\%$$

**15.2-6** (a) $H(\text{m}) = 3(\frac{1}{3}\log 3) - 1.585$ bits

(b) Ternary Code

| Message | Probability | Code |
|---|---|---|
| $m_1$ | 1/3 | 0 |
| $m_2$ | 1/3 | 1 |
| $m_3$ | 1/3 | 2 |

$$L = \frac{1}{3}(1) + \frac{1}{3}(1) + \frac{1}{3}(1) = 1 \quad \text{3-ary digits}$$

$$H(\text{m}) = 1.585 \text{ bits } = \frac{1.585}{\log_2 3} = 1 \quad \text{3-ary unit}$$

$$\text{Efficiency } \eta = \frac{H(\text{m})}{L} \times 100 = 100\%$$

$$\text{Redundancy } \gamma = (1-\eta)100 = 0\%$$

(c) Binary Code

| Message | Probability | Code | $s_1$ | |
|---|---|---|---|---|
| $m_1$ | 1/3 | 1 | 2/3 | 0 |
| $m_2$ | 1/3 | 00 | 1/3 | 1 |
| $m_3$ | 1/3 | 01 | | |

$$L = \frac{1}{3}(1) + (2)\frac{1}{3}(2) = \frac{5}{3} = 1.667 \text{ binary digits}$$

$$\text{Efficiency } \eta = \frac{H(\text{m})}{L} \times 100 = \frac{1.585}{1.667} \times 100 = 95.08\%$$

$$\text{Redundancy } \gamma = (1-\eta)100 = 4.92\%$$

**(d)** Second extension – binary code

$$L = \frac{1}{2}\left[(7)\left(\frac{1}{9}\right)(3) + (2)\left(\frac{1}{9}\right)(4)\right] = \frac{29}{18} = 1.611 \text{ binary digits}$$

$$H(\mathrm{m}) = 1.585 \text{ bits}$$

$$\text{Efficiency } \eta = \frac{H(\mathrm{m})}{L} \times 100 = \frac{1.585}{1.611} \times 100 = 98.39\%$$

$$\text{Redundancy } \gamma = (1 - \eta)100 = 1.61\%$$

| Message | Prob | Code | $S_1$ | | $S_2$ | | $S_3$ | | $S_4$ | | $S_5$ | | $S_6$ | | $S_7$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $m_1m_1$ | 1/9 | 001 | 2/9 | 01 | 2/9 | 01 | 2/9 | 01 | 2/9 | 01 | 1/3 | 00 | 4/9 | 1 | 5/9 | 0 |
| $m_1m_2$ | 1/9 | 0000 | 1/9 | 001 | 2/9 | 10 | 2/9 | 10 | 2/9 | 10 | 2/9 | 01 | 1/3 | 00 | 4/9 | 1 |
| $m_1m_3$ | 1/9 | 0001 | 1/9 | 0000 | 1/9 | 001 | 2/9 | 11 | 2/9 | 11 | 2/9 | 10 | 2/9 | 01 | | |
| $m_2m_1$ | 1/9 | 110 | 1/9 | 0001 | 1/9 | 0000 | 1/9 | 001 | 2/9 | 000 | 2/9 | 11 | | | | |
| $m_2m_2$ | 1/9 | 111 | 1/9 | 110 | 1/9 | 0001 | 1/9 | 0000 | 1/9 | 001 | | | | | | |
| $m_2m_3$ | 1/9 | 100 | 1/9 | 111 | 1/9 | 110 | 1/9 | 0001 | | | | | | | | |
| $m_3m_1$ | 1/9 | 101 | 1/9 | 100 | 1/9 | 111 | | | | | | | | | | |
| $m_3m_2$ | 1/9 | 010 | 1/9 | 101 | | | | | | | | | | | | |
| $m_3m_3$ | 1/9 | 011 | | | | | | | | | | | | | | |

**15.4-1** **(a)** The channel matrix can be represented as shown in Fig. S15.4-1

$$P(y_1) = P(y_1|x_1)P(x_1) + P(y_1|x_2)P(x_2)$$
$$= \frac{2}{3}\cdot\frac{1}{3} + \frac{1}{10}\cdot\frac{2}{3} = \frac{13}{45}$$
$$P(y_2) = 1 - P(y_1) = \frac{32}{45}$$

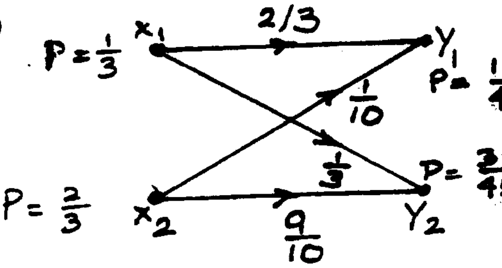



Fig. S15.4-1

**(b)**

$$H(\mathrm{x}) = P(x_1)\log\frac{1}{P(x_1)} + P(x_2)\log\frac{1}{P(x_2)}$$
$$= \frac{1}{3}\log_2 3 + \frac{2}{3}\log_2\frac{3}{2} = 0.918 \text{ bits}$$

To compute $H(\mathrm{x}|\mathrm{y})$, we find

$$P(x_1|y_1) = \frac{P(y_1|x_1)P(x_1)}{P(y_1)} = \frac{10}{13}, \qquad P(x_1|y_2) = \frac{P(y_2|x_1)P(x_1)}{P(y_2)} = \frac{5}{32}$$
$$P(x_2|y_1) = \frac{P(y_1|x_2)P(x_2)}{P(y_1)} = \frac{3}{13}, \qquad P(x_2|y_2) = \frac{P(y_2|x_2)P(x_2)}{P(y_2)} = \frac{54}{64}$$

$$H(\mathrm{x}|y_1) = P(x_1|y_1)\log\frac{1}{P(x_1|y_1)} + P(x_2|y_1)\log\frac{1}{P(x_2|y_1)}$$
$$= \frac{10}{13}\log_2\frac{13}{10} + \frac{3}{13}\log_2\frac{13}{3} = 0.779$$

$$H(\mathrm{x}|y_2) = P(x_1|y_1)\log\frac{1}{P(x_1|y_1)} + P(x_2|y_2)\log\frac{1}{P(x_2|y_2)}$$
$$= \frac{5}{32}\log_2\frac{32}{5} + \frac{54}{64}\log_2\frac{64}{54} = 0.624$$

and

$$H(\mathrm{x}|\mathrm{y}) = P(y_1)H(x|y_1) + P(y_2)H(x|y_2)$$
$$= \frac{13}{45}(0.779) + \frac{32}{45}(0.624) = 0.6687$$

Thus,

$$I(\mathrm{x}|\mathrm{y}) = H(\mathrm{x}) - H(\mathrm{x}|\mathrm{y}) = 0.918 - 0.6687 = 0.24893 \text{ bits / binit}$$
$$H(\mathrm{y}) = \sum_i P(y_i)\log\frac{1}{P(y_i)} = \frac{13}{45}\log\frac{45}{13} + \frac{32}{45}\log\frac{45}{32} = 0.8673 \text{ bits / symbol}$$

Also,

$$H(\mathrm{y}|\mathrm{x}) = H(\mathrm{y}) - I(\mathrm{x}|\mathrm{y}) = 0.8673 - 0.2493 = 0.618 \text{ bits / symbol}$$

15.4-2 The channel matrix $P(y_j|x_i)$ is

$$\begin{array}{cc} & x_i \\ y_j & \begin{bmatrix} 1 & 0 & 0 \\ 0 & p & 1-p \\ 0 & 1-p & p \end{bmatrix} \end{array}$$

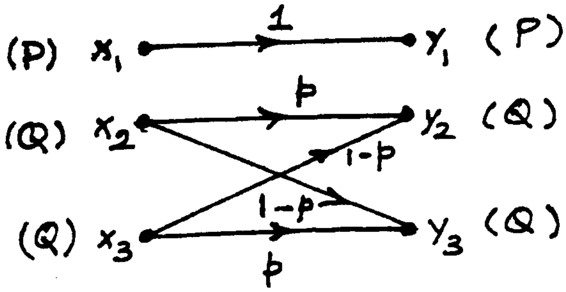



Fig. S15.4-2

Also, $P(y_1) = P,\ P(y_2) = P(y_3) = Q$

Now we use $P(x_i|y_j) = \dfrac{P(y_j|x_i)P(x_i)}{\sum_i P(x_i)P(y_j|x_i)}$ to obtain

$P(x_i|y_i)$ matrix as
$$\begin{array}{cc} & x_i \\ y_j & \begin{bmatrix} 1 & 0 & 0 \\ 0 & p & 1-p \\ 0 & 1-p & p \end{bmatrix} \end{array}$$

$$H(\mathrm{x}) = \sum P(x_i)\log\frac{1}{P(x_i)} = -P\log P - 2Q\log Q \quad \text{with } (2Q = 1-P)$$
$$= -\left[P\log P + (1-P)\log\left(\frac{1-P}{2}\right)\right] = \Omega(P) + (1-P)$$

$$H(\mathrm{x}|\mathrm{y}) = \sum_i\sum_j P(y_j)P(x_i|y_j)\log\frac{1}{P(x_i|y_j)}$$
$$= P\log 1 + Q\left[p\log\frac{1}{p} + (1-p)\log\frac{1}{1-p}\right] + Q\left[(1-p)\log\frac{1}{1-p} + p\log\frac{1}{p}\right]$$
$$= 0 + 2Q\Omega(p) = (1-P)\Omega(p)$$

$$I(\mathrm{x}|\mathrm{y}) = H(\mathrm{x}) - H(\mathrm{x}|\mathrm{y}) = \Omega(P) + (1-P) - (1-P)\Omega(p)$$
$$= \Omega(P) + (1-P)[1 - \Omega(p)]$$

Letting $\beta = 2^{\Omega(p)}$ or $\Omega(p) = \log\beta$

$$I(\mathrm{x}|\mathrm{y}) = \Omega(P) + (1-P)(1-\log\beta)$$

$\dfrac{d}{dP}I(\mathrm{x}|\mathrm{y}) = 0$ or $\dfrac{d}{dP}[\Omega(P) + (1-P)(1-\log\beta)] = 0$. This means

$$\frac{d}{dP}\left[P\log P + (1-P) - (1-P)\log(1-P)(1-\log\beta)\right] = 0$$
$$\log P - \log(1-P) + [1 - \log\beta] = 0$$

Therefore $$\log\frac{P}{1-P} = -1+\log\beta$$

Note: $-1+\log_2\beta = -\log_2 2+\log_2\beta = \log_2\frac{\beta}{2}$

$$\frac{P}{1-P}=\frac{\beta}{2} \quad\Rightarrow\quad P=\frac{\beta}{\beta+2} \text{ and } 1-P=\frac{2}{\beta+2}$$

so

$$C = MAX\, I(\mathrm{x}|\mathrm{y}) = \frac{\beta}{\beta+2}\log\frac{\beta+2}{\beta}+\frac{2}{\beta+2}\log\frac{\beta+2}{2}+\frac{2}{\beta+2}\underbrace{(1-\log\beta)}_{\log\frac{2}{\beta}} = \log\frac{\beta+2}{\beta}$$

**15.4-3** Consider the cascade of 2 BSCS shown in Fig. S15.4-3. In this case

$$P_{\mathrm{y|x}}(1|1) = (1-P_1)(1-P_2)+P_1P_2 = 1-P_1-P_2-2P_1P_2$$
$$P_{\mathrm{y|x}}(0|1) = (1-P_1)P_2+P_1(1-P_2) = P_1+P_2-2P_1P_2$$

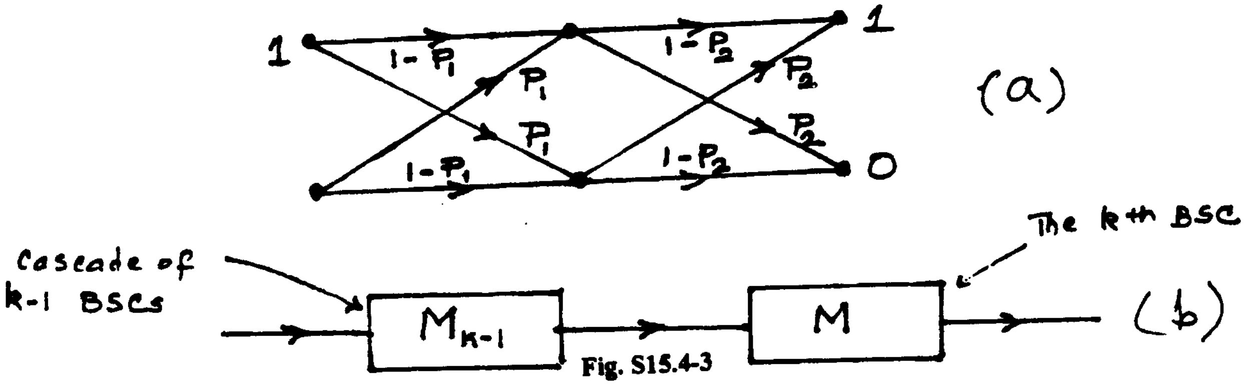



Fig. S15.4-3

Hence, the channel matrix of the cascade is

$$\begin{bmatrix}1-P_1-P_2-2P_1P_2 & P_1+P_2-2P_1P_2\\ P_1+P_2-2P_1P & 1-P_1-P_2-2P_1P\end{bmatrix}=\begin{bmatrix}1-P_1 & P_1\\ P_1 & 1-P_1\end{bmatrix}\begin{bmatrix}1-P_2 & P_2\\ P_2 & 1-P_2\end{bmatrix}$$

This result will prove everything in this problem.

**(a)** With $P_1=P_2=P_e$, from the above result it follows that the channel matrix is indeed $M^2$.

**(b)** We have already shown that the channel matrix of two cascaded channels is $M_1M_2$.

**(c)** Consider a cascade of $k$ identical channels broken up as $k-1$ channel cascaded with the $k^{th}$ channel. If $M_{k-1}$ is the channel matrix of the first $k-1$ channels in cascade, then from the results derived in part (b), the channel matrix of the $k$ cascaded channels is $M_k = M_{k-1}M$. This is valid for any $k$. We have already proved it for $k=2$, that $M_2 = M^2$. Using the process of induction it is clear that $M_k = M^k$. We can verify these results from the development in Example 10.7. From the results in Example 10.7, we have, for a cascade of 3 channels

$$\begin{aligned}1-P_E &= (1-P_e)^3+3P_e^2(1-P_e)\\ &= 1-3P_e+3P_e^2-P_e^3+3P_e^2-3P_e^3\\ &= 1-3P_e+6P_e^2-4P_e^3\end{aligned}$$

and

$$P_E = 3P_e-6P_e^2+4P_e^3$$

Now

$$M^3 = \begin{bmatrix} 1-P_e & P_e \\ P_e & 1-P_e \end{bmatrix}^3 = \begin{bmatrix} 1-(3P_e-6P_e^2+4P_e^3) & 3P_e-6P_e^2+4P_e^3 \\ 3P_e-6P_e^2+4P_e^3 & 1-(3P_e-6P_e^2+4P_e^3) \end{bmatrix}$$

Clearly

$$P_E = 3P_e - 6P_e^2 + 4P_e^3$$

which confirms the results in Example 10.7 for $k = 3$.

(d) From Equation 15.25

$$C_s = 1 - \left[ P_E \log \frac{1}{P_E} + (1-P_E) \log \frac{1}{1-P_E} \right]$$

where $P_E$ is the error probability of cascade of $k$ identical channel.

We have shown in Example 10.7 that

$$P_E = 1 - \left[ (1-P_e)^k + \sum_{j=2,4,6}^{k} \frac{k!}{j!(k-j)!} P_e^j (1-P_e)^{k-j} \right]$$

If $kP_e << 1$, $\quad P_E \cong kP_e$

and

$$C_s = 1 - \left[ kP_e \log \frac{1}{kP_e} + (1-kP_e) \log \frac{1}{1-kP_e} \right]$$

15.4-4 The channel matrix is

| | $y_1$ | $y_2$ | $y_3$ |
|---|---|---|---|
| $x_1$ | $q$ | 0 | $p$ |
| $x_2$ | 0 | $q$ | $p$ |

$q = 1 - p$

Let

$x_1 = 0 \quad y_1 = 0$

$x_2 = 1 \quad y_2 = 1$

$y_3 = E$

$$P(y_1) = P(x_1, y_1) + P(x_2, y_1) = P(x_1)P(y_1|x_1) + P(x_2)P(y_1|x_2)$$
$$= \frac{1}{2}q + \frac{1}{2}(0) = \frac{q}{2}$$

$$P(y_2) = P(x_1, y_2) + P(x_2, y_2) = P(x_1)P(y_2|x_1) + P(x_2)P(y_2|x_2)$$
$$= \frac{1}{2}(0) + \frac{1}{2}q = \frac{q}{2}$$

$$P(y_3) = 1 - P(y_1) - P(y_2) = 1 - q = p$$

Also,

$$P(x_1|y_1) = \frac{P(y_1|x_1)P(x_1)}{P(y_1)} = \frac{q/2}{q/2} = 1$$

$$P(x_2|y_1) = \frac{P(y_1|x_2)P(x_2)}{P(y_1)} = 0$$

$$P(x_1|y_2) = \frac{P(y_2|x_1)P(x_1)}{P(y_2)} = 0$$

$$P(x_2|y_2) = \frac{P(y_2|x_2)P(x_2)}{P(y_2)} = \frac{q/2}{q/2} = 1$$

$$P(x_1|y_3) = \frac{P(y_3|x_1)P(x_1)}{P(y_3)} = \frac{p/2}{p} = \frac{1}{2}$$

$$P(x_2|y_3) = \frac{P(y_3|x_2)P(x_2)}{P(y_3)} = \frac{p/2}{p} = \frac{1}{2}$$

$$P(x_1,y_1) = P(x_1)P(y_1|x_1) = \frac{q}{2}$$
$$P(x_1,y_2) = P(x_1)P(y_2|x_1) = 0$$
$$P(x_1,y_3) = P(x_1)P(y_3|x_1) = \frac{p}{2}$$
$$P(x_2,y_1) = P(x_2)P(y_1|x_2) = 0$$
$$P(x_2,y_2) = P(x_2)P(y_2|x_2) = \frac{q}{2}$$
$$P(x_2,y_3) = P(x_2)P(y_3|x_2) = \frac{p}{2}$$

Therefore,

$$\begin{aligned} H(\mathrm{x}) &= -P(x_1)\log P(x_1) - P(x_2)\log P(x_2) \\ &= \frac{1}{2} + \frac{1}{2} = 1 \end{aligned}$$

$$\begin{aligned} H(\mathrm{x}|\mathrm{y}) &= \sum_i \sum_j P(x_i,y_j)\log\frac{1}{P(x_i|y_j)} \\ &= \frac{q}{2}(0) + 0 + \frac{1}{2}p + 0 + \frac{1}{2}q \times 0 + \frac{1}{2}p = p \end{aligned}$$

$$\begin{aligned} I(\mathrm{x}|\mathrm{y}) &= H(\mathrm{x}) - H(\mathrm{x}|\mathrm{y}) \\ &= 1 - p \text{ bits / symbols} \end{aligned}$$

15.4-5

$$\begin{aligned} H(\mathrm{x}|\mathrm{z}) - H(\mathrm{x}|\mathrm{y}) &= \sum_x \sum_z P(x_i,z_k)\log\frac{1}{P(x_i|z_k)} - \sum_x \sum_y P(x_i,y_j)\log\frac{1}{P(x_i|y_j)} \\ &= \sum_x \sum_y \sum_z P(x_i,y_j,z_k)\log\frac{1}{P(x_i|z_k)} - \sum_x \sum_y \sum_z P(x_i,y_j,z_k)\log\frac{1}{P(x_i|y_j)} \\ &= \sum_x \sum_y \sum_z P(x_i,y_j,z_k)\log\frac{P(x_i|y_j)}{P(x_i|z_k)} \end{aligned}$$

Note that for cascaded channel, the output z depends only on y. Therefore,

$$P(z_k|y_j,x_i) = P(z_k|y_j)$$

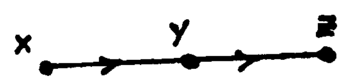



Fig. S15.4-5

By Bayes' rule

$$P(x_i|y_j,z_k) = P(x_i|y_j)$$

and

$$\begin{aligned} H(\mathrm{x}|\mathrm{z}) - H(\mathrm{x}|\mathrm{y}) &= \sum_x \sum_y \sum_z P(x_i,y_j,z_k)\log\frac{P(x_i|y_j,z_k)}{P(x_i|z_k)} \\ &= \sum_y \sum_z P(y_j,z_k)\left[\sum_x P(x_i|y_j,z_k)\log\frac{P(x_i|y_j,z_k)}{P(x_i|z_k)}\right] \end{aligned}$$

It can be shown that the summation over $x$ of the term inside the bracket is nonnegative. Hence, it follows that

$$H(\mathrm{x}|\mathrm{z}) - H(\mathrm{x}|\mathrm{y}) \ge 0$$

From the relationship for $I(\mathrm{x}|\mathrm{y})$ and $I(\mathrm{x}|\mathrm{z})$, it immediately follows that

$$I(\mathrm{x}|\mathrm{y}) \ge I(\mathrm{x}|\mathrm{z})$$

**15.5-1** We have $H(x) = \int_{-M}^{M} p \log \frac{1}{p} dx = \int_{-M}^{M} -p \log p dx$ and $\int_{-M}^{M} p dx = 1$

Thus,

$$F(x,p) = -p \log p \text{ and } \frac{\partial F}{\partial p} = -(1 + \log p)$$

$$\phi_1(x,p) = p \text{ and } \frac{\partial \phi_1}{\partial p} = 1$$

Substituting these quantities in Equation 15.37, we have

$$-(1 + \log p) + \alpha_1 = 0 \Rightarrow p = e^{\alpha_1 - 1}$$

and

$$\int_{-M}^{M} p dx = \int_{-M}^{M} e^{\alpha_1 - 1} dx = 2M\left(e^{\alpha_1 - 1}\right) = 1$$

Hence,

$$e^{\alpha_1 - 1} = \frac{1}{2M} \text{ and } p(x) = \frac{1}{2M}$$

Also,

$$H(x) = \int_{-M}^{M} p(x) \log \frac{1}{p(x)} dx = \int_{-M}^{M} \frac{1}{2M} \log 2M dx = \log 2M$$

**15.5-2** We have $H(x) = -\int_0^\infty p \log p dx$, $\quad A = \int_0^\infty xp dx$, $\quad 1 = \int_0^\infty p dx$

$$F(x,p) = -p \log p \text{ and } \frac{\partial F}{\partial p} = -(1 + \log p)$$

$$\phi_1(x,p) = px \text{ and } \frac{\partial \phi_1}{\partial p} = x$$

$$\phi_2(x,p) = p \text{ and } \frac{\partial \phi_2}{\partial p} = 1$$

Substituting these quantities in Equation 15.37, we have

$$-(1 + \log p) + \alpha_1 x + \alpha_2 = 0$$

or

$$p = e^{\alpha_1 x + \alpha_2 - 1} = (e^{\alpha_2 - 1}) e^{\alpha_1 x}$$

Substituting this relationship in earlier constraints, we get

$$1 = \int_0^\infty p dx = \int_0^\infty e^{\alpha_2 - 1} e^{\alpha_1 x} dx = \frac{-e^{\alpha_2 - 1}}{\alpha_1}; \begin{cases} \text{Hence,} \\ e^{\alpha_2 - 1} = -\alpha_1 \\ p(x) = -\alpha_1 e^{\alpha_1 x} \end{cases}$$

and

$$A = \int_0^\infty xp dx = \int_0^\infty -\alpha_1 x e^{\alpha_1 x} dx = -\frac{1}{\alpha_1}$$

Hence,

$$\alpha_1 = -\frac{1}{A} \text{ and } e^{\alpha_2 - 1} = -\alpha_1 = A$$

so

$$p(x) = \begin{cases} \frac{1}{A} e^{-x/A} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

To obtain $H(x)$

$$H(x) = -\int_0^\infty p(x)\log p(x)dx = -\int_0^\infty p\left[-\log A - \frac{x}{A}\log e\right]dx$$

$$= \log A\int_0^\infty p(x)dx + \frac{\log e}{A}\int_0^\infty xp(x)dx$$

$$= \log A + \log e = \log(eA)$$

**15.5-3** Information per picture frame $= 9.96\times10^5$ bits. (See Problem 15.1). For 30 picture frames per second, we need a channel with capacity C given by

$$C = 30\times9.96\times10^5 = 2.988\times10^7 \text{ bits/sec.}$$

But for a white Gaussian noise

$$C = B\log\left(1+\frac{S}{N}\right)$$

We are given $\frac{S}{N} = 50$ db $= 100{,}000$ (Note: $100{,}000 = 50$ db)

Hence,

$$B = 1.8 \text{ MHz}$$

**15.5-4** Consider a narrowband $\Delta f$ where $\Delta f \to 0$ so that we may consider both signal noise power density to be constant (bandlimited white) over the interval $\Delta f$. The signal and noise power $S$ and $N$ respectively are given by

$$S = 2S_s(\omega)\Delta f \quad\text{and}\quad N = 2S_n(\omega)\Delta f$$

The maximum channel capacity over this band $\Delta f$ is given by

$$C_{\Delta f} = \Delta f\log\left[\frac{S+N}{N}\right] = \Delta f\log\left[\frac{S_s(\omega)+S_n(\omega)}{S_n(\omega)}\right]$$

The capacity of the channel over the entire band $(f_1, f_2)$ is given by

$$C = \int_{f_1}^{f_2}\log\left[\frac{S_s(\omega)+S_n(\omega)}{S_n(\omega)}\right]df$$

We now wish to maximize C where the constraint is that the signal power is constant.

$$2\int_{f_1}^{f_2} S_s(\omega)\,df = S \quad \text{(a constant)}$$

Using Equation 15.37, we obtain

$$\frac{\partial}{\partial S_s}\log\left[\frac{S_s+S_n}{S_n}\right] + \alpha\frac{\partial S_s}{\partial S_s} = 0$$

or

$$S_s + S_n = -\frac{1}{\alpha} \quad \text{(a constant)}$$

Thus,

$$S_s(\omega) + S_n(\omega) = -\frac{1}{\alpha}$$

This shows that to attain the maximum channel capacity, the signal power density + noise power density must be a constant (white). Under this condition,

$$C = \int_{f_1}^{f_2}\log\left[\frac{S_s(\omega)+S_n(\omega)}{S_n(\omega)}\right]df = \int_{f_1}^{f_2}\log\left[-\frac{1}{\alpha S_n(\omega)}\right]df$$

$$= (f_2 - f_1)\log\left(-\frac{1}{\alpha}\right) - \int_{f_1}^{f_2}\log\left[S_n(\omega)\right]df$$

$$= B\log\left[S_s(\omega)+S_n(\omega)\right]-\int_{f_1}^{f_2}\log\left[S_n(\omega)\right]df$$

**15.5-5** In this problem, we use the results of Problem 15.5-4. Under the best possible conditions,

$$C = \underbrace{B\log\left[S_s(\omega)+S_n(\omega)\right]}_{\text{constant}}-\int_{f_1}^{f_2}\log\left[S_n(\omega)\right]df$$

We shall now show that the integral $\int_{f_1}^{f_2}\log\left[S_n(\omega)\right]df$ is maximum when $S_n(\omega) =$ constant if the noise is constrained to have a given mean square value (power). Thus, we wish to maximize

$$\int_{f_1}^{f_2}\log\left[S_n(\omega)\right]df$$

under the constraint

$$2\int_{f_1}^{f_2}\log\left[S_n(\omega)\right]df = N \quad \text{(a constant)}$$

Using Equation 15.37, we have

$$\frac{\partial}{\partial S_n}(\log S_n)+\alpha\frac{\partial S_n}{\partial S_n} = 0$$

or

$$\frac{1}{S_n}+\alpha = 0$$

and

$$S_n(\omega) = -\frac{1}{\alpha} \quad \text{(a constant)}$$

Thus, we have shown that for a noise with a given power, the integral

$$\int_{f_1}^{f_2}\log\left[S_n(\omega)\right]df$$

is maximized when the noise is white. This shows that white Guassian noise is the worst possible kind of noise.

# Chapter 16

**16.1-1** $2^{11} \geq \sum_{j=0}^{3} \binom{23}{j} = \binom{23}{0} + \binom{23}{1} + \binom{23}{2} + \binom{23}{3}$

$2048 \geq 1 + 23 + 23 \times 11 + 23 \times 77 = 2048$

**16.1-2** (a) There are $\binom{n}{j}$ ways in which $j$ positions can be chosen from $n$. But for a ternery code, a digit can be mistaken for two other digits. Hence the number of possible errors in $j$ places is

$$\binom{n}{j}(3-1)^j \quad \text{or} \quad 3^n \geq 3^k \sum_{j=0}^{t} \binom{n}{j} 2^j \rightarrow 3^{n-k} \geq \sum_{j=0}^{t} \binom{n}{j} 2^j$$

(b) $(11,6)$ code for $t = 2$

$$3^5 \geq \binom{11}{0} + \binom{11}{1} 2 + \binom{11}{2} 2^2 = 1 + 22 + 220 = 243$$

This is satisfied exactly.

**16.1-3** For $(18,7)$ code to correct up to 3 errors

$$2^{11} \geq \sum_{j=0}^{3} \binom{18}{j} \quad \text{or} \quad 2^{11} \geq \binom{18}{0} + \binom{18}{1} + \binom{18}{2} + \binom{18}{3}$$

$$= 1 + \frac{18!}{17!} + \frac{18!}{2!\,16!} + \frac{18!}{3!\,15!} = 1 + 18 + 153 + 816 = 988$$

$$2^{11} = 2048$$

Hence

$$2^{11} > \sum_{j=0}^{3} \binom{18}{j}$$

Clearly, there exists a possibility of 3 error correcting $(18,7)$ code. Since the Hamming bound is oversatisfied, this code could correct some 4 error patterns in addition to all patterns with up to 3 errors.

**16.2-1** $GH^T = [I_k \;\; P] \begin{bmatrix} P \\ I_m \end{bmatrix}$

$= P \oplus P$

$= 0$

**16.2-2** $c = dG$ where $d$ is a single digit $(0 \text{ or } 1)$.

For $d = 0$

$$c = 0\,[1 \; 1 \; 1] = [0 \; 0 \; 0]$$

For $d = 1$

$$c = 1\,[1 \; 1 \; 1] = [1 \; 1 \; 1]$$

16.2-3 $c = dG$ where $d$ is a single digit (0 or 1).

For $d = 0$

$$c = 0\,[1\ 1\ 1\ 1\ 1] = [0\ 0\ 0\ 0\ 0]$$

For $d = 1$

$$c = 1\,[1\ 1\ 1\ 1\ 1] = [1\ 1\ 1\ 1\ 1]$$

Hence in this code a digit repeats 5 times. Such a code can correct up to two errors using majority rule for detection.

16.2-4 0 is transmitted by $[0\ 0\ 0]$ and 1 is transmitted by $[1\ 1\ 1]$

(a) This is clearly a systematic code with

$$G = [1\ 1\ 1]$$

16.2-5 (a)

$$G = \begin{bmatrix} 1\,0\,0 \cdots 0 & 1 \\ 0\,1\,0 \cdots 0 & 1 \\ \cdots\cdots & \cdot \\ \underbrace{0\,0\,0 \cdots 1}_{I_k} & \underbrace{1}_{P} \end{bmatrix} \qquad P = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \qquad \text{Note that } m = 1$$

(b)

| Data word | | | Code word | | | |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 | 0 | 1 | 1 |
| 0 | 1 | 0 | 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | 0 | 1 | 1 | 0 |
| 1 | 0 | 0 | 1 | 0 | 0 | 1 |
| 1 | 0 | 1 | 1 | 0 | 1 | 0 |
| 1 | 1 | 0 | 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 |

(c) This is a parity check code. If a single error occurs anywhere in the code word, the parity is violated. Therefore this code can detect a single error.

(d) Equation (16.9a) in the text shows that $cH^T = 0$.

Now

$$r = c \oplus e$$

and

$$rH^T = (c \oplus e)H^T = cH^T \oplus eH^T = eH^T$$

If there is no error $e = 0$ and

$$rH^T = eH^T = 0$$

Also

$$H^T = \begin{bmatrix} P \\ I_m \end{bmatrix}. \text{ But since } m = 1,\ I_m = [1]$$

and

$$H^T = \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \\ 1 \end{bmatrix}$$

If there is a single error in the received word $r$, $e$ has a single 1 element with all other elements being 0. Hence

$rH^T = eH^T = 1$ (for single error)

**16.2-6**

| Data word | | | Code word | | | | | |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 |
| 0 | 1 | 0 | 1 | 1 | 1 | 0 | 1 | 0 |
| 0 | 1 | 1 | 0 | 0 | 1 | 0 | 1 | 1 |
| 1 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 1 |
| 1 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 0 |
| 1 | 1 | 0 | 1 | 0 | 0 | 1 | 1 | 1 |
| 1 | 1 | 1 | 0 | 1 | 0 | 1 | 1 | 0 |

From this code we see that the distance between any two code words is at least 3. Hence $d_{\min} = 3$.

**16.2-7**

| Data word | | | Code word | | | | | |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 | 0 | 1 | 1 | 1 | 0 |
| 0 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 1 |
| 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 1 |
| 1 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 1 |
| 1 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 0 |
| 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 |

Observe that $d_{\min} = 3$

**16.2-8** $H^T$ is a $15 \times 4$ matrix with all distinct rows. One possible $H^T$ is:

$$H^T = \begin{bmatrix} 1\,1\,1\,1 \\ 1\,1\,1\,0 \\ 1\,1\,0\,1 \\ 1\,1\,0\,0 \\ 1\,0\,1\,1 \\ 1\,0\,1\,0 \\ 1\,0\,0\,1 \\ 0\,0\,1\,1 \\ 0\,1\,1\,1 \\ 0\,1\,1\,0 \\ 0\,1\,0\,1 \\ 1\,0\,0\,0 \\ 0\,1\,0\,0 \\ 0\,0\,1\,0 \\ 0\,0\,0\,1 \end{bmatrix} = \begin{bmatrix} P \\ I_m \end{bmatrix}$$

$$G=\begin{bmatrix} I_k & P \end{bmatrix}=\begin{bmatrix}
1&0&0&0&0&0&0&0&0&0&0&1&1&1&1\\
0&1&0&0&0&0&0&0&0&0&0&1&1&1&0\\
0&0&1&0&0&0&0&0&0&0&0&1&1&0&1\\
0&0&0&1&0&0&0&0&0&0&0&1&1&0&0\\
0&0&0&0&1&0&0&0&0&0&0&1&0&1&1\\
0&0&0&0&0&1&0&0&0&0&0&1&0&1&0\\
0&0&0&0&0&0&1&0&0&0&0&1&0&0&1\\
0&0&0&0&0&0&0&1&0&0&0&0&0&1&1\\
0&0&0&0&0&0&0&0&1&0&0&0&1&1&1\\
0&0&0&0&0&0&0&0&0&1&0&0&1&1&0\\
0&0&0&0&0&0&0&0&0&0&1&0&1&0&1
\end{bmatrix}$$

For $d = 1\,0\,1\,1\,1\,0\,1\,0\,1\,0\,1$

$$c = dG = [1\,0\,1\,1\,1\,0\,1\,0\,1\,0\,1]\,G = 1\,0\,1\,1\,1\,0\,1\,0\,1\,0\,1\,1\,1\,1\,0$$

**16.2-9** **(a)**

$$G=\begin{bmatrix}1&0&0&1&1&1\\0&1&0&1&1&0\\ \underbrace{0\;0\;1}_{I_k}&&&\underbrace{1\;0\;1}_{P}\end{bmatrix} \qquad \& \qquad H^T=\begin{bmatrix}1&1&1\\1&1&0\\1&0&1\\1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

**(b)**

| Data word | | | Code word | | | | | |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 1 |
| 0 | 1 | 0 | 0 | 1 | 0 | 1 | 1 | 0 |
| 0 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 1 |
| 1 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 1 |
| 1 | 0 | 1 | 1 | 0 | 1 | 0 | 1 | 0 |
| 1 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 1 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 |

**(c)** The minimum distance between any two code words is 3. Hence, this is a single error correcting code. Since there are 6 single errors and 7 syndromes, we can correct all single errors and one double error.

**(d)** $s = eH^T$

| $e$ | | | | | | $s$ | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 |
| 0 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 0 |
| 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 |
| 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 |
| 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 |
| 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 |
| 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 1 |

(e)

| $r$ | $s$ | $e$ | $c$ | $d$ |
|---|---|---|---|---|
| 101100 | 110 | 010000 | 111100 | 111 |
| 000110 | 110 | 010000 | 010110 | 010 |
| 101010 | 000 | 000000 | 101010 | 101 |

**16.2-10** (a) done in Prob. 16.2-7

(b) $$H^T = \begin{bmatrix} 0\,1\,1 \\ 1\,0\,1 \\ 1\,1\,0 \\ 1\,0\,0 \\ 0\,1\,0 \\ 0\,0\,1 \end{bmatrix}$$

| | $e$ | $s$ |
|---|---|---|
| | 100000 | 011 |
| | 010000 | 101 |
| six single errors | 001000 | 110 |
| | 000100 | 100 |
| | 000010 | 010 |
| | 000001 | 001 |
| 1 double error | 100100 | 111 |

**16.2-11**

$$G = [I_k\ P] = \begin{bmatrix} 1\,0\,0\,0\,1\,0\,1 \\ 0\,1\,0\,0\,1\,1\,1 \\ 0\,0\,1\,0\,0\,1\,1 \\ 0\,0\,0\,1\,1\,1\,0 \end{bmatrix}$$

$$c = dG$$

$$\begin{array}{cc} \underline{d} & \underline{c} \\ \begin{bmatrix} 0000 \\ 0001 \\ 0010 \\ 0011 \\ 0100 \\ 0101 \\ 0110 \\ 0111 \\ 1000 \\ 1001 \\ 1010 \\ 1011 \\ 1100 \\ 1101 \\ 1110 \\ 1111 \end{bmatrix} & \begin{bmatrix} 0000000 \\ 0001110 \\ 0010011 \\ 0011101 \\ 0100111 \\ 0101001 \\ 0110100 \\ 0111010 \\ 1000101 \\ 1001011 \\ 1010110 \\ 1011000 \\ 1100010 \\ 1101100 \\ 1110001 \\ 1111111 \end{bmatrix} \end{array} \qquad H^T = \begin{bmatrix} 101 \\ 111 \\ 011 \\ 110 \\ 100 \\ 010 \\ 001 \end{bmatrix}$$

$$s = eH^T$$

$$\begin{array}{cc} \underline{e} & \underline{s} \\ \begin{bmatrix} 0000001 \\ 0000010 \\ 0000100 \\ 0001000 \\ 0010000 \\ 0100000 \\ 1000000 \end{bmatrix} & \begin{bmatrix} 001 \\ 010 \\ 100 \\ 110 \\ 011 \\ 111 \\ 101 \end{bmatrix} \end{array}$$

$s = rH^T$ where $r$ = received code

$c = r \oplus e$

$c$ = corrected code

**16.2-12** We observe that the syndrome for all the three 2-error patterns **100010**, **010100**, or **001001** have the same syndrome namely **111**. Since the decoding table specifies $s = 111$ for $e = 100010$ whenever $e = 100010$ occurs, it will be corrected. The other two patterns will not be corrected. If for example $e = 010100$ occurs, $s = 111$ and we shall read from the decoding table $e = 100010$ and the error is not corrected.

If we wish to correct the 2-error pattern **010100** (along with six single error patterns), the new decoding table is identical to that in Table 16.3 except for the last entry which should be

| $\underline{e}$ | $\underline{s}$ |
|---|---|
| 010100 | 111 |

**16.2-13** From Eq. on P.737, for a simple error correcting code

$$2^{n-k} \geq n+1 \quad \text{or} \quad 2^{n-8} \geq n+1 \rightarrow n-8 \geq \log_2(n+1)$$

This is satisfied for $n \geq 12$. Choose $n = 12$. This gives a $(12, 8)$ code. $H^T$ is chosen to have 12 distinct rows of four elements with the last 4 rows forming an identity matrix. Hence,

$$G = [I_4\ P]$$

$$H^T = \left|\begin{array}{c} 0011 \\ 0101 \\ 0110 \\ 0111 \\ 1001 \\ 1010 \\ 1011 \\ 1100 \\ 1000 \\ 0100 \\ 0010 \\ 0001 \end{array}\right| \begin{array}{l} \left.\begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \\ \end{array}\right\} P \\ \left.\begin{array}{c} \\ \\ \\ \\ \end{array}\right\} I \end{array} \qquad G = \left|\begin{array}{c} 100000000011 \\ 010000000101 \\ 001000000110 \\ 000100000111 \\ 000010001001 \\ 000001001010 \\ 000000101011 \\ 000000011100 \end{array}\right|.$$

The number of non-zero syndromes $= 16 - 1 = 15$. There are 12 single error patterns. Hence we may be able to correct 3 double-error patterns.

| $\underline{s}$ | $\underline{e}$ |
|---|---|
| 0000 | 000000000000 |
| 0011 | 100000000000 |
| 0101 | 010000000000 |
| 0110 | 001000000000 |
| 0111 | 000100000000 |
| 1001 | 000010000000 |
| 1010 | 000001000000 |
| 1011 | 000000100000 |
| 1100 | 000000010000 |
| 1000 | 000000001000 |
| 0100 | 000000000100 |
| 0010 | 000000000010 |
| 0001 | 000000000001 |
| 1111 | 100000010000 |
| 1110 | 001000001000 |
| 1101 | 000000010001 |

**16.2-14**

| Data word | Code word |
|---|---|
| 00 | 000000 |
| 01 | 011011 |
| 10 | 101110 |
| 11 | 110101 |

The minimum distance between any two code words is $d_{\min} = 4$. Therefore, it can correct all 1-error patterns. Since the code oversatisfies Hamming bound it can also correct some 2-error and possibly some 3-error patterns.

$$G = [I_4\ P]$$

$$H^T = \begin{bmatrix} 0011 \\ 0101 \\ 0110 \\ 0111 \\ 1001 \\ 1010 \\ 1011 \\ 1100 \\ 1000 \\ 0100 \\ 0010 \\ 0001 \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}0\\0\\0\\0\\0\\0\\0\\0\end{matrix}}\right\} P \\ \left.\vphantom{\begin{matrix}0\\0\\0\\0\end{matrix}}\right\} I \end{matrix} \qquad G = \begin{bmatrix} 100000000011 \\ 010000000101 \\ 001000000110 \\ 000100000111 \\ 000010001001 \\ 000001001010 \\ 000000101011 \\ 000000011100 \end{bmatrix}.$$

The number of non-zero syndromes $= 16 - 1 = 15$. There are 12 single error patterns. Hence we may be able to correct 3 double-error patterns.

| $\underline{s}$ | $\underline{e}$ |
|---|---|
| 0000 | 000000000000 |
| 0011 | 100000000000 |
| 0101 | 010000000000 |
| 0110 | 001000000000 |
| 0111 | 000100000000 |
| 1001 | 000010000000 |
| 1010 | 000001000000 |
| 1011 | 000000100000 |
| 1100 | 000000010000 |
| 1000 | 000000001000 |
| 0100 | 000000000100 |
| 0010 | 000000000010 |
| 0001 | 000000000001 |
| 1111 | 100000010000 |
| 1110 | 001000001000 |
| 1101 | 000000010001 |

16.2-14

| Data word | Code word |
|---|---|
| 00 | 000000 |
| 01 | 011011 |
| 10 | 101110 |
| 11 | 110101 |

The minimum distance between any two code words is $d_{\min} = 4$. Therefore, it can correct all 1-error patterns. Since the code oversatisfies Hamming bound it can also correct some 2-error and possibly some 3-error patterns.

(b)

$$H^T = \begin{bmatrix} 1\,1\,1\,0 \\ 1\,0\,1\,1 \\ 1\,0\,0\,0 \\ 0\,1\,0\,0 \\ 0\,0\,1\,0 \\ 0\,0\,0\,1 \end{bmatrix} \quad \text{and} \quad s = eH^T$$

| | $e$ | $s$ |
|---|---|---|
| 6 single error patterns | 100000 | 1110 |
| | 010000 | 1011 |
| | 001000 | 1000 |
| | 000100 | 0100 |
| | 000010 | 0010 |
| | 000001 | 0001 |
| 7 double-error patterns | 110000 | 0101 |
| | 101000 | 0110 |
| | 100100 | 1010 |
| | 100010 | 1100 |
| | 100001 | 1111 |
| | 011000 | 0011 |
| | 010010 | 1001 |
| 2 triple-error patterns | 000111 | 0111 |
| | 001101 | 1101 |

16.3-1 Systematic $(7,\,4)$ cyclic code

$g(x) = x^3 + x + 1$

For data 1111 $\quad d(x) = x^3 + x^2 + x + 1$

$x^3\left(x^3 + x^2 + x + 1\right) = x^6 + x^5 + x^4 + x^3$

$$\begin{array}{r l}
 & \quad\; x^3 + x^2 + 1 \\
x^3 + x + 1 \,\big)\!\! & \overline{x^6 + x^5 + x^4 + x^3} \\
 & \underline{x^6 \qquad + x^4 + x^3} \\
 & \qquad x^5 \\
 & \qquad \underline{x^5 + x^3 + x^2} \\
 & \qquad\quad x^3 + x^2 \\
 & \qquad\quad \underline{x^3 + x + 1} \\
 & \qquad\qquad x^2 + x + 1
\end{array}$$

$c(x) = \left(x^3 + x + 1\right)\left(x^3 + x + 1\right) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$

The code word is 11111111

For data 1110 $\quad d(x) = x^3 + x^2 + x$

$$
\begin{array}{r|l}
 & x^3 + x^2 \\
x^3 + x + 1 & x^6 + x^5 + x^4 \\
 & \underline{x^6 \quad + x^4 + x^3} \\
 & \quad x^5 \quad + x^3 \\
 & \quad \underline{x^5 \quad + x^3 + x^2} \\
 & \qquad\qquad x^2
\end{array}
$$

The code word is **1110100**
A similar procedure is used to find the remaining codes (see Table 1).

**(b)** From Table 1 it can be seen that the minimum distance between any two codes is 3. Hence this is a single error correcting code.

| *d* | *c* |
|---|---|
| 1111 | 1111111 |
| 1110 | 1110100 |
| 1101 | 1101001 |
| 1100 | 1100010 |
| 1011 | 1011000 |
| 1010 | 1010011 |
| 1001 | 1001110 |
| 1000 | 1000101 |
| 0111 | 0111010 |
| 0110 | 0110001 |
| 0101 | 0101100 |
| 0100 | 0100111 |
| 0011 | 0011101 |
| 0010 | 0010110 |
| 0001 | 0001011 |
| 0000 | 0000000 |

Table 1

**(c)** There are seven possible non-zero syndromes.

for $\quad e = 1000000$

$$
\begin{array}{r|l}
 & x^3 + x + 1 \\
x^3 + x + 1 & x^6 \\
 & \underline{x^6 + x^4 + x^3} \\
 & \quad x^4 + x^3 \\
 & \quad \underline{x^4 \quad + x^2 + x} \\
 & \qquad x^3 + x^2 + x \\
 & \qquad \underline{x^3 \quad + x + 1} \\
 & \qquad\quad x^2 \quad + 1
\end{array}
$$

$s = 101$

The remaining syndromes are shown in Table 2.

| e | s |
|---|---|
| 1 0 0 0 0 0 0 | 1 0 1 |
| 0 1 0 0 0 0 0 | 1 1 1 |
| 0 0 1 0 0 0 0 | 1 1 0 |
| 0 0 0 1 0 0 0 | 0 1 1 |
| 0 0 0 0 1 0 0 | 1 0 0 |
| 0 0 0 0 0 1 0 | 0 1 0 |
| 0 0 0 0 0 0 1 | 0 0 1 |

Table 2

(d) The received data is 1101100

$r(x) = x^6 + x^5 + x^3 + x^2$

$$
\begin{array}{r|l}
x^3+x+1 & x^6+x^5 \quad +x^3+x^2 \\
 & \underline{x^6 \quad +x^4+x^3} \\
 & \quad x^5+x^4 \quad +x^2 \\
 & \quad \underline{x^5 \quad +x^3+x^2} \\
 & \qquad x^4+x^3 \\
 & \qquad \underline{x^4 \quad +x^2+x} \\
 & \qquad\quad x^3+x^2+x \\
 & \qquad\quad \underline{x^3 \quad\quad x+1} \\
 & \qquad\qquad x^2 \quad +1
\end{array}
$$

$s(x) = x^2 + 1$

$s = 1\,0\,1$

From Table 2

$e = 1\,0\,0\,0\,0\,0\,0$

$c = r \oplus e = 1\,1\,0\,1\,1\,0\,0 \oplus 1\,0\,0\,0\,0\,0\,0 = 0\,1\,0\,1\,1\,0\,0$

Hence $d = 0\,1\,0\,1$

16.3-2 $g(x) = x^{11} + x^9 + x^7 + x^6 + x^5 + x + 1$

$c(x) = d(x)g(x)$

1.

$d_1 = 0\,0\,0\,0\,1\,1\,1\,1\,0\,0\,0\,0$, $d_1(x) = x^7 + x^6 + x^5 + x^4$

$d_2 = 1\,0\,1\,0\,1\,0\,1\,0\,1\,0\,1\,0$, $d_2(x) = x^{11} + x^9 + x^7 + x^5 + x^3 + x$

$c_1(x) = d_1(x)g(x) = x^{18} + x^{17} + x^{13} + x^{12} + x^{11} + x^9 + x^8 + x^7 + x^4$

and

$c_1 = 0\,0\,0\,0\,1\,1\,0\,0\,0\,1\,1\,1\,0\,1\,1\,1\,0\,0\,1\,0\,0\,0\,0$

$c_2(x) = d_2(x)g(x) = x^{22} + x^{18} + x^{17} + x^{15} + x^{13} + x^8 + x^5 + x^4 + x^3 + x^2 + x$

and

$c_2 = 1\,0\,0\,0\,1\,1\,0\,1\,0\,1\,0\,0\,0\,0\,1\,0\,0\,1\,1\,1\,1\,1\,0$

**16.3-3**

$$\begin{array}{r|l} & x^2+1 \\ \hline x+1 & x^3+x^2+x+1 \\ & \underline{x^3+x^2} \\ & \qquad x+1 \\ & \qquad \underline{x+1} \\ & \qquad\quad 0 \end{array}$$

Hence $x^3+x^2+x+1=(x+1)(x^2+1)=(x+1)(x+1)(x+1)=(x+1)^3$

**16.3-4**

$$\begin{array}{r|l} & x^4+x+1 \\ \hline x+1 & x^5+x^4+x^2+1 \\ & \underline{x^5+x^4} \\ & \qquad x^2+1 \\ & \qquad \underline{x^2+x} \\ & \qquad\quad x+1 \\ & \qquad\quad \underline{x+1} \\ & \qquad\qquad 0 \end{array}$$

Hence $x^5+x^4+x^2+1=(x+1)(x^4+x+1)$

Now try dividing $x^4+x+1$ by $x+1$, we get a remainder 1. Hence $(x+1)$ is not a factor of $(x^4+x+1)$. The 2nd-order prime factors not divisible by $x+1$ are $x^2$ and $x^2+x+1$. Since $(x^4+x+1)$ is not divisible by $x^2$, we try dividing by $(x^2+x+1)$. This also yields a remainder 1. Hence $x^4+x+1$ does not have either a first or a second order factor. This means it cannot have a third order factor either. Hence

$$x^5+x^4+x^2+1=(x+1)(x^4+x+1)$$

**16.3-4** Try dividing $x^7+1$ by $x+1$

$$\begin{array}{r|l} & x^6+x^5+x^4+x^3+x^2+x+1 \\ \hline x+1 & x^7+1 \\ & \underline{x^7+x^6} \\ & \quad x^6+1 \\ & \quad \underline{x^6+x^5} \\ & \qquad x^5+1 \\ & \qquad \underline{x^5+x^4} \\ & \qquad\quad x^4+1 \\ & \qquad\quad \underline{x^4+x^3} \\ & \qquad\qquad x^3+1 \\ & \qquad\qquad \underline{x^3+x^2} \\ & \qquad\qquad\quad x^2+1 \\ & \qquad\qquad\quad \underline{x^2+x} \\ & \qquad\qquad\qquad x+1 \\ & \qquad\qquad\qquad \underline{x+1} \\ & \qquad\qquad\qquad\quad 0 \end{array}$$

Therefore $\left(x^7+1\right)=(x+1)\left(x^6+x^5+x^4+x^3+x^2+x+1\right)$

Now try dividing $\left(x^6+x^5+x^4+x^3+x^2+x+1\right)$ by $(x+1)$. It does not divide. So try dividing by $\left(x^2+1\right)$. It does not divide. Try dividing by $\left(x^2+x+1\right)$. It does not divide. Next try dividing by $\left(x^3+1\right)$. It does not divide either. Now try dividing by $\left(x^3+x+1\right)$. It divides. We find

$$\left(x^6+x^5+x^4+x^3+x^2+x+1\right)=\left(x^3+x+1\right)\left(x^3+x^2+1\right)$$

Since $\left(x^3+x^2+1\right)$ is not divisible by $x$ or $x+1$ (the only two first-order prime factors), it must be a third-order prime factor. Hence

$$x^7+1=(x+1)\left(x^2+x+1\right)\left(x^2+x^2+1\right)$$

16.3-6 For a single error correcting $(7,\ 4)$ cyclic code with a generator polynomial

$$g(x)=x^3+x^2+1$$

$k=4 \quad n=7$

$$\begin{bmatrix} x^{k-1}\ g(x) \\ x^{k-2}\ g(x) \\ \cdot \\ \cdot \\ \cdot \\ g(x) \end{bmatrix}=\begin{bmatrix} x^3\ g(x) \\ x^2\ g(x) \\ x\ g(x) \\ g(x) \end{bmatrix}=\begin{bmatrix} x^6+x^5+x^3 \\ x^5+x^4+x^2 \\ x^4+x^3+x \\ x^3+x^2+1 \end{bmatrix}$$

Hence

$$G'=\begin{bmatrix} 1\,1\,0\,1\,0\,0\,0 \\ 0\,1\,1\,0\,1\,0\,0 \\ 0\,0\,1\,1\,0\,1\,0 \\ 0\,0\,0\,1\,1\,0\,1 \end{bmatrix}$$

Each code word is found by matrix multiplication $\quad c=dG'$

$$c=[0\,0\,0\,0]\begin{bmatrix} 1101000 \\ 0110100 \\ 0011010 \\ 0001101 \end{bmatrix}=0000000$$

$$c=[0\,0\,0\,1]\begin{bmatrix} 1101000 \\ 0110100 \\ 0011010 \\ 0001101 \end{bmatrix}=0001101$$

The remaining codes are found in a similar manner. See table below.

| $d$ | $c$ |
|---|---|
| 0000 | 0000000 |
| 0001 | 0001101 |
| 0010 | 0011010 |
| 0011 | 0010111 |
| 0100 | 0110100 |
| 0101 | 0111001 |
| 0110 | 0101110 |
| 0111 | 0100001 |
| 1000 | 1101000 |
| 1001 | 1100101 |
| 1010 | 1110010 |
| 1011 | 1111111 |
| 1100 | 1011100 |
| 1101 | 1010001 |
| 1110 | 1000110 |
| 1111 | 1001011 |

16.3-7 $g(x) = x^3 + x^2 + 1$

The desired form is

$$G' = \begin{bmatrix} 1\,0\,0\,0 \cdots 0 & h_{11}\; h_{21}\; h_{31} \cdots h_{m_1} \\ 0\,1\,0\,0 \cdots 0 & h_{12}\; h_{22}\; h_{32} \cdots h_{m_2} \\ 0\,0\,1\,0 \cdots 0 & h_{13}\; h_{23}\; h_{33} \cdots h_{m_3} \\ \cdots & \cdots \\ \cdots & \cdots \\ \cdots & \cdots \\ \underbrace{0\,0\,0\,0 \cdots 1}_{I_k} & \underbrace{h_{1k}\; h_{2k}\; h_{3k} \cdots h_{m_k}}_{P} \end{bmatrix}$$

$$(k \times k) \qquad\qquad (k \times m)$$

The code is found by using $\qquad c = dG$

Proceeding with matrix multiplication, and noting that

$$0+0=0, \quad 0+1=1+0=1, \quad 1+1=0 \text{ and } 0\times 0=0, \quad 0\times 1=1\times 0=0, \quad 1\times 1=1$$

we get

$$c_{15} = [1\,1\,1\,1] \begin{bmatrix} 1000110 \\ 0100011 \\ 0010111 \\ 0001101 \end{bmatrix} = [1\,1\,1\,1\,1\,1\,1]$$

$$c_{14} = [1\,1\,1\,0]\, G = [1\,1\,1\,0\,0\,1\,0]$$

and so on.

| d | c |
|---|---|
| 1111 | 1111111 |
| 1110 | 1110010 |
| 1101 | 1101000 |
| 1100 | 1100101 |
| 1011 | 1011100 |
| 1010 | 1010001 |
| 1001 | 1001011 |
| 1000 | 1000110 |
| 0111 | 0111001 |
| 0110 | 0110100 |
| 0101 | 0101110 |
| 0100 | 0100011 |
| 0011 | 0011010 |
| 0010 | 0010111 |
| 0001 | 0001101 |
| 0000 | 0000000 |

These results agree with those of Table 16.5

**16.3-8** (a)

$$G' = \begin{bmatrix} 1\,0\,1\,1\,0\,0\,0 \\ 0\,1\,0\,1\,1\,0\,0 \\ 0\,0\,1\,0\,1\,1\,0 \\ 0\,0\,0\,1\,0\,1\,1 \end{bmatrix}$$

(b) The code is found by matrix multiplication. $c = dG'$

In general $g(x) = g_1 x^{n-k} + g_2 x^{n-k-1} + \cdots g_{n-k+1}$

For this case $g_1 = 1,\ \ g_2 = 1,\ \ g_3 = 0,\ \ g_4 = 1$

Since $h_{1k} = g_2,\ h_{2k} = g_3,\ h_{3k} = g_4,$ the fourth row is immediately found. Thus, so far we have

$$G = \begin{bmatrix} \\ \\ \\ 0\,0\,0\,1\,1\,0\,1 \end{bmatrix}$$

Next, to get row 3, use row 4 with one left shift.

$$\begin{bmatrix} \\ \\ 0\,0\,1\,1\,0\,1\,0 \\ 0\,0\,0\,1\,1\,0\,1 \end{bmatrix}$$

The $\dot{1}$ is eliminated by adding row 4 to row 3.

$$\begin{bmatrix} \\ \\ 0\,0\,1\,0\,1\,1\,1 \\ 0\,0\,0\,1\,1\,0\,1 \end{bmatrix}$$

Next for row 2, use row 3 with 1 left shift.

$$\begin{bmatrix} 0101110 \\ 0010111 \\ 0001101 \end{bmatrix}$$

The 1 is eliminated by adding row 4 to row 2.

$$\begin{bmatrix} 0100011 \\ 0010111 \\ 0001101 \end{bmatrix}$$

Next for row 1, use row 2 with 1 left shift.

$$\begin{bmatrix} 1000110 \\ 0100011 \\ 0010111 \\ 0001101 \end{bmatrix}$$

This is the desired form.

| $c$ | $d$ |
|---|---|
| 0000 | 0000000 |
| 0001 | 0001011 |
| 0010 | 0010110 |
| 0011 | 0011101 |
| 0100 | 0101100 |
| 0101 | 0100111 |
| 0110 | 0111010 |
| 0111 | 0110001 |
| 1000 | 1011000 |
| 1001 | 1010011 |
| 1010 | 1001110 |
| 1011 | 1000101 |
| 1100 | 1110100 |
| 1101 | 1111111 |
| 1110 | 1100010 |
| 1111 | 1101001 |

(c) All code words are at a minimum distance of 3 units. Hence this is a single error correcting code.

16.3-9 $g(x) = x^3 + x + 1$. Hence row 4 is 0 0 0 1 0 1 1.

$$G' = \begin{bmatrix} 1011000 \\ 0101100 \\ 0010110 \\ 0001011 \end{bmatrix}$$

| | | |
|---|---|---|
| Row 4 is ok. | 0001011 | ← row 4 |
| Row 3 is left shift of row 4. | 0010110 | ← row 3 |
| For row 2, left shift row 3. | 0101100 | |
| And add row 1 to obtain row 2. | 0100111 | ← row 2 |
| For row 1, left shift row 2. | 1001110 | |
| And add row 1 to obtain row 1. | 1000101 | ← row 1 |

$$G = \begin{bmatrix} 1 0 0 0 1 0 1 \\ 0 1 0 0 1 1 1 \\ 0 0 1 0 1 1 0 \\ 0 0 0 1 0 1 1 \end{bmatrix}$$

**16.4-1** The burst (of length 5) detection ability is obvious. The single error correcting ability can be demonstrated as follows. If in any segment of $b$ digits a single error occurs, it will violate the parity in that segment. Hence we locate the segment where the error exists. This error will also cause parity violation in the augmented segment. By checking which bit in the augmented segment violates the parity, we can locate the wrong bit position exactly.

**16.5-1** The code can correct any 3 bursts of length 10 or less. It can also correct any 3 random errors in each code word.

**16.7-1** $P_{Eu} = kQ\left(\sqrt{2E_b/\mathcal{N}}\right) = 12Q\left(\sqrt{2 \times 9.12}\right) = 9.825 \times 10^{-6}$

$$P_{EC} = \binom{23}{4}\left[Q\left(\sqrt{\frac{2kE_b}{n\mathcal{N}}}\right)\right]^4 = \binom{23}{4}\left[Q\left(\sqrt{9.5165}\right)\right]^4 = 9.872 \times 10^{-9}$$

To achieve a value $9.872 \times 10^{-9}$ for $P_{Eu}$, we need new value $E_b/\mathcal{N}$ say $E_b'/\mathcal{N}$. Then

$$9.872 \times 10^{-9} = kQ\left(\sqrt{\frac{2E_b'}{\mathcal{N}}}\right) = 12Q\left(\sqrt{\frac{2E_b'}{\mathcal{N}}}\right)$$

Hence

$$Q\left(\sqrt{\frac{2E_b'}{\mathcal{N}}}\right) = 0.8227 \times 10^{-9}$$

and

$$\sqrt{\frac{2E_b'}{\mathcal{N}}} = 6.03 \Rightarrow \frac{E_b'}{\mathcal{N}} = 18.18$$

This means $E_b/\mathcal{N}$ must be increased from 9.12 to 18.18 (nearly doubled).