Geometric Brownian Motion Model in Financial Market

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In the modeling of financial market, especially stock market, Brownian Motion play a significant role in building a statisitcal model. In this section, I will explore some of the technique to build financial model using Brownian Motion and write my own code for simulation and model building.

Before the building our code in R, let's first introduce some concepts here in order to understand the inddepth process of Brownian Motion in the financial model.

One of most important concept in building such financial model is to understand the geometric brownian motion, which is a special case of Brownian Motion Process. Its importance is self-evident by its pratical definition. The definition is given by the following:

Definition of Geometric Brownian Motion

A stochastic process S_t is said to follow a Geometric Brownian Motion if it satisfies the following stochstic differential equation:

$$dS_t = uS_t dt + \sigma S_t dW_t$$

Where W_t is a Wiener process (Brownian Motion) and u, σ are constants.

Let's first understand this definition, normally, u is called the percentage drift and σ is called the percentage volatility. So, consdier a Brownian motion trajectory that satisfy this differential equation, the right hand side term $uS_t dt$ controls the "trend" of this trajctory and the term $\sigma S_t dW_t$ controls the "random noise" effect in the trajectory.

Since it is a differential equation, we want to find a solution.
Let's apply the technique of separation of varibles, then the equation becomes:

$$\frac{dS_t}{S_t} = udt + \sigma dW_t$$

Then take the integration of both side

$$\int \frac{dS_t}{S_t} = \int (udt + \sigma dW_t)dt$$

Since $\frac{dS_t}{S_t}$ relates to derivative of $In(S_t)$, so the next step involving the $It\bar{o}$ caluculus and arriving the following equaitons

$$ln(\frac{dS_t}{S_t}) = (u - \frac{1}{2}\sigma^2)t + \sigma W_t$$

Taking the exponential of both side and plugging the intial condition S_0, we obtain the solution. The analytical solution of this geometric brownian motion is given by:

$$S_t = S_0 exp((u - \frac{\sigma^2}{2})t + \sigma W_t)$$

In general, the process above is of solving a stochastic differential equation, and in fact, geometric brownian motion is defined as a stochastic differential equation, interested readers can refer to any book about stochastic differential equation.

Thus given the constant u and σ, we are able to produce a Geometric Brownian Motion solution through out time interval.

Before we start our computer simulation, let explore more mathematical aspects of the geometric brownian motion. First note that given drift rate and volatility rate, we can represent GBM solution in the form

$$S(t) = S_0 e^{X(t)}$$

where $X(t) = (u - \frac{\sigma^2}{2})t + \sigma W_t)$.

Markov Chain Properties of Geometric Brownian Motion

Now we show that Geometric Brownian Motion satisfies the Markov Chain property, given the $S(t) = S_0 e^{X(t)}$ definition above, we have

$$\begin{aligned} S(t+h) &= S_0 e^{X(t+h)} \\ &= S_0 e^{X(t)+X(t+h)-X(t)} \\ &= S_0 e^{X(t)} e^{X(t+h)-X(t)} \\ &= S(t) e^{X(t+h)-X(t)} \end{aligned}$$

Thus the future states $S(t+h)$ depends only on the future increment of the Brownian Motion, namely $X(t+h) - X(t)$, which is independent, so the markov property is proved here.

Expectation and Variance of Geometric Brownian Motion

Now we start to find the expectation and variance of GBM. Given the drift and volatility, the moement generating function is given by

$$M_{X(t)}(s) = E(e^{sX(t)}) = e^{uts + \frac{\sigma^2 ts^2}{2}}$$

Thus the expectation is given by

$$E(S(t)) = E(S_0 e^{X(t)}) = S_0 M_{X(t)}(1) = S_0 e^{(u + \frac{\sigma^2}{2})t}$$

And the variance is given by

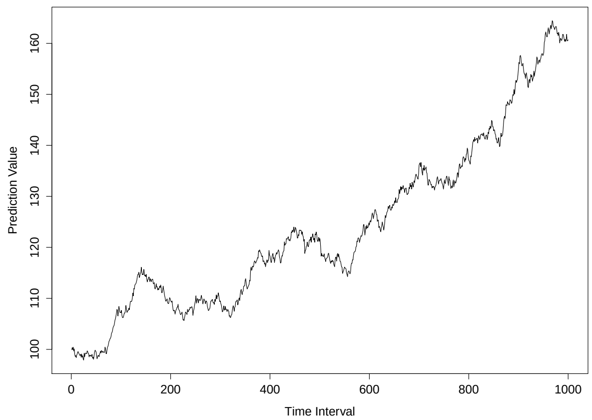
$$Var(S(t)) = E(S^2(t)) - E(S(t))^2) = S_0^2 e^{2ut+\sigma^2 t}(e^{\sigma^2 t} - 1)$$

Now given the Expectation and Variance of the Geometric distribution, we are able to predicted the value mathematically for given problem. But before that let us start our simulation first.

A Simple Simulation of Geometric Brownian Motion

Given the solution of the stochastic differential equation of geometry browanian motion, we are able to define a generating function of it. Note I define the time interval to be divided by number of days in a year.

Now let's try to plot a graph using the function defined above. Note that we choose our σ to be 0.005 and u to be 0.1, the initial price is chosen to be 100. The graph is produced as followings:

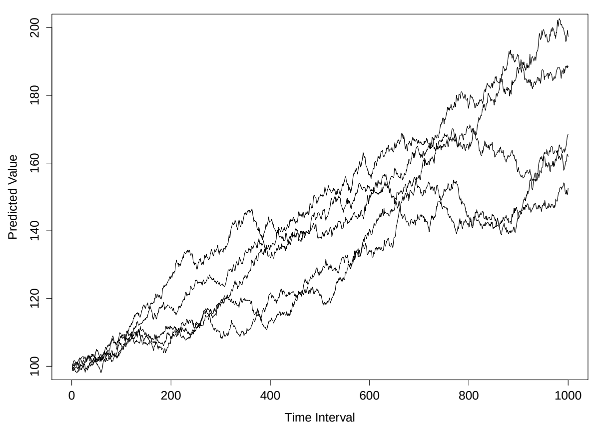


We see this is a pretty nice simulation of future value of some commodity. The time is adjustfied in terms of days. In the above graph, we see that after 1000 days, this commodity have a current price value of about 160.

Unlike the our usual Brownian motion, Geometric Brownian motion is controlled by the "trend". That is if we do hundreds of simulation of Geometric Brownian Motion simulation, most of the graph will "heading toward a direction" with some deviation. To verify this, let's plot several graph of Geometric Brownian Motion with the same constant in a same graph.

For example, We can use the following code to plot several GBM in the same graph to see the trend, where I choose to plot 5 GBMs with interest rate u to be 0.2, and volatility rate to be 0.005 in the graph.

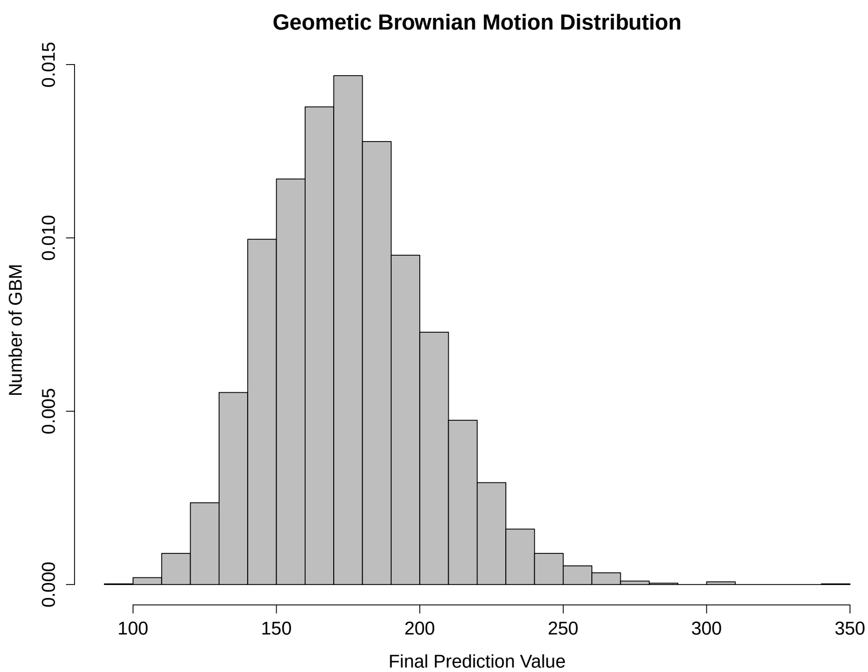
As we see from the graph, the trend of those GBM are roughly matches, it is the volatility factor and the internal random noise of Wiener Process cause these grpah to have different shape.



One of the things interested readers can try out is that try to change the constant of the interested rate factor u and the volatility rate factor σ in the code to see how these input affects the final prediction value. We expect that for any given u and σ, there is an interval of range for each the final predcition value hit into. If we can find this interval of range, we can have a roughly idea about how the price of our commodity will be in the future despite of the random fluctation that affects the price. In the next example, we are going to approach this problem.

Prediction Value Distribution and Interval

For simplicity let's define a Geometric Brownian Motion generating function only produce the final value of the commodity, and use that to estimate our variation of the GBMs.
We First give a histogram about the final predict value of our 5000 samples of GBMs.



We see the most of the predict values of this given commodity after 1000 days targets at 150 - 200 percent in the histogram, this confirms with our intuition. Now let's compute the confidence interval using our simulated sample.

Now we are able to compute the 90 percentage confidence interval for our simulated data set. After computing our simulation code see appendix. Thus we conclude that 90 percent of the time, the value of this given commodity will be in the $(139.3498, 211.7303)$ range after 1000 days.

Thus given a financial product's volatility rate and drift rate, we are able to build its confidence interval of predict value using this method.

We immediately see that the range of this interval is determined by input factors, the interest rate "determines" the mean value, the volatility "determines" the range of the interval, and of course the sample size affects both. Next , I will explore this point.

We first explore the reationship bewteen interest rate and the mean of the predicted value, keeping all other factors the same as in our previous examples. Using the function GBMP defined in the previous example, we are able to obtain the result based on following, Note the R is a vector-oriented language, we take advantage of this to write the following code.

Let's summaries the result using same volatility rate of 0.005, and diferent drift rateand the result is summarized in the following chart

drift rate	0.05	0.1	0.2	0.3	0.5	0.7
sample mean	116.2375	133.3026	175.3168	230.5729	398.8208	689.8385 1193.2103
sample deviation	18.38402	21.08302	27.72795	36.46721	63.07715	109.10424
drift rate	0.9	1	1.5	2	3	4
sample mean	1193.2103	1569.2848	6174.8402	24296.8332	376181.6157	5801592.3335
sample deviation	188.71709	248.19671	976.60729	3842.76573	59496.55292	916363.59795

The result is given above which corresponding the interest rate factor in the first line. Now the realtionship is self-evident. The target mean of the predicted value is of some constant times the expnential of the drift rate. This confirms with our ituitions that the solution of the stochastic differential equation is an exponential function. After doing the same proceure to the deviation of our predicted sample.Here we see not only the mean of our predicted sample have a exponential function relationship with the interest rate factor, but also the standarded deviation of the sample data.

Now let's do the same for the volatility factor σ, we apply the same code with little adjustment, for given drift rate of 0.2 using different volatility rate and the result is summaries in the following chart.

volatility rate	0.001	0.002	0.003	0.005	0.007	0.009	0.015
sample mean	173.1484	173.4942	174.0081	175.5470	177.7840	180.7469	194.4087
sample deviation	5.37930	10.79108	16.25980	27.46857	39.21437	51.72321	96.50318
volatility rate	0.02	0.03	0.04	0.05	0.1	0.15	0.2
sample mean	212.1958	272.1676	385.3663	601.5589	18650.59	1893718.91	292767745.02
sample deviation	147.33295	323.49073	722.70865	1693.58487	227396.1	42935936	8865575359.9

We see that the volatility rate have a much large affect on the predicted result relative to its change. It is might still be some sort of exponential realtionship between mean, standarded deviation and the volatility rate, but the scale effeect is much larger in this case. Normally, the volatility rate is taken to be small to keep the deviation not going "wild".

Recall that the expectation of GBM is given by

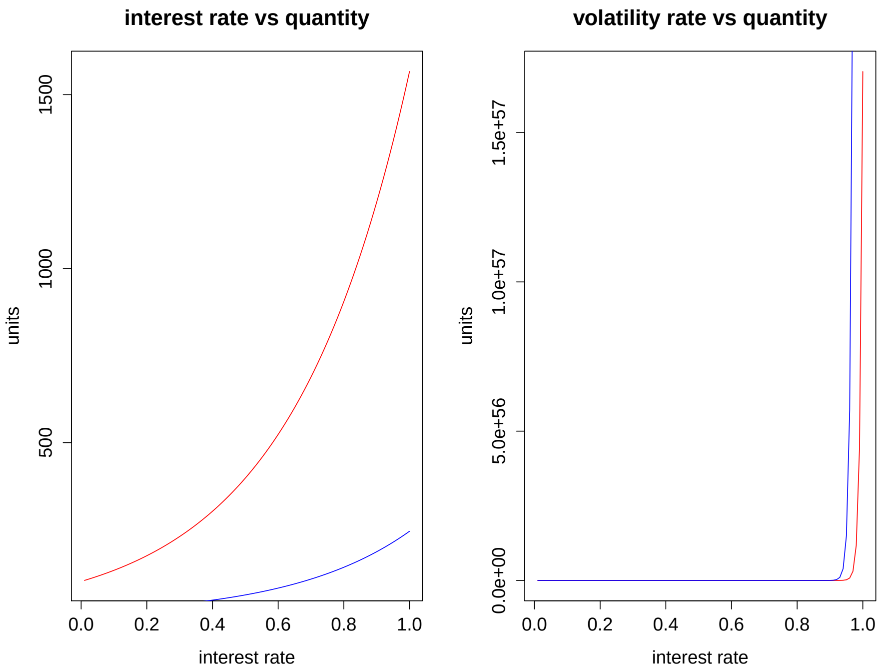
$$E(S(t)) = S_0 e^{(u+\frac{\sigma^2}{2})t}$$

And the variance of GBM is given by

$$Var(S(t)) = S_0^2 e^{2ut+\sigma^2 t}(e^{\sigma^2 t} - 1)$$

if we plug the data into our mathematical formula, we see the patterns we obtain from simulation is quiet match with our mathematical model of the geometric brownian motion. This verifies the accuracy of our simulated prediction results.

For a better understanding of the picture, let's plot a comparison of two rate as the following graph, where the red line representing the mean of our predicted sample (with size 5000), and the blue line representing the standard deviation of our predicted sample. The following graph give a rough idea about how those two factors affects our model prediction, given the other factor being fixe as in the previous examles. For interested readers, you can build a general linear regression model to explore more about the realtionship.



Now we have explored the Geometric Brownian Motion model in financial market in some details. We have done the simulation that quiet realistic to model the prediction value in future time given the parameters for draft and volatility rate of the commodity. However, a general question arises often in statistics is that given a set of data, how can we estimate the model's parameter. in our case, the quesiton here is that given a unknown price trend for a commodity, if we assumes it follows a geometric brownian motion model, how can we estimate the parameters in order to predict the future value.

In the recent years, there are a lot of published papers based on how to estimate the parameters (volatility and drift rate) with different methods. Interested readers can find those papers online without any difficult. Before reading any of those papers, one of the general method come to my mind is to do the maximal likelihood estimation. Other estimation based on those new published papers may have more accurate results, but here I will explore the maximal likelihood method for our case.

Recall that the Geometric Brownian Motion can be expressed as

$$S_t = S_0 exp((u - \frac{\sigma^2}{2})t + \sigma W_t)$$

We first denote that the density of Geometric Brownian Motion is given by:

$$f(t,x) = \frac{1}{\sigma x (2\pi t)^{\frac{1}{2}}} e^{(-(logx - logx_0 - ut)^2)/(2\sigma^2 t)}$$

Then recall the concept of Maximal Likelihood Estimation (MLE)

Definition of Maximal Likelihood Estimation

Suppose we have a sample $x_1, x_2, ..., x_n$ of n independent and identically distributed observations, coming from a distribution with an unknown probability density function $f(x_1, x_2, ..., x_n|\beta_1, \beta_2, ...)$

The joint density function is given by

$$f(x_1, x_2, ..., x_n|\beta_1, \beta_2, ...) = f(x_1|\beta_1, \beta_2, ...) \times f(x_2|\beta_1, \beta_2, ...) \times .... \times f(x_n|\beta_1, \beta_2, ...)$$

Now we want to find a set of parameters $\beta_1, \beta_2, ...$ that gives the function L maximal value.

$$L(\beta_1, \beta_2, ...|x_1, x_2, ..., x_n) = \prod_{i=1}^{n} f(x_i|\beta_1, \beta_2, ...)$$

Now let's take the logarithm of the left hand side.

$$ln(L(\beta_1, \beta_2, ...|x_1, x_2, ..., x_n)) = \sum_{i=1}^{n} ln(f(x_i|\beta_1, \beta_2, ...))$$

Now taking the derivative of the right hand side of the last equation, we can arrives a set of equations that gives us the critical points (optimal point) for the function, those values are called the maximal likelihood estimation.

We will develop this method into our estimation of Geometric Brownian Motion model in this section.

GBM Model Maximal Likelihood Estimation

Given the density function

$$f(t, x) = \frac{1}{\sigma x(2\pi t)^{\frac{1}{2}}} e^{(-(logx - logx_0 - ut)^2)/(2\sigma^2 t)}$$

Where S_t is a functon that takes t and output S.

Suppose that we have a set of input: $t_1, t_2, ...$ and a set of corresponding output: $S_1, S_2, ...$ And suppose the set of data is maximalized likelihood in the function $L(\Theta)$

We take advantage that Geometric Brownian Motion is a Markov Chain Process and we notice that we can write the probability density function as

$$L(\Theta) = f_\Theta(x_1, x_2, ...) = \prod_{i=1}^{n} f_\Theta(x_i)$$

And taken the logarithm of above we obtain

$$L(\Theta) = \sum_{i=1}^{n} log f_\Theta(x_i)$$

And recall that

$$m = (u - \tfrac{1}{2}\sigma^2) \triangle t \qquad v = \sigma^2 \triangle t$$

Instead of directing doing maximal likelihood for u and σ, where the mle of m and v are easily estimated. and since we have $X_i = logS(t_i) - logS(t_{i-1})$, and the mle of m, v is given by

$$\bar{m} = \sum_{i=1}^{n} x_i/n$$

$$\bar{v} = \sum_{i=1}^{n} (x_i - m)^2/n$$

Thus the corresponding mle of u and v can be obtained after that very easily, and givne the formula above our example to estimate a geometric brownian motion parameters using R is fairly simple, so I will leave that for interested readers.

Geometric Brownian Motion Model with Jumps

We first introduce to the jump process in stochastic models

we define a jump process as a type of stochastic process that has discrete movements, called jumps, rather than small continuous movements, where the notion of jump is common in mathematics. Where we denote our jump process by N_t.

The simplest jump process we have is the standard Possion Process, for which the process is given by

$$N_t = \sum_{k=1}^{\infty} 1_{[T_k,\infty)}(t), where t \in \Re^+$$

Where note the $N_{t_1} - N_{t_2}$ has Possion Distribution with parameters $\lambda(t-s)$.

In general, we have the following

$$P(N_t = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

Now we modify our GBM model to adjustify it to fit some realistic situation. Sometimes in stock market the assets price might be affected by some factors periodically or follow some distribution. In this case, we consider the geometric brownian motion distribution with jumps.

The stochastic differential solution of our GBM with Jumps is defined by

$$\frac{dS_t}{S_t} = udt + \sigma dW_t - \delta dN_t$$

Let's solve the stochastic differential equations, it must satisfy the following

$$S_t - S_0 = \int_0^t uS_u du + \int_0^t \sigma S_u dW - \int_0^t \delta S_u dN_u$$

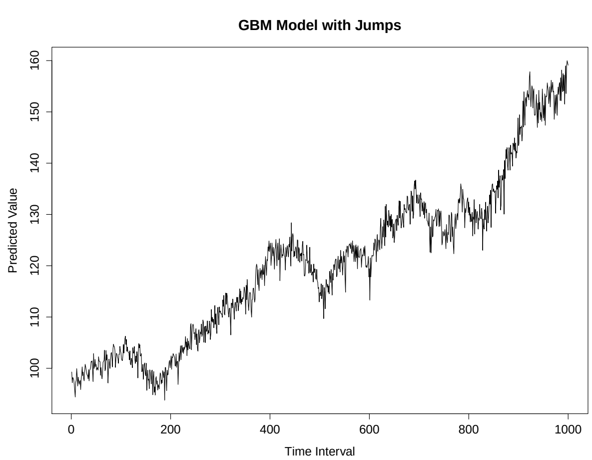
The first parts can be solved conventionally, and the second and third integral we using the same techniques in the previous chapter about $It\bar{o}$ lemma to deal with stochastic differential equation. Thus the genreral solution is given by the following.

$$S_t = S_0 exp((u - \frac{1}{2}\sigma^2)t + \sigma W_t)(1-\delta)^{N_t}$$

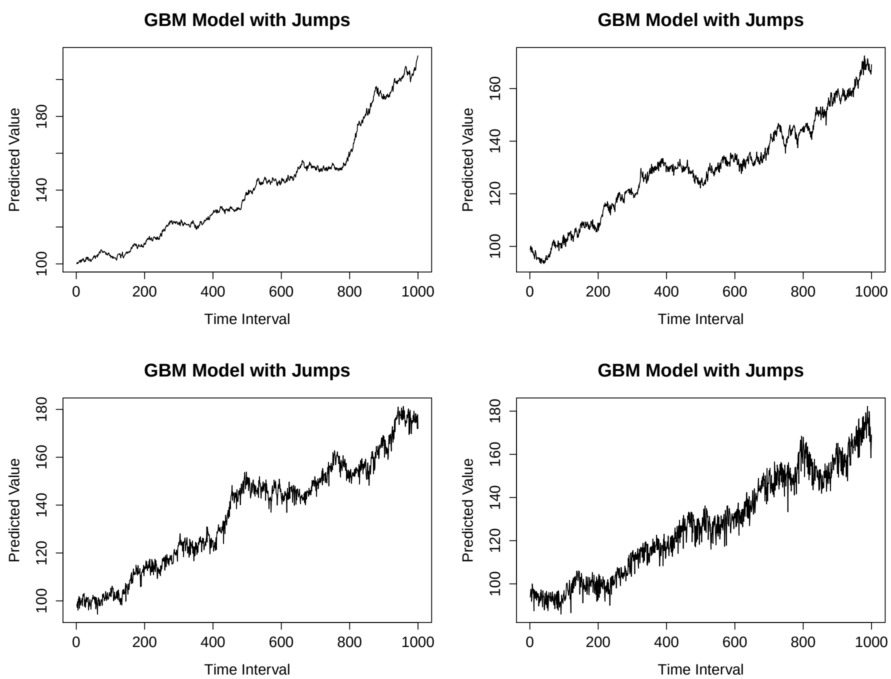
Now given the above information about geometric brownian motion model with jumps, we are able to write ourcode in R and simulate the results.

Given the mathematical formula above , I write the function that produce the solution to GBM Model with Jumps in R see appendix and we plot the graph with following parameters in appendix.

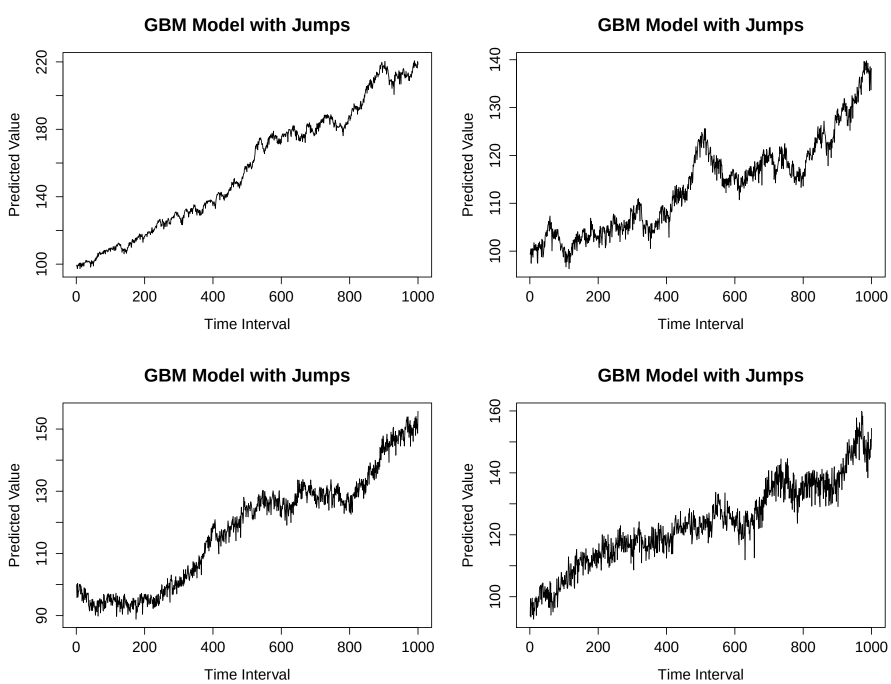
And In order to compare different δ, the fraction rate, we are comparing 4 different jump fraction factor and simulated them to see what effect it has on our model. And simliar we write the following code the plot them to see what effect of the Jump Process (Poisson Process) rate factor λ has on our GBM Model.



The graph that compare the different parameters of the fraction rate with parameters values of 0.002, 0.005, 0.01, 0.02 keeping all ohter factors the same is given as following, it is evidently that the larger the fraction rate, the larger the fluctation of our assets predicted value.



The graph that compare the different parameters of the Jumping process with Possion Process with parameters values of 0.5, 1, 2, 5 keeping all ohter factors the same is given as following, it is evidently that the larger the rate of Poisson Processs, the more frequently the fluctation of our assets predicted value.



Thus given a commodity's volatility and drift rate with the market's fraction rate, we are able to simualted the possible predicted value using our code above. We will not explore more relatinship about those factors in this setion, however interested readers can find more information in financial model predictin papars.

Appendix: Code

1. Geometric Brownian Motion Code

```
GBM = function(N,sigma,u,S0){
Wt = cumsum(rnorm(N,0,1));
t = (1:N)/365;
p1 = (u − 0.5 ∗ (sigma^2)) ∗ t;
p2 = sigma ∗ Wt;
St = S0 ∗ exp(p1 + p2);
return (St);
}
```

2. Geometric Brownian Motion Code with multiply plots

```
P = 5
GBMs = matrix(nrow = P, ncol = N)
for(i in 1:P){
GBMs[i,] = GBM(N,sigma,u,S0);
}

plot(t,GBMs[1,],type= "l", xlim = c(0,1000), ylim=c(100,200), xlab = "Time Interval",ylab="Predicted
Value")
for(i in 2:5){
lines(t,GBMs[i,])
}
```

3. Geometric Brownian Motion Ditribution confidence Interval

```
GBMP = function(N,sigma,u,S0){
Wt = (cumsum(rnorm(N,0,1)));
WN = Wt[N];
p1 = (u - 0.5 ∗ (sigma^2)) ∗ N/365;
p2 = sigma ∗ WN;
St = S0*exp(p1+p2);
return (St);
}

CI = function(data, percent){
m = mean(data);
s = sd(data);
p = qnorm(percent);
ul = m + s*p; ll = m - s*p;
return(c(ll,ul))
}
```

4. Votalility rate and drift rate code

```
us = c(0.05,0.1,0.2,0.3,0.5,0.7,0.9,1,1.5,2,3,4)
l = length(us)
M = 5000; N = 1000; t = 1:N; sigma = 0.005; S0 = 100
GBMPs = matrix(nrow = M, ncol = l);
```

```
for(i in 1:M){
GBMPs[i,] = GBMP(N,sigma,us,S0)
}
apply(GBMPs,2,mean)
apply(GBMPs,2,sd)
```

5. Geometric Brownian Motion Model with Jump

```
GBMJ = function(N,sigma,u,delta,lambda,S0){
Wt = cumsum(rnorm(N,0,1));
Nt = rpois(N,lambda)
t = (1:N)/365;
p1 = (u - 0.5 * (sigma^2)) * t;
p2 = sigma*Wt;
p3 = (1 - delta)^Nt
St = S0 * exp(p1 + p2) * p3;
return (St);
}

N = 1000; t = 1:N; sigma = 0.005; u = 0.2; S0 = 100; delta = 0.01; lambda = 2;
G = GBMJ(N,sigma,u,delta,lambda,S0)
plot(t,G,type= "l",main = "GBM Model with Jumps", xlab = "Time Interval",ylab="Predicted Value")
```

Reference

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