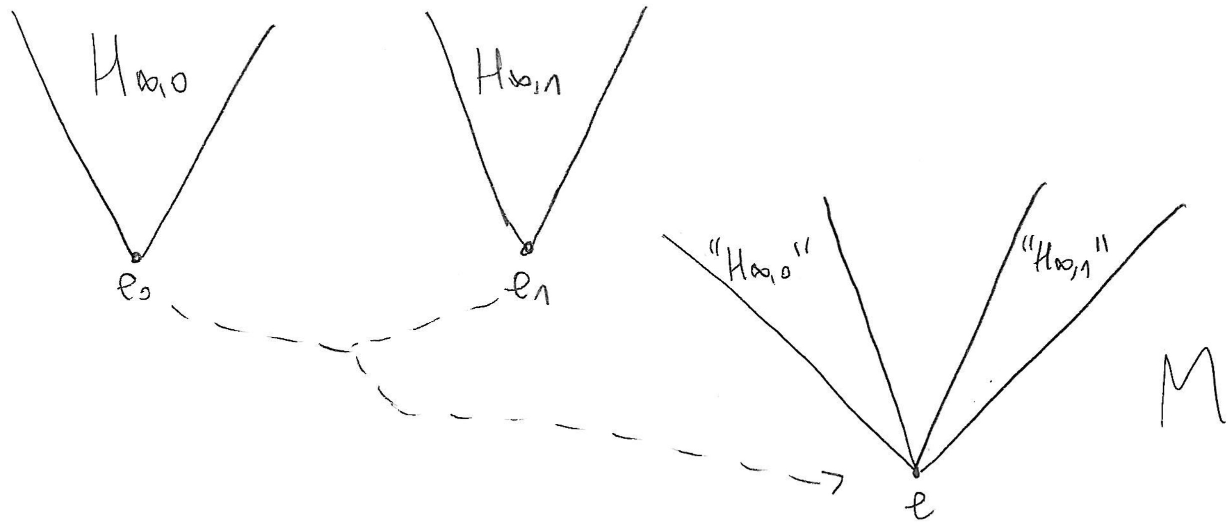

(Un) Let $x \in V$, and let us prove

$$\exists u \in V\ \forall y \in V (y E u \leftrightarrow \exists z \in V (z E x \wedge y E x))$$

- If $x = e$ or x has e as it's only E-predecessor, then $u = e$ works:

$$\forall y \in V (\underbrace{y E e}_{\text{always false}} \longleftrightarrow \underbrace{\exists z \in V (\underbrace{z E x}_{\substack{\text{either always false} \\ \text{or only true for } z=e \text{, so}}} \wedge y E x)}_{\text{always false}}$$

- Otherwise, $x \in V_{\infty,i} \setminus \{e_i\}$ for some i.
Since $H_{\infty,i} \models (\mathrm{Un})$, we have

(1) $$\exists u \in V_{\infty,i}\ \forall y \in V_{\infty,i} (y E_{\infty,i} u \longleftrightarrow \underbrace{\exists z \in V_{\infty,i} (z E_{\infty,i} x \wedge y E_{\infty,i} x))}_{\circledast}$$

Since $\circledast$ is true for at least one $y \in V_{\infty,i} \setminus \{e\}$ (since we are in the "otherwise" case), we have $u \neq e_i$.

Claim: $\forall y \in V (y E u \leftrightarrow \exists z \in V (z E x \wedge y E z))$.

Let $y \in V$.

Case 1: $y = e$

Then $e E u \iff e_i E_{\infty,i} u$

$\overset{(1)}{\iff} \exists z \in V_{\infty,i} (z E_{\infty,i} x \wedge e_i E_{\infty,i} z)$

(in particular $z \neq e_i$)

$\iff \exists z \in V (z E x \wedge e E z)$

(here we use that no E-relation exists between elements of $V_{\infty,0}$ and $V_{\infty,1}$)

Case 2: $y \in V_{\infty,1-i} \setminus \{e_{1-i}\}$ (ie $y \in V_{\infty,j} \setminus \{e_j\}$ for $j \neq i$)

Then $y E u \iff$ never

$\iff \underbrace{\exists z \in V (z E x \wedge y E z)}_{\text{impossible since } y, x \text{ are on different "sides" of } M.}$

Case 3: $y \in V_{\infty,i} \setminus \{e_i\}$

Then $y E u \iff y E_{\infty,i} u$

$\overset{(1)}{\iff} \exists z \in V_{\infty,i} (z E_{\infty,i} x \wedge y E_{\infty,i} z)$

(in particular $z \neq e_i$)

$\iff \exists z \in V (z E x \wedge y E z)$

(again, this uses the fact that no E relation "crosses sides" in M)

So we are done.

(Sing) Let $x \in V$, and let us prove
$$\exists s \in V\ \forall y \in V\ (y E s \leftrightarrow y = x)$$
• If $x = e$, we know from $H_{\infty,0} \models (\text{Sing})$ that there exists $s \in V_{\infty,i}$ satisfying

(2) $$\forall y \in V_{\infty,i}\ (y E_{\infty,0} s \leftrightarrow y = e_0)$$

(intuitively, $s = \{e_0\}$)

Hence $s \neq e_0$, thus $s \in V_{\infty,0} \setminus \{e_0\}$

<u>Claim:</u> $\forall y \in V\ (y E s \leftrightarrow y = e)$

Indeed, $e E s$ since $e_0 E_{\infty,0} s$, and for $y \neq e$ we have two cases:

i) $y \in V_{\infty,0} \setminus \{e_0\}$

Then $y E s \iff y E_{\infty,0} s \overset{(2)}{\iff} y = e_0 \iff$ false!

Hence $y \not\!E s$

ii) $y \in V_{\infty,1} \setminus \{e_1\}$

Then $y \not\!E s$ since these vertices are on different "sides" of M.

So we are done with the case $x = e$

• If $x \neq e$, then again $x \in V_{\infty,i} \setminus \{e_i\}$ and $H_{\infty,i} \models (\text{Sing})$ gives
$$\exists s \in V_{\infty,i}\ \forall y \in V_{\infty,i}\ (y E_{\infty,i} s \leftrightarrow y = x) \qquad (3)$$

Again, we have $s \in V_{\infty,i} \setminus \{e_i\}$, so we can make the following...

<u>Claim</u>: $\forall y \in V (y E s \leftrightarrow y = x)$

Indeed, for $y = x$ we have

$$x E s \iff x E_{\infty,i} s \overset{(3)}{\iff} \text{true!}$$

and for $y \neq x$ there are three cases:

(i) $y = e$

Then $y E s \iff e_i E_{\infty,i} s \overset{(3)}{\iff} \text{false!}$

(ii) $y \in V_{\infty,j} \setminus \{e_j\}$ for $j \neq i$

Then $y E s \iff \text{false}$

(iii) $y \in V_{\infty,i} \setminus \{e_i\}$

Then $y E s \iff y E_{\infty,i} s \overset{(3)}{\iff} y = x$

So we are done with the case $x \neq e$ as well.

(Sep) We can prove something stronger: every subclass of a vertex is coextensional to a vertex.

Let C be a subclass of $v \in V$.

If $v=e$, then C is empty, and v and C are therefore coextensional.

If $v \neq e$, then $v \in V_{\infty,i} \setminus \{e_i\}$ for some i.
In this case, note that C is contained in $(V_{\infty,i} \setminus \{e_i\}) \cup \{e\}$

Let C' be $\begin{cases} C, & \text{if } e \text{ is not in } C \\ (C \setminus \{e\}) \cup \{e_i\}, & \text{if } e \text{ is in } C. \end{cases}$ (4)

Note that we have that

$$C \text{ is } \begin{cases} C', & \text{if } e_i \text{ is not in } C' \\ (C' \setminus \{e_i\}) \cup \{e\}, & \text{if } e_i \text{ is in } C' \end{cases}$$

So C' is a subclass of v in $H_{\infty,i}$, so as we saw in class there exists $s \in V_{\infty,i}$ which is coextensional to C' (in $H_{\infty,i}$) (5)
Since C' is not empty (because C isn't), we must have $s \neq e_i$, i.e., $s \in V_{\infty,i} \setminus \{e_i\}$.

Claim: s is coextensional to C (in M)
Indeed $e \, E \, s \iff e_i \, E_{\infty,i} \, s$
$\overset{(5)}{\iff} e_i$ is in C'
$\overset{(4)}{\iff} e$ is in C,

and for $y \neq e$ we have

$$
\begin{aligned}
y E s &\Longleftrightarrow y \in V_{\infty,i} \setminus \{e_i\} \text{ and } y E_{\infty,i} s \\
&\overset{(5)}{\Longleftrightarrow} \quad " \quad \text{ and } y \text{ is in } C' \\
&\overset{(4)}{\Longleftrightarrow} y \text{ is in } C
\end{aligned}
$$

So we are done.

$\boxed{\neg(\mathrm{Ext})}$ Since $H_{\infty,0}$ and $H_{\infty,1}$ both satisfy (Sing), there exist $s_0 \in V_{\infty,0}$ and $s_1 \in V_{\infty,1}$ such that

$$\forall y \in V_{\infty,0} \, (y E_{\infty,0} s_0 \longleftrightarrow y = e_0)$$

and $$\forall y \in V_{\infty,1} \, (y E_{\infty,1} s_1 \longleftrightarrow y = e_1)$$

In particular $s_0 \neq e_0$ and $s_1 \neq e_1$, so $s_0, s_1 \in V$, and $s_0 \neq s_1$ since $H_{\infty,0}$ and $H_{\infty,1}$ are disjoint. However, we have

$$\forall y \in V \, (y E s_0 \longleftrightarrow y = e)$$

and $$\forall y \in V \, (y E s_1 \longleftrightarrow y = e),$$

so $$\forall y \in V \, (y E s_0 \longleftrightarrow y E s_1).$$

$\boxed{\neg \mathrm{BinUn}}$ Take $x \in V_{\infty,0} \setminus \{e_0\}$ $y \in V_{\infty,1} \setminus \{e_1\}$ such that there exist $x_0 \in V_{\infty,0} \setminus \{e_0\}$ and $y_0 \in V_{\infty,1} \setminus \{e_1\}$ with $x_0 E x$ and $y_0 E y$.

Now, suppose for a contradiction that there exists $b \in V$ satisfying

$$\forall z \in V \, (z E b \longleftrightarrow z E x \vee z E y).$$

Thus, we get $x_0 E b$ and $y_0 E b$, which is impossible since at least one of these E-relation "switches sides" in M. ↯

$\boxed{\neg(\text{Pair})}$ We know that (Un) + (Pair) imply (BinUn), and $M \models$ (Un) but $M \not\models$ (BinUn), hence

$$M \not\models (\text{Pair})$$

$\boxed{\neg(\text{Pow})}$ Take s_0, s_1 from the proof of $\neg$(Ext) above. If, for a contradiction, the powerset p of s_0 existed, we would have

$$s_0 \mathrel{E} p \qquad (6)$$
$$\text{and } s_1 \mathrel{E} p,$$

since both $s_0 \subseteq s_0$ and $s_1 \subseteq s_0$ are true. But at least one of the E-relations in (6) "switches sides" in M. $\lightning$