Homework Set #5

Axiomatische Verzamelingentheorie
2013/14: 2nd Semester; block b
Universiteit van Amsterdam

`http://hugonobrega.github.io/teaching/AxVT/`

The *preferred method* for submitting homework solutions is by handing them in before the start of the *werkcollege* on Wednesday morning. Electronic submissions are also possible, by email to `ilin.juli (at) gmail.com` or `hugonobrega (at) gmail.com` before the deadline. These must be in a single, legible PDF file. It is also possible to hand in your homework by putting it into Julia's or Hugo's mailboxes at the ILLC at Science Park 107, but in this case the deadline for submission is at 10:45am on the day of the submission deadline (before Julia and Hugo leave for the *werkcollege*). It is important to respect the strict deadlines stated above; late homework will not be accepted.
This homework set is due on **Wednesday 7 May 2014, before the morning *werkcollege*.**

1. If $(X,<)$ is a strict linear order, we define the *order topology* as follows: for $x \leq y$, we let $\mathrm{Interval}(x,y) := \{z \in X\,;\, x < z < y\}$ (called the *open interval between x and y*) and we call a set $Z \subseteq X$ *open* if it is a union of open intervals (not necessarily a finite union). Then the *order topology* τ is the set of open sets in X. As usual in topology, a point $x \in X$ is called a *τ-limit point* if for every $U \in \tau$ with $x \in U$ there is some $y \neq x$ such that $y \in U$.

   Remember from homework set #4 that we called a function $s : X \to X$ is a *successor function* if for all $x \in X$, we have $x < s(x)$ and there is no x' such that $x < x' < s(x)$. The elements of $\mathrm{ran}(s)$ are called *s-successors*, the elements of $X \backslash \mathrm{ran}(s)$ are called *s-limits*.

   (a) Prove that if $(X,<)$ is a strict wellorder with order topology τ and a successor function s, then a point x is a τ-limit point if and only if it is an s-limit.

   (b) Prove that if $(X,<)$ is a wellorder, then it has a unique successor function.

2. Remember that an ordinal α was called an *initial ordinal* if for no $\beta < \alpha$, we have $\beta \sim \alpha$. Furthermore, an ordinal α was called an *aleph* if there is some ordinal γ such that $\alpha = \aleph_\gamma$. Prove that for infinite ordinals α, α is an initial ordinal if and only if it is an aleph.

3. Let F be an *ordinal operation* (i.e., a formula Φ that behaves like a function and assigns an ordinal to every ordinal). We call F *monotone* if for all α and β, we have that $\alpha \leq \beta$ implies $F(\alpha) \leq F(\beta)$. We call F *continuous* if for all limit ordinals λ, we have that $F(\lambda) = \bigcup\{F(\alpha)\,;\, \alpha < \lambda\}$.

   (a) Show that every monotone and continuous ordinal operation has arbitrarily large fixed points, i.e., for every α there is a $\gamma > \alpha$ such that $F(\gamma) = \gamma$.

   (b) An ordinal γ is called a *principal number of addition* (also: *gamma-number*) if for all $\alpha, \beta < \gamma$, we have $\alpha + \beta < \gamma$. Show that there are arbitrarily large principal numbers of addition, i.e., for every η there is a $\gamma > \eta$ such that γ is a gamma-number.