

[Un] Let  $x \in V$ , and let us prove  $\exists u \in V \ \forall y \in V (y \in u \hookrightarrow \exists z \in V (z \in x \land y \in x))$ If x = e or x has e as its only E-predicesson, then u = e works:

Hyer (y Ee ) FZE V(Z Ex 1 y Ex)

eithur always false
or only true for z=e, so,

alwaystalse always false

Otherwise,  $x \in V_{\infty,i} \setminus \{e_i\}$  for some i. Since  $H_{\infty,i} \models (U_n)$ , we have

(1) JUEVoo, i YyEVo, i (y En, i h > JZEVo, i (Z En, i x 1 y En, i k))

Since Distrue for at least one yeVoirles (since we are in the "other wise" case), we have u + ei.

Let yeV.  Cose 1: y=e  Then e Eu  ei Eo, u  (in padicular 2 + ci)  (here we use that no E-relation 2)  (here we use that no E-relation 2)  (are 2: y & Voin-i len-i) (i.e. y & Voin lent lent nots of Voin and Voin)  Case 2: y & Voin-i len-i) (i.e. y & Voin lent lent lent lent lent lent lent len	Claim: YyEV (YEU => FREV(ZER ~ YEZ)).
Then eEu (1) = Ei Eo, iu  (1) = 32 < Voi (2 Erg ix 1 e; Eo, iz)  (in particular 2 + ex)  (here we use that no E-relation exists bothwar ellements of Voo, o and Voir)  Case 2; y & Voo, in ten, f (se y & Voo, g tej) for j + i)  Then y Eu (3) me ver  (3) = 32 < V(2 Ex 1 y Ez)  impossible since y ix are on different sods of M.  Case 3: y & Voo; tei]  Then y Eu (3) y Eoo, iu  (3) = 32 < V(2 Ex 1 y Ez)  (in particular 2 + e;)  (again, this was the fact that no E relation "crosses sides" in M)	
(in padicular Z = cx)  (here we use that no E-relation exists between elements of Voro and Voro)  (ax2; y ∈ Voo, -x) fer-x) (ine y ∈ Voo, y) feij for J ≠ i)  Then y ∈ u ⇔ nuver  (Szev(Z ∈ x n y ∈ Z)  impossible since y ne are on different since y for are on different since y ∈ No.  (ax2 3: y ∈ Voo, y feij)  Then y ∈ u ⇔ y ∈ No,	Case 1: y=e
(here we use that no E-relation exasts between elements of Voiso and Vois)  Case 2: y \( \text{Voiso} \) \(	= JZEVOII (ZEMIX NE, EMIZ)
Then yEu ( NVen ) JZEV(ZER NYEZ)  impossible since yire are on diffusion  "sides" of M.  Care 3: y < Vori 'Peil  Then yEu ( YEorilu  JZEVori ( ZEorix Ny Eoriz)  (in particular Z +ei)  (again, this uses the fact that no E  relation "crosses sides" in M)	(here we use that no E-relation exists between elements of Vo,0
impossible since yire are on different "sides" of M.  Care 3: y \in Vori 'rei)  Then y \in \in y \in \in y \in	Case 2: y & Vonni lenis (ie y + Vonis leis to j + i)
Case 3: y \( \forall \) y \( \xi_0 \) y \( \xi_0 \) y \( \xi_0 \) y \( \xi_0 \) \( \xi_0 \	Then yEu   mever => Fzev(zErnyEz)
(again, this uses the fact that no E relation "crosses sides" in M)	
relation "chosses sides" in M)	then yEu ( ) yEon u  Fze Voin (ZEon v Ay Eon Z)  (in ponticular Z +ei)  FZEV (ZER N yEZ)
	(again, this uses the fact that no E

(sing) (Let x & V, and let us prove JSEV HYEV (YES > Y=2) •If re=e, we know from Hop, o = (Sing) that there exists SEVoni satisfying (2)  $\forall y \in V_{m,n} (y \in \mathbb{R}_{n,0} s \iff y = e_{0})$ (intrutively, s= feo) Hence S \ e0 , thus S \ V \ \ o , \ \ leo} Claim: tyev (yEs - y=e) Indeed, EES since eo Expos, and for y = e we have two coses: i) y & Vo,0 120} Then yEs & y Enos y = eo € false! Hence yys is) y = Von ?en} Then y \$\forall since these vertices are on different "Sides" of M. So we are done with the case x=e · If x = e, then again x ∈ Vo, i heig and Hor, i = (Sing)

gives JseVooji tyeVooji (yEoo,is ) y= 2) (3)

Again , we have $S \in V_{\infty,i}$ }eif, so we can make the following
Claim: YyEV (yEs = ze) Indeed, for y=ze we have
ntes (3) re Faxis s
and for y + re there are twee cases:
$(\lambda) = e$
Then yES \estantise eitais  (3) false!
(iii) y \ Voij lejj for j\i
Then yEs \ false
(Mi) y & Vo, i leit
Then yts (3) y=x
So we are done with the case sete as well
(Sep) We can prove something stronger: every subclass of a vertex is coextensional to a vertex.
Let C be a subclass of GEV.

If 5=e, then C is empty, and 5 and C are therefore coextensional If  $v \neq e_i$  then  $v \in V_{\infty}$ , lend for some i. In this case, note that C is contained in (Voir Yeif) Usez Let C' be { C, if e is not in C. (4) (C-{eq}) usei], if e is in C. Note that we have that C is {C', if eins not in C' (C', leis) Web, if Ci is in C' So C'is a Subclass of v in Hopi, so as ins saw in class there exists  $S \in V_{0,i}$  which is coextensional to  $C^{1}(\underline{in}, \underline{H_{0,i}})$  (5) Since C' is not empty (because C isn't), we must have s≠ei, i.e., s∈ V∞,i 4eig. Claim: 5 is coextensional to C (in M) e Es (5.) ei Emis ei is im C' e is im C' and for y te we have

yEs (5) ye Vo, i '30,3 and y Ex, is c' (a) y is in C So we are done. T(Ext) Since Ho,0 and Ho,1 both satisfy (Sing), there exist so E Vopo and si E Vooin such that YyeVono (y Eon,o So ) and  $\forall y \in V_{\infty,n} (y \in S_{\infty,n} S_{n} \iff y = e_{n})$ In particular so + eo and s, + e,, so So,  $S_1 \in V_1$  and  $S_0 \neq S_1$  Since  $H_{\omega_1}$ 0 and  $H_{\omega_2}$ 1 are disjoint However, we have  $Yy \in V(y \in S_0 \iff y = e)$ and tyev(yEs1 = y=e), 50 YyEV (YESO > YES). TBinUn) Take RE Voio Yes ye Voon lend such that there exist no E Vojo led and yo E Voon lend with no Ex and yo Ey. Now Suppose for a contradiction that there exists bev satisfying YZEV (ZEb ←>ZER VZEY). Thus, we get 20 Eb and 40 Eb, which is impossible since at least one of these E-relation

"switches sides" in M. V

7(Pain) We know that (Un) + (Pain) imply (BimUn), and M = (Un) but M # (Bin Un), hence M H (Pain) Thow I Take so, so from the proof of n(Ext) above. If I for a contradiction, the powerset p of so existed, we would have and SIEP) since both socso and sieso are true.

But at least one of the E-relations in (6)

"switches sides" in M. M.