Exercise 4   Sheet #3

Claim: (Ext) + (Sep) + (Repl) + (pow) $\Rightarrow$ (pair)

By (Ext) + (Sep) there is a unique empty set. Let's call it $\emptyset$

By (pow) there is a powerset of $\emptyset$. Call it $\mathcal{P}(\emptyset)$. It is easy to see that $\mathcal{P}(\emptyset) = \{\emptyset\}$.

Again by (pow) there is a set $\mathcal{P}\mathcal{P}(\emptyset)$, the powerset of $\mathcal{P}(\emptyset)$ Again, it is easy to see that $\mathcal{P}\mathcal{P}(\emptyset) = \{\emptyset, \{\emptyset\}\}$

Note that by (Ext), we have $\emptyset \neq \{\emptyset\}$.

Let $\varphi(x, y, p_1, c_1, c_2)$

$$:= (x = p \wedge y = c_1) \vee (x \neq p \wedge y = c_2)$$

Then it is easy to see that.

$$\forall p \forall c_1 \forall c_2 \ \forall x \ \exists y \ \Psi(x, y, p, c_1, c_2)$$

and

$$\forall p \forall c_1 \forall c_2 \left(\forall x \left(\Psi(x, y, p, c_1, c_2) \wedge \Psi(x, y', p, c_1, c_2) \longrightarrow y = y' \right) \right),$$

ie. Ψ behaves like a function on the universe; that maps p to c_1 and and everything else to c_2.

Now let a and b be sets.

We want to show that the pair of a and b exists.

Istantiate Ψ with $p = \emptyset$ $c_1 = a$ $c_2 = b$.

By applying (rep') to $z = \{\emptyset, \{\emptyset\}\}$

We get a set $p(a,b)$

s.t.

$$\forall v \left(v \in p(a,b) \longleftrightarrow \exists w \left(w \in z \wedge \Psi(w, v, \emptyset, a, b) \right) \right)$$

i.e.: $\forall v \left(v \in p(a,b) \leftrightarrow \exists w (w \in z \wedge [(w = \emptyset \wedge v = a) \vee (w \neq \emptyset \wedge v = b)] \right)$

ie.: $\forall v \left(v \in p(a,b) \leftrightarrow (v = a) \vee (v = b) \right)$ (since z contains $\emptyset$ and $\{\emptyset\} \neq \emptyset$).

So $p(a,b)$ is the pair of a and b.