MATH3090/7039: Financial mathematics
Assignment 3
Semester I 2024

Due Tuesday May 21 1pm

MATH3090/7039 total marks **30 marks**

Submission:

- Submit onto Blackboard softcopy (i.e. scanned copy) of your assignment solutions. Hardcopies are not required.
- Include all your answers, numerical outputs, figures, tables and comments as required into one single PDF file.

Notation: "Lx.y" refers to [Lecture x, Slide y]

Assignment questions - all students

1. (6 marks) In the one-period binomial model specified in L6.32, suppose $r = 0$, $S_0 = 100$, $S_u = 120$ and $S_d = 80$.

a. (1 marks) Find an equivalent martingale measure (i.e. p_u in L6.36) using $\mathbb{E}(S_T) = S_0$.

b. (2 marks) To this market, suppose we add an additional asset $U = \{U_0, U_T\}$ with $U_0 = 10$ and

$$U_T = \begin{cases} U_u = x & \text{up} \\ U_d = 5 & \text{down} \end{cases}$$

for some $x > 0$. Find the value of x such that the market is arbitrage-free. With this selection of x, is the market complete?

c. (3 marks) Suppose x is different from what you obtained in part (b). Find a Type 1 arbitrage.

2. (6 marks) We revisit the example in L7.6, where $r = 0$, $S_0 = 100$, $S_u = 130$, $S_m = 100$ and $S_d = 80$.

a. (2 marks) Suppose we add an additional asset $U = \{U_0, U_T\}$ with $U_0 = 2$ and

$$U_T = \begin{cases} U_u = 10 & \text{up} \\ U_m = 0 & \text{middle} \\ U_d = 0 & \text{down} \end{cases}$$

to the market. Is the new market arbitrage-free? If so, is it complete? If it is complete, find the (unique) equivalent martingale measure.

b. (2 marks) Repeat part (a) with $U_0 = 5$ (keeping all other parameters the same as in part (a)).

c. (2 marks) Repeat part (a) with

$$U_T = \begin{cases} U_u = 5 & \text{up} \\ U_m = 2 & \text{middle} \\ U_d = 0 & \text{down} \end{cases}$$

(keeping all other parameters the same as in part (a)).

3. (4 marks) In the 2-period model described in L7.18, suppose $r > 0$ such that $e^{-r} = 0.9$.

a. (2 marks) Find the equivalent martingale measure so that $(e^{-rt}S_t)_{t=0,1,2}$ is a martingale.

b. (2 marks) Compute the arbitrage-free price of a 90-call option.

4. (4 marks) Suppose $W = (W_t)_{t\geq 0}$ is a standard Brownian motion. For the following stochastic processes X and Y, derive dX_t and dY_t. Are they martingales with respect to the filtration generated by W?

a. (2 marks) $X_t = e^{\frac{1}{2}t}\sin W_t$, $t \geq 0$.

b. (2 marks) $Y_t = (W_t + t)\exp(-W_t - \frac{1}{2}t)$, $t \geq 0$.

5. (10 marks) a. (2 marks) Let $x_1, x_2, \ldots, x_n$ be a sequence of positive numbers. The geometric average G is defined by

$$G = (x_1 x_2 \cdots x_n)^{1/n}.$$

Show that

$$G = \exp\left(\frac{1}{n}\sum_{j=1}^{n} \log x_j\right).$$

b. (2 marks) Suppose stock price $(S_t)_{t\geq 0}$ follows

$$\frac{dS_t}{S_t} = rdt + \sigma dW_t, \quad t \geq 0,$$

where $(W_t)_{t\geq 0}$ is a standard Brownian motion under risk-neutral measure $\mathbb{P}$, and $S_0 > 0$ is the today's stock price. Here $r > 0$ is the risk-free interest rate.
A geometrically-averaged Asian option pays, at $T > 0$,

$$(\exp(I_T/T) - K)^+,$$

where $K > 0$ is the strike price and

$$I_T = \int_0^T \log S_t dt.$$

Show that

$$I_T = T\log S_0 + (r - \frac{1}{2}\sigma^2)T^2/2 + \sigma\int_0^T W_t dt.$$

c. (3 marks) For every positive integer N, let $\Delta t = T/N$ and $t_k = k\Delta t$, $k = 0, \ldots, N$. We can write

$$\int_0^T W_t dt = \lim_{N\to\infty} A_N$$

where

$$A_N := \sum_{k=0}^{N-1} W_{t_k}\Delta t, \quad N \geq 1.$$

What is the distribution of A_N? Assuming that the distribution of A_N converges to that of $\int_0^T W_t dt$, what is the distribution of I_T?

d. (3 marks) Find the time-zero price of the geometrically-averaged Asian option in terms of the standard normal cumulative distribution function $\mathcal{N}$.