# MATH3090 A3

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## Question 1 [6 Marks]

In the one-period binomial model specified in L6.32, suppose $r = 0, S_0 = 100, S_u = 120$ and $S_d = 80$.

### Part A [1 Mark]

Find an equivalent martingale measure (i.e. $p_u$ in L6.36), using $\mathbb{E}(S_T) = S_0$.
We can compute $p_u$ as

$$\begin{aligned}
S_0 &= \mathbb{E}\left(\frac{S_T}{B_T}\right) && (1)\\
&= \mathbb{E}(S_T) && (2)\\
&= e^{-rT}\left[p_u S_u + (1 - p_u) S_d\right] && (3)\\
100 &= e^{-0\cdot T}\left[p_u 120 + (1 - p_u) 80\right] && (4)\\
100 &= 120p_u + 80 - 80p_u && (5)\\
20 &= 40p_u && (6)\\
p_u &= \frac{1}{2} && (7)
\end{aligned}$$

And hence $p_d = 1 - p_u = 1 - \frac{1}{2} = \frac{1}{2}$. Therefore, we have shown there exist non-zero probabilities of each scenario, and these probabilities have no degree of freedom $\Rightarrow$ there exists an equivalent martingale measure.

## Part B [2 Marks]

To this market, suppose we add an additional asset $U = \{U_0, U_T\}$ with $U_0 = 10$ and

$$U_T = \begin{cases} U_u = x & \text{up} \\ U_d = 5 & \text{down} \end{cases}$$

for some $x > 0$. Find the value of $x$ such that the market is arbitrage-free. With this selection of $x$, is the market complete?

By the no arbitrage assumption, we have that for each asset in the portfolio,

$$X_0^i = e^{-rT}\mathbb{E}\left[X_T^i\right]$$

So then

$$\begin{aligned} U_0 &= e^{-rT}\mathbb{E}\left[U_T\right] && (8) \\ 10 &= e^{-0\cdot T}\left[p_u U_u + (1-p_u)\,U_d\right] && (9) \\ 10 &= \frac{1}{2}x + \frac{1}{2}\cdot 5 && (10) \\ 7.5 &= x\cdot\frac{1}{2} && (11) \\ x &= 15 && (12) \end{aligned}$$

A value of $x = 15$ ensures the market is arbitrage free as we have already shown in (a) a value of $p_u = \frac{1}{2}$ gives an equivalent martingale measure. And by the first fundamental theorem of asset pricing, the existence of an equivalent martingale measure is equivalent to the lack of an arbitrage.

By the second fundamental theorem of asset pricing, an arbitrage free market is complete if and only if there exists a unique martingale measure. We have shown above already that the market is arbitrage free. Furthermore, our solution for $p_u$ is constrained by the asset $S$ to be $p_u = \frac{1}{2}$, and for the asset $U$, $x$ is constrained to $x = 15$. Hence, the martingale we have found

$$\left(\mathbb{E}\left(\frac{S_T}{S_0}\right) = \frac{S_u}{S_0}\cdot p_u + \frac{S_d}{S_0}\cdot(1-p_u)\right) = \frac{S_u}{S_0}\cdot\frac{1}{2} + \frac{S_d}{S_0}\cdot\left(1-\frac{1}{2}\right) = 1$$

This indicates that the market is complete, as there exists a unique martingale measure, satisfying the conditions of the second fundamental theorem of asset pricing.

## Part C [3 Marks]

Suppose $x$ is different from what you obtained in part (b). Find a Type 1 arbitrage.

First consider when $x = 15$. Construct a portfolio consisting of $S$ and $B$ which replicates $U_T$.

$$\Theta = \left(\theta^1, -\theta^2\right)$$

The replicating portfolio's time $T$ value is given by

$$V_T = \theta^1 \cdot S_T - \theta^2 B_T$$

Also set the time-0 value to be equal to $V_0$:

$$U_0 = V_0$$

We then solve

$$10 = \theta^1 \cdot S_0 - \theta^2 \beta_0 \tag{13}$$
$$10 = 100\theta^1 - \theta^2 \tag{14}$$
$$\theta^2 = 100\theta^1 - 10 \tag{15}$$

Consider the up case

$$U_T = \theta^1 \cdot S_T - \theta^2 \cdot B_T$$

Substituting in we obtain

$$U_u = \theta^1 \cdot S_U - \theta^2 \cdot B_u \tag{16}$$
$$15 = 120\theta^1 - \theta^2 \tag{17}$$
$$\theta^2 = 120\theta^1 - 15 \tag{18}$$

then

$$100\theta^1 - 10 = 120\theta^1 - 15 \tag{19}$$
$$20\theta^1 = 5 \tag{20}$$
$$\theta^1 = \frac{1}{4} \tag{21}$$

and

$$\theta^2 = 100\theta^1 - 10 \tag{22}$$
$$= 100\frac{1}{4} - 10 \tag{23}$$
$$= 25 - 10 \tag{24}$$
$$= 15 \tag{25}$$

Now consider the down case:

$$U_d = \theta^1 \cdot S_d - \theta^2 \cdot B_d \tag{26}$$
$$5 = 80\theta^1 - \theta^2 \tag{27}$$

Substituting in we obtain

$$5 = 80\frac{1}{4} - 15 \tag{28}$$
$$5 = 20 - 15 \tag{29}$$
$$5 = 5 \tag{30}$$
$$:) \tag{31}$$

Which is correct. Therefore, in a non-arbitrage market, our portfolio is

$$\Theta = \left(\frac{1}{4}, -15\right)$$

First consider when $x < 15$. In this case we long our replicating portfolio $\boldsymbol{\Theta}$ and short the new asset $U$. At time-0 we have

$$V_0 = \frac{1}{4}S_0 - 15B_0 - U_0 \tag{32}$$
$$= \frac{1}{4}100 - 15 \cdot 1 - 10 \tag{33}$$
$$= 25 - 15 - 10 \tag{34}$$
$$= 10 - 10 \tag{35}$$
$$= 0 \tag{36}$$

And at time-$T$ we have a random outcome (for $U_T$):

$$V_T = \left(\frac{1}{4}S_T - 15 \cdot B_T\right) - U_T$$

In the up case we have

$$V_{T,\text{up}} = \frac{1}{4}S_u - 15B_u - U_u \tag{37}$$
$$= \frac{1}{4} \cdot 120 - 15 - U_u \tag{38}$$
$$= 15 - U_u \tag{39}$$
$$= 15 - x > 0 \tag{40}$$

and in the down case we have

$$V_{T,\text{down}} = \frac{1}{4}S_d - 15B_d - U_d \tag{41}$$
$$= \frac{1}{4} \cdot 80 - 15 \cdot 1 - 5 \tag{42}$$
$$= 20 - 15 - 5 \tag{43}$$
$$= 0 \tag{44}$$

Therefore, we have shown when $x < 15$, it is possible to construct a type-1 arbitrage portfolio which has properties $V_0 = 0$ and $\mathbb{P}(V_T \geq 0) = 1$ and $\mathbb{P}(V_T > 0) > 0$.

Now consider when $x > 15$. We can simply invert / reserve our portfolio such that we long asset $U$ and short our replicating portfolio $\mathbf{\Theta}$. At time-0 we have

$$\begin{aligned} V_0 &= U_0 - \frac{1}{4}S_0 - 15B_0 && (45)\\ &= 10 - \frac{1}{4}100 - 15 \cdot 1 && (46)\\ &= 10 - 25 - 15 && (47)\\ &= 10 - 10 && (48)\\ &= 0 && (49) \end{aligned}$$

and at time $T$ we have a random outcome (again for $U_T$):

$$V_T = U_T - \left(\frac{1}{4}S_T - 15 \cdot B_T\right)$$

In the up case we have

$$\begin{aligned} V_{T,\text{up}} &= U_u - \frac{1}{4}S_u - 15B_u && (50)\\ &= U_u - \frac{1}{4}120 - 15 \cdot 1 && (51)\\ &= U_u - 30 - 15 && (52)\\ &= U_u - 15 > 0 && (53)\\ &= x - 15 > 0 && (54) \end{aligned}$$

since we assumed $x > 15$. And in the down case we have

$$\begin{aligned} V_{T,\text{down}} &= U_d - \left(\frac{1}{4}S_d - 15B_d\right) && (55)\\ &= 5 - \frac{1}{4} \cdot 80 + 15 \cdot 1 && (56)\\ &= 5 - 20 + 15 && (57)\\ &= 0 && (58) \end{aligned}$$

Therefore, we have shown when $x > 15$, it is also possible to construct a type-1 arbitrage portfolio. Note in both cases, we only make money in the up case and remain equal in the down case. Therefore, when $x! = 15$, it is possible to construct a type-1 arbitraging portfolio.

## Question 2 [6 Marks]

We revisit the example in L7.6, where $r = 0, S_0 = 100, S_u = 130, S_m = 100$ and $S_d = 80$.

### Part A [2 Marks]

Suppose we add an additional asset $U = \{U_0, U_T\}$ with $U_0 = 2$ and

$$U_T = \begin{cases} U_u = 10 & \text{up} \\ U_m = 0 & \text{middle} \\ U_d = 0 & \text{down} \end{cases}$$

to the market. Is the new market arbitrage-free? If so, is it complete? If it is complete, find the (unique) equivalent martingale measure.

First consider the asset $S$ and solve so that it's (discounted) expectation equals it's initial value $S_0 = 100$.

$$\begin{aligned} S_0 &= \mathbb{E}\left(\frac{S_T}{B_T}\right) && (59) \\ &= \mathbb{E}\left(S_T\right) && (60) \\ S_0 &= e^{-rT}\mathbb{E}\left[S_T\right] && (61) \\ 100 &= e^{-0 \cdot T}\left[130p_u + 100p_m + 80p_d\right] && (62) \\ 100 &= 130p_u + 100p_m + 80p_d && (63) \end{aligned}$$

Then consider the second asset $U$, and solve for the same.

$$\begin{aligned} U_0 &= e^{-rT}\mathbb{E}\left[U_T\right] && (64) \\ 2 &= e^{-0 \cdot T}\left[10p_u + 0p_m + 0p_d\right] && (65) \\ 2 &= 10p_u && (66) \\ p_u &= \frac{1}{5} && (67) \end{aligned}$$

Then since $\sum_\omega p_\omega = 1$, we have

$$\begin{aligned} 1 &= p_d + p_m + p_u && (68) \\ 1 &= p_d + p_m + \frac{1}{5} && (69) \\ \frac{4}{5} &= p_d + p_m && (70) \\ p_d &= \frac{4}{5} - p_m && (71) \end{aligned}$$

Finally, substituting back in

$$\begin{aligned} 100 &= 130p_u + 100p_m + 80p_d && (72) \\ 100 &= 130 \cdot \frac{1}{5} + 100p_m + 80 \cdot \left(\frac{4}{5} - p_m\right) && (73) \end{aligned}$$

$$100 = 26 + 100p_m + 64 - 80p_m \tag{74}$$
$$10 = 20p_m \tag{75}$$
$$p_m = \frac{1}{2} \tag{76}$$

and

$$p_d = \frac{4}{5} - p_m \tag{77}$$
$$= \frac{4}{5} - \frac{1}{2} \tag{78}$$
$$= \frac{8}{10} - \frac{5}{10} \tag{79}$$
$$= \frac{3}{10} \tag{80}$$

We therefore have

$$(p_d, p_m, p_u) = \left(\frac{3}{10}, \frac{1}{2}, \frac{1}{5}\right)$$

And hence all of $p_d, p_m, p_u$ are non-zero probabilities, meaning there exists an equivalent martingale measure. By the first fundamental theorem of asset pricing, there then exists no arbitrage in the market.

By the second fundamental theorem of asset pricing, the market is complete because the market contains a **unique** equivalent martingale measure by the fact that there is no degree of freedom in the solution to the probabilities above.

## Part B [2 Marks]

Repeat part (a) with $U_0 = 5$ (keeping all other parameters the same as in part (a)).

From part (a), we have the (unchanged) equation

$$100 = 130p_u + 100p_m + 80p_d$$

Now consider the (modified) asset $U$

$$\begin{aligned} U_0 &= e^{-rT}\mathbb{E}\left[U_T\right] && (81)\\ 5 &= e^{-0\cdot T}\left[10p_u + 0p_m + 0p_d\right] && (82)\\ 5 &= 10p_u && (83)\\ p_u &= \frac{1}{2} && (84) \end{aligned}$$

Then since $\sum_\omega p_\omega = 1$, we have

$$\begin{aligned} 1 &= p_d + p_m + p_u && (85)\\ 1 &= p_d + p_m + \frac{1}{2} && (86)\\ \frac{1}{2} &= p_d + p_m && (87)\\ p_d &= \frac{1}{2} - p_m && (88) \end{aligned}$$

Finally, substituting back in

$$\begin{aligned} 100 &= 130p_u + 100p_m + 80p_d && (89)\\ 100 &= 130\cdot\frac{1}{2} + 100p_m + 80\cdot\left(\frac{1}{2} - p_m\right) && (90)\\ 100 &= 65 + 100p_m + 40 - 80p_m && (91)\\ -5 &= 20p_m && (92)\\ p_m &= \frac{-1}{4} && (93) \end{aligned}$$

and (yes technically this part is not necessary given we already determined $p_m \notin (0,1)$ but:)

$$\begin{aligned} p_d &= \frac{1}{2} - p_m && (94)\\ &= \frac{1}{2} - \frac{-1}{4} && (95)\\ &= \frac{2}{4} + \frac{1}{4} && (96)\\ &= \frac{3}{4} && (97) \end{aligned}$$

We therefore have

$$(p_u, p_m, p_d) = \left(\frac{3}{4}, \frac{-1}{4}, \frac{1}{2}\right)$$

Since $p_m \notin (0,1)$, we say $\nexists$ an equivalent martingale measure. By the first fundamental theorem of asset pricing, then $\exists$ arbitrage in the market.

### Part C [2 Marks]

Repeat part (a) with

$$U_T = \begin{cases} U_u = 5 & \text{up} \\ U_m = 2 & \text{middle} \\ U_d = 0 & \text{down} \end{cases}$$

(keeping all other parameters the same as in part (a)).

First consider the asset $S$ and solve so that it's (discounted) expectation equals it's initial value $S_0 = 100$.

$$S_0 = \mathbb{E}\left(\frac{S_T}{B_T}\right) \tag{98}$$
$$= \mathbb{E}\left(S_T\right) \tag{99}$$
$$S_0 = e^{-rT}\mathbb{E}\left[S_T\right] \tag{100}$$
$$100 = e^{-0\cdot T}\left[130p_u + 100p_m + 80p_d\right] \tag{101}$$
$$100 = 130p_u + 100p_m + 80p_d \tag{102}$$

Then consider the second asset $U$, and solve for the same.

$$U_0 = e^{-rT}\mathbb{E}\left[U_T\right] \tag{103}$$
$$2 = e^{-0\cdot T}\left[5p_u + 2p_m + 0p_d\right] \tag{104}$$
$$2 = 5p_u + 2p_m \tag{105}$$
$$2p_m = 2 - 5p_u \tag{106}$$
$$p_m = 1 - \frac{5}{2}p_u \tag{107}$$

Then since $\sum_\omega p_\omega = 1$, we have

$$1 = p_d + p_m + p_u \tag{108}$$
$$1 = p_d + \left(1 - \frac{5}{2}p_u\right) + p_u \tag{109}$$
$$0 = p_d - \frac{3}{2}p_u \tag{110}$$
$$p_d = \frac{3}{2}p_u \tag{111}$$

Finally, substituting back in

$$100 = 130p_u + 100p_m + 80p_d \tag{112}$$
$$100 = 130p_u + 100\left(1 - \frac{5}{2}p_u\right) + 80\cdot\left(\frac{3}{2}p_u\right) \tag{113}$$
$$100 = 130p_u + 100 - 250p_u + 120p_u \tag{114}$$
$$100 = 100 + \left(130 - 250 + 120\right)p_u \tag{115}$$
$$100 = 100 \tag{116}$$

Which is under-determined. We therefore have

$$(p_u, p_m, p_d) = \left(\alpha, 1 - \frac{5}{2}\alpha, \frac{3}{2}\alpha\right)$$

Therefore, for values $\alpha \in \left(0, \frac{2}{5}\right)$, we do obtain an equivalent martingale measure, and by the first fundamental theorem of asset pricing, $\nexists$ arbitrage in the market. However, by the second fundamental theorem of asset pricing, we do not have a **unique** equivalent martingale measure, hence we say the market is **not** complete.

## Question 3 [4 Marks]

In the 2-period model described in L7.18, suppose $r > 0$ such that $e^{-r} = 0.9$.

### Part A [2 Marks]

Find the equivalent martingale measure so that $(e^{-rt}S_t)_{t=0,1,2}$ is a martingale.

From L7.18, we have

$$S_0 = 100$$

$$S_1(D) = 75 \qquad S_1(U) = 115$$

$$S_2(DD) = 50 \qquad S_2(DU) = S_2(UD) = 100 \qquad S_2(UU) = 150$$

Now solve using conditional expectation for the probabilities, remembering to work backwards

$$\begin{aligned}
S_1(U) &= e^{-rT} \cdot \mathbb{E}\left(S_2 | \text{first move is } U\right) && (117)\\
115 &= 0.9\left[p_u(U)S_2(UU) + \left(1 - p_u(U)\right)S_2(UD)\right] && (118)\\
115 &= 0.9\left[p_u(U) \cdot 150 + \left(1 - p_u(U)\right) \cdot 100\right] && (119)\\
\frac{115}{0.9} &= 150p_u(U) + 100 - 100p_u(U) && (120)\\
\frac{115}{0.9} - 100 &= 50p_u(U) && (121)\\
\frac{250}{9} &= 50p_u(U) && (122)\\
p_u(U) &= \frac{25}{9 \cdot 50} && (123)\\
&= \frac{5}{9} && (124)\\
p_u(U) &\approx 0.5556 && (125)
\end{aligned}$$

and

$$\begin{aligned}
S_1(D) &= e^{-rT} \cdot \mathbb{E}\left(S_2 | \text{first move is } D\right) && (126)\\
75 &= 0.9\left[p_u(D)S_2(DU) + \left(1 - p_u(D)\right)S_2(DD)\right] && (127)\\
75 &= 0.9\left[p_u(D) \cdot 100 + \left(1 - p_u(D)\right) \cdot 50\right] && (128)\\
\frac{75}{0.9} &= (100 - 50)\,p_u(D) + 50 && (129)\\
\frac{75 - 40.5}{0.9} &= 50p_u(D) && (130)\\
p_u(D) &= \frac{115}{3 \cdot 50} && (131)\\
&= \frac{23}{30} && (132)\\
p_u(D) &\approx 0.7667 && (133)
\end{aligned}$$

Finally,

$$
\begin{aligned}
S_0 &= e^{-rT} \cdot \mathbb{E}\left(S_1\right) && (134)\\
100 &= 0.9\left[p_u S_1(U) + \left(1-p_u\right) S_1(D)\right] && (135)\\
\frac{1000}{9} &= p_u \cdot 115 + \left(1-p_u\right) \cdot 75 && (136)\\
\frac{1000}{9} &= \left(115-75\right) p_u + 75 && (137)\\
\frac{325}{9} &= 40 p_u && (138)\\
p_u &= \frac{325}{9 \cdot 40} && (139)\\
&= \frac{325}{360} && (140)\\
&= \frac{65}{72} && (141)\\
p_u &\approx 0.9028 && (142)
\end{aligned}
$$

Therefore, the equivalent martingale measure ($\mathbb{E}\left[S_T\right] = S_0$) is given by

$$
\begin{aligned}
\left(p_u, p_u(U), p_u(D)\right) &= \left(\frac{65}{72}, \frac{5}{9}, \frac{23}{30}\right) && (143)\\
&\approx (0.9028, 0.5556, 0.7667) && (144)
\end{aligned}
$$

## Part B [2 Marks]

Compute the arbitrage-free price of a 90-call option.

Remembering from part (a) we have

$$(p_u, p_u(U), p_u(D)) = \left(\frac{65}{72}, \frac{68}{81}, \frac{23}{27}\right)$$

we can work backwards starting from $T = 2$. We have

$$C_2(UU) = (150 - 90)^+ = 60 \tag{145}$$
$$C_2(UD) = C_2(DU) = (100 - 90)^+ = 10 \tag{146}$$
$$C_2(DD) = (50 - 90)^+ = 0 \tag{147}$$

Then

$$\begin{aligned}
C_1(U) &= e^{-r\cdot 2} \cdot \mathbb{E}\left[C_2 | \text{first move is } U\right] && (148)\\
&= 0.9^2 \left[p_u(U) C_2(UU) + (1 - p_u(U))\, C_2(UD)\right] && (149)\\
&= 0.81 \left[\frac{68}{81} \cdot 60 + \left(1 - \frac{68}{81}\right) 10\right] && (150)\\
&= 0.81 \left[\frac{4080}{81} + \frac{13 \cdot 10}{81}\right] && (151)\\
&= 0.81 \frac{4210}{81} && (152)\\
&= \frac{81}{100} \cdot \frac{4210}{81} && (153)\\
&= \frac{4210}{100} && (154)\\
&= 42.1 && (155)
\end{aligned}$$

and

$$\begin{aligned}
C_1(D) &= e^{-r\cdot 2} \cdot \mathbb{E}\left[C_2 | \text{first move is } D\right] && (156)\\
&= 0.9^2 \left[p_u(D) C_2(DU) + (1 - p_u(U))\, C_2(DD)\right] && (157)\\
&= 0.81 \left[\frac{23}{27} \cdot 10 + \left(1 - \frac{23}{27}\right) \cdot 0\right] && (158)\\
&= 0.81 \cdot \frac{230}{27} && (159)\\
&= 6.9 && (160)
\end{aligned}$$

Then finally

$$
\begin{aligned}
C_0 &= e^{-r\cdot 1} \cdot \mathbb{E}\left[C_1\right] && (161)\\
&= 0.9\left[p_u C_1(U) + (1-p_u)\, C_1(D)\right] && (162)\\
&= 0.9\left[\frac{65}{72}\cdot 42.1 + \left(1 - \frac{65}{72}\right)\cdot 6.9\right] && (163)\\
&= 0.9\left[\frac{65\cdot 42.1}{72} + \frac{7\cdot 6.9}{72}\right] && (164)\\
&= 0.9\cdot\frac{3481}{90} && (165)\\
&= \frac{90}{100}\cdot\frac{3481}{90} && (166)\\
&= \frac{3481}{100} && (167)\\
&= 34.81 && (168)
\end{aligned}
$$

Hence the arbitrage free price of a 90-call is \$34.81.

# 1 Question 4 [4 Marks]

Suppose $W = (W_t)_{t\geq 0}$ is a standard Brownian motion. For the following stochastic processes $X$ and $Y$, derive $dX_t$ and $dY_t$. Are they martingales with respect to the filtration generated by $W$?

## Part A [2 Marks]

$$X_t = e^{\frac{1}{2}t}\sin W_t, \qquad t \geq 0$$

To determine if these stochastic processes are martingales, we need to apply Itô's formula to see if they have any drift. Zero drift means they are a martingale by definition.

In this case, compute partial derivatives for $f(t, W_t) = e^{\frac{1}{2}t}\sin W_t$:

$$\frac{\partial f(t,W_t)}{\partial W_t} = e^{\frac{1}{2}t}\cos(W_t) \tag{169}$$

$$\frac{\partial f(t,W_t)}{\partial t} = \frac{1}{2}e^{\frac{1}{2}t}\sin(W_t) \tag{170}$$

$$\frac{\partial^2 f(t,W_t)}{\partial W_t^2} = -e^{\frac{1}{2}t}\sin(W_t) \tag{171}$$

Observe since $\sigma_t = 1$, then $(dW_t)^2 = dt$. Now apply Itô's formula

$$\begin{aligned}
dX_t &= df\,(t, W_t) && (172)\\
&= \frac{\partial f(t,W_t)}{\partial W_t}dW_t + \frac{\partial f(t,W_t)}{\partial t}dt + \frac{1}{2}\frac{\partial^2 f(t,W_t)}{\partial W_t^2}(dW_t)^2 && (173)\\
&= \left[e^{\frac{1}{2}t}\cos(W_t)\right]dW_t + \left[\frac{1}{2}e^{\frac{1}{2}t}\sin(W_t)\right]dt + \frac{1}{2}\left[-e^{\frac{1}{2}t}\sin(W_t)\right]dt && (174)\\
&= e^{\frac{1}{2}t}\cos(W_t)dW_t + \frac{1}{2}\left[e^{\frac{1}{2}t}\sin(W_t)\right]dt - \frac{1}{2}\left[e^{\frac{1}{2}t}\sin(W_t)\right]dt && (175)\\
&= e^{\frac{1}{2}t}\cos(W_t)dW_t && (176)
\end{aligned}$$

Observe $dX_t$ has no $dt$ (drift) coefficient. In other words, it's gradient is not expected to change with time. Therefore, we conclude the stochastic process $X_t$ is a martingale.

## Part B [2 Marks]

$$Y_t = (W_t + t)\exp\left(-W_t - \frac{1}{2}t\right), \qquad t \geq 0$$

Same as before: apply Itô's formula for a function $f : \mathbb{R} \mapsto \mathbb{R}$ with $f(t, W_t) = (W_t + t)\exp\left(-W_t - \frac{1}{2}t\right)$. Partial derivatives

$$\frac{\partial f(t, W_t)}{\partial W_t} = \exp\left(-W_t - \frac{1}{2}t\right) - (W_t + t)\exp\left(-W_t - \frac{1}{2}t\right) \tag{177}$$

$$= \exp\left(-W_t - \frac{1}{2}t\right)[1 - (W_t + t)] \tag{178}$$

$$= \exp\left(-W_t - \frac{1}{2}t\right)[1 - W_t - t] \tag{179}$$

$$\frac{\partial f(t, W_t)}{\partial t} = \exp\left(-W_t - \frac{1}{2}t\right) - \frac{1}{2}(W_t + t)\exp\left(-W_t - \frac{1}{2}t\right) \tag{180}$$

$$= \exp\left(-W_t - \frac{1}{2}t\right)\left[1 - \frac{1}{2}(W_t + t)\right] \tag{181}$$

$$\frac{\partial^2 f(t, W_t)}{\partial W_t^2} = -\exp\left(-W_t - \frac{1}{2}t\right) - \left[\exp\left(-W_t - \frac{1}{2}t\right) - (W_t + t)\exp\left(-W_t - \frac{1}{2}t\right)\right] \tag{182}$$

$$= \exp\left(-W_t - \frac{1}{2}t\right)[(W_t + t) - 2] \tag{183}$$

$$= \exp\left(-W_t - \frac{1}{2}t\right)[W_t + t - 2] \tag{184}$$

Now apply Itô's formula.

$$dY_t = df(t, W_t) \tag{185}$$

$$= \frac{\partial f(t, W_t)}{\partial W_t}dW_t + \frac{\partial f(t, W_t)}{\partial t}dt + \frac{1}{2}\frac{\partial^2 f(t, W_t)}{\partial W_t^2}(dW_t)^2 \tag{186}$$

$$\begin{aligned} = &\left[\exp\left(-W_t - \frac{1}{2}t\right)[1 - W_t - t]\right]dW_t + \left[\exp\left(-W_t - \frac{1}{2}t\right)\left[1 - \frac{1}{2}(W_t + t)\right]\right]dt \\ &+ \frac{1}{2}\left[\exp\left(-W_t - \frac{1}{2}t\right)[W_t + t - 2]\right]dt \end{aligned} \tag{187}$$

$$\begin{aligned} = &\left[\exp\left(-W_t - \frac{1}{2}t\right)[1 - W_t - t]\right]dW_t \\ &+ \left[\exp\left(-W_t - \frac{1}{2}t\right)\left[1 - \frac{1}{2}(W_t + t)\right] + \frac{1}{2}\exp\left(-W_t - \frac{1}{2}t\right)[W_t + t - 2]\right]dt \end{aligned} \tag{188}$$

$$= \left[\exp\left(-W_t - \frac{1}{2}t\right)[1 - W_t - t]\right]dW_t + \left[\exp\left(-W_t - \frac{1}{2}t\right)\left(\left[1 - \frac{1}{2}(W_t + t)\right] + \left[\frac{1}{2}(W_t + t) - 1\right]\right)\right]dt \tag{189}$$

$$= \left[\exp\left(-W_t - \frac{1}{2}t\right)[1 - W_t - t]\right]dW_t + \left[\exp\left(-W_t - \frac{1}{2}t\right)\left(\left[1 - \frac{1}{2}(W_t + t)\right] - \left[1 - \frac{1}{2}(W_t + t)\right]\right)\right]dt \tag{190}$$

$$= \left[\exp\left(-W_t - \frac{1}{2}t\right)[1 - W_t - t]\right]dW_t \tag{191}$$

Observe $dY_t$ has no drift coefficient (of $dt$), therefore, $Y_t$ is an Itô process with zero drift and hence is also a martingale.

## Question 5 [10 Marks]

### Part A [2 Marks]

Let $x_1, x_2, ..., x_n$ be a sequence of positive numbers. The geometric average $G$ is defined by

$$G = (x_1 x_2 \cdots x_n)^{\frac{1}{n}}$$

Show that

$$G = \exp\left(\frac{1}{n}\sum_{j=1}^{n}\log x_j\right).$$

$$\begin{aligned}
G &= (x_1 x_2 \cdots x_n)^{\frac{1}{n}} && (192)\\
&= \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}} && (193)\\
&= \exp\log\left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}} && (194)\\
&= \exp\frac{1}{n}\log\left(\prod_{i=1}^{n} x_i\right) && (195)\\
&= \exp\left(\frac{1}{n}\sum_{i=1}^{n}\log x_i\right) && (196)
\end{aligned}$$

■

## Part B [2 Marks]

Suppose stock price $(S_t)_{t\geq 0}$ follows

$$\frac{dS_t}{S_t} = rdt + \sigma dW_t, \qquad t \geq 0$$

where $(W_t)_{t\geq 0}$ is a standard Brownian motion under risk-neutral measure $\mathbb{P}$, and $S_0 > 0$ is the today's stock price. Here $r > 0$ is the risk-free interest rate.
A geometrically-averaged Asian option pays, at $T > 0$,

$$\left(\exp\left(I_T/T\right) - K\right)^+,$$

where $K > 0$ is the strike price and

$$I_T = \int_0^T \log S_t dt$$

Show that

$$I_T = T\log S_0 + \left(r - \frac{1}{2}\sigma^2\right)T^2/2 + \sigma\int_0^T W_t dt$$

Observe that the payoff of an Asian option is a function of the (continuous version of the) geometric average of the stock price over the duration of the option, and $K$. We can derive this as

$$G = \exp\left(\frac{I_T}{T}\right) \tag{197}$$
$$\left(\prod_{t=1}^{T} S_t\right)^{\frac{1}{T}} = \exp\left(\frac{I_T}{T}\right) \tag{198}$$
$$\exp\left(\frac{1}{T}\sum_{t=1}^{T}(\log S_t)\right) = \exp\left(\frac{I_T}{T}\right) \tag{199}$$
$$\frac{1}{T}\sum_{t=1}^{T}(\log S_t) = \frac{I_T}{T} \tag{200}$$
$$\sum_{t=1}^{T}(\log S_t) = I_T \tag{201}$$
$$\iff \tag{202}$$
$$I_T = \int_0^T \log S_t dt \tag{203}$$

Hence the stock price $S_t$ can be considered as geometric Brownian motion:

$$\frac{dS_t}{S_t} = rdt + \sigma dW_t \tag{204}$$
$$dS_t = rS_t dt + \sigma S_t dW_t \tag{205}$$

Observe that this is now in the form of an Itô process with $\mu = r$. The solution to this stochastic differential equation (SDE) is given in L8.22 as

$$S_t = S_0 \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right\}, \qquad t > 0$$

Then substituting for $S_t$ into $I_t$ we obtain

$$\begin{aligned}
I_t &= \int_0^t \log S_k dk && (206)\\
&= \int_0^t \log S_0 \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)k + \sigma W_t\right\} dk && (207)\\
&= \int_0^t \log S_0 + \left(r - \frac{1}{2}\sigma^2\right)k + \sigma W_t dk && (208)\\
&= \log S_0 \int_0^t dk + \left[\left(r - \frac{1}{2}\sigma^2\right)\frac{k^2}{2}\right]_{k=0}^{k=t} + \sigma \int_0^t W_k dk && (209)\\
&= \log S_0 \left[t - 0\right] + \left(r - \frac{1}{2}\sigma^2\right)\left[\frac{t^2}{2} - \frac{0^2}{2}\right] + \sigma \int_0^t W_k dk && (210)\\
&= t \log S_0 + \left(r - \frac{1}{2}\sigma^2\right)\frac{t^2}{2} + \sigma \int_0^t W_k dk && (211)
\end{aligned}$$

Hence substituting $t = T$, we obtain

$$I_T = T \log S_0 + \left(r - \frac{1}{2}\sigma^2\right)\frac{T^2}{2} + \sigma \int_0^T W_t dt$$

as given $\blacksquare$.

## Part C [3 Marks]

For every positive integer $N$, let $\Delta t = T/N$ and $t_k = k\Delta t, k = 0, ..., N$. We can write

$$\int_0^T W_t dt = \lim_{N\to\infty} A_N$$

where

$$A_N := \sum_{k=0}^{N-1} W_{t_k}\Delta t, \qquad N \geq 1.$$

What is the distribution of $A_N$? Assuming that the distribution of $A_N$ converges to that of $\int_0^T W_t dt$, what is the distribution of $I_T$?

Observe that $A_N$ is an approximation of the integral $\int_0^T W_t dt$. We know from (b) that $W_t$ is standard Brownian motion, meaning it has $\mu = 0$ (no drift), and dispersion factor $\sigma = 1$. Also note that since $\Delta t = \frac{T}{N}$, we have the result that

$$\begin{aligned}\sum_{k=0}^{N-1} \Delta t &= \sum_{k=1}^{N} \Delta t && (212)\\ &= N\Delta t && (213)\end{aligned}$$

Furthermore, standard Brownian motion tells us that

$$W_t \sim \mathcal{N}(0,1)$$

Now observe that when splitting up $W_t$ into $N$ slices we obtain the difference between each consecutive $W_{t_k}$ and $W_{t_{k+1}}$ to be distributed i.i.d. normally with. Therefore, we can rewrite the sum of $W_{t_k}$ as

$$\begin{aligned}A_N &= \sum_{k=0}^{N-1} W_{t_k}\Delta t && (214)\\ &= \Delta t \sum_{k=0}^{N-1} W_{t_k} && (215)\\ \frac{A_N}{\Delta t} &= \sum_{k=0}^{N-1} W_{t_k} && (216)\\ &= \underbrace{NW_{t_0}}_{=0} + \sum_{k=1}^{N-1}\sum_{i=k}^{N-1}\left(W_{t_i} - W_{t_{i-1}}\right) && (217)\\ &= \sum_{k=1}^{N-1}\sum_{i=k}^{N-1}\left(W_{t_i} - W_{t_{i-1}}\right) && (218)\end{aligned}$$

where each

$$\left(W_{t_i} - W_{t_{i-1}}\right) \overset{\text{i.i.d.}}{\sim} \mathcal{N}\left(0, t_i - t_{i-1}\right), \qquad i \in \{1, 2, ..., N-1\}$$

And $t_i - t_{i-1}, \forall i \in \{1, 2, ..., N-1\} = \Delta t$ since $W_t$ is partitioned equally into $N = \frac{T}{\Delta t}$ segments. Hence

$$\left(W_{t_i} - W_{t_{i-1}}\right) \overset{\text{i.i.d.}}{\sim} \mathcal{N}\left(0, \Delta t\right), \qquad i \in \{1, 2, ..., N-1\}$$

Consider the triangle we are growing

$$\frac{A_N}{\Delta t} = \sum_{k=0}^{N-1} \sum_{i=k+1}^{N} \left(W_{t_i} - W_{t_{i-1}}\right) \tag{219}$$

$$= \sum_{i=1}^{N} \left(W_{t_i} - W_{t_{i-1}}\right) \sum_{k=i}^{N} 1 \tag{220}$$

$$= \sum_{i=1}^{N} (N-i) \left(W_{t_i} - W_{t_{i-1}}\right) \tag{221}$$

Observe by Faulhaber's formula

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Using this observe the distribution of $\frac{A_N}{\Delta t}$ is given by

$$\frac{A_N}{\Delta t} \sim \sum_{i=1}^{N} (N-i)\mathcal{N}(0, \Delta t) \tag{222}$$

$$= \mathcal{N}\left(0, \sum_{i=1}^{N} (N-i)^2 \cdot \Delta t\right) \tag{223}$$

$$= \mathcal{N}\left(0, \frac{N(N-1)(2N-1)}{6} \Delta t\right) \tag{224}$$

$$\Rightarrow A_N \sim \Delta t \cdot \mathcal{N}\left(0, \frac{N(N-1)(2N-1)}{6} \Delta t\right) \tag{225}$$

$$= \mathcal{N}\left(0, \frac{N(N-1)(2N-1)}{6} (\Delta t)^3\right) \tag{226}$$

$$= \mathcal{N}\left(0, \frac{NT^3(N-1)(2N-1)}{6N^3}\right) \tag{227}$$

$$= \mathcal{N}\left(0, \frac{T^3(N-1)(2N-1)}{6N^2}\right) \tag{228}$$

Take the limit to get

$$\lim_{N\to\infty} A_N \sim \lim_{N\to\infty} \mathcal{N}\left(0, \frac{T^3(N-1)(2N-1)}{6N^2}\right) \tag{229}$$

$$\sim \lim_{N\to\infty} \mathcal{N}\left(0, T^3 \frac{(N)(2N)}{6N^2}\right) \tag{230}$$

$$= \mathcal{N}\left(0, \frac{T^3}{3}\right) \tag{231}$$

To compute the distribution of $I_T$, we can simply take the other terms in the expression to be constant. Therefore

$$I_T \sim T \log S_0 + \left(r - \frac{1}{2}\sigma^2\right) \frac{T^2}{2} + \sigma \mathcal{N}\left(0, \frac{T^3}{3}\right) \tag{232}$$

$$= T \log S_0 + \left(r - \frac{1}{2}\sigma^2\right) \frac{T^2}{2} + \mathcal{N}\left(0, \frac{\sigma^2 \cdot T^3}{3}\right) \tag{233}$$

$$= \mathcal{N}\left(T \log S_0 + \left(r - \frac{1}{2}\sigma^2\right) \frac{T^2}{2}, \frac{\sigma^2 \cdot T^3}{3}\right) \tag{234}$$

### Part D [3 Marks]

Find the time-zero price of the geometrically-averaged Asian option in terms of the standard normal cumulative distribution function $\mathcal{N}$.

From part (b) we know

$$I_T = T\log S_0 + \left(r - \frac{1}{2}\sigma^2\right)\frac{T^2}{2} + \sigma\int_0^T W_t \cdot dt \tag{235}$$

$$= T\log S_0 + \left(r - \frac{1}{2}\sigma^2\right)\frac{T^2}{2} + \sigma A_N \tag{236}$$

Where $A_N := \sum_{k=0}^{N-1} W_{t_k}\Delta t \underset{N\to\infty}{\longrightarrow} \int_0^T W_t dt$. Our payoff function is then given by

$$P_T = \left(\exp\left(\frac{I_T}{T}\right) - K\right)^+$$

We also know from part (c) that

$$I_T \sim \mathcal{N}\left(T\log S_0 + \left(r - \frac{1}{2}\sigma^2\right)\frac{T^2}{2}, \frac{\sigma^2 T^3}{3}\right)$$

Therefore

$$\frac{I_T}{T} \sim \frac{1}{T}\mathcal{N}\left(T\log S_0 + \left(r - \frac{1}{2}\sigma^2\right)\frac{T^2}{2}, \frac{\sigma^2 T^3}{3}\right) \tag{237}$$

$$= \mathcal{N}\left(T\log S_0 + \left(r - \frac{1}{2}\sigma^2\right)\frac{T^2}{2}, \frac{\sigma^2 T^3}{3T^2}\right) \tag{238}$$

$$= \mathcal{N}\left(T\log S_0 + \left(r - \frac{1}{2}\sigma^2\right)\frac{T^2}{2}, \frac{\sigma^2 T}{3}\right) \tag{239}$$

$$= \underbrace{T\log S_0 + \left(r - \frac{1}{2}\sigma^2\right)\frac{T^2}{2}}_{\alpha} + \underbrace{\sigma\sqrt{\frac{T}{3}}}_{\beta} \cdot \mathcal{N}(0,1) \tag{240}$$

$$= \alpha + \beta \cdot Z \tag{241}$$

(with the last two lines expressing the distribution of $\frac{I_T}{T}$ as a standard normal.)

Now we can solve the **discounted** expected price at time $t = 0$ as

$$C_0 = e^{-rT}\mathbb{E}\left[\left(e^{\alpha+\beta z} - K\right)^+\right] \tag{242}$$

$$= e^{-rT}\int_{-\infty}^{\infty}\left(e^{\alpha+\beta z} - K\right)^+ \cdot \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{z^2}{2}\right)dz \tag{243}$$

We need to handle this $(\cdot)^+ := \min(\cdot, 0)$ function. Consider solving the root (w.r.t. z) when $e^{\alpha+\beta z} - K = 0$

$$0 = e^{\alpha+\beta z} - K \tag{244}$$

$$K = \exp\left\{T\log S_0 + \left(r - \frac{1}{2}\sigma^2\right)\frac{T^2}{2} + \sigma\sqrt{\frac{T}{3}}z\right\} \tag{245}$$

$$\log K = T\log S_0 + \left(r - \frac{1}{2}\sigma^2\right)\frac{T^2}{2} + \sigma\sqrt{\frac{T}{3}}z \tag{246}$$

$$\sigma\sqrt{\frac{T}{3}}z = \log K - T\log S_0 - \left(r - \frac{1}{2}\sigma^2\right)\frac{T^2}{2} \tag{247}$$

$$z^* = \frac{\sqrt{3}}{\sigma\sqrt{T}}\left[\log K - T\log S_0 - \left(r - \frac{1}{2}\sigma^2\right)\frac{T^2}{2}\right] \tag{248}$$

I'll label this point as $z^*$ ($z$-star) from here.

We now return to rewriting (243) as

$$\begin{aligned}
C_0 &= \int_{z^*}^{\infty}\left(e^{\alpha+\beta z} - K\right)\cdot\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{z^2}{2}\right\}dz && (249)\\
&= \int_{z^*}^{\infty} e^{\alpha+\beta z}\cdot\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{z^2}{2}\right\}dz - \int_{z^*}^{\infty} K\cdot\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{z^2}{2}\right\}dz && (250)\\
&= \int_{z^*}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left\{\alpha+\beta z - \frac{z^2}{2}\right\}dz - K\int_{z^*}^{\infty}\cdot\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{z^2}{2}\right\}dz && (251)\\
&= \int_{z^*}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left\{\alpha+\frac{1}{2}\beta^2 - \frac{(z-\beta)^2}{2}\right\}dz - K\left[1 - \int_{-\infty}^{z^*}\cdot\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{z^2}{2}\right\}dz\right] && (252)\\
&= \int_{z^*}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left\{\alpha+\frac{1}{2}\beta^2\right\}\exp\left\{-\frac{(z-\beta)^2}{2}\right\}dz - K\left[1-\Phi(z^*)\right] && (253)\\
&= \exp\left\{\alpha+\frac{1}{2}\beta^2\right\}\int_{z^*-\beta}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{z^2}{2}\right\}dz - K\left[1-\Phi(z^*)\right] && (254)\\
&= \exp\left\{\alpha+\frac{1}{2}\beta^2\right\}\left[1 - \int_{-\infty}^{z^*-\beta}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{z^2}{2}\right\}dz\right] - K\left[1-\Phi(z^*)\right] && (255)\\
&= \exp\left\{\alpha+\frac{1}{2}\beta^2\right\}\left[1-\Phi(z^*)\right] - K\left[1-\Phi(z^*-\beta)\right] && (256)\\
&= \exp\left\{T\log S_0 + \left(r-\frac{1}{2}\sigma^2\right)\frac{T^2}{2} + \frac{1}{2}\left(\sigma\sqrt{\frac{T}{3}}\right)^2\right\}\left[1-\Phi(z^*)\right] - K\left[1-\Phi(z^*-\beta)\right] && (257)\\
&= \exp\left\{T\log S_0 + \left(r-\frac{1}{2}\sigma^2\right)\frac{T^2}{2} + \sigma^2\frac{T}{6}\right\}\left[1-\Phi(z^*)\right] - K\left[1-\Phi(z^*-\beta)\right] && (258)
\end{aligned}$$

Where $\Phi(\cdot)$ is the CDF of the standard normal variable $Z$ and

$$z^* = \frac{\sqrt{3}}{\sigma\sqrt{T}}\left[\log K - T\log S_0 - \left(r - \frac{1}{2}\sigma^2\right)\frac{T^2}{2}\right].$$