MATH3090/7039: Financial mathematics
Assignment 2
Semester I 2024

Due Tuesday April 23 1pm

MATH3090 total marks **24 marks**

MATH7039 total marks **30 marks**

Submission:

- Submit onto Blackboard softcopy (i.e. scanned copy) of (i) your assignment solutions, as well as (ii) Matlab/Python code for Problem 2. Hardcopies are not required.
- Include all your answers, numerical outputs, figures, tables and comments as required into one single PDF file.
- You also need to upload all Matlab/Python files onto Blackboard.

General coding instructions:

- You are allowed to reuse any code provided/developed in lectures and tutorials.

Notation: "Lx.y" refers to [Lecture x, Slide y]

Assignment questions - all students

1. (3 marks) You have just invested in a 3-year coupon paying bond with 8% semi-annual coupons and a face value $F = \$100,000$. Suppose the coupon-paying bond yield curve is *flat* at 9%.

a. (1 mark) Calculate the present value and the (absolute value of) duration |D| of the bond.

b. (1 mark) Now calculate the value of the bond in |D| years time.

c. (1 mark) Suppose that, immediately after buying the bond, the yield curve shifted up to be flat at 10%. Now calculate the value of the bond again in |D| years time under the new yield curve (don't calculate D again). Compare your answer with what you obtained in (b).

You are welcome to reuse Excel sheets provided in class/tutorial. You are not required to submit Excel files.

2. (7 marks) Assume that you observe the following yield curve for government's coupon paying bonds.

- There are a total of 20 bonds.
- For the k-th bond, $k = 1, \ldots, 20$, the maturity is k years.
- The face value is $F = \$100,000$ and the coupon rate for the k-th bond, $k = 1, \ldots, 20$, is $c = 4\%$. Let $C = cF$.

- The prices of the bonds ($P(k)$, $k = 1, 2, \ldots, 20$) are given by

$$\begin{aligned}&[P(1), P(2), \ldots, P(20)]\\&= [99412, 97339, 94983, 94801, 94699, 94454, 93701, 93674, 93076, 92814,\\&\quad 91959, 91664, 87384, 87329, 86576, 84697, 82642, 82350, 82207, 81725].\end{aligned}$$

- Denote by $y_{0,k}$ the spot zero-coupon bond yield curve, and by $y_{k-1,k}$ the implied one-year forward rates.

Assume that all the coupon payments are made annually. Use continuous compounding.

a. (1 mark) Show that

$$y_{0,k} = \frac{1}{k} \log \left(\frac{C+F}{P(k) - C \sum_{j=1}^{k-1} e^{-y_{0,j} \times j}} \right), \quad 1 \le k \le 20.$$

b. (2 marks) Implement a Matlab/Python program to compute spot zero-coupon bond yield curve $y_{0,k}$ and the implied one-year forward rates $y_{k-1,k}$. Submit Table 1 filled with computed values.

Table 1: Table for Question 2 (b)

period k	spot $y_{0,k}$	forward $y_{k-1,k}$
1	$\ldots$	$\ldots$
2	$\ldots$	$\ldots$
$\vdots$	$\vdots$	$\vdots$
20	$\ldots$	$\ldots$

c. (3 marks) Suppose you enter into a 20-year vanilla fixed-for-floating swap on a notional principal of \$1,000,000 where you pay the fixed rate of 6.5% and the counter-party pays the yield curve plus 1%.

Code in Matlab/Python a program to compute the swap value. Submit a table of results, similar to the table on L5.15.

e. (1 marks) Test with different fixed rates and provide a better approximation of the swap rate so that the swap value is near zero (you do not need to develop a new code).

3. (6 marks) Assume annual time periods, $T = 3$, a binomial model of the yield curve, and $y_{0,1} = 2\%$. Suppose over the whole forward rate lattice that the next period's forward rates can either go up by a factor of $u = 1.3$ with a probability of $p = 60\%$ or down by a factor of $d = 0.9$. (For example $y(u) = y_{0,1} \times u = 0.02 \times 1.3 = 0.026$ or 2.6%, $y(uu) = y_{0,1} \times u \times u$ and so on.) Use discrete compounding.

a. (4 marks) Construct the forward rate lattice and the zero coupon bond yield curve $y_{0,2}$ and $y_{0,3}$.

b. (2 marks) Construct the 1-period forward rates $y_{1,2}$ and $y_{2,3}$, which are embedded in this zero coupon bond yield curve (we already have $y_{0,1}$).

4. (3 marks) Consider the payoff at maturity T in Figure 1. Show how to construct this payoff using European calls with the same maturity only (you can use any combination of European calls with any strike price). You must state long/short, strike prices as well as the number of units. In addition, express the current value of the (replicating) portfolio in terms of the current prices of strike-K European calls $C_0(K)$, $K > 0$.

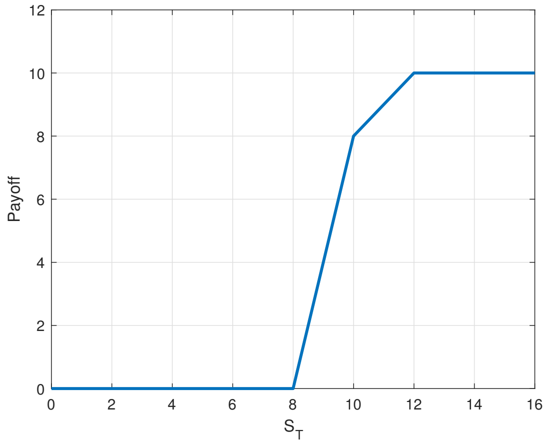

Figure 1: Payoff diagram.

5. (5 marks) Given a stock whose time-t price is S_t, consider a derivative that pays e^{S_T} at maturity T (the writer pays e^{S_T} to the holder; the holder pays nothing to the writer). We assume that there is also a (risk-free) zero-coupon bond with maturity T and face value 1, whose time-0 price is Z_0. Let C_0 be the arbitrage-free time-0 price of the derivative. Answer the following.

a. (2 marks) Suppose S_T can take any positive value with a strictly positive probability (under the physical probability measure P), and hence $\mathrm{P}(S_T > M) > 0$ for any $M > 0$. Show that the considered derivative cannot be super-replicated if only the stock and bond are available in the market.

b. (3 marks) Show that

$$C_0 \geq e^{\frac{S_0}{Z_0}} Z_0.$$

Assignment questions - MATH7039 students only

6. (3 marks) A binary call option with strike $K > 0$ and maturity $T > 0$ pays \$1 if the terminal stock price satisfies $S_T \geq K$ and it pays nothing otherwise. Similarly a binary put option with strike $K > 0$ and maturity $T > 0$ pays \$1 if $S_T \leq K$ and pays nothing otherwise.

Consider the payoff in Figure 2. Show how to construct this payoff using European calls, European puts, binary calls and binary puts only (you can use any combination of these options with any strike price with the same maturity T). You must state long/short, strike prices as well as the number of units.

7. (3 marks) In Problem 5, suppose S_T only takes values on $[0, 1]$ (i.e., $\mathrm{P}(0 \leq S_T \leq 1) = 1$). Show that

$$C_0 \leq \min(eZ_0, Z_0 + (e-1)S_0).$$

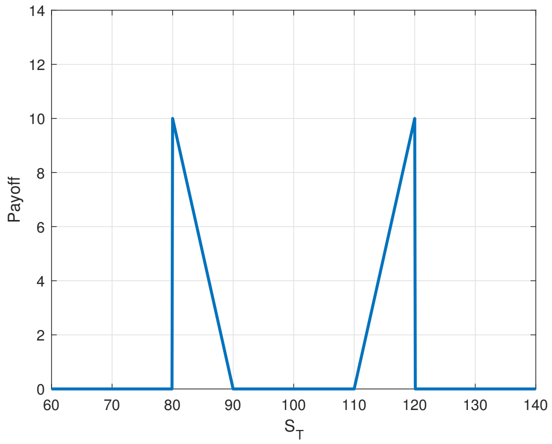

Figure 2: Payoff diagram.