Benders Decomposition

General Benders Decomposition Integer Programming Perspective

Master problem

$$\min f^T y + \Theta(y)$$

Subject to:

$$Dy \leq d$$
$$y \geq 0 \ (integer, binary)$$

- $\Theta(y)$ is a complicated (non-linear) function of y. Another optimisation problem, a simulation problem, checking special legality conditions, etc.
- $\Theta(y^*) = \infty \Leftrightarrow y^*$ is not a feasible solution to the whole problem

Benders Master Problem (BMP)

$$\min f^T y + \theta$$

Subject to:

$$Dy \leq d$$
$$\theta \geq \Gamma_k + \gamma_k^T y \;\forall\; k \in OptCuts$$
$$0 \geq \Gamma_k + \gamma_k^T y \;\forall\; k \in FeasCuts$$
$$y \geq 0 \;(integer, binary)$$

$OptCuts = FeasCuts = \emptyset$ (or more likely some problem specific initial cuts)

while True:

 Solve BMP to get solution y^*
 Calculate $\Theta(y^*)$
 If $\Theta(y^*) \leq \theta + \epsilon$, break
 If $\Theta(y^*) = \infty$, calculate a feasibility cut (Γ_k, γ_k), otherwise calculate an optimality cut (Γ_k, γ_k).

Benders Cuts

- A valid feasibility cut must satisfy:
  - $0 < \Gamma_k + \gamma_k^T y^*$ (i.e. the current BMP solution violates the cut)
  - $0 < \Gamma_k + \gamma_k^T y \Longrightarrow \Theta(y) = \infty$ (i.e. all solutions cut off are sub-problem infeasible)
- A valid optimality cut must satisfy:
  - $\Theta(y^*) = \Gamma_k + \gamma_k^T y^*$ (i.e. the cut is exact at the current BMP solution)
  - $\Theta(y) \geq \Gamma_k + \gamma_k^T y$ (i.e. the cut under estimates the sub-problem objective everywhere)
- In general, the "stronger" the cut the better
  - Stronger feasibility cuts will cut off more solutions to the BMP
  - Stronger optimality cuts will give better bounds at alternative solutions

This seems too simple!

Benders decomposition works best when:

- $\Theta(y)$ is difficult to calculate but easy to approximate – strong cuts
- $\Theta(y)$ can be written as $\sum_{i=1}^{n} \Theta_i(y)$
  - Disaggregated Benders decomposition
- A good set of initial cuts is available
  - Feasibility and/or optimality, as the context requires
  - e.g. must touch one cell of same type in snake eggs
- Good cuts can be generated by solving the LP relaxation of the BMP
- The cuts are added as lazy constraints in a callback
  - i.e. the BMP is solved just once
- "Primal heuristics" are used to find good feasible solutions – upper bounds

Duality Theory (in one slide)

Primal:

$$\min c^T x$$

Subject to:

$$Ax \leq b$$
$$x \geq 0$$

Has dual:

$$\max b^T y$$

Subject to:

$$A^T y \leq c$$
$$y \leq 0$$

If these both have optimal solutions then they are equal.

For all legal solution $c^T x \geq b^T y$

Classical Benders

$$\min c^T x + f^T y$$

Subject to:

$$Ax + By \leq b$$
$$Dy \leq d$$
$$x \geq 0, y \geq 0\ (integer, binary)$$

This is a problem in our general form with $\Theta(y^*)$ given by:

Benders Sub-Problem (BSP)

$$\min c^T x$$

Subject to:

$$Ax \leq b - By^*$$
$$x \geq 0$$

Classical Benders (cont)

Dual BSP

$$\max z^* = (b - By^*)^T u$$

Subject to:

$$A^T u \le c$$
$$u \le 0$$

Assume we have an optimal solution u^*. Then for any y, u^* is a feasible solution of Dual BSP and $(b - By)^T u^*$ is a lower bound on its solution and hence a lower bound on BSP.

Therefore:

$$\theta \ge (b - By)^T u^*$$

Satisfies the conditions for a Benders optimality cut.