Markowitz or Mean/Variance models

Sets

N set of assets

Data

r_i return on asset $i \in N$

W covariance matrix

We assume the returns have a multivariate normal distribution.

Variables

$x_i$is the fraction of out portfolio invested in each asset

$$max\ \lambda \sum_{i \in N} r_i x_i - (1-\lambda) x^T W x$$

Subject to:

$$\sum_{i \in N} x_i = 1$$

$$x_i \geq 0 \ \ \forall\ i \in N$$

General second order cone constraint

$$\boldsymbol{x}^T W \boldsymbol{x} \leq z$$

Where $\boldsymbol{x} \geq 0$ (vector) and $z \geq 0$ are variables and $W$is a positive semi-definite matrix.

Value at Risk (Var) and Conditional VaR (CvaR)

Sample Average Approximation (SAA)

Assume we want VaR and CVaR at the α level (e.g. bottom 5%)

Set

S set of samples

Data

r_{is} return for investment $i \in N$ in sample $s \in S$

Variables

β_s return of sample s

Var estimate of VaR from samples

$CVar$ estimate of CvaR from samples

β_s^- amount that the return of scenario s is below Var

$$\max \frac{\lambda}{|S|} \sum_{s \in S} \beta_s + (1 - \lambda) CVar$$

Subject to:

$$\sum_{i \in I} x_i = 1$$

$$\beta_s = \sum_{i \in N} r_{is} x_i \quad \forall\, s \in S$$

$$\beta_s + \beta_s^- \geq Var \quad \forall\ s \in S$$

$$CVar = Var - \frac{1}{\alpha |S|} \sum_{s \in S} \beta_s^-$$

Chance Constrained Portfolio Model

$$\max \sum_{i \in N} r_i x_i$$

Subject to:

$$\sum_{i \in N} x_i = 1$$

Chance constraint (for small values of alpha)

$$pr\left(\sum_{i \in I} r_i x_i \geq 1\right) \geq (1 - \alpha)$$

$$F^{-1}(1 - \alpha)^2 x^T W x \leq \left(1 - \sum_{i \in I} r_i x_i\right)^2$$

What if the data are not normally distributed, but we can generate a lot of samples?

Use Campi & Garatti formula to calculate k: the number of failures permitted for given sample size, α and problem degrees of freedom. Replace W with the sample variance and r with the sample average returns. Replace $F^{-1}(1 - \alpha)$ in the above with a parameter, say ω, with initial value of $\omega = F^{-1}\left(1 - \frac{k}{N}\right)$, and repeatedly solve and count the number of failures, varying ω to get exactly k failures. (Line search/interpolation technique).

Water network design problem

Multiperiod deterministic demand

Sets

N water infrastructure nodes (dams, water treatment, desalination, etc)

A pipelines

T time periods

Data

d_{nt} demand (supply is negative) at node n

c_a cost of operating arc a (pumping or production cost)

cap_n maximum storage capacity at node n

f_a fixed cost of arc a

u_a upper bound on flow on arc a

Variables

$y_a \in \{0,1\}$ if we build arc a

$x_{at} \geq 0$ flow on arc a in time period t

$z_{nt} \geq 0$ storage at node n at the end of time period t

$$\min \sum_{a \in A} f_a y_a + \sum_{\substack{a \in A \\ t \in T}} c_a x_{at}$$

Subject to:

$$x_{at} \leq u_a y_a \quad \forall\, a \in A, t \in T$$

$$z_{nt} \leq z_{n(t-1)} - d_{nt} + \sum_{a\, flows\, to\, n} x_{at} - \sum_{a\, flows\, from\, n} x_{at} \quad \forall\, n \in N, t > 1$$

$$z_{nt} \leq cap_n \quad \forall\, n \in N, t \in T$$

Initialise z_{n0}

Water network design problem

Multiperiod stochastic demand – scenario tree

Sets

N water infrastructure nodes (dams, water treatment, desalination, etc)

A pipelines

T time periods

S scenarios (built as a tree and indexed by the leaf nodes)

Data

d_{nts} demand (supply is negative) at node n in scenario s. These values are all the same for common nodes in the scenario tree. An observation of a random variable, time and space correlation.

c_a cost of operating arc a (pumping or production cost)

cap_n maximum storage capacity at node n

f_a fixed cost of arc a

u_a upper bound on flow on arc a

p_s probability of scenario s

Variables

$y_a \in \{0,1\}$ if we build arc a

$x_{ats} \geq 0$ flow on arc a in time period t

$z_{nts} \geq 0$ storage at node n at the end of time period t

$$\min \sum_{a \in A} f_a y_a + \sum_{\substack{a \in A \\ t \in T \\ s \in S}} p_s c_a x_{ats}$$

Subject to:

$$x_{ats} \leq u_a y_a \quad \forall\, a \in A, t \in T, s \in S$$

$$z_{nts} \leq z_{n(t-1)s} - d_{nts} + \sum_{a\, flows\, to\, n} x_{ats} - \sum_{a\, flows\, from\, n} x_{ats} \quad \forall\, n \in N, t > 1, s \in S$$

$$z_{nts} \leq cap_n \quad \forall\, n \in N, t \in T, s \in S$$

Initialise z_{n0s}

Non-anticipating constraints

$b_{st} \in S$ the lowest index scenario that passes through the same scenario tree node as scenarios s at time t

$$x_{ats} = x_{atb_{st}}\, \forall\, a \in A, t \in T, s \in S, s \neq b_{st} LOL$$