Multiple Machine Scheduling

A company has a set of jobs that need to be completed today, and an array of machines able to process those jobs. Each machine can process more than one job at a time. But each machine has a capacity, and each job has a fixed resource consumption for each machine – in other words, the total resource consumption of concurrent jobs on a single machine is bounded. Each job has a release time (*the earliest time it may start*) and a deadline (*the latest time it may finish*). Our goal is to decide when and by which machine each job will be processed. The goal is to minimize the makespan (the maximum completion time of any job).

Sets

J = Set of jobs
M= Set of machines
T = Set of minutes, T = {0, 1, … , |T| - 1}

T_{jmt} = All the start times of job *j* on machine *m* that would cause it to be being processed at time *t*

Data

p_{jm} the processing time of job j on machine m
C_m the capacity of machine m
c_{jm} the resource consumption of job j on machine m
D = Deadline

Variables

$X_{jmt} \in \{0,1\}$ job j starts on machine at time t
μ the makespan

Objective

min μ

Constraints

$$\mu \geq \sum_{m \in M} \sum_{t \in T} (t + p_{jm}) X_{jmt} \ for\ all\ j \in J$$

$$\sum_{t \in T, m \in M} X_{jmt} = 1\ for\ all\ j \in J$$

$$\sum_{j \in J} \sum_{t \in T_{ijk}} c_{jm} X_{jmt} \leq C_j \ for\ all\ m \in M, k \in T$$

Variables

$$X_{jm} \in \{0, 1\}\ job\ j\ allocated\ to\ machine\ m$$

$$\theta_m\ the\ estimated\ makespan\ of\ machine\ m$$

$$\theta\ estimated\ total\ makespan$$

Master problem

$$\min \theta$$

Constraints

$$\theta \geq \theta_m\ for\ all\ m \in M$$

$$\sum_{m \in M} X_{jm} = 1\ \ for\ all\ j \in J$$

Subproblem *m* = Calculate the makespan for machine *m*

X'_{jm} fixed, $J_m = \{\, j \in J : X'_{jm} = 1\}$

Subproblem variables:

$$\sigma_{jt} \in \{0, 1\} \text{ job } j \text{ started by time } t$$

$$\sigma_{j,-1} = 0\ for\ all\ j \in J$$

$$\sigma_{j,D-p_{jm}} = 1$$

$$\sigma_{jt} \geq \sigma_{j,t-1}$$

$$\sum_{j \in J} c_{jm}\left(\sigma_{jt} - \ \sigma_{jk}\right) \leq C_m\ \ where\ k = \max(t - p_{jm}\,, -1)$$

Job j *starts* at time *t* if $s_{jt} = \sigma_{jt} - \ \sigma_{j,t-1} = 1$.