PROBABILITY & RANDOM PROCESSES

① $X(t)$ is a WSS process, i.e. its mean and autocorrelation is independent of time.
μ is independent of t and $R_X(t_1, t_2)$ is a function of time difference $(t_2 - t_1)$.

Now, $X(t) = \sum_{k=-\infty}^{\infty} A_k\, g(t-k)$

where A_0, A_1, A_{-2}, A_{-1} are iid random variables.

$$g(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & \text{otherwise} \end{cases}$$

~~$\hat{\mu}_X = \frac{1}{2T} \int_{-T}^{T} x(t)\, dt$~~

if $\tilde{\mu}_x$ converges to μ_x as $T \to \infty \Rightarrow X(t)$ is mean ergodic.

Now, $\hat{\mu}_x = \frac{1}{2T} \int_{-T}^{T} X(t)\, dt$

$$E(\hat{\mu}_x) = \frac{1}{2T} \int_{-T}^{T} E(X(t))\, dt = \frac{1}{2T} \int_{-T}^{T} \mu_x\, dt$$

$$= \frac{\mu_x}{2T} [T - (-T)]$$

$$= \frac{\mu_x}{2T} \times 2T = \mu_x$$

Since the expectation of $\hat{\mu}_x$ converges to $\mu_x \Rightarrow \hat{\mu}_x$ will also converge to μ_x when $T \longrightarrow \infty$ (more samples are considered).

(2)

a. $X(t)$ is a gaussian process.

$\mu_x(t) = 2t$

$R_x(t_1, t_2) = 1 + 4t_1t_2$

Let $x(1) + x(2) = Y$; Y is a normal random variable.

$E[Y] = E[x(1)] + E[x(2)]$

$= 2 + 2(2)$

$= \underline{\underline{6}}$

$Var[Y] = Var[x(1)] + Var[x(2)] + 2Cov[x(1), X(2)]$

$Var[x(1)] = E[x(1)^2] - (E[x(1)])^2$

$= R_{xx}(1) - (2)^2$

$= 5 - 4$

$= 1$

$Var[x(2)] = E[x(2)^2] - (E[x(2)])^2$

$= R_x(2) - (4)^2$

$= 17 - 16$

$= \underline{\underline{1}}$

$Cov[x(1), X(2)] = E[x(1)x(2)] - E[x_1]E[x_2]$

$= R_x(1,2) - \cancel{8}\ 2(4)$

$= (1 + 4(2)) - 8$

$= 9 - 8$

$= \underline{1}$

$\therefore var[Y] = 1 + 1 + 2(1)$

$= \underline{4}.$

$\therefore P(Y < 3) = \phi\left(\frac{3-6}{2}\right)$

$= \phi(-1.5)$

$= \underline{\underline{0.0668}}$

b. $X(t)$ is a WSS gaussian random process.

$\mu_x(t) = 1$

$R_x(\tau) = 1 + 4sinc(\tau)$

Let $Y = X(1)$ and $Z = X(2)$

then Y and Z are jointly gaussian $\rightarrow Y \sim N(1, 4)$

$Z \sim N(1, 4)$

$[Var[X(1)] = Var[X(2)] = 1 + 4sinc(0) - (1)^2 = 4].$

~~Since Y and Z are independent (they are not correlated)~~

Finding covariance of Y and Z:

$Cov(Y, Z) = E[YZ] - E[Y]E[Z]$

$= R_x(1-2) - 1 \times 1$

$= R_x(-1) - 1$

$= 1 - 1 = \underline{0}$

Since Y & Z are not correlated $\Longrightarrow$ Y & Z are independent.

$$\therefore P(1 < Y < 2,\ Z < 3) = P(1 < Y < 2)\, P(Z < 3)$$

$$= \left(\phi\left(\frac{2-1}{2}\right) - \phi\left(\frac{1-1}{2}\right) \right) \phi\left(\frac{3-1}{2}\right)$$

$$= [\phi(0.5) - \phi(0)]\, \phi(1)$$

$$= (0.69 - 0.5)\, 0.84$$

$$= 0.19 \times 0.84$$

$$= \underline{\underline{0.1596}}$$