04/09/23

RECAP: → $\hat{\omega}\theta \longrightarrow R = e^{[\hat{\omega}]\theta}$

→ $\dot{p}(t) = e^{[\hat{\omega}]t}\, p(0)$

→ Rodrigues' formula: $e^{[\hat{\omega}]\theta} = I + \sin\theta\,[\hat{\omega}] + (1-\cos\theta)\,[\hat{\omega}]^2$.

→

(i) if $R = I$; $\theta = 0, \pm 2\pi, \pm 4\pi, \dots$ and $\hat{\omega}$ is undefined.

(ii) if $\text{Trace}(R) = -1$; $\theta = \pm\pi, \pm 3\pi, \dots$ and set $\hat{\omega}$ to:

$$R(\hat{\omega}, \pi) = I + 2[\hat{\omega}]^2$$

(iii) $$\theta = \cos^{-1}\left(\frac{1}{2}(\text{tr}\,R - 1)\right) \in [0, \pi)$$

$$\hat{\omega} = \frac{1}{2\sin\theta}(R - R^T).$$

— • — • — • —

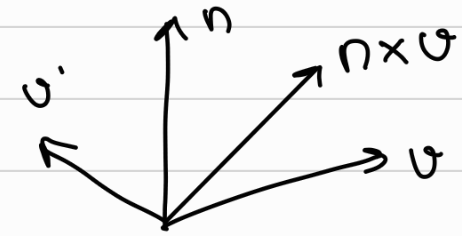
• we can represent some arbitrary rotation as a unit axis rotation.

AXIS - ANGLE THEOREM

• $z' = e^{i\theta} z$

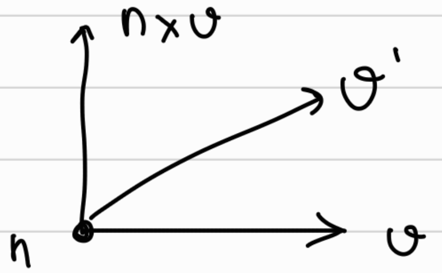
• any arbitrary vector $v \in \mathbb{R}^3$ can be rotated about normal vector $v \cdot n = 0$

now, our aim is to find v' using n, v, θ



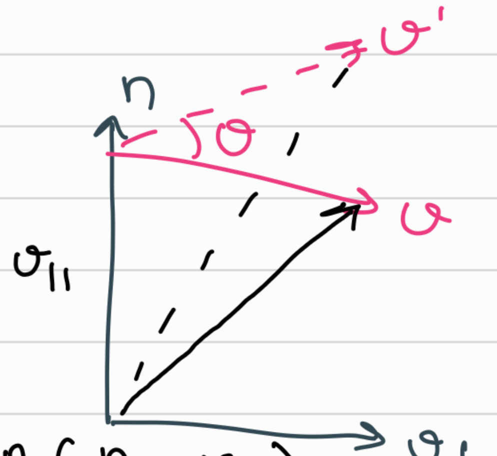
$v \cdot n = 0$

$(n \times v) \perp v$

- 

$\leftarrow$ Taking the $(n \times v)$ & v plane

- $v = v_{\perp} + v_{\parallel}$



$v_{\parallel}' = v_{\parallel}$

$\therefore v' = v_{\parallel} + v_{\perp}'$

$v' = v_{\parallel} + \cos\theta \, v_{\perp} + \sin\theta \, (n \times v_{\perp})$

$= (1 - \cos\theta) \, v_{\parallel} + \cos\theta \, v_{\perp} + \sin\theta \, (n \times v)$

$$\boxed{v' = (1 - \cos\theta)(v \cdot n) n + \cos\theta \, v + \sin\theta \, (n \times v)}$$

Rodrigues' rotation formula

After simplifying with identeties,

$R(n, \theta) = (I + \sin\theta \, \hat{n} + (1 - \cos\theta) \, \hat{n}^2)$

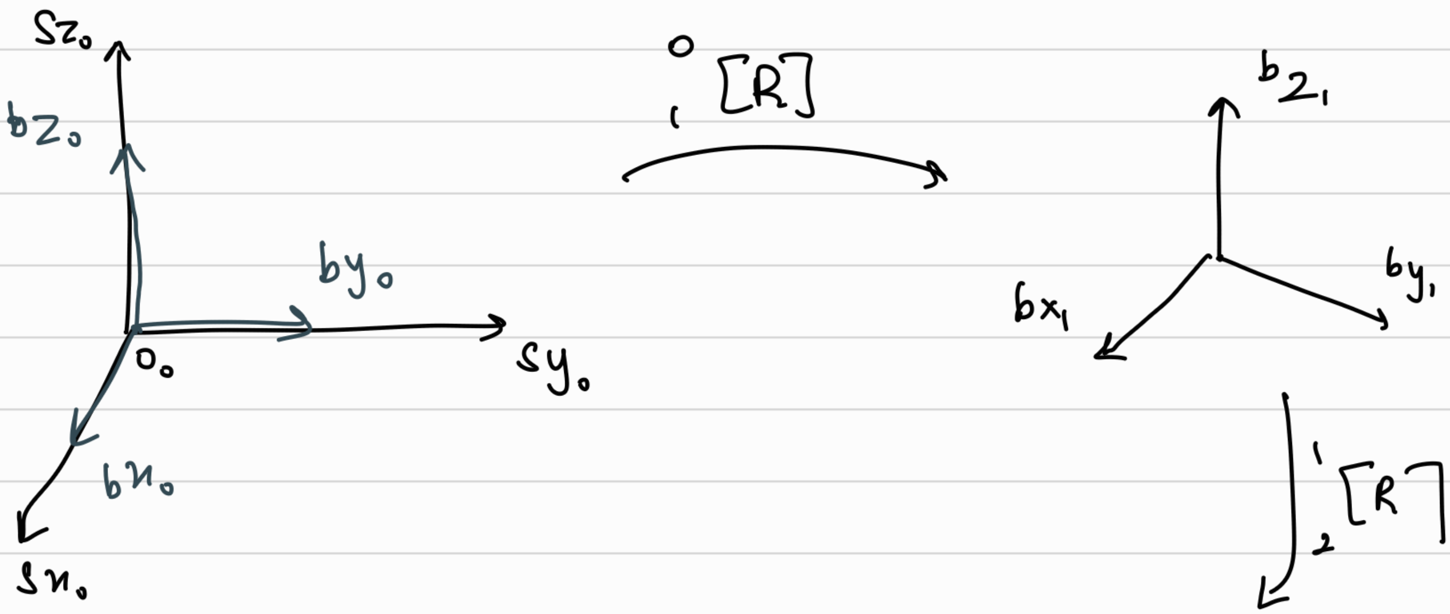
$\hookrightarrow$ Angle-axis theorem.

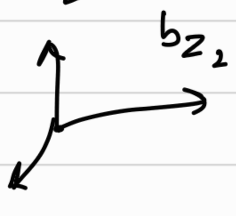
- we would need four variables → 3 for n + 1 for θ

- <u>Rotation vector notation</u> : $k = n\theta$

$$R(k) = R\left(\underbrace{\frac{k}{|k|}}_{\hat{n}}, \underbrace{|k|}_{\theta}\right)$$

COMPOSITION OF ROTATION





- body fixed frame

$${}^1_0[R] \longrightarrow {}^1_2[R] \longrightarrow {}^2_3[R]$$

- in frame 1, there is some rotation occuring

$$p' = A' q'$$

how would we observe the rotation from another frame?

→ convert the vectors to frame 0.

$${}^{0}p = {}^{0}_{1}[R]\,{}^{1}p$$

$${}^{0}q = {}^{0}_{1}[R]\,{}^{1}q$$

now, $${}^{1}p = {}^{1}A\,{}^{1}q$$
$${}^{0}p = R\;{}^{1}A\;R^{T}\,{}^{0}q \qquad R = {}^{0}_{1}[R]$$

now, $${}^{0}({}^{1}A) = {}^{0}_{1}R\,{}^{1}A\,{}^{0}_{1}R^{T}$$ 40 more min :(((

(SVD decomposition) → similarity transformation

${}^{1}A$ as viewed from frame 0.

In general,

$${}^{0}\left({}^{1}_{2}[R]\right) = {}^{0}_{1}[R]\;{}^{1}_{2}[R]\left({}^{0}_{1}[R]\right)^{T}$$

$$\boxed{{}^{a}\left({}^{b}_{c}[R]\right) = {}^{a}_{b}[R]\;{}^{b}_{c}[R]\;\left({}^{a}_{b}[R]\right)^{T}}$$

→ $${}^{0}_{1}[R] = {}^{S}R_{1} = {}^{b}R_{1}$$
(body fixed & space fixed superimposed)

$${}^{S}R_{2}\,{}^{S}R_{1} = {}^{0}_{1}R\,{}^{1}_{2}R\;\cancel{{}^{0}_{1}R^{T}\;{}^{0}_{1}R}\;I$$

$$= {}^{0}_{1}R\,{}^{1}_{2}R$$

Q) $R(n, \theta)$

- rotating by θ about current n axis
  $R(\hat{n}, \theta)$
- z axis by ϕ (fixed z axis)
  $R(\hat{n}, \theta) \cdot R(z, \phi)$
- z axis by α
  $R(z, \alpha)\left[R(\hat{n}, \theta) \cdot R(z, \phi)\right]$

→ post-multiply for body fixed
pre-multiply for space fixed

EIGEN VECTORS

- vectors that are invariant under some transformation.

For $SO(2)$: eigen values $\lambda_1, \lambda_2 \Rightarrow e^{\pm i\theta}$
(complex conjugate pairs)

For $SO(3)$: $\lambda_1, \lambda_2, \lambda_3$

- $R\vec{n} = \lambda \vec{n}$
  $\lambda_1 = 1$ (unit axis of rotation)
  $\lambda_2, \lambda_3 \rightarrow$ complex conjugate

- For identity matrix as R,
  eigen value $= 1$
  eigen vectors → can be anything

- if $\theta = \pi$, $\lambda_1 = 1$; $\lambda_2 = -1$, $\lambda_3 = -1$

- in general, $\lambda_1 = 1$; $\lambda_2 = \lambda_3 = e^{\pm i\theta}$

- to find the axis of rotation $\Longrightarrow$ we find the eigen vector corresponding to $\lambda = 1$.