Recap:

EULER'S THEOREM

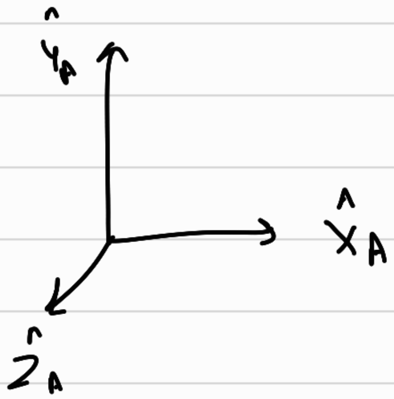
any two independent orthonormal coordinate frames can be represented by a sequence of rotations (not more than 3)

EULER ZYX CONVENTION

Gimbal lock

- first rotate by z-axis (ψ)
  then y – (ϕ)
  then x – (θ)

- So instead of 9 elements we could use 3 elements (ψ, ϕ, θ)



1- $R = R(\hat{z}_A, \alpha)$

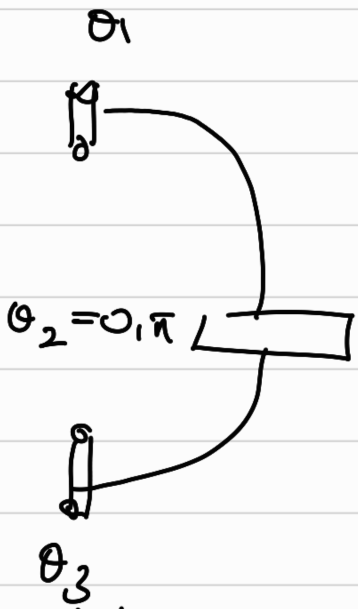
2- $R = R(\hat{z}_A, \alpha) * R(\hat{y}, \beta)$

3- $* \ R(\hat{z}_c, \gamma)$

(body fixed frame – post multiply).

- Since this uses minimal number of elements $\Rightarrow$ explicit parametrization (leading to singularities)

SINGULARITY



- eg - if $\theta_2 = 0, \pi$

the matrix becomes :-

$$R = \begin{bmatrix} c(\theta_1+\theta_3) & -s(\theta_1+\theta_3) & 0 \\ s(\theta_1+\theta_3) & c(\theta_1+\theta_3) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

now if $\theta = \theta_1 + \theta_3$ → reduces to rotation about one axis → infinite solutions.

GIMBAL LOCK

- when two axes line up, leading to a loss in the degrees of freedom.

→ if $r_{33} \neq \pm 1$ $[\theta_2 \neq 0, \pi]$

then $\theta_2 = \text{arctan2}\left(\pm\sqrt{r_{31}^2 + r_{32}^2}, r_{33}\right)$

$$\theta_1 = \text{arctan2}\left(\frac{r_{23}}{s\theta_2}, \frac{r_{13}}{s\theta_2}\right)$$

$$\theta_3 = \text{arctan2}\left(\frac{r_{32}}{s\theta_2}, \frac{-r_{31}}{s\theta_2}\right)$$

→ if $r_{33} = 1$

$\theta_1 = \theta_2 = 0$; $\theta_3 = \text{arctan2}(-r_{12}, r_{11})$

ij $r_{33} = -1$

$\theta_1 = 0, \quad \theta_2 = \pi, \quad \theta_3 = \text{arctan2}(r_{12}, -r_{11})$

$\Rightarrow$ So far all the representation we have studied they all suffer from singularities. To fix this we use quaternions.

FIELD: closed under two binary operations.
Eg: real numbers, complex numbers.

QUATERNIONS

- Broom Bridge — Wikipedia!

- $z = x + iy$

now let $z = x + iy + jz$

$$\boxed{i^2 = -1 \; ; \; j^2 = -1, \; k^2 = -1 ; \; ijk = -1}$$

- $q = q_0 + q_1 \hat{i} + q_2 \hat{j} + q_3 \hat{k}$

lives in 4-D space and is non-commutative.

- $q_0 \perp q_1, q_2, q_3$

(orthogonal to each other)

- compact representation: (q_0, q) ; $\tilde{q} = [q_1 \; q_2 \; q_3]$

addition : element-wise addition

multiplication : $p * q = (p_0 + p_1 \hat{i} + p_2 \hat{j} + p_3 \hat{k}) * (p_0 + q_1 \hat{i} + q_2 \hat{j} + q_3 \hat{k})$

$= p_0 q_0 - p \cdot q + p_0 q + q_0 p + p \times q$

$= p_0 q_0 - (p_1 q_1 + p_2 q_2 + p_3 q_3) + p_0 q + q_0 p$

conjugate : $q^* = q_0 - p$

magnitude : $|q|^2 = q q^*$

inverse : $q^{-1} = \frac{q^*}{|q|^2}$

ROTATION & QUATERNIONS

- $p' = \mathbb{R} p$
  - pure quaternion $\rightarrow$ $q_0 = 0$ and here $q \in \mathbb{R}^3$
- pure quaternion is similar to vectors in $\mathbb{R}^3$.
- similar to complex number rotation, we need to find a unit quaternion.

For unit norm,

$q_0^2 + |q|^2 = 1$

$\cos^2\theta + \sin^2\theta = 1 \implies q_0 = \cos\theta$
$q = \sin\theta$.

- suppose $q = \cos\alpha + u \sin\alpha$
  $p = \cos\beta + u \sin\beta$

$\therefore pq = \cos\alpha\cos\beta - \sin\alpha\sin\beta + u\sin\alpha\cos\alpha + \cdots$

$\therefore pq = \cos(\alpha+\beta) + u\sin(\alpha+\beta)$
$= r$

(new quanternion with same unit vector and angle as sum of α & β).

$\implies$ correspondance b/w pure quaternion & $\mathbb{R}^3$

$qv = (q_0 + q)(0 + v)$
$= -q \cdot v + q_0 v + q \times v$

[$q_0 + q \rightarrow$ unit quaternion
$(0+v) \rightarrow v$ is $\mathbb{R}^3$ vector expressed as a quaternion]

- however their direct multiplication, doesn't give us a pure quaternion.

- multiplying two pure quanternions doesn't ensure a pure quaternion.

$\Longrightarrow$ Q_0 is not closed under multiplication.

THE TRICK ?

- $p' = q p q^*$

unit quaternion $\rightarrow$ $c\frac{\theta}{2} + \hat{u} \sin\frac{\theta}{2}$

where $\theta \rightarrow$ angular displacement
$\hat{u} \rightarrow$ axis of rotation.

$\rightarrow$ $p' = q p q^*$

$$= \left(q_0 + \frac{q}{|q|}\right)(0+p)\left(q_0 - \frac{q}{|q|}\right)$$

$$= q_0^2 + \ldots$$

Eg: $p = (0,0,1)$ rotate by $\pi/2$ about the y-axis.

COMPOSITION OF QUATERNIONS

$v' = q_1 v q_1^*$

- another conjugate, $v'' = q_2 v' q_2^*$

$\therefore$ $v'' = q_2 q_1 v q_1^* q_2^*$

$$= q_2 q_1 \, v \, (q_2 q_1)^*$$

→ unit quaternion lives in S^3 space (hyper sphere)

Tell him its Krishna Jayanti and ask him to leave

PLANAR TRANSLATION

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \phi & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

PLANAR ROTATION

$$p' = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} p$$

- pose → position and orientation in a plane.

$$^{C}p = {}^{C}_{B}[R] \, {}^{B}p$$

- vector addition – only applicable if vectors are in same or parallel frame.

$$^{A}p = {}^{A}_{B}[T]$$

$${}^{A}_{B}[T] = \begin{bmatrix} r_{11} & r_{12} & {}^{A}x_B \\ r_{21} & r_{22} & {}^{A}y_B \\ 0 & 0 & 1 \end{bmatrix}$$

$\mathbb{R}^2 \in$ [redacted]

$$= \begin{bmatrix} {}^{A}_{B}[R] & {}^{A}O_{B} \\ 0 & 1 \end{bmatrix}$$

- $SE(2) = \{(p, R) \mid p \in \mathbb{R}^2, R \in SO(2)\}$
  ↳ special Euclidian group

- identity of this group SE is I.