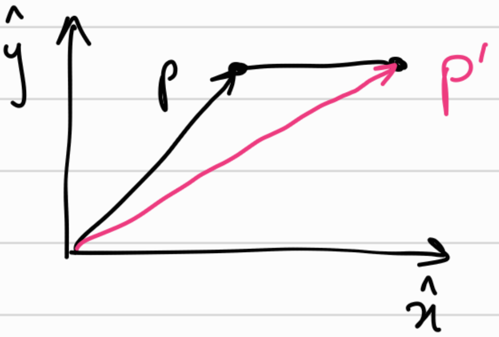
Recap :- • task space, work space
• workspace is a subset of task space

RIGID BODY TRANSFORMATION

• For rigid body transformation :
- length should be preserved
- cross product must be preserved

Translation

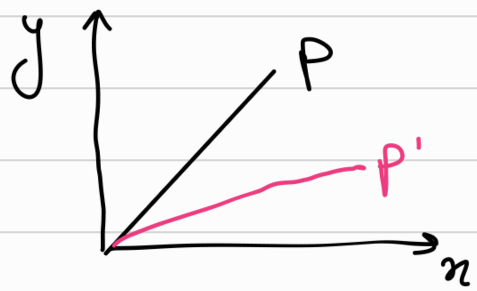


$p' = p + t$
just vector addition

Orientation

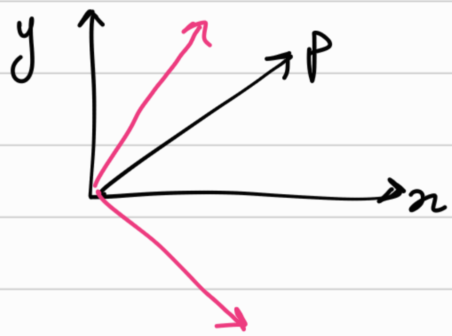
• How to represent 3D orientation :-

ACTIVE TRANSFORMATION : move the object, keep the coordinates fixed.

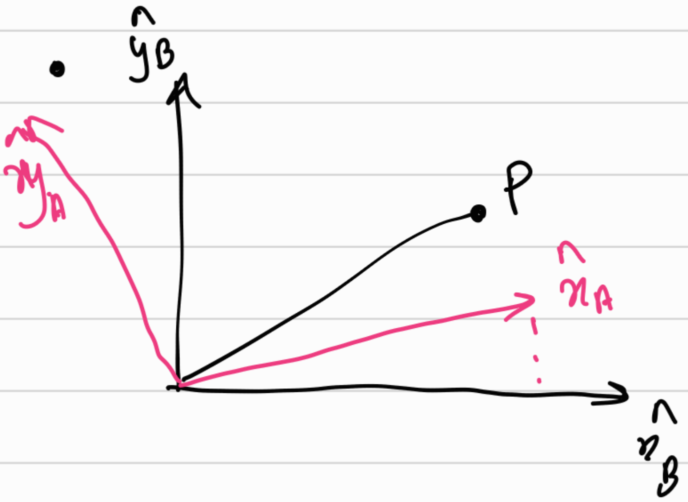


→ just moving object.

PASSIVE TRANSFORMATION : move the coordinates, object is fixed.



$\rightarrow$ moving the whole coordinate system.



A frame : $A_x \hat{x}_A + A_y \hat{y}_A$

B frame : $B_x \hat{x}_B + B_y \hat{y}_B$

$$\hat{x}_B = (\hat{x}_B \cdot \hat{x}_A)\,\hat{x}_A + (\hat{x}_B \cdot \hat{y}_A)\,\hat{y}_A$$

$$\hat{y}_B = (\hat{y}_B \cdot \hat{x}_A)\,\hat{x}_A + (\hat{y}_B \cdot \hat{y}_A)\,\hat{y}_A$$

now, $\hat{x}_B = \cos\theta\,\hat{x}_A - \sin\theta\,\hat{y}_A$

$\hat{y}_B = -\sin\theta\,\hat{x}_A + \cos\theta\,\hat{y}_A$

we know A_p and B_p so by substituting & comparing.

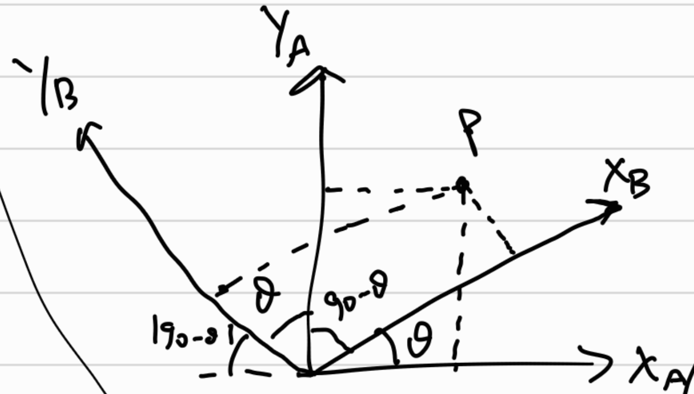
$$\boxed{\begin{bmatrix} A_x \\ A_y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} B_x \\ B_y \end{bmatrix}}$$

This is called a transformation matrix.

$$ {}^{A}p = \underbrace{{}^{A}_{B}[R]}_{} \, {}^{B}p $$

↳ rotation matrix (from frame B to frame A)

- this is implicit parametrization (representing one variable θ with 4 elements)



$$\hat{p} = {}^{B}p_x \, \hat{x}_B + {}^{B}p_y \, \hat{y}_B$$

$$\hat{x}_B = (\hat{x}_B \cdot \hat{x}_A)\hat{x}_A + (\hat{x}_B \cdot \hat{y}_A)\hat{y}_A$$

$$\hat{y}_B = (\hat{y}_B \cdot \hat{x}_A)\hat{x}_A + (\hat{y}_B \cdot \hat{y}_A)\hat{y}_A$$

→ $\hat{x}_B = \cos\theta \, \hat{x}_A + \sin\theta \, \hat{y}_A$

$\hat{y}_B = -\sin\theta \, \hat{x}_A + \cos\theta \, \hat{y}_A$

→ Combine to get

CONSTRAINTS:

1. orthogonal axes (dot product is zero)
2. Scaling is axes not allowed - unit vector columns (2 constraints)

Transformation matrix properties:

① $R \rightarrow$ transformation matrix

$R^T R = 1$

$R^{-1} = R^T$

② $GL(n) \rightarrow$ General linear group

$GL(G) \rightarrow$ all non-singular matrices

$O(G) \rightarrow$ orthogonal matrix set:

$SO(G) \rightarrow$ special orthogonal set.

$$\boxed{SO(G) \subset O(G) \subset GL(G)}$$

③ $\det(R) = \det(R^{-1})$

$\Longrightarrow$ Group properties:

(i) closure : $R_1 \in SO(2)$ $\quad R_2 \in SO(2)$

$R_1 \cdot R_2 \in SO(2)$

(ii) associative : $(R_1 \cdot R_2) \cdot R_3 = R_1 \cdot (R_2 \cdot R_3)$

(iii) identity : $I_{2\times 2}$ (ideal rotation matrix)

(iv) $RR^{-1} = I \in SO(2)$

RELATIONSHIP WITH COMPLEX NUMBERS

$z = x + iy$

$z' = e^{i\theta} z$

$= (\cos\theta + i\sin\theta)(x + iy)$

$= (x\cos\theta - y\sin\theta) + i(x\sin\theta + y\cos\theta)$

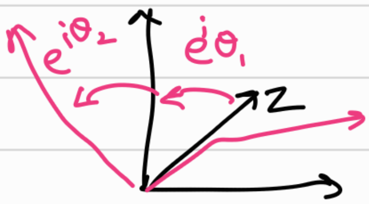
this can be expressed in the form of a matrix.

$$zz' = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- constraint $\Rightarrow$ $\det(R) = (a^2+b^2) = 1$

Quaternions – move from a two dimensional no. to a four dimensional numbers.

- $z'' = e^{i\theta_2} e^{i\theta_1} z$
  $= e^{i(\theta_1+\theta_2)} z$



using rotation matrix,

$P' = R(\theta_1) P$

$P'' = R(\theta_2) R(\theta_1) P$

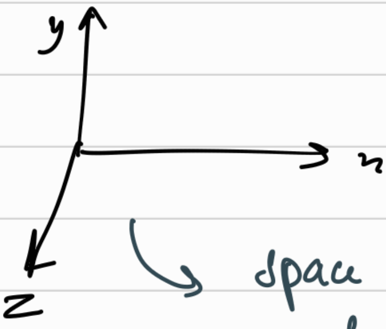
$\therefore e^{i(\theta_1+\theta_2)} = R(\theta_1)R(\theta_2)$

- $R_1 = \begin{bmatrix} c\theta_1 & -s\theta_1 \\ s\theta_1 & c\theta_1 \end{bmatrix}$

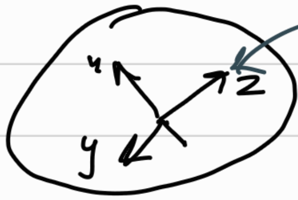
$\rightarrow \theta = \arctan(y, x)$
$= \arctan(r_{21}, r_{11})$
OR
$\arctan(-r_{12}, r_{11})$

- order of rotation doesn't matter in $SO(2)$

3-D ROTATION



space fixed frame



Body frame or moving frame

$${}^{A}_{B}[R] = R(\hat{z}_A, \theta),$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(rotating wrt single axis)

$${}^{A}_{B}[R] = R(\hat{x}, \theta)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

- when doing a complex rotation, 9 elements are required (3 for each axis)
- For orientation we only require 3 variables but we actually use 9 $\Rightarrow$ 6 constraints.

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{31} \\ r_{21} & r_{22} & r_{32} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$q \rightarrow$ columns.

Constraints :

$\|q_1\| = \|q_2\| = \|q_3\| = 1$ [no scaling]

$q_1 \cdot q_2 = q_2 \cdot q_3 = q_1 \cdot q_3 = 0$ [orthogonal]

$\} 6$

PROPERTIES:

- $R^T R = I$ & $R^T = R^{-1}$

- Special orthogonal
  $SO(3) = \{ R \in GL(n) \mid \det(R) = 1 , R^T R = I \}$

- $\det(R) = q_1^T (q_2 \times q_3)$
  $= q_1^T q_1$
  $= \underline{\underline{1}}$

- RH $\Longrightarrow \det(R) = 1$ (right hand frame)
  LH $\Longrightarrow \det(R) = -1$

- $R_1 R_2 \neq R_2 R_1$ $(SO(3))$

— — TILL HERE FOR — —
QUIZ