Robotics: Dynamics and Control

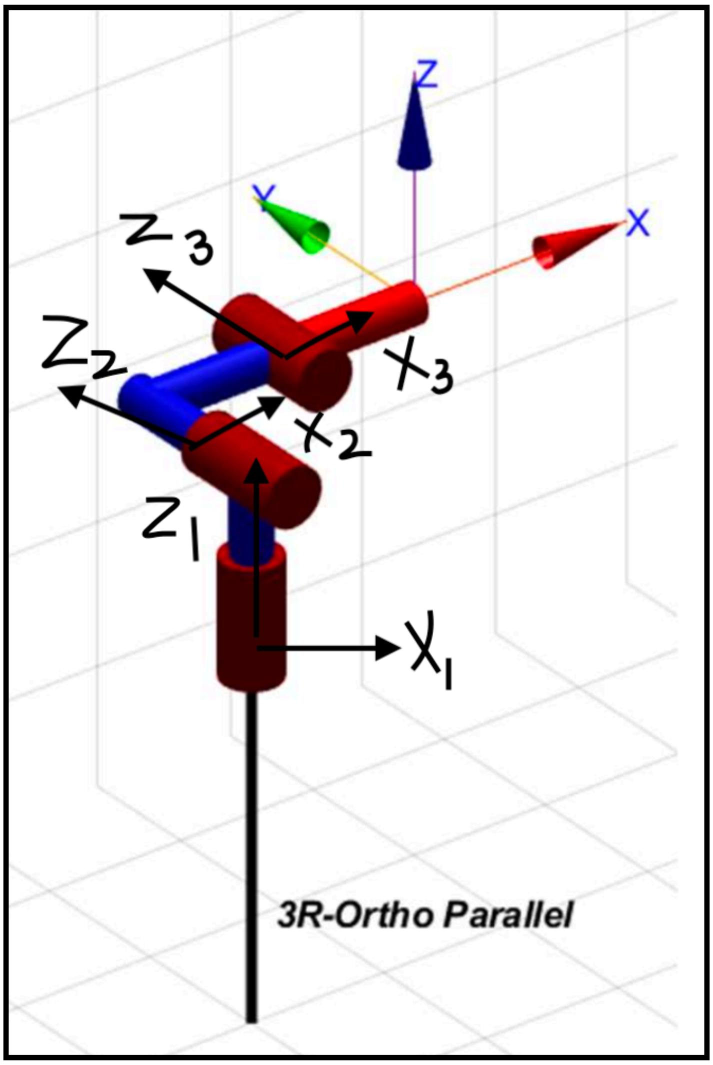
Assignment 3 - Report

Sreeja Guduri (2021102007)

QUESTION - 1: Forward Kinematics

1.1)

A 3R spatial ortho-parallel manipulator is taken to be similar to a 2R manipulator embedded in a rotating plane (the plane is able to rotate due to the rotation of the first joint).



Frame assignment for 3R spatial manipulator

DH parameter table

Frame number (i)	Link length	Link twist	LInk offset	Joint angle
1	0	-π/2	L1 + L2	θ1
2	L2 + L1	0	L2 + L3	θ2
3	L3	π/2	0	θ3

1.2)

The function 'DH_matrices' takes in the DH parameters as input and outputs the transformation matrices for each frame. This is calculated using the below formula.

$$
{}^{i-1}_{i}[\mathbf{T}] = \begin{pmatrix} \mathrm{Cos}[\theta_i] & -\mathrm{Sin}[\theta_i] & 0 & a_{i-1} \\ \mathrm{Cos}[\alpha_{i-1}]\mathrm{Sin}[\theta_i] & \mathrm{Cos}[\alpha_{i-1}]\mathrm{Cos}[\theta_i] & -\mathrm{Sin}[\alpha_{i-1}] & -\mathrm{Sin}[\alpha_{i-1}]d_i \\ \mathrm{Sin}[\alpha_{i-1}]\mathrm{Sin}[\theta_i] & \mathrm{Cos}[\theta_i]\mathrm{Sin}[\alpha_{i-1}] & \mathrm{Cos}[\alpha_{i-1}] & \mathrm{Cos}[\alpha_{i-1}]d_i \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

1.3)

The equations obtained from the DH convention transformation matrices have been verified with the code output.

The forward kinematics function makes use of the DH matrices that have been created to find the final position of the end-effector, by post-multiplying all of the obtained transformation matrices.

1.4)

The manipulators home configuration is figured out by setting all the joint angles to zero. This is then simulated in MATLAB using the specified toolbox.

1.5)

DERIVATION OF FORWARD KINEMATICS:

The forward kinematics of the 3R spatial manipulator is obtained using the DH convention.

$$ {}^{i-1}_{i}[T] = R(\hat{x}_{i-1}, \alpha_{i-1})\ T(\hat{x}_{i-1}, a_{i-1})\ T(\hat{z}_i, d_i)\ R(z_i, \theta_i) $$

Thus, $$ {}^{i-1}_{i}[T] = \begin{bmatrix} \cos[\theta_i] & -\sin[\theta_i] & 0 & \alpha_{i-1} \\ \cos[\alpha_{i-1}]\sin[\theta_i] & \cos[\alpha_{i-1}]\cos[\theta_i] & -\sin[\alpha_{i-1}] & -\sin[\alpha_{i-1}]d_i \\ \sin[\alpha_{i-1}]\sin[\theta_i] & \cos[\theta_i]\sin[\alpha_{i-1}] & \cos[\alpha_{i-1}] & \cos[\alpha_{i-1}]d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Thus, substituting the DH parameters from the table,

$$ {}^{0}_{1}[T] = \begin{bmatrix} c1 & -s1 & 0 & 0 \\ s1 & c1 & 0 & 0 \\ 0 & 0 & 1 & l_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{1}_{2}[T] = \begin{bmatrix} c_2 & -s_2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{2}_{3}[T] = \begin{bmatrix} c_3 & -s_3 & 0 & l_2 \\ s_3 & c_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{3}_{4}[T] = \begin{bmatrix} 1 & 0 & 0 & l_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Now,

$$ {}^{0}_{2}[T] = \begin{bmatrix} c_1 c_2 & -s_2 c_1 & s_1 & 0 \\ s_1 c_2 & -s_1 s_2 & -c_1 & 0 \\ s_2 & c_2 & 0 & l_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \left\{ {}^{0}_{2}[T] = {}^{0}_{1}[T]\,{}^{1}_{2}[T] \right\} $$

Similarly, ${}^{2}_{4}[T] = {}^{2}_{3}[T]\ {}^{3}_{4}[T]$

$$ = \begin{bmatrix} c_3 & -s_3 & 1 & l_3 c_3 + l_2 \\ s_3 & c_3 & 0 & l_3 s_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Hence, final end-effector transformation matrix :-

$$ {}^{0}_{4}[T] = {}^{0}_{2}[T]\ {}^{2}_{4}[T] $$

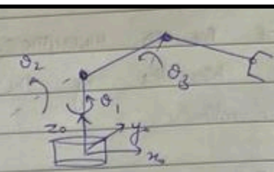
$$ = \begin{bmatrix} c_1 c_{23} & -s_{23} c_1 & s_1 & (l_2 c_2 + l_3 c_{23}) c_1 \\ s_1 c_{23} & -s_1 s_{23} & -c_1 & (l_2 c_2 + l_3 c_{23}) s_1 \\ s_{23} & c_{23} & 0 & l_1 + l_2 s_2 + l_3 s_{23} \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Hence the end-effector position : $P = \begin{bmatrix} (l_2 c_2 + l_3 c_{23}) c_1 \\ (l_2 c_2 + l_3 c_{23}) s_1 \\ l_1 + l_2 s_2 + l_3 s_{23} \end{bmatrix}$

QUESTION - 2: Inverse Kinematics

2.1)

For a 3R spatial manipulator, given:



$x = (l_2 c\theta_2 + l_3 c\theta_{23})\, c\theta_1$

$y = (l_2 c\theta_2 + l_3 c\theta_{23})\, s\theta_1$

$z = l_1 + l_2 s\theta_2 + l_3 s\theta_{23}$

Now in the orthoparallel configuration,

$x^2 = (l_2 c\theta_2 + l_3 c\theta_{23})^2\, c^2\theta_1$

$y^2 = (l_2 c\theta_2 + l_3 c\theta_{23})^2\, s^2\theta_1$

$\Longrightarrow x^2 + y^2 = (l_2 c\theta_2 + l_3 c\theta_{23})^2$

So, $x^2 + y^2 = r^2$

Thus $r = \pm\sqrt{x^2+y^2}$

Now, $x = r\cos\theta_1$ and $y = r\sin\theta_1$

Thus, $\tan\theta_1 = \dfrac{y}{x}$

$$\Longrightarrow \boxed{\theta_1 = \tan^{-1}\left(\frac{y}{x}\right)}$$

Now let, $x' = \dfrac{x}{\cos\theta_1} = l_2 c\theta_2 + l_3 c\theta_{23}$

$z' = z - d_1$

$= l_2 s\theta_2 + l_2 s\theta_{23}$

This is equivalent to a 2R manipulator with $x = x'$ and $y = z'$ and we are required to find θ_2 and θ_3.

Thus,
$$\boxed{\theta_3 = \cos^{-1}\left(\frac{x'^2 + z'^2 - (l_1^2 + l_2^2)}{2 l_1 l_2}\right)}$$

$$\boxed{\theta_2 = \tan^{-1}\left(\frac{z'}{x'}\right) - \tan^{-1}\left(\frac{l_3 s\theta_3}{l_1 + l_2 c\theta_3}\right)}$$

Hence, the inverse kinematics of a 3R orthoparallel spatial manipulator have been derived.

2.2)

This inverse kinematic function takes the end effector position as input and outputs the joint angles, using the formula derived above.

QUESTION - 3: Velocity Kinematics

3.1)

The linear and angular velocities have been computed using a recursive algorithm.

The formula for the velocities depends on the previous values computed, hence we require a recursive algorithm.

$$
\boxed{\begin{aligned}
{}^{i}\boldsymbol{\omega}_i &= {}_{i-1}^{\ i}[\mathbf{R}]\left({}^{i\text{-}1}\boldsymbol{\omega}_{i-1}\right) + \hat{\boldsymbol{k}}_i\dot{\theta}_i \\
{}^{i}\mathbf{v}_i &= {}_{i-1}^{\ i}[\mathbf{R}]\left({}^{\mathrm{i-1}}\mathbf{v}_{i-1} + {}^{\mathrm{i-1}}\boldsymbol{\omega}_{i-1} \times {}^{i-1}\boldsymbol{O}_i\right) + d_i\hat{\boldsymbol{k}}_i
\end{aligned}}
$$

3.2)

Singularity occurs when the ~~????~~ Jacobian matrix is Rank-deficient. i.e, its determinant is zero.

Now, $\dot{p} = v$

$$= J_v \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}$$

Thus, from the formula of the final end-effector position, we can compute the Jacobian as:-

$$p = \begin{bmatrix} (l_1 c_2 + l_3 c_{23})\, c_1 \\ (l_2 c_2 + l_3 c_{23})\, s_1 \\ l_1 + l_2 s_2 + l_3 s_{23} \end{bmatrix}$$

$$\therefore \dot{p} = J_v \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} \implies J_v = \begin{bmatrix} -(l_2 c_2 + l_3 c_{23}) s_1 & -(l_2 s_2 + l_3 s_{23}) c_1 & l_3 s_{23} c_1 \\ -(l_2 c_2 + l_3 c_{23}) c_1 & -(l_2 s_2 + l_2 s_{23}) s_1 & l_3 s_1 s_{23} \\ 0 & l_2 c_2 + l_3 c_{23} & l_3 c_{23} \end{bmatrix}$$

now $\det(J_v) = 0$

Since it is not a square matrix,

$$|J^T J| = 0$$

The two types of singularities that arise are called elbow and wrist singularity.

3.3)

The velocity Jacobian is computed using the below mentioned formula which is obtained using the relation between the derivative of the final position of the manipulator and the derivatives of the joint angles.

$$\boxed{\mathbf{J_v} = \begin{bmatrix} -(l_2c_2 + l_3c_{23})s_1 & -(l_2s_2 + l_3s_{23})c_1 & l_3s_{23}c_1 \\ (l_2c_2 + l_3c_{23})c_1 & -(l_2s_2 + l_3s_{23})s_1 & l_3s_1s_{23} \\ 0 & l_2c_2 + l_3c_{23} & l_3c_{23} \end{bmatrix}}$$

The angular velocity Jacobian is computed in a similar manner by using the below relation:

$$\boxed{\mathbf{J}_\omega = \begin{bmatrix} {}_0\mathrm{Z}_1 & {}_0\mathrm{Z}_2 & {}_0\mathrm{Z}_3 \end{bmatrix} = \begin{bmatrix} {}^0_1\mathbf{R}\,\mathrm{Z} & {}^0_2\mathbf{R}\,\mathrm{Z} & {}^0_3\mathbf{R}\,\mathrm{Z} \end{bmatrix}}$$