Recap: $SO(2)$; rotation matrix

- $SO(2) \rightarrow$ group of orthogonal matrices.
  - $SO(2) = \{ R \in GL(2) \; ; \; R^T R = I \; ; \; \det(R) = 1 \}$
  - also satisfies commutative law (Abelian group)

- $R : \begin{bmatrix} | & | & | \\ a & b & c \\ | & | & | \end{bmatrix}$

$$\det(R) = a^T (b \times c) = a^T a = \underline{\underline{1}}$$

if $\det(R) = -1$ [Left hand convention]

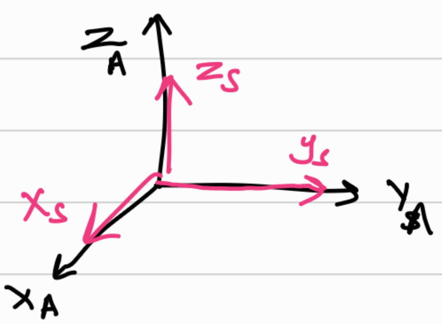
USES OF ROTATION MATRIX:

- represent orientation
- rotate a vector
- change coordinates of vector from one frame to another.

REPRESENTING ORIENTATION

Say we have a moving frame & a fixed frame

$$ {}^{S}_{A}[R] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$



(cause no change)

$$ {}^{S}_{B}[R] = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{S}_{C}[R] = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} $$

Now, $\quad {}^{S}P = {}^{S}_{a}[R]\, P^{a}$

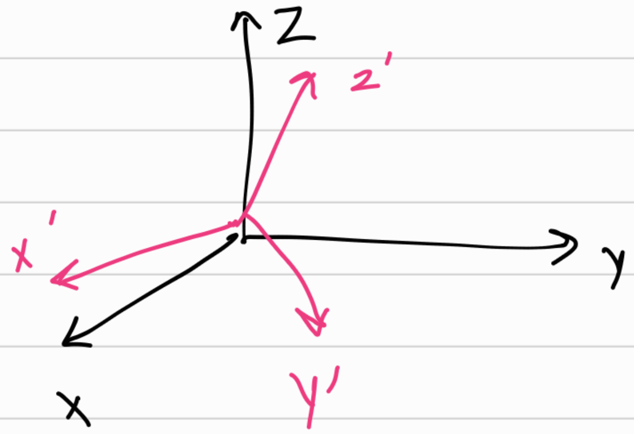
$$ {}^{S}_{A}[R] = \begin{bmatrix} x_A \cdot x_s & y_A \cdot x_s & z_A \cdot x_s \\ x_A \cdot y_s & y_A \cdot y_s & z_A \cdot y_s \\ x_A \cdot z_s & y_A \cdot z_s & z_A \cdot z_s \end{bmatrix} $$

- to change reference frame :

$$ {}^{S}_{b}[R] = {}^{S}_{a}[R]\, {}^{a}_{b}[R] $$

SUBSCRIPT CANCELLATION RULE

- $\{c\}$ is aligned with $\{s\}$



$\{c\}$ is rotated by θ. about some arbitrary axis

- the rotation matrix we get will be the general form.

- $R(\hat{\omega}, \theta) = R(-\hat{\omega}, -\theta)$

- $\{b\} \rightarrow \{s\}$
  ${}^{s}_{b}[R]$

- rotate $\{b\}$ about $\hat{\omega}$ by θ.
  [we need to know whether $\hat{\omega}$ is defined wrt $\{s\}$ or $\{b\}$].

- $\{b'\}$ is the new frame obtained; when $\hat{\omega} = \hat{\omega}_s$
  (space fixed frame)

Hz caught me tryna sleep in 1st bench
did you close eyes?
Not fully
but was v. sleepy

- $\{b''\}$ is the new frame when $\hat{\omega} = \hat{\omega}_b$
  (body fixed frame)

→ now $R(\hat{z}, \pi/2)$

$$\boxed{\begin{array}{ll} & \qquad\qquad R\,{}^{s}_{b}[R] \\ \cancel{\text{two}}\quad \hat{\omega} = \hat{\omega}_s & : \text{premultiply with } [R] \\ \hat{\omega} = \hat{\omega}_b & : \text{postmultiply with } [R] \\ & \qquad\qquad {}^{s}_{b}[R]\,R \end{array}}$$

- cartesian coordinates ⟷ polar coordinates
  for 2-D $(r, \theta) \Rightarrow re^{i\theta}$
  (polar coordinates are preferred due to geometric meaning)

3-D polar coordinates

- $R = e^{[\hat{\omega}]\theta}$ I ate your mom's coochie and cream (pied her)

$\hat{\omega} \rightarrow$ axis of rotation
$\theta \rightarrow$ angle of rotation

MATRIX BASICS:

- $\vec{a} \times \vec{b} = [\hat{a}]\,\vec{b}$
  ↳ skew symmetric matrix

- $\hat{\omega} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \Longleftrightarrow [\hat{\omega}] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$

- Skew symmetric matrix $A = -A^T$
  diagonal elements $= 0$

- set of all skew symmetric matrix forms a group : so(3) called Lie algebra.

→ SO(3) is called the lie group & so(3) is the lie algebra of the Lie group.

Now, $\dot{x} = ax$

$\Rightarrow x(t) = e^{at} x(t_0)$

$x(t_0) \rightarrow$ initial cond^n

now if a is matrix
$x(t) = e^{[a]t} x(t_0)$

$\frac{dx}{x} = a\, dt$

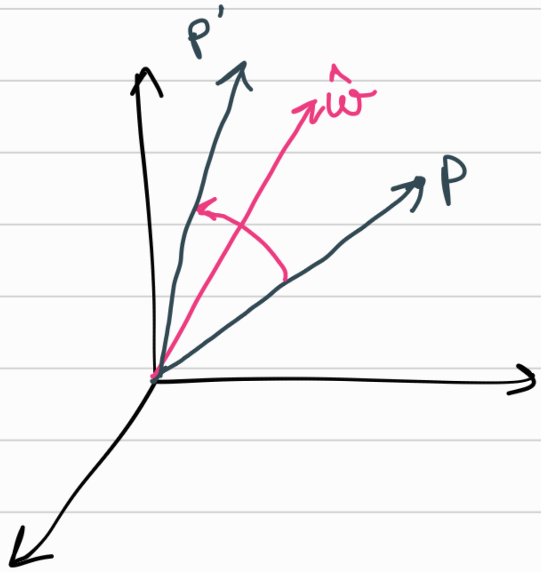
$\ln x = at$

$x(t) = e^{at} \cdot x(t_0)$

- we use Taylor series expansion to simplify the eq^n.

- Eigen value decomposition : $A = P \underbrace{\Lambda}_{\text{eigen value matrix}} P^{-1}$

HW to learn about this



velocity → differentiation.

$$\dot{p} = \hat{\omega} \times p$$
$$= [\hat{\omega}]\, p$$

<u>the sol^n is :</u>

$$p(t) = e^{[\hat{\omega}]t}\, p(0)$$

now $p(t) = p(\theta) \implies p(\theta) = e^{[\hat{\omega}]\theta}\, p(0)$

Expanding $e^{[\hat{\omega}]\theta}$,

$$e^{[\hat{\omega}]\theta} = I + [\hat{\omega}]\theta + \frac{[\hat{\omega}]^2\theta^2}{2!} + \cdots$$

(taylor's series)

now, $[\hat{\omega}]^3 = -[\hat{\omega}]$

$$\therefore\ e^{[\hat{\omega}]\theta} = I + [\hat{\omega}]\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$

which simplifies to :-

$$\boxed{R(\hat{\omega}, \theta) = e^{[\hat{\omega}]\theta} = I + \sin\theta\,[\hat{\omega}] + (1-\cos\theta)\,[\hat{\omega}]^2}$$

Rodrigues' formula for rotation

$$I + \sin\theta\,[\hat{\omega}] + (1-\cos\theta)\,[\hat{\omega}]^2 = R$$

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} +$$

$$\begin{bmatrix} 0 & -\omega_3 & +\omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \sin\theta$$

$$+$$

$$\begin{bmatrix} \; \end{bmatrix} (1-\cos\theta)$$

$$= 1 + (1-\cos\theta)(-\omega_1^2 - \omega_2^2) \ldots$$

[TB → Modern Robotics has the proof].

NOTE :- $(\omega_1^2 + \omega_2^2 + \omega_3^2) = 1$

- if $\sin\theta \neq 0$ and not an integer multiple of π.

$$\hat{\omega}_1 = \frac{1}{2\sin\theta}(r_{32} - r_{23})$$

$$\hat{\omega}_2 = \frac{1}{2\sin\theta}(r_{13} - r_{31})$$

$$\hat{\omega}_3 = \frac{1}{2\sin\theta}(r_{21} - r_{12})$$

- we get θ from the trace of the matrix:
$r_{11} + r_{22} + r_{33} = 1 + 2\cos\theta$

→ we notice that when $\theta = 0 \implies \hat{\omega}$ = undefined
This is because we are now using explicit parameterization (θ & $\hat{\omega}$) ⇒ leads to singularities.
Hence, we will have certain constraints.

CONSTRAINTS:

a) $R = I$; Trace $(R) = 3$
$\theta = \pm 2\pi, \pm 4\pi \ldots$ and $\hat{\omega}$ is undefined.

b) Trace $(R) = -1$
$\theta = \pm \pi, \pm 3\pi,$ and $\hat{\omega}$ would be one of three defined configs.

c) $\theta = \cos^{-1}\left(\frac{1}{2}\left[\text{trace }(R) - 1\right]\right) \in [0, \pi)$

and
$$[\hat{\omega}] = \frac{1}{2\sin\theta}(R - R^T)$$