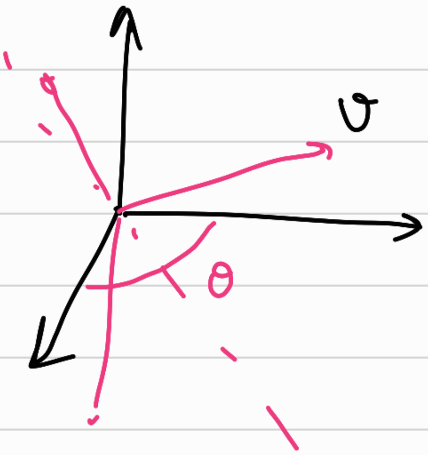
Recap:



v is a pure quaternion

$q_0 = (0, v)$

$q = \cos\left(\frac{\theta}{2}\right) + \hat{n} \sin\left(\frac{\theta}{2}\right)$

$\downarrow$

unit quaternion

$$v' = q_{n_1} q_0 q_{n_1}^* = q v q^*$$

$$v'' = q_{n_2} v' q_{n_2}^*$$

$$= q_{n_2} q_{n_1} v q_{n_1}^* q_{n_2}^*$$

$$= q_{n_2} q_{n_1} v (q_{n_2} q_{n_1})^*$$

HOMOGENEOUS TRANSFORMATION MATRIX

- equation is homogeneous if $Ax = 0$
- for rigid bodies :- (x, y, w) but $w = 1$ (no scaling).

- $p' = p + t$ $\qquad p' = -t + p$

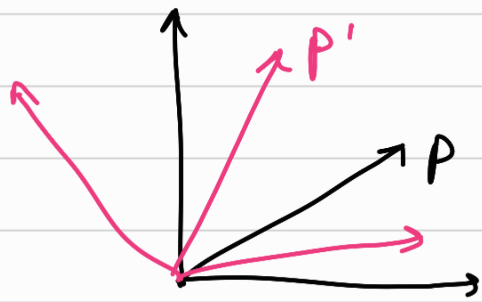
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

• inverse of t is just $T(-p)$

$$T(p)^{-1} = T(-p)$$

only for pure translation

$\rightarrow$ $p' = \begin{bmatrix} c\theta & -\sin\theta \\ s\theta & \cos\theta \end{bmatrix} p$



$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

this is for pure rotation

Vector addition :- can only be added if vectors are in same frames or expressed in parallel frames.

$${}^{A}p = {}^{A}O_B + {}^{C}p$$

$$= {}^{A}O_B + {}^{C}_{B}[R]\,{}^{C}p$$

$\Longrightarrow$ $\begin{bmatrix} x_A \\ y_A \\ 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & {}^{A}x_B \\ r_{21} & r_{22} & {}^{A}y_B \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_B \\ y_B \\ 1 \end{bmatrix}$

$\downarrow$

${}^{A}_{B}[T] \rightarrow$ transformation matrix

- $T \in SE(3)$
  (special eucledian)

  $R \in SO(3)$
  (special orthogonal)

- $SE(3) = \mathbb{R}^3 \times SO(3)$

- $SE(3)$: forms a group - identity, closure, associative, inverse.

$$\longrightarrow \quad T = \begin{bmatrix} R & P \\ 0 & 1 \end{bmatrix}_{4\times 4}$$

$$TT^{-1} = I$$

$$\begin{bmatrix} R & P \\ 0 & 1 \end{bmatrix} \begin{bmatrix} Q & q \\ 0 & 1 \end{bmatrix} = I$$

now, $RQ = I$

$$\cancel{R}\, Q = R^{-1} I$$
$$= R^T I$$
$$= \underline{\underline{R^T}}$$

$$q = -R^T p$$

$$\therefore T^{-1} = \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix}$$

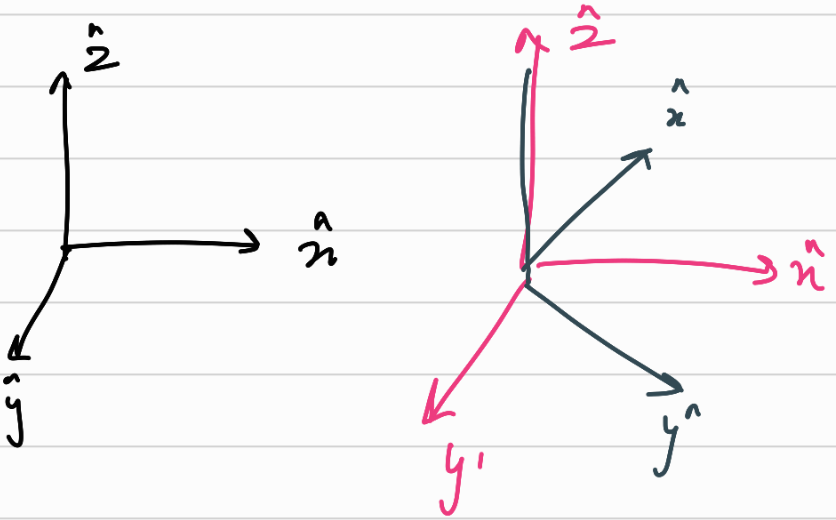
$\Longrightarrow$ pure translation : ${}^{A}_{B}[T] = \begin{bmatrix} I & {}^{A}_{B}P \\ 0 & 1 \end{bmatrix}$

pure rotation : ${}^{A}_{B}[T] = \begin{bmatrix} {}^{A}_{B}[R] & 0 \\ 0 & 1 \end{bmatrix}$

- from now, $T \rightarrow$ pure translation
  $R \rightarrow$ pure rotation

Now,

- $T_1 = T(\hat{x}, L_1) R(\hat{z}, \theta_1)$ $R(\hat{z}, \theta_1)$



post multiplication
so wrt translated frame.

- $T_2 = R(\hat{z}, \theta_1) T(\hat{x}, L_1)$

first we rotate along $\hat{z}$ and then move frame by L_1 wrt rotated frame.

$$T_1 = \begin{bmatrix} \cos\theta & -s\theta & 0 & L_1 \\ s\theta & c\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

pose

origin is this

$$T_2 = \begin{bmatrix} c\theta & -s\theta & 0 & c\theta_1 L_1 \\ s\theta & c\theta & 0 & \sin\theta_1 L_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

origin of transformed matrix.

USES OF T:

- represent the configuration of a rigid body
- to change the reference frame of a vector
- displace a vector or frame.

eg -

$$^{s}_{b}R = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\hat{n}_B = \hat{z}_A$$

(take dot product with all axes)

now, $^{b}_{c}T = {}^{b}_{s}T \, {}^{s}_{c}T$

$$= \left({}^{s}_{b}T\right)^{-1} {}^{s}_{c}T$$

$$\Longrightarrow \quad T = (R, p)$$
$$= (R(\hat{\omega}, \theta), \; p)$$

now, $\quad \text{trans}(p) = \begin{bmatrix} I & p \\ 0 & 1 \end{bmatrix}$

$$R(\hat{\omega}, \theta) = \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix}$$

$$^{s}_{b}[T] = \begin{bmatrix} ^{s}_{b}[R] & ^{s}_{b}p \\ 0 & 1 \end{bmatrix}$$

(i) premultiply with T :- transformation occurs wrt space fixed frame.

rotation of $\{b\}$ with respect to $\{s\}$ and then translation by p in $\longrightarrow \{s\}$ to get $\{b'\}$

$$^{s}_{b'}[T] = T \; ^{s}_{b}[T]$$
$$= \text{Trans}(p) \; R(\hat{\omega}, \theta) \; ^{s}_{b}[T]$$

relation b/w $\{b\}$ & $\{s\}$

(ii) postmultiply with T :- with respect to body fixed frame.

first translation about p in $\{b\}$ then rotation by θ wrt $\{b\}$ $\rightarrow \{b''\}$.

$$^{s}_{b''}[T] = {}^{s}_{b}[T] \; \text{Trans}(p) \; R(\hat{\omega}, \theta)$$

COMPOSITION

- $$ {}^A_B T = \begin{bmatrix} {}^A_B R & {}^A_B O \\ 0 & 1 \end{bmatrix} $$

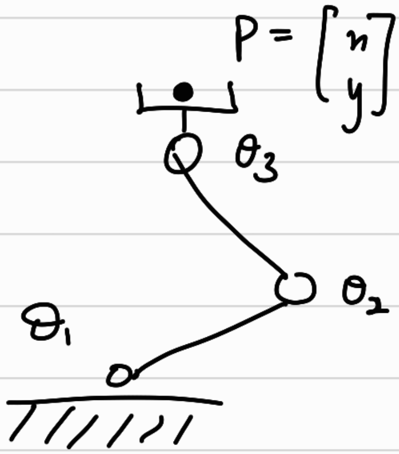
$$ {}^B_C T = \begin{bmatrix} {}^B_C R & {}^B_C O \\ 0 & 1 \end{bmatrix} $$

$$ \Rightarrow {}^A_C T = {}^A_B T \; {}^B_C T $$

$$ = \begin{bmatrix} {}^A_B R \; {}^B_C R & {}^A_B R \; {}^B O_C + {}^A O_B \\ 0 & 1 \end{bmatrix} $$

KINEMATICS

- study of motion without considering forces.



<u>Forward</u> <u>kinematics</u> :-

$p = f(\theta)$

$p, \theta \rightarrow$ vectors

$f \rightarrow$ vector valued non-linear function

$p \in R^m$ $\qquad m = 3$ or 6

$\theta \in R^n$ $\qquad n =$ depends on joints & configura-

$f : R^n \rightarrow R^m$

For forward kinematics, given $\theta \to$ find P.