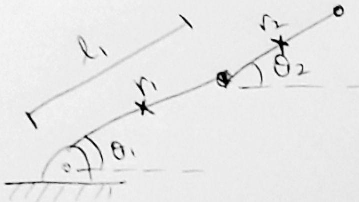
06/09/23 Langragian Statistics

$\mathcal{L} = KE_T - PE_T$



$q = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$

${}^0p_{cm_1} = \begin{pmatrix} r_1 c\theta_1 \\ r_1 s\theta_1 \end{pmatrix}$ [posn of CoM]

${}^0p_{cm_2} = \begin{pmatrix} l_1 c\theta_1 + r_2 c\theta_{12} \\ l_1 s\theta_1 + r_2 s\theta_{12} \end{pmatrix}$

$KE_T = KE_1 + KE_2$.

$KE_1 = \frac{1}{2} m_1 \, {}^0v_1^T \, {}^0v_1 + \frac{1}{2} \omega^T \, {}_1I \, {}^0_1\omega$

$KE_2 = \frac{1}{2} m_2 \, {}^0v_2^T \, {}^0v_2 + \frac{1}{2} \, {}^1\omega_2 \, I \cdot {}^0\omega_2$

${}^0\dot{p}_{cm_1} = \begin{pmatrix} -r_1 s\theta_1 \\ r_1 c\theta_1 \end{pmatrix} \dot{\theta}_1$

${}^0\dot{p}_{cm_2} = \begin{pmatrix} -l_1 s\theta_1 \dot{\theta}_1 + r_2 s\theta_{12} \dot{\theta}_{12} \\ l_1 c\theta_1 \dot{\theta}_1 + r_2 c\theta_{12} \dot{\theta}_{12} \end{pmatrix}$

$KE_T = KE_1 + KE_2$

$= \frac{1}{2} m_1 r_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 (l_1^2 \dot{\theta}_1^2 + r_2^2 \dot{\theta}_{12}^2$

$+ 2 l_1 r_2 c\theta_2 \dot{\theta}_1 \theta_{12})$

Now, $KE^T = \frac{1}{2} m_1 r_1^2 \dot{\theta}_{12}^2$ . . .

$PE^T =$

So, $\frac{d}{dt}\left(\frac{d\mathcal{L}}{d\dot{\theta}_1}\right) - \frac{dL}{d\theta_1} = \tau_1$

(gives torque for 1st joint)

Similarly for second joint, $\frac{d}{dt}\left(\frac{dL}{d\dot{\theta}_2}\right) - \frac{dL}{d\theta_2} = \tau_2$

$$\boxed{\tau = M(q)\,\ddot{q} + C(q,\dot{q}) + G(q)}$$

equation of motion (for 2R)

- $M(q)$ – $(n \times n)$
  $\ddot{q}$ – $(n \times 1)$

  τ – $(n \times 1)$
  $C(q,\dot{q})$ – $(n \times 1)$
  $G(q)$ – $(n \times 1)$

JOINT SPACE DYNAMIC MODEL

- inverse dynamics – find joint torques based on given inertia matrix M, centripetal & coriolis
- forward dynamics – find $\ddot{q}$ given the rest.

$\Rightarrow$ $KE = \frac{1}{2}\dot{q}^T M \dot{q}$ [$q \rightarrow$ generalised pos^n, $\dot{q} \rightarrow$ generalised vel]

$PE = V(q)$

$$\frac{d}{dt}\left(\frac{dL}{dq}\right) = \frac{d}{dt}\left(M\dot{q}\right)$$

$$- \dot{M}\dot{q} + M\ddot{q}$$

$$\frac{\partial L}{\partial q} = \frac{1}{2} \dot{q}^T \frac{\partial M}{\partial q} \dot{q} - \frac{\partial V}{\partial q}$$

$$\dot{M}\dot{q} + M\ddot{q} + \left(\frac{1}{2} \dot{q}^T \frac{\partial M}{\partial q} \dot{q} \right) - \frac{\partial V}{\partial q} = \tau$$

$$\Longrightarrow \frac{1}{2} \dot{q}^T M \dot{q} = \frac{1}{2} \sum_i \left(m_i v_{c_i}^T \cdot v_{c_i} + \omega_i^T I_{c_i} \omega_i \right)$$

KE $\nearrow$

now, $v = J_v \dot{\theta} = J_v \dot{q}$
$\omega = J_\omega \dot{\theta} = J_\omega \dot{q}$

substituting in the above formula :-

$$M = \sum_i \left(m_i J_{v_i}^T J_{v_i} + J_{\omega_i}^T I_{c_i} J_{\omega_i} \right)$$

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- $$M = \begin{bmatrix} m_{11} & \cdots & m_{1n} \\ \vdots & & \vdots \\ m_{n1} & \cdots & m_{nn} \end{bmatrix}$$

$m_{11} \rightarrow$ inertia of arm perceived at first joint.

④ $$C_{ij} = \frac{1}{2}\sum_{k=1}^{n}\left(\frac{\partial M_{ij}}{\partial q_k} + \frac{\partial M_{ik}}{\partial q_j} - \frac{\partial M_{kj}}{\partial q_i}\right)\dot{q}_k$$

$i, j = 2 \cdots n$

③ $$G = g\begin{bmatrix} (m_1 r_1 + (m_2 + m_p) l_1)\cos\theta_1 + (m_p l_2 + m_2 r_2)\cos(\theta_1 + \theta_2) \\ (m_2 r_2 + m_p l_2)\cos(\theta_1 + \theta_2) \end{bmatrix}$$

STEPS:

① locate base frame
② find position of CoM
③ find J_v and J_ω
④ find M, C and G

Thus eq^n of motion $M(q)\ddot{q} + C(q, \dot{q}) + G(q) = \tau$