CHAPTER 14

Special Theory of Relativity

14.1 Introduction

In Section 2.7, it was pointed out that the Newtonian idea of the complete separability of space and time and the concept of the absoluteness of time break down when they are subjected to critical analysis. The final overthrow of the Newtonian system as the ultimate description of dynamics was the result of several crucial experiments, culminating with the work of Michelson and Morley in 1881–1887. The results of these experiments indicated that the speed of light is independent of any relative uniform motion between source and observer. This fact, coupled with the finite speed of light, required a fundamental reorganization of the structure of dynamics. This was provided during the period 1904–1905 by H. Poincaré, H. A. Lorentz, and A. Einstein,* who formulated the **theory of relativity** in order to provide a consistent description of the experimental facts. The basis of relativity theory is contained in two postulates:

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*Although Albert Einstein (1879–1955) is usually accorded the credit for the formulation of relativity theory (see, however, Wh53, Chapter 2), the basic *formalism* had been discovered by Poincaré and Lorentz by 1904. Einstein was unaware of some of this previous work at the time (1905) of the publication of his first paper on relativity. (Einstein's friends often remarked that "he read little, but thought much.") The important contribution of Einstein to special relativity theory was the replacement of the many *ad hoc* assumptions made by Lorentz and others with but two basic postulates from which all the results could be derived. [The question of precedence in relativity theory is discussed by G. Holton, *Am. J. Phys.* **28**, 627 (1960); see also Am63.] In addition, Einstein later provided the fundamental contribution to the formulation of the *general* theory of relativity in 1916. His first publication on a topic of importance in general relativity—speculations on the influence of gravity on light—was in 1907. It is interesting to note that Einstein's 1921 Nobel Prize was awarded, not for contributions to relativity theory, but for his work on the photoelectric effect.

I. *The laws of physical phenomena are the same in all inertial reference frames (that is, only the relative motion of inertial frames can be measured; the concept of motion relative to "absolute rest" is meaningless).*

II. *The velocity of light (in free space) is a universal constant, independent of any relative motion of the source and the observer.*

Using these postulates as a foundation, Einstein was able to construct a beautiful, logically precise theory. A wide variety of phenomena that take place at high velocity and cannot be interpreted in the Newtonian scheme are accurately described by relativity theory.

Postulate I, which Einstein called the *principle of relativity*, is the fundamental basis for the theory of relativity. Postulate II, the law of propagation of light, follows from Postulate I if we accept, as Einstein did, that Maxwell's equations are fundamental laws of physics. Maxwell's equations predict the speed of light in vacuum to be c, and Einstein believed this to be the case in all inertial reference frames.

We do not attempt here to give the experimental background for the theory of relativity; such information can be found in essentially every textbook on modern physics and in many others concerned with electrodynamics.* Rather, we simply accept as correct the above two postulates and work out some of their consequences for the area of mechanics.† The discussion here is limited to the case of **special relativity,** in which we consider only inertial reference frames, that is, frames that are in uniform motion with respect to one another. The more general treatment of accelerated reference frames is the subject of the **general theory of relativity.**

14.2 Galilean Invariance

In Newtonian mechanics, the concepts of space and time are completely separable; furthermore, time is assumed to be an absolute quantity susceptible of precise definition independent of the reference frame. These assumptions lead to the invariance of the laws of mechanics under coordinate transformations of the following type. Consider two inertial reference frames K and K', which move along their x_1- and x_1'-axes with a uniform relative velocity v (Figure 14-1). The transformation of the coordinates of a point from one system to the other is clearly of the form

$$\left.\begin{aligned} x_1' &= x_1 - vt \\ x_2' &= x_2 \\ x_3' &= x_3 \end{aligned}\right\} \tag{14.1a}$$

Also, we have

$$t' = t \tag{14.1b}$$

*A particularly good discussion of the experimental necessity for relativity theory can be found in Panofsky and Phillips (Pa62, Chapter 15).

†Relativistic effects in electrodynamics are discussed in Heald and Marion (He95, Chapter 14).

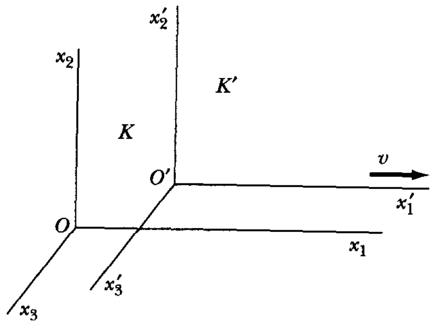

FIGURE 14-1 Two inertial reference frames K and K' move along their x_1- and x_1'-axes with a uniform relative velocity v.

Equations 14.1 define a **Galilean transformation.** Furthermore, the element of length in the two systems is the same and is given by

$$\begin{aligned} ds^2 &= \sum_j dx_j^2 \\ &= \sum_j dx_j'^2 = ds'^2 \end{aligned} \tag{14.2}$$

The fact that Newton's laws are invariant with respect to Galilean transformations is termed **the principle of Newtonian relativity** or **Galilean invariance.** Newton's equations of motion in the two systems are

$$\begin{aligned} F_j &= m\ddot{x}_j \\ &= m\ddot{x}_j' = F_j' \end{aligned} \tag{14.3}$$

The form of the law of motion is then *invariant* to a Galilean transformation. The individual terms are not invariant, however, but they transform according to the same scheme and are said to be *covariant.*

We can easily show that the Galilean transformation is inconsistent with Postulate II. Consider a light pulse emanating from a flashbulb positioned in frame K'. The velocity transformation is found from Equation 14.1a, where we consider the light pulse only along x_1:

$$\dot{x}_1' = \dot{x}_1 - v \tag{14.4}$$

In system K', the velocity is measured as $\dot{x}_1' = c$; Equation 14.4 therefore indicates the speed of the light pulse to be $\dot{x}_1 = c + v$, clearly in violation of Postulate II.

14.3 Lorentz Transformation

The principle of Galilean invariance predicts that the velocity of light is different in two inertial reference frames that are in relative motion. This result is in contradiction to the second postulate of relativity. Therefore, a new transformation law that renders physical laws *relativistically* covariant must be found. Such a transformation law is the **Lorentz transformation.** The original use of the

Lorentz transformation preceded the development of Einsteinian relativity theory,* but it also follows from the basic postulates of relativity; we derive it on this basis in the following discussion.

If a light pulse from a flashbulb is emitted from the common origin of the systems K and K' (see Figure 14-1) when they are coincident, then according to Postulate II, the wavefronts observed in the two systems must be described† by

$$\left.\begin{aligned} \sum_{j=1}^{3} x_j^2 - c^2t^2 &= 0 \\ \sum_{j=1}^{3} x_j'^2 - c^2t'^2 &= 0 \end{aligned}\right\} \tag{14.5}$$

We can already see that Equations 14.5, which are consistent with the two postulates of the theory of relativity, cannot be reconciled with the Galilean transformations of Equations 14.1. The Galilean transformation allows a spherical light wavefront in one system but requires the center of the spherical wavefront in the second system to move at velocity v with respect to the first system. The interpretation of Equations 14.5, according to Postulate II, is that each observer believes that his spherical wavefront has its center fixed at his own coordinate origin as the wavefront expands.

We are faced with a quandary. We must abandon either the two relativity postulates or the Galilean transformation. Much experimental evidence, including the Michelson-Morley experiment and the aberration of starlight, requires the two postulates. However, the belief in the Galilean transformation is entrenched in our minds by our everyday experience. The Galilean transformation had produced satisfactory results, including those of the preceding chapters of this book, for centuries. Einstein's great contribution was to realize that the Galilean transformation was *approximately* correct, but that we needed to reexamine our concepts of space and time.

Notice that we do not assume $t = t'$ in Equations 14.5. Each system, K and K', has its own clocks, and we assume that a clock may be located at any point in space. These clocks are all identical, run the same way, and are synchronized. Because the flashbulb goes off when the origins are coincident and the systems move only in the x_1-direction with respect to each other, by direct observation we have

$$\left.\begin{aligned} x_2' &= x_2 \\ x_3' &= x_3 \end{aligned}\right\} \tag{14.6}$$

At time $t = t' = 0$, when the flashbulb goes off, the motion of the origin O' of K' is measured in K to be

$$x_1 - vt = 0 \tag{14.7}$$

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*The transformation was originally postulated by Hendrik Anton Lorentz (1853–1928) in 1904 to explain certain electromagnetic phenomena, but the formulas had been set up as early as 1900 by J. J. Larmor. The complete generality of the transformation was not realized until Einstein *derived* the result. W. Voigt was actually the first to use the equations in a discussion of oscillatory phenomena in 1887.

†See Appendix G.

and in system K', the motion of O' is

$$x_1' = 0 \tag{14.8}$$

At time $t = t' = 0$ we have $x_1' = x_1 - vt$, but we know that Equation 14.1a is incorrect. Let us assume the next simplest transformation, namely,

$$x_1' = \gamma(x_1 - vt) \tag{14.9}$$

where γ is some constant that may depend on v and some constants, but not on the coordinates x_1, x_1', t, or t'. Equation 14.9 is a linear equation and assures us that each event in K corresponds to one and only one event in K'. This additional assumption in our derivation will be vindicated if we can produce a transformation that is consistent with *all* the experimental results. Notice that γ must normally be very close to 1 to be consistent with the classical results discussed in earlier chapters.

We can use the preceding arguments to describe the motion of the origin O of system K in both K and K' to also determine

$$x_1 = \gamma'(x_1' + vt') \tag{14.10}$$

where we only have to change the relative velocities of the two systems.

Postulate I demands that the laws of physics be the same in both reference systems such that $\gamma = \gamma'$. By substituting x_1' from Equation 14.9 into Equation 14.10, we can solve the remaining equation for t':

$$t' = \gamma t + \frac{x_1}{\gamma v}(1 - \gamma^2) \tag{14.11}$$

Postulate II demands that the speed of light be measured to be the same in both systems. Therefore, in both systems we have similar equations for the position of the flashbulb light pulse:

$$\left.\begin{aligned} x_1 &= ct \\ x_1' &= ct' \end{aligned}\right\} \tag{14.12}$$

Algebraic manipulation of Equations 14.9–14.12 gives (see Problem 14-1)

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{14.13}$$

The complete transformation equations can now be written as

$$\left.\begin{aligned} x_1' &= \frac{x_1 - vt}{\sqrt{1 - v^2/c^2}} \\ x_2' &= x_2 \\ x_3' &= x_3 \\ t' &= \frac{t - \dfrac{vx_1}{c^2}}{\sqrt{1 - v^2/c^2}} \end{aligned}\right\} \tag{14.14}$$

These equations are known as the Lorentz (or Lorentz-Einstein) transformation in honor of the Dutch physicist H. A. Lorentz, who first showed that the equations are necessary so that the laws of electromagnetism have the same form in all inertial reference frames. Einstein showed that these equations are required for all the laws of physics.

The inverse transformation can easily be obtained by replacing v by $-v$ and exchanging primed and unprimed quantities in Equations 14.14.

$$\left.\begin{aligned} x_1 &= \frac{x_1' + vt'}{\sqrt{1 - v^2/c^2}} \\ x_2 &= x_2' \\ x_3 &= x_3' \\ t &= \frac{t' + \dfrac{vx_1'}{c^2}}{\sqrt{1 - v^2/c^2}} \end{aligned}\right\} \tag{14.15}$$

As required, these equations reduce to the Galilean equations (Equations 14.1) when $v \to 0$ (or when $c \to \infty$).

In electrodynamics, the fields propagate with the speed of light, so Galilean transformations are never allowed. Indeed, the fact that the electrodynamic field equations (**Maxwell's equations**) are not covariant to Galilean transformations was a main factor in the realization of the need for a new theory. It seems rather extraordinary that Maxwell's equations, which are a complete set of equations for the electromagnetic field and are *covariant to Lorentz transformations*, were deduced from experiment long before the advent of relativity theory.

The velocities measured in each of the systems are denoted by u.

$$\left.\begin{aligned} u_i &= \frac{dx_i}{dt} \\ u_i' &= \frac{dx_i'}{dt'} \end{aligned}\right\} \tag{14.16}$$

Using Equations 14.14, we determine

$$u_1' = \frac{dx_1'}{dt'} = \frac{dx_1 - v\,dt}{dt - \dfrac{v}{c^2}dx_1}$$

$$u_1' = \frac{u_1 - v}{1 - \dfrac{u_1 v}{c^2}} \tag{14.17a}$$

Similarly, we determine

$$u_2' = \frac{u_2}{\gamma\left(1 - \dfrac{u_1 v}{c^2}\right)} \tag{14.17b}$$

$$u_3' = \frac{u_3}{\gamma\left(1 - \dfrac{u_1 v}{c^2}\right)} \tag{14.17c}$$

Now we can determine whether Postulate II is satisfied directly. An observer in system K measures the speed of the light pulse from the flashbulb to be $u_1 = c$ in the x_1-direction. From Equation 14.17a, an observer in K' measures

$$u_1' = \frac{c - v}{1 - \dfrac{v}{c}} = c\left(\frac{c - v}{c - v}\right) = c$$

as required by Postulate II, independent of the relative system speed v.

EXAMPLE 14.1

Determine the relativistic length contraction* using the Lorentz transformation.

Solution. Consider a rod of length l lying along the x_1-axis of an inertial frame K. An observer in system K' moving with uniform speed v along the x_1-axis (as in Figure 14-1) measures the length of the rod in the observer's own coordinate system by determining *at a given instant of time* t' the difference in the coordinates of the ends of the rod, $x_1'(2) - x_1'(1)$. According to the transformation equations (Equations 14.14),

$$x_1'(2) - x_1'(1) = \frac{[x_1(2) - x_1(1)] - v[t(2) - t(1)]}{\sqrt{1 - v^2/c^2}} \tag{14.18}$$

where $x_1(2) - x_1(1) = l$. Note that times $t(2)$ and $t(1)$ are the times in the K system at which the observations are made; they do not correspond to the instants in K' at which the observer measures the rod. In fact, because $t'(2) = t'(1)$, Equations 14.14 give

$$t(2) - t(1) = [x_1(2) - x_1(1)]\frac{v}{c^2}$$

The length l' as measured in the K' system is therefore

$$l' = x_1'(2) - x_1'(1)$$

Equation 14.18 now becomes

$$\text{length contraction} \qquad \boxed{l' = l\sqrt{1 - v^2/c^2}} \tag{14.19}$$

and, to a stationary observer in K, objects in K' also appear contracted. Thus, to an observer in motion relative to an object, the dimensions of objects are contracted by a factor $\sqrt{1 - \beta^2}$ in the direction of motion, in which $\beta \equiv v/c$.

An interesting consequence of the FitzGerald-Lorentz contraction of length was reported in 1959 by James Terrell.† Consider a cube of side l moving with uniform velocity $\mathbf{v}$ with respect to an observer some distance away. Figure 14-2a

*The contraction of length in the direction of motion was proposed by G. F. FitzGerald (1851–1901) in 1892 as a possible explanation of the Michelson-Morley ether-drift experiment. This hypothesis was adopted almost immediately by Lorentz, who proceeded to apply it in his theory of electrodynamics.

†J. Terrell, *Phys. Rev.* **116**, 1041 (1959).

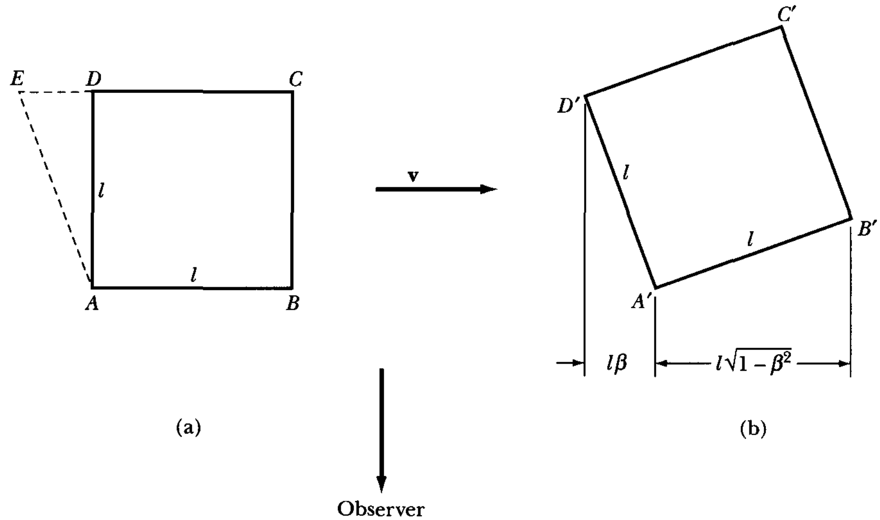


FIGURE 14-2 (a) An observer far away sees a cube of sides l at rest in system K. (b) Terrell pointed out that surprisingly, the same cube appears to be rotated if it is moving to the right with velocity **v** relative to system K.

shows the projection of the cube on the plane containing the velocity vector **v** and the observer. The cube moves with its side AB perpendicular to the observer's line of sight. We wish to determine what the observer "sees"; that is, at a given instant of time in the observer's rest frame, we wish to determine the relative orientation of the corners A, B, C, and D. The traditional view (which went unquestioned for more than 50 years!) was that the only effect is a foreshortening of the sides AB and CD such that the observer sees a distorted tube of height l but of length $l\sqrt{1-\beta^2}$. Terrell pointed out that this interpretation overlooks certain facts: For light from corners A and D to reach the observer at the same instant, the light from D, which must travel a distance l farther than that from A, must have been emitted when corner D was at position E. The length DE is equal to $(l/c)v = l\beta$. Therefore, the observer sees not only face AB, which is perpendicular to the line of sight, but also face AD, which is *parallel* to the line of sight. Also, the length of the side AB is foreshortened in the normal way to $l\sqrt{1-\beta^2}$. The net result (Figure 14-2b) corresponds exactly to the view the observer would have if the cube were rotated through an angle $\sin^{-1}\beta$. Therefore, the cube is not distorted; it undergoes an *apparent* rotation. Similarly, the customary statement* that a moving sphere appears as an ellipsoid is incorrect; it appears still as a sphere.† Computers can be used to show extremely interesting results of the type‡ we have been discussing (Figure 14-3).

*See, for example, Joos and Freeman (Jo50, p. 242).

†An interesting discussion of apparent rotations at high velocity is given by V. F. Weisskopf, *Phys. Today* **13,** no. 9, 24 (1960), reprinted in Am63.

‡See also an interesting website at the Australian National University, www.anu.edu.au/Physics/Searle/index.html, Antony C. Searle (2003) and an article by M. C. Chang, F. Lai, and W. C. Chen, *ACM Transactions on Graphics* **15,** No. 4, 265 (1996).

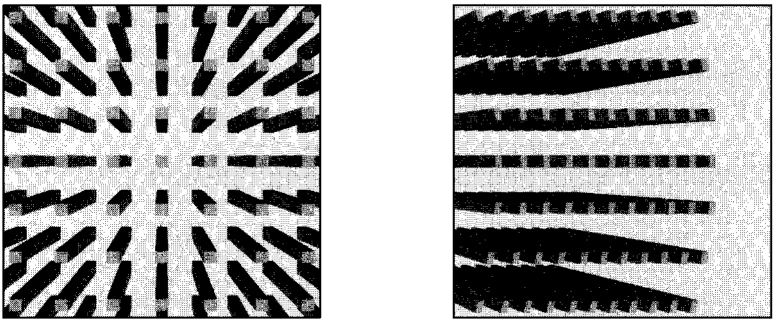


FIGURE 14-3 An array of rectangular bars is seen from above at rest in the figure on the left. In the right figure the bars are moving to the right with $v = 0.9c$. The bars appear to contract and rotate. Quoted from P.-K. Hsiung and R.H.P. Dunn, see *Science News* **137,** 232 (1990).

EXAMPLE 14.2

Use the Lorentz transformation to determine the time dilation effect.

Solution. Consider a clock fixed at a certain position (x_1) in the K system that produces signal indications with the interval

$$\Delta t = t(2) - t(1)$$

According to the Lorentz transformation (Equations 14.14), an observer in the moving system K' measures a time interval $\Delta t'$ (on the same clock) of

$$\Delta t' = t'(2) - t'(1)$$
$$= \frac{\left[t(2) - \dfrac{vx_1(2)}{c^2}\right] - \left[t(1) - \dfrac{vx_1(1)}{c^2}\right]}{\sqrt{1 - v^2/c^2}}$$

Because $x_1(2) = x_1(1)$ and because the clock is fixed in the K system, we have

$$\Delta t' = \frac{t(2) - t(1)}{\sqrt{1 - v^2/c^2}}$$

$$\boxed{\Delta t' = \frac{\Delta t}{\sqrt{1 - v^2/c^2}}} \qquad \textbf{(14.20)}$$

Thus, to an observer in motion relative to the clock, the time intervals appear to be lengthened. This is the origin of the phrase "moving clocks run more slowly." Because the measured time interval on the moving clock is lengthened, the clock actually ticks slower. Notice that the clock is fixed in the K system, $x_1(1) = x_1(2)$, but not in the K' system, $x_1'(1) \neq x_1'(2)$.

The argument in the previous example can be reversed and the clock fixed in the K' system. The same result occurs; moving clocks run slower. The effect is called **time dilation.** It is important to note that the physical system is unimportant. The same effect occurs for a tuning fork, an hourglass, a quartz crystal, and a heartbeat. The problem is one of simultaneity. Events simultaneous in one system may not be simultaneous in another one moving with respect to the first. The same clock may be viewed from n different reference frames and found to be running at n different rates, simultaneously. Space and time are intricately interwoven. We shall return to this point later.

The time measured on a clock fixed in a system present at two events is called the **proper time** and given the symbol τ. For example, $\Delta t = \Delta\tau$ when a clock fixed in system K is present for both events, $x_i(1)$ and $x_i(2)$. Equation 14.20 becomes

$$\Delta t' = \gamma\Delta\tau \tag{14.21}$$

Notice that the proper time is always the minimum measurable time difference between two events. Moving observers always measure a longer time period.

14.4 Experimental Verification of the Special Theory

The special theory of relativity explains the difficulties existing before 1900 with optics and electromagnetism. For example, the problems with stellar aberration and the Michelson-Morley experiment are solved by assuming no ether but requiring the Lorentz transformation.

But what about the new startling predictions of the special theory—length contraction and time dilation? These topics are addressed every day in the accelerator laboratories of nuclear and particle physics, where particles are accelerated to speeds close to that of light, and relativity must be considered. Other experiments can be performed with natural phenomena. We examine two of these.

Muon Decay

When cosmic rays enter the earth's outer atmosphere, they interact with particles and create cosmic showers. Many of the particles in these showers are π-mesons, which decay to other particles called muons. Muons are also unstable and decay according to the radioactive decay law, $N = N_0 \exp(-0.693\, t/t_{1/2})$, where N_0 and N are the number of muons at time $t = 0$ and t, respectively, and $t_{1/2}$ is the half-life. However, enough muons reach the earth's surface that we can detect them easily.

Let us assume that we mount a detector on top of a 2,000-m mountain and count the number of muons traveling at a speed near $v = 0.98c$. Over a given period of time, we count 10^3 muons. The half-life of muons is known to be 1.52×10^{-6} s in their own rest frame (system K'). We move our detector to sea level and measure the number of muons (having $v = 0.98c$) detected during an equal period of time. What do we expect?

Determined classically, muons traveling at a speed of 0.98c cover the 2,000 m in 6.8×10^{-6} s, and 45 muons should survive the flight from 2,000 m to sea level according to the radioactive decay law. But experimental measurement indicates that 542 muons survive, a factor of 12 more.

This phenomenon must be treated relativistically. The decaying muons are moving at a high speed relative to the experimenters fixed on the earth. We therefore observe the muons' clock to be running slower. In the muons' rest frame, the time period of the muons' flight is not $\Delta t = 6.8 \times 10^{-6}$ s but rather $\Delta t/\gamma$. For $v = 0.98c$, $\gamma = 5$, so we measure the flight time on a clock at rest in the muons' system to be 1.36×10^{-6} s. The radioactive decay law predicts that 538 muons survive, much closer to our measurement and within the experimental uncertainties. An experiment similar to this has verified the time dilation prediction.*

EXAMPLE 14.3

Examine the muon decay just discussed from the perspective of an observer moving with the muon.

Solution. The half-life of the muon according to its own clock is 1.52×10^{-6} s. But an observer moving with the muon would not measure the distance from the top of the mountain to sea level to be 2,000 m. According to that observer, the distance would be only 400 m. At a speed of $0.98c$, it takes the muon only 1.36×10^{-6} s to travel the 400 m. An observer in the muon system would predict 538 muons to survive, in agreement with an observer on the earth.

Muon decay is an excellent example of a natural phenomenon that can be described in two systems moving with respect to each other. One observer sees time dilated and the other observer sees length contracted. Each, however, predicts a result in agreement with experiment.

Atomic Clock Time Measurements

An even more direct confirmation of special relativity was reported by two American physicists, J. C. Hafele and Richard E. Keating, in 1972.† They used four extremely accurate cesium atomic clocks. Two clocks were flown on regularly scheduled commercial jet airplanes around the world, one eastward and one westward; the other two reference clocks stayed fixed on the earth at the U.S. Naval Observatory. A well-defined, hyperfine transition in the ground state of the ^{133}Cs atom has a frequency of 9,192,631,770 Hz and can be used as an accurate measurement of a time period.

*The experiment was reported by B. Rossi and D. B. Hall in the *Phys. Rev.*, **59,** 223 (1941). A film entitled "Time Dilation—An Experiment with μ-Mesons" by D. H. Frisch and J. H. Smith is available from the Education Development Center; Newton, Mass. See also D. H. Frisch and J. H. Smith, *Am. J. Phys.*, **31,** 342 (1963).

†See J. C. Hafele and Richard E. Keating, *Science*, **177,** 166–170 (1972).

The time measured on the two moving clocks was compared with that of the two reference clocks. The eastward trip lasted 65.4 hours with 41.2 flight hours. The westward trip, a week later, took 80.3 hours with 48.6 flight hours. The predictions are complicated by the rapid rotation of the earth and by a gravitational effect from the general theory of relativity.

We can gain some insight to the expected effect by neglecting the corrections and calculating the time difference as if the earth were not rotating. The circumference of the earth is about 4×10^7 m, and a typical jet airplane speed is almost 300 m/s. A clock fixed on the ground measures a flight time T_0 of

$$T_0 = \frac{4 \times 10^7\,\text{m}}{300\,\text{m/s}} = 1.33 \times 10^5\,\text{s}(\approx 37\,\text{hr}) \tag{14.22}$$

Because the moving clock runs more slowly, the observer on the earth would say that the moving clock measures only $T = T_0\sqrt{1 - \beta^2}$. The time difference is

$$\begin{aligned} \Delta T &= T_0 - T = T_0(1 - \sqrt{1 - \beta^2}) \\ &\approx \frac{1}{2}\beta^2 T_0 \end{aligned} \tag{14.23}$$

where only the first and second terms of the power series expansion for $\sqrt{1 - \beta^2}$ are kept because β^2 is so small.

$$\begin{aligned} \Delta T &= \frac{1}{2}\left(\frac{300\,\text{m/s}}{3 \times 10^8\,\text{m/s}}\right)^2 (1.33 \times 10^5\,\text{s}) \\ &= 6.65 \times 10^{-8}\,\text{s} = 66.5\,\text{ns} \end{aligned} \tag{14.24}$$

This time difference is greater than the uncertainty of the measurement. Notice that in this case, the clock left on the earth actually measures more time in seconds than the moving clock. This seems at variance with our earlier comments (see Equation 14.21 and discussion). But the time period referred to in Equation 14.21 is the time between two ticks, in this case, a transition in ^{133}Cs, which we measure in seconds. It is easy to remember that moving clocks run more slowly, so that in seconds the measured time difference involves fewer ticks and, according to the definition of a second, fewer seconds.

The actual predictions and observations for the time difference are

Travel	Predicted	Observed
Eastward	-40 ± 23 ns	-59 ± 10 ns
Westward	275 ± 21 ns	273 ± 7 ns

Again, the special theory of relativity is verified within the experimental uncertainties. A negative sign indicates that the time on the moving clock is less than the earth reference clock. The moving clocks lost time (ran slower) during the eastward trip and gained time (ran faster) during the westward trip. This difference is caused by the rotation of the earth, indicating that the flying clocks actually ticked faster or slower than the reference clocks on the earth. The overall positive time difference is a result of the gravitational potential effect (which we do not discuss here).

We have only briefly described two of the many experiments that have verified the special theory of relativity. There are no known experimental measurements that are inconsistent with the special theory of relativity. Einstein's work in this regard has so far withstood the test of time.

14.5 Relativistic Doppler Effect

The Doppler effect in sound is represented by an increased pitch of sound as a source approaches a receiver and a decrease of pitch as the source recedes. The change in frequency of the sound depends on whether the source or receiver is moving. This effect seems to violate Postulate I of the theory of relativity until we realize that there is a special frame for sound waves because there is a medium (e.g., air or water) in which the waves travel. In the case of light, however, there is no such medium. Only relative motion of source and receiver is meaningful in this context, and we should therefore expect some differences in the relativistic Doppler effect for light from the normal Doppler effect of sound.

Consider a source of light (e.g., a star) and a receiver approaching one another with relative speed v (Figure 14-4a). First, consider the receiver fixed in system K and the light source in system K' moving toward the receiver with speed v. During time Δt as measured by the receiver, the source emits n waves. During that time Δt, the total distance between the front and rear of the waves is

$$\text{length of wave train} = c\Delta t - v\Delta t \tag{14.25}$$

The wavelength is then

$$\lambda = \frac{c\Delta t - v\Delta t}{n} \tag{14.26}$$

and the frequency is

$$\nu = \frac{c}{\lambda} = \frac{cn}{c\Delta t - v\Delta t} \tag{14.27}$$

According to the source, it emits n waves of frequency ν_0 during the proper time $\Delta t'$:

$$n = \nu_0 \Delta t' \tag{14.28}$$

This proper time $\Delta t'$ measured on a clock in the source system is related to the time Δt measured on a clock fixed in system K of the receiver by

$$\Delta t' = \frac{\Delta t}{\gamma} \tag{14.29}$$

The clock moving with the source measures the proper time, because it is present at both the beginning and end of the waves.

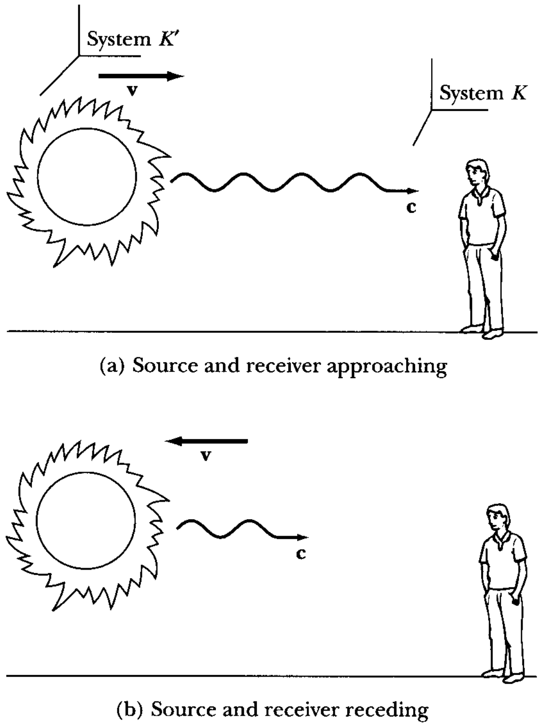


FIGURE 14-4 (a) An observer in system K sees light coming from a source fixed in system K'. System K' is moving toward the observer with speed v. The frequency of the light is observed in K to be increased over the value observed in K'. (b) When system K' is moving away from the observer, the frequency of the light decreases (the wavelength increases). This is the source of the term *redshifted.*

Substituting Equation 14.29 into Equation 14.28, which in turn is substituted for n in Equation 14.27, gives

$$\begin{aligned} \nu &= \frac{1}{(1 - v/c)} \frac{\nu_0}{\gamma} \\ &= \frac{\sqrt{1 - v^2/c^2}}{1 - v/c} \nu_0 \end{aligned} \tag{14.30}$$

which can be written as

$$\nu = \frac{\sqrt{1 + \beta}}{\sqrt{1 - \beta}} \nu_0 \quad \text{source and receiver approaching} \tag{14.31}$$

It is left for the reader (Problem 14-14) to show that Equation 14.31 is also valid when the source is fixed and the receiver approaches it with speed v.

Next, we consider the case in which the source and receiver recede from each other with velocity v (Figure 14-4b). The derivation is similar to the one just presented—with one small exception. In Equation 14.25, the distance between the beginning and end of the waves becomes

$$\text{length of wave train} = c\Delta t + v\Delta t \tag{14.32}$$

This change in sign is propagated through Equations 14.30 and 14.31, giving

$$\nu = \frac{\sqrt{1 - v^2/c^2}}{1 + v/c}\nu_0$$

$$\nu = \frac{\sqrt{1-\beta}}{\sqrt{1+\beta}}\nu_0 \quad \text{source and receiver receding} \tag{14.33}$$

Equations 14.31 and 14.33 can be combined into one equation,

$$\nu = \frac{\sqrt{1+\beta}}{\sqrt{1-\beta}}\nu_0 \quad \text{relativistic Doppler effect} \tag{14.34}$$

if we agree to use a + sign for $\beta(+v/c)$ when the source and receiver are approaching each other and a − sign for β when they are receding.

The relativistic Doppler effect is important in astronomy. Equation 14.34 indicates that, if the source is receding at high speed from an observer, then a lower frequency (or longer wavelength) is observed for certain spectral lines or characteristic frequencies. This is the origin of the term *red shift*; the wavelengths of visible light are shifted toward longer wavelengths (red) if the source is receding from us. Astronomical observations indicate that the universe is expanding. The farther away a star is, the faster it appears to be moving away (or the greater its red shift). These data are consistent with the "big bang" origin of the universe, which is estimated to have occurred some 13 billion years ago.

EXAMPLE 14.4

During a spaceflight to a distant star, an astronaut and her twin brother on the earth send radio signals to each other at annual intervals. What is the frequency of the radio signals each twin receives from the other during the flight to the star if the astronaut is moving at $v = 0.8c$? What is the frequency during the return flight at the same speed?

Solution. We use Equation 14.34 to determine the frequency of radio signals that each receives from the other. The frequency $\nu_0 = 1$ signal/year. On the leg of the trip away from the earth, $\beta = -0.8$ and Equation 14.34 gives

$$\nu = \frac{\sqrt{1-0.8}}{\sqrt{1+0.8}}\nu_0$$
$$= \frac{\nu_0}{3}$$

The radio signals are received once every 3 years.

On the return trip, however, $\beta = +0.8$ and Equation 14.34 gives $\nu = 3\nu_0$, so the radio signals are received every 4 months. In this way, the twin on the earth can monitor the progress of his astronaut twin.

14.6 Twin Paradox

Consider twins who choose different career paths. Mary becomes an astronaut, and Frank decides to be a stockbroker. At age 30, Mary leaves on a mission to a planet in a nearby star's system. Mary will have to travel at a high speed to reach the planet and return. According to Frank, Mary's biological clock will tick more slowly during her trip, so she will age more slowly. He expects Mary to look and appear younger than he does when she returns. According to Mary, however, Frank will appear to be moving rapidly with respect to her system, and she thinks Frank will be younger when she returns. This is the paradox. Which twin, if either, is younger when Mary (the moving twin) returns to the earth where Frank (the fixed twin) has remained? Because the two expectations are so contradictory, doesn't Nature have a way to prove they will be the same age?

This paradox has existed almost since Einstein first published his special theory of relativity. Variations of the argument have been presented many times. The correct answer is that Mary, the astronaut, will return younger than her twin brother, Frank, who remains busy on Wall Street. The correct analysis is as follows. According to Frank, Mary's spaceship blasts off and quickly reaches a coasting speed of $v = 0.8c$, travels a distance of 8 ly (ly = a light year, the distance light travels in 1 year) to the planet, and quickly decelerates for a short visit to the planet. The acceleration and deceleration times are negligible compared with the total travel time of 10 years to the planet. The return trip also takes 10 years, so on Mary's return to Earth, Frank will be $30 + 10 + 10 = 50$ years old. Frank calculates that Mary's clock is ticking slower and that each leg of the trip takes only $10\sqrt{1 - 0.8^2} = 6$ years. Mary therefore is only $30 + 6 + 6 = 42$ years old when she returns. Frank's clock is (almost) in an inertial system.

When Mary performs the time measurements on her clock, they may be invalid according to the special theory because her system is not in an inertial frame of reference moving at a constant speed with respect to the earth. She accelerates and decelerates at both the earth and the planet, and to make valid time measurements to compare with Frank's clock, she must account for this acceleration and deceleration. The instantaneous rate of Mary's clock is still given by Equation 14.20, because the instantaneous rate is determined by the instantaneous speed v.* Thus, there is no paradox if we obey the two postulates of the

*See the clock hypothesis of W. Rindler (Ri82, p. 31).

special theory. It is also clear which twin is in the inertial frame of reference. Mary will actually feel the forces of acceleration and deceleration. Frank feels no such forces. When Mary returns home, her twin brother has invested her 20 years of salary, making her a rich woman at the young age of 42. She was paid a 20-year salary for a job that took her only 12 years!

EXAMPLE 14.5

Mary and Frank send radio signals to each other at 1-year intervals after she leaves Earth. Analyze the times of receipt of the radio messages.

Solution. In Example 14.4, we calculated that such radio signals are received every 3 years on the trip out and every $\frac{1}{3}$ year on the trip back. First, we examine the signals Mary receives from Frank. During the 6-year trip to the planet, Mary receives only two radio messages, but on the 6-year return trip, she receives eighteen signals, so she correctly concludes that her twin brother Frank has aged 20 years and is now 50 years old.

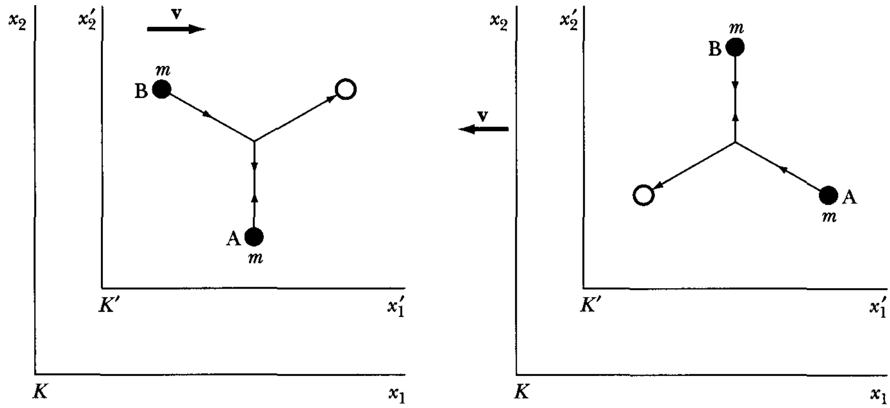
In Frank's system, Mary's trip to the planet takes 10 years. By the time Mary reaches the planet, Frank receives 10/3 signals (i.e., three signals plus one-third of the time to the next one). However, Frank continues to receive a signal every 3 years for the 8 years it takes the last signal Mary sends when she reaches the planet to travel to Frank. Thus, Frank receives signals every 3 years for 8 more years (total of 18 years) for a total of six radio signals from the period of travel to the planet. Frank has no way of knowing that Mary has stopped and turned around until the radio message, which takes 8 years, is received. Of the remaining 2 years of Mary's journey according to Frank ($20 - 18 = 2$), Frank receives signals every $\frac{1}{3}$ year, or six more signals. Frank correctly determines that Mary has aged $6 + 6 = 12$ years during her journey because he receives a total of 12 signals.

Thus, both twins agree about their own ages and about each other's. Mary is 42 and Frank is 50 years old.

14.7 Relativistic Momentum

Newton's Second Law, $\mathbf{F} = d\mathbf{p}/dt$, is covariant under a Galilean transformation. Therefore, we do not expect it to keep its form under a Lorentz transformation. We can foresee difficulties with Newton's laws and the conservation laws unless we make some necessary changes. According to Newton's Second Law, for example, an acceleration at high speeds might cause a particle's velocity to exceed c, an impossible condition according to the special theory of relativity.

We begin by examining the conservation of linear momentum in a force-free (no external forces) collision. There are no accelerations. Observer A at rest in system K holds a ball of mass m, as does observer B in system K' moving to



(a) Collision according to system K (b) Collision according to system K'

FIGURE 14-5 Observer A, at rest in fixed system K, throws a ball straight up in system K. Observer B, at rest in system K', which is moving to the right with velocity **v**, throws a ball straight down so that the two balls collide. (a) The collision according to observer A in system K. (b) The collision according to observer B in system K'. Each observer measures the speed of his or her ball to be u_0. We examine the linear momentum of the ball.

the right with relative speed v with respect to system K, as in Figure 14-1. The two observers throw their (identical) balls along their respective x_2-axes, which results in a perfectly elastic collision. The collision, according to observers in the two systems, is shown in Figure 14-5. Each observer measures the speed of his or her ball to be u_0.

We first examine the conservation of momentum according to system K. The velocity of the ball thrown by observer A has components

$$\left.\begin{aligned} u_{A1} &= 0 \\ u_{A2} &= u_0 \end{aligned}\right\} \tag{14.35}$$

The momentum of ball A is in the x_2-direction:

$$p_{A2} = mu_0 \tag{14.36}$$

The collision is perfectly elastic, so the ball returns down with speed u_0. The change in momentum observed in system K is

$$\Delta p_{A2} = -2mu_0 \tag{14.37}$$

Does Equation 14.37 also represent the change in momentum of the ball thrown by observer B in the moving system K'? We use the inverse velocity transformation of Equations 14.17 (i.e., we interchange primes and unprimes and let $v \rightarrow -v$) to determine

$$\left.\begin{aligned} u_{B1} &= v \\ u_{B2} &= -u_0\sqrt{1 - v^2/c^2} \end{aligned}\right\} \tag{14.38}$$

where $u'_{B1} = 0$ and $u'_{B2} = -u_0$. The momentum of ball B and its change in momentum during the collision become

$$p_{B2} = -mu_0\sqrt{1 - v^2/c^2} \tag{14.39}$$

$$\Delta p_{B2} = +2mu_0\sqrt{1 - v^2/c^2} \tag{14.40}$$

Equations 14.37 and 14.40 do not add to zero: *Linear momentum is not conserved according to the special theory if we use the conventions for momentum of classical physics.* Rather than abandoning the law of conservation of momentum, we look for a solution that allows us to retain both it and Newton's Second Law.

As we did for the Lorentz transformation, we assume the simplest possible change. We assume that the classical form of momentum $m\mathbf{u}$ is multiplied by a constant that may depend on speed $k(u)$:

$$\mathbf{p} = k(u)m\mathbf{u} \tag{14.41}$$

In Example 14.6, we show that the value

$$k(u) = \frac{1}{\sqrt{1 - u^2/c^2}} \tag{14.42}$$

allows us to retain the conservation of linear momentum. Notice that the *form* of Equation 14.42 is the same as that found for the Lorentz transformation. In fact, the constant $k(u)$ is given the same label: γ. However, this γ contains the speed of the particle u, whereas the Lorentz transformation contains the relative speed v between the two inertial reference frames. This distinction must be kept in mind; it often causes confusion.

We can make a plausible calculation for the relativistic momentum if we use the proper time τ (see Equation 14.21) rather than the normal time t. In this case,

$$\mathbf{p} = m\frac{d\mathbf{x}}{d\tau} = m\frac{d\mathbf{x}}{dt}\frac{dt}{d\tau} \tag{14.43}$$

$$= m\frac{d\mathbf{x}}{dt}\frac{1}{\sqrt{1 - u^2/c^2}} \tag{14.44}$$

$$\boxed{\mathbf{p} = \frac{m\mathbf{u}}{\sqrt{1 - u^2/c^2}} = \gamma m\mathbf{u}} \quad \text{relativistic momentum} \tag{14.45}$$

where we retain $\mathbf{u} = d\mathbf{x}/dt$ as used classically. Although all observers do not agree as to $d\mathbf{x}/dt$, they do agree as to $d\mathbf{x}/d\tau$, where the proper time $d\tau$ is measured by the moving object itself. The relation $dt/d\tau$ is obtained from Equation 14.21, where the speed u has been used in γ to represent the speed of a reference frame fixed in the object that is moving with respect to a fixed frame.

Equation 14.45 is our new definition of momentum, called **relativistic momentum.** Notice that it reduces to the classical result for small values of u/c. It was fashionable in past years to call the mass in Equation 14.45 the **rest mass** m_0 and to call the term

$$m = \frac{m_0}{\sqrt{1 - u^2/c^2}} \quad \textbf{(old-fashioned notation)} \tag{14.46}$$

the **relativistic mass.** The term *rest mass* resulted from Equation 14.46 when $u = 0$, and the classical form of momentum was thus retained: $\mathbf{p} = m\mathbf{u}$. Scientists spoke of the mass increasing at high speeds. We prefer to keep the concept of mass as an invariant, intrinsic property of an object. The use of the two terms *relativistic* and *rest mass* is now considered old-fashioned, although the terms are still sometimes used. *We always refer to the mass m, which is the same as the rest mass.* The use of relativistic mass often leads to mistakes when using classical expressions.

EXAMPLE 14.6

Show that linear momentum is conserved in the x_2-direction for the collision shown in Figure 14-5 if relativistic momentum is used.

Solution. We can modify the classical expressions for momentum already obtained for the two balls. The momentum for ball A becomes (from Equation 14.36)

$$p_{A2} = \frac{mu_0}{\sqrt{1 - u_0^2/c^2}} \tag{14.47}$$

and

$$\Delta p_{A2} = \frac{-2mu_0}{\sqrt{1 - u_0^2/c^2}} \tag{14.48}$$

Before modifying Equation 14.39 for the momentum of ball B, we must first find the speed of ball B as measured in system K. We use Equation 14.38 to determine

$$\begin{aligned} u_B &= \sqrt{u_{B1}^2 + u_{B2}^2} \\ &= \sqrt{v^2 + u_0^2(1 - v^2/c^2)} \end{aligned} \tag{14.49}$$

The momentum p_{B2} is found by modifying Equation 14.39:

$$p_{B2} = -mu_0\gamma\sqrt{1 - v^2/c^2}$$

where

$$\gamma = \frac{1}{\sqrt{1 - u_B^2/c^2}}$$

$$p_{B2} = \frac{-mu_0\sqrt{1 - v^2/c^2}}{\sqrt{1 - u_B^2/c^2}} \tag{14.50}$$

Using u_B from Equation 14.49 gives

$$\begin{aligned} p_{B2} &= \frac{-mu_0\sqrt{1 - v^2/c^2}}{\sqrt{(1 - u_0^2/c^2)(1 - v^2/c^2)}} \\ &= \frac{-mu_0}{\sqrt{1 - u_0^2/c^2}} \end{aligned} \tag{14.51}$$

$$\Delta p_{B2} = \frac{+2mu_0}{\sqrt{1 - u_0^2/c^2}} \tag{14.52}$$

Equations 14.48 and 14.52 add to zero, as required for the conservation of linear momentum.

14.8 Energy

With a new definition of linear momentum (Equation 14.45) in hand, we turn our attention to energy and force. We keep our former definition (Equation 2.86) of kinetic energy as being the work done on a particle. The work done is defined in Equation 2.84 to be

$$W_{12} = \int_1^2 \mathbf{F} \cdot d\mathbf{r} = T_2 - T_1 \tag{14.53}$$

Equation 2.2 for Newton's Second Law is modified to account for the new definition of linear momentum:

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d}{dt}(\gamma m\mathbf{u}) \tag{14.54}$$

If we start from rest, $T_1 = 0$, and the velocity $\mathbf{u}$ is initially along the direction of the force.

$$W = T = \int \frac{d}{dt}(\gamma m\mathbf{u}) \cdot \mathbf{u}\, dt \tag{14.55}$$

$$= m\int_0^u u\, d(\gamma u) \tag{14.56}$$

Equation 14.56 is integrated by parts to obtain

$$T = \gamma m u^2 - m\int_0^u \frac{u\,du}{\sqrt{1 - u^2/c^2}}$$

$$= \gamma m u^2 + mc^2\sqrt{1 - u^2/c^2}\,\Bigg|_0^u$$

$$= \gamma m u^2 + mc^2\sqrt{1 - u^2/c^2} - mc^2 \tag{14.57}$$

With algebraic manipulation, Equation 14.57 becomes

$$\boxed{T = \gamma mc^2 - mc^2} \quad \text{relativistic kinetic energy} \tag{14.58}$$

Equation 14.58 seems to resemble in no way our former result for kinetic energy, $T = \frac{1}{2}mu^2$. However, Equation 14.58 must reduce to $\frac{1}{2}mu^2$ for small values of velocity.

EXAMPLE 14.7

Show that Equation 14.58 reduces to the classical result for small speeds, $u \ll c$.

Solution. The first term of Equation 14.58 can be expanded in a power series:

$$\begin{aligned} T &= mc^2(1 - u^2/c^2)^{-1/2} - mc^2 \\ &= mc^2\left(1 + \frac{1}{2}\frac{u^2}{c^2} + \cdots\right) - mc^2 \end{aligned} \tag{14.59}$$

where all terms of power $(u/c)^4$ or greater are neglected because $u \ll c$.

$$\begin{aligned} T &= mc^2 + \frac{1}{2}mu^2 - mc^2 \\ &= \frac{1}{2}mu^2 \end{aligned} \tag{14.60}$$

which is the classical result.

It is important to note that neither $\frac{1}{2}mu^2$ nor $\frac{1}{2}\gamma mu^2$ gives the correct relativistic value for the kinetic energy.

The term mc^2 in Equation 14.58 is called the **rest energy** and is denoted by E_0.

$$\boxed{E_0 \equiv mc^2} \quad \text{rest energy} \tag{14.61}$$

Equation 14.58 is rewritten

$$\gamma mc^2 = T + mc^2$$

Thus,

$$E = T + E_0 \tag{14.62}$$

where

$$\boxed{E \equiv \gamma mc^2 = T + E_0} \quad \text{total energy} \tag{14.63}$$

The total energy, $E = \gamma mc^2$, is defined as the sum of kinetic energy and the rest energy. Equations 14.58–14.63 are the origin of Einstein's famous relativistic result of the equivalence of mass and energy (energy $= mc^2$). These equations are consistent with this interpretation. Note that when a body is not in motion ($u = 0 = T$), Equation 14.63 indicates that the total energy is equal to the rest energy.

If mass is simply another form of energy, then we must combine the classical conservation laws of mass and energy into one conservation law of mass-energy represented by Equation 14.63. This law is easily demonstrated in the atomic nucleus, where the mass of constituent particles is converted to the energy that binds the individual particles together.

EXAMPLE 14.8

Use the atomic masses of the particles involved to calculate the binding energy of a deuteron.

Solution. A deuteron is composed of a neutron and a proton. We use atomic masses, because the electron masses cancel.

$$\begin{aligned} \text{mass of neutron} &= 1.008665 \text{ u} \\ \text{mass of proton } (^1\text{H}) &= \underline{1.007825 \text{ u}} \\ \text{sum} &= 2.016490 \text{ u} \\ \text{mass of deuteron } (^2\text{H}) &= 2.014102 \text{ u} \\ \text{difference} &= 0.002388 \text{ u} \end{aligned}$$

This difference in mass-energy is equal to the binding energy holding the neutron and proton together as a deuteron. The mass units are atomic mass units (u), which can be converted to kilograms if necessary. However, the conversion of mass to energy is facilitated by the well-known relation between mass and energy:

$$1 \text{ u}c^2 = 931.5 \text{ MeV} \tag{14.64}$$

The binding energy of the deuteron is therefore

$$0.002388 \text{ u}c^2 \times 931.5 \frac{\text{MeV}}{\text{u}c^2} = 2.22 \text{ MeV}$$

Nuclear experiments of the form $\gamma + {}^2\text{H} \rightarrow \text{n} + \text{p}$ indicate that gamma rays of energy just greater than 2.22 MeV are required to break the deuteron apart into a neutron and a proton. Conversely, when a neutron and proton join at rest to form a deuteron, 2.22 MeV of energy is released in the form of kinetic energy of the deuteron and gamma ray.

Because physicists believe that momentum is a more fundamental concept than kinetic energy (for example, there is no general law of conservation of kinetic energy), we would like a relation for mass-energy that includes momentum rather than kinetic energy. We begin with Equation 14.45 for momentum:

$$\begin{aligned} p &= \gamma m u \\ p^2c^2 &= \gamma^2 m^2 u^2 c^2 \\ &= \gamma^2 m^2 c^4 \left(\frac{u^2}{c^2}\right) \end{aligned} \tag{14.65}$$

It is easy to show that

$$\frac{u^2}{c^2} = 1 - \frac{1}{\gamma^2} \tag{14.66}$$

so Equation 14.65 becomes

$$p^2c^2 = \gamma^2 m^2 c^4\left(1 - \frac{1}{\gamma^2}\right)$$
$$= \gamma^2 m^2 c^4 - m^2 c^4$$
$$= E^2 - E_0^2$$

$$\boxed{E^2 = p^2c^2 + E_0^2} \tag{14.67}$$

Equation 14.67 is a very useful kinematic relationship. It relates the total energy of a particle to its momentum and rest energy.

Notice that a photon has no mass, so that Equation 14.67 gives

$$E = pc \quad \text{photon} \tag{14.68}$$

There is no such thing as a photon at rest.

14.9 Spacetime and Four-Vectors

In Section 14.3 (Equation 14.5), we noticed that the quantities

$$\left.\begin{aligned} \sum_{j=1}^{3} x_j^2 - c^2t^2 = 0 \\ \sum_{j=1}^{3} x_j'^2 - c^2t'^2 = 0 \end{aligned}\right\}$$

are invariant because the speed of light is the same in all inertial systems in relative motion. Consider two events separated by space and time. In system K,

$$\Delta x_i = x_i(\text{event } 2) - x_i(\text{event } 1)$$
$$\Delta t = t(\text{event } 2) - t(\text{event } 1)$$

The interval Δs^2 is invariant in all inertial systems in relative motion (see Problem 14-34):

$$\Delta s^2 = \sum_{j=1}^{3} (\Delta x_j)^2 - c^2 \Delta t^2 \tag{14.69}$$

$$\Delta s^2 = \Delta s'^2 = \sum_{j=1}^{3} (\Delta x_j')^2 - c^2 \Delta t'^2 \tag{14.70}$$

Equation 14.69 can be written as a differential equation:

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 - c^2dt^2 \tag{14.71}$$

Consider the system K', where the particle is instantaneously at rest. Because $dx_1' = dx_2' = dx_3' = 0$ in this case, $dt' = d\tau$, the proper time interval discussed

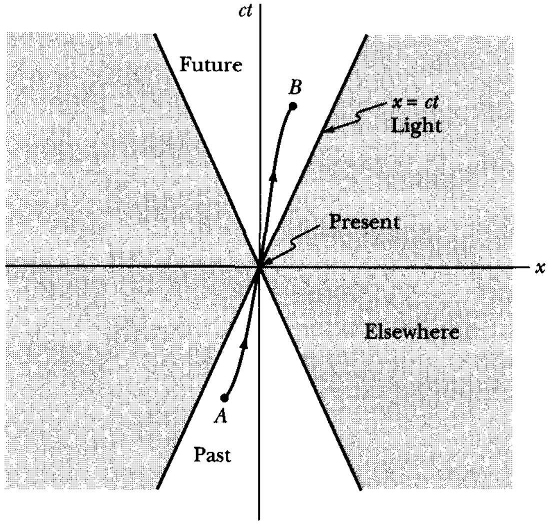


FIGURE 14-6 The variable ct is plotted versus x with the origin being the present. The heavy solid lines indicate the past and future paths of light and form a *light cone.* To the right and left of these lines is considered "elsewhere," because we cannot reach this region from the present. The path from A to B represents a *worldline,* a path that we can take traveling at speeds less than or equal to light.

above (Equation 14.21). Equation 14.70 becomes

$$-c^2 d\tau^2 = dx_1^2 + dx_2^2 + dx_3^2 - c^2 dt^2 \tag{14.72}$$

Using the Lorentz transformation, Equation 14.72 gives a similar result to Equation 14.21:

$$d\tau = \frac{dt}{\gamma} \tag{14.73}$$

The proper time τ is, along with the length quantity Δs^2, another Lorentz invariant quantity.

A useful concept in special relativity is that of the **light cone.** The invariant length Δs^2 suggests adding ct as a fourth dimension to the three space dimensions x_1, x_2, and x_3. In Figure 14-6, we plot ct versus one of the Euclidean space coordinates. The origin of (x, ct) is the present $(0, 0)$. The solid lines represent the paths taken in the past and in the future by light. A particle traveling the path from A to B is said to be moving along its **worldline.** For time $t < 0$, the particle has been in the lower cone, the past. Similarly, for $t > 0$ the particle will move in the upper cone, the future. It is not possible for us to know about events outside the light cone; this region, called "elsewhere," requires $v > c$.

There are two possibilities concerning the value of Δs^2. If $\Delta s^2 > 0$, the two events have a **spacelike interval.** One can always find an inertial frame traveling with $v < c$ such that the two events occur at different space coordinates but at the same time. When $\Delta s^2 < 0$, the two events are said to have a **timelike interval.** One can always find a suitable inertial frame in which the events occur at the same point in space but at different times. In the case $\Delta s^2 = 0$, the two events are separated by a light ray.

Only events separated by a timelike interval can be causally connected. The present event in the light cone can be causally related only to events in the past region of the light cone. Events with a spacelike interval cannot be causally connected. Space and time, although distinct, are nonetheless intricately related.

The previous discussion of space and time suggests using ct as a fourth dimensional parameter. We continue this line of thought by defining $x_4 \equiv ict$ and $x_4' \equiv ict'$. The use of the imaginary number $i(\sqrt{-1})$ does not indicate that this component is imaginary. The imaginary number simply allows us to represent the relations in concise, mathematical form. The rest of this section could just as well be carried out without the use of i (e.g., $x_4 = ct$), but the mathematics would be more cumbersome. The useful results are in terms of real, physical quantities.

Using $x_4 = ict$ and $x_4' = ict'$, we can write Equations 14.5 as*

$$\left.\begin{aligned} \sum_{\mu=1}^{4} x_\mu^2 &= 0 \\ \sum_{\mu=1}^{4} x_\mu'^2 &= 0 \end{aligned}\right\} \tag{14.74}$$

From these equations, it is clear that the two sums must be proportional, and because the motion is symmetrical between the systems, the proportionality constant is unity.† Thus,

$$\sum_\mu x_\mu^2 = \sum_\mu x_\mu'^2 \tag{14.75}$$

This relation is analogous to the three-dimensional, distance-preserving, orthogonal rotations we have studied previously (see Section 1.4) and indicates that the Lorentz transformation corresponds to a rotation in a *four-dimensional* space (called **world space** or **Minkowski space**‡). The Lorentz transformations are then orthogonal transformations in Minkowski space:

$$x_\mu' = \sum_\nu \lambda_{\mu\nu} x_\nu \tag{14.76}$$

*In accordance with standard convention, we use Greek indices (usually μ or ν) to indicate summations that run from 1 to 4; in relativity theory, Latin indices are usually reserved for summations that run from 1 to 3.

†A "proof" is given in Appendix G.

‡Herman Minkowski (1864–1909) made important contributions to the mathematical theory of relativity and introduced *ict* as a fourth component.

where the $\lambda_{\mu\nu}$ are the elements of the Lorentz transformation matrix. From Equations 14.14, the transformation $\boldsymbol{\lambda}$ is

$$\boldsymbol{\lambda} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \tag{14.77}$$

A quantity is called a **four-vector** if it consists of four components, each of which transforms according to the relation*

$$A'_\mu = \sum_\nu \lambda_{\mu\nu} A_\nu \tag{14.78}$$

where the $\lambda_{\mu\nu}$ define a Lorentz transformation. Such a four-vector† is

$$\mathbb{X} = (x_1, x_2, x_3, ict) \tag{14.79a}$$

or

$$\boxed{\mathbb{X} = (\mathbf{x}, ict)} \tag{14.79b}$$

where the notation of the last line means that the first three (space) components of $\mathbb{X}$ define the ordinary three-dimensional position vector $\mathbf{x}$ and that the fourth component is ict. Similarly, the differential of $\mathbb{X}$ is a four-vector:

$$d\mathbb{X} = (d\mathbf{x}, ic\, dt) \tag{14.80}$$

In Minkowski space, the four-dimensional element of length is invariant. Its magnitude is unaffected by a Lorentz transformation, and such a quantity is called a **four-scalar** or **world scalar.** Equation 14.71 can be written as

$$ds = \sqrt{\sum_\mu dx_\mu^2} \tag{14.81}$$

and Equation 14.72 as

$$d\tau = \frac{i}{c}\sqrt{\sum_\mu dx_\mu^2} = \frac{i}{c}\, ds \tag{14.82}$$

The proper time $d\tau$ is invariant because it is simply i/c times the element of length ds. The ratio of the four-vector $d\mathbb{X}$ to the invariant $d\tau$ is therefore also a four-vector, called the four-vector velocity $\mathbb{V}$:

$$\boxed{\mathbb{V} = \frac{d\mathbb{X}}{d\tau} = \left(\frac{d\mathbf{x}}{d\tau}, ic\frac{dt}{d\tau}\right)} \tag{14.83}$$

The components of the ordinary velocity $\mathbf{u}$ are

$$u_j = \frac{dx_j}{dt}$$

*We do not distinguish here between *covariant* and *contravariant* vector components; see, for example, Bergmann (Be46, Chapter 5).

†Four-vectors are denoted exclusively by openface capital letters.

so, using Equations 14.71 and 14.82, $d\tau$ can be expressed as

$$d\tau = dt\sqrt{1 - \frac{1}{c^2}\sum_j \frac{dx_j^2}{dt^2}}$$

or

$$d\tau = dt\sqrt{1 - \beta^2} \tag{14.84}$$

as we found in Equation 14.73. The four-vector velocity can therefore be written as

$$\boxed{\mathbb{V} = \frac{1}{\sqrt{1 - \beta^2}}(\mathbf{u}, ic)} \tag{14.85}$$

where $\mathbf{u}$ represents the three space components of ordinary velocity, u_1, u_2, u_3. (Remember that the particle's velocity is now denoted by $\mathbf{u}$ to distinguish it from the moving frame velocity $\mathbf{v}$.) The four-vector momentum is now simply the mass times four-vector velocity,* because mass is invariant:

$$\mathbb{P} = m\mathbb{V} \tag{14.86}$$

$$\boxed{\mathbb{P} = \left(\frac{m\mathbf{u}}{\sqrt{1 - \beta^2}}, ip_4\right)} \tag{14.87}$$

where

$$p_4 \equiv \frac{mc}{\sqrt{1 - \beta^2}} \tag{14.88}$$

The first three components of the four-vector momentum $\mathbb{P}$ are the components of the relativistic momentum (Equation 14.45):

$$P_j = p_j = \gamma m u_j, \quad j = 1, 2, 3 \tag{14.89}$$

Using Equation 14.63, the fourth component of the momentum is related to the total energy E:

$$p_4 = \gamma mc = \frac{E}{c} \tag{14.90}$$

The four-vector momentum can therefore be written as

$$\mathbb{P} = \left(\mathbf{p}, i\frac{E}{c}\right) \tag{14.91}$$

where $\mathbf{p}$ stands for the three space components of momentum. Thus, in relativity theory, momentum and energy are linked in a manner similar to that which joins the concepts of space and time. If we apply the Lorentz transformation matrix

*A four-vector multiplied by a four-scalar is also a four-vector.

(Equation 14.77) to the momentum $\mathbb{P}$, we find

$$\boxed{\begin{aligned} p_1' &= \frac{p_1 - (v/c^2)E}{\sqrt{1-\beta^2}} \\ p_2' &= p_2 \\ p_3' &= p_3 \\ E' &= \frac{E - vp_1}{\sqrt{1-\beta^2}} \end{aligned}} \tag{14.92}$$

EXAMPLE 14.9

Using the methods of this section, derive Equation 14.67.

Solution. If we place the origin of the moving system K' fixed on the particle, we have $u = v$. The square of the four-vector velocity (Equation 14.85) is invariant:

$$\mathbb{V}^2 = \sum_\mu V_\mu^2 = \frac{v^2 - c^2}{1 - \beta^2} = -c^2 \tag{14.93}$$

Hence, the square of the four-vector momentum is also invariant:

$$\mathbb{P}^2 = \sum_\mu P_\mu^2 = m^2\mathbb{V}^2 = -m^2c^2 \tag{14.94}$$

From Equation 14.91, we also have, using $\mathbf{p} \cdot \mathbf{p} = p^2 = p_1^2 + p_2^2 + p_3^2$,

$$\mathbb{P}^2 = p^2 - \frac{E^2}{c^2} \tag{14.95}$$

Combining the last two equations gives Equation 14.67.

$$E^2 = p^2c^2 + m^2c^4 = p^2c^2 + E_0^2$$

If we define an angle ϕ such that $\beta = \sin\phi$, the relativistic relations between velocity, momentum, and energy can be obtained by trigonometric relations involving the so-called "relativistic triangle" (Figure 14-7).

EXAMPLE 14.10

Derive the velocity addition rule.

Solution. Suppose that there are three inertial reference frames, K, K', and K'', which are in collinear motion along their respective x_1-axes. Let the velocity of K' relative to K be v_1 and let the velocity of K'' relative to K' be v_2. The speed of K'' relative to K cannot be $v_1 + v_2$, because it must be possible to propagate a signal between any two inertial frames, and if both v_1 and v_2 are greater than $c/2$ (but less than c), then $v_1 + v_2 > c$. Therefore, the rule for the addition of velocities in relativity must be different from that in Galilean theory. The relativistic

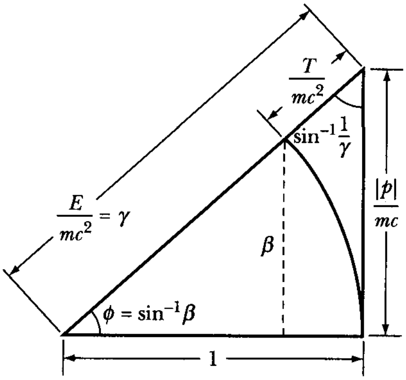


FIGURE 14-7 The relativistic triangle allows us to find relations between velocity, momentum, and energy by using trigonometric relations.

velocity addition rule can be obtained by considering the Lorentz transformation matrix connecting K and K''. The individual transformation matrices are

$$\boldsymbol{\lambda}_{K'\to K} = \begin{pmatrix} \gamma_1 & 0 & 0 & i\beta_1\gamma_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta_1\gamma_1 & 0 & 0 & \gamma_1 \end{pmatrix}$$

$$\boldsymbol{\lambda}_{K''\to K'} = \begin{pmatrix} \gamma_2 & 0 & 0 & i\beta_2\gamma_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta_2\gamma_2 & 0 & 0 & \gamma_2 \end{pmatrix}$$

The transformation from K'' to K is just the product of these two transformations:

$$\boldsymbol{\lambda}_{K''\to K} = \boldsymbol{\lambda}_{K''\to K'}\boldsymbol{\lambda}_{K'\to K} = \begin{pmatrix} \gamma_1\gamma_2(1+\beta_1\beta_2) & 0 & 0 & i\gamma_1\gamma_2(\beta_1+\beta_2) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\gamma_1\gamma_2(\beta_1+\beta_2) & 0 & 0 & \gamma_1\gamma_2(1+\beta_1\beta_2) \end{pmatrix}$$

So that the elements of this matrix correspond to those of the normal Lorentz matrix (Equation 14.77), we must identify β and γ for the $K'' \to K$ transformation as

$$\left.\begin{aligned} \gamma &= \gamma_1\gamma_2(1+\beta_1\beta_2) \\ \beta\gamma &= \gamma_1\gamma_2(\beta_1+\beta_2) \end{aligned}\right\} \tag{14.96}$$

from which we obtain

$$\beta = \frac{\beta_1+\beta_2}{1+\beta_1\beta_2} \tag{14.97}$$

If we multiply this last expression by c, we have the usual form of the velocity (speed) addition rule:

$$\boxed{v = \frac{v_1 + v_2}{1 + (v_1 v_2/c^2)}} \tag{14.98}$$

It follows that if $v_1 < c$ and $v_2 < c$, then $v < c$ also.

Even though *signal* velocities can never exceed c, there are other types of velocity that can be greater than c. For example, the *phase velocity* of a light wave in a medium for which the index of refraction is less than unity is greater than c, but the phase velocity does not correspond to the signal velocity in such a medium; the signal velocity is indeed less than c. Or consider an electron gun that emits a beam of electrons. If the gun is rotated, then the electron beam describes a certain path on a screen placed at some appropriate distance. If the angular velocity of the gun and the distance to the screen are sufficiently large, then the velocity of the spot traveling across the screen can be *any* velocity, arbitrarily large. Thus, the *writing speed* of an oscilloscope can exceed c, but again the writing speed does not correspond to the signal velocity; that is, information cannot be transmitted from one point on the screen to another by means of the electron beam. In such a device, a signal can be transmitted only from the gun to the screen, and this transmission takes place at the velocity of the electrons in the beam (i.e., $< c$).

EXAMPLE 14.11

Derive the relativistic Doppler effect if the angle between the light source and direction of relative motion of the observer is θ (Figure 14-8).

Solution. This example can easily be solved using the momentum-energy four-vector by treating the light as a photon with total energy $E = h\nu$. The light source is at rest in system K and emits a single frequency ν_0.

$$E = h\nu_0 \tag{14.99}$$

$$p = \frac{E}{c} = \frac{h\nu_0}{c} \tag{14.100}$$

The observer moving to the right in system K' measures the energy E' for a photon of frequency ν'. From Equation 14.92, we have

$$E' = \gamma(h\nu_0 - vp_1) \tag{14.101}$$

$$h\nu' = \gamma\left(h\nu_0 - \frac{vh\nu_0}{c}\cos\theta\right) \tag{14.102}$$

where $p_1 = p\cos\theta$. Equation 14.102 reduces to

$$\nu' = \gamma\nu_0(1 - \beta\cos\theta) \tag{14.103}$$

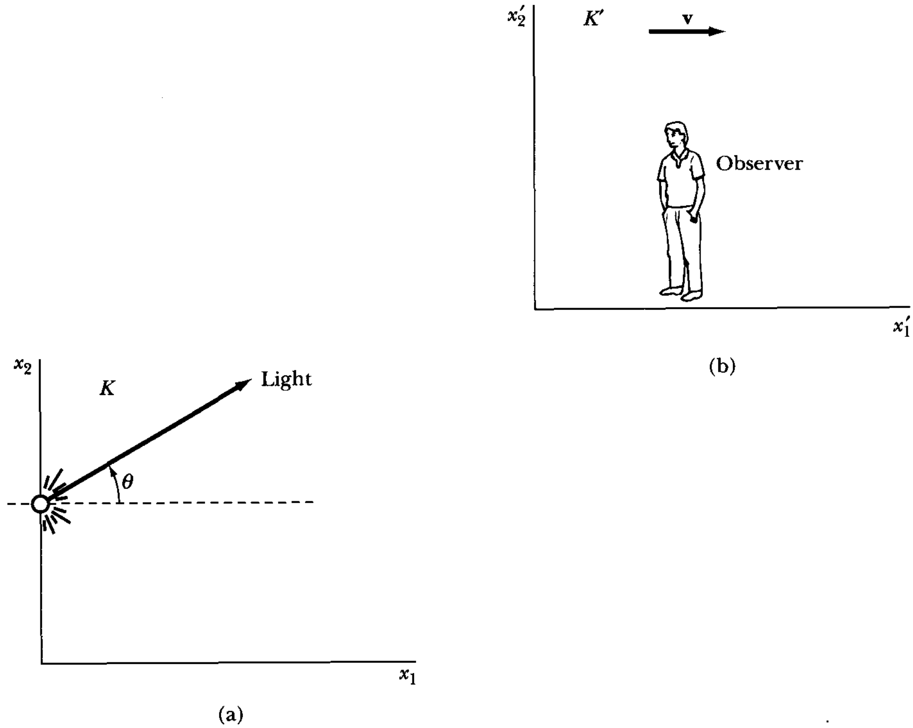


FIGURE 14-8 A light source fixed in system K emits light at a single frequency ν_0. An observer in system K', moving to the right at velocity **v** with respect to K, measures the light frequency to be ν'.

which is equivalent to Equation 14.34, depending on the value of θ. For an early time, the observer is far to the left of the source, and as the observer approaches the source ($\theta = \pi$),

$$\nu' = \nu_0 \frac{\sqrt{1 + \beta}}{\sqrt{1 - \beta}} \qquad \text{observer approaching source} \qquad \textbf{(14.104)}$$

as in Equation 14.31. At a much later time, the observer is receding ($\theta = 0$) and

$$\nu' = \nu_0 \frac{\sqrt{1 - \beta}}{\sqrt{1 + \beta}} \qquad \text{observer receding from source} \qquad \textbf{(14.105)}$$

as in Equation 14.33. When the observer just passes the source ($\theta = \pi/2$),

$$\nu' = \frac{\nu_0}{\sqrt{1 - \beta^2}} \qquad \text{observer passing source} \qquad \textbf{(14.106)}$$

We can also treat the case where the observer is at rest and the source is moving (see Problem 14-18). We still obtain Equations 14.104–14.106 because, according to the principle of relativity, it is not possible to distinguish between the motion of the observer and the motion of the source.

14.10 Lagrangian Function in Special Relativity

Lagrangian and Hamiltonian dynamics (discussed in Chapter 7) must be adjusted in light of the new concepts presented here. We can extend the Lagrangian formalism into the realm of special relativity in the following way. For a single (nonrelativistic) particle moving in a velocity-independent potential, the rectangular momentum components (see Equation 7.150) may be written as

$$p_i = \frac{\partial L}{\partial u_i} \tag{14.107}$$

According to Equation 14.87, the relativistic expression for the ordinary (i.e., space) momentum component is

$$p_i = \frac{mu_i}{\sqrt{1 - \beta^2}} \tag{14.108}$$

We now require that the *relativistic* Lagrangian, when differentiated with respect to u_i as in Equation 14.107, yield the momentum components given by Equation 14.108:

$$\frac{\partial L}{\partial u_i} = \frac{mu_i}{\sqrt{1 - \beta^2}} \tag{14.109}$$

This requirement involves only the *velocity* of the particle, so we expect that the velocity-*independent* part of the relativistic Lagrangian is unchanged from the nonrelativistic case. The velocity-*dependent* part, however, may no longer be equal to the kinetic energy. We therefore write

$$L = T^* - U \tag{14.110}$$

where $U = U(x_i)$ and $T^* = T^*(u_i)$. The function T^* must satisfy the relation

$$\frac{\partial T^*}{\partial u_i} = \frac{mu_i}{\sqrt{1 - \beta^2}} \tag{14.111}$$

It can be easily verified that a suitable expression for T^* (apart from a possible constant of integration that can be suppressed) is

$$T^* = -mc^2\sqrt{1 - \beta^2} \tag{14.112}$$

Hence, the relativistic Lagrangian can be written as

$$\boxed{L = -mc^2\sqrt{1 - \beta^2} - U} \tag{14.113}$$

and the equations of motion are obtained in the standard way from Lagrange's equations.

Notice that the Lagrangian is *not* given by $T - U$, because the relativistic expression for the kinetic energy (Equation 14.58) is

$$T = \frac{mc^2}{\sqrt{1 - \beta^2}} - mc^2 \tag{14.114}$$

The Hamiltonian (see Equation 7.153) can be calculated from

$$\begin{aligned} H &= \sum_i u_i p_i - L \\ &= \sum_i \frac{p_i^2 c^2}{\gamma mc^2} + \frac{mc^2}{\gamma} + U \end{aligned}$$

where we have used Equations 14.108 and 14.113 and changed $\sqrt{1-\beta^2}$ to γ^{-1}. Thus,

$$\begin{aligned} H &= \frac{p^2c^2}{\gamma mc^2} + \frac{mc^2}{\gamma} + U = \frac{1}{\gamma mc^2}(p^2c^2 + m^2c^4) + U \\ &= \frac{E^2}{\gamma mc^2} + U \\ &= E + U = T + U + E_0 \end{aligned} \tag{14.115}$$

The relativistic Hamiltonian is equal to the total energy defined in Section 14.8 *plus* the potential energy. It differs from the total energy used previously in Chapter 7 by now including the rest energy.

14.11 Relativistic Kinematics

In the event that the velocities in a collision process are not negligible with respect to the velocity of light, it becomes necessary to use *relativistic* kinematics. In the discussion in Chapter 9, we took advantage of the properties of the center-of-mass coordinate system in deriving many of the kinematic relations. Because mass and energy are interrelated in relativity theory, it no longer is meaningful to speak of a "center-of-mass" system; in relativistic kinematics, one uses a "center-of-momentum" coordinate system instead. Such a system possesses the same essential property as the previously used center-of-mass system—the total linear momentum in the system is zero. Therefore, if a particle of mass m_1 collides elastically with a particle of mass m_2, then in the center-of-momentum system we have

$$p_1' = p_2' \tag{14.116}$$

Using Equation 14.87, the space components of the momentum four-vector can be written as

$$m_1 u_1' \gamma_1' = m_2 u_2' \gamma_2' \tag{14.117}$$

where, as before, $\gamma \equiv 1/\sqrt{1-\beta^2}$ and $\beta \equiv u/c$.

In a collision problem, it is convenient to associate the laboratory coordinate system with the inertial system K and the center-of-momentum system with K' (see Figure 14-9). A simple Lorentz transformation then connects the two systems. To derive the relativistic kinematic expressions, the procedure is

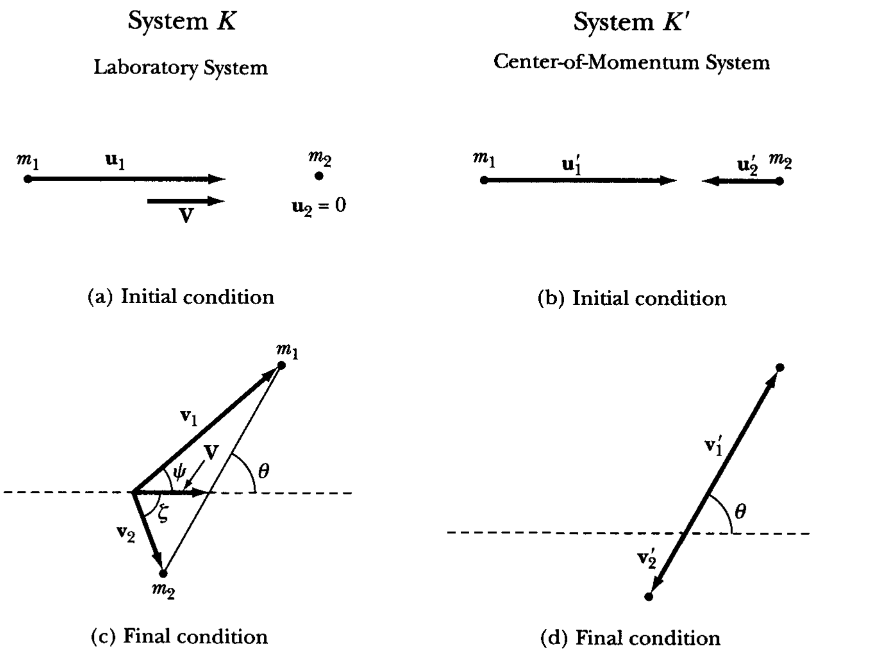


FIGURE 14-9 The elastic collision schematic of Figure 9-10 is redisplayed with systems K and K' indicated.

to obtain the center-of-momentum relations and then perform a Lorentz transformation back to the laboratory system. We choose the coordinate axes so that m_1 moves along the x-axis in K with speed u_1. Because m_2 is initially at rest in K, $u_2 = 0$. In K', m_2 moves with speed u_2' and so K' moves with respect to K also with speed u_2' and in the same direction as the initial motion of m_1.

Using the fact that $\beta\gamma = \sqrt{\gamma^2 - 1}$, we have

$$\begin{aligned} p_1' &= m_1 u_1' \gamma_1' = m_1 c \beta_1' \gamma_1' \\ &= m_1 c \sqrt{\gamma_1'^2 - 1} = m_2 c \sqrt{\gamma_2'^2 - 1} \\ &= p_2' \end{aligned} \tag{14.118}$$

which expresses the equality of the momenta in the center-of-momentum system.

According to Equation 14.92, the transformation of the momentum p_1 (from K to K') is

$$p_1' = \left(p_1 - \frac{u_2'}{c^2} E_1\right)\gamma_2' \tag{14.119}$$

We also have

$$\left.\begin{aligned} p_1 &= m_1 u_1 \gamma_1 \\ E_1 &= m_1 c^2 \gamma_1 \end{aligned}\right\} \tag{14.120}$$

so Equation 14.118 can be used to obtain

$$\begin{aligned} m_1 c\sqrt{\gamma_1'^2 - 1} &= (m_1 c\beta_1\gamma_1 - \beta_2' m_1 c\gamma_1)\gamma_2' \\ &= m_1 c\left(\gamma_2'\sqrt{\gamma_1^2 - 1} - \gamma_1\sqrt{\gamma_2'^2 - 1}\right) \\ &= m_2 c\sqrt{\gamma_2'^2 - 1} \end{aligned} \tag{14.121}$$

These equations can be solved for γ_1' and γ_2' in terms of γ_1:

$$\gamma_1' = \frac{\gamma_1 + \dfrac{m_1}{m_2}}{\sqrt{1 + 2\gamma_1\left(\dfrac{m_1}{m_2}\right) + \left(\dfrac{m_1}{m_2}\right)^2}} \tag{14.122a}$$

$$\gamma_2' = \frac{\gamma_1 + \dfrac{m_2}{m_1}}{\sqrt{1 + 2\gamma_1\left(\dfrac{m_2}{m_1}\right) + \left(\dfrac{m_2}{m_1}\right)^2}} \tag{14.122b}$$

Next, we write the equations of the transformation of the momentum components from K' back to K after the scattering. We now have both x- and y-components:

$$\begin{aligned} p_{1,x} &= \left(p_{1,x}' + \frac{u_2'}{c^2}E_1'\right)\gamma_2' \\ &= (m_1 c\beta_1'\gamma_1'\cos\theta + m_1 c\beta_2'\gamma_1')\gamma_2' \\ &= m_1 c\gamma_1'\gamma_2'(\beta_1'\cos\theta + \beta_2') \end{aligned} \tag{14.123a}$$

(Note that, because the transformation is from K' to K, a plus sign occurs before the second term, in contrast to Equation 14.119.) Also,

$$p_{1,y} = m_1 c\beta_1'\gamma_1'\sin\theta \tag{14.123b}$$

The tangent of the laboratory scattering angle ψ is given by $p_{1,y}/p_{1,x}$; therefore, dividing Equation 14.123b by Equation 14.123a, we obtain

$$\tan\psi = \frac{1}{\gamma_2'}\,\frac{\sin\theta}{\cos\theta + (\beta_2'/\beta_1')}$$

Using Equation 14.117 to express β_2'/β_1', the result is

$$\tan\psi = \frac{1}{\gamma_2'}\,\frac{\sin\theta}{\cos\theta + (m_1\gamma_1'/m_2\gamma_2')} \tag{14.124}$$

For the recoil particle, we have

$$\begin{aligned} p_{2,x} &= \left(p_{2,x}' + \frac{u_2'}{c^2}E_2'\right)\gamma_2' \\ &= (-m_2 c\beta_2'\gamma_2'\cos\theta + m_2 c\beta_2'\gamma_2')\gamma_2' \\ &= m_2 c\beta_2'\gamma_2'^2(1 - \cos\theta) \end{aligned} \tag{14.125a}$$

where a minus sign occurs in the first term because $p'_{2,x}$ is directed opposite to $p_{1,x}$. Also,

$$p_{2,y} = -m_2 c\beta'_2\gamma'_2 \sin\theta \tag{14.125b}$$

As before, the tangent of the laboratory recoil angle ζ is given by $p_{2,y}/p_{2,x}$:

$$\tan\zeta = -\frac{1}{\gamma'_2}\,\frac{\sin\theta}{1-\cos\theta} \tag{14.126}$$

The overall minus sign indicates that if m_1 is scattered toward positive values of ψ, then m_2 recoils in the negative ζ-direction.

A case of special interest is that in which $m_1 = m_2$. From Equations 14.122, we find

$$\gamma'_1 = \gamma'_2 = \sqrt{\frac{1+\gamma_1}{2}}, \qquad m_1 = m_2 \tag{14.127}$$

The tangents of the scattering angles become

$$\tan\psi = \sqrt{\frac{2}{1+\gamma_1}}\cdot\frac{\sin\theta}{1+\cos\theta} \tag{14.128}$$

$$\tan\zeta = -\sqrt{\frac{2}{1+\gamma_1}}\cdot\frac{\sin\theta}{1-\cos\theta} \tag{14.129}$$

The product is therefore

$$\tan\psi\tan\zeta = -\frac{2}{1+\gamma_1}, \qquad m_1 = m_2 \tag{14.130}$$

(The minus sign is of no essential importance; it only indicates that ψ and ζ are measured in opposite directions.)

We previously found that in the nonrelativistic limit there was always a right angle between the final velocity vectors in the scattering of particles of equal mass. Indeed, in the limit $\gamma_1 \to 1$, Equations 14.128 and 14.129 become equal to Equations 9.69 and 9.73, respectively, and so $\psi + \zeta = \pi/2$. Equation 14.130, however, shows that in the relativistic case $\psi + \zeta < \pi/2$; thus, the included angle in the scattering is always smaller than in the nonrelativistic limit. For equal scattering and recoil angles ($\psi = \zeta$), Equation 14.130 becomes

$$\tan\psi = \left(\frac{2}{1+\gamma_1}\right)^{1/2}, \qquad m_1 = m_2$$

and the included angle between the directions of the scattered and recoil particles is

$$\begin{aligned}\phi &= \psi + \zeta = 2\psi \\ &= 2\tan^{-1}\left(\frac{2}{1+\gamma_1}\right)^{1/2}, \qquad m_1 = m_2\end{aligned} \tag{14.131}$$

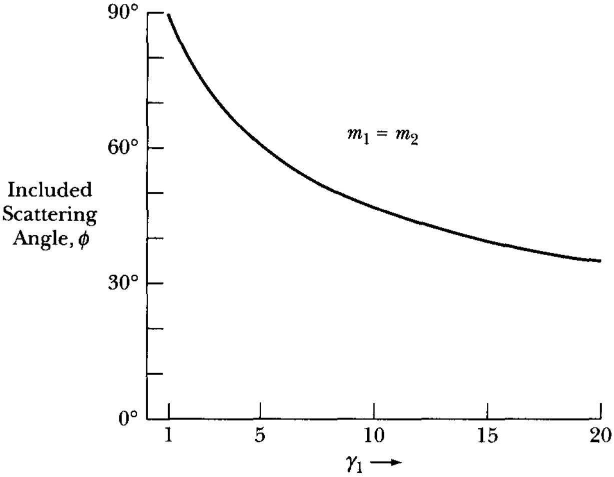


FIGURE 14-10 The included scattering angle, $\phi = \psi + \zeta$, is shown as a function of the relativistic parameter γ_1 for $m_1 = m_2$. For nonrelativistic scattering ($\gamma_1 = 1$), this angle is always 90°.

Figure 14-10 shows ϕ as a function of γ_1 up to $\gamma_1 = 20$. At $\gamma_1 = 10$, the included angle is approximately 46°. This value of γ_1 corresponds to an initial velocity that is 99.5% of the velocity of light. According to Equation 14.58, the kinetic energy is given by $T_1 = m_1c^2(\gamma_1 - 1)$; therefore, a proton with $\gamma_1 = 10$ would have a kinetic energy of approximately 8.4 GeV, whereas an electron with the same velocity would have $T_1 \cong 4.6$ MeV.*

By using the transformation properties of the fourth component of the momentum four-vector (i.e., the total energy), it is possible to obtain the relativistic analogs of all the energy equations we have previously derived in the nonrelativistic limit.

PROBLEMS

14-1. Prove Equation 14.13 by using Equations 14.9–14.12.

14-2. Show that the transformation equations connecting the K' and K systems (Equations 14.14) can be expressed as

$$x_1' = x_1 \cosh \alpha - ct \sinh \alpha$$
$$x_2' = x_2, \quad x_3' = x_3$$
$$t' = t \cosh \alpha - \frac{x_1}{c} \sinh \alpha$$

where $\tanh \alpha = v/c$. Show that the Lorentz transformation corresponds to a rotation through an angle $i\alpha$ in four-dimensional space.

*These units of energy are defined in Problem 14-39: 1 GeV = 10^3 MeV = 10^9 eV = 1.602×10^{-3} erg = 1.602×10^{-10} J.

14-3. Show that the equation

$$\nabla^2\Psi - \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} = 0$$

is invariant under a Lorentz transformation but not under a Galilean transformation. (This is the wave equation that describes the propagation of light waves in free space.)

14-4. Show that the expression for the FitzGerald-Lorentz contraction (Equation 14.19) can also be obtained if the observer in the K' system measures the time necessary for the rod to pass a fixed point in that system and then multiplies the result by v.

14-5. What is the apparent shape of a cube moving with a uniform velocity directly *toward* or *away from* an observer?

14-6. Consider two events that take place at different points in the K system at the same instant t. If these two points are separated by a distance Δx, show that in the K' system the events are not simultaneous but are separated by a time interval $\Delta t' = -v\gamma\,\Delta x/c^2$.

14-7. Two clocks located at the origins of the K and K' systems (which have a relative speed v) are synchronized when the origins coincide. After a time t, an observer at the origin of the K system observes the K' clock by means of a telescope. What does the K' clock read?

14-8. In his 1905 paper (see the translation in Lo23), Einstein states: "We conclude that a balance-clock at the equator must go more slowly, by a very small amount, than a precisely similar clock situated at one of the poles under otherwise identical conditions." Neglect the fact that the equator clock does not undergo uniform motion and show that after a century the clocks will differ by approximately 0.0038 s.

14-9. Consider a relativistic rocket whose velocity with respect to a certain inertial frame is v and whose exhaust gases are emitted with a constant velocity V with respect to the rocket. Show that the equation of motion is

$$m\frac{dv}{dt} + V\frac{dm}{dt}(1 - \beta^2) = 0$$

where $m = m(t)$ is the mass of the rocket in its rest frame and $\beta = v/c$.

14-10. Show by algebraic methods that Equations 14.15 follow from Equations 14.14.

14-11. A stick of length l is fixed at an angle θ from its x_1-axis in its own rest system K. What is the length and orientation of the stick as measured by an observer moving along x_1 with speed v?

14-12. A racer attempting to break the land speed record rockets by two markers spaced 100 m apart on the ground in a time of 0.4 μs as measured by an observer on the ground. How far apart do the two markers appear to the racer? What elapsed time does the racer measure? What speeds do the racer and ground observer measure?

14-13. A muon is moving with speed $v = 0.999c$ vertically down through the atmosphere. If its half-life in its own rest frame is 1.5 μs, what is its half-life as measured by an observer on Earth?

14-14. Show that Equation 14.31 is valid when a receiver approaches a fixed light source with speed v.

14-15. A star is known to be moving away from Earth at a speed of 4×10^4 m/s. This speed is determined by measuring the shift of the H_α line ($\lambda = 656.3$ nm). By how much and in what direction is the shift of the wavelength of the H_α line?

14-16. A photon is emitted at an angle θ' by a star (system K') and then received at an angle θ on Earth (system K). The angles are measured from a line between the star and Earth. The star is receding at speed v with respect to Earth. Find the relation between θ and θ'; this effect is called the *aberration of light.*

14-17. The wavelength of a spectral line measured to be λ on Earth is found to increase by 50% on a far distant galaxy. What is the speed of the galaxy relative to Earth?

14-18. Solve Example 14.11 for the case of the observer at rest and the source moving. Show that the results are the same as those given in Example 14.11.

14-19. Equation 14.34 indicates that a red (blue) shift occurs when a source and observer are receding (approaching) with respect to one another in purely radial motion (i.e., $\beta = \beta_r$). Show that, if there is also a relative tangential speed β_t, Equation 14.34 becomes

$$\frac{\lambda_0}{\lambda} = \frac{\nu}{\nu_0} = \frac{\sqrt{1 - \beta_r^2 - \beta_t^2}}{1 - \beta_r}$$

and that the condition for always having a red shift (i.e., no blue shift), $\lambda > \lambda_0$ or $\nu < \nu_0$, is*

$$\beta_t^2 > 2\beta_r(1 - \beta_r)$$

14-20. An astronaut travels to the nearest star system, 4 light years away, and returns at speed $0.3c$. How much has the astronaut aged relative to those people remaining on Earth?

14-21. The expression for the ordinary force is

$$\mathbf{F} = \frac{d}{dt}\left(\frac{m\mathbf{u}}{\sqrt{1 - \beta^2}}\right)$$

Take $\mathbf{u}$ to be in the x_1-direction and compute the components of the force. Show that

$$F_1 = m_l \dot{u}_1, \quad F_2 = m_t \dot{u}_2, \quad F_3 = m_t \dot{u}_3$$

where m_l and m_t are, respectively, the *longitudinal mass* and the *transverse mass*:

$$m_l = \frac{m}{(1 - \beta^2)^{3/2}}, \qquad m_t = \frac{m}{\sqrt{1 - \beta^2}}$$

*See J. J. Dykla, *Am. J. Phys.* **47**, 381 (1979).

14-22. The average rate at which solar radiant energy reaches Earth is approximately 1.4×10^3 W/m^2. Assume that all this energy results from the conversion of mass to energy. Calculate the rate at which the solar mass is being lost. If this rate is maintained, calculate the remaining lifetime of the Sun. (Pertinent numerical data can be found in Table 8-1.)

14-23. Show that the momentum and the kinetic energy of a particle are related by $p^2c^2 = 2Tmc^2 + T^2$.

14-24. What is the minimum proton energy needed in an accelerator to produce antiprotons $\bar{p}$ by the reaction

$$p + p \rightarrow p + p + (p + \bar{p})$$

The mass of a proton and antiproton is m_p.

14-25. A particle of mass m, kinetic energy T, and charge q is moving perpendicular to a magnetic field B as in a cyclotron. Find the relation for the radius r of the particle's path in terms of m, T, q, and B.

14-26. Show that an isolated photon cannot be converted into an electron-positron pair, $\gamma \rightarrow e^- + e^+$. (The conservation laws allow this to happen only near another object.)

14-27. Electrons and positrons collide from opposite directions head-on with equal energies in a storage ring to produce protons by the reaction

$$e^- + e^+ \rightarrow p + \bar{p}$$

The rest energy of a proton and antiproton is 938 MeV. What is the minimum kinetic energy for each particle to produce this reaction?

14-28. Calculate the range of speeds for a particle of mass m in which the classical relation for kinetic energy, $\frac{1}{2}mv^2$, is within one percent of the correct relativistic value. Find the values for an electron and a proton.

14-29. The 2-mile long Stanford Linear Accelerator accelerates electrons to 50 GeV (50×10^9 eV). What is the speed of the electrons at the end?

14-30. A free neutron is unstable and decays into a proton and an electron. How much energy other than the rest energies of the proton and electron is available if a neutron at rest decays? (This is an example of nuclear beta decay. Another particle, called a neutrino—actually an antineutrino $\bar{\nu}$ is also produced.)

14-31. A neutral pion π^0 moving at speed $v = 0.98c$ decays in flight into two photons. If the two photons emerge on each side of the pion's direction with equal angles θ, find the angle θ and energies of the photons. The rest energy of π^0 is 135 MeV.

14-32. In nuclear and particle physics, momentum is usually quoted in MeV/c to facilitate calculations. Calculate the kinetic energy of an electron and proton if each has a momentum of 1000 MeV/c.

14-33. A neutron (m_n = 939.6 MeV/c^2) at rest decays into a proton (m_p = 938.3 MeV/c^2), an electron (m_e = 0.5 MeV/c^2), and an antineutrino ($m_{\bar{\nu}} \approx 0$). The three particles emerge at symmetrical angles in a plane, 120° apart. Find the momentum and kinetic energy of each particle.

14-34. Show that Δs^2 is invariant in all inertial systems moving at relative velocities to each other.

14-35. A spacecraft passes Saturn with a speed of $0.9c$ relative to Saturn. A second spacecraft is observed to pass the first one (going in the same direction) at relative speed of $0.2c$. What is the speed of the second spacecraft relative to Saturn?

14-36. We define the four-vector force $\mathbb{F}$ (called the Minkowski force) by differentiating the four-vector momentum with respect to proper time.

$$\mathbb{F} = \frac{d\mathbb{P}}{d\tau}$$

Show that the four-vector force transformation is

$$\begin{aligned} F_1' &= \gamma(F_1 + i\beta F_4) \\ F_2' &= F_2 \\ F_3' &= F_3 \\ F_4' &= \gamma(F_4 - i\beta F_1) \end{aligned}$$

14-37. Consider a one-dimensional, relativistic harmonic oscillator for which the Lagrangian is

$$L = mc^2\left(1 - \sqrt{1 - \beta^2}\right) - \frac{1}{2}kx^2$$

Obtain the Lagrange equation of motion and show that it can be integrated to yield

$$E = mc^2 + \frac{1}{2}ka^2$$

where a is the maximum excursion from equilibrium of the oscillating particle. Show that the period

$$\tau = 4\int_{x=0}^{x=a} dt$$

can be expressed as

$$\tau = \frac{2a}{\kappa c}\int_0^{\pi/2} \frac{1 + 2\kappa^2\cos^2\phi}{\sqrt{1 + \kappa^2\cos^2\phi}}\, d\phi$$

Expand the integrand in powers of $\kappa \equiv (a/2)\sqrt{k/mc^2}$ and show that, to first order in κ,

$$\tau \cong \tau_0\left(1 + \frac{3}{16}\frac{ka^2}{mc^2}\right)$$

where τ_0 is the nonrelativistic period for small oscillations, $2\pi\sqrt{m/k}$.

14-38. Show that the relativistic form of Newton's Second Law becomes

$$F = m\frac{du}{dt}\left(1 - \frac{u^2}{c^2}\right)^{-3/2}$$

14-39. A common unit of energy used in atomic and nuclear physics is the electron volt (eV), the energy acquired by an electron in falling through a potential difference of one volt: 1 MeV = 10^6 eV = 1.602×10^{-13} J. In these units, the mass of an electron is $m_e c^2 = 0.511$ MeV and that of a proton is $m_p c^2 = 938$ MeV. Calculate the kinetic energy and the quantities β and γ for an electron and for a proton each having a momentum of 100 MeV/c. Show that the electron is "relativistic" whereas the proton is "nonrelativistic."

14-40. Consider an inertial frame K that contains a number of particles with masses m_α, ordinary momentum components $p_{\alpha,j}$, and total energies E_α. The center-of-mass system of such a group of particles is defined to be that system in which the net ordinary momentum is zero. Show that the velocity components of the center-of-mass system with respect to K are given by

$$\frac{v_j}{c} = \frac{\sum_\alpha p_{\alpha,j}c}{\sum_\alpha E_\alpha}$$

14-41. Show that the relativistic expression for the kinetic energy of a particle scattered through an angle ψ by a target particle of equal mass is

$$\frac{T_1}{T_0} = \frac{2\cos^2\psi}{(\gamma_1 + 1) - (\gamma_1 - 1)\cos^2\psi}$$

The expression evidently reduces to Equation 9.89a in the nonrelativistic limit $\gamma_1 \rightarrow 1$. Sketch $T_1(\psi)$ for neutron-proton scattering for incident neutron energies of 100 MeV, 1 GeV, and 10 GeV.

14-42. The energy of a light quantum (or photon) is expressed by $E = h\nu$, where h is Planck's constant and ν is the frequency of the photon. The momentum of the photon is $h\nu/c$. Show that, if the photon scatters from a free electron (of mass m_e), the scattered photon has an energy

$$E' = E\left[1 + \frac{E}{m_e c^2}(1 - \cos\theta)\right]^{-1}$$

where θ is the angle through which the photon scatters. Show also that the electron acquires a kinetic energy

$$T = \frac{E^2}{m_e c^2}\left[\frac{1 - \cos\theta}{1 + \dfrac{E}{m_e c^2}(1 - \cos\theta)}\right]$$

"*Better is the end of a thing than the beginning thereof.*"—Ecclesiastes