Chapter 3

Newtonian Gravity

It is perfectly possible to discuss cosmology without having already learned general relativity. In fact, the most crucial equation, the Friedmann equation which describes the expansion of the Universe, turns out to be the same when derived from Newton's theory of gravity as it is when derived from the equations of general relativity. The Newtonian derivation is, however, some way from being completely rigorous, and general relativity is required to fully patch it up, a detail that need not concern us at this stage. Readers with some familiarity with general relativity may wish to study the relativistic derivation given in Advanced Topic 1.

In Newtonian gravity all matter attracts, with the force exerted by an object of mass M on one of mass m given by the famous relationship

$$F = \frac{GMm}{r^2}\,, \tag{3.1}$$

where r is the distance between the objects and G is Newton's gravitational constant. That is, gravity obeys an inverse square law. Because a force on an object induces an acceleration which is also proportional to its mass, via $F = ma$, the acceleration an object feels under gravity is *independent* of its mass.

The force exerted means there is a gravitational potential energy

$$V = -\frac{GMm}{r}\,, \tag{3.2}$$

with the force being in the direction which decreases the potential energy the fastest. Like the electric potential of two opposite charges, the gravitational potential is negative, favouring the two objects being close together. But with gravity there is no analogue of the repulsion of like charges. Gravity always attracts.

The derivation of the Friedmann equation requires a famous result due originally to Newton, which I won't attempt to prove here. This result states that in a spherically-symmetric distribution of matter, a particle feels no force at all from the material at greater radii, and the material at smaller radii gives exactly the force which one would get if all the

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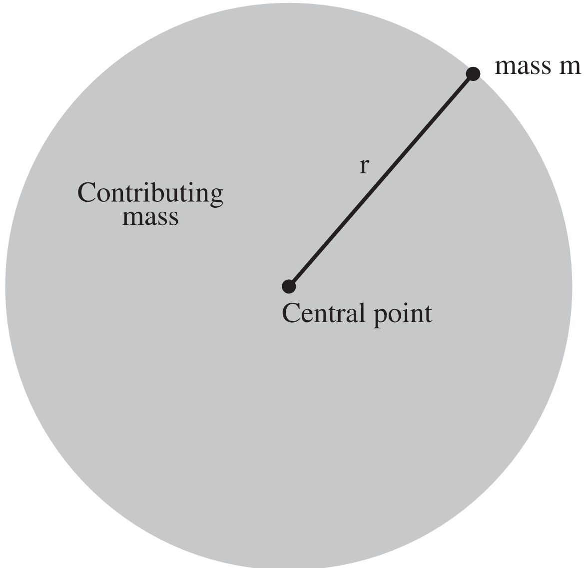

Figure 3.1 The particle at radius r only feels gravitational attraction from the shaded region. Any gravitational attraction from the material outside cancels out, according to Newton's theorem.

material was concentrated at the central point. This property arises from the inverse square law; the same results exist for electromagnetism. One example of its use is that the gravitational (or electromagnetic) force outside a spherical object of unknown density profile depends only on the total mass (charge). Another is that an 'astronaut' inside a spherical shell feels no gravitational force, not only if they are at the centre but if they are at any position inside the shell.

3.1 The Friedmann equation

The Friedmann equation describes the expansion of the Universe, and is therefore the most important equation in cosmology. One of the routine tasks for a working cosmologist is solving this equation under different assumptions concerning the material content of the Universe. To derive the Friedmann equation, we need to compute the gravitational potential energy and the kinetic energy of a test particle (it doesn't matter which one, since everywhere in the Universe is the same according to the cosmological principle), and then use energy conservation.

Let's consider an observer in a uniform expanding medium, with mass density ρ, the mass density being the mass per unit volume. We begin by realizing that because the Universe looks the same from anywhere, we can consider any point to be its centre. Now consider a particle a distance r away with mass m, as shown in Figure 3.1. (By 'particle', I really mean a small volume containing the mass m.) Due to Newton's theorem, this particle only feels a force from the material at smaller radii, shown as the shaded region. This material has total mass given by $M = 4\pi\rho r^3/3$, contributing a force

$$F = \frac{GMm}{r^2} = \frac{4\pi G\rho\, rm}{3}\,, \tag{3.3}$$

and our particle has a gravitational potential energy

$$V = -\frac{GMm}{r} = -\frac{4\pi G\rho\, r^2 m}{3}\,. \tag{3.4}$$

The kinetic energy is easy; the velocity of the particle is $\dot{r}$ (I'll always use dots to mean time derivatives) giving

$$T = \frac{1}{2}m\dot{r}^2\,. \tag{3.5}$$

The equation describing how the separation r changes can now be derived from energy conservation for that particle, namely

$$U = T + V\,, \tag{3.6}$$

where U is a constant. Note that U need not be the same constant for particles separated by different distances. Substituting gives

$$U = \frac{1}{2}\,m\dot{r}^2 - \frac{4\pi}{3}\,G\rho\, r^2 m\,. \tag{3.7}$$

This equation gives the evolution of the separation r between the two particles.

We now make a crucial step in this derivation, which is to realize that this argument applies to any two particles, because the Universe is homogeneous. This allows us to change to a different coordinate system, known as **comoving coordinates**. These are coordinates which are carried along with the expansion. Because the expansion is uniform, the relationship between real distance $\vec{r}$ and the comoving distance, which we can call $\vec{x}$, can be written

$$\vec{r} = a(t)\,\vec{x}\,, \tag{3.8}$$

where the homogeneity property has been used to ensure that a is a function of time alone. Note that these distances have been written as vector distances. What you should think of when studying this equation is a coordinate grid which expands with time, as shown in Figure 3.2. The galaxies remain at fixed locations in the $\vec{x}$ coordinate system. The original $\vec{r}$ coordinate system, which does not expand, is usually known as **physical coordinates**.

The quantity $a(t)$ is a crucial one, and is known as the **scale factor of the Universe**. It measures the universal expansion rate. It is a function of time alone, and it tells us how physical separations are growing with time, since the coordinate distances $\vec{x}$ are by definition fixed. For example, if, between times t_1 and t_2, the scale factor doubles in value, that tells us that the Universe has expanded in size by a factor two, and it will take us twice as long to get from one galaxy to another.

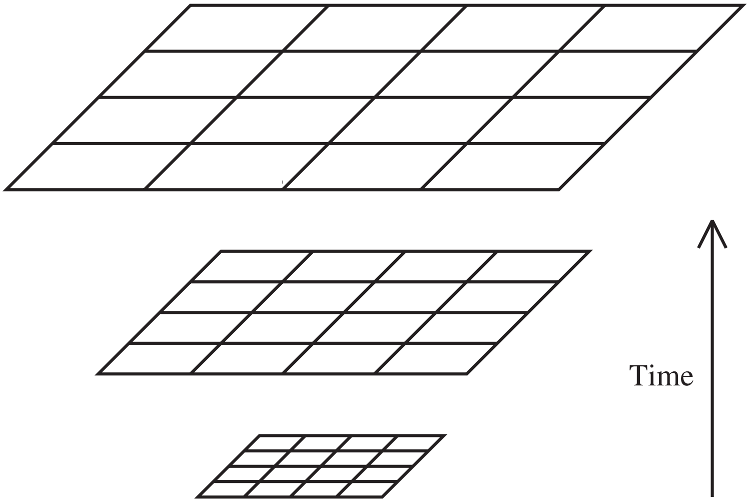

Figure 3.2 The comoving coordinate system is carried along with the expansion, so that any objects remain at fixed coordinate values.

We can use the scale factor to rewrite Equation (3.7) for the expansion. Substituting Equation (3.8) into it, remembering $\dot{x} = 0$ by definition as objects are fixed in comoving coordinates, gives

$$U = \frac{1}{2} m \dot{a}^2 x^2 - \frac{4\pi}{3} G \rho\, a^2 x^2 m\,. \tag{3.9}$$

Multiplying each side by $2/ma^2x^2$ and rearranging the terms then gives

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}\,, \tag{3.10}$$

where $kc^2 = -2U/mx^2$. This is the standard form of the **Friedmann equation**, and it will appear frequently throughout this book. In this expression k must be independent of x since all the other terms in the equation are, otherwise homogeneity will not be maintained. So in fact we learn that homogeneity requires that the quantity U, while constant for a given particle, does indeed change if we look at different separations x, with $U \propto x^2$.

Finally, since $k = -2U/mc^2x^2$ which is time independent (as the total energy U is conserved, and the comoving separation x is fixed), we learn that k is just a constant, unchanging with either space or time. It has the units of [length]$^{-2}$. An expanding Universe has a unique value of k, which it retains throughout its evolution. In Chapter 4 we will see that k tells us about the geometry of the Universe, and it is often called the curvature.

3.2 On the meaning of the expansion

So what does the expansion of the Universe mean? Well, let's start with what it does *not* mean. It does not mean that your body is gradually going to get bigger with time (and certainly isn't an excuse if it does). It does not mean that the Earth's orbit is going to get further from the Sun. It doesn't even mean that the stars within our galaxy are going to become more widely separated with time.

But it does mean that distant galaxies are getting further apart.

The distinction is whether or not the motion of objects is governed by the cumulative gravitational effect of a homogeneous distribution of matter between them, as shown in Figure 3.1. The atoms in your body are not; their separation is dictated by the strength of chemical bonds, with gravity playing no significant role. So molecular structures will not be affected by the expansion. Likewise, the Earth's motion in its orbit is completely dominated by the gravitational attraction of the Sun (with a minor contribution from the other planets). And even the stars in our galaxy are orbiting in the common gravitational potential well which they themselves create, and are not moving apart relative to one another. The common feature of these environments is that they are ones of considerable excess density, very different from the smooth distribution of matter we used to derive the Friedmann equation.

But if we go to large enough scales, in practice tens of megaparsecs, the Universe does become effectively homogeneous and isotropic, with the galaxies flying apart from one another in accordance with the Friedmann equation. It is on these large scales that the expansion of the Universe is felt, and on which the cosmological principle applies.

3.3 Things that go faster than light

A common question that concerns people is whether faraway galaxies are receding from us faster than the speed of light. That is to say, if velocity is proportional to distance, then if we consider galaxies far enough away can we not make the velocity as large as we like, in violation of special relativity?

The answer is that indeed in our theoretical predictions distant objects can *appear* to move away faster than the speed of light. However, it is space itself which is expanding. There is no violation of causality, because no signal can be sent between such galaxies. Further, special relativity is not violated, because it refers to the relative speeds of objects passing each other, and cannot be used to compare the relative speeds of distant objects.

One way to think of this is to imagine a colony of ants on a balloon. Suppose that the fastest the ants can move is a centimetre per second. If any two ants happen to pass each other, their fastest relative speed would be two centimetres per second, if they happened to be moving in opposite directions. Now start to blow the balloon up. Although the ants wandering around the surface still cannot exceed one centimetre per second, the balloon is now expanding under them, and ants which are far apart on the balloon could easily be moving apart at faster than two centimetres per second if the balloon is blown up fast enough. But if they are, they will never get to tell each other about it, because the balloon is pulling them apart faster than they can move together, even at full speed. Any ants that start close enough to be able to pass one another must do so at no more than two centimetres per second even if the Universe is expanding.

The expansion of space is just like that of the balloon, and pulls the galaxies along with it.

3.4 The fluid equation

Fundamental though it is, the Friedmann equation is of no use without an equation to describe how the density ρ of material in the Universe is evolving with time. This involves the pressure p of the material, and is called the fluid equation. (Unfortunately the standard symbol p for pressure is the same as for momentum, which we've already used. Almost always in this book, p will be pressure.) As we'll shortly see, the different types of material which might exist in our Universe have different pressures, and lead to different evolution of the density ρ.

We can derive the fluid equation by considering the first law of thermodynamics

$$dE + p dV = T dS\,, \tag{3.11}$$

applied to an expanding volume V of unit comoving radius.[1] This is exactly the same as applying thermodynamics to a gas in a cylinder. The volume has physical radius a, so the energy is given, using $E = mc^2$, by

$$E = \frac{4\pi}{3} a^3 \rho\, c^2\,. \tag{3.12}$$

The change of energy in a time dt, using the product rule, is

$$\frac{dE}{dt} = 4\pi a^2 \rho\, c^2 \,\frac{da}{dt} + \frac{4\pi}{3} a^3 \,\frac{d\rho}{dt}\, c^2\,, \tag{3.13}$$

while the rate of change in volume is

$$\frac{dV}{dt} = 4\pi a^2 \,\frac{da}{dt}\,. \tag{3.14}$$

Assuming a reversible adiabatic expansion $dS = 0$, putting these into Equation (3.11) and rearranging gives

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0\,, \tag{3.15}$$

where as always dots are shorthand for time derivatives. This is the **fluid equation**. As we see, there are two terms contributing to the change in the density. The first term in the brackets corresponds to the dilution in the density because the volume has increased, while the second corresponds to the loss of energy because the pressure of the material has done work as the Universe's volume increased. This energy has not disappeared entirely of course; energy is always conserved. The energy lost from the fluid via the work done has gone into gravitational potential energy.

[1] Don't confuse V for volume with V for gravitational potential energy.

Let me stress that there are no pressure forces in a homogeneous universe, because the density and pressure are everywhere the same. A pressure *gradient* is required to supply a force. So pressure does not contribute a force helping the expansion along; its effect is solely through the work done as the Universe expands.

We are still not in a position to solve the equations, because now we only know what ρ is doing if we know what the pressure p is. It is in specifying the pressure that we are saying what kind of material our model universe is filled with. The usual assumption in cosmology is that there is a unique pressure associated with each density, so that $p \equiv p(\rho)$. Such a relationship is known as the **equation of state**, and we'll see two different examples in Chapter 5. The simplest possibility is that there is no pressure at all, and that particular case is known as (non-relativistic) matter.

Once the equation of state is specified, the Friedmann and fluid equations are all we need to describe the evolution of the Universe. However, before discussing this evolution, I am going to spend some time exploring some general properties of the equations, as well as devoting Chapter 4 to consideration of the meaning of the constant k. If you prefer to immediately see how to solve these equations, feel free to jump straight away to Sections 5.3 to 5.5, and come back to the intervening material later. On the way, you might want to glance at Section 3.6 to find out why a factor of c^2 mysteriously vanishes from the Friedmann equation between here and there.

3.5 The acceleration equation

The Friedmann and fluid equations can be used to derive a third equation, not independent of the first two of course, which describes the acceleration of the scale factor. By differentiating Equation (3.10) with respect to time we obtain

$$2\,\frac{\dot{a}}{a}\,\frac{a\ddot{a}-\dot{a}^2}{a^2} = \frac{8\pi G}{3}\dot{\rho} + 2\frac{kc^2\dot{a}}{a^3}\,. \tag{3.16}$$

Substituting in for $\dot{\rho}$ from Equation (3.15) and cancelling the factor $2\dot{a}/a$ in each term gives

$$\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 = -4\pi G\left(\rho + \frac{p}{c^2}\right) + \frac{kc^2}{a^2}\,, \tag{3.17}$$

and finally, using Equation (3.10) again, we arrive at an important equation known as the **acceleration equation**

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)\,. \tag{3.18}$$

Notice that if the material has any pressure, this *increases* the gravitational force, and so further decelerates the expansion. I remind you that there are no forces associated with pressure in an isotropic universe, as there are no pressure gradients.

The acceleration equation does not feature the constant k which appears in the Friedmann equation; it cancelled out in the derivation.

3.6 On mass, energy and vanishing factors of c^2

You should be aware that cosmologists have a habit of using mass density ρ and energy density ϵ interchangeably. They are related via Einstein's most famous equation as $\epsilon = \rho c^2$, and if one chooses so-called 'natural units' in which c is set equal to one, the two become the same. For clarity, however, I will be careful to maintain the distinction. Note that the phrase 'mass density' is used in Einstein's sense – it includes the contributions to the mass from the energy of the various particles, as well as any rest mass they might have.

The habit of setting $c = 1$ means that the Friedmann equation is normally written without the c^2 in the final term, so that it reads

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}\,. \tag{3.19}$$

The constant k then appears to have units $[\text{time}]^{-2}$ – setting $c = 1$ makes time and length units interchangeable. Since the practice of omitting the c^2 in the Friedmann equation is widely adopted in other cosmology textbooks, I will drop it for the remainder of this book too. In practice, it is a rare situation indeed where one has to be careful about this.