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UNIVERSE ACROSS SCALES - ASSIGNMENT 2

(1) Wavefunction, $\psi(x,0) = \begin{cases} Ax/a, & 0 \le x \le a \\ A(b-x)/(b-a), & a \le x \le b \\ 0, & \text{otherwise} \end{cases}$

(a) Normalising : $\int_{-\infty}^{\infty} |\psi(x,0)|^2 \, dx = 1$

$$\int_0^a |\psi(x,0)|^2 \, dx + \int_a^b \left| \frac{A(b-x)}{(b-a)} \right|^2 dx = 1$$

$$\int_0^a \frac{A^2 x^2}{a^2} \, dx + \int_a^b \frac{A^2 (b-x)^2}{(b-a)^2} \, dx = 1$$

$$\frac{A^2}{a^2} \int_0^a x^2 \, dx + \frac{A^2}{(b-a)^2} \int_a^b (b-x)^2 \, dx = 1$$

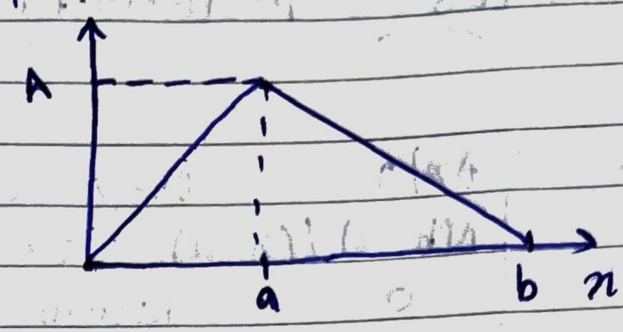
$$\frac{A^2}{a^2} \left[\frac{x^3}{3} \right]_0^a + \frac{A^2}{(b-a)^2} \left[\frac{-(b-x)^3}{3} \right]_a^b = 1$$

$$\frac{A^2}{\cancel{a^2}\, 3} (\cancel{a^3} - 0) + \frac{A^2}{\cancel{(b-a)^2}} \times \frac{(b-a)^{\cancel{3}}}{3} = 1$$

$$\frac{a A^2}{3} + \frac{A^2 (b-a)}{3} = 1$$

$$\frac{b}{3} A^2 = 1 \implies A^2 = 3/b \implies A = \underline{\underline{\sqrt{3/b}}}$$

(b) $\psi(x,0)$



(c) From the above graph, we can see that the particle is most likely to be found at $x=a$.

(d) Probability of finding particle b/w a and b:

$$\int_a^b |\psi(x,0)|^2 \, dx$$

$$P(-\infty < x \leq a) = \int_{-\infty}^{0} |\psi(x,0)|^2 \, dx + \int_0^a |\psi(x,0)|^2 \, dx$$

$$= 0 + \int_0^a \frac{A^2 x^2}{a^2} \, dx$$

$$= \frac{A^2}{a^2}\left[\frac{x^3}{3}\right]_0^a$$

$$= \frac{A^2}{3a^{\cancel{2}}}(a^3)^0$$

$$= \frac{aA^2}{3}$$

We know, $A = \sqrt{\frac{3}{b}}$

$$P(-\infty < x \le a) = \frac{a}{\cancel{3}} \times \frac{\cancel{3}}{b}$$

$$= \frac{a}{b}$$

CASE 1: $b = a$

$$P(-\infty < x \le a) = \frac{a}{a} = 1$$

CASE 2: $b = 2a$

$$P(-\infty < x \le a) = \frac{a}{2a} = \frac{1}{2}$$

(2) The Schrodinger equation is given by :-

$$\frac{\partial \psi}{\partial t} = \frac{i\hbar}{2m} \frac{\partial^2 \psi}{\partial x^2} - \frac{i}{\hbar} V(x,t)\, \psi(x,t)$$

Let us now take the wave function $\psi(x,t)$ to have a potential energy of $V(x,t) + V_0$

$$\frac{\partial \psi}{\partial t} = \frac{i\hbar}{2m} \frac{\partial^2 \psi}{\partial x^2} - \frac{i}{\hbar} \left[V(x,t) + V_0 \right] \psi(x,t)$$

$$= \frac{i\hbar}{2m} \frac{\partial^2 \psi}{\partial x^2} - \frac{i}{\hbar} V(x,t)\, \psi(x,t) - \frac{i}{\hbar} V_0\, \psi(x,t)$$

$$\frac{\partial \psi}{\partial t} + \frac{i}{\hbar} V_0 \psi(x,t) = \frac{i\hbar}{2m}\frac{\partial^2 \psi}{\partial x^2} - \frac{i}{\hbar} V(x,t)\, \psi(x,t)$$

This is a differential equation, so the integrating factor

$$I = \exp\left(\int \frac{iV_0}{\hbar}\, dt\right)$$

$$= e^{iV_0 t/\hbar}$$

Multiplying both sides of the DE by I:-

$$I\frac{d\psi}{dt} + I\frac{i}{\hbar} V_0 \psi(x,t) = I\frac{i\hbar}{2m}\frac{\partial^2 \psi}{\partial x^2} - I\frac{i}{\hbar} V(x,t)\, \psi(x,t)$$

$$e^{iV_0 t/\hbar}\frac{d\psi}{dt} + e^{iV_0 t/\hbar}\frac{i}{\hbar} V_0 \psi(x,t) = e^{iV_0 t/\hbar}\left(\frac{i\hbar}{2m}\frac{\partial^2 \psi}{\partial x^2} - \frac{i}{\hbar} V \psi(x,t)\right)$$

The LHS can be seen as a case of product rule for differentiation.

$$\therefore \frac{\partial}{\partial t}\left(e^{iV_0 t/\hbar}\psi\right) = e^{iV_0 t/\hbar}\frac{i\hbar}{2m}\frac{\partial^2 \psi}{\partial x^2} - e^{iV_0 t/\hbar}\frac{i}{\hbar} V \psi(x,t)$$

Bringing $e^{iV_0 t/\hbar}$ into the partial derivative,

$$\frac{\partial}{\partial t}\left(e^{iV_0 t/\hbar}\psi\right) = \frac{i\hbar}{2m}\frac{\partial^2}{\partial x^2}\left(e^{iV_0 t/\hbar}\psi\right) - \frac{i}{\hbar} V(x,t)\, e^{iV_0 t/\hbar}\psi$$

Comparing this to Schrodinger's equation,

$$\phi = \psi(x,t)\, e^{iV_0 t/\hbar}$$

$$\Rightarrow \psi(x,t) = \phi(x,t)\, e^{-iV_0 t/\hbar}$$

Hence, the wave function picked up a time dependent $[e^{-iV_0 t/\hbar}]$ factor.

(3) $$\frac{d}{dt}\int_{-\infty}^{\infty} \psi_1^* \psi_2 \, dx = 0 \qquad \text{[TO BE PROVEN]}$$

$$\frac{d}{dt}\int_{-\infty}^{\infty} \psi_1^* \psi_2 \, dx = \int_{-\infty}^{\infty} \frac{\partial}{\partial t}(\psi_1^* \psi_2)\, dx$$

$$= \int_{-\infty}^{\infty} \left(\psi_1^* \frac{\partial}{\partial t}\psi_2 + \psi_2 \frac{\partial}{\partial t}\psi_1^* \right) dx$$

We know ψ_1 and ψ_2 are solutions to the Schrodinger equation.

$$\therefore \frac{\partial \psi_2}{\partial t} = \frac{i\hbar}{2m} \frac{\partial^2 \psi_2}{\partial x^2} - \frac{i}{\hbar} V \psi_2$$

$$\frac{\partial \psi_1^*}{\partial t} = \frac{-i\hbar}{2m} \frac{\partial^2 \psi_1^*}{\partial t} + \frac{i}{\hbar} V \psi_1^*$$

$$\int_{-\infty}^{\infty}\left[\psi_1^*\frac{\partial\psi_2}{\partial t}+\frac{\partial\psi_1^*}{\partial t}\psi_2\right]dx =$$

$$\int_{-\infty}^{\infty}\psi_1^*\left[\frac{i\hbar}{2m}\frac{\partial^2}{\partial x^2}\psi_2-\frac{i}{\hbar}V\psi_2\right]+\psi_2\left[\frac{-i\hbar}{2m}\frac{\partial^2}{\partial x^2}\psi_1^*+\frac{i}{\hbar}V\psi_1^*\right]dx$$

$$\int_{-\infty}^{\infty}\frac{i\hbar}{2m}\left[\psi_1^*\frac{\partial^2\psi_2}{\partial x^2}-\psi_2\frac{\partial^2\psi_1^*}{\partial x^2}\right]-\cancel{\frac{iV\psi_1^*\psi_2}{\hbar}}+\cancel{\frac{iV\psi_1^*\psi_2}{\hbar}}$$

$$=\int_{-\infty}^{\infty}\frac{i\hbar}{2m}\left[\psi_1^*\frac{\partial^2\psi_2}{\partial x^2}-\psi_2\frac{\partial^2\psi_1^*}{\partial x^2}\right]$$

$$=\frac{i\hbar}{2m}\left\{\left[\psi_1^*\frac{\partial^2\psi^2}{\partial x^2}\Big|_{-\infty}^{\infty}-\cancel{\frac{\partial\psi_1^*}{\partial x}\frac{\partial\psi_2}{\partial x}}\right.-\left[\psi_2\frac{\partial^2\psi_1^*}{\partial x^2}\Big|_{-\infty}^{\infty}+\cancel{\frac{\partial\psi_2}{\partial x}\frac{\partial\psi_1^*}{\partial x}}\right.\right\}$$

$$0$$

because $\left|\psi_1^*\frac{\partial^2\psi^2}{\partial x^2}\right|_{-\infty}^{\infty}=\left|\psi_2\frac{\partial^2\psi_1^*}{\partial x^2}\right|_{-\infty}^{\infty}=0$

(solutions are orthogonal to each other)

7(a) $\psi(x,0) = \begin{cases} A(a^2 - x^2), & -a \le x \le a \\ 0, & \text{otherwise} \end{cases}$

9(a) To normalize $\int_{-\infty}^{\infty} |\psi(x,0)|^2 = 1$

$$\int_{-\infty}^{\infty} A^2 (a^2 - x^2)^2 \, dx = 1$$

$$\int_{-a}^{a} A^2 (a^4 + x^4 - 2a^2x^2) \, dx = 1$$

$$2A^2 \int_{0}^{a} (a^4 + x^4 - 2a^2x^2) \, dx = 1$$ [changing limits from $(-a, a)$ to $(0, a)$]

$$2A^2 \left[a^4 x + \frac{x^5}{5} - \frac{2a^2x^3}{3} \right]_0^a = 1$$

$$2A^2 \left[a^5 + \frac{a^5}{5} - \frac{2a^5}{3} - 0 \right] = 1$$

$$2A^2 \left[\frac{15a^5 + 3a^5 - 10a^5}{15} \right] = 1$$

$$\frac{2A^2 \, 8a^5}{15} = 1$$

$$A^2 = \frac{15}{2} \times \frac{1}{8a^5}$$

$$= \frac{15}{16a^5}$$

$$\Longrightarrow A = \frac{1}{4}\sqrt{\frac{15}{a^5}} \;//$$

(b)

$$\langle x \rangle = \int_{-\infty}^{\infty} x\,|\psi(x,0)|^2\,dx$$

$$= \int_{-a}^{a} x\,|A(a^2-x^2)|^2\,dx$$

$$= \int_{-a}^{a} A^2\,x(a^2-x^2)^2\,dx$$

$$= A^2 \int_{-a}^{a} x(a^2-x^2)^2\,dx$$

now, $f(x) = x(a^2-x^2)^2$

$f(-x) = -x(a^2-x^2)^2 = -f(x)$

Thus, $f(x)$ is an odd function and the integration of an odd integral over a symmetric interval is zero.

$$\langle x \rangle = 0$$

(c)

$$\langle p \rangle = \langle mv \rangle = m\langle v \rangle = m\frac{d}{dt}\langle x \rangle = 0$$

(d)

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 |\psi(x,0)|^2 \, dx$$

$$= \int_{-a}^{a} x^2 A^2 (a^2 - x^2)^2 \, dx$$

$$= A^2 \int_{-a}^{a} x^2 (a^2 - x^2)^2 \, dx$$

$$= A^2 \int_{-a}^{a} (a^4 x^2 + x^6 - 2a^2 x^4) \, dx$$

$$= A^2 \left[\frac{a^4 x^3}{3} + \frac{x^7}{7} - \frac{2a^2 x^5}{5} \right]_{-a}^{a}$$

7:00 $= 2A^2\left[\frac{a^4x^3}{3} + \frac{x^7}{7} - \frac{2a^2x^5}{5}\right]_0^a$ ٧:٠٠

8:00 ٨:٠٠

$= 2\times\frac{1}{16}\times\frac{15}{a^4}\left(\frac{a^7}{3}+\frac{a^7}{7}-\frac{2a^7}{5}\right)$

9:00 ٩:٠٠

10:00 $= \frac{15}{8a^4}\left(\frac{35+15-42}{105}\right)a^7$ ١٠:٠٠

11:00 ١١:٠٠

$= \frac{\cancel{15}\,3}{\cancel{8a^4}}\times\frac{a^7}{\cancel{105}\,\cancel{21}\,7}\times\cancel{4}\,8$

12:00 ١٢:٠٠

$= \cancel{\frac{a^7}{a^4\,7}} \quad = \frac{a^7}{a^5\,7}$

13:00 ١٣:٠٠

14:00 $= \underline{\underline{a^2/7}}$ ١٤:٠٠

(e) $\langle p^2 \rangle = \int_{-\infty}^{\infty} \psi^* \left(-\hbar^2 \frac{\partial^2}{\partial x^2}\right) \psi \, dx$ [using operators]

$$= -\int_{-a}^{a} A(a^2-x^2)\, \hbar^2 \frac{\partial^2}{\partial x^2}\left(A(a^2-x^2)\right) dx$$

$$= -A^2\hbar^2 \int_{-a}^{a} (a^2-x^2)(-2)\, dx$$

$$= 2A^2\hbar^2 \int_{-a}^{a} (a^2-x^2)\, dx$$

$$= 2A^2\hbar^2 \left[a^2x - \frac{x^3}{3}\right]_{-a}^{a}$$

$$= 4A^2\hbar^2 \left[a^2x - \frac{x^3}{3}\right]_{0}^{a}$$

$$= 4A^2\hbar^2 \left(a^3 - \frac{a^3}{3}\right)$$

$$= \frac{2a^3}{3} \times 4A^2\hbar^2$$

$$= \cancel{\frac{2a^3}{3}} \times \cancel{4}\hbar^2 \times \frac{1}{\underset{2}{\cancel{16\cdot 4}}} \times \frac{\overset{5}{\cancel{15}}}{a^5} = \frac{5\hbar^2}{2a^2}$$

7(f) The uncertainity in x is

$$\sigma_x = \sqrt{\langle x^2\rangle - (\langle x\rangle)^2}$$

$$= \sqrt{\frac{a^2}{7} - 0}$$

$$= \frac{a}{\sqrt{7}} \;//$$

(g) The uncertainity in p is

$$\sigma_y = \sqrt{\langle p^2\rangle - (\langle p\rangle)^2}$$

$$= \sqrt{\frac{5\hbar^2}{2a^2} - 0}$$

$$= \sqrt{\frac{5}{2}}\,\frac{\hbar}{a}$$

(h) The uncertainity principle states: $\sigma_x \sigma_y \geq \frac{\hbar}{2}$

$$\sigma_x \sigma_p = \sqrt{\frac{5}{2}}\,\frac{\hbar}{\cancel{a}} \times \frac{\cancel{a}}{\sqrt{7}}$$

$$= \sqrt{\frac{5}{14}}\,\hbar$$

$$= 2\sqrt{\frac{5}{14}}\,\frac{\hbar}{2} \geq \frac{\hbar}{2}$$

Thus, the uncertainity principle is satisfied.

(5) The time independent Schrodinger equation is

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x) = E\psi(x)$$

$$\frac{d^2}{\partial x^2}\psi(x) = -\frac{2m}{\hbar^2}E\psi(x)$$

For infinite square well, $\psi(0) = \psi(a) = 0$

$\therefore E=0 \Rightarrow \frac{\partial^2\psi(x)}{\partial x^2} = 0$ [$V=0$ inside well]

Thus, $\psi(x)$ will be a linear function of the form $Ax + B$.

now, $\psi(0) = B = 0$

$\psi(a) = Aa = 0$

$\Rightarrow A = 0$

Thus, $\psi = 0$

$\Rightarrow$ There is no acceptable solution for the time-independent Schrodinger equation for $E=0$ for the infinite well.

(6) $$\psi(x,0) = \begin{cases} A, & 0 \le x \le a/2 \\ 0, & a/2 \le x \le a \end{cases}$$

To normalize, $\int_{-\infty}^{\infty} |\psi(x,0)|^2 = 1$

$$\int_0^{a/2} A^2 \, dx + \int_{a/2}^{a} 0 \, dx = 1$$

$$A^2 \left[x\right]_0^{a/2} = 1$$

$$A^2 \left[\frac{a}{2}\right] = 1$$

$$A = \sqrt{\frac{2}{a}}$$

We know any wave function $\psi(x,0) = \sum_{n=1}^{\infty} c_n \psi_n(x)$

Using Fourier Series formula,

$$c_n = \sqrt{\frac{2}{a}} \int_0^{a/2} \sin\left(\frac{n\pi x}{a}\right) \psi(x,0)\, dx$$

$$= \sqrt{\frac{2}{a}} \int_0^{a/2} A \sin\left(\frac{n\pi x}{a}\right) dx$$

$$= A\sqrt{\frac{2}{a}} \int_0^{a/2} \sin\left(\frac{n\pi x}{a}\right) dx$$

$$= \sqrt{\frac{2}{a}} \cdot \sqrt{\frac{2}{a}} \left[-\cos\left(\frac{n\pi x}{a}\right)\right]_0^{a/2} \times \left(\frac{n\pi}{a}\right)^{-1}$$

For $n=1$, $c_1 = \frac{2}{a} \times \left(\frac{\pi}{a}\right)^{-1} \times \left[-\cos(\pi/2) + \cos(0)\right]$

$$= \frac{2}{\pi}(1)$$

$$= \frac{2}{\pi}$$

The probability of measuring E_n is $|c_n|^2$.

7:00 $E_n = \dfrac{n^2 \pi^2 \hbar^2}{2ma^2}$

8:00

Hence, $E_1 = |C_1|^2 = \dfrac{4}{\pi^2}$ //

9:00

(7) The general solution for a particle in an infinite square well is

$$\psi(x,t) = \sum_{n=1}^{\infty} c_n \sqrt{\frac{2}{a}} \exp\left(-i \frac{n^2\pi^2\hbar}{2ma^2} t\right) \sin\left(\frac{n\pi x}{a}\right)$$

TP: $\psi(x,0) = \psi(x,T)$

$$\sum_{n=1}^{\infty} c_n \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) = \sum_{n=1}^{\infty} c_n \sqrt{\frac{2}{a}} \exp\left(\frac{-i n^2\pi^2\hbar T}{2ma^2}\right)$$

$$\sin\left(\frac{n\pi x}{a}\right)$$

comparing both sides, $\exp\left(-\frac{in^2\pi^2\hbar}{2ma^2} T\right) = 1$

$$\cos\left(\frac{n^2\pi^2\hbar}{2ma^2} T\right) - i \sin\left(\frac{n^2\pi^2\hbar T}{2ma^2}\right) = 1$$

comparing real and imaginary parts,

$$\cos\left(\frac{n^2\pi^2\hbar}{2ma^2} T\right) = 1$$

$$\sin\left(\frac{n^2\pi^2\hbar T}{2ma^2}\right) = 0$$

7:00 Both sides of the equation are satisfied if

$$\frac{n^2\pi^2\hbar}{2ma^2}T = n^2 2\pi$$

8:00

9:00

$$\Rightarrow T = 2\pi \times \frac{2ma^2}{\pi^2\hbar}$$

10:00

$$= \frac{4ma^2}{\pi\hbar}$$

11:00

Hence, proved.

12:00

(8) Let us assume that there is a unitary operator that copies two quantum states $|\psi\rangle$ and $|\phi\rangle$

$$|\psi\rangle \otimes |0\rangle \longrightarrow |\psi\rangle \otimes |\psi\rangle$$
$$|\phi\rangle \otimes |0\rangle \longrightarrow |\phi\rangle \otimes |\phi\rangle$$

Now let us consider the state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ which when cloned becomes :-

$$|\psi\rangle|0\rangle \longrightarrow |\psi\rangle|\psi\rangle = (\alpha|0\rangle + \beta|1\rangle)(\alpha|0\rangle + \beta|1\rangle)$$

$$= \alpha^2|00\rangle + \beta^2|11\rangle + \beta\alpha|10\rangle + \alpha\beta|01\rangle$$

However, due to linearity of the cloning operator which preserves inner product, we get.

$$(\alpha|0\rangle + \beta|1\rangle)|0\rangle \longrightarrow \alpha|0\rangle|0\rangle + \beta|1\rangle|0\rangle$$

Now, $\alpha|0\rangle|0\rangle \rightarrow \alpha^2|00\rangle$ $\beta|1\rangle|0\rangle \rightarrow \beta^2|11\rangle$

This cloning misses the $|10\rangle$ and $|01\rangle$ terms. Thus, we have a contradiction and the operator cannot copy the linear superpositions of any two quantum states though it can copy them individually. This is because the operator is a unitary and linear operator.

(9) In the Stern Gerlach experiment, the probability of spin-up is $\cos^2(\theta/2)$ and that of spin-down is $\sin^2(\theta/2)$.

Where, $\theta \rightarrow$ angle between the two magnetic fields

Now, $P(up) = 2\,P(down)$

$$\cos^2(\theta/2) = 2\sin^2(\theta/2)$$

$$\Rightarrow \tan^2(\theta/2) = 1/2$$

$$\tan\theta/2 = \sqrt{1/2} \Rightarrow \frac{\theta}{2} = \tan^{-1}\left(\sqrt{\frac{1}{2}}\right)$$

$$\theta = 2\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

(10) According to the quantum entanglement theory, the particle follows a probabilistic curve until the moment of measurement.

~~After this~~ Alice measures the wave which forces the wave function to collapse and simultaneously produces Bob's spin.

To ensure that there is no causal influence, we need to make sure that light also doesn't reach

$c = 3\times10^8$ m/s

$t = 1$ ms $= 10^{-3}$ s

distance, $= c\times t$

$= 3\times10^5$ m

$= \underline{\underline{300 \text{ km}}}$

Thus, they need to be atleast 300km for there to be no causal interference.

(11)

(a) m (neutron) $= 1.674\times10^{-27}$ kg

$v = 10^6$ m/s

$\Delta p = m\Delta v = 1.67\times10^{-27}\times10^6$

$= 1.67\times10^{-21}$ kgm/s

Uncertainity principle, $\Delta x\,\Delta p \geq \frac{\hbar}{2}$

$$\Delta x \geq \frac{1.054\times10^{-34}}{2\times1.67\times10^{-21}}$$

$\Delta x \geq 0.315\times10^{-13}$ m

$\Delta x \geq \underline{\underline{3.15\times10^{-14}}}$ m

(B) $m = 50\,kg$

$v = 2\,m/s$

Uncertainity principle, $\Delta x \, \Delta p \geq \frac{\hbar}{2}$

$$\Delta x \, m \Delta v \geq \frac{\hbar}{2}$$

$$\Delta x \times 50 \times 2 \geq \frac{1.054 \times 10^{-34}}{2}$$

$$\Delta x \geq \frac{1.054 \times 10^{-34}}{2 \times 50 \times 2}$$

$$\Delta x \geq 0.527 \times 10^{-34}\,m$$

$$\Delta x \geq \underline{5.27 \times 10^{-37}\,m}$$

(15) The reason for why an electron cannot reside inside the nucleus can be explained using the uncertainity principle.

The size of the nucleus is of the order of $10^{-15}\,m$. Thus Δx should be less than $10^{-15}\,m$ for an e^-.

we also know, $m_e = 9.31 \times 10^{-31}\,kg$

$\therefore \Delta x \, \Delta p \geq \frac{\hbar}{2}$ (uncertainity principle)

$$\Delta x \cdot m\Delta v \ge \frac{\hbar}{2}$$

$$\Delta v \ge \frac{1.054 \times 10^{-34}}{2 \times 10^{-15} \times 9.1 \times 10^{-31}}$$

$$\Delta v \ge 0.057 \times 10^{12} \text{ m/s}$$

$$\Delta v \ge 5.7 \times 10^{10} \text{ m/s}$$

This implies that the e^- will need to have a speed much higher than the speed of light [$c = 3 \times 10^8$ m/s]; which is impossible. Hence, the e^- cannot reside inside the nucleus.