4 Spacetime and its curveture

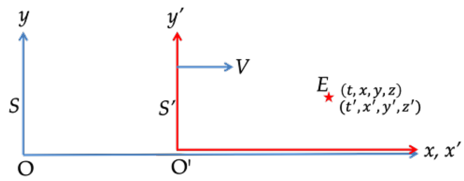
Farmet's Principle: The path taken by a ray of light between two points is the path that can be traveled in the least time.

4.1 Spectime

The Galelian Relativity states *the laws of motion are same in all inertial frames*. In 1861 James Clerck Maxwell proposed a theory that showed that the speed of light (electromagnetic waves) is constant. It was unclear in which frame the speed is measured. While some people doubted the correctness of Maxwell's theory, others thought that there must be a universal frame in which the light's speed is constant. This universal frame was called aether – a medium that fills all space and light propagates through it. Experiments were soon designed to detect aether. One such experiment by Michelson and Morley. To everybody's surprise Michelson and Morley could not detect any aether. Rather the experiments showed that the speed of light is constant and is independent of the relative speed between source and the observer. While confusions grew, taking the result of the Michaelson and Morley experiment at face value, Einstein proposed a new theory, the Special Theory of Relativity, to make sense of it all. The STR is based on the following two postulates by Einstein

- *laws of physics are invariant in all inertial frames*
- *the speed of light c is same for all observers, irrespective of the relative velocity with the source*

Einstein's task was to find a set of transformations between inertial frames that keeps the above two statements true. Such coordinate transformations were already worked out by Hendrik Lorentz, but it was Einstein who realized their significance.



With reference to the figure, consider two frames $O \equiv (x, y, z)$ and $O' \equiv (x', y', z')$, such that the axes of the two frames are parallel. At $t = 0$ the origin of the two frames were coincide. At some other time t the O' has a speed v with respect to the O frame along the x-axis. In the S frame, an event is observed to takes place at (t, x, y, z), and the same is observed to take place at (t', x', y', z') in the S' frame. The Lorentz Transformation (LT)

$$\begin{aligned} t' &= \gamma\big(t - vx\big)\,, \\ x' &= \gamma\big(x - vt\big) \\ y' &= y\,, \quad z' = z \end{aligned}$$

where

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}\,.$$

In STR time and space are on the same footing – they are coordinates of the four-dimensional *Minkowskian spacetime*. A set of four coordinates (t, x, y, z) is called an event – this is similar to specify a location in the three-dimensional space. In three-dimensional space, the distance^2 between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is $(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2$. This distance^2 is invariant – *i.e.,* it is same in all coordinate system. The concept of invariant distance^2 can be extended to four-dimensional spacetime as well. It is called *spacetime interval* or simply interval. If (t_1, x_1, y_1, z_1) and (t_2, x_2, y_2, z_2) are two events then the interval

$$(t_2 - t_1)^2 c^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2 ,$$

is invariant under LT. Therefore, infinitesimally small interval is

$$ds^2 = c^2 dt^2 - (dx^2 + dy^2 + dz^2) .$$

It is also known as the spacetime *metric*. Finally, let say the distance $d\ell = \sqrt{dx^2 + dy^2 + dz^2}$ traversed by a light ray in time dt. Then

$$\frac{d\ell}{dt} = c , \qquad \Longrightarrow \sqrt{dx^2 + dy^2 + dz^2} = c dt .$$

Hence the metric

$$ds^2 = 0 .$$

So the path of light ray in spacetime is represented by $ds^2 = 0$, or

$$c dt = \pm \sqrt{dx^2 + dy^2 + dz^2} . \tag{120}$$

This equation gives the trajectory of light in a 4d-spacetime and is called the *world line* of light. Since it is not possible to draw in more than two dimension, we define $dX = \sqrt{dx^2 + dy^2 + dz^2}$ where X accounts for the three space coordinates. So $cdt = \pm dX$ which is noting but a straight line in $X - cdt$ plane. This is shown in figure 10 where the horizontal axis is space and the vertical axis is cdt. A world line of an

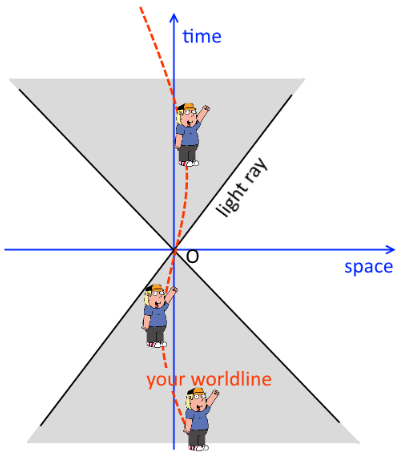


Figure 10: Light cone and your world line.

ordinary object that travels slower than light is also shown.

There are many interesting predictions of STR all of which are verified by experiments. One of the most well known is the mass-energy equivalence relation – if a particle of mass m has momentum p then the energy is

$$E = \sqrt{m^2 c^4 + p^2 c^2} .$$

For a particle at rest $p = 0$ and $E = mc^2$.

4.2 Describing gravity: Newton vs Einstein

Read the thought experiment from Ryden's Cosmology book chapter 3.

The experiment shows that gravity can be simulated by accelerated frame. In the absence of gravity light travels in a straight line according to Fermet's principle. In the presence of gravity light still travels the shortest path between two points, but this path is no longer a straight line. It means that in the presence of gravity the spacetime is curved (also called non-Euclidean). So according to Einstein *the presence of mass-energy causes the spacetime to curve. Light as well as "freely falling" particles follow a straight line called "geodesics". Gravitational acceleration is simply the consequence of the curvature spacetime.*. So Einstein's theory of gravity also known as the *general theory of relativity* is radically different from that of Newton's. Interestingly, all of Newton's theory can be obtained from general relativity under certain approximations.

Mathematically, Einstein's idea of relating mass-energy with geometry is summarized by the equation

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} \,,$$

where the l.h.s is geometry and the r.h.s is mass-energy of the universe ($R_{\mu\nu}$ is Ricci tensor, $g_{\mu\nu}$ is spacetime metric, and $T_{\mu\nu}$ is stress-tensor). Einstein later added a terms $\Lambda g_{\mu\nu}$

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \,,$$

It is believed that Λ is the dark energy that is tearing the universe apart.

4.3 Metric of the universe: FRW metric

The metric of curved spacetime or the universe is

$$\boxed{ds^2 = -(cdt)^2 + a(t)^2 \left[\frac{dr^2}{1 - Kr^2} + r^2 (d\theta^2 + \sin^2\theta d\phi^2) \right]} \,. \tag{121}$$

The factor K is called the curvature of the universe, and $K = -1, 0, +1$ correspond to closed, flat, and open universe. What is meant by open, closed, flat etc, will be explained later.

The reference frame in which the metric has the form (121) is called a *comoving frame*. An observer who is at rest in this frame is called the *comoving observer*. A comoving observer is the only observer who sees the universe to be isotropic and homegeneous. The scale factor $a(t)$ in (121) accounts for the expansion of the universe – it is universal and depends on time only. You can therefore imagine the comoving frame as a grid that is expanding as shown in figure 11. Stars/galaxies/observers located at the intersections of the grid are called comoving. Distance between comoving observers/objects is called the *comoving distance – it is simply the coordinate distance that remains constant.*

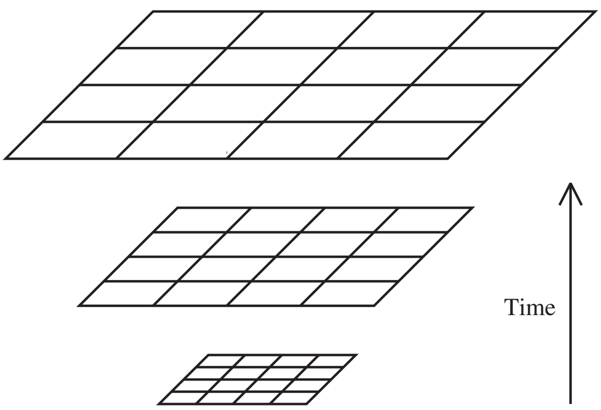


Figure 11: Comoving coordinate system.

Whereas the comoving distance remain constant, it is the *proper distance* that scales with $a(t)$. It is define as follows: Suppose you are observing a galaxy. The comoving coordinate of the galaxy is (r, θ, ϕ). You can always align the coordinate system in a way that $\theta = 0, \phi = 0$ and its location is simply given by r. Then from the FRW metric proper distance is

$$\dot{d} = \frac{da(t)}{dt} \int_0^r \frac{dr}{\sqrt{1 - Kr^2}} = \frac{\dot{a}}{a} d \ , \tag{122}$$

The speed of the galaxy is

$$v = \frac{d}{dt} a(t) \int_0^r \frac{dr}{\sqrt{1 - Kr^2}} = \frac{\dot{a}}{a} d \ , \tag{123}$$

This is nothing but Hubble's law.

As the universe is expanding will the size of atoms, everyday objects, radius of planets in the solar system get bigger? The answer is No. Expansion is only felt at large scale $\sim$ 100Mpc or so. Locally the universe is dominated by different physical laws.

4.4 Cosmological redshift

Whether or not the scale factor $a(t)$ is increasing of decreasing with time can be determined from redshift of spectral lines of distant galaxies. Consider a galaxy with comoving corrdinate $(r, 0, 0)$, and a coving observer at the origin. The light rays emitted by the galaxy is observed by the observer. As a light rays travel along the null geodesic, for it

$$dt^2 = \pm a(t) \frac{dr}{\sqrt{1 - Kr^2}} \ . \tag{124}$$

We must keep the negative sign because, a light ray starts its journey at $t = t_e$ from a distance r and reaches $r = 0$ at $t = t_o$ when it is observed. In other words, the negative sign reflects the fact that the distance of an emitted ray to the observer is decreasing. We can then write

$$\int_{t_o}^{t_e} \frac{dt}{a(t)} = - \int_r^0 \frac{dr}{\sqrt{1 - Kr^2}} = \int_0^r \frac{dr}{\sqrt{1 - Kr^2}} \tag{125}$$

The right-hand side of the equation is a constant, obviously.

Now consider two rays consecutively emitted by the galaxy at times t_e and $t_e + \delta t_e$, and detected by the observer at times t_0 and $t_0 + \delta t_0$. We can write

$$\int_{t_e}^{t_0} \frac{dt}{a(t)} = \int_{t_e + \delta t_e}^{t_0 + \delta t_0} \frac{dt}{a(t)} \ .$$

From both sides we subtract $\int_{t_e + \delta t_e}^{t_0} (dt/a(t))$ which gives

$$\int_{t_e}^{t_e + \delta t_e} \frac{dt}{a(t)} = \int_{t_0}^{t_0 + \delta t_0} \frac{dt}{a(t)} \tag{126}$$

As the intervals δt_e and δt_o are small, the scale parameter remains constant in these intervals: $a(t_e) = a(t_e + \delta t_e)$ and $a(t_0) = a(t_0 + \delta t_0)$. So the above integration gives

$$\frac{\delta t_e}{a(t_e)} = \frac{\delta t_0}{a(t_0)} \ . \tag{127}$$

The emitted and detected rays are subsequent. If the emitted and detected rays have frequency and wavelengths as ν_e, λ_e and ν_0, λ_0, respectively, then

$$\nu_e = \frac{1}{\delta t_e} \ , \qquad \nu_0 = \frac{1}{\delta t_0} \ .$$

Substituting in (147) we get

$$\boxed{\frac{\nu_0}{\nu_e} = \frac{a(t_e)}{a(t_0)}\,, \qquad \text{or,} \qquad \frac{\lambda_e}{\lambda_0} = \frac{a(t_e)}{a(t_0)}}\,. \tag{128}$$

According to Hubble's observations, $\nu_e > \nu_0$ or $\lambda_e < \lambda_0$, *i.e.*, the spectral lines are redshifted. From the definition of redshift we get

$$z = \frac{\lambda_0 - \lambda_e}{\lambda_e} = \frac{\lambda_0}{\lambda_e} - 1 = \frac{a(t_0)}{a(t_e)}\,,$$

We have already defined redshift z as

$$z = \frac{\lambda_{\text{obs}} - \lambda_{\text{em}}}{\lambda_{\text{em}}}\,.$$

Suppose a galaxy emits λ_{em} at time t_e and λ_{obs} is observed at present time t_0. Then it can be shown that

$$\boxed{1 + z = \frac{a(t_0)}{a(t_e)}}\,. \tag{129}$$

If the scale factor is increasing then $z > 0$ and it is redshift. If the scale factor is decreasing then $z < 0$ and it is called blueshift. Red and blue shifts are interpreted as Doppler effect. If the light is redshifted then the galaxy is receding from us. In case of blueshift it is moving towards us.

4.5 Other consequences of expansion

Suppose a particle of mass m has a momentum p. Due to the expansion of the universe the momentum decreases as

$$p \propto \frac{1}{a(t)}\,,$$

For this reason, particles that were relativistic ($p >> mc^2$) at the beginning of the universe are non-relativistic now.

The energy of a photon also scale inversely with the scale factor

$$E \propto \frac{1}{a(t)} \tag{130}$$

So the wavelength of photon is stretched by the expansion of the universe.

If n number of particles/galaxis occupy a certain comoving volume then the density n scales as

$$n \propto \frac{1}{a(t)^3}$$