Fast and Small

Quantum Mechanics

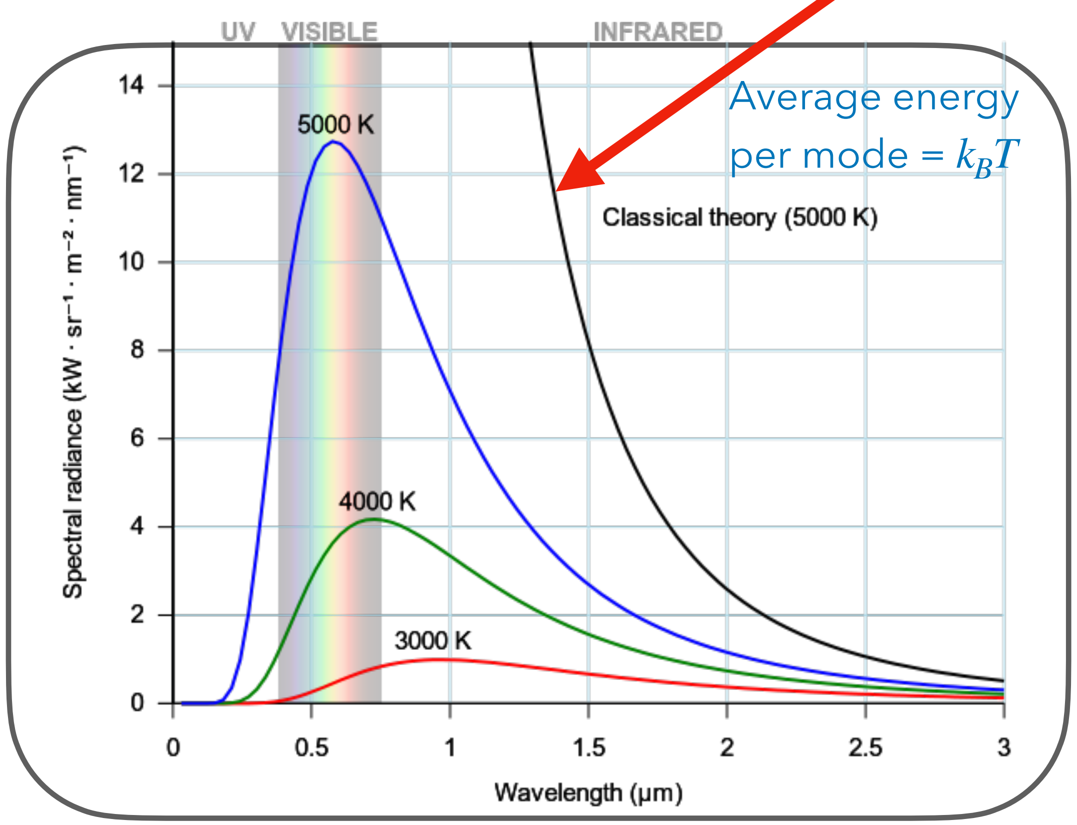
Imaging Bell-type nonlocal behavior, Moreau et al., Science Advances, Vol. 5, no. 7, eaaw2563 (2019)
https://advances.sciencemag.org/content/5/7/eaaw2563

Why Quantum?

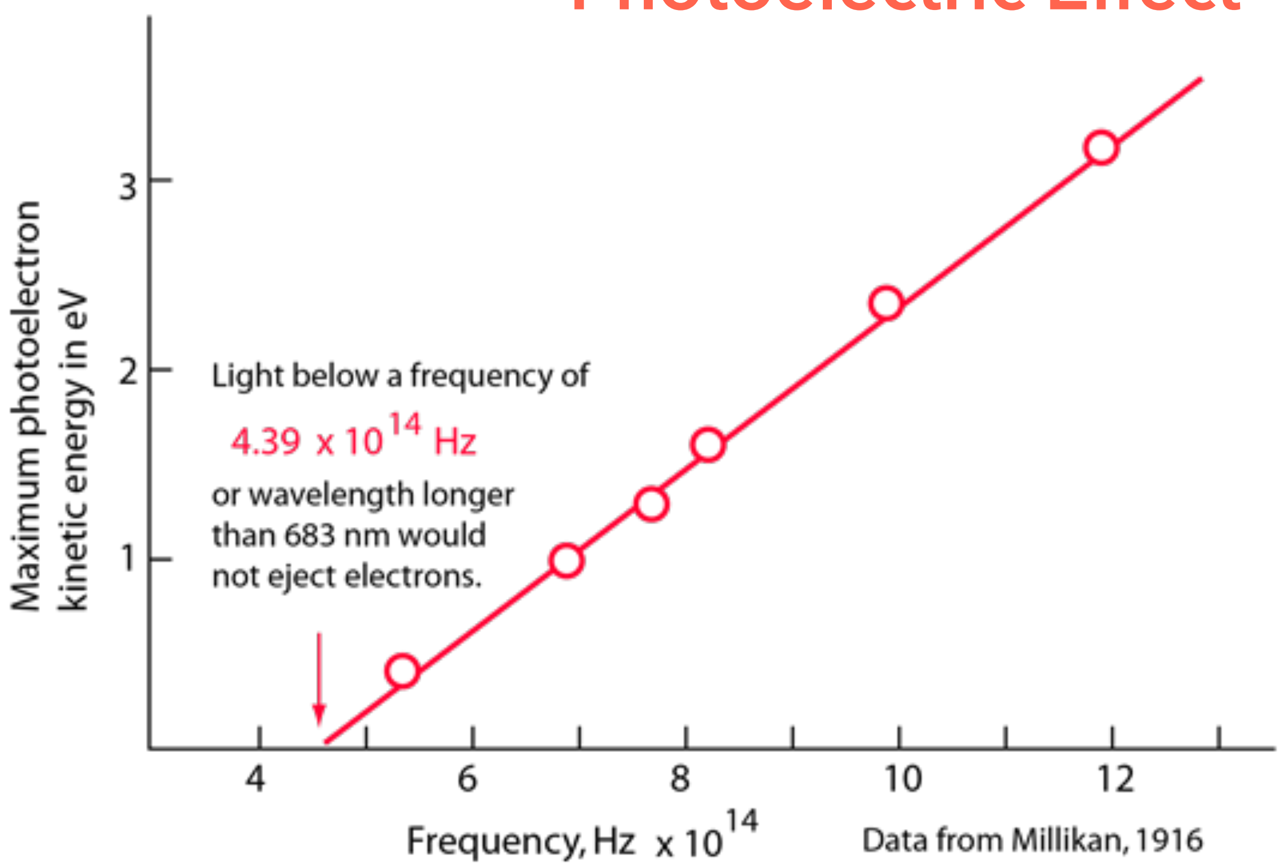
Blackbody Radiation: Ultraviolet Catastrophe

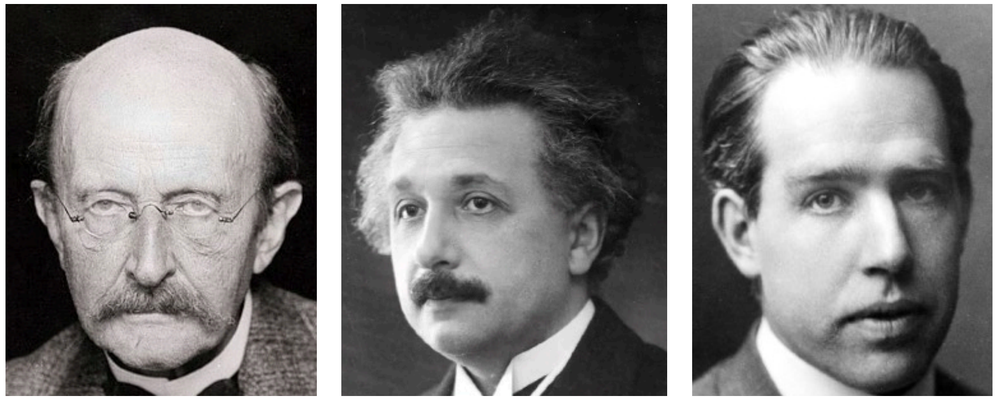
The number of independent standing waves inside a cavity

$$g(\nu)d\nu = \frac{8\pi\nu^2}{c^3}d\nu$$ → Rayleigh–Jeans Formula



Photoelectric Effect



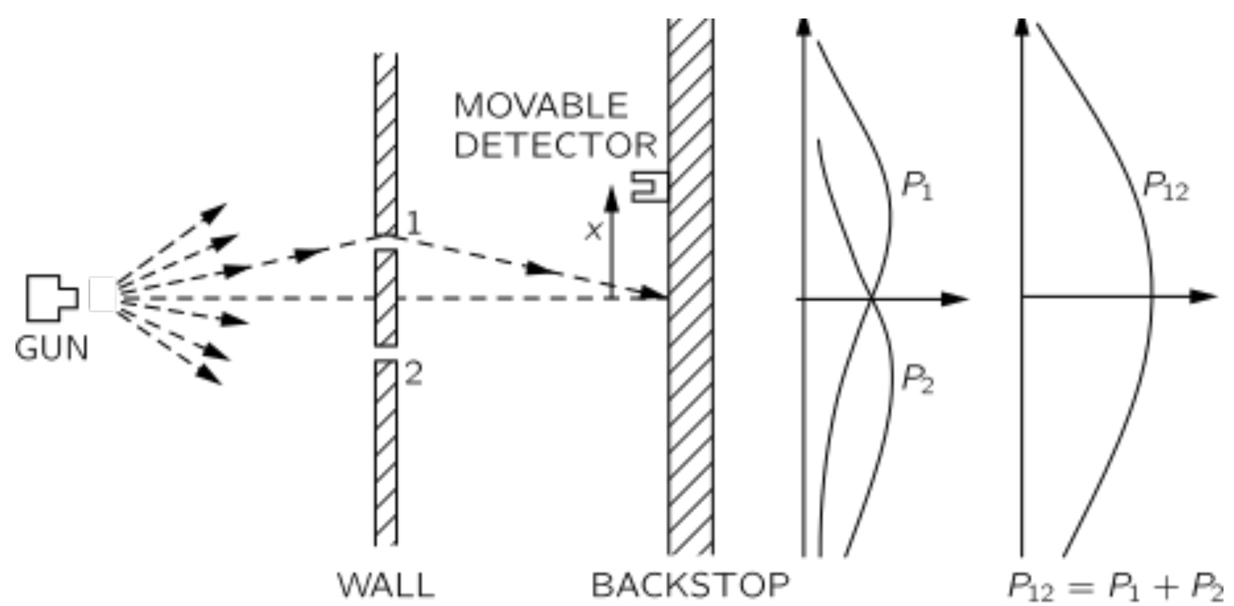


- Stability of atomic orbitals
- Davisson–Germer electron diffraction experiment
- …

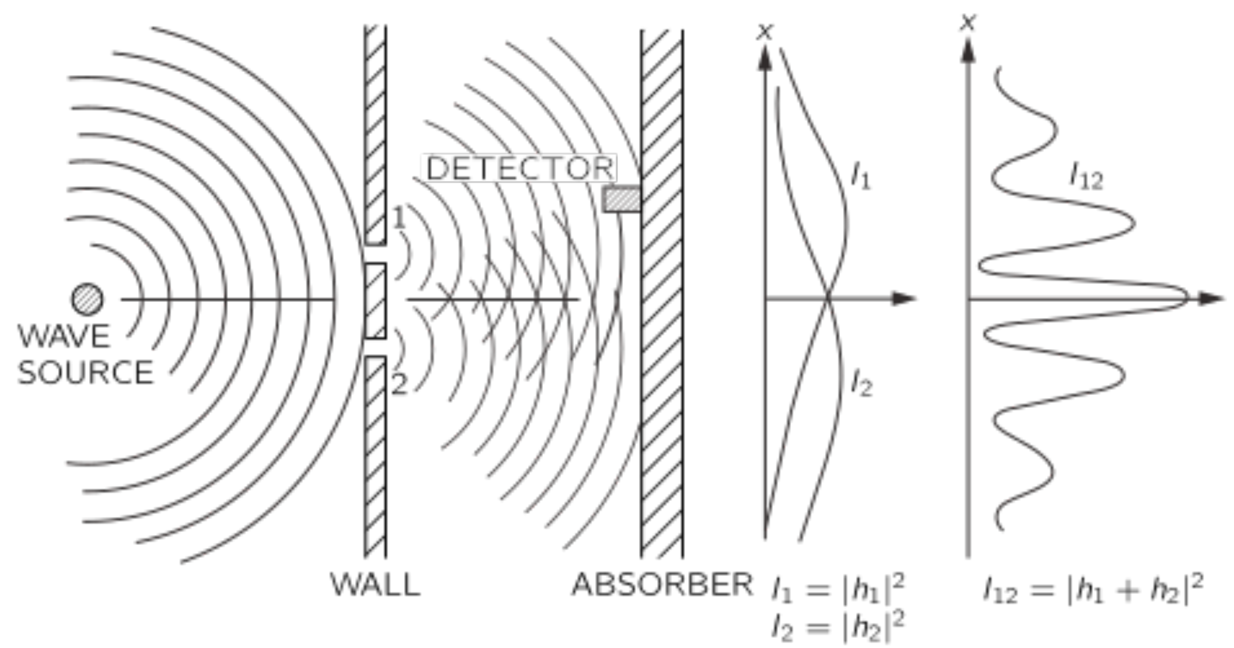
Bullets vs Waves

Feynman Lectures

An experiment with bullets

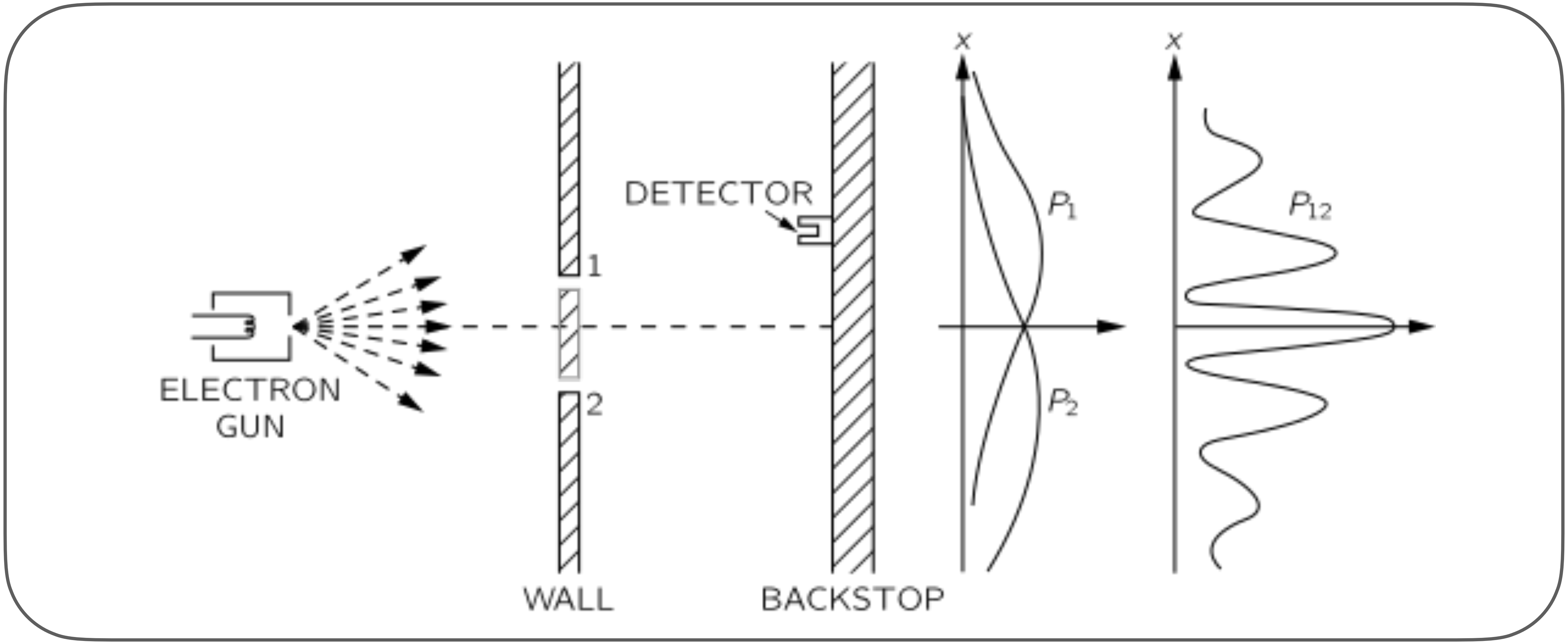


The same experiment with waves



Enter Electrons

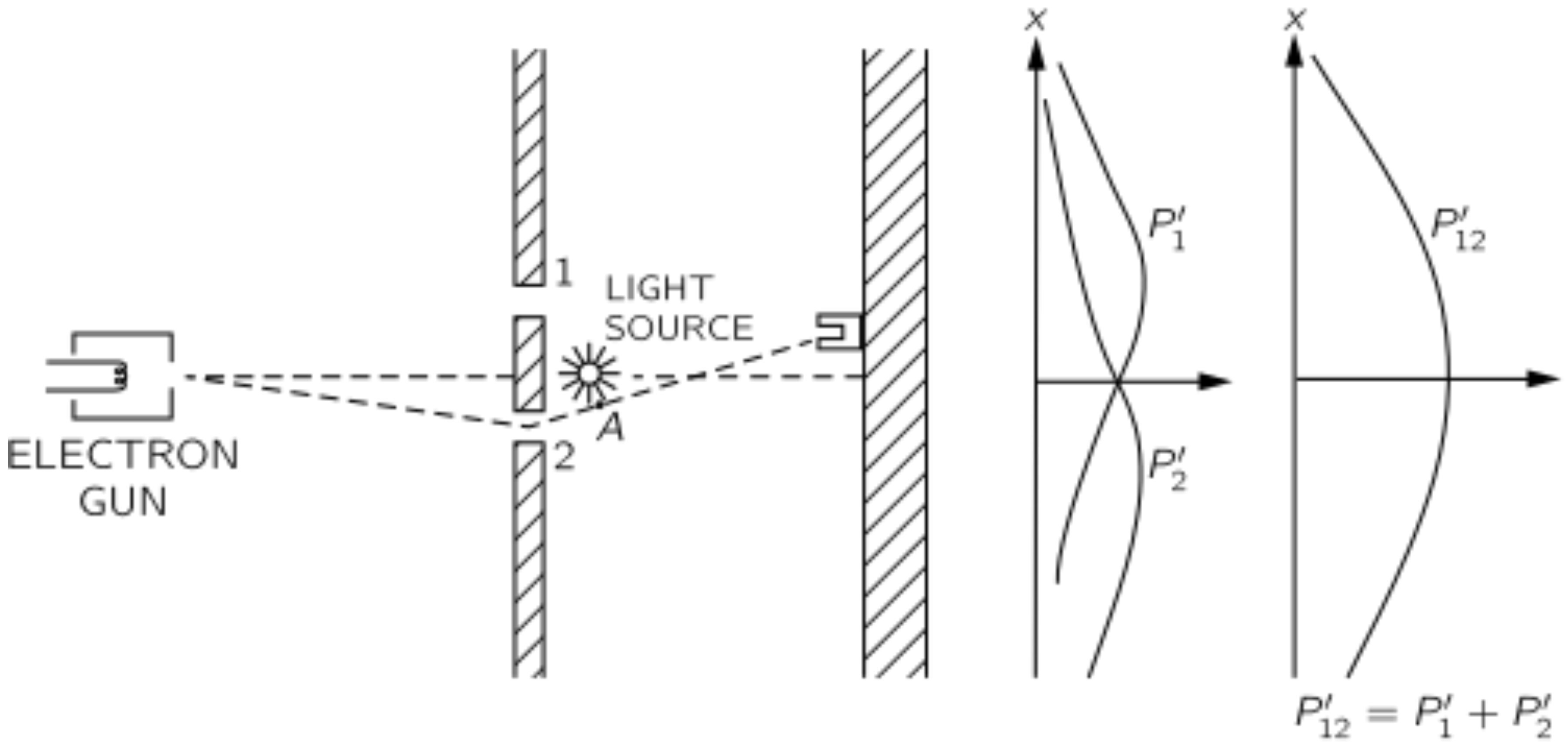
The same experiment with electrons



$P_{12} \neq P_1 + P_2$. The two probabilities, P_1 and P_2 are related to two complex numbers.

The definition of probability is not changed, but the method to calculate it is completely different

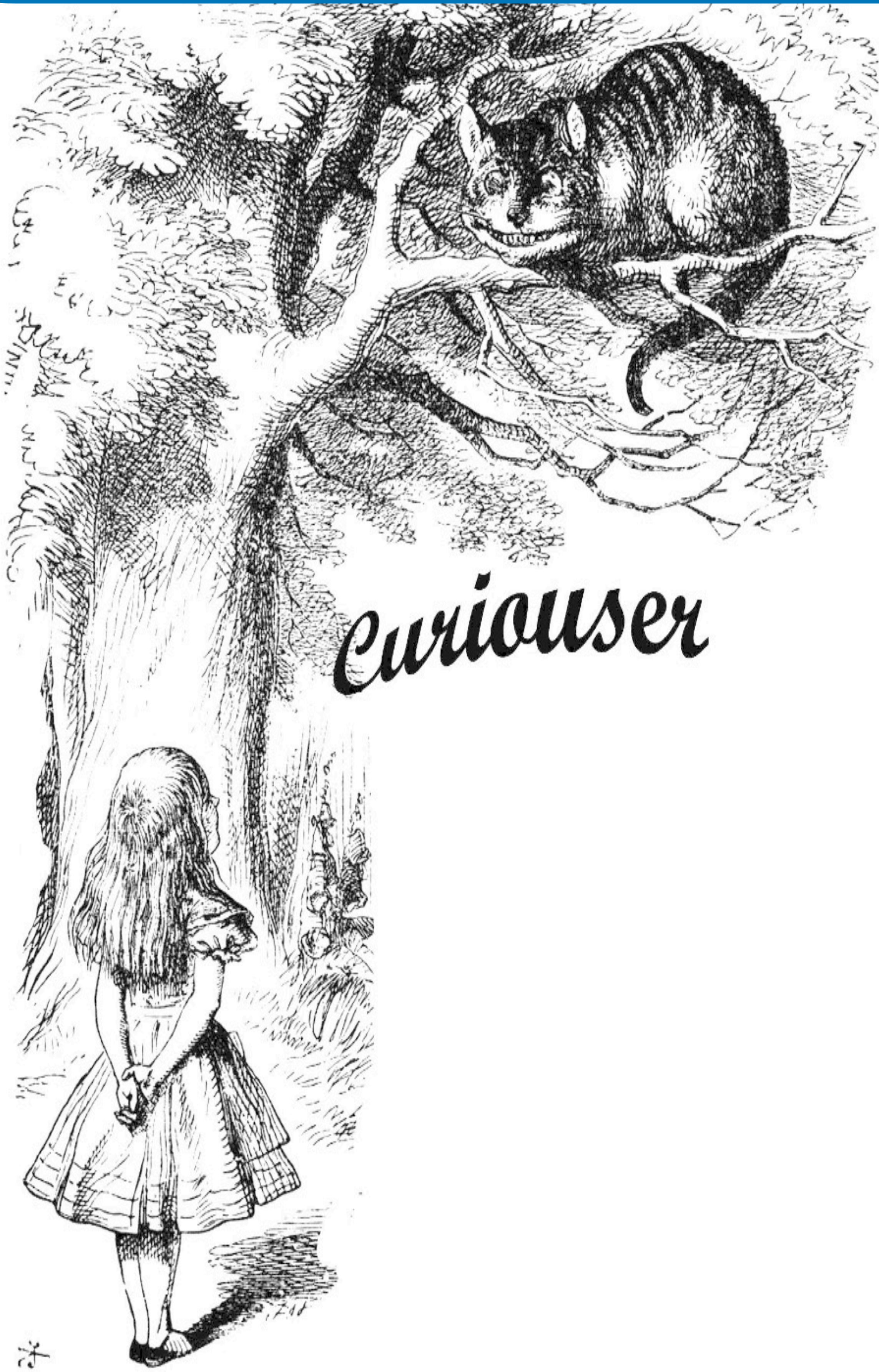
Which hole?



When we watch them $P_{12} = P_1 + P_2$

Curiouser

Feynman Lectures / Feynman, Hibbs



We are changing the pattern on the screen just by watching the electrons. How is this possible?

To watch them we used light. When an electron collides with a photon its chance of arrival at the detector is possibly altered.

So, can we use weaker light and thus expect a weaker effect? A negligible disturbance certainly cannot be presumed to produce the finite change in the distribution.

Curiouser

Light comes in photons of energy $\hbar\nu$ or of momentum $\hbar/\lambda$. Weakening the light just means using fewer photons so that we may miss seeing an electron.

The electrons we miss are distributed according to the interference law, while those we do see (and which therefore have scattered a photon) arrive at the detector with the probability $P_{12} = P_1 + P_2$.

It might still be suggested that weaker effects could be produced by using light of longer wavelength. But there is a limit to this. If light of too long a wavelength is used, we will not be able to tell whether it was scattered from hole 1 or hole 2.

& Curiouser

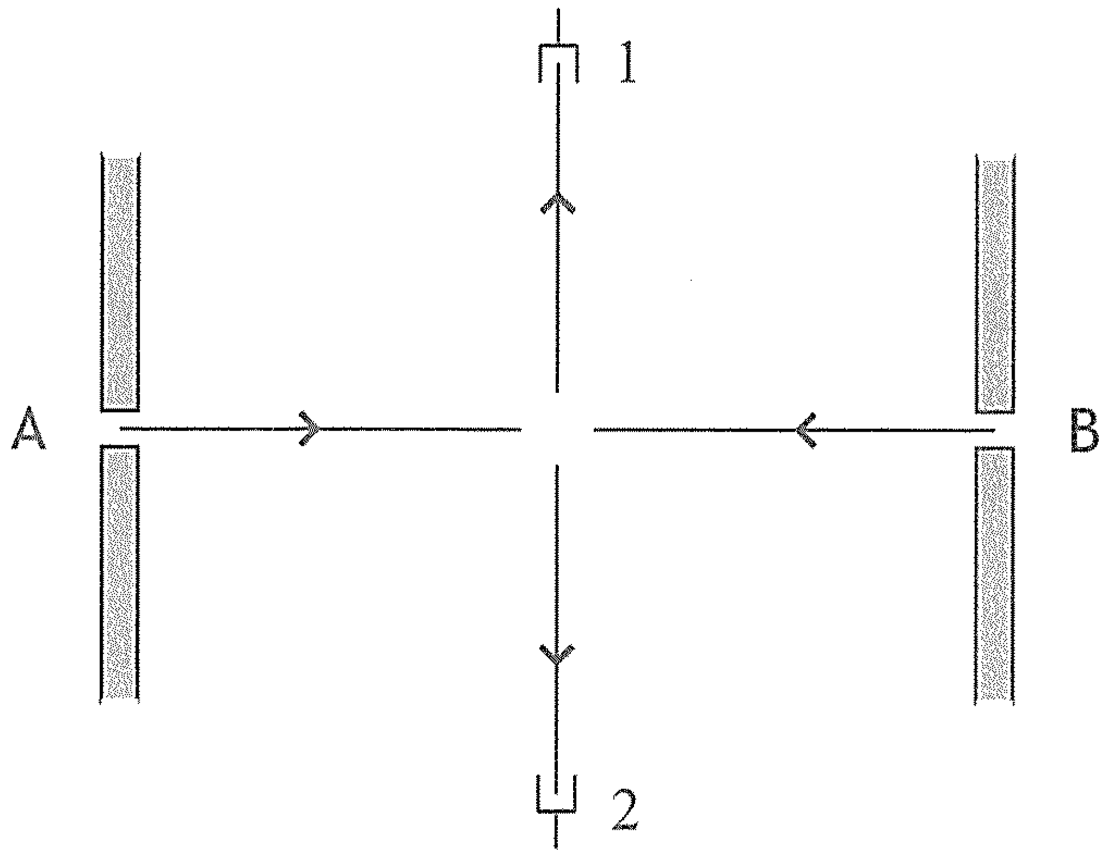
Feynman Lectures / Feynman, Hibbs

The uncertainty principle

Any determination of the alternative taken by a process capable of taking more than one alternatives destroys the interference between alternatives. (Feynman's qualitative formulation of the uncertainty principle)

& Curiouser

Feynman Lectures / Feynman, Hibbs



Let the amplitude of scattering for the particle to start from A and end up at 1 (and the one to start from B and end up at 2) be $\alpha\,(1{,}A)$ so that the probability is $p = |\,\alpha\,(1{,}A)\,|^2$. This is also equal to $|\,\alpha\,(2{,}A)\,|^2$ as the scattering is by 90^o.

If the particles are different (like two different nuclei or two electrons with different spins – assuming the scattering is soft, i.e., can not flip the spins)

$$p(1, A \text{ or } B) = |\,\alpha(1{,}A)\,|^2 + |\,\alpha(1{,}B)\,|^2 = 2p$$

If we wish, we can distinguish the two cases by measuring.

If the particles are alpha particles (i.e., no way to tell them apart)

$$p(1, A \text{ or } B) = |\,\alpha(1{,}A) + \alpha(1{,}B)\,|^2 = 4p$$

If the particles are electrons with the same spin orientation, i.e., both up or both down

$$p(1, A \text{ or } B) = |\,\alpha(1{,}A) - \alpha(1{,}B)\,|^2 = 0$$

90^o scattering not possible.

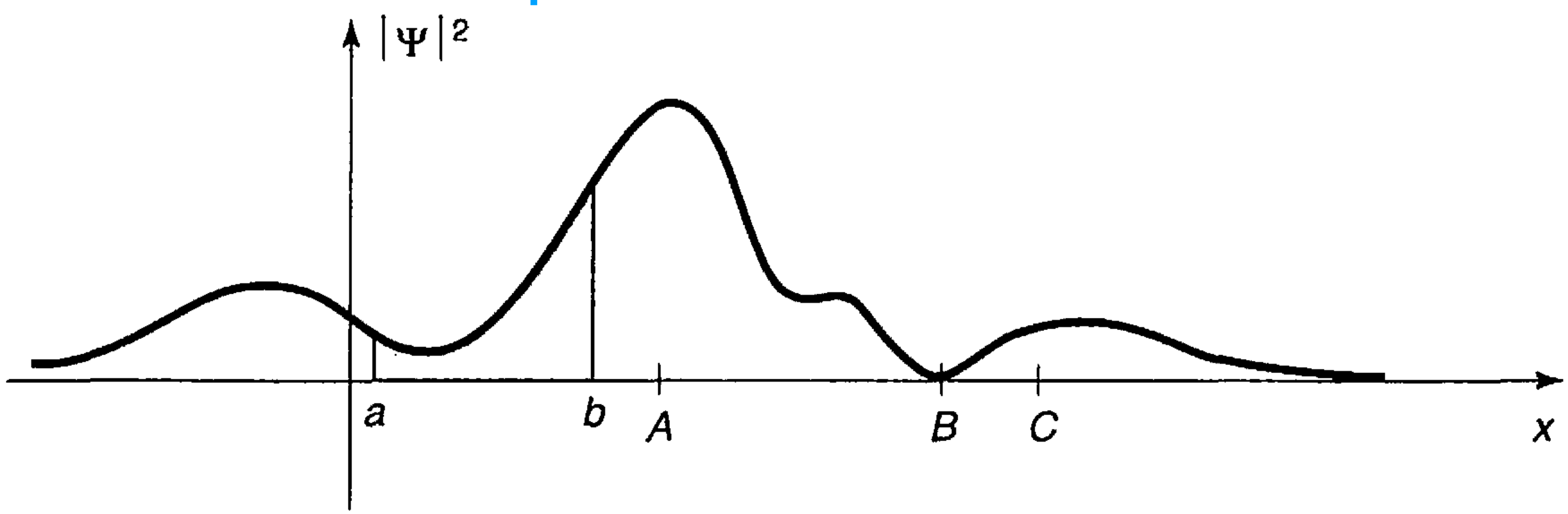
The Schrödinger Equation

In Quantum Mechanics, unlike classical physics, we solve the Schrödinger equation to get the "wave function" of a particle:

$$\boxed{i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi}$$

$$\hbar = \frac{h}{2\pi} = 1.054572 \times 10^{-34}\ \text{J s}$$

Born's statistical interpretation

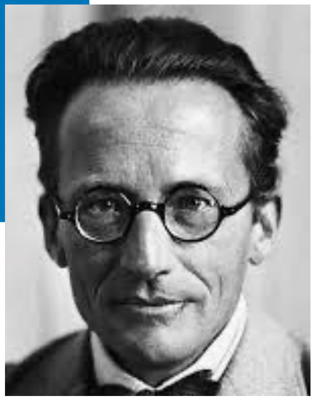


The probability of finding the particle between point a and b at time t is given by

$$\int_a^b \boxed{\left|\Psi(x,t)\right|^2} dx$$

PDF

Normalisation



Since $|\Psi(x,t)|^2$ is a PDF, $\int_{-\infty}^{+\infty} |\Psi(x,t)|^2 dx = 1$, i.e., the Schrödinger equation keeps the wave function normalised.

$$\frac{d}{dt}\int_{-\infty}^{+\infty} |\Psi(x,t)|^2 dx = \int_{-\infty}^{+\infty} \frac{\partial}{\partial t}|\Psi(x,t)|^2 dx = \int_{-\infty}^{+\infty} \left\{ \Psi^* \frac{\partial \Psi}{\partial t} + \frac{\partial \Psi^*}{\partial t}\Psi \right\} dx$$

$$\frac{\partial \Psi}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2 \Psi}{\partial x^2} - \frac{i}{\hbar}V\Psi \qquad \text{and} \qquad \frac{\partial \Psi^*}{\partial t} = -\frac{i\hbar}{2m}\frac{\partial^2 \Psi^*}{\partial x^2} + \frac{i}{\hbar}V\Psi^*$$

$$\frac{d}{dt}\int_{-\infty}^{+\infty} |\Psi(x,t)|^2 dx = \frac{i\hbar}{2m}\left\{ \Psi^* \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x}\Psi \right\}\Bigg|_{-\infty}^{+\infty}$$

Prove this!

However, since $\Psi(x,t) \to 0$ as $x \to \pm\infty$,

$$\frac{d}{dt}\int_{-\infty}^{+\infty} |\Psi(x,t)|^2 dx = 0$$

Where was the Particle?

If, suppose, we locate a particle at point c, what will happen if we measure it immediately again?

We will find it at c

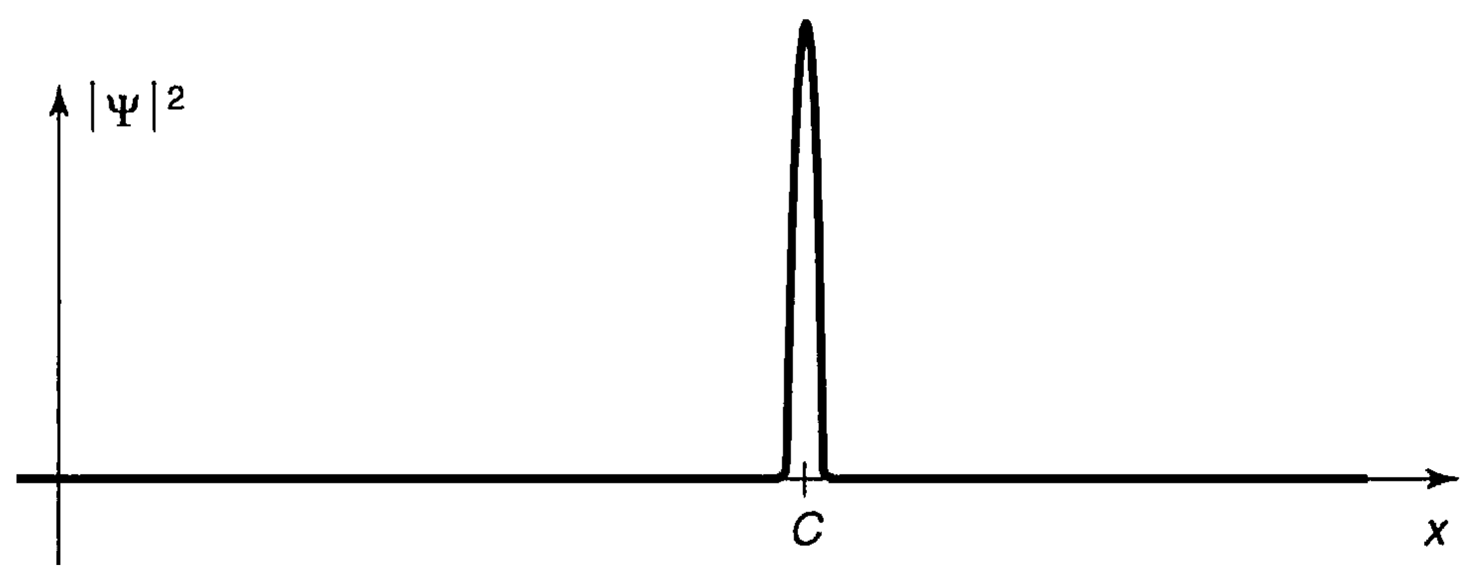
But where was it before we located it for the first time?

It was at c. This implies QM is an incomplete theory. There are hidden variables. If we know them, we get back the deterministic picture.

Is the moon really there when nobody looks? Reality and the quantum theory
N. David Mermin, **Physics Today,** April 1985

It was everywhere. The act of measurement is a physical process that forced it to take a definite position. After the first measurement the wave function collapses to a spike at c.

Bell's test has (almost) ruled out the first option.



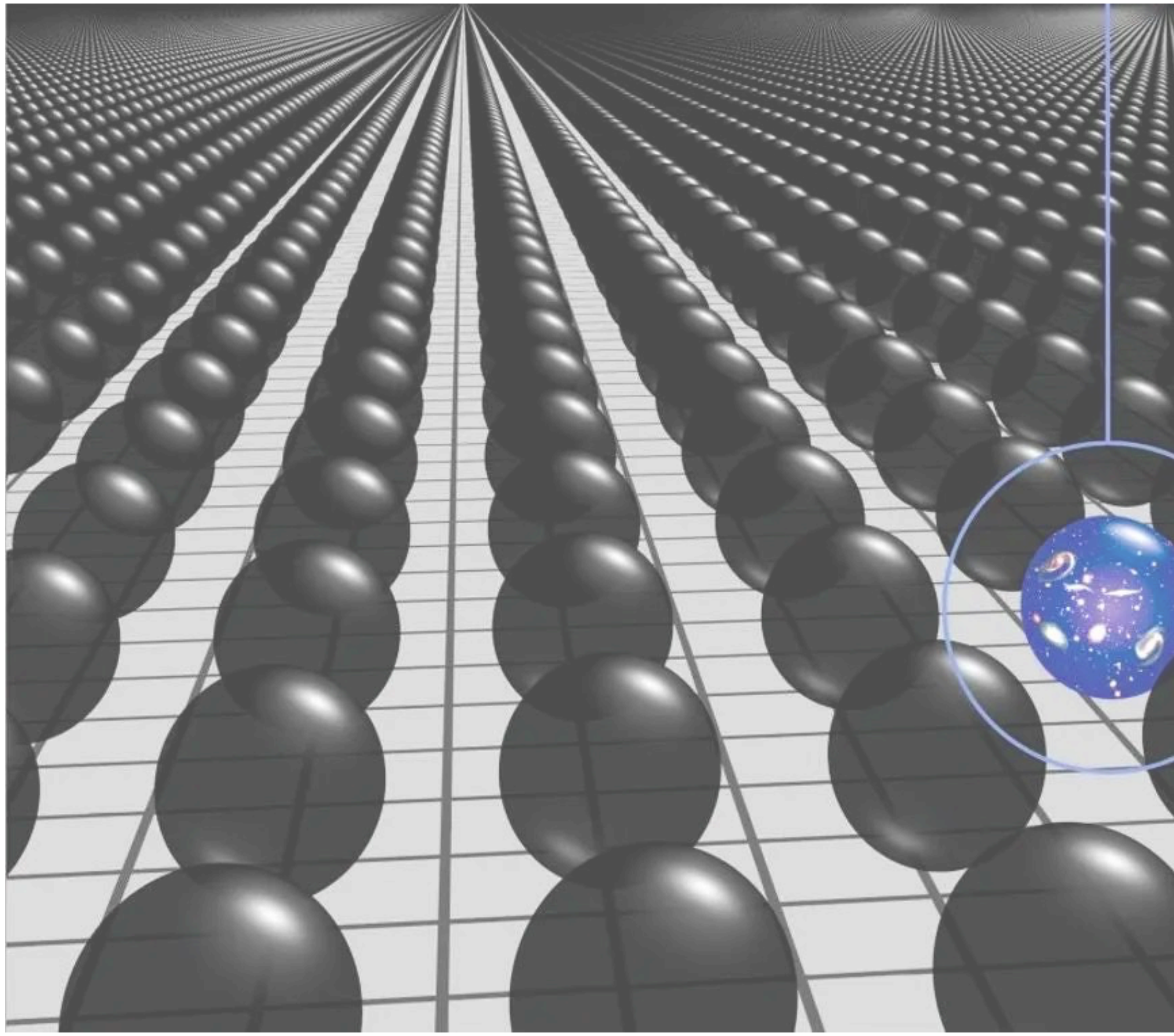
$|\Psi(x,t)|^2$ is a PDF

Since $|\Psi(x,t)|^2$ is a PDF of x, if we measure the position of a particle repeatedly a large number of times, what would be the expected outcome?

$$\langle x \rangle = \int_{-\infty}^{+\infty} x\,|\Psi(x,t)|^2 dx = \int_{-\infty}^{+\infty} \Psi^*(x,t)\,x\,\Psi(x,t)dx$$

But, there is collapse! The wave function collapses after the first measurement. Hence, the repeated measurement would not be given by the expectation value.

Instead, we should think that if we start with an ensemble of identically prepared systems [all with the same wave function $\Psi(x,t)$–we say, all in the same state $\Psi(x,t)$] and perform the measurement on all of them at time t, we would get the expected value.



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But what use has this?

The classical objects are made up of lots of quantum objects. So, roughly, we can think the classical objects as ensemble of quantum states. Then, if we measure the location of a classical object we expect to get the average or the "expected value", i.e., $\langle x \rangle$. This could work provided the average quantities obey the classical laws.

$\boxed{\text{Ehrenfest's Theorem}}$

How to compute $\langle p \rangle$?

We need to compute dynamical quantities like $\langle p \rangle$ or $\langle \vec{L} \rangle$ etc.

$$\langle p \rangle = m\langle v \rangle = m\frac{d\langle x \rangle}{dt} = m\int x\frac{\partial}{\partial t}|\Psi(x,t)|^2 dx = \frac{i\hbar}{2}\int x\frac{\partial}{\partial x}\left\{\Psi^*\frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x}\Psi\right\}dx$$
$$= -\frac{i\hbar}{2}\int\left\{\Psi^*\frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x}\Psi\right\}dx = -i\hbar\int \Psi^*\frac{\partial \Psi}{\partial x}dx$$

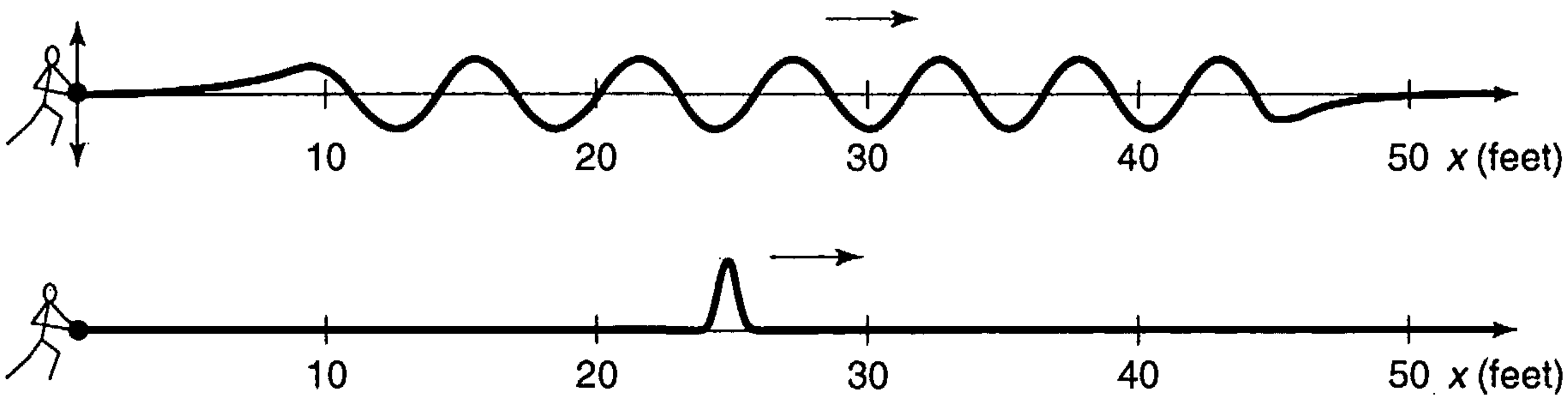
Every dynamical quantity has its operator. We can compute their expectation values by operating with the corresponding operators.

$$\langle x \rangle = \int \Psi^* \underbrace{\boxed{x}}_{\hat{x}} \Psi\, dx \qquad \langle p \rangle = \int \Psi^* \underbrace{\boxed{\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)}}_{\hat{p}} \Psi\, dx$$

$$\langle T \rangle = \frac{\langle p \rangle^2}{2m} = \int \Psi^* \underbrace{\boxed{\frac{1}{2m}\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)^2}}_{\hat{T}} \Psi\, dx$$

$$\langle Q(x,p) \rangle = \int \Psi^* \underbrace{\boxed{Q\left(x, \frac{\hbar}{i}\frac{\partial}{\partial x}\right)}}_{\hat{Q}(x,t)} \Psi\, dx$$

The Uncertainty Principle



Every measurement on a state yields some definite answer. However, measurements over identically prepared states vary:

$$\sigma_x \sigma_p \geq \frac{\hbar}{2}$$

Here, σ_q is the standard deviation, i.e., $\sigma_q = \sqrt{\langle q^2 \rangle - \langle q \rangle^2}$

Solving the Schrödinger Equation

Time-independent Potential, $V = V(x)$

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V(x)\Psi$$

Can we separate the variables?

$$\Psi(x,t) = \psi(x)\,\phi(t) \quad \longrightarrow \quad \frac{\partial\Psi}{\partial t} = \psi\frac{d\varphi}{dt}, \qquad \frac{\partial^2\Psi}{\partial x^2} = \frac{d^2\psi}{dx^2}\varphi$$

The Schrödinger equation can now be written as

$$i\hbar\frac{1}{\varphi}\frac{d\varphi}{dt} = -\frac{\hbar^2}{2m}\frac{1}{\psi}\frac{d^2\psi}{dx^2} + V$$

The only way this could be true is if both sides are equal to a constant.

So, we get two equations:

Steady states or stationary states

$$\frac{d\varphi}{dt} = -\frac{iE}{\hbar}\varphi \quad \longrightarrow \quad \varphi(t) = e^{-iEt/\hbar} \quad \longrightarrow$$

Time-independent probabilities

$$|\Psi(x,t)|^2 = |\psi(x)|^2\phi^*(t)\phi(t) = |\psi(x)|^2$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi \quad \longrightarrow \quad \hat{H}\psi = E\psi$$

Time-independent Schrödinger Equation

The stationary states are eigenfunctions of the Hamiltonian operator.

Stationary States

The entire ensemble has the same energy, E

$$\langle H \rangle = \int \Psi^* \hat{H} \Psi \, dx = \int (\phi^* \phi)(\psi^* \hat{H} \psi) \, dx = E \int (\psi^* \psi) \, dx = E$$

$$\langle H^2 \rangle = \int (\psi^* \hat{H}\hat{H} \psi) \, dx = E^2 \int (\psi^* \psi) \, dx = E^2 \Rightarrow \sigma_H = 0$$

General solutions are linear combinations

$$\Psi(x, t) = \sum_{n=1}^{\infty} c_n \psi_n(x) e^{-iE_n t/\hbar} = \sum_{n=1}^{\infty} c_n \Psi_n(x, t)$$

Example 2.1 Suppose a particle starts out in a linear combination of just *two* stationary states:

$$\Psi(x, 0) = c_1 \psi_1(x) + c_2 \psi_2(x).$$

(To keep things simple I'll assume that the constants c_n and the states $\psi_n(x)$ are *real*.) What is the wave function $\Psi(x, t)$ at subsequent times? Find the probability density, and describe its motion.

$$\Psi(x, t) = c_1 \psi_1(x) e^{-iE_1 t/\hbar} + c_2 \psi_2(x) e^{-iE_2 t/\hbar},$$

where E_1 and E_2 are the energies associated with ψ_1 and ψ_2. It follows that

$$|\Psi(x, t)|^2 = (c_1 \psi_1 e^{iE_1 t/\hbar} + c_2 \psi_2 e^{iE_2/\hbar})(c_1 \psi_1 e^{-iE_1 t/\hbar} + c_2 \psi_2 e^{-iE_2/\hbar})$$

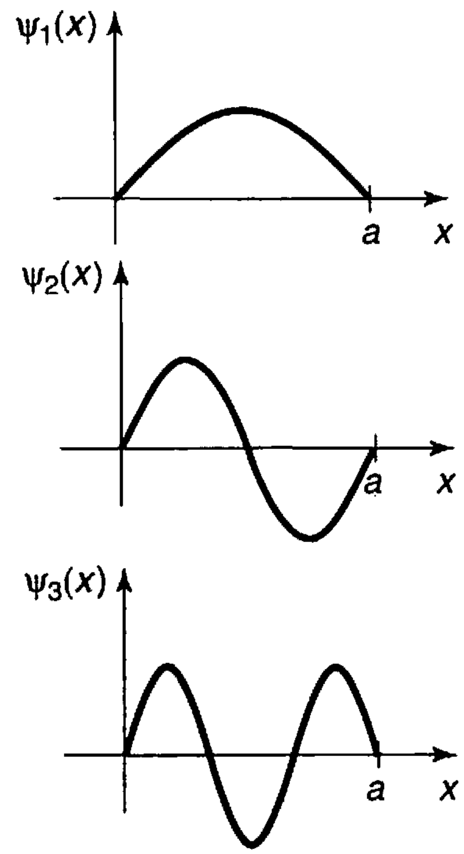
$$= c_1^2 \psi_1^2 + c_2^2 \psi_2^2 + 2c_1 c_2 \psi_1 \psi_2 \cos[(E_2 - E_1)t/\hbar].$$

Combinations can have dynamics

Particle in a Box

$$V(x) = \begin{cases} 0, & \text{if } 0 \leq x \leq a, \\ \infty, & \text{otherwise} \end{cases}$$

If $E < 0$, there is no normalisable solution to the Schrödinger equation



Outside the walls $\psi(x) = 0$. Inside,

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \quad \rightarrow \quad \frac{d^2\psi}{dx^2} = -k^2\psi \quad \text{where} \quad k \equiv \frac{\sqrt{2mE}}{\hbar}$$

$$\psi(x) = A \sin kx + B \cos kx$$

Boundary conditions

ψ and $d\psi/dx$ are continuous unless $V = \infty$. In that case, only ψ is continuous

$$\psi(0) = B = 0$$

$$\psi(a) = A \sin(ka) = 0$$

$$ka = \cancel{0}, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$$

We can absorb the minus sign in A

$$k_n = \frac{n\pi}{a} \quad \text{with } n = 1, 2, 3, \ldots$$

Discrete

$$E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{n^2\pi^2\hbar^2}{2ma^2}$$

Normalise to get

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right)$$

General solution

$$\Psi(x,t) = \sum_{n=1}^{\infty} c_n \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right) e^{-i(n^2\pi^2\hbar/2ma^2)t}$$

Particle in a Box

$$E_n = \frac{n^2 h^2}{8ma^2} \qquad \psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right) \qquad P_n(x) = \frac{2}{a}\sin^2\left(\frac{n\pi x}{a}\right)$$

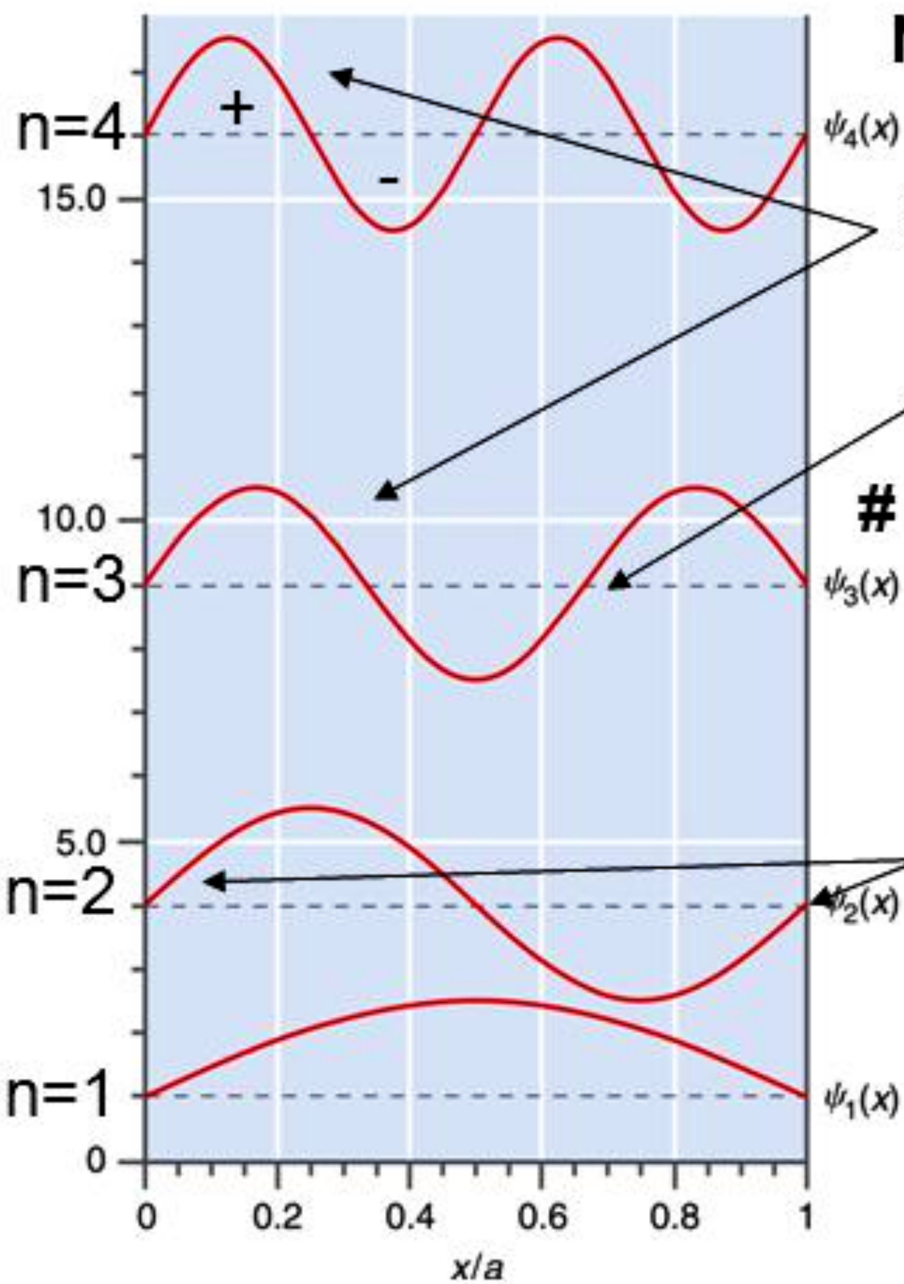
$$E_4 = \frac{16h^2}{8ma^2}$$

$$E_3 = \frac{9h^2}{8ma^2}$$

$$E_2 = \frac{4h^2}{8ma^2}$$

$$E_1 = \frac{h^2}{8ma^2} \neq 0$$

Ground state



Normalized

Orthogonal

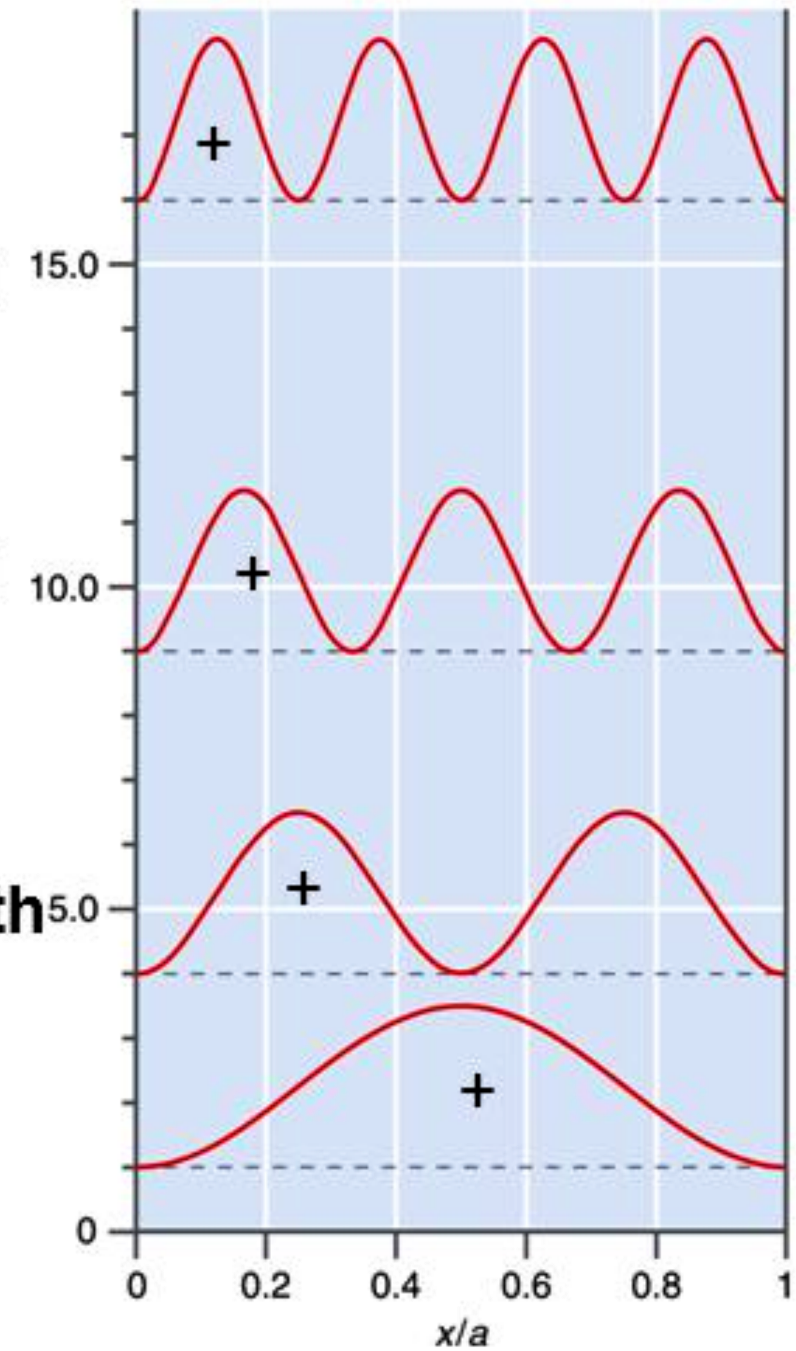
Node

 # nodes= n-1

n > 0

Wavelength

$$\lambda = \frac{2a}{n}$$



Mathematics

Wave functions are all square integrable (normalised).

Such functions ($\int |f(x)|^2 dx < \infty$) form a vector space called Hilbert space or L_2 space.

Inner product

A complex number

$$\langle\Phi|\Psi\rangle = \int \Phi^*(x,t)\,\Psi(x,t)\,dx$$

Dual vector space
Bra space

Vector space
Ket space

Complex conjugation

Dirac's bra(c)ket notation

Normalisation: $\langle\Psi|\Psi\rangle = 1$

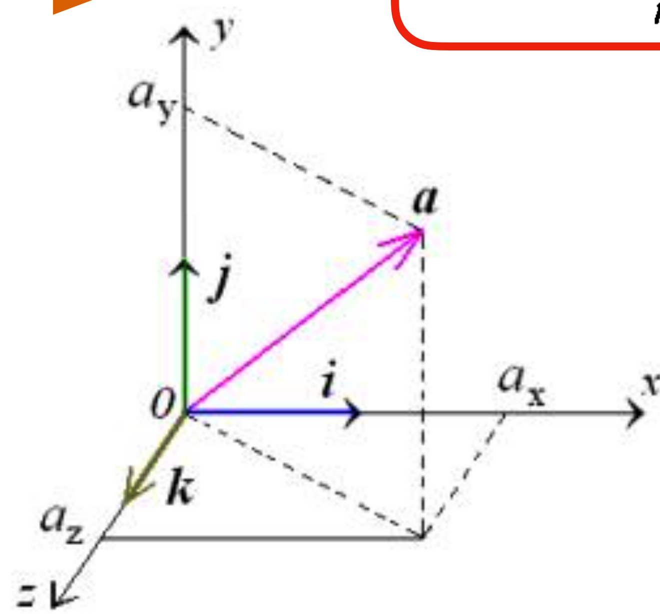
Resolving $\Psi(x,t)$ into mutually orthogonal components

$$\Psi(x,t) = \sum_{n=1}^{\infty} c_n \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right) e^{-i(n^2\pi^2\hbar/2ma^2)t}$$

n^{th} unit vector $|\Psi_n\rangle$

If $|\Psi\rangle = c_0|\Psi_0\rangle + c_1|\Psi_1\rangle$

$\langle\Phi|\Psi\rangle = c_0\langle\Phi|\Psi_0\rangle + c_1\langle\Phi|\Psi_1\rangle$



Orthogonality

$\equiv \langle\Psi_m|\Psi_n\rangle = \delta_{mn}$

$c_n = \langle\Psi_n|\Psi\rangle$

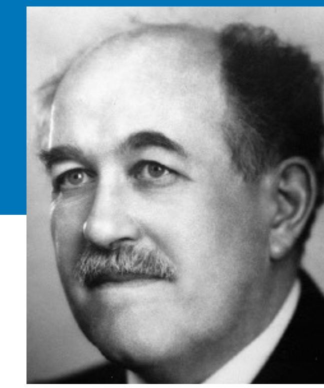
If $|\Phi\rangle = c_0|\Phi_0\rangle + c_1|\Phi_1\rangle$

$\langle\Phi| = c_0^*\langle\Phi_0| + c_1^*\langle\Phi_1|$

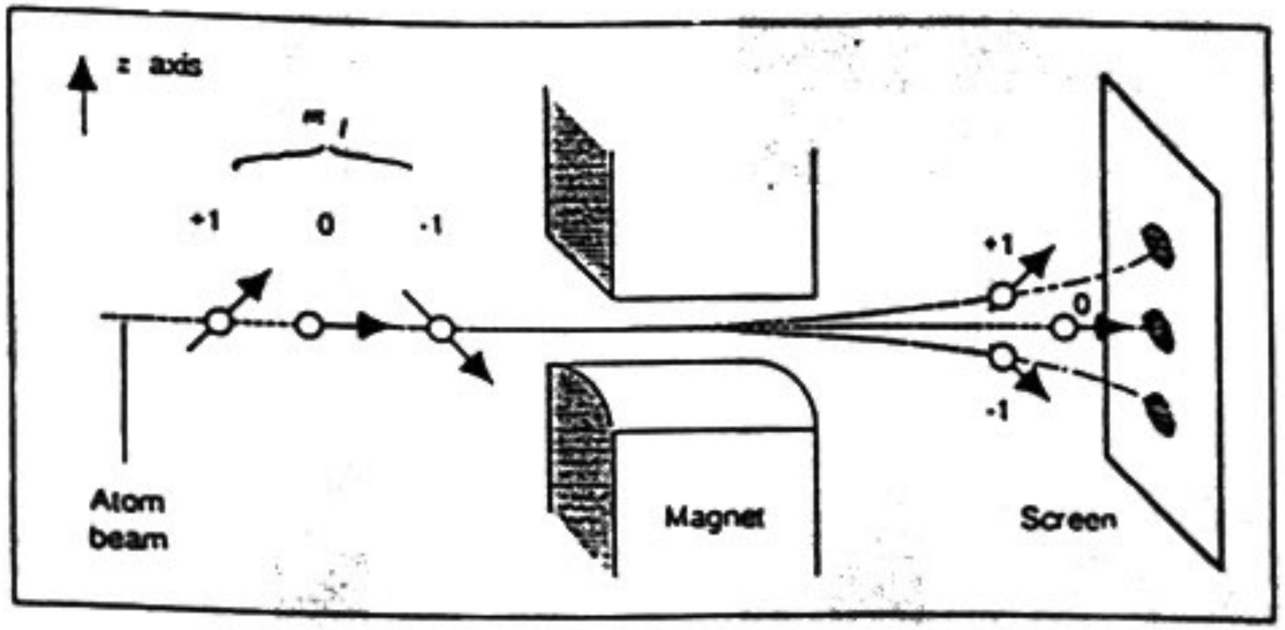
$\langle\Phi|\Psi\rangle = c_0^*\langle\Phi_0|\Psi\rangle + c_1^*\langle\Phi_1|\Psi\rangle$

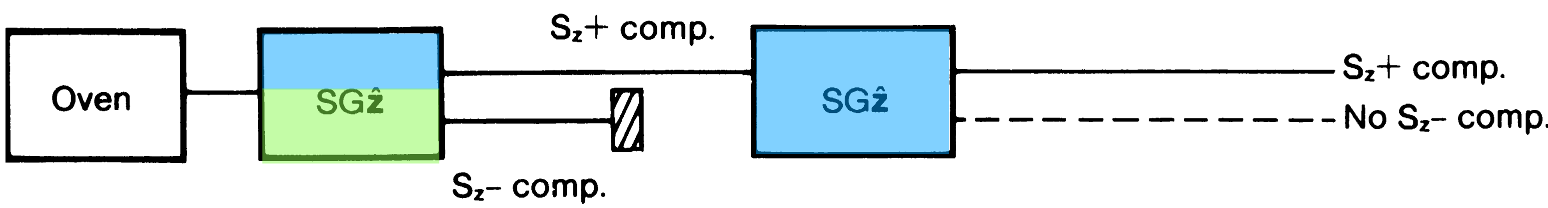
$\langle\Psi|\Psi\rangle = |c_0|^2\langle\Psi_0|\Psi_0\rangle + |c_1|^2\langle\Psi_1|\Psi_1\rangle$

Stern-Gerlach Experiment: Spin


Otto Stern


Walther Gerlach





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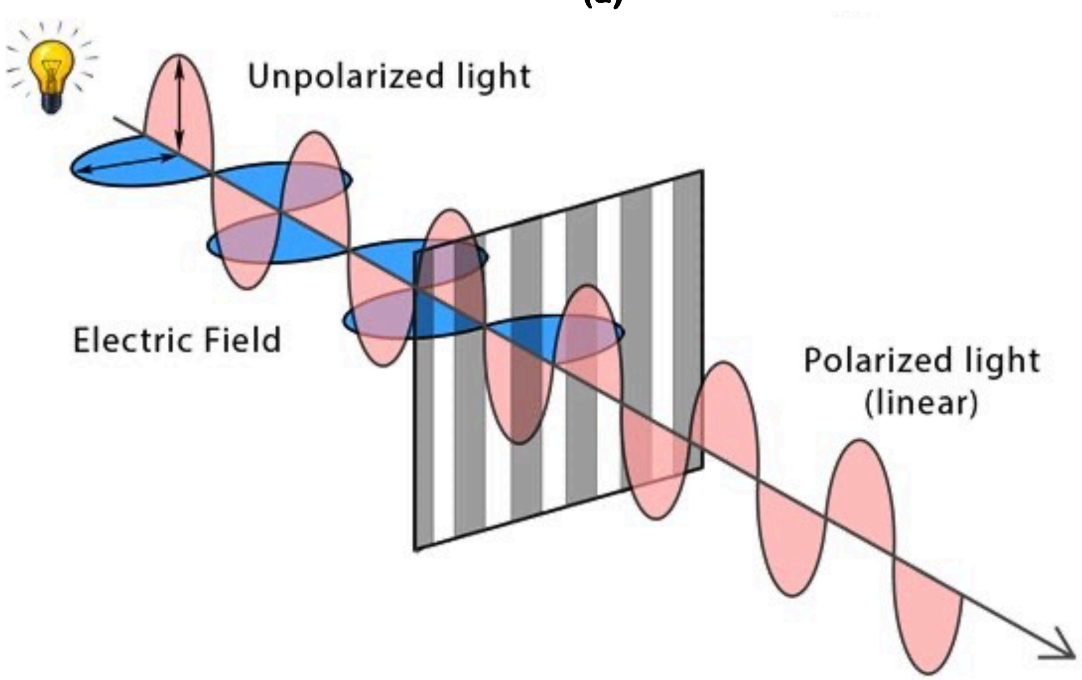
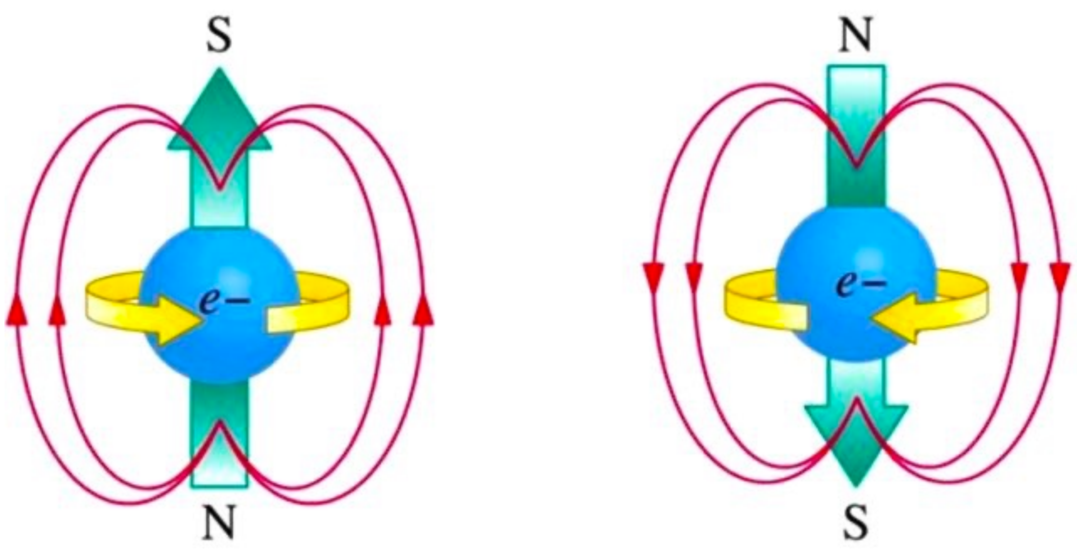


What is spin?



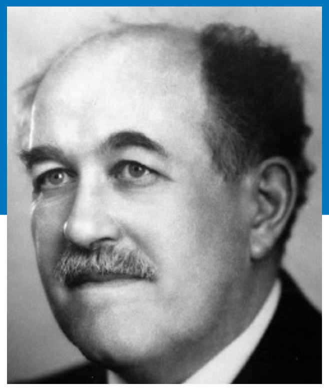
Suppose electrons are tiny solid spheres of radius $r_{\mathrm{cl}} = \dfrac{e^2}{4\pi\epsilon_0 mc^2}$ with $= \left|\vec{L}\right| = \dfrac{\hbar}{2}$.

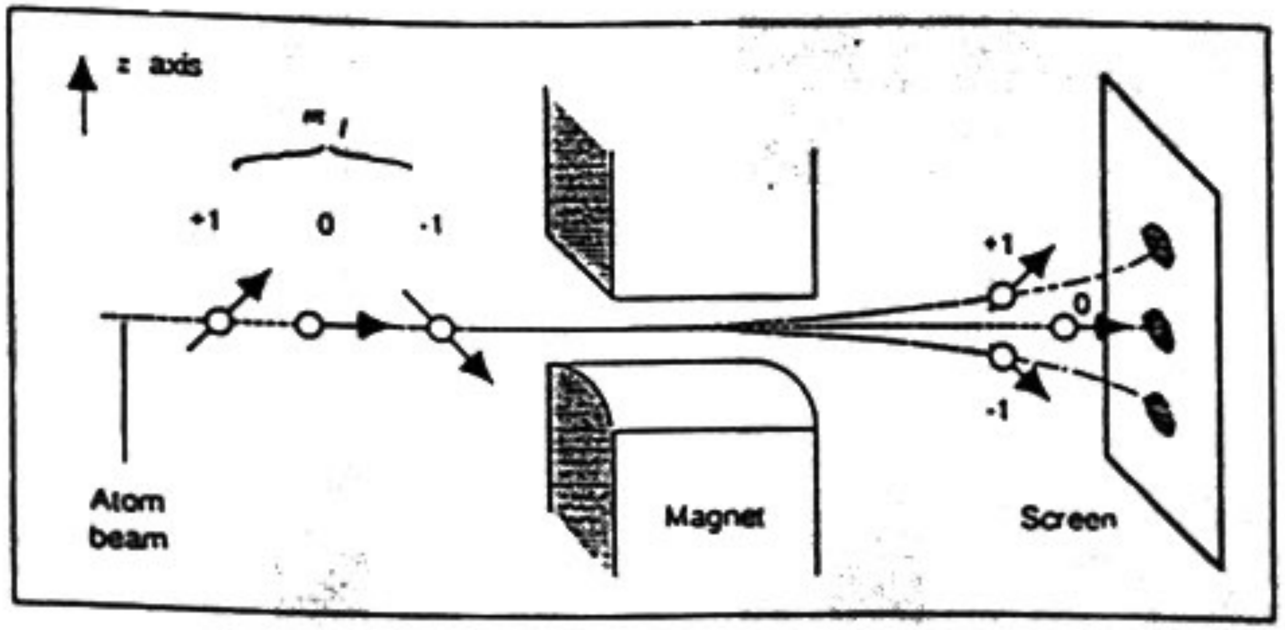
$$r_{\mathrm{cl}} = \frac{(1.6\times10^{-19})^2}{4\pi(8.85\times10^{-12})(9.11\times10^{-31})(3\times10^8)^2} = 2.8\times10^{-15}\ \mathrm{m}$$

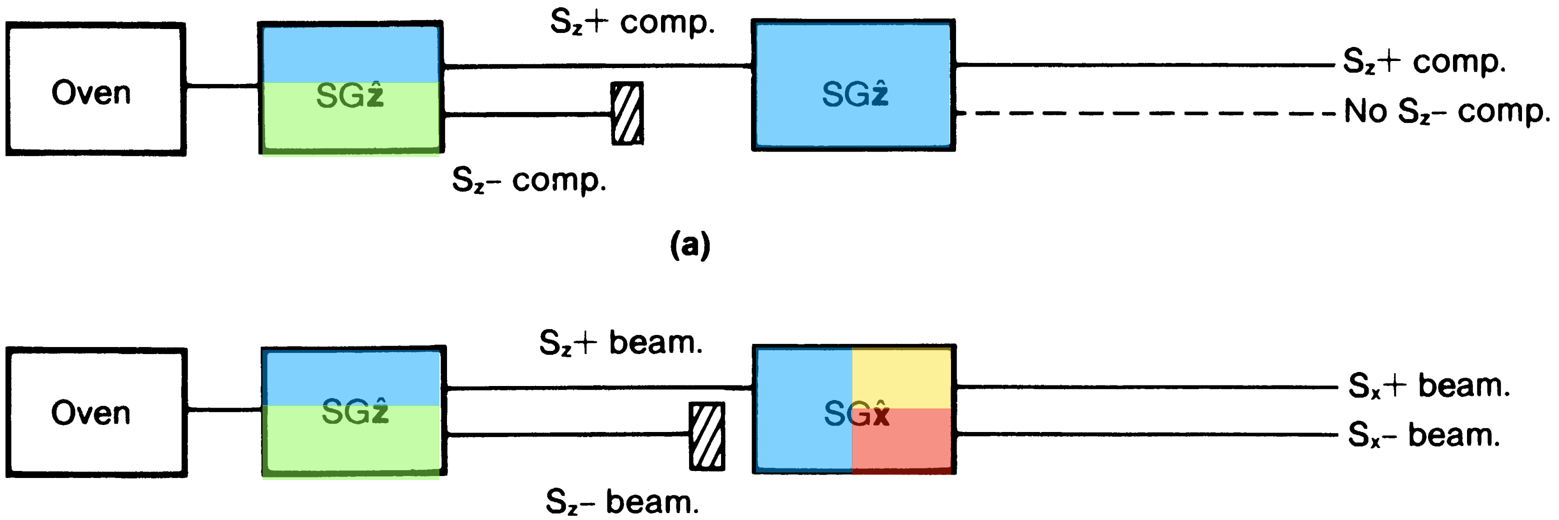
$$\left|\vec{L}\right| = \frac{\hbar}{2} = I\omega = \left(\frac{2}{5}mr_{cl}^2\right)\left(\frac{v}{r_{cl}}\right) = \frac{2}{5}mr_{cl}v$$

$$v = \frac{5\hbar}{4mr_{cl}} = \frac{5\times1.055\times10^{-34}}{4\times9.11\times10^{-31}\times2.8\times10^{-15}} \approx 5\times10^{10}\ \mathrm{m/s}$$

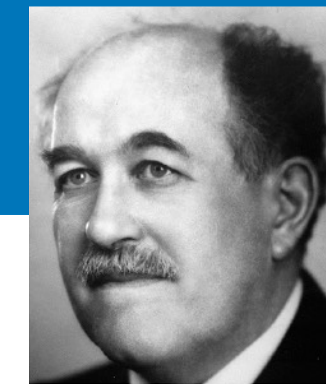
Stern-Gerlach Experiment: Spin

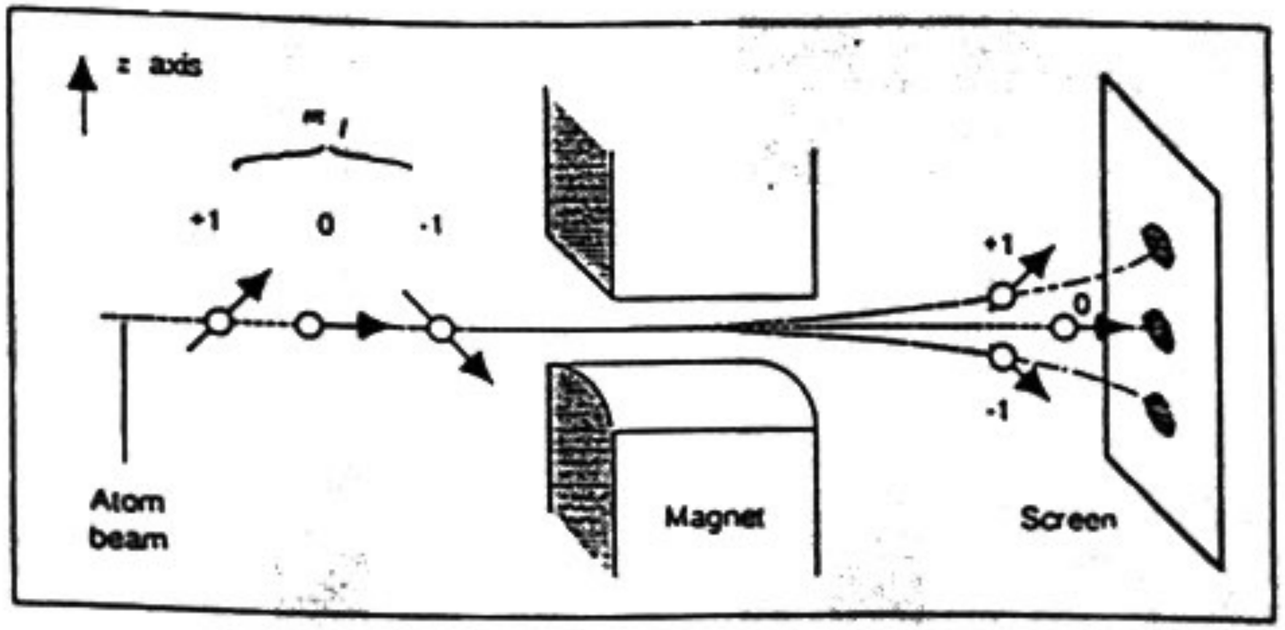


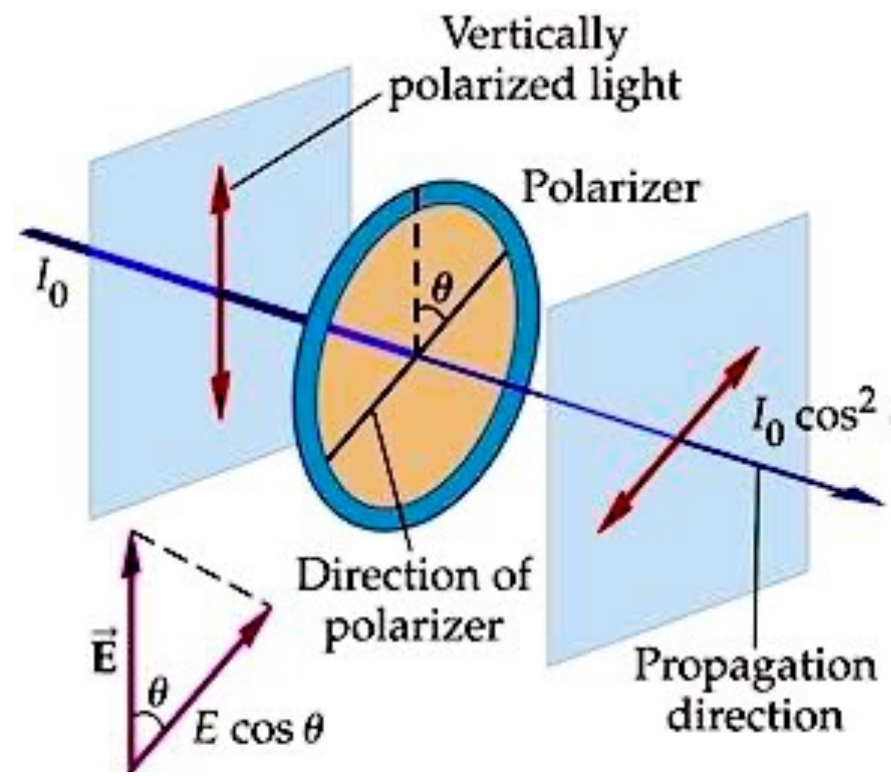


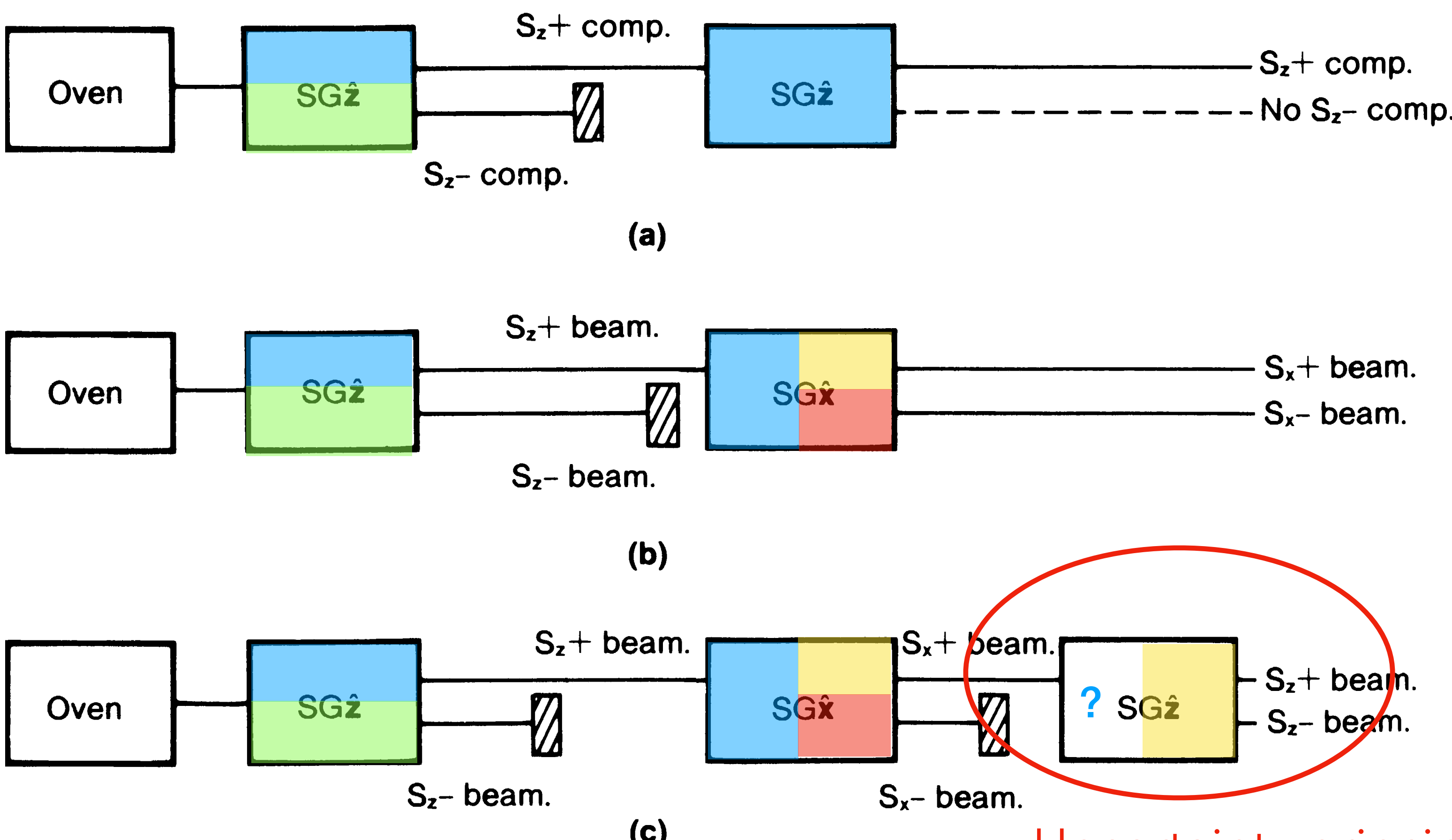


Stern-Gerlach Experiment: Spin







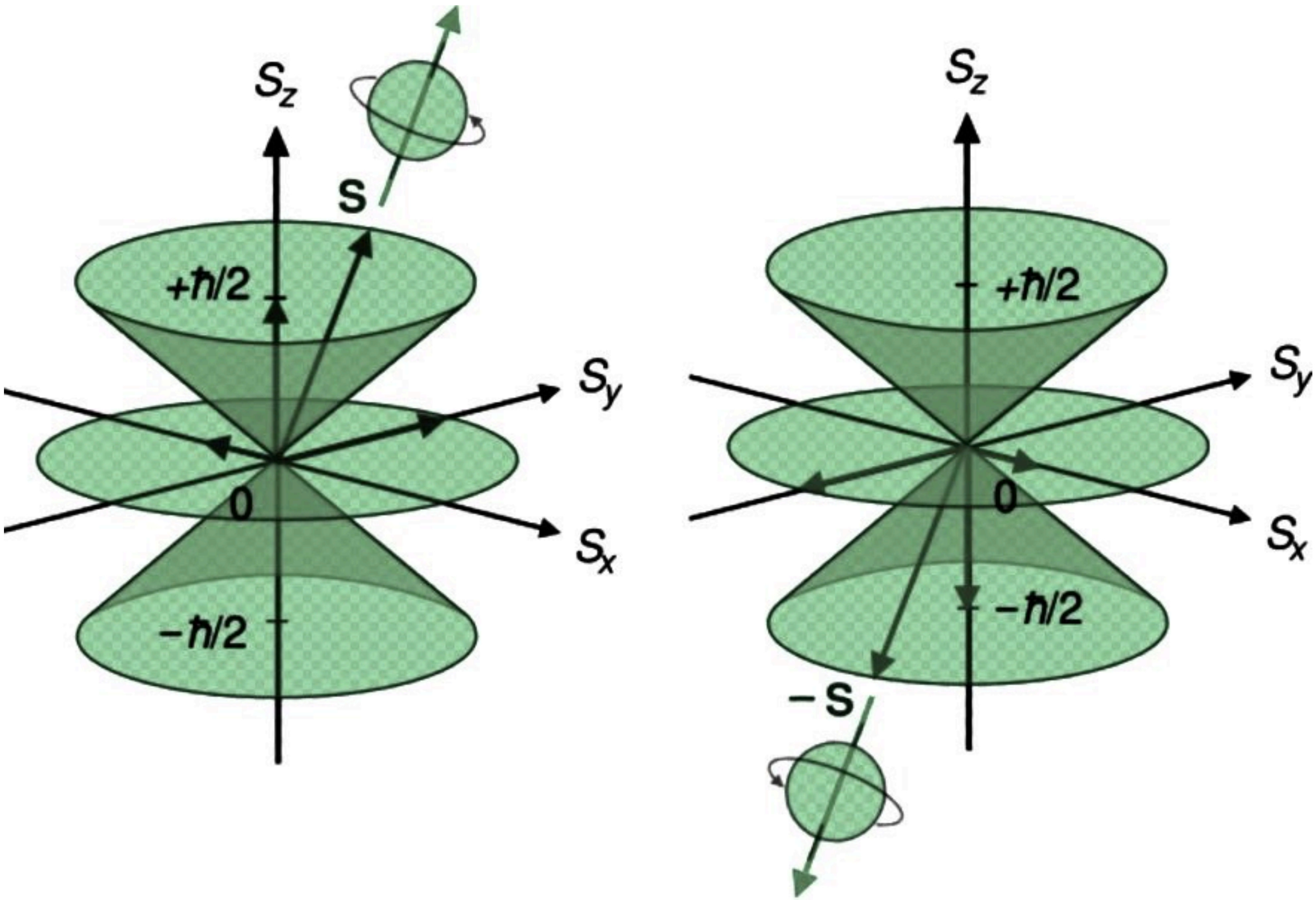


Uncertainty principle

Conditional probability

In general, if the second magnetic field (SG$\hat{n}$) makes an angle θ with the first one, one gets S_n+ and S_n- particles with probabilities $\cos^2(\theta/2)$ and $\sin^2(\theta/2)$.

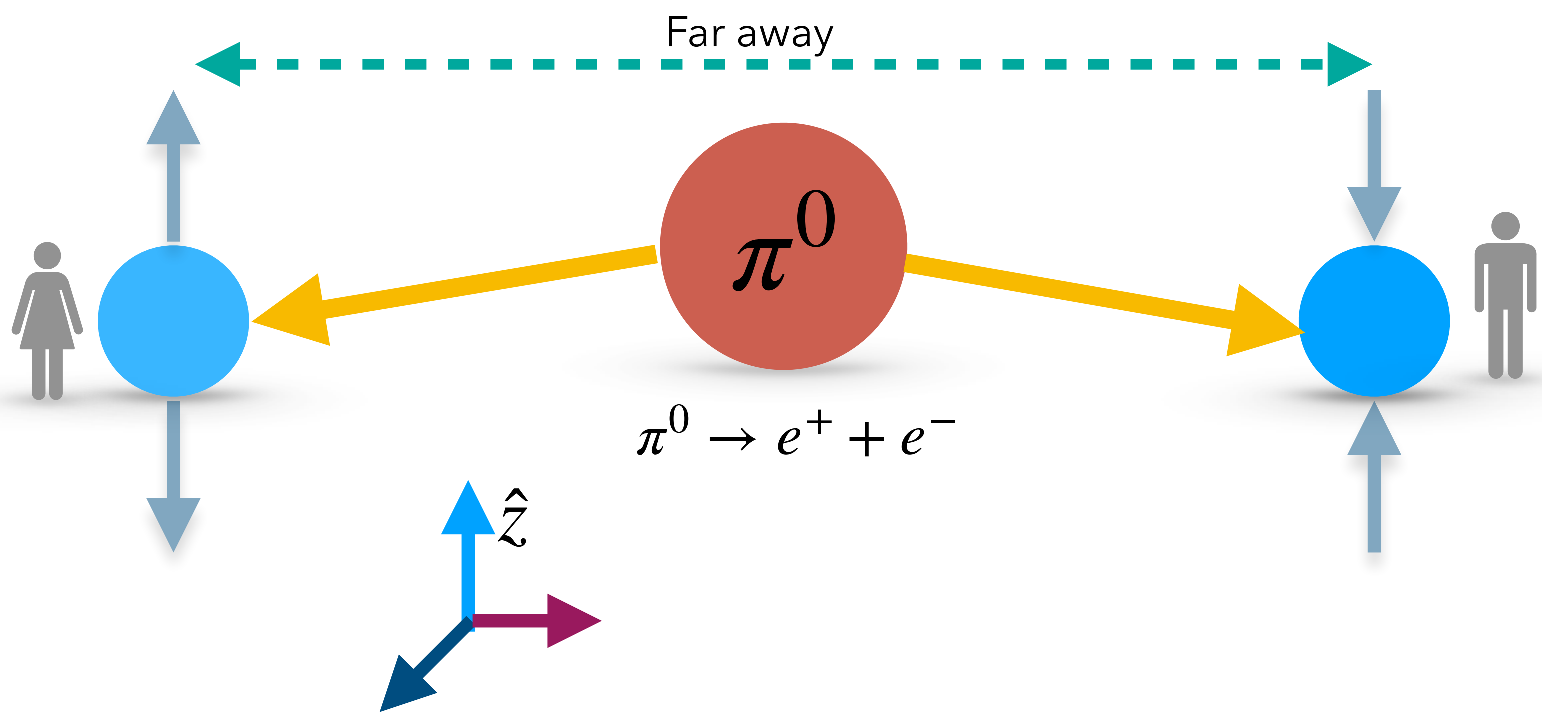
What is spin?



$|\uparrow\rangle \equiv \left|+\frac{1}{2}\right\rangle$ $\quad$ $|\downarrow\rangle \equiv \left|-\frac{1}{2}\right\rangle$

Intrinsic angular momentum

The EPR Paradox

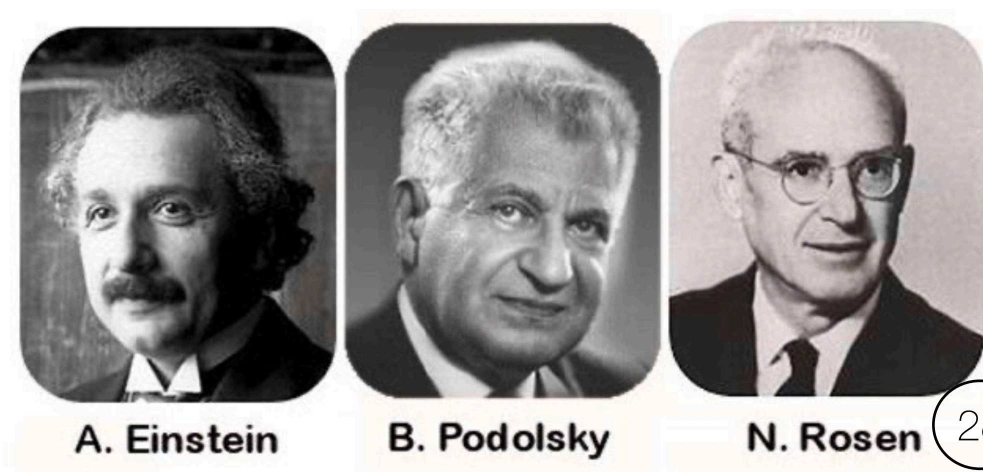


If Alice's electron has spin up, Bob's one must have spin down, and vice versa. QM does not predict the combination, but says that the measurements will be correlated, and you'll get each combination half the time (on average).

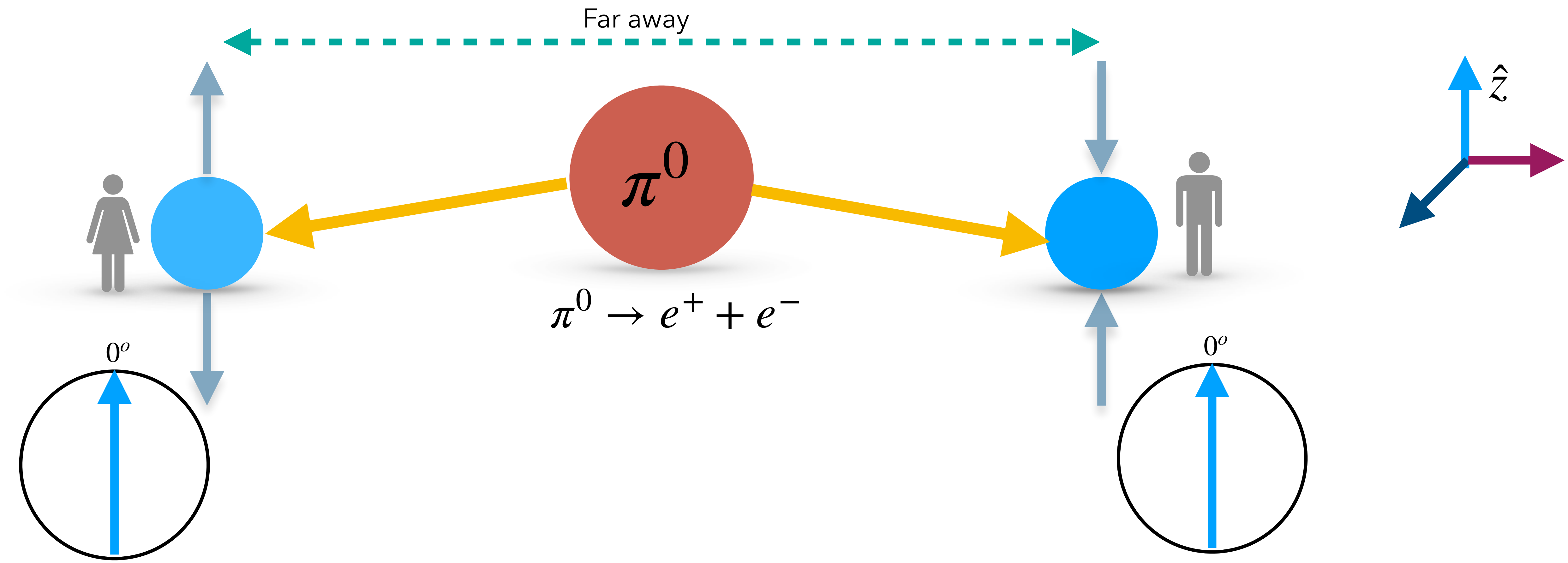
If neither particle had either spin up or spin down until the act of measurement intervened, Alice's measurement of the electron collapsed the wave function, and instantaneously "produced" the spin of the positron far away. Einstein, Podolsky, and Rosen considered such "spooky action-at-a- distance" (Einstein's delightful term) preposterous.

Hidden Variables (λ)

The fundamental assumption on which the EPR argument rests is that no influence can propagate faster than the speed of light. We call this the principle of locality.



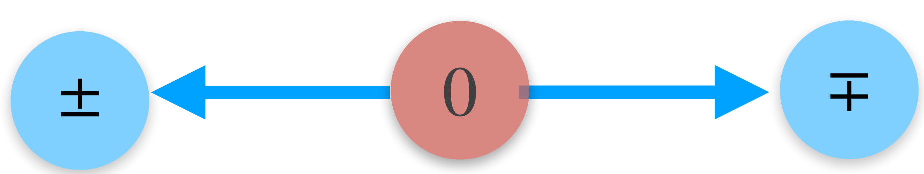
The EPR Paradox



If Alice's and Bob's magnetic fields are along $\hat{z}$, the spins of their electrons will point opposite to each other, i.e., Bob will see a flipped spin w.r.t. Alice's.

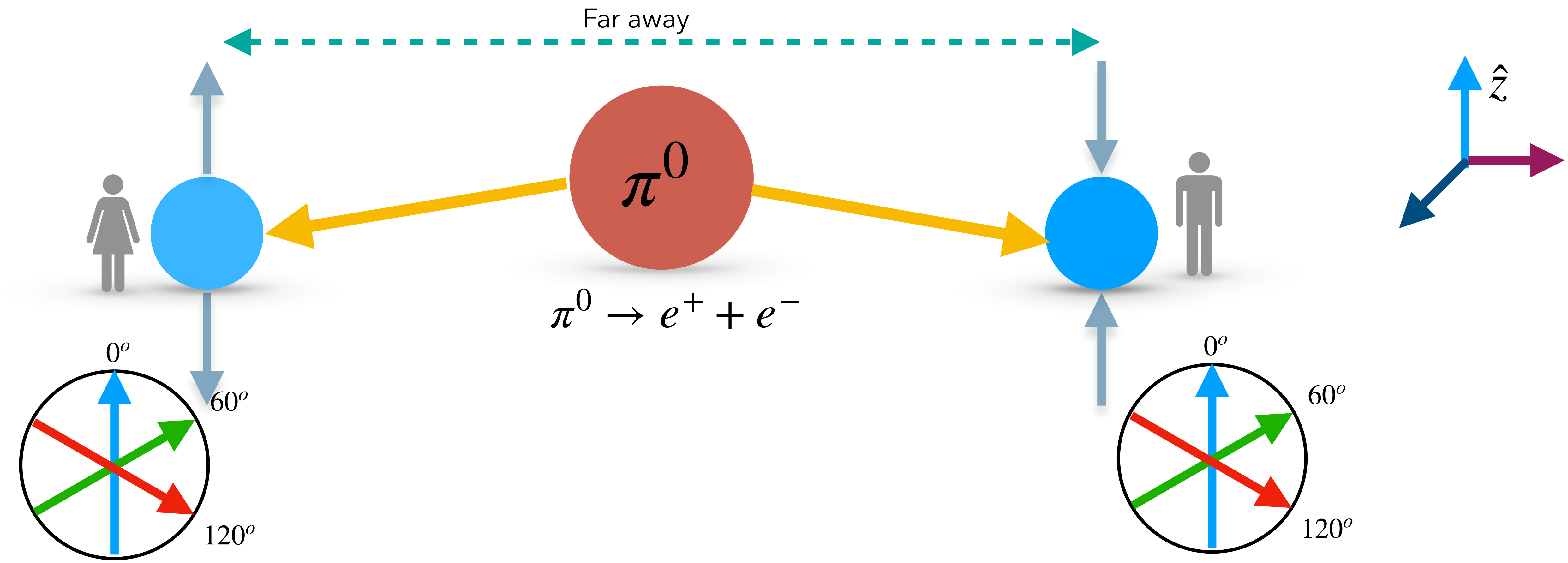
So, if the outcome of Alice's spin measurement is α, Bob's is $\beta = -\alpha$.

Expectation:



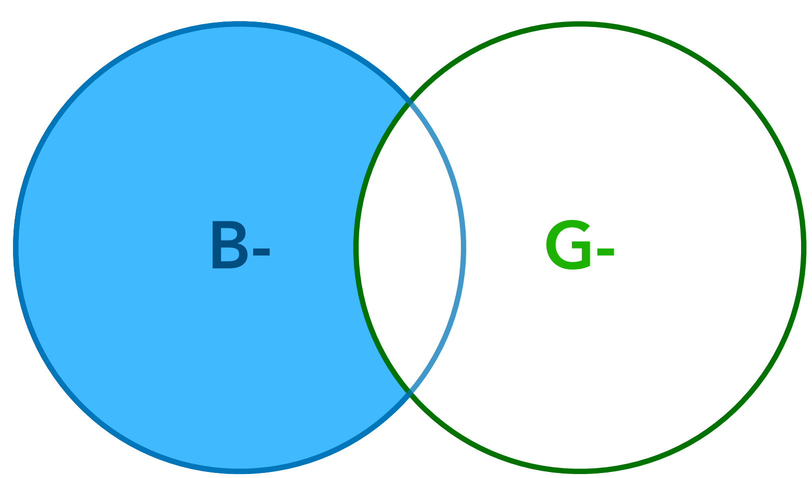
$$P(\alpha, \beta) = \int \rho(\lambda)\, f(\hat{B}_{Alice}, \lambda)\; g(\hat{B}_{Bob}, \lambda)\; d\lambda \qquad \text{with} \int \rho(\lambda) d\lambda = 1$$

The EPR Paradox



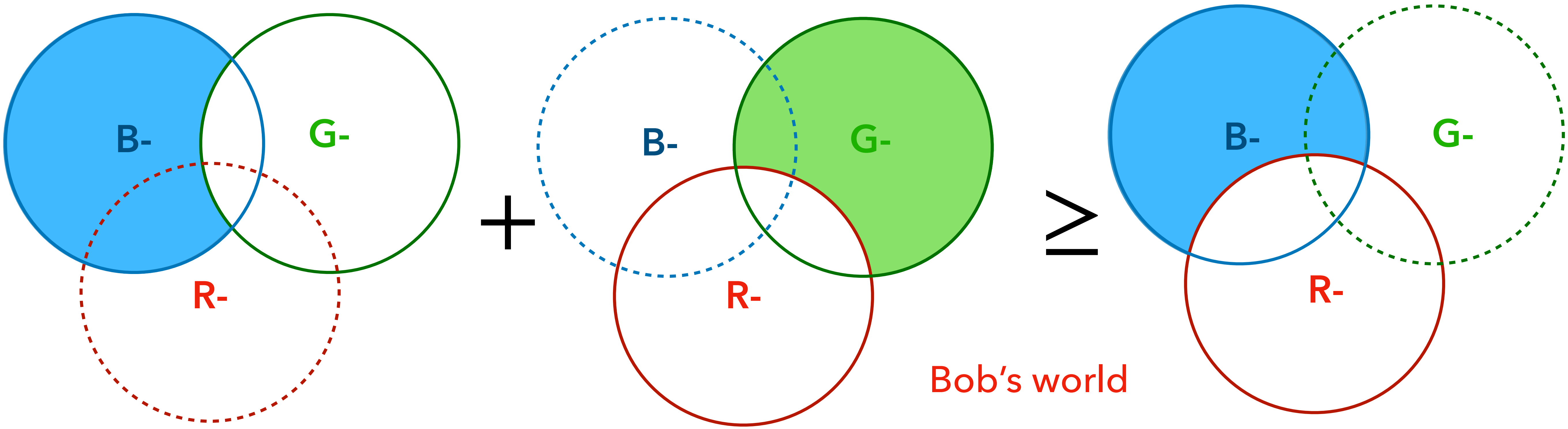
The situation becomes more interesting if they use different measurements axes.

$$P(\text{Alice} : B_+; \text{Bob} : G_+) \equiv P(\text{Bob} : B_-; \text{Bob} : G_+)$$

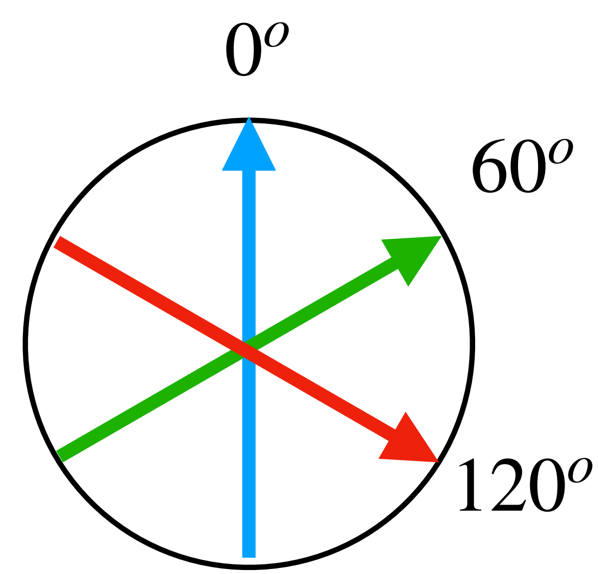


Bob's world

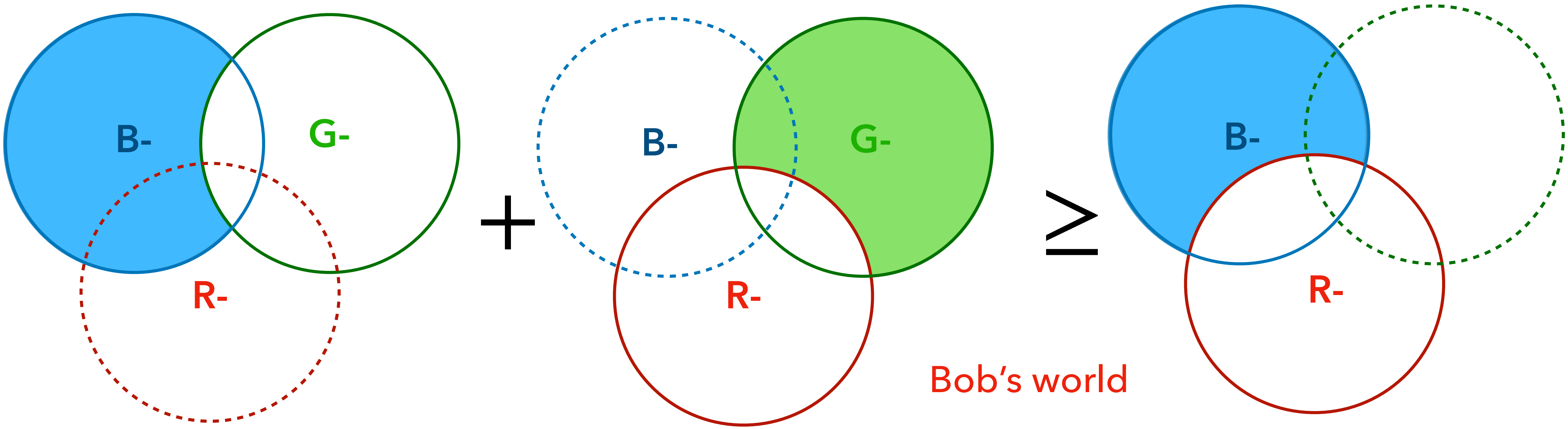
Bell’s Inequality

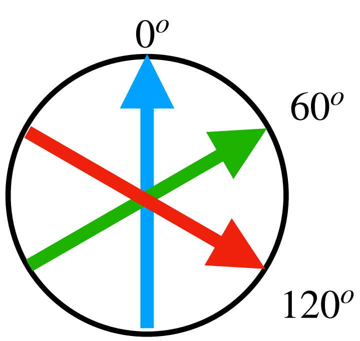


$P(\text{Bob}; \text{Bob})$	$P(B_-; G_+) + P(G_-; R_+) \geq P(B_-; R_+)$
$P(\text{Alice}; \text{Bob})$	$P(B_+; G_+) + P(G_+; R_+) \geq P(B_+; R_+)$



Bell's Inequality Violation






Alain Aspect (Nobel 2022)

$P(\text{Alice}; \text{Bob})$	$P(\text{Bob}; \text{Bob})$	$\frac{1}{2} \times \sin^2(\Delta\theta/2)$
$P(B_+; G_+)$	$P(B_-; G_+)$	$\frac{1}{8}$
$P(G_+; R_+)$	$P(G_-; R_+)$	$\frac{1}{8}$
$P(B_+; R_+)$	$P(B_-; R_+)$	$\frac{3}{8}$

No Quantum "Xerox" Machine

Suppose there is a machine: $|\psi\rangle|X\rangle \rightarrow |\psi\rangle|\psi\rangle$

It cannot copy a linear superposition:

$(a|\psi\rangle + b|\phi\rangle)|X\rangle \rightarrow a|\psi\rangle|\psi\rangle + b|\phi\rangle|\phi\rangle$

As we want:

$(a|\psi\rangle + b|\phi\rangle)|X\rangle \rightarrow (a|\psi\rangle + b|\phi\rangle)(a|\psi\rangle + b|\phi\rangle)$

Mathematics

$\langle f | g \rangle = \langle g | f \rangle^*$

What about operators?

The expectation value of some observable $Q(x,p)$: $\langle Q \rangle = \int \Psi^* (\hat{Q}\Psi)\, dx = \langle \Psi | \hat{Q}\Psi \rangle$

The dual vector of $|\hat{Q}\Psi\rangle$ is $\boxed{\langle \Psi \hat{Q}^\dagger |}$. Hence, $\langle \Psi \hat{Q}^\dagger | \Psi \rangle = \int (\hat{Q}^\dagger \Psi^*)\, \Psi\, dx = \int (\hat{Q}\Psi)^*\, \Psi\, dx = \langle Q \rangle^*$

Also written as $\boxed{\langle \hat{Q}^\dagger \Psi |}$

Observables ≡ Hermitian operators

Since measurement outcomes are real, $\langle \Psi \hat{Q}^\dagger | \Psi \rangle = \langle \Psi | \hat{Q}\Psi \rangle$
or, simply, $\langle \Psi \hat{Q}^\dagger \Psi \rangle = \langle \Psi \hat{Q} \Psi \rangle$, i.e., $\hat{Q}^\dagger = \hat{Q}$

Momentum operator

$$\langle f | \hat{p} g \rangle = \int_{-\infty}^{\infty} f^* \frac{\hbar}{i}\frac{dg}{dx}\, dx = \frac{\hbar}{i} f^* g \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \left(\frac{\hbar}{i}\frac{df}{dx}\right)^* g\, dx = \langle \hat{p} f | g \rangle$$

Measuring an observable ≡ Operating on the state vector with an (hermitian) operator

$\hat{Q}|\Psi\rangle = q_i |\Psi\rangle$ Normally, if measurements are done on an ensemble, different members return different outcomes

If $|\Psi\rangle$ is an eigenvector/eigenfunction of $\hat{Q}$ then we would get the same value, the eigenvalue, from all members

$\hat{Q}|\Psi\rangle = q|\Psi\rangle$ ⇒ $\langle Q \rangle = \langle \Psi | \hat{Q} | \Psi \rangle = q \langle \Psi | \Psi \rangle = q$ and $\sigma_Q = 0$

Operator | Number

E.g., stationary states are eigenfunctions of the Hamiltonian operator with e.v. E_n

Mathematics

Two theorems for Hermitian operators ($\hat{Q} = \hat{Q}^\dagger$) in linear algebra

Eigenvalues of Hermitian operators are real

Let $\hat{Q}\,|\Psi\rangle = q\,|\Psi\rangle$. Now, $q\,\langle\Psi\,|\,\Psi\rangle = \langle\Psi\,|\,\hat{Q}\Psi\rangle = \langle\Psi\hat{Q}^\dagger\,|\,\Psi\rangle = q^*\langle\Psi\,|\,\Psi\rangle$ ***QED***

Eigenvectors/eigenfunctions with different eigenvalues are orthogonal

Let $\hat{Q}\,|\Phi\rangle = q'\,|\Phi\rangle$. Now, $q'\,\langle\Psi\,|\,\Phi\rangle = \langle\Psi\,|\,\hat{Q}\Phi\rangle = \langle\Psi\hat{Q}^\dagger\,|\,\Phi\rangle = q^*\langle\Psi\,|\,\Phi\rangle = q\,\langle\Psi\,|\,\Phi\rangle$

QED

In many cases it is possible to prove that eigenvectors of a Hermitian operator form a complete basis (e.g. particle in a box). In some infinite dimensional cases it is not. There we consider only those operators as our observables for which this is true. Hence, we can span the Hilbert space with these vectors/functions, i.e., express any function in the Hilbert space with them.