Chapter 5

Simple Cosmological Models

In Chapter 3 we derived the equations satisfied by an expanding isotropic gas. They are the Friedman equation[1]

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}\,, \tag{5.1}$$

which governs the time evolution of the scale factor $a(t)$, and the fluid equation

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0\,, \tag{5.2}$$

which gives us the evolution of the mass density $\rho(t)$. I stress that, despite their Newtonian derivation, these are the real equations used by cosmologists, more traditionally derived via the equations of general relativity as described in Advanced Topic 1.

This chapter finds and discusses some simple solutions to these equations, which will be used extensively during the book. However, while they have wide applicability during the early stages of the Universe, it will turn out that these are not sufficient to describe the present state of the Universe, for which an extra ingredient, the cosmological constant, will be needed. It is introduced in Chapter 7.

Before finding solutions to these equations, we can study two of their implications.

5.1 Hubble's law

The Friedmann equation allows us to explain Hubble's discovery that recession velocity is proportional to the distance. The velocity of recession is given by $\vec{v} = d\vec{r}/dt$ and is in the

[1] In accord with the discussion of Section 3.6, the c^2 on the final term has been dropped, so that its appearance matches that of other cosmology textbooks.

An Introduction to Modern Cosmology, Third Edition. Andrew Liddle.

same direction as $\vec{r}$, allowing us to write

$$\vec{v} = \frac{|\dot{\vec{r}}|}{|\vec{r}|}\,\vec{r} = \frac{\dot{a}}{a}\,\vec{r}. \tag{5.3}$$

The last step used $\vec{r} = a\vec{x}$, remembering that the comoving position $\vec{x}$ is a constant by definition. Consequently, Hubble's law $\vec{v} = H\vec{r}$ tells us that the proportionality constant, the Hubble parameter, should be identified as

$$H = \frac{\dot{a}}{a}, \tag{5.4}$$

and the value as measured today can be denoted with a subscript '0' as H_0. Because we measure Hubble's constant to be positive rather than negative, we know that the Universe is expanding rather than contracting.

We notice from this that the phrase Hubble's *constant* is a bit misleading. Although certainly it is constant in space due to the cosmological principle, there is no reason for it to be constant in time. In fact, using it as a more compact notation, we can write the Friedmann equation as an evolution equation for $H(t)$, as

$$H^2 = \frac{8\pi G}{3}\,\rho - \frac{k}{a^2}\,. \tag{5.5}$$

It is best to use the phrase 'Hubble parameter' for this quantity as a function of time, reserving 'Hubble constant' for its present value. Normally the Hubble parameter decreases with time, for instance as the expansion is slowed by the gravitational attraction of the matter in the Universe.

5.2 Expansion and redshift

The redshift of spectral lines that we used to justify the assumption of an expanding universe can also be related to the scale factor. In this derivation I'll make the simplifying assumption that light is passed between two objects which are very close together, separated by a small distance dr, as shown in Figure 5.1. I've drawn the objects as galaxies, but I really mean two nearby points. According to Hubble's law, their relative velocity dv will be

$$dv = H\,dr = \frac{\dot{a}}{a}\,dr\,. \tag{5.6}$$

As the points are nearby we can directly apply the Doppler law to say that the change in wavelength between emission and reception, $d\lambda \equiv \lambda_{\rm r} - \lambda_{\rm e}$, is

$$\frac{d\lambda}{\lambda_{\rm e}} = \frac{dv}{c}\,, \tag{5.7}$$

where $d\lambda$ is going to be positive since the wavelength is increased. The time between emission and reception is given by the light travel time $dt = dr/c$, and putting all that

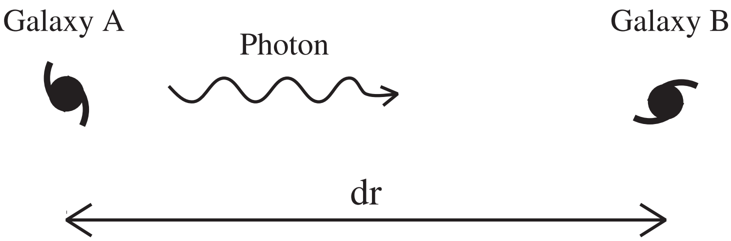

Figure 5.1 A photon travels a distance dr between two galaxies A and B.

together gives

$$\frac{d\lambda}{\lambda_{\rm e}} = \frac{\dot{a}}{a}\frac{dr}{c} = \frac{\dot{a}}{a}dt = \frac{da}{a}\,. \tag{5.8}$$

Integrate and we find that $\ln \lambda = \ln a + \text{constant}$, i.e.

$$\lambda \propto a\,, \tag{5.9}$$

where λ is now the instantaneous wavelength measured at any given time.

Although as I've derived it this result only applies to objects which are very close to each other, it turns out that it is completely general (a rigorous treatment is given in Advanced Topic 2). It tells us that as space expands, wavelengths become longer in direct proportion. One can think of the wavelength as being stretched by the expansion of the Universe, and its change therefore tells us how much the Universe has expanded since the light began its travels. For example, if the wavelength has doubled, the Universe must have been half its present size when the light was emitted.

The redshift as defined in Equation (2.1) is related to the scale factor by

$$1 + z = \frac{\lambda_{\rm r}}{\lambda_{\rm e}} = \frac{a(t_{\rm r})}{a(t_{\rm e})}\,. \tag{5.10}$$

and is normally only used to refer to light received by us at the present epoch.

5.3 Solving the equations

In order to discover how the Universe might evolve, we need some idea of what is in it. In a cosmological context, this is done by specifying the relationship between the mass density ρ and the pressure p. This relationship is known as the **equation of state**. At this point, we shall only consider two possibilities.

Matter: In this context, the term 'matter' is used by cosmologists as shorthand for 'non-relativistic matter', and refers to any type of material which exerts negligible pressure, $p = 0$. Occasionally care is needed to avoid confusion between 'matter' used in this sense, and used to indicate all types of matter whether non-relativistic or not. A pressureless universe is the simplest assumption that can be made. It is a good approximation to use for the atoms in the Universe once it has cooled down, as they are quite well separated and seldom interact, and it is also a good description of a collection of galaxies in the Universe, as they have no interactions other than gravitational ones. Occasionally the term 'dust' is used instead of 'matter'.

Radiation: Particles of light move, naturally enough, at the speed of light. Their kinetic energy leads to a pressure force, the radiation pressure, which using the standard theory of radiation can be shown to be $p = \rho c^2/3$. Problem 5.2 gives a rather hand-waving derivation of this result. More generally, any particles moving at highly-relativistic speeds have this equation of state, neutrinos being an obvious example.

I will concentrate on the case where the constant k in the Friedmann equation is set equal to zero, corresponding to a flat geometry.

5.3.1 Matter

We start by solving the fluid equation, having set $p = 0$ for matter. One way to solve it is to notice a clever way of rewriting it, as follows

$$\dot{\rho} + 3\frac{\dot{a}}{a}\rho = 0 \quad \Longrightarrow \quad \frac{1}{a^3}\frac{d}{dt}\left(\rho a^3\right) = 0 \quad \Longrightarrow \quad \frac{d}{dt}\left(\rho a^3\right) = 0\,, \tag{5.11}$$

though one could also solve it more formally by noting that it is a separable equation. Integrating tells us that ρa^3 equals a constant, i.e.

$$\rho \propto \frac{1}{a^3}\,. \tag{5.12}$$

This is not a surprising result. It says that the density falls off in proportion to the volume of the Universe. It is very natural that if the volume of the Universe increases by a factor of say two, then the density of the matter must fall by the same factor. After all, material cannot come from nowhere, and there is no pressure to do any work.

The equations we are solving (with $k = 0$) have one very useful symmetry; their form is unchanged if we multiply the scale factor a by a constant, since only the combination $\dot{a}/a$ appears. This means that we are free to rescale $a(t)$ as we choose, and the normal convention is to choose $a = 1$ at the present time. With this choice physical and comoving coordinate systems coincide at the present, since $\vec{r} = a\,\vec{x}$. Throughout this book I will use the subscript '0' to indicate the present value of quantities. Denoting the present density by ρ_0 fixes the proportionality constant

$$\rho = \frac{\rho_0}{a^3}\,. \tag{5.13}$$

Having solved for the evolution of the density in terms of a, we must now find how a varies with time by using the Friedmann equation. Substituting in for ρ, and remembering we are assuming $k = 0$, gives

$$\dot{a}^2 = \frac{8\pi G\rho_0}{3}\frac{1}{a}\,. \tag{5.14}$$

Faced with an equation like this, one can use formal techniques to solve it (this equation is separable, allowing it to be integrated), or alternatively make an educated guess as to the solution and confirm it by substitution. In cosmology, a good educated guess is normally a power-law $a \propto t^q$. Substituting this in, the left-hand side has time dependence t^{2q-2} and the right-hand side t^{-q}. This can only be a solution if these match, which requires $q = 2/3$, and so the solution is $a \propto t^{2/3}$. As we have fixed $a = 1$ at the present time $t = t_0$, the full solution is therefore

$$a(t) = \left(\frac{t}{t_0}\right)^{2/3} \quad ; \quad \rho(t) = \frac{\rho_0}{a^3} = \frac{\rho_0 t_0^2}{t^2} \,. \tag{5.15}$$

In this solution, the Universe expands forever, but the rate of expansion $H(t)$ decreases with time

$$H \equiv \frac{\dot{a}}{a} = \frac{2}{3t} \,, \tag{5.16}$$

becoming infinitely slow as the Universe becomes infinitely old. Notice that despite the pull of gravity, the material in the Universe does not recollapse but rather expands forever.

This is one of the classic cosmological solutions, and will be much used throughout this book.

5.3.2 Radiation

Radiation obeys $p = \rho c^2/3$. Consequently the fluid equation is changed from the matter-dominated case, now reading

$$\dot{\rho} + 4\frac{\dot{a}}{a}\rho = 0 \,. \tag{5.17}$$

This is amenable to the same trick as before, with the a^3 replaced by a^4 in Equation (5.11), giving

$$\rho \propto \frac{1}{a^4} \,. \tag{5.18}$$

Carrying out the same analysis we did in the matter-dominated case gives

$$a(t) = \left(\frac{t}{t_0}\right)^{1/2} \quad ; \quad \rho(t) = \frac{\rho_0}{a^4} = \frac{\rho_0 t_0^2}{t^2} \,. \tag{5.19}$$

This is the second classic cosmological solution.

Notice that the Universe expands more slowly if radiation dominated than if matter dominated, a consequence of the extra deceleration that the pressure supplies – see Equation (3.18). So it is definitely wrong to think of pressure as somehow 'blowing' the Universe apart. However, in each case the density of material falls off as t^2.

We'd better examine the fall off of the radiation density with volume more carefully. It drops as the fourth power of the scale factor. Three of those powers we have already identified as the increase in volume, leading naturally to a drop in the density. The final power arises from a different effect, the stretching of the wavelength of the light. Since the stretching is proportional to a, and the energy of radiation proportional to its frequency via $E = hf$, this results in a further drop in energy by the remaining power of a. This lowering of energy is exactly the redshifting effect we use to measure distances.

The rate of decrease of the radiation density also has an explanation in terms of thermodynamics, which is macroscopic rather than microscopic. Since the Universe in this case has a pressure, when it expands there is work done which is given by $p\,dV$, in exactly the same way as work is done on a piston when the gas is allowed to expand and cool. This work done corresponds to the extra diminution of the radiation density by the final factor of a.

5.3.3 Mixtures

A more general situation is when one has a mixture of both matter and radiation. Then there are two separate fluid equations, one for each of the two components. The trick which allows us to write ρ as a function of a still works, so we still have

$$\rho_{\mathrm{mat}} \propto \frac{1}{a^3} \quad ; \quad \rho_{\mathrm{rad}} \propto \frac{1}{a^4} \,. \tag{5.20}$$

However, there is still only a single Friedmann equation (after all, there is only one Universe!), which now has

$$\rho = \rho_{\mathrm{mat}} + \rho_{\mathrm{rad}} \,. \tag{5.21}$$

This means that the scale factor will have a more complicated behaviour, and so to convert $\rho(a)$ into $\rho(t)$ is much harder. It is possible to obtain exact solutions for this situation, but they are quite messy so I won't include them here. Instead, I'll consider the simpler situation where one or other of the densities is by far the larger.

In that case, we can say that the Friedmann equation is accurately solved by just including the dominant component. That is, we can use the expansion rates we have already found. For example, suppose radiation is much more important. Then one would have

$$a(t) \propto t^{1/2} \quad ; \quad \rho_{\mathrm{rad}} \propto \frac{1}{t^2} \quad ; \quad \rho_{\mathrm{mat}} \propto \frac{1}{a^3} \propto \frac{1}{t^{3/2}} \,. \tag{5.22}$$

Notice that the density in matter falls off more slowly than that in radiation. So the situation of radiation dominating cannot last forever; however small the matter component might be originally it will eventually come to dominate. We can say that domination of the Universe by radiation is an unstable situation.

In the opposite situation, where it is the matter which is dominant, we obtain the solution

$$a(t) \propto t^{2/3} \quad ; \quad \rho_{\mathrm{mat}} \propto \frac{1}{t^2} \quad ; \quad \rho_{\mathrm{rad}} \propto \frac{1}{a^4} \propto \frac{1}{t^{8/3}} \,. \tag{5.23}$$

Matter domination is a stable situation, the matter becoming increasingly dominant over the radiation as time goes by.

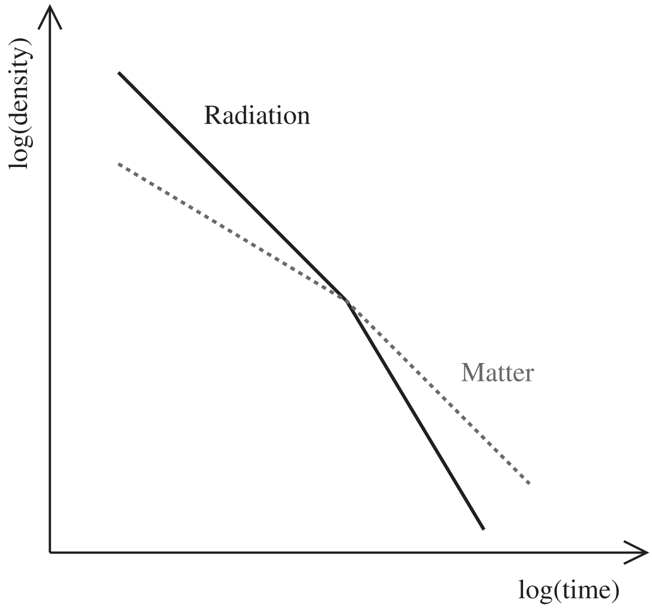

Figure 5.2 A schematic illustration of the evolution of a universe containing radiation and matter. Once matter comes to dominate the expansion rate speeds up, so the densities fall more quickly with time.

Figure 5.2 shows the evolution of a universe containing matter and radiation, with the radiation initially dominating. Eventually the matter comes to dominate, and as it does so the expansion rate speeds up from $a(t) \propto t^{1/2}$ to the $a(t) \propto t^{2/3}$ law. It is likely that this is the situation that applied in our Universe until fairly recently, as we'll see in Chapter 11.

5.4 Particle number densities

An important alternative view of the evolution of particles, which will be much used later in the book, is that of the **number density** n of particles rather than of their mass or energy density.

The number density is simply the number of particles in a given volume. If the mean energy per particle (including any mass–energy) is E, then the number density is related to the energy density by

$$\epsilon = n \times E . \tag{5.24}$$

The number density is useful because in most circumstances particle number is conserved. For example, if particle interactions are negligible, you wouldn't expect an electron to suddenly vanish into oblivion, and the same is true of a photon of light. The particle number can change through interactions, for example an electron and positron could annihilate and create two photons. However, if the interaction rate is high we expect the Universe to be in a state of thermal equilibrium. If so, then particle number is conserved even in a highly-interacting state, since by definition thermal equilibrium means that any interaction, which may change the number density of a particular type of particle, must proceed at the same rate in both forward and backward directions so that any change cancels out.

So, barring brief periods where thermal equilibrium does not hold, we expect the number of particles to be conserved. The only thing that changes the number *density*, therefore, is that the volume is getting bigger, so that these particles are spread out in a larger volume. This implies

$$n \propto \frac{1}{a^3}\,. \tag{5.25}$$

This looks encouragingly like the behaviour we have already seen for matter, but it's also true for radiation as well!

How does this relate to our earlier results? The energy of non-relativistic particles is dominated by their rest mass–energy which is constant, so

$$\rho_{\rm mat} \propto \epsilon_{\rm mat} \propto n_{\rm mat} \times E_{\rm mat} \propto \frac{1}{a^3} \times {\rm const} \propto \frac{1}{a^3}\,. \tag{5.26}$$

But photons lose energy as the Universe expands and their wavelength is stretched, so their energy is $E_{\rm rad} \propto 1/a$ as we have already seen. So

$$\rho_{\rm rad} \propto \epsilon_{\rm rad} \propto n_{\rm rad} \times E_{\rm rad} \propto \frac{1}{a^3} \times \frac{1}{a} \propto \frac{1}{a^4}\,. \tag{5.27}$$

These are exactly the results we saw before, equations (5.12) and (5.18).

Although the energy densities of matter and radiation evolve in different ways, their particle numbers evolve in the same way. So, apart from epochs during which the assumption of thermal equilibrium fails, the relative number densities of the different particles (e.g. electrons and photons) do not change as the Universe expands.

5.5 Evolution including curvature

We can now re-introduce the possibility that the constant k is non-zero, corresponding to spherical or hyperbolic geometry. Rather than seeking precise solutions, I will concentrate on the qualitative properties of the solutions. These are actually of rather limited use in describing our own Universe, because as we will see the cosmological models discussed so far are not general enough and we will need to consider a cosmological constant (see Chapter 7). Nevertheless, studying the possible behaviours of these simple models is a useful exercise, even if one should be cautious about drawing general conclusions.

In analysing the possible dynamics, I will assume that the Universe is dominated by non-relativistic matter always, which in practice is not a restrictive assumption. We have already seen that if we assume that the constant k in the Friedmann equation is zero, then the Universe expands for ever, $a \propto t^{2/3}$, but slows down arbitrarily at late times. So we know the fate of the Universe in that case. But what happens if $k \neq 0$?

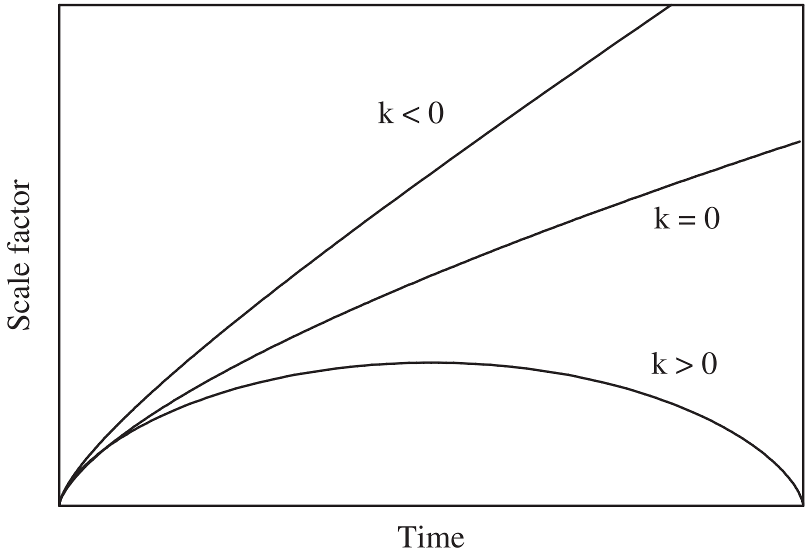

Figure 5.3 Three possible evolutions for the Universe, corresponding to the different signs of k. The middle line corresponds to the $k = 0$ case where the expansion rate approaches zero in the infinite future. During the early phases of the expansion the lines are very close and so observationally it can be difficult to distinguish which path the actual Universe will follow.

The principal question to ask is whether it is possible for the expansion of the Universe to stop, which since $H = \dot{a}/a$ corresponds to $H = 0$. Looking at the Friedmann equation

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}\,, \tag{5.28}$$

it is immediately apparent that this is not possible if k is negative, for then both the terms on the right-hand side of the Friedmann equation are positive. Consequently, such a universe must expand forever. That enables us to study the late-time behaviour, because we can see that the term k/a^2 falls off more slowly with the expansion than does $\rho_{\mathrm{mat}} \propto 1/a^3$. Since a becomes arbitrarily large for the matter-dominated solution for negligible k, the k/a^2 term must eventually come to dominate. When it does, the Friedmann equation becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = -\frac{k}{a^2}\,. \tag{5.29}$$

Cancel off the a^2 terms and you'll find the solution is $a \propto t$. So when the last term comes to dominate, the expansion of the Universe becomes yet faster. In this case, the velocity does not tend to zero at late times, but instead becomes constant. This is sometimes known as free expansion.

Things are very different if k is positive. It then becomes possible for H to be zero, by the two terms on the right-hand side of the Friedmann equation cancelling each other out. Indeed, this is inevitable, because the negative influence of the k/a^2 term will become more and more important relative to the $\rho_{\rm mat}$ term as time goes by. In such a universe therefore, the expansion must come to an end after a finite amount of time. As gravitational attraction persists, the recollapse of the Universe becomes inevitable.

The collapse of the Universe is fairly easy to describe, because the equations governing the evolution are time reversible. That is, if one substitutes $-t$ for t, they remain the same. The collapse phase is therefore just like the expansion in reverse, and so after a finite time the Universe will come to an end in a Big Crunch. Problem 5.5 investigates this in more detail.

These three behaviours, illustrated in Figure 5.3, can be related to the particle energy U in our Newtonian derivation of the Friedmann equation. If the particle energy is positive, then it can escape to infinity, with a final kinetic energy given by U. If the total energy is zero, then the particle can just escape but with zero velocity. Finally, if the energy is negative, it cannot escape the gravitational attraction and is destined to recollapse inwards.

There is a fairly precise analogy with escape velocity from the Earth (or the Moon, if you want to worry about the atmosphere). If you throw a rock up in the air hard enough, gravity will be unable to stop it and eventually it will sail off into space at a constant velocity. If your throw is too puny, it will rapidly fall back. And in between is the escape velocity, where the rock is just able to escape the gravitational field and no more.

Problems

5.1. Is the total energy of the Universe conserved as it expands?

5.2. This problem indicates the origin of the equation of state $p = \rho c^2/3$ for radiation. An ideal gas has pressure

$$p = \frac{1}{3} n \langle \mathbf{v}.\mathbf{p} \rangle \, ,$$

where $\langle \cdots \rangle$ indicates an average over the direction of particle motions. Here n is the number density, and be careful not to confuse the unfortunate notation p for pressure and $\mathbf{p}$ for momentum. Using Equation (2.4) to relate the photon energy and momentum, show that this gives

$$p = \frac{1}{3} n \langle E \rangle \, ,$$

where $\langle E \rangle$ is the mean photon energy. Hence demonstrate the equation of state for radiation.

5.3. During this chapter we examined solutions for the expansion when the Universe contained either matter ($p = 0$) or radiation ($p = \rho c^2/3$). Suppose we have a more general equation of state, $p = (\gamma - 1)\rho c^2$, where γ is a constant in the range $0 < \gamma < 2$. Find solutions for $\rho(a)$, $a(t)$ and hence $\rho(t)$ for universes containing such matter. Assume $k = 0$ in the Friedmann equation.

What is the solution if $p = -\rho c^2$?

5.4. Using your answer to Problem 5.3, what value of γ would be needed so that ρ has the same time dependence as the curvature term k/a^2? Find the solution $a(t)$ to the full Friedmann equation for a fluid with this γ, assuming negative k.

5.5. The full Friedmann equation is

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\,\rho - \frac{k}{a^2}\,.$$

Consider the case $k > 0$, with a universe containing only matter ($p = 0$) so that $\rho = \rho_0/a^3$. Demonstrate that the parametric solution

$$a(\theta) = \frac{4\pi G\rho_0}{3k}\,(1 - \cos\theta) \quad ; \quad t(\theta) = \frac{4\pi G\rho_0}{3k^{3/2}}\,(\theta - \sin\theta)$$

solves this equation, where θ is a variable which runs from 0 to 2π.

Sketch a and t as functions of θ. Describe qualitatively the behaviour of the Universe. Attempt to sketch a as a function of t.

5.6. Now consider the case $k < 0$, with a universe containing only matter ($p = 0$) so that $\rho = \rho_0/a^3$. What is the solution $a(t)$ in a situation where the final term of the Friedmann equation dominates over the density term? How does the density of matter vary with time? Is domination by the curvature term a stable situation that will continue forever?