The Universe Across Scales

Fast and Small
Patterns in the Middle
Slow and Big

The Universe Across Scales

Fast and Small - Subhadip Mitra

The Special Theory of Relativity and the Elementary World

Slow and Big - Diganta Das

Cosmology

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Statistical Mechanics

Fast and Small

Special Theory of Relativity

What is Relativity?

"The motions of bodies included in a given space are the same among themselves, whether that space is at rest or moves uniformly forward in a straight line."

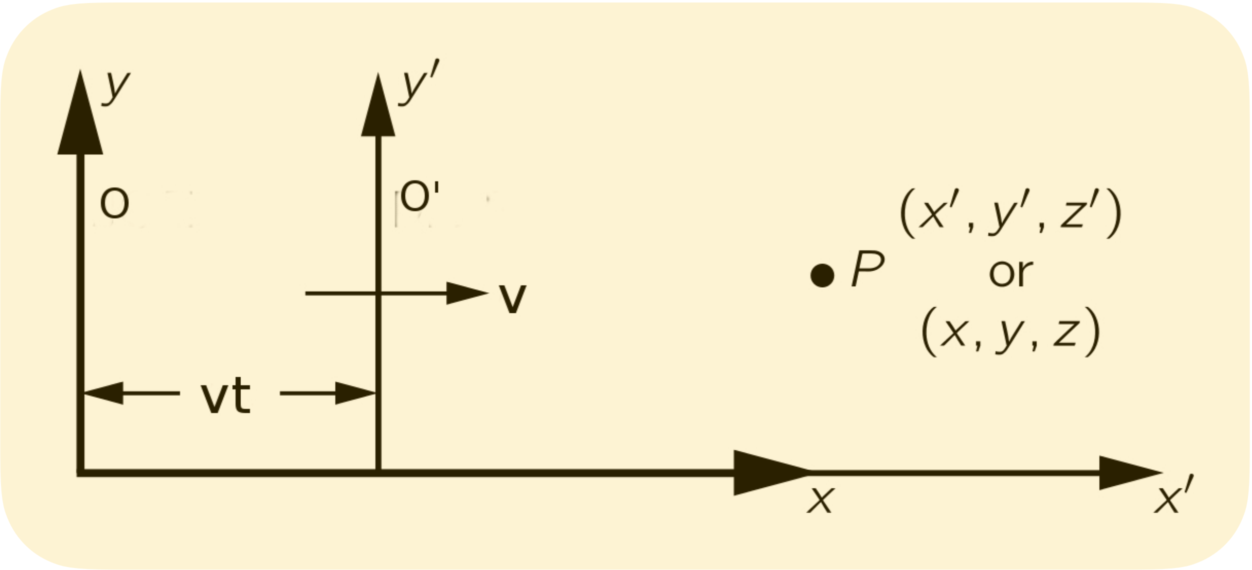
Newton, in one of his corollaries to the laws of motion (the idea goes back to Galileo).

If you suddenly wake up in a spaceship, you won't be able to decide (by performing any mechanical experiment) whether the ship is drifting along at a uniform speed or standing in a port without looking outside.

This is not true if the ship is accelerating. We can tell without looking outside when a train suddenly starts moving as we all feel a force pulling us back

This lets us formulate the laws of mechanics without referring to any particular frame of reference.

What is Relativity?



$$x' = x - v\,t\,, \quad y' = y\,, \quad z' = z\,, \quad t' = t\,.$$

$$dx'/dt' = dx/dt - v, \quad dy'/dt' = dy/dt, \quad dz'/dt' = dz/dt\,.$$

The law of inertia works in both frames: inertial frames

$$d^2x'/dt'^2 = d^2x/dt^2, \quad d^2y'/dt'^2 = d^2y/dt^2, \quad d^2z'/dt'^2 = d^2z/dt^2\,.$$

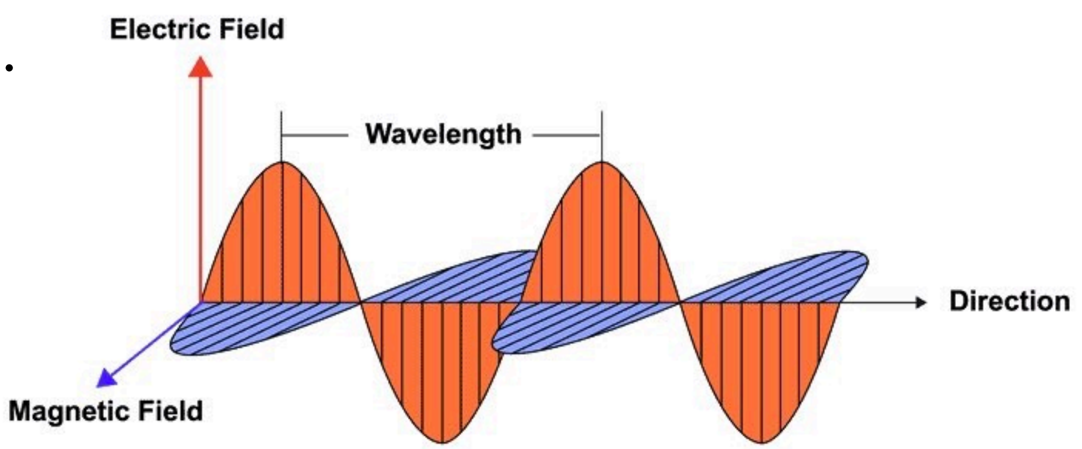
Acceleration (hence, force) is not affected

Troubling Light

In free space, the electromagnetic wave equation (that describes the propagation of light and follow directly from Maxwell's equations) can be written either in terms of the electric field $\vec{E}$ or magnetic field $\vec{B}$ as (taking the x-axis along the direction of propagation):

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E} = 0$$

where its speed $c = 1/\sqrt{\mu_0\epsilon_0} = 3.0 \times 10^8$m/s is fixed by the permeability (μ_0) and the permittivity (ϵ_0) of free space.



Which frame?

$$\left(\left(1 - \frac{v^2}{c^2}\right)\frac{\partial^2}{\partial x'^2} + \frac{2v}{c^2}\frac{\partial}{\partial x'}\frac{\partial}{\partial t'} - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2}\right)\vec{E}' = 0$$

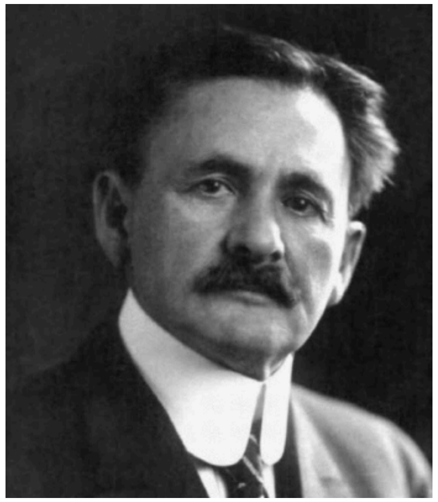
A wave of speed $c \pm v$

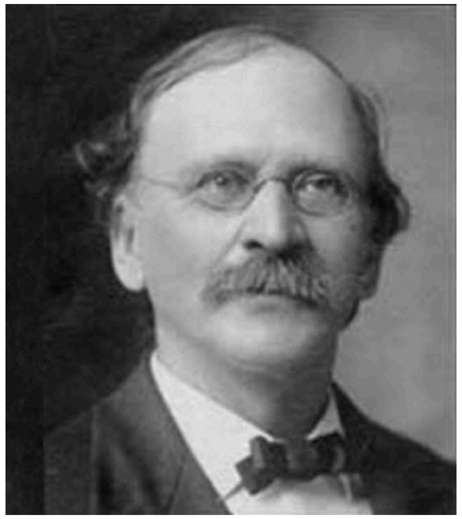
Those days, people thought that space is filled with something called "æther" through which light moves (just like other waves require mediums for propagation), so the wave equation must be valid in this æther rest frame.

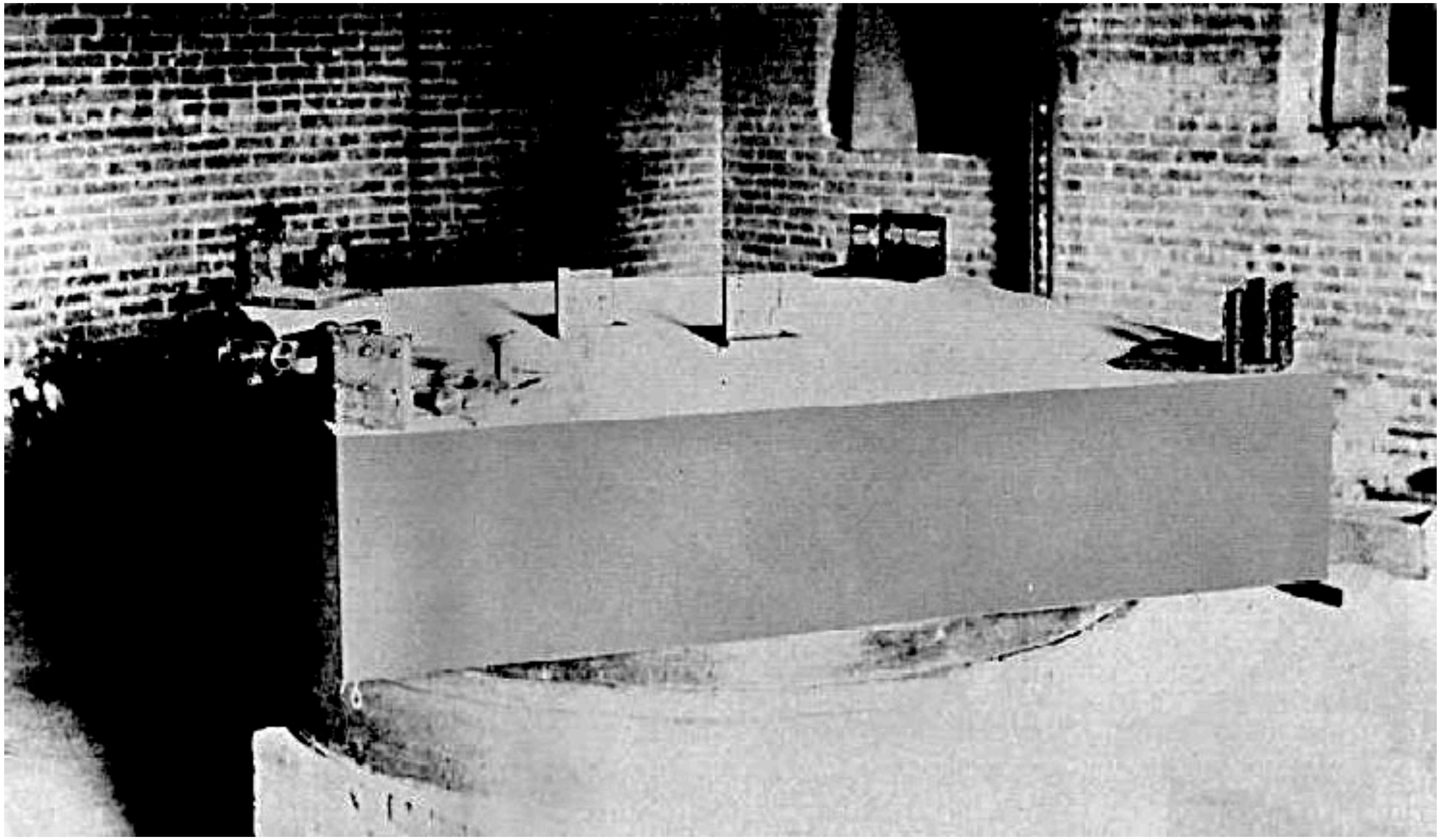
Michelson-Morley Experiment

Since the earth is moving through æther, some people tried to measure the speed of the earth w.r.t. the æther rest frame.

The most famous of these attempts was the experiment performed by Albert Michelson and Edward Morley in 1887, exploiting the wave nature of the light.



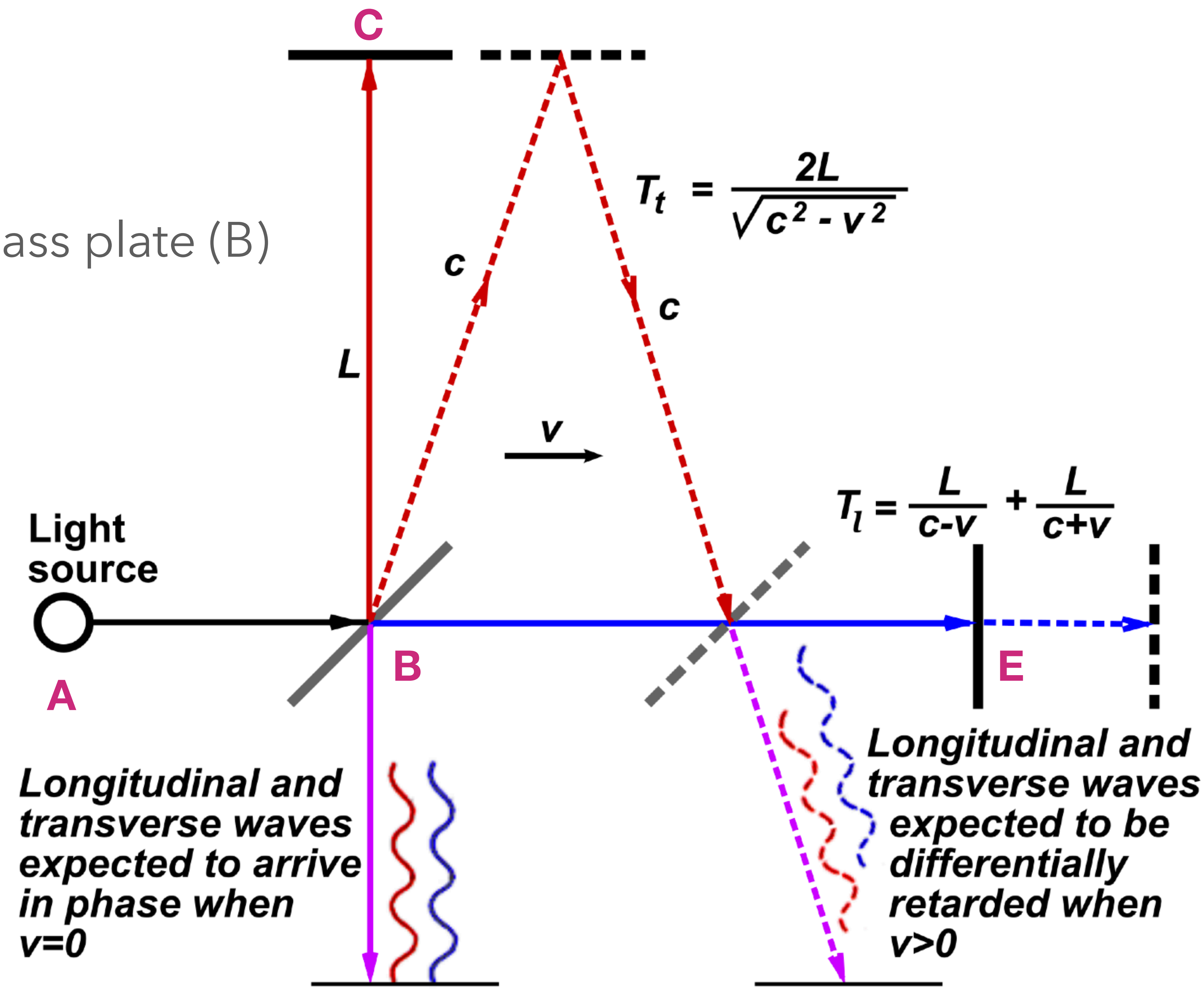




Michelson-Morley Experiment

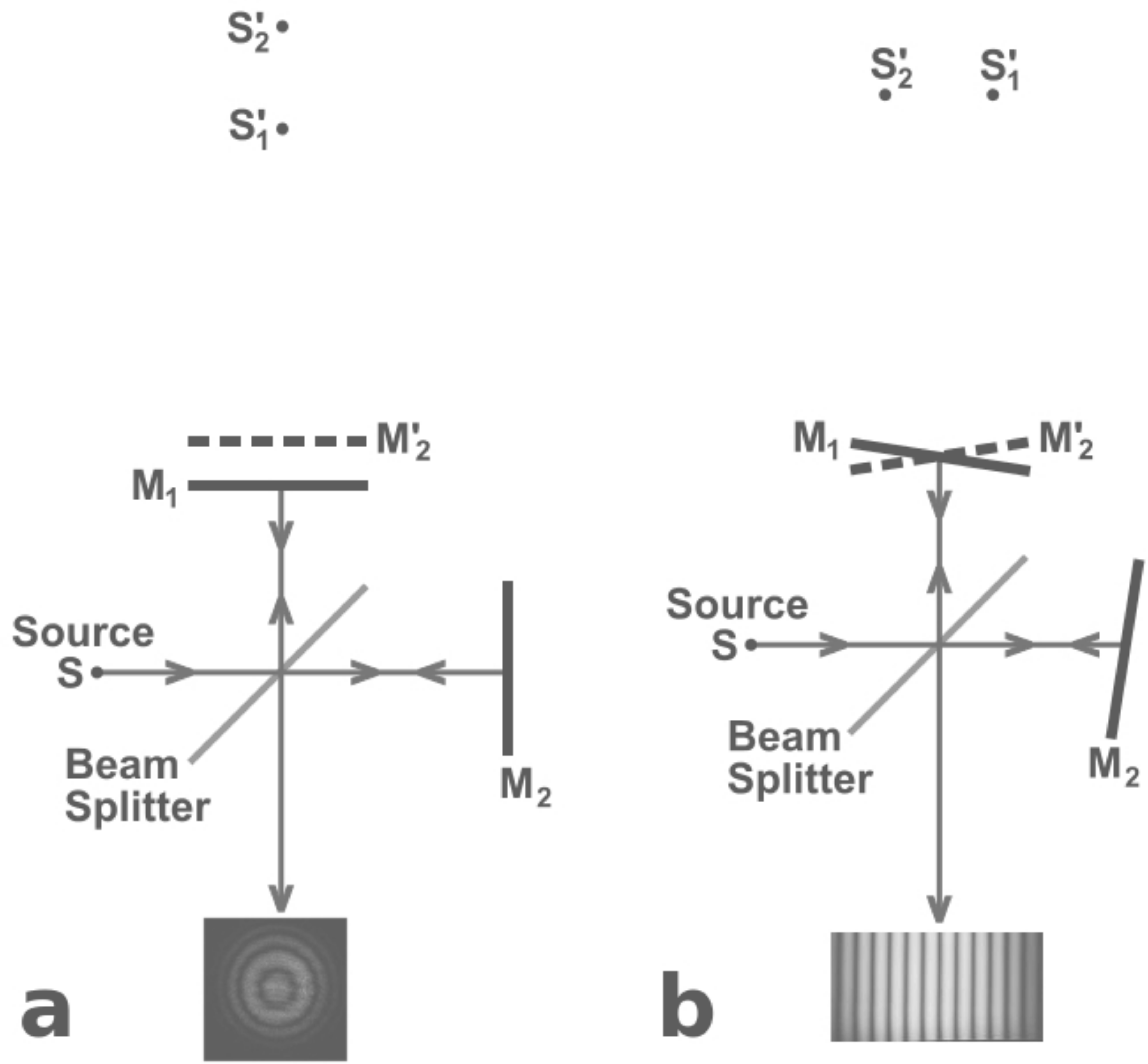
- ➡ A light source (A)
- ➡ A partially reflective glass plate (B)
- ➡ Two mirrors (C and E)

The whole set-up was mounted on a horizontal base that could easily be rotated about a vertical axis. The mirrors are placed at equal distances (L) from B.

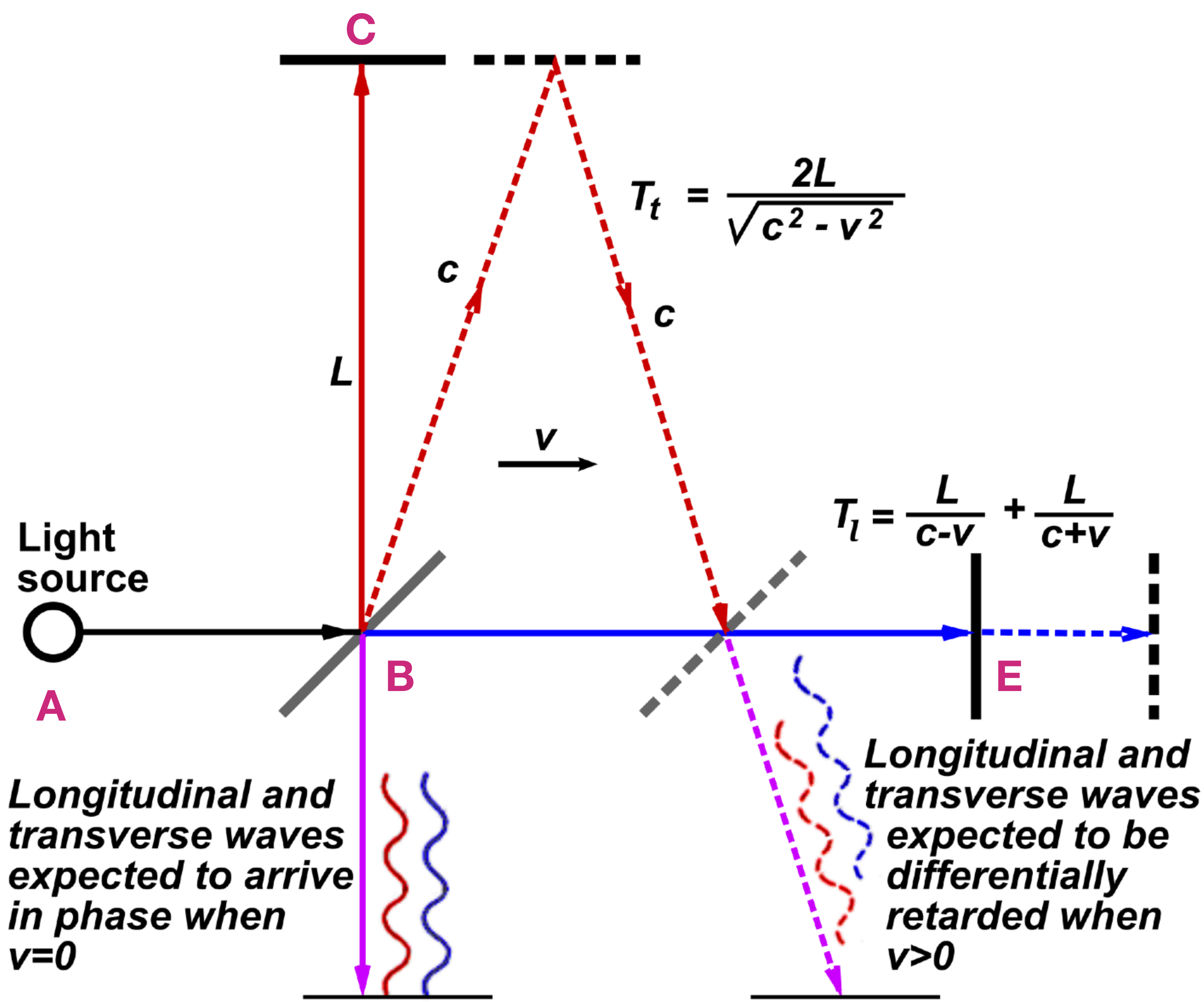


Michelson-Morley Experiment

The mirrors are kept slightly tilted so that the two beams produce interference fringes on returning to B. There will be a shift in the fringe pattern depending on whether the time taken for the light to go from B to E and back (the "||" direction) is the same as the time from B to C and back (the "⊥" direction) or not.



Michelson-Morley Experiment

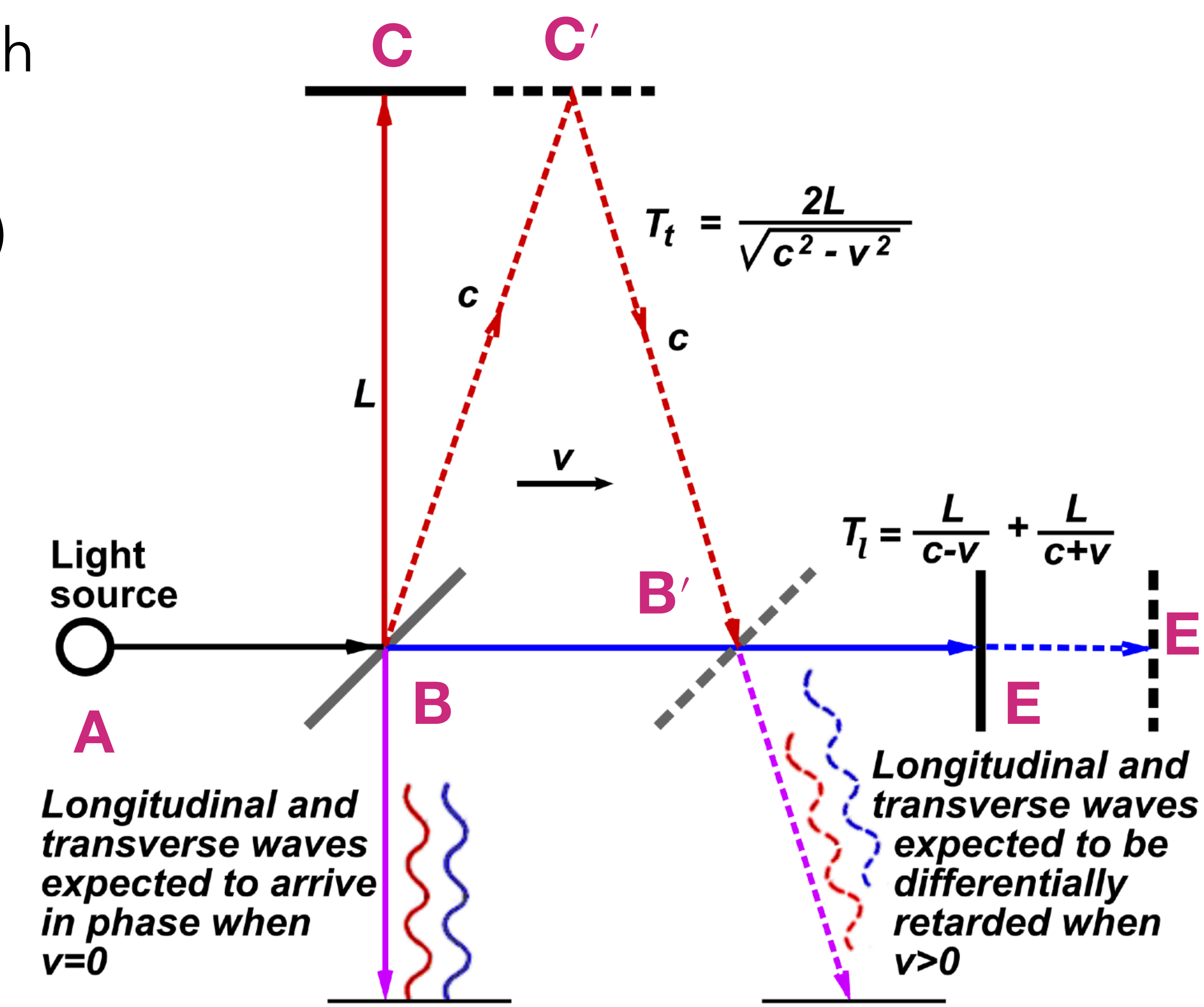


If the earth is at rest in the æther, the times should be precisely equal, but if it is moving toward the right with a velocity v, there should be a difference in the times.

Calculate $T_{||}$

In the rest frame of æther, light moves with speed c but since the apparatus is also moving with velocity v, by the time (say t_1) light reaches the mirror E, it shifts by a distance vt_1 in the same direction so that light has to cover a distance $ct_1 = L + vt_1$.

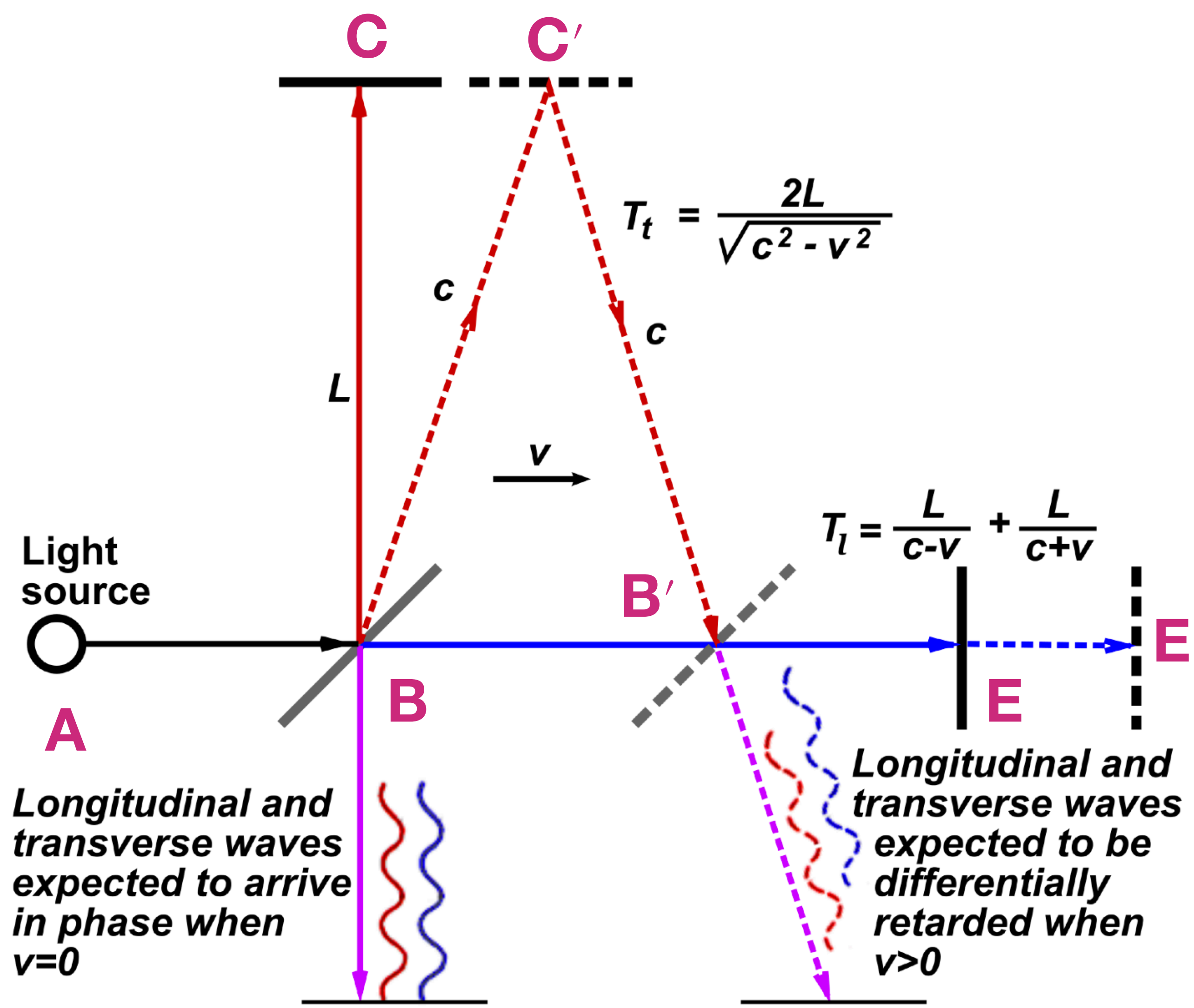
Similarly, while coming back, light takes time t_2 to reach B (now at B′) and has to cover a distance $ct_2 = L - vt_2$.



Hence, the total time taken for light to go from B to E and back is

$$T_{||} = t_1 + t_2 = \frac{L}{c - v} + \frac{L}{c + v}.$$

Calculate $T_{||}$

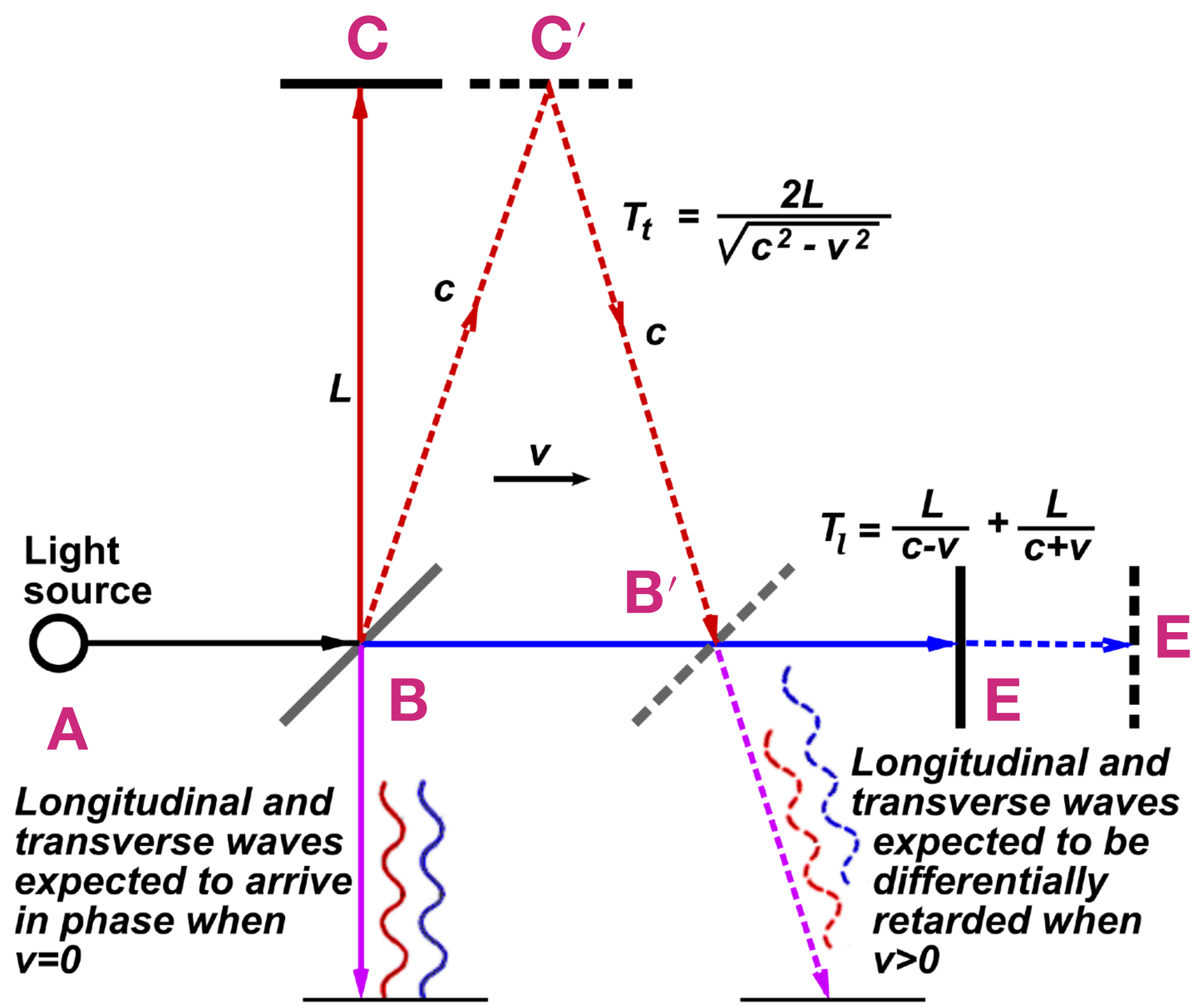


From the lab frame perspective, however, light will go from B to E with a speed of $c - v$ and return with $c + v$ but the total distance will remain $2L$. Hence, we recover the same result,

$$T_{||} = \frac{L}{c-v} + \frac{L}{c+v} = \frac{2cL}{c^2 - v^2} = \frac{2L/c}{1-\beta^2}$$

where $\beta = v/c$.

Calculate $T_\perp$



By the time light reaches C (say t_0), C moves by a distance vt_0 to reach C′ and when it comes back to B, it moves by $2vt_0$ to B′. So instead of $2L$ light has to cover some extra path along the $\perp$ direction.

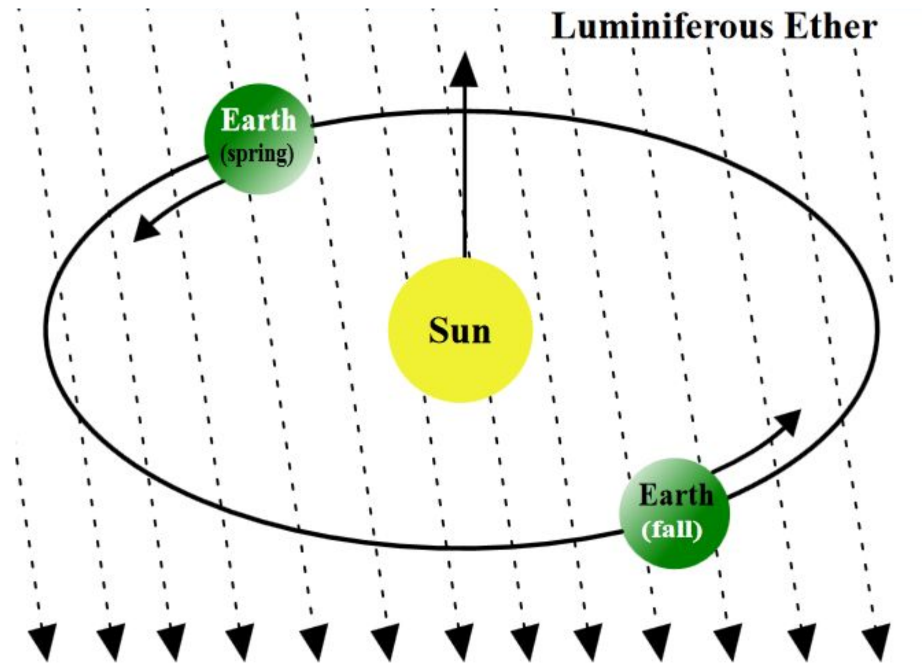
Simple geometry tells us

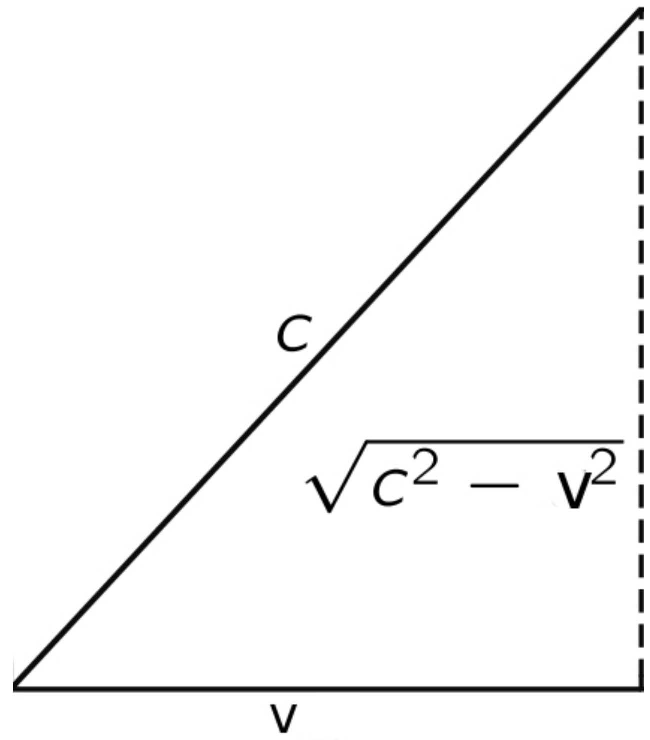
$$(ct_0)^2 = (vt_0)^2 + L^2 \Rightarrow \qquad T_\perp = 2t_0 = \frac{2L}{\sqrt{c^2 - v^2}} = \frac{2L/c}{\sqrt{1 - \beta^2}}$$

Calculate $T_\perp$

To arrive at this result in the lab frame first notice that the situation is similar to when a man is swimming perpendicularly across a river that is flowing with a speed v.

Since the earth is moving through æther there is an "æther wind" present in the lab frame in the opposite direction. Let's assume in the lab frame light moves with speed $v_\perp$ from B to C and back. But to cancel the drag of the æther wind there must be a || component to its velocity, $v_{||} = v$. So, we arrive at the same answer:





$$v_\perp^2 + v^2 = c^2 \Rightarrow \qquad T_\perp = \frac{2L}{v_\perp} = \frac{2L}{\sqrt{c^2 - v^2}} = \frac{2L/c}{\sqrt{1-\beta^2}}$$

Expectation

	Æther Frame (S)	Lab Frame (S')
$T_{\perp}$	$\dfrac{2L}{\sqrt{c^2 - v^2}}$	$= \dfrac{2L/c}{\sqrt{1 - \beta^2}}$
$T_{\vert\vert}$	$\dfrac{2cL}{c^2 - v^2}$	$= \dfrac{2L/c}{1 - \beta^2}$

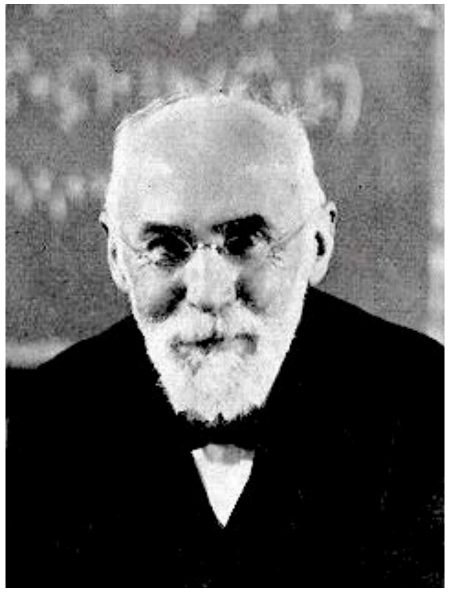
The Failed Experiment

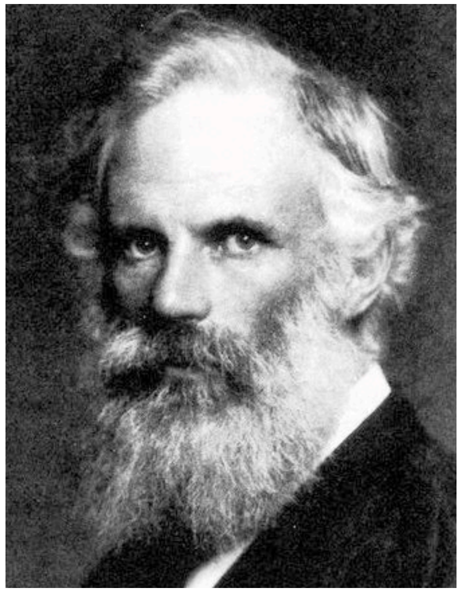
We can compute the difference in the travel times

$$\Delta T = T_{\parallel} - T_{\perp} = \frac{2L/c}{1-\beta^2} - \frac{2L/c}{\sqrt{1-\beta^2}} = \frac{L}{c}\beta^2 + \mathcal{O}\left(\beta^4\right).$$

Michelson and Morley oriented their apparatus so that the line BE was nearly parallel to the earth's motion in its orbit (in some particular times). It was sufficiently sensitive to observe any “æther wind” of the order of the earth's orbital speed. Still, no time difference was found even after many repetitions of the experiment at different times. The experiment became probably the most famous null experiment.

FitzGerald-Lorentz Hypothesis - Length Contraction





George FitzGerald (1889) and Hendrik Lorentz (1892) proposed an interesting resolution of the puzzle (now known as Lorentz contraction)—all objects contract by a factor $\sqrt{1-v^2/c^2}$ when they are moving with speed v, and this contraction is only in the direction of the motion. Suppose we have a stick of length L_0 measured at rest, its length will become

$$L_0 \quad \rightarrow \quad L_0\sqrt{1-v^2/c^2} = L_0\sqrt{1-\beta^2}$$

when it moves with speed v parallel to its length.

Then, in the æther rest frame,

$$T_\parallel \;=\; \frac{L_\parallel}{c-v} + \frac{L_\parallel}{c+v} \;=\; \frac{2L/c}{\sqrt{1-\beta^2}} \;=\; T_\perp$$

Later, people realised that this argument does not really need æther.

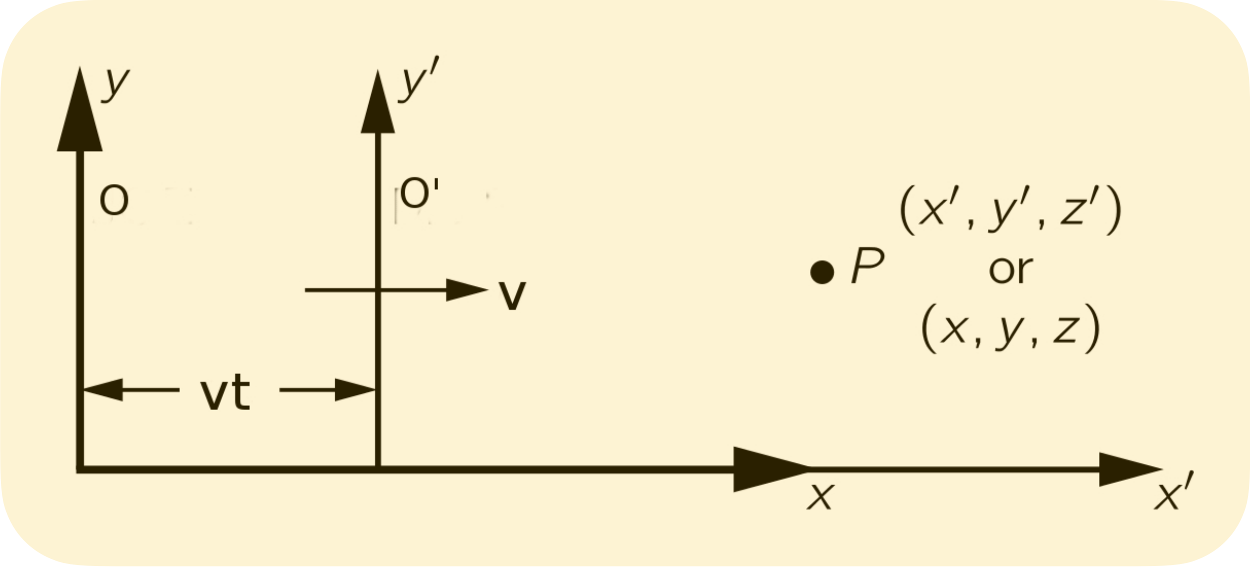
Lorentz Transformations

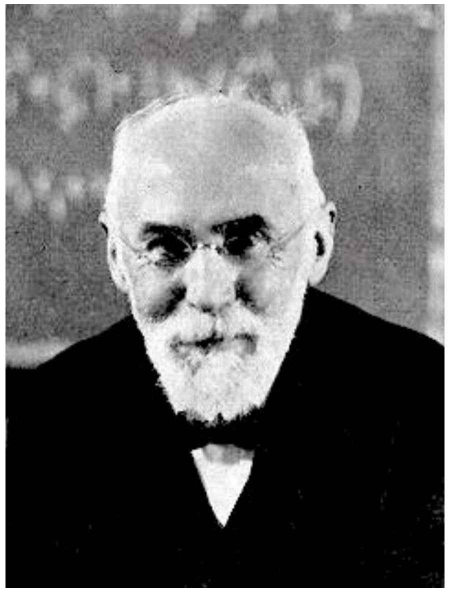
Lorentz and Henri Poincaré arrived at a set of transformation rules (among O and O′ coordinate systems) that preserves the Maxwell Equations (and in turn the electromagnetic wave equations):

$$x' = \gamma(x - vt)\ , \quad y' = y\ , \quad z' = z\ , \quad t' = \gamma\left(t - \frac{xv}{c^2}\right)$$

where $\gamma = 1/\sqrt{1-\beta^2} = 1/\sqrt{1 - v^2/c^2}$ is the Lorentz factor. They can be easily inverted by changing $v \to -v$ as O is moving with same speed v but in the opposite direction w.r.t. O′.

$$x = \gamma(x' + vt')\ , \quad y = y'\ , \quad z = z'\ , \quad t = \gamma\left(t' + \frac{x'v}{c^2}\right)$$





Lorentz Transformations

The presence of the γ factor indicates that length contraction is inbuilt in Lorentz transformations.

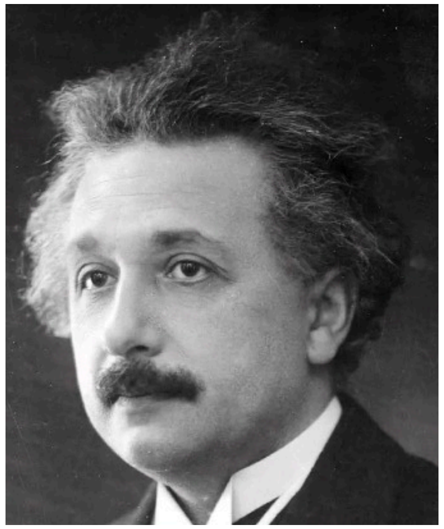
Suppose an insect is sitting in the O′ frame at some distance away along the x'-coordinate. The observer in O asks the observer in O′ to measure how far the insect is sitting from the origin of her coordinate system.

The observer in O′ starts from its origin, puts a measuring scale x' times to arrive at the insect (which then flies away of course!) and declares the distance to be x'.

However, to the observer in O it appears that the measuring scale she is using has shortened by a factor $1/\gamma = \sqrt{1 - v^2/c^2}$ and so the length x' is actually x'/γ and by the time (t) she reaches the insect, her origin has shifted by a distance vt. So he concludes that if he were to mark the point from where the insect flew off, it would be $x = x'/\gamma + vt$ distance away from his origin. This is noting but the x-relation of the Lorentz transformations: $x = x'/\gamma + vt \Rightarrow x' = \gamma(x - vt)$.

Notice that $t' \neq t$, i.e., time can change from observer to observer.

Special Theory of Relativity



Is relativity applicable only to the laws of mechanics?
What about Maxwell's electrodynamics?

If you move a wire loop through the field of a fixed magnet, then a current is generated by the motion of the loop. But if you see this from the loop rest frame, you'll still see the same current but think that a changing magnetic field is generating it. So, the effect remains the same, but the interpretation is relative.

These observations and the failure to find any absolute æther frame prompted Einstein to write down the postulates of the Special Theory of Relativity (1905):

1. **Principle of relativity:** The laws of physics are the same in all inertial frames of reference.
2. **Invariance of c:** The speed of light in free space has the same value, c, in all inertial frames of reference.

Special Theory of Relativity

Physics laws: With these, he promoted the Newtonian idea of relativity to include Maxwell's electrodynamics.

The invariance of c means there is no sense in defining an absolute “æther rest frame” anymore, and Galilean transformations are no longer the correct transformation rules as they do not support an “absolute” speed of any object.

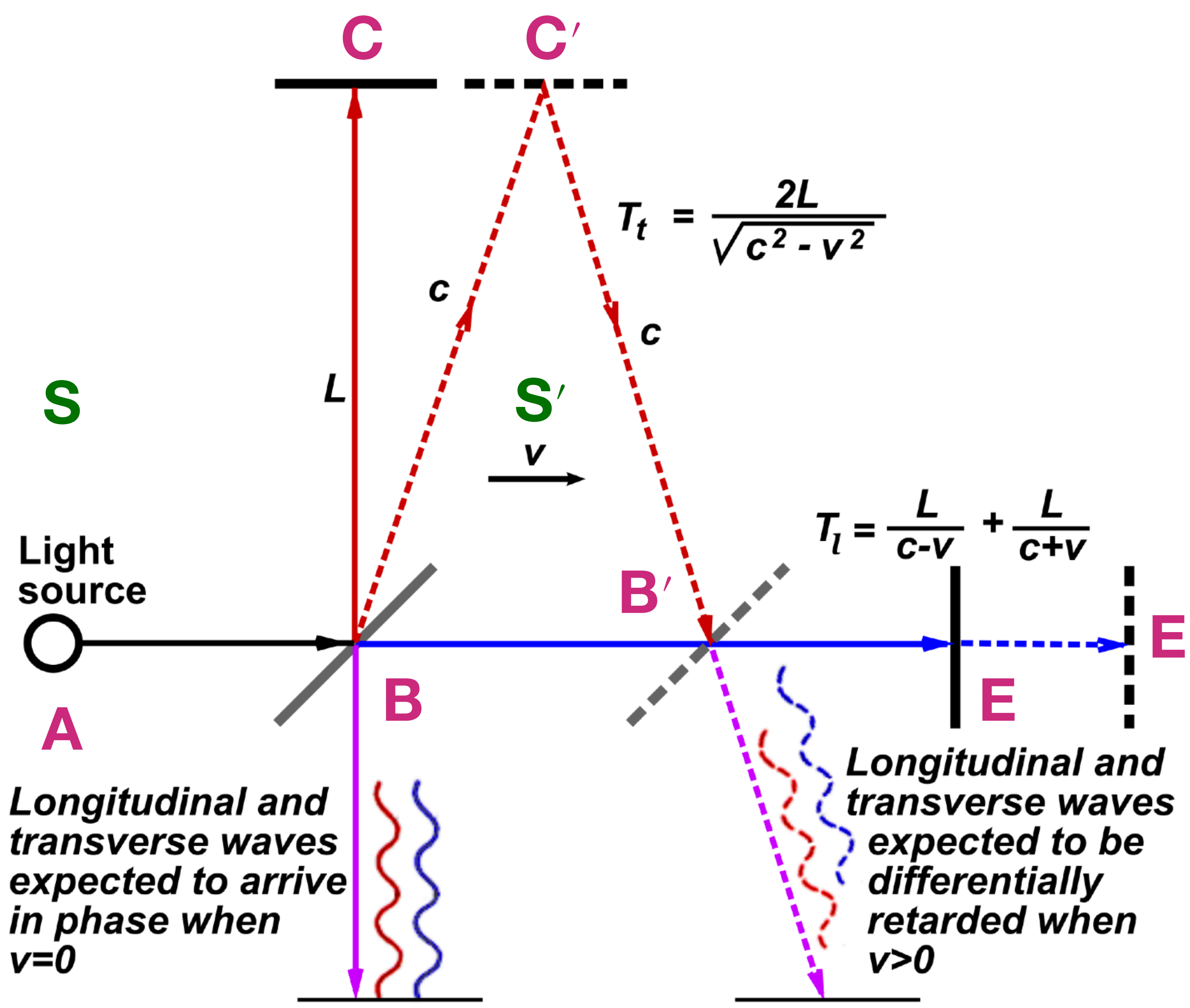
It is not true that there is nothing absolute in the Special Theory of Relativity—the “absolute rest frame” of Newtonian relativity is replaced by the “absolute velocity of light in vacuum” in any inertial frame—it is not relative.

However, since Maxwell's equations do not change under Lorentz transformations, with the introduction of these postulates, Lorentz transformations replaced the Galilean ones.

Time Changes

It is clear that $T_\perp$ obtained in the "æther frame" does not depend on the existence of æther in any way and must be correct as the speed of light was set to c in that frame.

We call it an inertial frame (S) in which the apparatus moves with velocity v. However, when computing $T_\parallel$, we must consider Lorentz contraction of the arm BE. Once that is done, we get the same time in both "||" and "⊥" directions.



$$T_S = T_\parallel\Big|_S = T_\perp\Big|_S = \frac{2L/c}{\sqrt{1-\beta^2}} = \gamma\, 2L/c$$

What happens in the lab frame? In the lab frame (S') length of BE and BC remain unchanged (L) and light moves with speed c in both directions. We get,

$$T_{S'} = T_\parallel\Big|_{S'} = T_\perp\Big|_{S'} = 2L/c$$

After Einstein

	Æther Frame (S)	Lab Frame (S')
$T_\perp$	$\dfrac{2L/c}{\sqrt{1-\beta^2}} = \gamma\dfrac{2L}{c}$	$\dfrac{2L}{c}$
$T_{\|\|}$	$\dfrac{2L/c}{\sqrt{1-\beta^2}} = \gamma\dfrac{2L}{c}$	$\dfrac{2L}{c}$

So in both frames, $T_{||} = T_\perp$ (null result is explained), but the two frames disagree about the actual time passed for light to go from point B and come back.

The difference is tiny if $v \ll c$: $\gamma = \dfrac{1}{\sqrt{1-\beta^2}} \approx 1 + \dfrac{1}{2}\dfrac{v^2}{c^2} + \dfrac{3}{8}\dfrac{v^4}{c^4} + \cdots$

Time Dilation

This is not completely unexpected as time changes from frame to frame under the Lorentz transformations and this is a consequence of that.

If you attach two clocks with two frames, they will differ just as the measuring scale contracted when viewed from the other frame. The time measured in the lab frame ($T_{S'}$) is also called the proper (own) time for the events (light coming at B, reflecting from C or E and coming back to B) are happening are at rest w.r.t. the clock.

Whereas viewed from the frame S, the time taken for the same set of events will appear larger as $T_S > T_{S'}$.

If we generate light pulses from source A and say the time light takes to go from B to C (or E) and back is the new unit time, then the apparatus will effectively behave as a clock.

Since $T_S > T_{S'}$, the time unit in S frame will be longer (dilated), so that's why people say "moving clock runs slow" and the whole effect is known as “time dilation”.

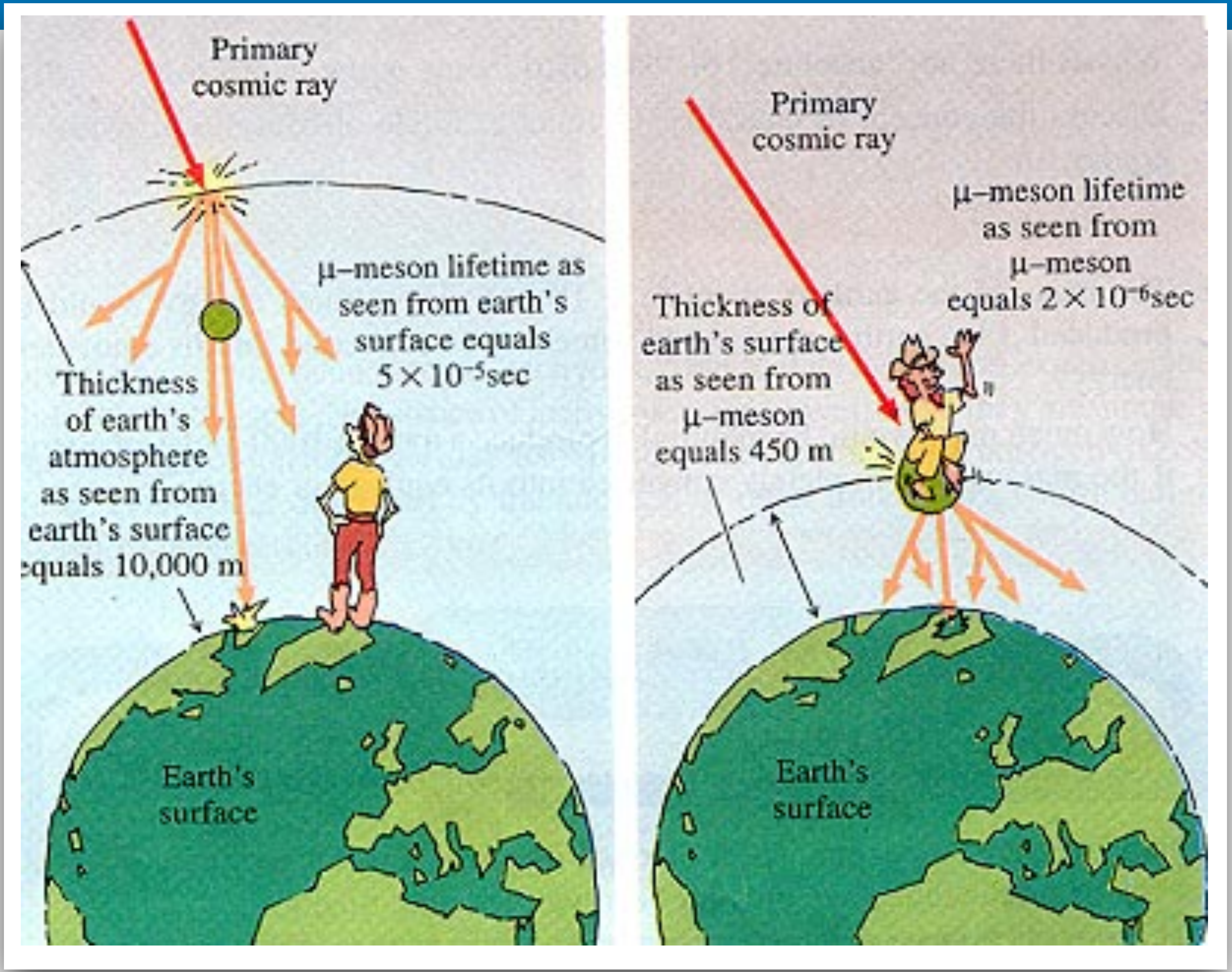
Time Dilation

There is nothing special about frame S or S'. If you observe some events in happening S rest frame, it will appear to take longer when observed from S'—after all, that's the principle of relativity.

This is not a mathematical trick, time dilation is as physical as the Lorentz contraction.

There are plenty of other experiments whose outcomes demonstrate time dilation. Consider the examples of muons (μ). They are generated in the outer layer of our atmosphere when cosmic rays hits the Earth's atmosphere. They move almost at the speed of light ($\sim 0.98c$ or $\gamma = 5$), but even at that speed, the average muon should travel less than a kilometre before they decay (their average lifetime is about 2.2 μs). But plenty of muons make it to the surface of the earth.

Muons



This is because of time dilation —they are moving and so to us their time gets stretched.

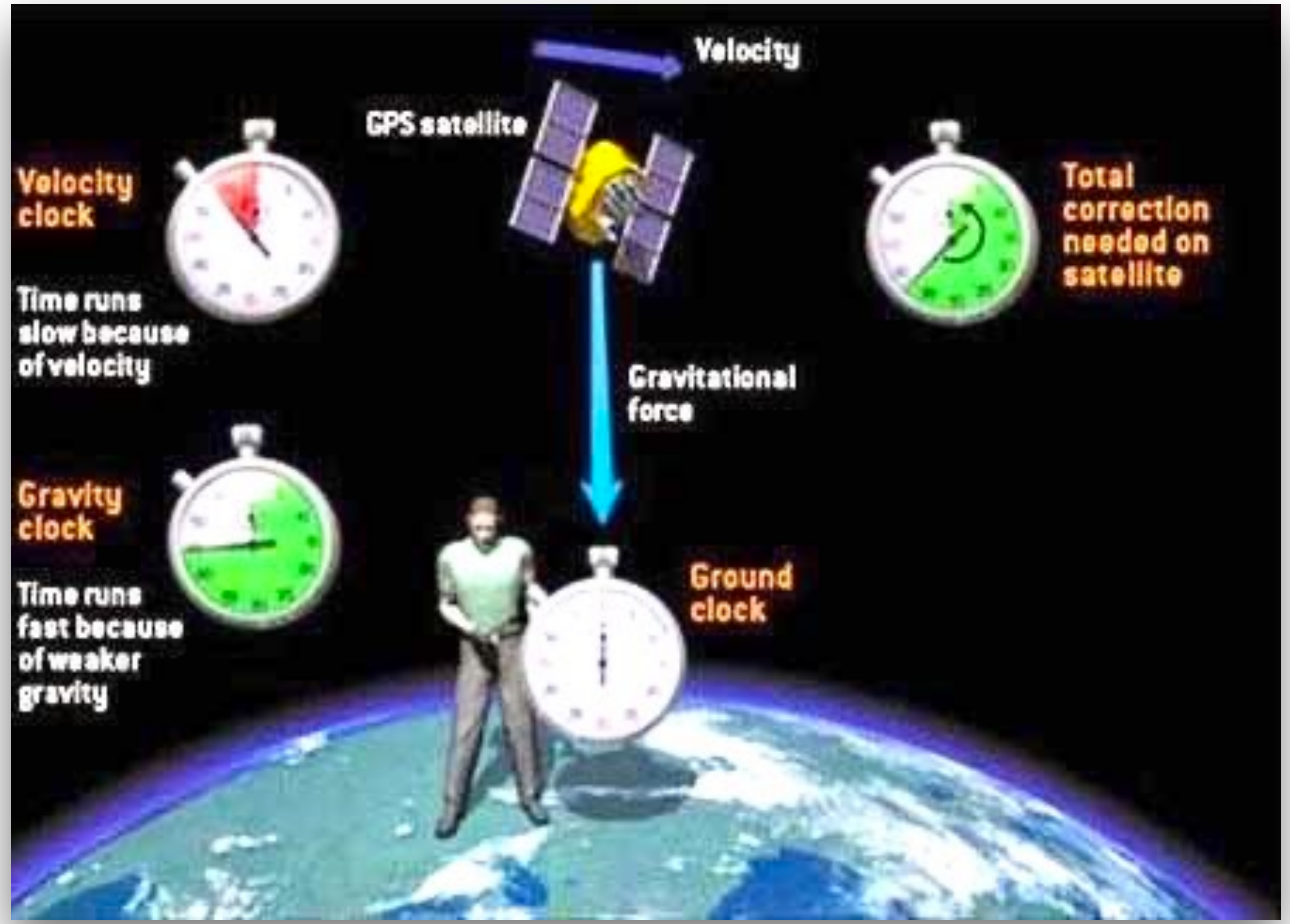
Seen from the point of view of muons, they of course live only 2.2 μs on an average, but the length to reach the earth gets contracted.

Time Dilation

Then consider how time goes in the artificial satellites.

The GPS satellites are moving at 14,000 km/hr in orbits that circle the Earth twice per day, much faster than clocks on the surface of the Earth. The atomic clocks kept in these get delayed by about seven microseconds per day.

·Actually, this is not the complete story—the satellites are miles above the Earth where gravity is only one-fourth of what it is on the ground. General relativity says that gravity affects time, making the orbiting clocks tick slightly faster, by about 45 microseconds per day. The net result is that time on a GPS satellite clock advances faster than a clock on the ground by about 38 microseconds per day.

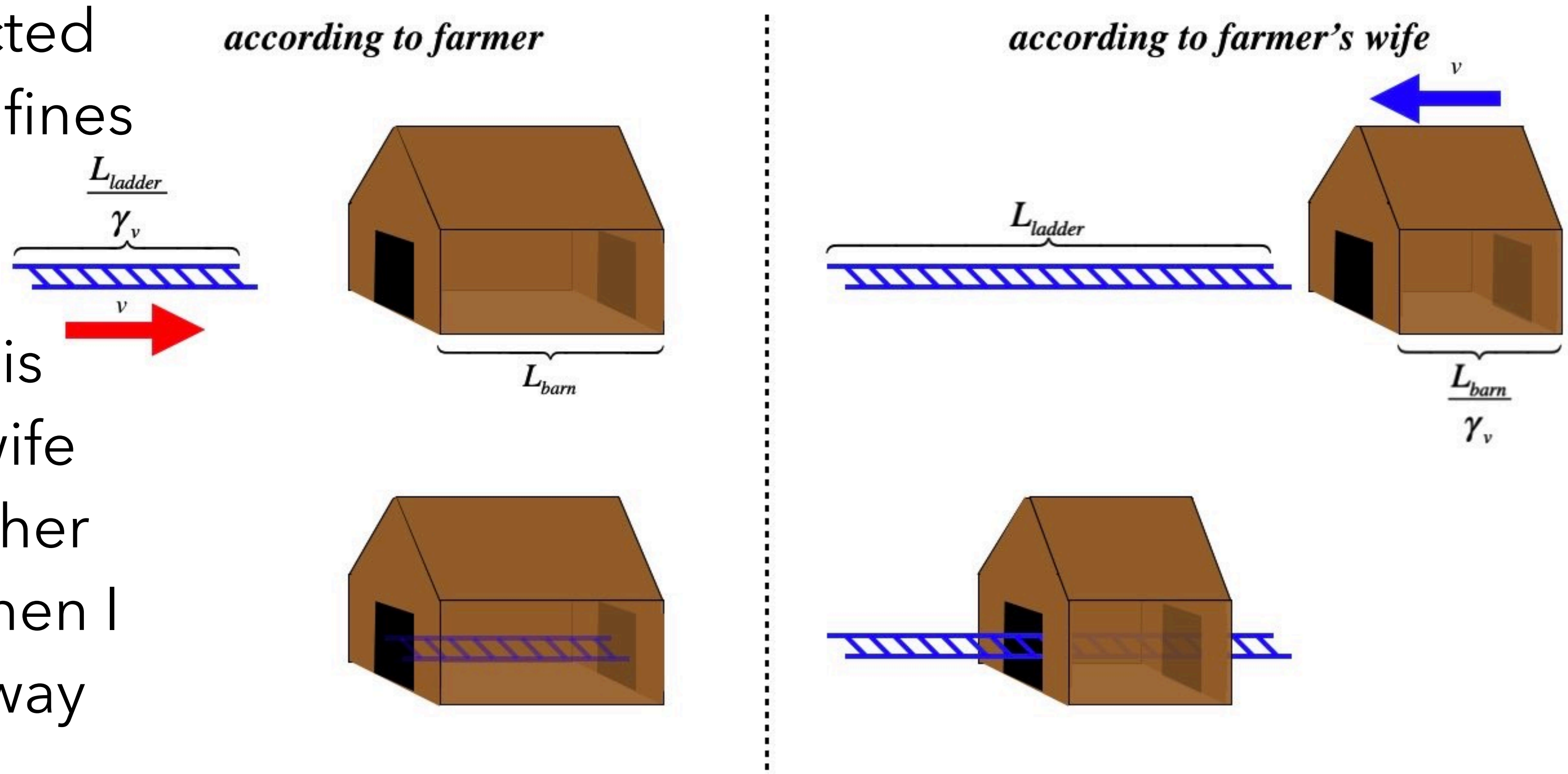


Paradoxes

A farmer wishes to store a long ladder that he owns inside his barn, but is frustrated to discover that his ladder is too long to fit. Specifically, he finds that is ladder is 50ft long, while his barn is only 40ft long. But like every good farmer, this fellow is well-versed in special relativity, and decides that if he can get the ladder moving fast enough relative to the barn, then the ladder's length will contract enough so that both the front and back barn doors can be closed at the same time while the ladder is inside.

Sure enough, as his wife drives their souped-up tractor, he notes that both ends of the length-contracted ladder are briefly within the confines of the two doors.

Though overjoyed that his ploy is going to work, he notices that wife looks distraught. When he asks her what is the matter, she says, "When I am zooming into the barn, it is way too short to fit the ladder."

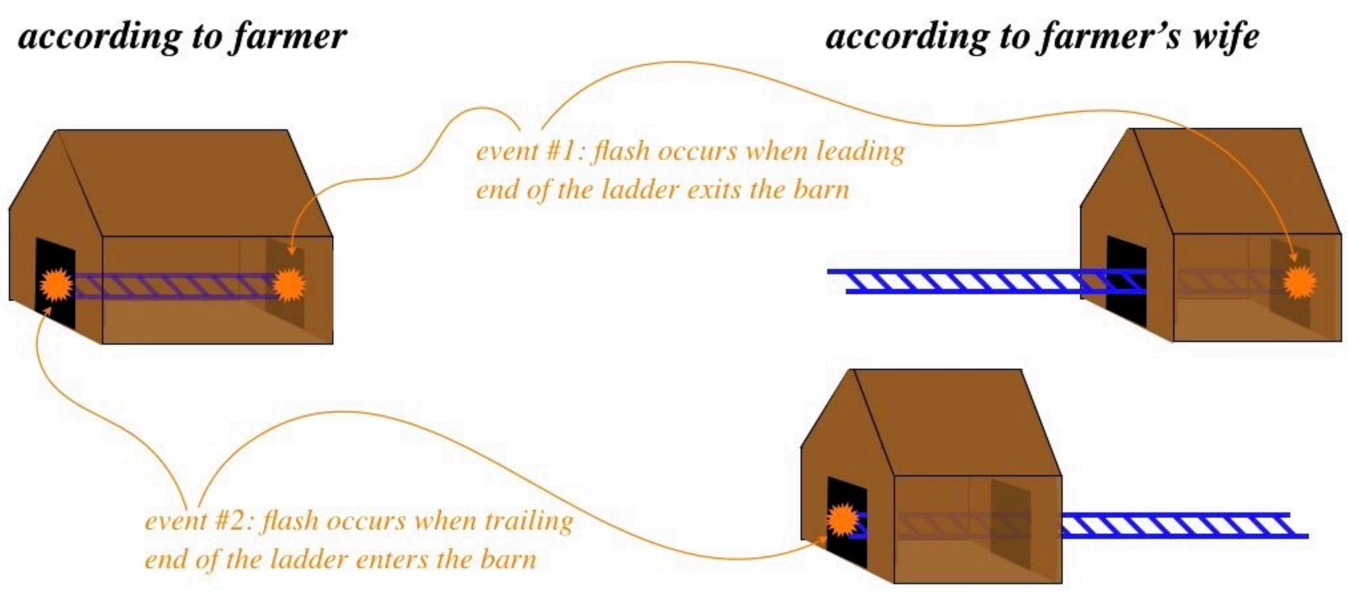


Paradoxes

As with everything else in relativity, we can't trust our logic without converting everything into the language of spacetime events. In terms of events, what does it mean for the ladder to be "entirely within the barn?"

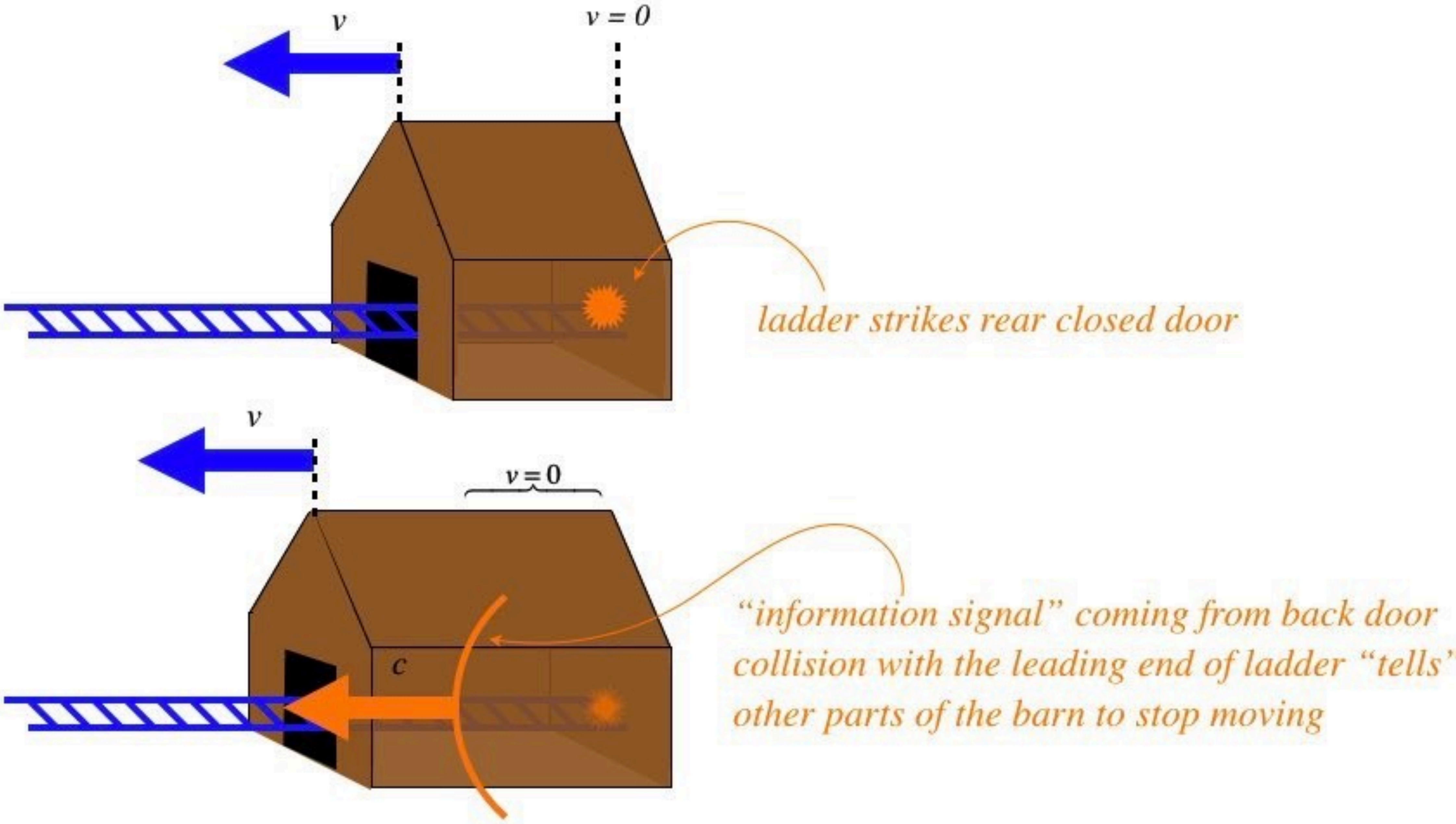
Well, imagine that each end of the ladder is equipped with a flashbulb that flashes whenever it is in a doorway of the barn. We can say that the ladder is just barely "completely enclosed" if the light at the front of the ladder flashes in the exit doorway at the same instant that the light at the rear of the ladder flashes in the entrance doorway. Then we can say that both ends of the ladder are inside the barn at the same time.

But what one observer sees as simultaneous events will not be seen as simultaneous by another. So the farmer sees the ladder as being within the barn because he can declare both ends to be within the confines of the barn at the same time, even as his wife claims that the front of the ladder exits the barn well in advance of the rear of the ladder entering it.



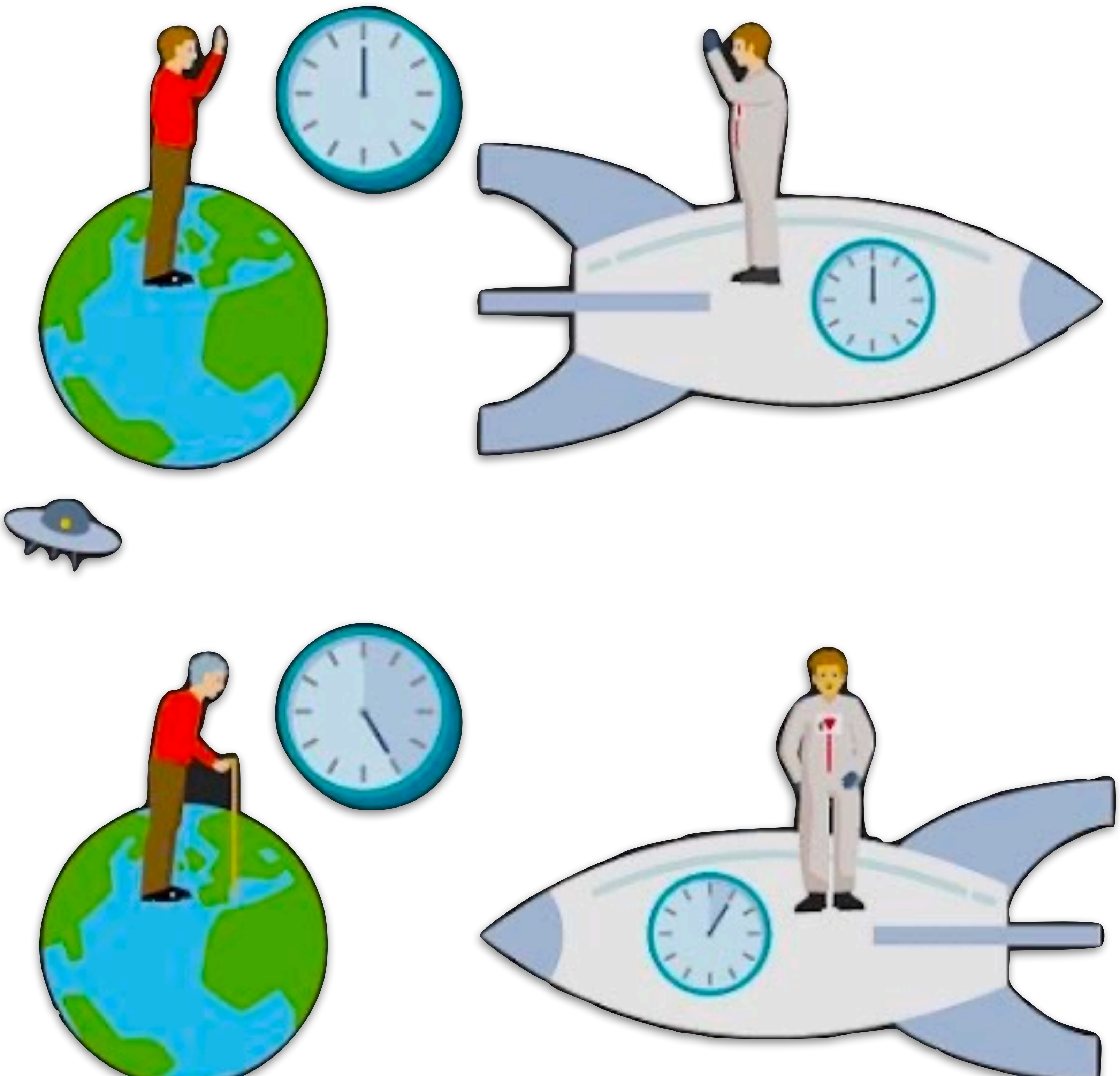
Paradoxes

Can Both Barn Doors Be Closed?



Paradoxes

Twin Paradox



Mark and Scott Kelly

Velocity Addition

Suppose an object has velocity $\vec{u}'$ in the frame S′. We want to know what is the velocity of this object in frame S. After a time t' (we started counting time when it was at the origin),

$$x' = u'_x t' \,, \quad y' = u'_y t' \,, \quad z' = u'_z t' \,.$$

From the inverse Lorentz transformations we get,

$$x = \frac{\left(u'_x t'\right) + v t'}{\sqrt{1 - v^2/c^2}} \,, \quad y = u'_y t' \,, z = u'_z t' \,, \quad t = \frac{t' + \left(u'_x t'\right) v/c^2}{\sqrt{1 - v^2/c^2}} \,.$$

Hence,

$$u_x = \frac{x}{t} = \frac{u'_x + v}{1 + v u'_x/c^2} \,, \quad u_y = \frac{y}{t} = \frac{u'_y \sqrt{1 - v^2/c^2}}{1 + v u'_x/c^2} \,, u_z = \frac{z}{t} = \frac{u'_z \sqrt{1 - v^2/c^2}}{1 + v u'_x/c^2} \,.$$

Velocity Addition

Again the inverse transformations can be found by changing $v \to -v$:

$$u'_x = \frac{x}{t} = \frac{u_x - v}{1 - vu_x/c^2}\,, \quad u'_y = \frac{y}{t} = \frac{u_y\sqrt{1 - v^2/c^2}}{1 - vu_x/c^2}\,, \quad u'_z = \frac{z}{t} = \frac{u_z\sqrt{1 - v^2/c^2}}{1 - vu_x/c^2}\,.$$

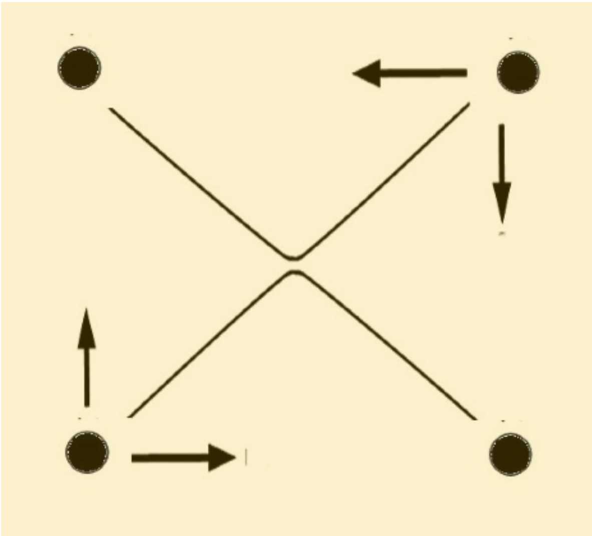
In the limit $v \ll c$, the Lorentz transformations become essentially same as the Galilean transformations. It is easy to see that so do the velocity transformation rules.

Now, just for fun, let's suppose the object is a light pulse and it is moving with $u' = c$ along the x' direction, i.e, $u'_x = c$, $u'_y = u'_z = 0$.

This gives us, $u_x = \dfrac{v + c}{1 + vc/c^2} = c$, i.e., speed c is indeed special. Anything that moves with speed c in one inertial frame, it does so in all inertial frames.

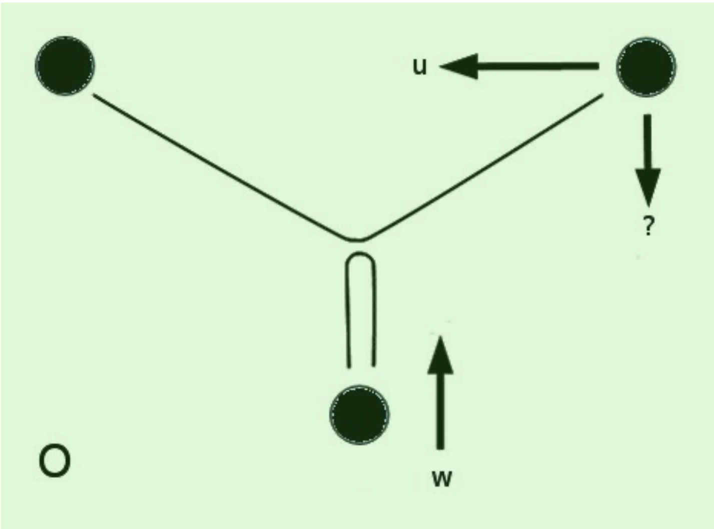
Relativistic Momentum and Energy

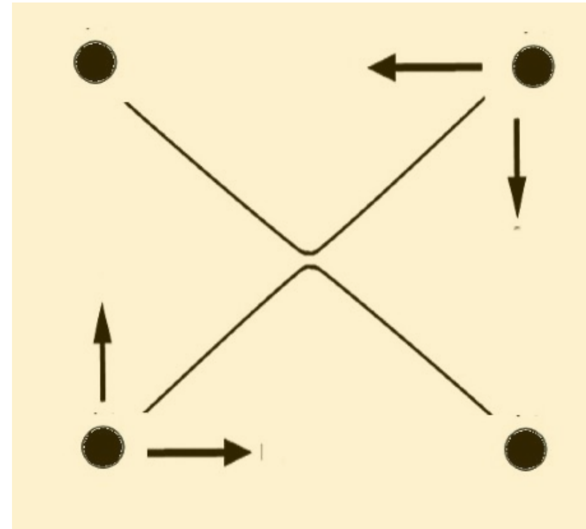
Since now we know velocity adds in a peculiar way, let's find out what happens to momentum as it depends on velocity by definition. We can investigate a collision among particles since we know momentum is conserved in collisions—we shall find out the relativistic expression of momentum by figuring out what is conserved. However, before we start, we can actually guess the form of the answer: $\vec{p} = m\vec{v} \rightarrow \vec{p} = m_v\vec{v}$ in such a way that $m_v \rightarrow m$ when $v \rightarrow 0$.



Let us consider a perfectly symmetric elastic collision between two objects with equal masses coming towards each other with same velocity when viewed from the centre of momentum frame.

Relativistic Momentum and Energy



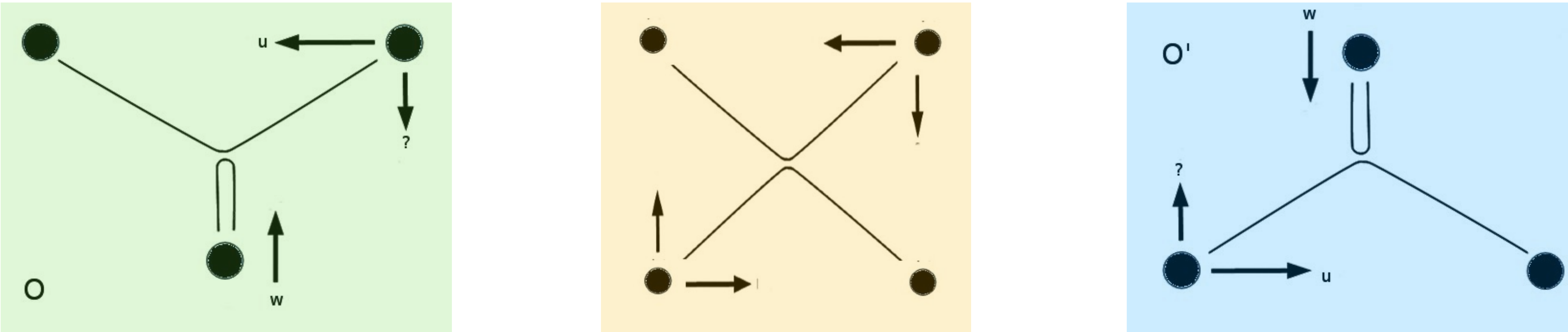


Now, let us view this from a frame (O) that is moving in the x direction with the same speed as the x component of the velocity of the lower object. In this frame, the lower object will have only transverse velocity (say along y-axis)—let's denote this velocity w so that the change of momentum of the lower object is,

$$\left[\Delta p^{\perp}_{lower} = |p^{final}_{lower} - p^{initial}_{lower}| = 2m_w w\right]_O .$$

To balance the momentum of this collision we need to know the y component of the velocity of the upper object. Let's denote the x-component of the velocity of the upper object as $-u$ in this frame (O). To find out the y component of the velocity of the upper object (denoted as $u^{\perp}_{upper}$ say) let's do a trick (Feynman).

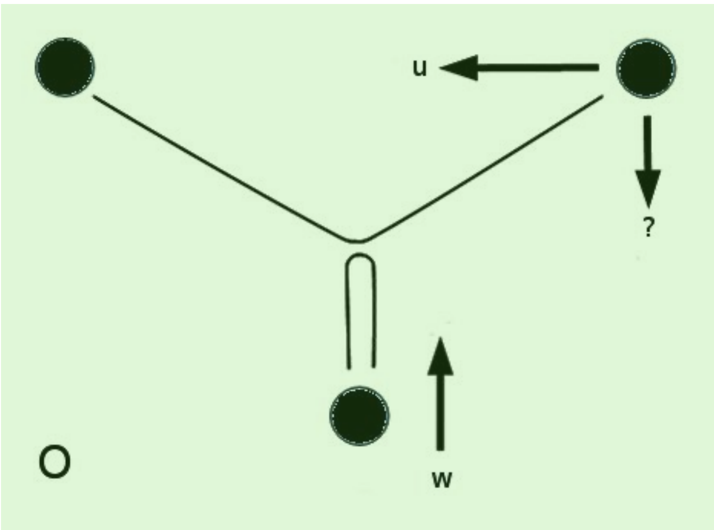
Relativistic Momentum and Energy

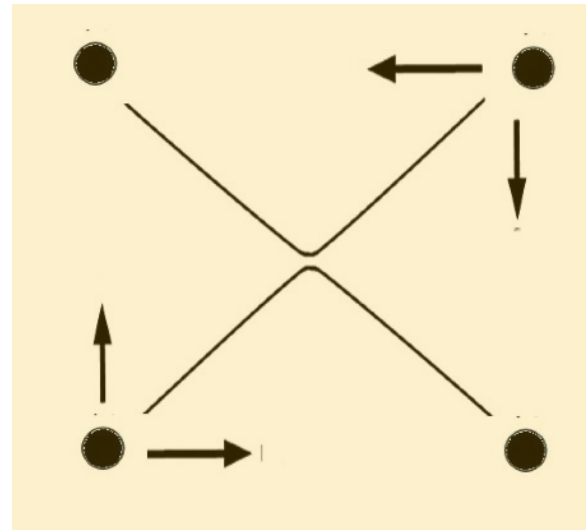


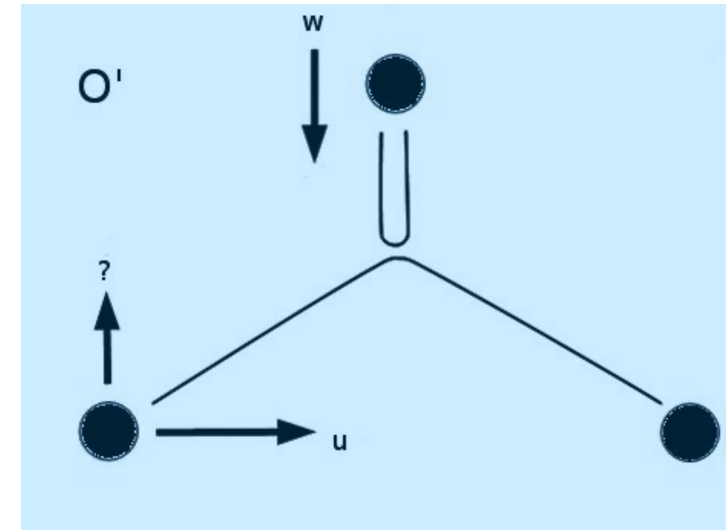
We go to a frame (O′) that is moving with velocity u in the $-x$ direction. What is this frame? In this frame the upper object has no velocity in the x direction and the lower object is moving with a velocity whose x component is u. Since we started from a symmetric picture of the collision we know that in O′, the y component of the velocity of the upper object ($u^{'\perp}_{upper}$) is $-w$ initially and w finally. Now we can use the appropriate velocity transformation relation to get,

$$u^{\perp}_{upper} = \frac{u^{'\perp}_{upper}\sqrt{1-u^2/c^2}}{1-u\,u^{'\parallel}_{upper}/c^2} = w\sqrt{1-u^2/c^2}\ .$$

Relativistic Momentum and Energy







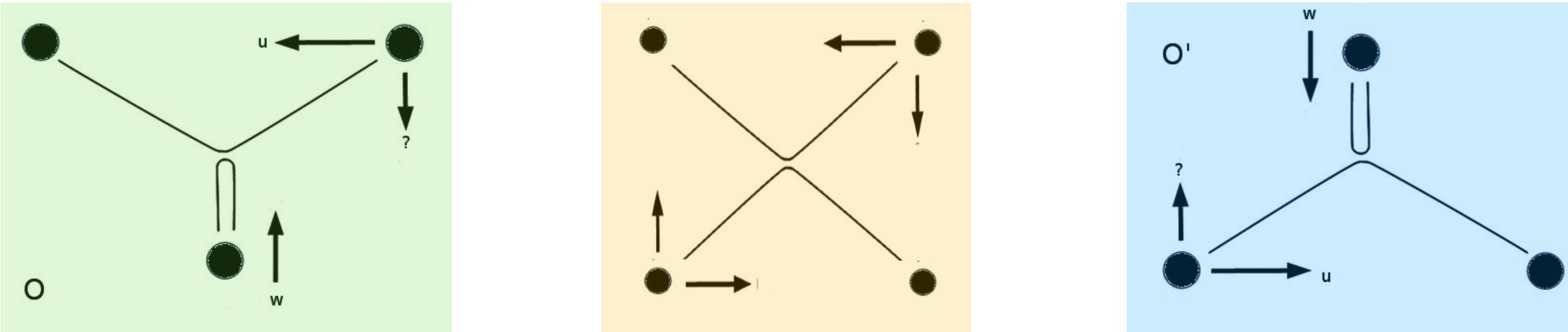
Now we can compute the change of momentum in the $\perp$ direction of the upper object in the O frame as,

$$\left[\Delta p^{\perp}_{upper} = |p^{\perp\, final}_{upper} - p^{\perp\, initial}_{upper}| = 2m_q w\sqrt{1 - u^2/c^2}\right]_O$$

where q is the speed of the upper particle in O,

$$q^2 = \left(u^{\perp}_{upper}\right)^2 + u^2 = w^2\left(1 - \frac{u^2}{c^2}\right) + u^2\,.$$

Relativistic Momentum and Energy



If momentum is conserved then,

$$m_w = m_q\sqrt{1 - u^2/c^2} \quad or, \quad m_q = \frac{m_w}{\sqrt{1 - u^2/c^2}}.$$

In the limit $w \to 0$, $q \to u$. Hence,

$$m_u = \frac{m_0}{\sqrt{1 - u^2/c^2}}.$$

Note in the limit $u \to 0$, $m_u \to m_0$.

So now we have our answer, the relativistic expression for momentum is

$$\vec{p} = m_v\,\vec{v} = \frac{m}{\sqrt{1 - v^2/c^2}}\,\vec{v}$$

where we have identified m_0 with m which is the Newtonian mass or the mass measured in the rest frame—the rest mass also called the invariant mass.

Relativistic Momentum and Energy

We can define the force on an object with velocity $\vec{v}$ as,

$$\vec{F} = \frac{d\vec{p}}{dt} = m\frac{d}{dt}\left(\frac{\vec{v}}{\sqrt{1 - v^2/c^2}}\right).$$

Let's suppose a constant force acts on an object initially at rest and increases its speed and kinetic energy, T. Now, we can compute the increase in kinetic energy as

$$dT = \vec{F}\cdot d\vec{x} = \frac{d\vec{p}}{dt}\cdot\vec{v}\,dt \quad \Rightarrow \quad T = \int \vec{v}\cdot d\vec{p} = m\int \vec{v}\cdot d\left(\frac{\vec{v}}{\sqrt{1 - v^2/c^2}}\right)$$

Doing integration by parts, we get,

$$T = m\left(\frac{\vec{v}\cdot\vec{v}}{\sqrt{1 - v^2/c^2}} - \int \frac{\vec{v}\cdot d\vec{v}}{\sqrt{1 - v^2/c^2}}\right)$$

Now, $2\vec{v}\cdot d\vec{v} = d(\vec{v}\cdot\vec{v}) = d(v^2) = -c^2 d(1 - v^2/c^2)$. So,

$$T = \frac{mv^2}{\sqrt{1 - v^2/c^2}} + \frac{mc^2}{2}\int \frac{d\left(1 - v^2/c^2\right)}{\sqrt{1 - v^2/c^2}} = \frac{mv^2}{\sqrt{1 - v^2/c^2}} + mc^2\sqrt{1 - v^2/c^2} - C = \frac{mc^2}{\sqrt{1 - v^2/c^2}} - C$$

where C is the integration constant.

Relativistic Momentum and Energy

Since, $T = 0$ when $v = 0$, we get, $C = mc^2$. Hence,

$$T = \frac{mc^2}{\sqrt{1 - v^2/c^2}} - mc^2 = m_v c^2 - mc^2 \,.$$

For $v \ll c$,

$$T = mc^2 \left(1 + \frac{1}{2}\frac{v^2}{c^2} + \cdots \right) - mc^2 \approx \frac{1}{2} mv^2 \,,$$

i.e., we recover the Newtonian expression as expected.

Einstein defined the energy of a particle as,

$$E = T + mc^2 \;=\; \frac{mc^2}{\sqrt{1 - v^2/c^2}} \quad \text{or,} \quad E = m_v c^2 \,,$$

which we immediately recognise as the most famous equation of physics. This can be written in terms of momentum as,

$$E^2 = p^2c^2 + m^2c^4,$$

which is actually more useful to physicists.