Universe Across Scales - A3

① The co-moving coordinates of the galaxy are $(r, 0, 0)$. A light ray is emitted from the galaxy at t_1 and $(t_1 + dt_1)$ and is observed at earth at t_0 and $(t_0 + dt_0)$

(a)

We know that the light rays travel along a null geodesic, $\Rightarrow dt^2 = \pm\, a(t) \dfrac{dr}{\sqrt{1-kr^2}}$

$$\Rightarrow \int_{t_1}^{t_0} \frac{dt}{a(t)} = \int_0^r \frac{dr}{\sqrt{1-kr^2}}$$

Thus, using the above relation, we can write :-

$$\int_{t_1}^{t_0} \frac{dt}{a(t)} = \int_{t_1+dt_1}^{t_0+dt_0} \frac{dt}{a(t)}$$

we subtract $\int_{t_1+dt_1}^{t_0} \frac{dt}{a(t)}$ from both sides,

$$\int_{t_1}^{t_1+dt_1} \frac{dt}{a(t)} = \int_{t_0}^{t_0+dt_0} \frac{dt}{a(t)}$$

Date:

The intervals dt_e and dt_o are small and so the scale parameter remains constant.

$\therefore\ a(t_1) = a(t_1 + dt_1)$

$a(t_0) = a(t_0 + dt_0)$

Thus integration gives, $\boxed{\dfrac{dt_1}{a(t_1)} = \dfrac{dt_0}{a(t_0)}}$

Hence, proved.

(b) From the definition of redshift,

$$z = \frac{\lambda_0 - \lambda_1}{\lambda_1}$$

[$\lambda_0 \rightarrow$ observed wavelength

$\lambda_1 \rightarrow$ emitted wavelength]

we know, $\nu_1 = \dfrac{1}{dt_1}$ and $\nu_0 = \dfrac{1}{dt_0}$

From part (a) $\Rightarrow \dfrac{\nu_0}{\nu_1} = \dfrac{a(t_1)}{a(t_0)}$

now, $\nu \propto \dfrac{1}{\lambda}$

$$\Rightarrow \frac{\lambda_1}{\lambda_0} = \frac{a(t_1)}{a(t_0)}$$

$$\therefore\ z = \frac{\lambda_0}{\lambda_1} - 1$$

$$\boxed{z = \frac{a(t_0)}{a(t_1)} - 1}$$

Date:

② From (1b), we know that $z = \frac{a(t_0)}{a(t_1)} - 1$.

$$z+1 = \frac{a(t_0)}{a(t_1)}$$

Differentiating wrt t_0,

$$\frac{dz}{dt_0} = \frac{\frac{da(t_0)}{dt_0}\, a(t_1) - \frac{da(t_1)}{dt_0}\, a(t_0)}{(a(t_1))^2}$$

[we get the above using chain rule]

$$\therefore \frac{dz}{dt_0} = \frac{da(t_0)}{dt_0}\,\frac{1}{a(t_1)} - \frac{a(t_0)}{a(t_1)^2}\,\frac{da(t_1)}{dt_0}$$

$$= \left[\frac{1}{a(t_0)}\,\frac{da(t_0)}{dt_0}\right]\frac{a(t_0)}{a(t_1)} - \frac{a(t_0)\,dt_1}{a(t_1)\,dt_0}\left[\frac{1}{a(t_1)}\,\frac{da(t_1)}{dt_1}\right]$$

now, the Hubble parameter $H(t)$ is defined as $H(t) = \frac{\dot{a}(t)}{a(t)}$ where $\dot{a}(t)$ is the derivative of the scale factor $a(t)$ at time 't'.

$$\Rightarrow H(t_0) = \frac{\dot{a}(t_0)}{a(t_0)} = \frac{1}{a(t_0)}\,\frac{da(t_0)}{dt_0}$$

$$H(t_1) = \frac{\dot{a}(t_1)}{a(t_1)} = \frac{1}{a(t_1)}\,\frac{da(t_1)}{d(t_1)}$$

$$\frac{a(t_0)}{a(t_1)} = z+1 \quad \text{[From (1b)]}$$

Date:

Substituting all of the above we get,

$$\frac{dz}{dt_0} = H(t_0)(1+z) - \frac{a(t_0)}{a(t_1)}\frac{dt_1}{dt_0} H(t_1)$$

now from (1a) $\Rightarrow \frac{dt_1}{a(t_1)} = \frac{dt_0}{a(t_0)} \Rightarrow \frac{dt_1}{dt_0} = \frac{a(t_1)}{a(t_0)}$

$$\therefore \frac{dz}{dt_0} = H(t_0)(1+z) - \frac{a(t_0)}{a(t_1)}\frac{a(t_1)}{a(t_0)} H(t_1)$$

$$\Rightarrow \boxed{H(t_1) = H(t_0)(1+z) - \frac{dz}{dt_0}}$$

Hence, proved.

③ The constants to acquire the required plots have been mentioned in the code.
The plot is constant at two extremes, and has a steep increase for some values of temperature (K).

(b) From the plot, the temperature corresponding to $X = 0.8$ is $T = 4000k$.