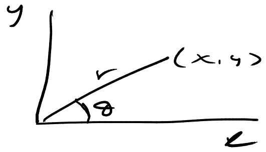
Polar Coordinates.



$r^2 = x^2 + y^2$
$x = r\cos\theta$
$y = r\sin\theta$.

① $df = \left(\frac{\partial f}{\partial x}\right)_y dx + \left(\frac{\partial f}{\partial y}\right)_x dy = f_x\, dx + f_y\, dy$

② $x = r\cos\theta \Rightarrow dx = dr\cos\theta - r\sin\theta\, d\theta$

③ $y = r\sin\theta \Rightarrow dy = dr\sin\theta + r\cos\theta\, d\theta$.

④ Put ② and ③ in ①

$df = f_x(dr\cos\theta - r\sin\theta\, d\theta) + f_y(dr\sin\theta + r\cos\theta\, d\theta)$

⑤ when r = constant ie $dr = 0$, we get ④ as

$dr = 0 \overset{④}{\Longrightarrow} df = f_x(0 - r\sin\theta\, d\theta) + f_y(0 + r\cos\theta\, d\theta)$

So that $\left(\frac{\partial f}{\partial \theta}\right)_r = f_\theta = -r\sin\theta f_x + r\cos\theta f_y = -y f_x + x f_y$.

⑥ when θ = constant ie $d\theta = 0$, we see ④ as

$d\theta = 0 \overset{④}{\Longrightarrow} df = f_x(dr\cos\theta + 0) + f_y(dr\sin\theta + 0)$

So that $\left(\frac{\partial f}{\partial r}\right)_\theta = f_r = \cos\theta f_x + \sin\theta f_y = \frac{1}{r}(x f_x + y f_y)$

⑦ combining ⑤ and ⑥ into matrix form

$$\begin{pmatrix} f_r \\ f_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} f_x \\ f_y \end{pmatrix}$$

which upon inverting the matrix gives

$$\begin{pmatrix} f_x \\ f_y \end{pmatrix} = \frac{1}{\cos\theta \cdot r\cos\theta - (-r\sin\theta)(\sin\theta)} \begin{pmatrix} r\cos\theta & -\sin\theta \\ +r\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} f_r \\ f_\theta \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} f_x \\ f_y \end{pmatrix} = \frac{1}{r} \begin{pmatrix} r\cos\theta & -\sin\theta \\ r\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} f_r \\ f_\theta \end{pmatrix}$$

operators

$\Rightarrow f_x = \cos\theta f_r - \frac{\sin\theta}{r} f_\theta \longrightarrow$ (7a) $\Rightarrow \frac{\partial}{\partial x} f = \left(\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right) f$.

$f_y = \sin\theta f_r + \frac{\cos\theta}{r} f_\theta \longrightarrow$ (7b) $\Rightarrow \frac{\partial}{\partial y} f = \left(\sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right) f$.

⑧ We are interested in Laplacian

$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, which can be obtained operators in (7a) and (7b)

$$\frac{\partial^2}{\partial x^2} f = \frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}\left(\cos\theta \frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial \theta}\right) = \left(\frac{\partial}{\partial x}\right)\left(\cos\theta f_r - \frac{\sin\theta}{r} f_\theta\right)$$

$$= \left(\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta f_r - \frac{\sin\theta}{r} f_\theta\right)$$

$$\Rightarrow \frac{\partial^2}{\partial x^2} f = \cos^2\theta f_{rr} - \cos\theta\sin\theta \frac{\partial}{\partial r}\left(\frac{f_\theta}{r}\right) - \frac{\sin\theta}{r}\left[\frac{\partial}{\partial \theta}(\cos\theta f_r) - \frac{1}{r}\frac{\partial}{\partial \theta}(\sin\theta f_\theta)\right] \text{ — (8a)}$$

Similarly

Lets do now $\frac{\partial^2 f}{\partial y^2} = \left(\frac{\partial}{\partial y}(f_y)\right) = \underbrace{\left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)}_{\frac{\partial}{\partial y}}\underbrace{\left(\sin\theta\frac{\partial f}{\partial r} + \frac{\cos\theta}{r}\frac{\partial f}{\partial\theta}\right)}_{f_y}$

$$\frac{\partial^2 f}{\partial y^2} = \sin^2\theta f_{rr} + \sin\theta\cos\theta\frac{\partial}{\partial r}\left(\frac{1}{r}f_\theta\right) + \frac{\cos\theta}{r}\left\{\frac{\partial}{\partial\theta}(\sin\theta f_r) + \frac{1}{r}(\cos\theta f_\theta)\right\} \quad \text{— (8b)}$$

from (8a) and (8b)

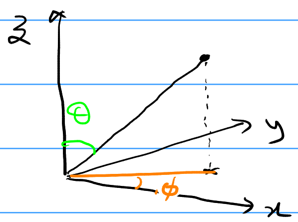
$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \overbrace{(\cos^2\theta + \sin^2\theta)}^{1}f_{rr} + \frac{\partial}{\partial r}\left(\frac{f_\theta}{r}\right)\overbrace{[-\cos\theta\sin\theta + \sin\theta\cos\theta]}^{0}$$
$$+\frac{1}{r}\left[\sin\theta\frac{\partial}{\partial\theta}(\cos\theta f_r) + \cos\theta\frac{\partial}{\partial\theta}(\sin\theta f_r)\right]$$
$$+\frac{1}{r^2}\left[-\sin\theta\frac{\partial}{\partial\theta}(\sin\theta f_\theta) + \cos\theta\frac{\partial}{\partial\theta}(\cos\theta f_\theta)\right]$$

$$\Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2}{\partial y^2} = f_{rr} + \frac{1}{r}\left(-\sin\theta[(-\sin\theta)f_r + \cos\theta f_{r\theta}] + \cos\theta[\cos\theta f_r + \sin\theta f_{r\theta}]\right)$$
$$+\frac{1}{r^2}\left[\sin\theta[\cos\theta f_\theta + \sin\theta f_{\theta\theta}] + \cos\theta[(-\sin\theta)f_\theta + \cos\theta f_{\theta\theta}]\right]$$
$$= f_{rr} + \frac{1}{r}[f_r]$$
$$+\frac{1}{r^2}[f_{\theta\theta}]$$

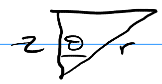
$$\Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = f_{rr} + \frac{1}{r}f_r + \frac{1}{r^2}f_{\theta\theta}$$

LAPLACIAN IN POLAR COORDINATES.

(10) 3-d. Spherical Coordinates

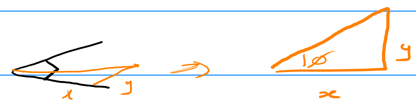


$r^2 = x^2 + y^2 + z^2$



$\theta = \cos^{-1}\left(\frac{z}{r}\right)$

$\tan\phi = y/x$



Equivalently
$$\boxed{\begin{aligned} x^2+y^2+z^2 &= r^2 \\ \frac{z}{\sqrt{x^2+y^2+z^2}} &= \cos\theta \\ \frac{y}{x} &= \tan\phi. \end{aligned}}$$

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta.$

(11)

$$df = \left(\frac{\partial f}{\partial x}\right)_{y,z}dx + \left(\frac{\partial f}{\partial y}\right)_{x,z}dy + \left(\frac{\partial f}{\partial z}\right)_{x,y}dz = f_x\,dx + f_y\,dy + f_z\,dz$$

(12) (a) $x = r\sin\theta\cos\phi \Rightarrow dx = dr\cdot\sin\theta\cos\phi + d\theta\cdot r\cos\theta\cos\phi - d\phi\, r\sin\theta\sin\phi$

(b) $y = r\sin\theta\sin\phi \Rightarrow dy = dr\sin\theta\sin\phi + d\theta\cdot r\cos\theta\sin\phi + d\phi\, r\sin\theta\cos\phi$

(c) $z = r\cos\theta \Rightarrow dz = dr\cos\theta - d\theta\cdot r\sin\theta.$

(13) Substituting (12)(a),(b),(c) in (11) and then for LHS using $\left(\frac{\partial f}{\partial r}\right) = \left(\frac{df}{dr}\right)_{\phi, \theta \text{ const}}$

(13a) $f_r = \left(\frac{df}{dr}\right)_{(\theta, \phi) = \text{const}} = f_x (\sin\theta \cos\phi) + f_y (\sin\theta \sin\phi) + f_z \cos\theta$

(13b) $f_\theta = \left(\frac{df}{d\theta}\right)_{(r, \phi) = \text{const}} = f_x (r\cos\theta \cos\phi) + f_y (r\cos\theta \sin\phi) + f_z (-r\sin\theta)$

(13c) $f_\phi = \left(\frac{df}{d\phi}\right)_{(r, \theta) = \text{const}} = f_x (-r\sin\theta \sin\phi) + f_y (r\sin\theta \cos\phi) + f_z (0)$

Hence
$$\begin{pmatrix} f_r \\ f_\theta \\ f_\phi \end{pmatrix} = \underbrace{\begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ r\cos\theta\cos\phi & r\cos\theta\sin\phi & -r\sin\theta \\ -r\sin\theta\sin\phi & r\sin\theta\cos\phi & 0 \end{pmatrix}}_{\text{Matrix } M.} \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}$$

$$\det(M) = \sin\theta\cos\phi(+r^2\sin^2\theta\cos\phi) - \sin\theta\sin\phi(-r^2\sin^2\theta\sin\phi) + \cos\theta(r^2\cos\theta\sin\theta\cos^2\phi + r^2\cos\theta\sin\theta\sin^2\phi)$$

$$= r^2\sin^3\theta + r^2\cos^2\theta\sin\theta = \boxed{r^2\sin\theta(1) = \det(M)}$$

Since $\det(M)$ is non-zero, it is invertable and is

$$M^{-1} = \frac{1}{\det(M)} \cdot \begin{bmatrix} -r^2\sin^2\theta\cos\phi & +r^2\sin^2\theta\sin\phi & r\cos\theta\sin\theta(1) \\ -r\sin\theta\cos\theta\cos\phi & +r\sin\theta\cos\theta\sin\phi & r^2\sin^2\theta\,(1) \\ -r\sin\phi\cdot(1) & -r\cos\phi\,(1) & 0 \end{bmatrix}$$

use $\det(M) = r^2\sin\theta$ $\Rightarrow M^{-1} = \frac{1}{r}\begin{bmatrix} -r\sin\theta\cos\phi & r\sin\theta\sin\phi & r\cos\theta \\ -\cos\theta\cos\phi & \cos\theta\sin\phi & r\sin\theta \\ -\sin\phi/\sin\theta & -\cos\phi/\sin\theta & 0 \end{bmatrix}$

We have $\begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} = M^{-1}\begin{pmatrix} f_r \\ f_\theta \\ f_\phi \end{pmatrix}$

Then get
$$\left.\begin{aligned} f_{xx} &= (f_x)_x \\ f_{yy} &= (f_y)_y \\ f_{zz} &= (f_z)_z \end{aligned}\right\}$$
and then get $f_{xx} + f_{yy} + f_{zz}$ in terms of spherical coordinates.

SEE NEXT PAGE FOR SLIGHTLY EASIER METHOD.

Spherical Coordinates

(14)

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta.$$

Introduce new variable $s = r\sin\theta. \Rightarrow$

$$x = s\cos\phi$$
$$y = s\sin\phi$$
$$z = r\cos\theta.$$

ie $s = \sqrt{x^2+y^2}$, $s = r\sin\theta$

(14a) Consider xy planes (ie z = constant); we have already shown that

$$\partial^2_{xx} + \partial^2_{yy} = \partial^2_{ss} + \frac{1}{s}\partial_s + \frac{1}{s^2}\partial^2_{\phi\phi}. \qquad \left[\begin{matrix} \circ\circ & x = s\cos\phi \\ \circ & y = s\sin\phi \end{matrix}\right]$$

(14b) Consider sz planes (ie ϕ = constant), we have already shown that

$$\partial^2_{zz} + \partial^2_{ss} = \partial^2_{rr} + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial^2_{\theta\theta} \qquad \left[\begin{matrix} \circ\circ & z = r\cos\theta \\ \circ & s = r\sin\theta \end{matrix}\right]$$

From (14a) we have to eliminate 's' variable.

First step is to use (14b) to get ∂^2_{ss} to put in RHS of (14a)

$$\Rightarrow \partial^2_{xx} + \partial^2_{yy} + \partial^2_{zz} = \left(\partial^2_{rr} + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial^2_{\theta\theta}\right) + \underbrace{\frac{1}{s}\partial_s}_{} + \frac{1}{s^2}\partial^2_{\phi\phi} \longrightarrow (14c)$$

we only need to get ∂_s!

(14d) For any function f, we have $df = f_r\,dr + f_\theta\,d\theta + f_\phi\,d\phi$

ie $\dfrac{\partial f}{\partial s} = f_r \cdot \dfrac{\partial r}{\partial s} + f_\theta \cdot \dfrac{\partial \theta}{\partial s} + f_\phi \dfrac{\partial \phi}{\partial s}$ and since $s = r\sin\theta \Rightarrow \boxed{\dfrac{\partial \phi}{\partial s} = 0}$.

$$\Rightarrow \frac{\partial f}{\partial s} = f_r\frac{\partial r}{\partial s} + f_\theta\frac{\partial \theta}{\partial s} \quad \text{—— (14d)}$$

for $\dfrac{\partial r}{\partial s}$ (14e) $r = \sqrt{s^2+z^2} \Rightarrow \dfrac{\partial r}{\partial s} = \dfrac{1}{2r}\cdot 2s = \dfrac{s}{r} \Rightarrow \boxed{\dfrac{\partial r}{\partial s} = \dfrac{s}{r}}$ —— (14e)

for $\dfrac{\partial \theta}{\partial s}$ (14f) $s = r\sin\theta \Rightarrow \dfrac{\partial s}{\partial s} = \dfrac{\partial r}{\partial s}\cdot\sin\theta + r\cos\theta\dfrac{\partial\theta}{\partial s}$.

$$1 = \left(\frac{s}{r}\right)\sin\theta + r\cos\theta\cdot\frac{\partial\theta}{\partial s} \Rightarrow$$

$$1 = \sin^2\theta + r\cos\theta\frac{\partial\theta}{\partial s} \Rightarrow \boxed{\frac{\partial\theta}{\partial s} = \frac{\cos\theta}{r}} \text{ —— (14f)}$$

Substituting (14f) and (14e) in (14d),

we get $\partial_s f = f_r\cdot\dfrac{s}{r} + f_\theta\dfrac{\cos\theta}{r} \Rightarrow \boxed{\partial_s f = \left(\dfrac{s}{r}\partial_r + \dfrac{\cos\theta}{r}\partial_\theta\right)f}$ (14g)

Substituting $s = r\sin\theta$ and (14g) in (14c) we get

$$\partial^2_{xx} + \partial^2_{yy} + \partial^2_{zz} = \partial^2_{rr} + \frac{2}{r}\partial_r + \frac{1}{r^2}\left[\partial^2_{\theta\theta} + \frac{1}{\sin^2\theta}\partial^2_{\phi\phi} + \frac{\cos\theta}{\sin\theta}\partial^2_{\phi\phi}\right]$$

THIS IS THE LAPLACIAN IN SPHERICAL COORDINATES