Random variables.

Probability space (Ω, F, P)

Sample ~~step~~ space: Set of all possible outcomes.

Event space: Set of (set of outcomes in Ω)

Probability measure: Function from event F to $[0,1]$

- Contains both empty set ϕ & Ω
- closed under complement: If $A \in F$, then $A^c \in F$
- closed under countable union: If $A_1, A_2, \ldots \in F$, then $\bigcup_{i=1}^{\infty} A_i \in F$.

Eg. Rolling a dice $\rightarrow \Omega = \{1, 2, 3, 4, 5, 6\}$.

$A = \{1 \cup 2 \cup 3\}$, $B = \{1\}$.

Event space F_A generated by A will be

$F_A = \{\Omega, \phi, A, A^c\}$

$= \{\Omega, \phi, \{1 \cup 2 \cup 3\}, \{4 \cup 5 \cup 6\}\}$

Event space F_B generated by B will be

$F_B = \{\Omega, \phi, B, B^c\}$

$= \{\Omega, \phi, \{1\}, \{2 \cup 3 \cup 4 \cup 5 \cup 6\}\}$.

Event space generated by A & B will be.

$F_{AB} = \{\Omega, \phi, A, A^c, B, B^c, A \cup B, A \cup B^c, B \cup A^c, B^c \cup A^c, (A \cup B)^c, \ldots A \cap B, A \cap B^c, \ldots\}$.

Power set $\rightarrow$ all ~~elem~~ possible subsets of a set.

Power set is always a valid event set.

No. of elements in a power set = 2^n.

Eg. choose any real numbers from $[0,1)$

$\Omega \to$ infinite set.

$F \to$ infinite

$P \to$ any event (biased / unbiased).

we are interested ~~and~~ in addition of 2 numbers

$\Omega' \to$ infinite

$F' \to$

$P' \to$ It will not be uniform.

$Y \to$ mapping

$$(\Omega, F, P) \xrightarrow{Y} (\Omega', F', P')$$

when Ω has infinite (countable / uncountable), we used random variables.

To comeup with general platform which is independent of the choice of particular Ω, random variables are used.

Definition:

$Y \to$ Random variable.

For random variables, we consider "special" Ω', F' & corresponding induced probability measure P'.

$\Omega' \to$ the set of real numbers, denoted by $\mathbb{R}$.

$F' \to$ will be Borel σ-Algebra, denoted by $B(\mathbb{R})$.

P' will be the corresponding induced probability measure, denoted by P_X.

A random variable X is a map given by

$$X : (\Omega, F, P) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}), P_X)$$

Map X needs to satisfy some conditions.

Borel σ-Algebra $\mathcal{B}(\mathbb{R})$:

$$(\Omega, F, P) \longrightarrow \{\Omega', F', P'\}$$

$$\Omega' = \mathbb{R}, \quad F' = \mathcal{B}(\mathbb{R})$$

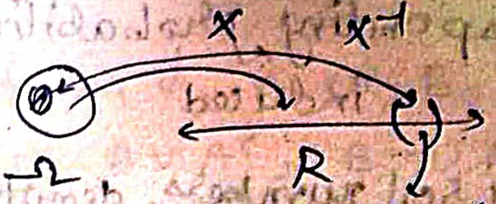
* If $\Omega = \mathbb{R}$, then $\mathcal{B}(\mathbb{R})$ is the event set generated by open sets of the form (a, b) where $a < b$ & $a, b \in \mathbb{R}$

* $\mathcal{B}(\mathbb{R})$ contains intervals of the form.

$$[a, b], [a, b), (a, \infty), [a, \infty), (-\infty, b],$$
$$(-\infty, b), (a, b], \{[a, b]\}, (-\infty, \infty)$$

Some examples of random variables.

→ A single coin toss
→ Three coin toss
→ choose a real number between $[0, 100]$.

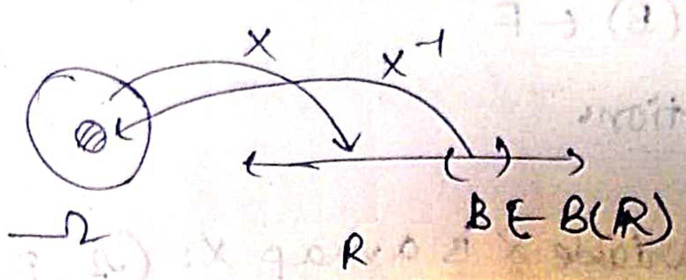


$$X^{-1}(B) := \{\omega \in \Omega : X(\omega) \in B\} \in F.$$

$$\Omega \xrightarrow{X} \mathbb{R}$$

$$F \xrightarrow{X} B(\mathbb{R}) \text{, and } P(\cdot) \xrightarrow{X} P_X(\cdot)$$

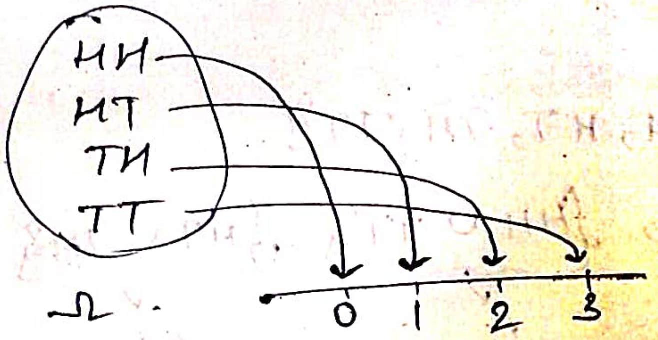
→ care must be taken such that the events considered in the new event space $B(\mathbb{R})$ are also valid events included in F



Inverse image/ Preimage ← $X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\}$.

all entries in Ω such that the selected interval is formed by those entries.

Eg.



$X^{-1}([-\infty, -5]) = \phi \in F$.

Let interval be $[-0.3, 2.5]$.

$$X^{-1}([-0.3, 2.5]) = \{HH, HT, TH\}$$

* $X^{-1}(B)$ is called as the preimage or inverse image of B.

$\boxed{\text{Map } X \text{ is called a random variable if } X^{-1}(B) \in F.}$

$$F \xrightarrow{X} B(\mathbb{R})$$

F ↓ old event space

$B(\mathbb{R})$ ↓ new event space (entries in $B(\mathbb{R})$ are intervals)

Definition: A random variable X is a map $X: (\Omega, F, P) \rightarrow (\mathbb{R}, B(\mathbb{R}), P_X)$ such that for each $B \in B(\mathbb{R})$, the inverse image $X^{-1}(B) := \{\omega \in \Omega : X(\omega) \in B\}$,

satisfies the condition

$$X^{-1}(B) \in F$$

General definition:

A random variable X is a map $X: (\Omega, F, P) \rightarrow (\Omega', F', P')$ s.t for each $f \in F'$, $X^{-1}(f) \in F$.

P_X denotes the probability measure induced by X on $B(\mathbb{R})$ & is given by

$$P_X(B) = Pr(\omega \in \Omega : X(\omega) \in B).$$

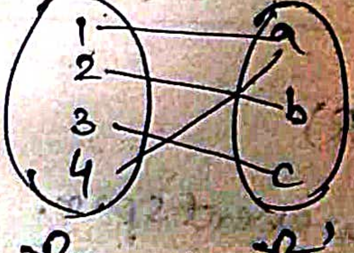
Eg: 1) $\Omega = \{HH, HT, TH, TT\}$.

$$F = \{\phi, \Omega, \underbrace{\{HH \cup TT\}}_{A}, \underbrace{\{HT \cup TH\}}_{A^c}\}$$

$$B = (-\infty, 2.5]$$

from previous example $\quad X^{-1}(B) = \{HH, HT, TH\} \notin F$

Eg.

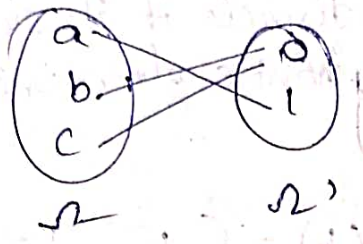


$\Omega \qquad \Omega'$

$F = 2^{\Omega} \qquad F' = 2^{\Omega'}$

$X^{-1}(\{b \cup c\}) = \{2 \cup 3\} \in F = 2^{\Omega}$

$X^{-1}(\{a \cup b\}) = \{1 \cup 4 \cup 2\} \in F.$



$F = \{\phi, \Omega, \{a\}, \{b \cup c\}\}$

$F' = 2^{\Omega'} = \{\phi, \Omega, 0, 1\}$

$X^{-1}(0) = (b \cup c) \in F.$ | $X^{-1}(\phi) = \phi$

$X^{-1}(1) = (a) \in F.$ | $X^{-1}(\Omega') = \Omega.$

F is not a power set but still ~~...~~ it satisfies the conditions.

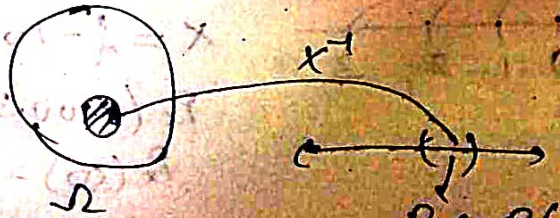
$X : (\Omega, F, P) \longrightarrow (\Omega', F', P')$

This is an indicator random variable.

Discrete Random Variable

A random variable X is a map $X : (\Omega, F, P) \rightarrow (R, B(R), P_X)$ such that for each $B \in B(R)$, the inverse image $X^{-1}(B) \in F.$

collection of intervals.



$B \in B(R)$

$X^{-1}(B) = \{w \in \Omega : X(w) \in B\}.$

Eg: $P[HH] = 0.2$, $P[HT] = 0.3$, $P[TH] = 0.35$,

$P[TT] = 0.15$, where $\Omega = [HH, TH, HT, TT]$

→ The range or support set of a random variable X is the set of values that it can take.

→ A random variable is called discrete if its support set consists of finite or countably infinite elements (discrete elements).

Eg. $\Omega : b \in [-1, 1]$

$X(b) = b^2$

$X(b) \in [0, 1]$

Supp $[0, 1]$

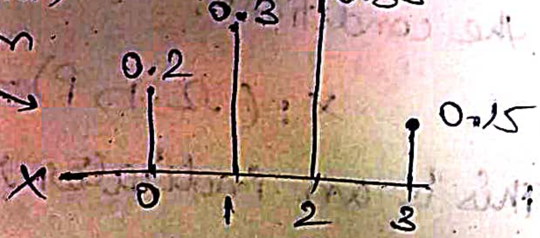
Infinite elements
Continuous random variable.

$$X(b) = \begin{cases} 1, & b > 0 \\ 0, & b = 0 \\ -1, & b < 0 \end{cases}$$

$Supp(X) = \{1, 0, -1\}$.

finite elements
Discrete random variable.

(PMF)
Probability Mass Function of a discrete random variable

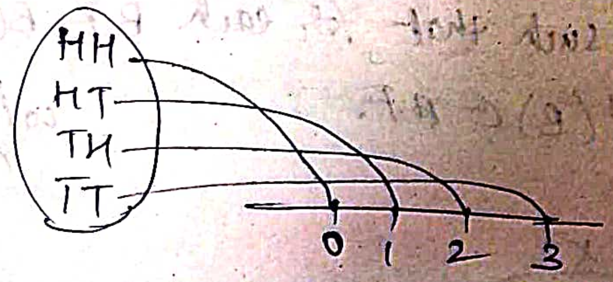


$X \in [0.2 \;\; 0.3 \;\; 0.15 \;\; 0.35]$

$\{P_X(X = x)\}$, $x \in X$

PMF

support set of r.v X.



$P_X(X=1) = P(HT)$
$= 0.3$

$X = \{-100.3, 50, ..\}$

$P_X(-100.3) = ..$

$P_X(50) = ...$

$P_X(X = x)$

Capital

realization of r.v. X

X, Y → collection of values the rv can take

$P_X(x_i) = P(X = x_i)$ → $P(w \in \Omega, X(w) = x)$

$x_i \in$ Range of X,

Eg. Bernoulli Random variable (p) → parameter

Bernoulli (p) ⇒ $X = \begin{cases} 0 & \text{with prob } p \text{ (head)} \\ 1 & \text{with prob } (1-p) \text{ (tail).} \end{cases}$

$$X \quad \underset{0}{p} \quad \underset{1}{1-p}$$

Binomial random variable (n) → parameter

Toss a coin n times. $k = 0, 1, 2, \ldots n$

$n = 4$. $\Omega = 2^n$ outcomes.

$P_x(H) = p$, $P_x(T) = 1-p$.

P_x (Head occurs exactly 3 times)

$= P\{HHHT, HHTH, HTHH, THHH\}$.

$= \binom{4}{3} p^3(1-p) = {}^4C_3\, p^3(1-p)$

P(Heads/ ~~Tails~~)

$= \binom{n}{k} p^k (1-p)^{n-k}$.

X = Number of heads.

$X = \{0, 1, 2, \ldots n\}$.
k

$$\boxed{P_x(X = k) = \binom{n}{k} p^k (1-p)^{n-k}}$$

$\sum_k P_x(X=k) = ?$

$\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k}$. → (total probability should always be 1.)

$= [p + (1-p)]^n = 1^n = 1 \Rightarrow$ probability

Cummulative Distribution Function (CDF) of a random variable.

Consider a random variable $X : (\Omega, F, P)$
$$\downarrow$$
$$(\mathbb{R}, B(\mathbb{R}), P_X).$$

Random variable X translates the probability law defined on the events in the sample space to a probability law on corresponding to events on the real line R.

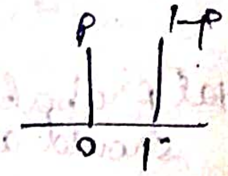
$B(\mathbb{R})$ consists of intervals of the type (a, b), $(-\infty, a)$, $(a, \infty]$ and so on.

Definition: In CDF of a r.v, we focus on the events of the form $(-\infty, x)$, where $x \in \mathbb{R}$. We are interested in events of the form $\{X \leq x, x \in \mathbb{R}\}$.

CDF of a random variable X, denoted by $F_X(\cdot)$ is defined as.

$$F_X(x) := P_X(X \leq x)$$
$$= P(\{\omega \in \Omega \text{ such that } X(\omega) \leq x\})$$

Eg. 1) Bernoulli



$x = -2$

$F_X(-2) = P_X(X \leq -2) = 0.$

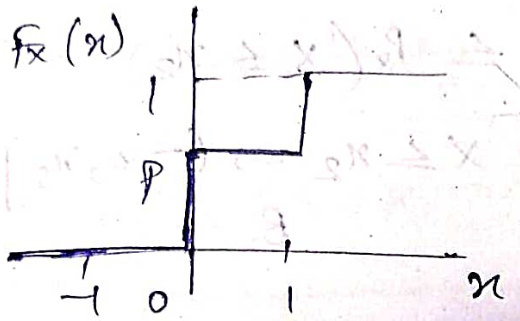
$x = 0.7$

$F_X(0.7) = P_X(X \leq 0.7) = P(0) = p$

$x = 3$

$F_X(3) = P_X(X \leq 3) = 1$

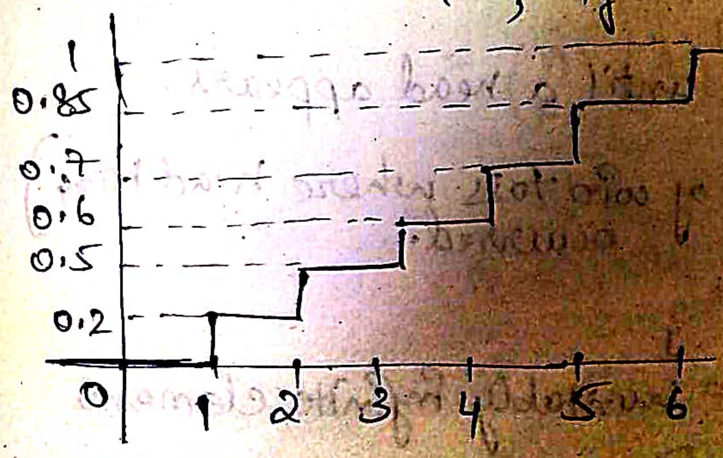
$$F_X(x) = \begin{cases} 0, & \text{if } x < 0 \\ p, & \text{if } 0 \le x < 1 \\ 1, & \text{if } x \ge 1 \end{cases}$$



<u>Eg. 2.</u> $X = \{1, 2, 3, 4, 5, 6\}$.

$P(1) = 0.2$, 0.3, 0.1, 0.1, 0.15, 0.15

$$F_X(x) = \begin{cases} 0, & \text{if } x < 1 \\ 0.2, & \text{if } 1 \le x < 2 \\ 0.5, & \text{if } 2 \le x < 3 \\ 0.6, & \text{if } 3 \le x < 4 \\ 0.7, & \text{if } 4 \le x < 5 \\ 0.85, & \text{if } 5 \le x < 6 \\ 1, & \text{if } x \ge 6 \end{cases}$$



<u>Properties of CDF $F_X(\cdot)$</u>

→ 1.) $F_X(\cdot)$ is monotonically nondecreasing

If $x \le y \Rightarrow F(x) \le F(y)$

Show that: If $x_1, x_2 \in \mathbb{R}$ such that $x_1 \le x_2$ then $F_X(x_1) \le F_X(x_2)$.

$$P_X(X \le x_1) \le P_X(X \le x_2)$$

$X \le x_1 \Rightarrow \underset{A}{(-\infty, x_1]}$, $X \le x_2 \Rightarrow \underset{B.}{(-\infty, x_2]}$

$$A \subset B$$
$$\Rightarrow P(A) \le P(B)$$

(or)

$$P_X(X \le x_2) = P_X(X \le x_1) + \underbrace{P_X(x \in (x_1, x_2))}_{\ge 0}$$

2.) $F_X(-\infty) = 0$, $F_X(\infty) = 1$.

3.) $F_X(\cdot)$ is right continuous.

$$\lim_{x \to x_0+} F(x) = F(x_0)$$

Geometric random variable

Eg. H, TH, TTH, . —

coins are tossed until a head appears.

~~X~~ X = [index of coin toss where head has occurred.]

$\mathcal{X} = \{1, 2, 3, \dots\}$ → countably infinite elements

$\boxed{P_X(X=k) = (1-p)^{k-1} \cdot p}$ $\Rightarrow \underbrace{TTT\dots}_{(k-1)}\cdot H$ → kth position.

$$\sum_k P_X(x=k) = \sum_{k=1}^{\infty} (1-p)^{k-1} p = p \cdot \frac{1}{1-(1-p)} = p \times \frac{1}{p} = 1$$

→ $1 \le k < \infty$

$$\left[\because \frac{1}{1-a} = \sum_{i=1}^{\infty} a^i\right.$$

used for countably rare events.

<u>Poisson</u> <u>random</u> <u>variable</u> (λ) → parameter.

↳ Queuing theory

$$\boxed{P_X(X=K) = e^{-\lambda}\frac{\lambda^K}{K!}} \quad , K=0,1,2,\ldots$$

$$\sum_{K=0}^{\infty} e^{-\lambda}\frac{\lambda^K}{K!} = e^{-\lambda}\underbrace{\left(\sum_{K=0}^{\infty}\frac{\lambda^K}{K!}\right)}_{\text{Taylor series of } e^{\lambda}}$$

$$= e^{-\lambda} \times e^{\lambda}$$

$$= 1$$

Poisson r.v can be used to approximate Binomial r.v

$n \to$ large
$p \to$ small

$\lambda = np$. ; $p = \frac{\lambda}{n}$

$$P_K(X=K) = \binom{n}{K} p^K (1-p)^{n-K}$$

$$= \binom{n}{K}\left(\frac{\lambda}{n}\right)^K\left(1-\frac{\lambda}{n}\right)^{n-K}$$

$$= \frac{n!}{(n-K)! \times K!}\left(\frac{\lambda^K}{n^K}\right)\frac{\left(1-\frac{\lambda}{n}\right)^{n}}{\left(1-\frac{\lambda}{n}\right)^K}$$

$1 \rightarrow$; $\left(1-\frac{\lambda}{n}\right)^{n} \approx e^{-\lambda}$; $\left(1-\frac{\lambda}{n}\right)^K \approx 1$

$$= \frac{\lambda^K}{K!} \times e^{-\lambda}$$

A random variable is not discrete does not mean it is continuous, and vice versa.

3.) $F_X(\cdot)$ is right-continuous.

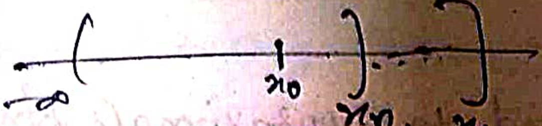
$$\lim_{x_n \to x_0} F_X(x_n) = \lim_{x_n \to x_0} P(X \leq x_n)$$

Continuity of probability

$A_1, A_2, \ldots A_n$ are events and then they converge into another sequence of event A, then.

$$\lim_{n\to\infty} P_X(A_n) = P_X\left(\lim_{n\to\infty} A_n\right) = P_X(A).$$

$$\Rightarrow \lim_{x_n \to x_0} P(X \leq x_n) = P_X\left(\lim_{x_n \to x_0} (-\infty, x_n]\right)$$



$$= P_X((-\infty, x_0])$$

$$= F_X(x_0)$$

$A_i = (-\infty, x_i]$

$A = (-\infty, x_0]$.

Mean of a discrete R.V → mean is the weighted average

Consider a discrete r.v X with support set $\mathcal{X}$.

Expected value (or mean) of X, denoted by $E[X]$ is defined as $E[X] := \sum_{x \in \mathcal{X}} x P_X(X = x)$.

1.) Find $E[X]$ for Bernoulli (p) r.v ($P_X(X=1) = p$ and $P_X(X=0) = 1-p$).

$E[X] = 0 \times (1-p) + 1 \times p = p$

2) Find $\mathbb{E}[X]$ of a r.v with $\mathcal{X} = \{1,2,3\}$ and PMF $\{0.25, 0.5, 0.25\}$

$$\mathbb{E}[X] = 1\times\frac{1}{4} + 2\times\frac{1}{2} + 3\times\frac{1}{4} = \underline{\underline{2}}$$

Expected value of function $f(x)$: $= f \circ X$ of X defined as

$$\mathbb{E}[f(X)] := \sum_{x \in \mathcal{X}} f(x)\, P_X(X=x)$$

Find $\mathbb{E}[X^2]$

1.) $\mathbb{E}[X^2] = 0^2\times(1-p) + 1^2\times p = \underline{p}$.

2.) $\mathbb{E}[X^2] = 1^2\times\frac{1}{4} + 2^2\times\frac{1}{2} + 3^2\times\frac{1}{4} = 2.5 + 2 = \underline{4.5}$

For r.v.s X, Y and constants a, b, we have

$$\mathbb{E}[aX+bY] = a\mathbb{E}[X] + b\mathbb{E}[Y].$$

<u>Variance of a discrete R.V.</u>

consider a discrete r.v X with support set $\mathcal{X}$. Variance of X, denoted by $V[X]$ is defined as:

$$V[X] := \mathbb{E}\left[(X - \mathbb{E}[X])^2\right]$$
$$= \sum_{x\in\mathcal{X}} (x - \mathbb{E}[X])^2\, P_X(X=x)$$

$$\boxed{V[X] = (\mathbb{E}[X^2]) - (\mathbb{E}[X])^2}$$

$\Rightarrow V[X] = \mathbb{E}\left((X - \mathbb{E}[X])^2\right)$

<u>Proof:</u> $\mu = \mathbb{E}[X]$.

$$V[X] = \mathbb{E}\left[(X-\mu)^2\right]$$

$\mathbb{E}[1] = 1$

$$= \mathbb{E}\left[X^2 - 2\mu X + \mu^2\right]$$
$$= \mathbb{E}[X^2] - 2\mu\,\mathbb{E}[X] + \mu^2\,\mathbb{E}[1]$$

$$= \mathbb{E}[X^2] - 2\mu\,\mathbb{E}[X] + \mu^2$$
$$= \mathbb{E}[X^2] - 2(\mathbb{E}[X])^2 + (\mathbb{E}[X])^2$$
$$= \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

1.) $V[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$
$= p - p^2 = p(1-p)$

2.) $V[X] = E[X^2] - (E[X])^2$
$= 4.5 - 4 = \underline{0.5}$

Continuous Random Variables

for continuous random variable, $P_X(X=x) = 0$

$X = \{x_1, x_2, \dots\}$ as it is continuous, it cannot take a single/unique value.

$PMF = \sum_{i=1}^{\infty} P_X(X = x_i) = 1.$

Definition: A random variable X is called continuous if its probability law can be described in terms of a nonnegative function $f_X(\cdot)$, called the probability density function (PDF) of X, which satisfies.

$$P_X[X \in B] = \int_B f_X(x)\,dx.$$

for every subset B of the real line.

Thus, the probability that $X \in [a, b]$ will be

$$P_X[X \in [a,b]] = \int_a^b f_X(x)\,dx.$$

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

PDF $\Rightarrow P_X[X \in [a,b)] = \int_a^b f_X(x)\,dx$

CDF of X will be

$$F_X(x) := P_X[X \le x] = P_X[X \in (-\infty, x]]$$

PDF can be expressed in terms of CDF as

$$f_X(x) = \frac{dF_X(x)}{dx}.$$

Mean & Variance

Consider a continuous r.v X with PDF $f_X(x)$.

Expected value of X is defined as

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

Variance of X is defined as

$$V[X] := \mathbb{E}[(X - \mathbb{E}[X])^2]$$

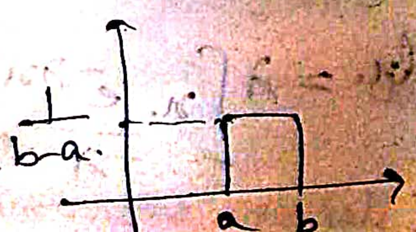
$$= \int_{-\infty}^{\infty} (x - \mathbb{E}[X])^2 f_X(x)\,dx$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 f_X(x)\,dx = (\mathbb{E}[X^2]) - (\mathbb{E}[X])^2$$

Uniform random variable

PDF of a uniform r.v with the support set $[a, b)$ is given by

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & \text{if } a \le x \le b. \\ 0, & \text{otherwise.} \end{cases}$$



uniform r.v is used $\longrightarrow$ Quantisation noise is modelled as a uniform r.v

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx.$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx.$$

$$E[X] = \int_a^b x \times \frac{1}{(b-a)}\, dx = \frac{1}{(b-a)} \int_a^b x\, dx$$

$$= \frac{1}{(b-a)} \times \frac{(b^2-a^2)}{2} = \frac{b+a}{2}$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx = \int_a^b x^2 \times \frac{1}{(b-a)}\, dx$$

$$= \frac{1}{(b-a)} \times \frac{(b^3-a^3)}{3} = \frac{b^2+a^2+ab}{3}$$

$$V[X] = E[X^2] - (E[X])^2$$

$$= \left(\frac{b^2+a^2+ab}{3}\right) - \left(\frac{b+a}{2}\right)^2$$

$$= \frac{(b-a)^2}{12}$$

Exponential random variable (like waiting line)

- PDF of an exponential r.v with parameter λ is given by

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

$$\Rightarrow E[X] = \int_0^{\infty} x.\lambda.e^{-\lambda x}\, dx.$$

$$= \lambda \int_0^{\infty} x.e^{-\lambda x}\, dx. = \lambda \left[x.\frac{e^{-\lambda x}}{-\lambda} \right) \Big|_0^{\infty} - \lambda \times \frac{1}{(-\lambda)} \int_0^{\infty} e^{-\lambda x}$$

$$= -(0-0) + \left[\frac{e^{-\lambda x}}{-\lambda}\right]_0^\infty$$

$$= -(0-0) - \frac{1}{\lambda}(0-1) = \frac{1}{\lambda}$$

$\left[\because \int f(x)\cdot g(x)\,dx = f(x)\int g(x)\,dx - \int\left(f'(x)\int g(x)\,dx\right)dx\right]$

$$E[X^2] = \int_0^\infty x^2 \lambda \cdot e^{-\lambda x}\,dx$$

$$= \lambda\left[\int_0^\infty x^2 e^{-\lambda x}\,dx\right]$$

$$= \lambda\left[\left(\frac{x^2 e^{-\lambda x}}{-\lambda}\right)_0^\infty - \int \frac{2x e^{-\lambda x}}{-\lambda}\,dx\right]$$

$$= \lambda\left[0 + \frac{1}{\lambda}\times 2\int_0^\infty x e^{-\lambda x}\,dx\right]$$

$$= 2\int_0^\infty x e^{-\lambda x}\,dx = \frac{2}{\lambda}$$

$$V(X) = E[X^2] - (E[X])^2$$

$$= \frac{2}{\lambda} - \left(\frac{1}{\lambda}\right)^2 = \frac{2}{\lambda} - \frac{1}{\lambda^2}$$

as $\sigma \downarrow$, the graph shrinks & $\sigma \uparrow$ graph expands, (the graph is in bell shape.)

Gaussian random variable

PDF of a Gaussian r.v with parameters μ and σ is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\mu = 0 \Rightarrow f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}$$

Gaussian r.v is more accurate than other r.vs.

$$P_X(x) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} dx.$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}} dx$$

let $y = \frac{x}{\sigma}$

$dx = \sigma dy$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy = I$$

$$I = \int_{-\infty}^{\infty} e^{-y^2/2} dy.$$

$$I^2 = I.I = \int_{-\infty}^{\infty} e^{-x^2/2} dx . \int_{-\infty}^{\infty} e^{-y^2/2} dy.$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{(x^2+y^2)}{2}} dx . dy.$$

$$= \int_{r=0}^{\infty} \int_{\theta=0}^{2\pi} e^{-r^2/2} r dr . d\theta$$

$z = \frac{r^2}{2} \Rightarrow dz = r dr.$

$x^2+y^2 = r^2 \Rightarrow 2x dx + 2y dy = 2r dr$

$\Rightarrow x dx + y dy = r dr$

$x = r\cos\theta$
$y = r\sin\theta$

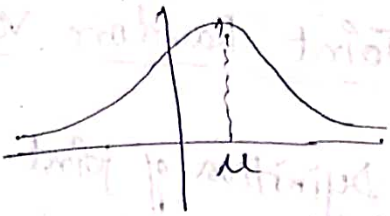
$(r\cos\theta)(dr\cos\theta - r\sin\theta d\theta)$
$+ (r\sin\theta)(dr\sin\theta + r\cos\theta d\theta) = r dr$

$r dr \cos^2\theta - r^2\cos\theta\sin\theta d\theta + r dr \sin^2\theta + r^2 \sin\theta\cos\theta d\theta = r dr$

Noise $\rightarrow$ Thermal noise is sum of large no. of independent r.v

Central Probability Theorem $\rightarrow$ If there ~~exist~~ is a sum of large no.

$$I^2 = \int_{z=0}^{\infty} \int_{\theta=0}^{2\pi} (e^{-z} dz)\, d\theta.$$



Bell curve

$$= 2\pi \Rightarrow I = \sqrt{2\pi}$$

$$E[X] = \frac{1}{2} \int_{-\infty}^{\infty} \frac{2x}{\sqrt{2\pi\sigma^2}} \times e^{-\frac{x^2}{2\sigma^2}}\, dx.$$

$$x^2 = t \Rightarrow 2x\,dx = dt.$$

$$= \frac{1}{2} \int_{0}^{\infty} \frac{e^{-t/2\sigma^2}}{\sqrt{2\pi\sigma^2}}\, dt$$

$$= \frac{1}{2\sqrt{2\pi\sigma^2}} \times \left[\frac{e^{-t/2\sigma^2}}{\left(\frac{-1}{2\sigma^2}\right)} \right]_0^{\infty} = 0$$

of independent r.v.s, then the resulting r.v is a Gaussian r.v, so Gaussian r.v is used to model noise

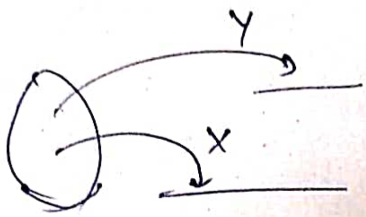
Joint Random variables.

Definition of joint random variables X & Y.

Joint PMFs / PDFs

↙ for discrete r.v.s

↘ for continuous r.v.s.



$$P_{X,Y}(X=x, Y=y) \triangleq P\{\omega \mid X(\omega)=x, Y(\omega)=y\}.$$

Marginal PMFs/PDFs from joint PMFs/PDFs

↙

for not specifying whether a r.v is discrete or continuous, we just mention as distribution.

$$P_X(X=x) = \sum_{y \in Y} P_{X,Y}(X=x, Y=y)$$

↙

Marginal PMF of X.

Find the marginal PMFs for the following eg.

1.)

$X \backslash Y$	0	1
0	$\frac{1}{3}$	$\frac{1}{3}$
1	0	$\frac{1}{3}$.

$P(X=0, Y=0) = 1/3$

$P(X=0, Y=1) = 0$

$P(X=1, Y=0) = 1/3$

$P(X=1, Y=1) = 1/3$

So $P_X(X=0) = \sum_{y \in \{0,1\}} P_{X,Y}(X=0, Y=y)$

$= \frac{1}{3} + \frac{1}{3} = \frac{2}{3}.$

$P_X(X=1) = 0 + \frac{1}{3} = \frac{1}{3}$

$\Rightarrow X \sim \{\frac{2}{3}, \frac{1}{3}\}$

$P_Y(Y=0) = \frac{1}{3} + 0 = 1/3$

$P_Y(Y=1) = \frac{1}{3} + \frac{1}{3} = 2/3.$

Joined density for continuous r.v.

$$f_X(x) = \int_y f_X(x,y)\, dy.$$

we treat x as constant and integrate w.r.t y.

Eg. 2.

Y \ X	1	2	3	4
1	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{32}$
2	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{32}$	$\frac{1}{32}$
3	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$
4	$\frac{1}{4}$	0	0	0

$P_X(X=1) = \frac{1}{2}$

$P_X(X=2) = \frac{1}{4}$

$P_X(X=3) = \frac{1}{8}$

$P_X(X=4) = \frac{1}{8}$

$P_Y(Y=1) = \frac{1}{4}$

$P_Y(Y=2) = \frac{1}{4}$

$P_Y(Y=3) = \frac{1}{4}$, $P_Y(Y=4) = \frac{1}{4}$

Properties of joint CDF

→ $F_{X,Y}(\cdot)$ should be non-decreasing and right-continuous in both variables.

→ $F_{X,Y}(\infty,\infty) = 1$, $F_{X,Y}(-\infty,-\infty) = 0$

$F_{X,Y}(-\infty, y) = 0$, $F_{X,Y}(x, -\infty) = 0$

$F_{X,Y}(\infty, y) = F_Y(y)$, $F_{X,Y}(x, \infty) = F_X(x)$

conditioning one ~~ct~~ random variable on another

conditional distribution of X given $Y=y$:

$$P_{X|Y}(X=x \mid Y=y) = \frac{P_{X,Y}(X=x, Y=y)}{P_Y(Y=y)}$$

Note: $P_{X|Y}(x|y)$ is defined only for those values of y such that $P_Y(Y=y) > 0$.

$$\sum_{x \in \mathcal{X}} P_{X|Y}(X=x \mid Y=y) = 1.$$

Conditional Expectation

support set of conditional expectation is the same as the ~~cond~~ support set of the r.v.

$$= \sum_{x \in \mathcal{X}} x\, P_{X|Y}(X=x \mid Y=y).$$

Eg. Find the conditional ~~probabi~~ distribution of $X|Y=0$ and $X|Y=1$. (given $Y=1$, find

X \ Y	0	1
0	$\frac{1}{3}$	$\frac{1}{3}$
1	0	$\frac{1}{3}$

$P(X=0 \mid Y=0)$

$$= \frac{P(X=0, Y=0)}{P(Y=0)} = \frac{\frac{1}{3}}{\frac{1}{3}+0} = 1$$

$P(X=1 \mid Y=0) = 0$

$$P(X=0 \mid Y=1) = \frac{P(X=0, Y=1)}{P(Y=1)} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$$

$P(X=1 \mid Y=1) = \cancel{P(X} \; \frac{1}{2}$

we take any r.v, then any function of a r.v. is also a r.v

Eg: X is a r.v, then $f(X)$ is also a r.v.

$$\int_{-\infty}^{\infty} e^{-x^2/2}\,dx = \sqrt{2\pi}$$

Egs..

1.) Random variables X and Y are independent & identically distribution according to Bernoulli (p) distribution. Find joint PMF.

$X \sim Ber(p) \Rightarrow P(X=1)=p$

$\cancel{\&}\ Y \sim Ber(p) \Rightarrow P(Y=1)=p$

$P(X=1, Y=1) = P(X=1)\cdot P(Y=1) = p\cdot p = p^2.$

IID $\rightarrow$ independent & identically distribution

2) Find. $\mathbb{E}[X+Y]$ let $X+Y=Z$ another r.v.

y \ x	1	2	3	4	
4	0	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{20}$	$\rightarrow 3/20$
3	4/20	2/20	3/20	4/20	$\rightarrow 7/20$
2	4/20	2/20	3/20	4/20	$\rightarrow 7/20$
1	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{20}$	0	$\rightarrow 3/20$
	$\downarrow$ 3/20	$\downarrow$ 6/20	$\downarrow$ 8/20	$\downarrow$ 3/20.	

① $Z = \{2, 3, \ldots 8\}.$

$P_Z(Z=2) = P(X=1, Y=1) = \frac{1}{20}$

$P_Z(Z=3) = P(X=1, Y=2) + P(Y=1, X=2)$

$= \frac{1}{20} + \frac{1}{20} = \frac{1}{10}$

Z $\rightarrow$	2	3	4	5	6	7	8
	$\frac{1}{20}$	$\frac{2}{20}$	$\frac{4}{20}$	$\frac{5}{20}$	$\frac{5}{20}$	$\frac{2}{20}$	$\frac{1}{20}$

$$E_Z = \sum_{z \in Z} z \cdot P(Z = z)$$

$$= 2 \times \frac{1}{20} + 3 \times \frac{2}{20} + 4 \times \frac{4}{20} + 5 \times \frac{5}{20} + 6 \times \frac{5}{20}$$

$$+ 7 \times \frac{2}{20} + 8 \times \frac{1}{20}$$

$$= \frac{101}{20}$$

3.) 4 dice are rolled independently. Let X be the number of 1's and Y be the number of 2's. Find the joint PMF of X and Y.

$X = \{0, 1, 2, 3, 4\}$

$Y = \{0, 1, 2, 3, 4\}$.

$$P(X = x, Y = y) = P(Y = y)\, P(X = x \mid Y = y).$$

$\square \quad \square \quad \square \quad \boxed{}$

$$P(Y = 2) = \binom{4}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^2$$

$$P(Y = y) = \binom{4}{y}\left(\frac{1}{6}\right)^y\left(\frac{5}{6}\right)^{4-y}$$

$\boxed{2} \quad \square \quad \square \quad \square$

~~$P(X = 2) =$~~ Given $Y = 1$

so let first dice show 2,

so ~~now~~ now $X = 2 \rightarrow P(X = 2 \mid Y = 1)$

$$= \binom{3}{2}\left(\frac{1}{5}\right)^2\left(\frac{4}{5}\right)^1$$

$\frac{1}{5}$ because

the other dice have

only 5 options $\rightarrow$ 1, 3, 4, 5, 6.

$$P(X=x \mid Y=y) = \cancel{\binom{4-xy}{x}}$$

$$P(X=x \mid Y=y) = \binom{4-y}{x}\left(\frac{1}{5}\right)^{x}\left(\frac{4}{5}\right)^{4-y-x}.$$

$$p(x) = P_X(X=x)$$

$$p(x,y) = P_{XY}(X=x,\ Y=y)$$

$$P_X(X=x) = \sum_{y} P(X=x, Y=y).$$

cdf $\rightarrow$ Discrete (pmf)

cdf $\rightarrow$ Continuous (pdf)

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx.$$ pdf

Integration of ~~fx~~ gives cpdf and differentiation of cdf gives pdf.

Function of discrete r.v is always discrete.

Function of continuous r.v can be discrete/continuous.

$$E[X] = \sum_{y} P_Y(y)\, E[X/Y=y]$$

$$E[Y] = \sum_{x} P_X(x)\, E[Y/X=x].$$