Verification Tests for FreeFem code

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1 Modelling equations and weak form

We remind the modelling equationns for the porous medium:

$$w = -K\nabla p \tag{1}$$
$$\nabla \cdot w = \beta_g \left(p_g - p\right) \tag{2}$$
$$+BC\, conditions \tag{3}$$

According to the parameters defined in the document "Modelling and parameterisation of governing equations in a porous medium domain" (Doc1 for future reference).

We also provide the variational (weak) form for the governing in equations. The proof and calculations are detailed in a separate document.

Variational form for Inhomogenous Neumann boundary conditions ($w \cdot \mathbf{n} = f_b$ on boundaries):

$$\int_\Omega K\nabla p \nabla v + \int_\Omega \beta_g p v = \int_\Omega \beta_g p_g v - \int_{\partial\Omega} f_g v \tag{4}$$

Variational form for Mixed boundary conditions ($p = p_b$ on Γ_p and $w \cdot \mathbf{n} = f_b$ on Γ_f):

$$\int_\Omega K\nabla p \nabla v + \int_\Omega \beta_g p v = \int_\Omega \beta_g p_g v - \int_{\Gamma_f} f_g v \tag{5}$$

2 Parameter Information

The following table presents the initial values used for the porous medium model, before the iteration process. For each verification test, we will state the values that differ from this table and we will consider that the rest of the parameters follow the values of the table. We also report the different boundary conditions used for each test, as well as the expected pressure solution.

Parameter	**Value**		**Units**	**Description**
K	0.0025		$\mathrm{mm^2\,Pa^{-1}\,s^{-1}}$	Permeability div. by blood viscosity ($\mu = 0.0035$)
p_{out}	0		Pa	Venous tree pressure (Reference pressure)
β_{out}	0.0031145		$\mathrm{Pa^{-1}\,s^{-1}}$	Venous tree beta
p_{cap}	2666.45		Pa	Average capillary tree pressure (20 mmHg)
	tv 15	**tv 19**		
p_i	12386.8062	12384.3253	Pa	Source pressure for term. vessel i
β_i	$8.3737\cdot10^{-4}$	$8.7159\cdot10^{-4}$	$\mathrm{Pa^{-1}\,s^{-1}}$	Beta of term. vessel i
q_i	1037.3879	1035.9284	$\mathrm{mm^3\,s^{-1}}$	Flow rate of term. vessel i
R_i	11.9403	11.9548	$\mathrm{Pa}\cdot s\,\mathrm{mm^{-3}}$	Resistance of term. vessel i
V_i	127.45	122.306	$\mathrm{mm^3}$	Perfusion volume of term. vessel i

3 Verification Test 1

3.1 Assumptions and analytical results

We set the permeability of the myocardium $K = 0$, while imposing a zero flux BC on the boundaries. Thus the analytical solution for p will be $p = p_g$, with p_g being defined in eq. (31) (Doc1). To further test our model, we distinguish 2 cases. In case 1, we include the effect of the venous tree, while in case 2, we completely ignore it. Substituting to the governing equations:

$$w = 0 \tag{6}$$

$$p = p_g \tag{7}$$

$$-\nabla p \cdot \mathbf{n} = 0 \; on \; \partial\Omega \tag{8}$$

The expected pressure values are:
Case 1: $p = 2624,6662\, \mathbb{1}_{\Omega_{15}} + 2707,9303\, \mathbb{1}_{\Omega_{19}}$
Case 2: $p = 12386,8062\, \mathbb{1}_{\Omega_{15}} + 12384,3253\, \mathbb{1}_{\Omega_{19}}$

3.2 Numerical results

Case 1:

Variable	Numerical value	
	tv 15	tv 19
p	2624.64	2707.89
q_k^i (for coupling)	1041.850355	1031.510643

Remarks: We have achieved perfect agreement with the analytical solution for the pressure. The values for the flow, corresponding to each perfusion volume, are extremely close to the ones provided by the CFD model. This result is valid from a physical perspective, since the fluid remains stationary at its original position ($w = 0$ and $w \cdot \mathbf{n} = 0$ at $\partial\Omega$).(I think the last sentence is completely false!)

Case 2:

Variable	Numerical value	
	tv 15	tv 19
p	12386.8	12384.3
q_k^i (for coupling)	$2.93695319 \cdot 10^{-14}$	$-7.932110617 \cdot 10^{-14}$

Remarks: We have achieved perfect agreement with the analytical solution for the pressure. The values for the flow, corresponding to each perfusion volume, are equal to 0. This result is expected, since the iso pressure and the pressure corresponding to each source are identical. The validity of the result becomes apparent when we consider the way q_k^i is calculated, that is $q_k^i = \int_{\Omega_i} \beta_i (p_i - p)$.

4 Verification Test 2

4.1 Assumptions and analytical results

We set the global source term $s_g = 0$. We distinguish 3 cases. In case 1, we impose a zero flux BC on the boundaries. In case 2, we impose $p = 0$ on the epicardium, a zero flux BC on the endocardium and a constant velocity of $f_b = 3\ mm/s$ on the top surface of the myocardium, while in case 3, we impose $p = 0$ on the epicardium, a zero flux BC on the on the top surface of the myocardium and a constant velocity of $f_b = 3\ mm/s$ on the endocardium.
Substituting to the governing equations:

$$w = -K\nabla p \tag{9}$$

$$\nabla \cdot w = 0 \tag{10}$$

With BCs for **Case 1**:

$$-\nabla p \cdot \mathbf{n} = 0 \; on \; \partial\Omega \tag{11}$$

With BCs for **Case 2**:

$$p = 0 \; on \; \partial\Omega_{epi} \tag{12}$$

$$-\nabla p \cdot \mathbf{n} = 0 \; on \; \partial\Omega_{endo} \tag{13}$$

$$w \cdot \mathbf{n} = 3 \; on \; \partial\Omega_{top} \tag{14}$$

With BCs for **Case 3**:

$$p = 0 \; on \; \partial\Omega_{epi} \tag{15}$$

$$w \cdot \mathbf{n} = 3 \; on \; \partial\Omega_{endo} \tag{16}$$

$$-\nabla p \cdot \mathbf{n} = 0 \; on \; \partial\Omega_{top} \tag{17}$$

The expected pressure values are:
Case 1: $p = 0$
Case 2: We are interested in the value of pressure at the inlet, that is at $\partial\Omega_{top}$. This value will be given by the following formula: $p_{inlet} = \frac{f_b}{A_{top} \cdot K} =$
Case 3: Similarly with Case 2, we are interested in the value of pressure at the inlet, that is at $\partial\Omega_{endo}$. This value will be given by the following formula: $p_{inlet} = \frac{f_b}{A_{endo} \cdot K} =$

4.2 Numerical results

Case 1:

Variable	**Numerical value**		**Variable**	**Numerical value**	
	min	max		tv 15	tv 19
p	-1.26799e-019	-1.26799e-019	q_k^i (for coupling)	2.890557654e-036	2.77388092e-036

Remarks: We have achieved perfect agreement with the analytical solution for the pressure. The values for the flow, corresponding to each perfusion volume are also expected, since we have no flow in our system.

Case 2:

Variable	**Numerical value**		**Variable**	**Numerical value**	
	min	max		tv 15	tv 19
p	-1117.7	max 3.73779e-031	q_k^i (for coupling)	1.002729111e-015	9.973168199e-016

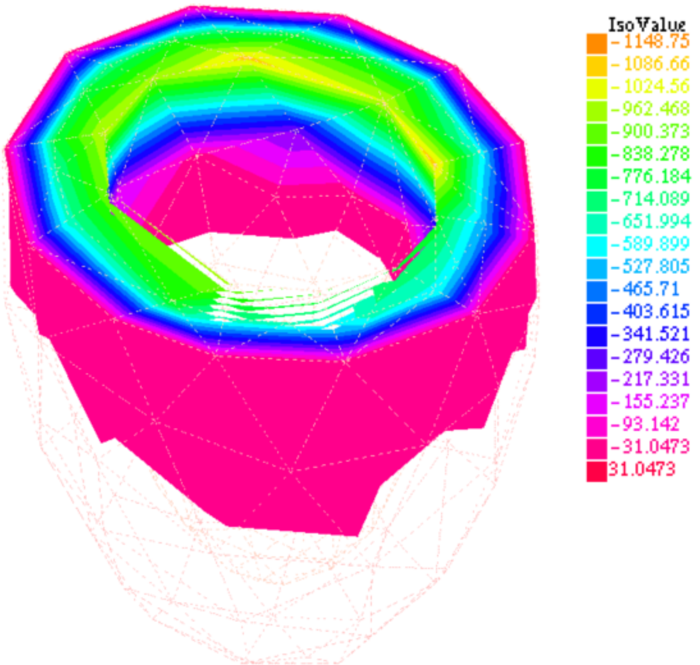

Figure 1: Verification Test 2 - Case 2 (p in Pa)

Case 3:

Variable	Numerical value min	Numerical value max	Variable	Numerical value tv 15	Numerical value tv 19
p	-1571.18	-5.26325e-031	q_k^i (for coupling)	6.958696594e-015	6.798967491e-015

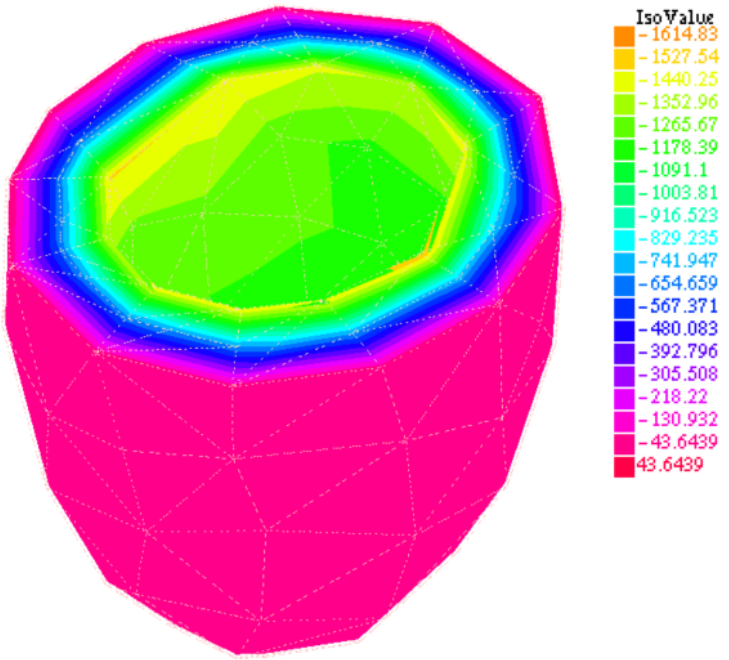

Figure 2: Verification Test 2 - Case 3 (p in Pa)

5 Verification Test 3

5.1 Assumptions and analytical results

This test is a replication of the one that was used to verify our model in a simple cube geometry. We set the global parameters p_g and b_g to the values 100 and $6.169 \cdot 10^{-3}$ respectively. We impose a zero flux conditions at the top surface of the myocardium, a constant pressure BC $p = 0$ at the epicardium and a constant pressure BC $p = 1000$ Pa at the endocardium. The expected pressure values will be given by the following analytical formula: ( )

5.2 Numerical results

Case 1:

Variable	**Numerical value**		**Variable**	**Numerical value**	
	min	max		tv 15	tv 19
p	-1.49473e-030	1000	q_k^i (for coupling)	1275.124966	1273.43676

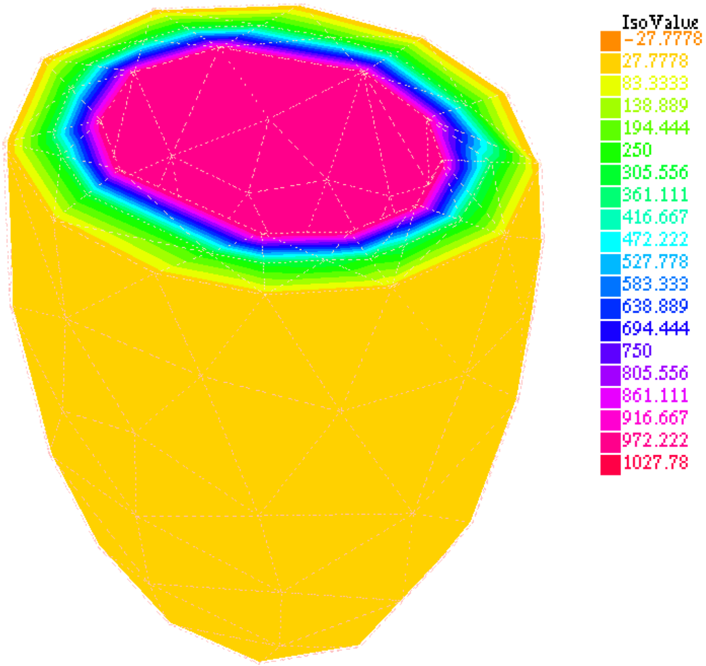

Figure 3: Verification Test 3 (p in Pa)

Remarks: Still haven't checked with the analytical solution, however the results seem correct. We have the same pressure profile as with the cube test. We can also notice that the min and max values of the pressure are exactly equal to the BCs imposed, as expected.