Distance and curvature

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1 Normals and curvature

There is a deep link between the evolution of the normal along a curve and the curvature.

Let C be a planar curve. To define the curvature at a point, we can consider the case of a straight line. We can admit that in this case the curvature is zero along the line. For a portion of a circle, the curvature is defined to be inversely proportional to the radius of the circle:

$$\kappa = \frac{1}{R} \tag{1}$$

More precisely, for any curve, we can locally approximate a sufficiently regular curve ($\mathcal{C}^2$ is enough) at a point by a circle that best approximates it locally (see Fig. 1. This circle is tangent to C and is called an *osculating circle*. The radius of this circle defines the curvature.

Another way to define the curvature is to consider a point moving at a constant speed along the curve C. The variation of the tangent vector along the curve defines the curvature. This is equivalent to specifying the acceleration of the point.

$$\kappa = \frac{d\mathbf{T}}{ds}, \tag{2}$$

where s is a parametrisation of the curve. Both definition of the curvature are in fact equivalent (see Fig. 2).

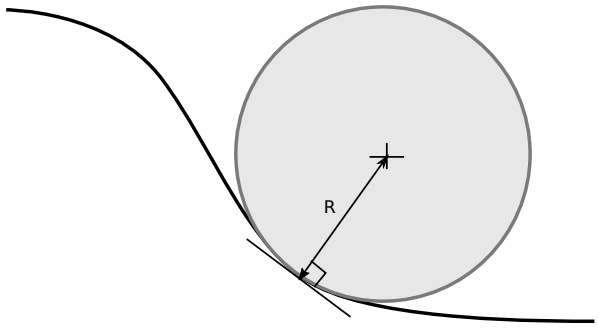

Figure 1: The notion of an osculating circle (best local approximation) defines the curvature as the inverse of the radius of the circle.

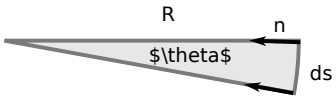

Figure 2: We can write $\sin d\theta \approx d\theta = \frac{ds}{R}$. As ds tends to zero, we have $R = \frac{d\theta}{ds} = \frac{d\mathbf{T}}{ds}$.

Parametrisation

Let $\gamma(t)$ be a parametrisation of the curve C, i.e.

$$\gamma(t) = (x(t), y(t)) \tag{3}$$

This defines the position of a point on the curve over time. We assume an injective parametrisation, i.e. such that the speed $\gamma'(t)$ is never zero. This means

$$\forall t, \|\gamma'(t)\|^2 = x'(t)^2 + y'(t)^2 > 0 \tag{4}$$

In this case, we can re-paramametrise the curve with curvilinear abcissa s in such a way that the speed is constant and equal to one.

$$\forall s, \gamma'(s)^2 = x'(s)^2 + y'(s)^2 = 1 \tag{5}$$

In this parametrisation, γ' is the unit tangent velocity vector $\mathbf{T}$. If $\mathbf{N}$ is the unit normal vector to the curve, we have

$$\mathbf{T}'(s) = \kappa(s)\mathbf{N}(s) \tag{6}$$

We note that instead of deriving the unit tangent vector, we can also consider deriving the unit normal vector. This is because $\mathbf{N}$ is $\mathbf{T}$ rotated by $\frac{\pi}{2}$, i.e. $\mathbf{N}(x, y) = (-y'(s), x'(s))$. This yields

$$\mathbf{N}'(s) = \kappa(s)\mathbf{T}(s) \tag{7}$$

We will make use of that fact in the next section.

Level sets

In imaging it can be difficult to represent a parametric curve because of discretization effects. It is common to represent it by a *level set* (see Fig. 3).

Let $\phi(x, y)$ be a $\mathcal{C}^2$ function in a domain Ω. We define the curve Γ as the zero-level-set of this function

$$\Gamma = \{(x, y), \phi(x, y) = 0\} \tag{8}$$

Γ is a set and no longer a parametrized curve, however we can manipulate it by working on the underlying ϕ function. This is the main idea behind the level-set method [**?**].

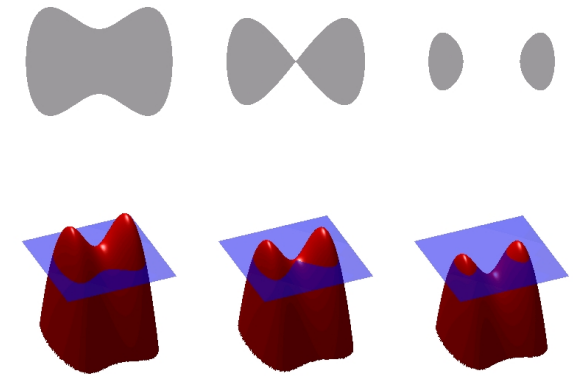

Figure 3: Representing a curve by a level set.

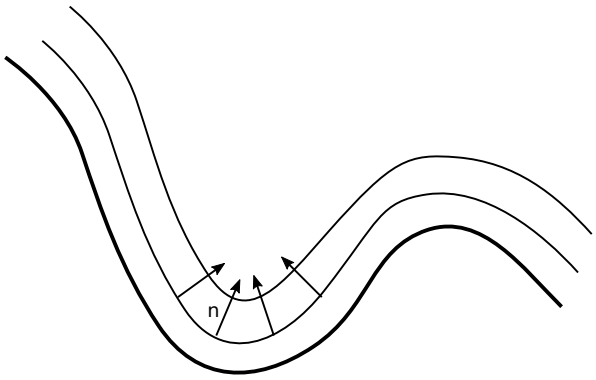

Figure 4: The faster the normal evolves along a curve, the higher the curvature.

Level sets and curvature

Curvature is easy to define in the level set case. For any point (x, y) in Ω, $\nabla\phi(x, y)$ is the gradient at (x, y). If we consider the level-set at $(x, y, \phi(x, y))$, i.e. the curve that passes through (x, y) at level $\phi(x, y)$, then $\nabla\phi(x, y)$ is the normal vector to this curve at (x, y). The unit normal is given by

$$\mathbf{n}(x, y) = \frac{\nabla\phi}{|\nabla\phi|}(x, y) \tag{9}$$

The curvature is given by the derivative of this expression. However this is a multidimensional derivative. Since $\nabla\phi$ is a vector, we must use the divergence operator:

$$\kappa = \nabla . \frac{\nabla\phi}{|\nabla\phi|} \tag{10}$$

This is in particular true for the computation of the curvature of Γ.

2 Application to our problem

We want to define how to sample a segment p_1p_2 so that checking at each sampling point if it is located outside the perfusion territory, we can conclude the segment belongs inside the territory or not. The number of checking $(n+1)$, shall be optimized in order to avoid too many tests (minimize computation time) but also to guarantee the result with a tolerance reasonable enough (maximize test accuracy).

2.1 First approach

We can estimate n, the number of tests along the segment, from:

$$n \propto \frac{|p_1 - p_0|}{R} \tag{11}$$

R is an approximation of the local curvature along the segment by estimating the divergence at each point p_1 and p_2, called R_1 and R_2 respectively :

$$R = max(|R_1|, |R_2|) \tag{12}$$

The figure 5 illustrates the relationship between $D = |p_1 - p_0|$, R_1, R_2 and n.

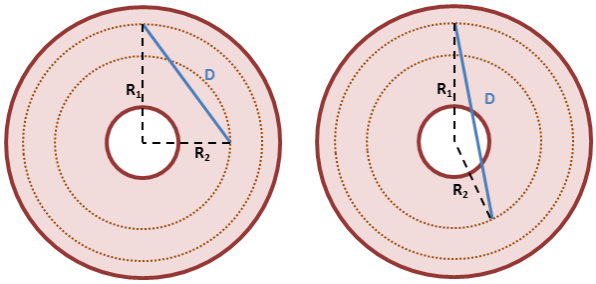

Figure 5: When D gets bigger relatively to $max(|R_1, R_2|)$, the value of n is also higher to detect if the segment crosses the perfusion territory.

The divergence of a a continuously differentiable vector field ω is equal to the scalar value function:

$$div(\omega) = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) \cdot (\omega_x, \omega_y) \tag{13}$$

$$div(\omega) = (\omega_x(x,y) - \omega_x(x-1,y)) + (\omega_y(x,y) - \omega_y(x,y-1)) \tag{14}$$

with

$$\omega_x = \nabla_x \omega = \omega(x+1,y) - \omega(x,y) \tag{15}$$

and

$$\omega_y = \nabla_y \omega = \omega(x,y+1) - \omega(x,y) \tag{16}$$

2.2 Second approach

We are in the case of measuring the curvature along a segment, so we could use the segment information within an iterative process.

We calculate the distances λ_i that solves:

$$\omega(p_1) + \lambda_1 < \nabla\omega(p_1), p_1p_2 >= 0 \tag{17}$$

$$\omega(p_2) + \lambda_2 < \nabla\omega(p_2), p_2p_1 >= 0 \tag{18}$$

and then select:

$$\lambda = min(\lambda_1, \lambda_2) \tag{19}$$

to obtain :

$$n = ceiling(\frac{1}{\lambda}) \tag{20}$$

We keep in mind that this method would fail for any case where the dot product $< \nabla\omega(p_1), p_1p_2 > simeq0$, and in some specific situations such as the figure x, so we define a global boundary:

$$n \geq \frac{||p_1p_2||}{2R_{max}} \tag{21}$$

with R_{max} the maximal curvature radius of both inner and outer surfaces.

This equation does not look correct to me: you are adding a potential value to a scalar which has nothing to do with a potential. You actually don't test if the line defined by the segment p_1p_2 crosses a concavity, which actually might not happen!

If we want to find the first point along the gradient direction that reaches the border of the perfusion territory we look for *lambda* that solves:

$$\omega(p_1 + \lambda\nabla w(p_1)) = 0 \tag{22}$$

This can also be written as:

$$\omega(p_1) + \lambda||\nabla w(p_1)|| = 0 \tag{23}$$

If λ is small, p_1 is close to a concavity. Then we can use its projection on the segment p_1p_2 to determine if the segment is how fast the segment is going toward the concavity.

$$d_{proj} = \frac{< \lambda\nabla\omega(p_1), p_1p_2 >}{||p_1p_2||} \tag{24}$$

This distance d_{proj} is efficient to sample the segment, so n is defined as:

$$n = ceiling(\frac{||p_1p_2||}{d_{proj}}) \tag{25}$$

$$n = ceiling(\frac{1}{< \lambda\nabla\omega(p_1), p_1p_2 >}) \tag{26}$$

This method is expected to be more accurate than the previous one 2.1, because it considers the potential along the segment (not only at the tips), but also both local and global curvature.