

Distance, curvature

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1 Normals and curvature

There is a deep link between the evolution of the normal vector along a curve and its curvature

Let C be a planar curve. To define the curvature at a point, we can consider the case of a straight line. We can admit that in this case the curvature is zero along the line. For a portion of a circle, the curvature is defined to be inversely proportional to the radius of the circle:

$$\kappa = \frac{1}{R} \quad (1)$$

More precisely, for any curve, we can locally approximate a sufficiently regular curve (\mathcal{C}^2 is enough) at a point by a circle that best approximates it locally (see Fig. 1. This circle is tangent to C and is called an *osculating circle*. The radius of this circle defines the curvature.

Another way to define the curvature is to consider a point moving at a constant speed along the curve C . The variation of the tangent vector along the curve defines the curvature. This is equivalent to specifying the acceleration of the point.

$$\kappa = \frac{d\mathbf{T}}{ds}, \quad (2)$$

where s is a parametrisation of the curve. Both definition of the curvature are in fact equivalent (see Fig. 2).

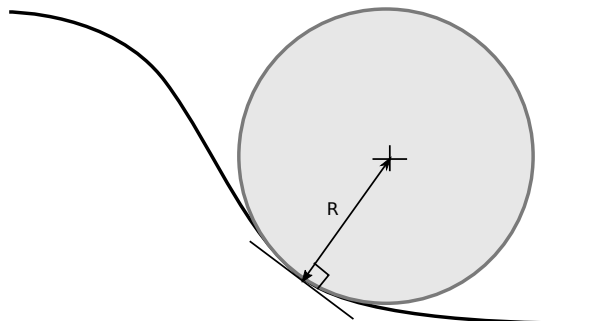


Figure 1: The notion of an osculating circle (best local approximation) defines the curvature as the inverse of the radius of the circle.

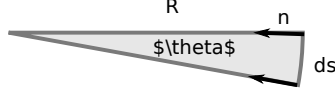


Figure 2: We can write $\sin d\theta \approx d\theta = \frac{ds}{R}$. As ds tends to zero, we have $R = \frac{d\theta}{ds} = \frac{d\mathbf{T}}{ds}$.

Parametrisation

Let $\gamma(t)$ be a parametrisation of the curve C , i.e.

$$\gamma(t) = (x(t), y(t)) \quad (3)$$

This defines the position of a point on the curve over time. We assume an injective parametrisation, i.e. such that the speed $\gamma'(t)$ is never zero. This means

$$\forall t, \|\gamma'(t)\|^2 = x'(t)^2 + y'(t)^2 > 0 \quad (4)$$

In this case, we can re-parametrise the curve with curvilinear abscissa s in such a way that the speed is constant and equal to one.

$$\forall s, \gamma'(s)^2 = x'(s)^2 + y'(s)^2 = 1 \quad (5)$$

In this parametrisation, γ' is the unit tangent velocity vector \mathbf{T} . If \mathbf{N} is the unit normal vector to the curve, we have

$$\mathbf{T}'(s) = \kappa(s)\mathbf{N}(s) \quad (6)$$

We note that instead of deriving the unit tangent vector, we can also consider deriving the unit normal vector. This is because \mathbf{N} is \mathbf{T} rotated by $\frac{\pi}{2}$, i.e. $\mathbf{N}(x, y) = (-y'(s), x'(s))$. This yields

$$\mathbf{N}'(s) = \kappa(s)\mathbf{T}(s) \quad (7)$$

We will make use of that fact in the next section.

Level sets

In imaging it can be difficult to represent a parametric curve because of discretization effects. It is common to represent it by a *level set* (see Fig. 3).

Let $\phi(x, y)$ be a \mathcal{C}^2 function in a domain Ω . We define the curve Γ as the zero-level-set of this function

$$\Gamma = \{(x, y), \phi(x, y) = 0\} \quad (8)$$

Γ is a set and no longer a parametrized curve, however we can manipulate it by working on the underlying ϕ function. This is the main idea behind the level-set method [?].

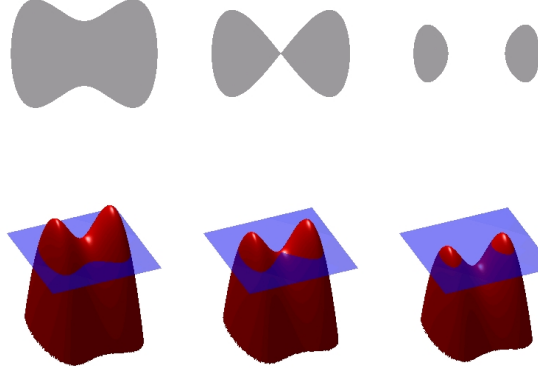


Figure 3: Representing a curve by a level set.

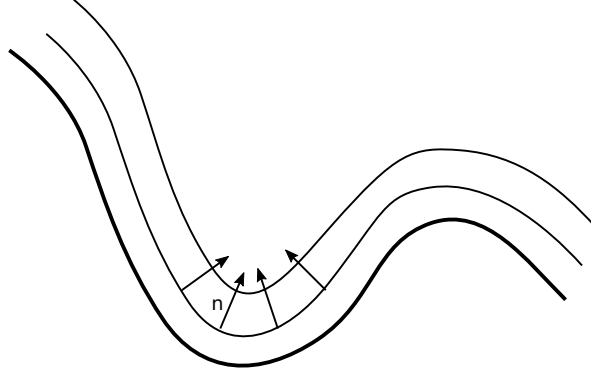


Figure 4: The faster the normal evolves along a curve, the higher the curvature.

Level sets and curvature

Curvature is easy to define in the level set case. For any point (x, y) in Ω , $\nabla\phi(x, y)$ is the gradient at (x, y) . If we consider the level-set at $(x, y, \phi(x, y))$, i.e. the curve that passes through (x, y) at level $\phi(x, y)$, then $\nabla\phi(x, y)$ is the normal vector to this curve at (x, y) . The unit normal is given by

$$\mathbf{n}(x, y) = \frac{\nabla\phi}{|\nabla\phi|}(x, y) \quad (9)$$

The curvature is given by the derivative of this expression. However this is a multidimensional derivative. Since $\nabla\phi$ is a vector, we must use the divergence operator $\nabla \cdot \mathbf{F} \equiv \sum_i^d \frac{\partial \mathbf{F}_i}{\partial x_i}$:

$$\kappa = \nabla \cdot \frac{\nabla\phi}{|\nabla\phi|} \quad (10)$$

This is in particular true for the computation of the curvature of Γ .

2 Application to our problem

Here we assume a non-convex territory with two nested surfaces as borders. Every point p of the territory is associated with a value $\omega(p)$ so that $\omega(p) = 0$ if p belongs to the inner surface, and $\omega(p) = 1$ on the outer surface. Numerically, ω can for example be computed from solving the electrostatic Poisson equation, i.e. a random walker. Our formulations and illustrations are in 2D but carry over to 3D without significant changes.

Let $[p_1 p_2]$ be the segment defined by the points $\{p_1, p_2\}$. We want to define how to sample this segment, so that we can conclude with high confidence whether or not the whole segment is located inside a non-convex territory. The number of samples $(n + 1)$, shall be optimized in order to avoid too many tests (minimize computation time) but also to guarantee the result within a reasonable tolerance (maximize test accuracy).

2.1 First approach

We can estimate n , the number of samples along the segment, from:

$$n \propto \frac{|p_2 - p_1|}{R} \quad (11)$$

R is an approximation of the local radius of curvature along the segment by estimating the divergence at each point p_1 and p_2 , called κ_1 and κ_2 respectively :

$$R = \max(|\frac{1}{\kappa_1}|, |\frac{1}{\kappa_2}|) \quad (12)$$

The figure 5 illustrates the relationship between $D = |p_2 - p_1|$, $R_1 = \frac{1}{|\kappa_1|}$, $R_2 = \frac{1}{|\kappa_2|}$ and n .

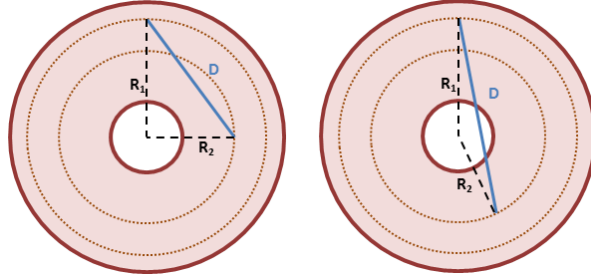


Figure 5: When D becomes larger relatively to $\max(|R_1, R_2|)$, the value of n is also higher to detect if the segment crosses the perfusion territory.

The divergence of a continuously differentiable vector field $\mathbf{w} = (w_x, w_y)$ is equal to the scalar value function:

$$\text{div}(\mathbf{w}) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \cdot (w_x, w_y) \quad (13)$$

$$\text{div}(\mathbf{w}) = (w_x(x, y) - w_x(x - 1, y)) + (w_y(x, y) - w_y(x, y - 1)). \quad (14)$$

Since we are interested in estimating a curvature, the field \mathbf{w} is the normalized gradient of ω .

$$w_x = \frac{\nabla_x \omega}{\|\nabla \omega\|} = \frac{\omega(x + 1, y) - \omega(x, y)}{\sqrt{(\nabla_x \omega)^2 + (\nabla_y \omega)^2}} \quad (15)$$

and

$$w_y = \frac{\nabla_y \omega}{\|\nabla \omega\|} = \frac{\omega(x, y+1) - \omega(x, y)}{\sqrt{(\nabla_x \omega)^2 + (\nabla_y \omega)^2}} \quad (16)$$

This approach does not take into account the position of p_1 and p_2 within the territory, and so it is easy to find counter-examples where this approach fails.

2.2 Second approach

Here we consider an iterative process to take into account the curvature along a segment.

2.2.1 Interpolating the point along the segment line that crosses the concavity

We calculate the distances λ_1, λ_2 that solves:

$$\omega(p_1) + \lambda_1 \langle \nabla \omega(p_1), \vec{p_1 p_2} \rangle = 0, \quad (17)$$

and symmetrically for λ_2 .

Then we select:

$$\lambda = \min(\lambda_1, \lambda_2) \quad (18)$$

to obtain :

$$n = \left\lceil \frac{1}{\lambda} \right\rceil, \quad (19)$$

where $\lceil \cdot \rceil$ denotes the ceiling operator, i.e. the function that maps a real number to the smallest integer that is larger or equal to it.

If $\lambda < 0$ or $\lambda > 1$, it means the line $(p_1 p_2)$ crosses the concavity outside of the segment $[p_1 p_2]$. We could consider that there is actually no need of refining the test along the segment. But we have to keep in mind that this interpretation comes only from a linear local approximation. In Fig. 6, we show a counter-example, where this approach fails.

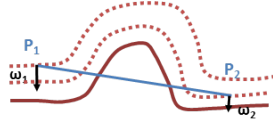


Figure 6: Specific situation where the local gradient doesn't help detecting the concavity.

Also, this method would fail for any case where the dot product $\langle \nabla \omega(p_1), \vec{p_1 p_2} \rangle \simeq 0$, and in some specific situations such as the figure 2.2.1. Hence we need to define a global boundary:

$$n \geq \frac{\|\vec{p_1 p_2}\|}{R_{\max}} \quad (20)$$

with R_{\max} the maximal curvature radius of both inner and outer surfaces.

Note: if the segmentation is noisy, we might actually measure non-physical very high curvatures, i.e. R_{\max} close to zero, hence it is necessary to consider a tolerance s so that

$$R_{\max}^* = R_{\max} + s \quad (21)$$

The semi-cord c defined by R_{\max}^* and s is calculated as:

$$c = \sqrt{(R_{\max}^* - s)^2 - R_{\max}^{*2}} \quad (22)$$

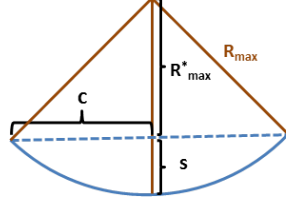


Figure 7: The maximally inscribed radius and tolerance s define a semi-cord c .

That can be simplified as:

$$c = \sqrt{R_{max}^2 - (R_{max} + s)^2} \quad (23)$$

The global boundary follows this tolerance:

$$n \geq \frac{\|\vec{p_1 p_2}\|}{c} \quad (24)$$

We take the largest n between maximal curvature and gradient method. This provides a robust sampling definition considering both local and global informations, see figure 2.2.1.

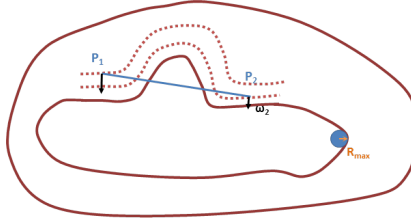


Figure 8: In this situation, considering the global maximum curvature will allow us to detect the concavity between p_1 and p_2 .

2.2.2 Estimating a sampling distance from the projection of the border

Given p_1 and p_2 , we want to find length M_1 in Fig. 9 from local information.

First we denote p_0 the intersection of the steepest descent from p_1 with the 0-surface, the surface where $\omega = 0$. since $\nabla\omega(p_1)$ and $\vec{p_0 p_1}$ are co-linear, to first order we can write

$$\nabla\omega(p_1) \approx \frac{\omega(p_0) - \omega(p_1)}{\vec{p_1 p_0}} \quad (25)$$

Point p_0 is actually unknown, but $\omega(p_0) = 0$, so we have:

$$\vec{p_1 p_0} \approx \frac{-\omega(p_1)}{\nabla\omega(p_1)} \quad (26)$$

Then we can project this vector $\vec{p_1 p_0}$ on the segment $\vec{p_1 p_2}$:

$$M_1 = \frac{\langle \vec{p_1 p_0}, \vec{p_1 p_2} \rangle}{\|\vec{p_1 p_2}\|} \quad (27)$$

