Distance and curvature

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1 Normals and curvature

There is a deep link between the evolution of the normal along a curve and the curvature.

Let C be a planar curve. To define the curvature at a point, we can consider the case of a straight line. We can admit that in this case the curvature is zero along the line. For a portion of a circle, the curvature is defined to be inversely proportional to the radius of the circle:

$$\kappa = \frac{1}{R} \tag{1}$$

More precisely, for any curve, we can locally approximate a sufficiently regular curve ($\mathcal{C}^2$ is enough) at a point by a circle that best approximates it locally (see Fig. 1. This circle is tangent to C and is called an *osculating circle*. The radius of this circle defines the curvature.

Another way to define the curvature is to consider a point moving at a constant speed along the curve C. The variation of the tangent vector along the curve defines the curvature. This is equivalent to specifying the acceleration of the point.

$$\kappa = \frac{d\mathbf{T}}{ds}, \tag{2}$$

where s is a parametrisation of the curve. Both definition of the curvature are in fact equivalent (see Fig. 2).

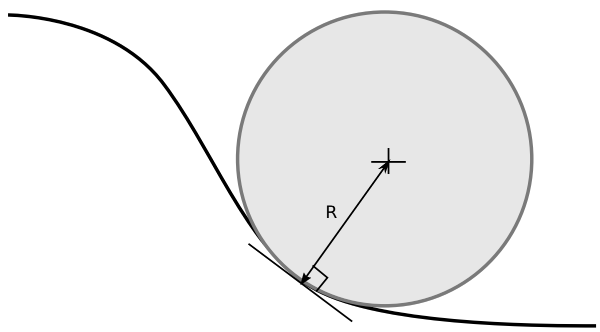

Figure 1: The notion of an osculating circle (best local approximation) defines the curvature as the inverse of the radius of the circle.

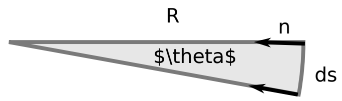

Figure 2: We can write $\sin d\theta \approx d\theta = \frac{ds}{R}$. As ds tends to zero, we have $R = \frac{d\theta}{ds} = \frac{d\mathbf{T}}{ds}$.

Parametrisation

Let $\gamma(t)$ be a parametrisation of the curve C, i.e.

$$\gamma(t) = (x(t), y(t)) \tag{3}$$

This defines the position of a point on the curve over time. We assume an injective parametrisation, i.e. such that the speed $\gamma'(t)$ is never zero. This means

$$\forall t, \|\gamma'(t)\|^2 = x'(t)^2 + y'(t)^2 > 0 \tag{4}$$

In this case, we can re-paramametrise the curve with curvilinear abcissa s in such a way that the speed is constant and equal to one.

$$\forall s, \gamma'(s)^2 = x'(s)^2 + y'(s)^2 = 1 \tag{5}$$

In this parametrisation, γ' is the unit tangent velocity vector $\mathbf{T}$. If $\mathbf{N}$ is the unit normal vector to the curve, we have

$$\mathbf{T}'(s) = \kappa(s)\mathbf{N}(s) \tag{6}$$

We note that instead of deriving the unit tangent vector, we can also consider deriving the unit normal vector. This is because $\mathbf{N}$ is $\mathbf{T}$ rotated by $\frac{\pi}{2}$, i.e. $\mathbf{N}(x, y) = (-y'(s), x'(s))$. This yields

$$\mathbf{N}'(s) = \kappa(s)\mathbf{T}(s) \tag{7}$$

We will make use of that fact in the next section.

Level sets

In imaging it can be difficult to represent a parametric curve because of discretization effects. It is common to represent it by a *level set* (see Fig. 3).

Let $\phi(x, y)$ be a $\mathcal{C}^2$ function in a domain Ω. We define the curve Γ as the zero-level-set of this function

$$\Gamma = \{(x, y), \phi(x, y) = 0\} \tag{8}$$

Γ is a set and no longer a parametrized curve, however we can manipulate it by working on the underlying ϕ function. This is the main idea behind the level-set method [1].

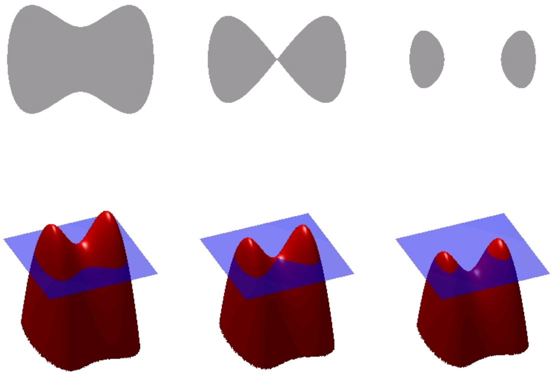

Figure 3: Representing a curve by a level set.

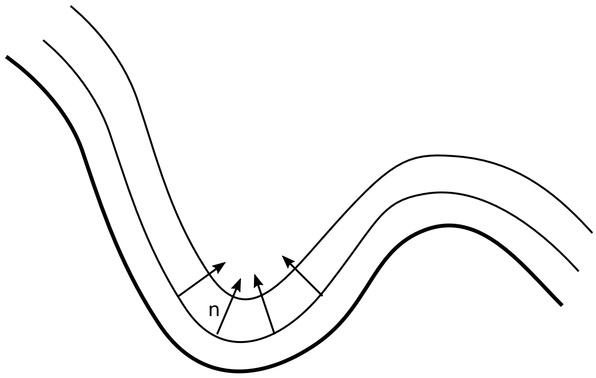

Figure 4: The faster the normal evolves along a curve, the higher the curvature.

Level sets and curvature

Curvature is easy to define in the level set case. For any point (x, y) in Ω, $\nabla\phi(x, y)$ is the gradient at (x, y). If we consider the level-set at $(x, y, \phi(x, y))$, i.e. the curve that passes through (x, y) at level $\phi(x, y)$, then $\nabla\phi(x, y)$ is the normal vector to this curve at (x, y). The unit normal is given by

$$\mathbf{n}(x, y) = \frac{\nabla\phi}{|\nabla\phi|}(x, y) \tag{9}$$

The curvature is given by the derivative of this expression. However this is a multidimensional derivative. Since $\nabla\phi$ is a vector, we must use the divergence operator:

$$\kappa = \nabla . \frac{\nabla\phi}{|\nabla\phi|} \tag{10}$$

This is in particular true for the computation of the curvature of Γ.

References

[1] J.A. Sethian. *Level set methods and fast marching methods*. Cambridge University Press, 1999. ISBN 0-521-64204-3.