MACS205

Monte Carlo methods

Project 3: Control variates with kernel smoothing: toward faster than root n rates

Each project must be done by groups of students of about $4-5$ persons. Each project must contain a theoretical development and a practical part as well (based on simulation). Each group has until April 12 to upload their documents on eCampus. Here is the list of the required documents:

- A report of (about) 10 pages (or more). The report must be clearly written, well structured and illustrated.
- A notebook (python or R might be used). The notebook should be well-structured with markdowns explaining what is done.
- Each student must keep records, i.e., the list of his action concerning his project (like a personal diary). An example is given in the eCampus outline.

Each student must read carefully the lecture notes (available on Ecampus) before working on the project. They will help the students to understand and to realize their projects.

1 The background

The background is the one of the chapter on control variates in the lecture notes. The aim is to estimate $\int gfd\lambda$ with $g:\mathbb{R}^d\to\mathbb{R}$ and f is a density function on $\mathbb{R}^d$. We have a sampler q. Given an iid sequence $(X_i)_i$ with distribution q the importance sampling estimate is given by

$$n^{-1}\sum_{i=1}^n g(X_i)w(X_i),$$

with $w(x)=f(x)/q(x)$. Let Φ be a class of control function such that $\int \phi d\lambda = 0$ for all $\phi\in\Phi$. Given $\phi\in\Phi$, the control variate estimate is then given by

$$\hat{I}_n(\phi)=n^{-1}\sum_{i=1}^n (f(X_i)-\phi(X_i))/q(X_i). \tag{1}$$

A recent article in relation with the project is Oates et al. (2017).

2 The project

This project consists in a study of the method introduced before.

- When ϕ is fixed, provide an analysis of $\hat{I}_n(\hat{\phi}_m)$ (bias, variance, MSE). Give explicit bounds on the rate of convergence.
- The problem that should be considered in practice could be the one of estimating mean value of the Gaussian $\mathcal{N}(0,1)$ when the sampler q is the density of $\mathcal{N}(\theta,1)$.
- The simulations should take into account the number of evaluations of f.

3 Ideas

Here is a list of ideas that might be used.

- Present kernel smoothing estimation. In particular achieve a complete study of the estimate

$$\tilde{\varphi}_m(x) = m^{-1} \sum_{j=1}^{m} (f(X'_j)/q(X'_j)) K_h(x - X'_j).$$

  Give its bias and its variance and derive conditions under which $\mathbb{E}[(\tilde{\varphi}_m(x) - \varphi(x))^2]$ goes to 0. Try also to implement the previous idea by optimizing the weights α in

$$\sum_{j=1}^{m} \alpha_i K_h(x - X'_j).$$

- In the practical implementation, it shall be important to put in relation the quality of $\tilde{\varphi}$ estimating φ with the one the accuracy of the control variate method.

References

Oates, C. J., M. Girolami, and N. Chopin (2017). Control functionals for Monte Carlo integration. *J. R. Statist. Soc. B 79*(3), 695–718.