Analysis [für Informatiker]
Übungsblatt 5

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Mittwoch 11:15-12:45 Drigalla, Stefan Gruppe a;
Montag 15:15-16:45 Drigalla, Stefan Gruppe b

1) Zeigen Sie durch vollständige Induktion, dass für alle $n \in \mathbb{N}$,

a) $\sum_{k=1}^{n} k^3 = (\frac{n(n+1)}{2})^2$,
Induktion:

IA $n = 1$:

$$\sum_{k=1}^{1} k^3 = 1^3 = 1$$

$$\left(\frac{1(1+1)}{2}\right)^2 = \left(\frac{2}{2}\right)^2 = 1$$

IV $n = n$:

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

IS $n = n+1$:

$$\begin{aligned}
\sum_{k=1}^{n+1} k^3 &= (n+1)^3 + \sum_{k=1}^{n} k^3 \\
&= (n+1)^3 + \left(\frac{n(n+1)}{2}\right)^2 \\
&= \frac{4(n+1)^3 + n^2(n+1)^2}{4} \\
&= \frac{4(n^3+3n^2+3n+1) + n^2(n^2+2n+1)}{4} \\
&= \frac{4n^3+12n^2+12n+4+n^4+2n^3+n^2}{4} \\
&= \frac{n^4+6n^3+13n^2+12n+4}{4} \\
&= \frac{(n^2+3n+2)^2}{2^2} \\
&= \left(\frac{n^2+3n+2}{2}\right)^2 \\
&= \left(\frac{(n+1)(n+2)}{2}\right)^2 \\
&= \left(\frac{(n+1)((n+1)+1)}{2}\right)^2
\end{aligned}$$

$\square$

b) $\sqrt{n} \leq \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \leq 2\sqrt{n} - 1$ [Hinweis: nutzen Sie die 2.Binomische Formel um $\sqrt{n+1} - \sqrt{n}$ als Bruch darzustellen.]

Induktion $\sqrt{n} \leq \sum_{k=1}^{n} \frac{1}{\sqrt{k}}$

IA $n = 1$:

$$\begin{aligned}
\sqrt{1} &\leq 1 \\
&\leq \frac{1}{1} \\
&\leq \frac{1}{\sqrt{1}} \\
&\leq \sum_{k=1}^{1} \frac{1}{\sqrt{k}}
\end{aligned}$$

IV $n = n$:

$$\sqrt{n} \leq \sum_{k=1}^{n} \frac{1}{\sqrt{k}}$$

IS $n = n + 1$:

$$\begin{aligned}
\sqrt{n+1} &\leq \sum_{k=1}^{n+1} \frac{1}{\sqrt{k}} \\
\sqrt{n+1} &\leq \frac{1}{\sqrt{n+1}} + \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \\
\sqrt{n+1} &\leq \frac{1}{\sqrt{n+1}} + \sqrt{n} && | \cdot \sqrt{n+1} \\
n+1 &\leq 1 + \sqrt{n} \cdot \sqrt{n+1} && | - 1 \\
n &\leq \sqrt{n(n+1)} \\
n &\leq \sqrt{n^2 + n} && |()^2 \\
n^2 &\leq n^2 + n && | - n^2 \\
0 &\leq n
\end{aligned}$$

Induktion $\sum_{k=1}^{n} \frac{1}{\sqrt{k}} \leq 2\sqrt{n} - 1$

IA $n = 1$:

$$\begin{aligned}
\sum_{k=1}^{1} \frac{1}{\sqrt{k}} &\leq 2\sqrt{1} - 1 \\
\frac{1}{\sqrt{1}} &\leq 2 \cdot 1 - 1 \\
\frac{1}{1} &\leq 2 - 1 \\
1 &\leq 1
\end{aligned}$$

IV $n = n$:

$$\sum_{k=1}^{n} \frac{1}{\sqrt{k}} \leq 2\sqrt{n} - 1$$

IS $n = n + 1$:

$$\begin{aligned}
\sum_{k=1}^{n+1} \frac{1}{\sqrt{k}} &\leq 2\sqrt{n+1} - 1 \\
\frac{1}{n+1} + \sum_{k=1}^{n} \frac{1}{\sqrt{k}} &\leq 2\sqrt{n+1} - 1 \\
\frac{1}{n+1} + 2\sqrt{n} - 1 &\leq 2\sqrt{n+1} - 1 && | + 1 \\
\frac{1}{n+1} + 2\sqrt{n} &\leq 2\sqrt{n+1} && | \cdot \sqrt{n+1} \\
1 + 2\sqrt{n(n+1)} &\leq 2(n+1) \\
1 + 2\sqrt{n^2+n} &\leq 2n + 2 && | - 1 \\
2\sqrt{n^2+n} &\leq 2n + 1 && | ()^2 \\
4n^2 + 4n &\leq 4n^2 + 4n + 1 && | - (4n^2 + 4n) \\
0 &\leq 1
\end{aligned}$$

$\square$

2)

a) Gegeben seien die komplexen Zahlen $z = 1 - i$ und $w = 2 + 3i$. Stellen Sie die folgenden komplexen Zahlen in der Form $a + bi, a, b \in \mathbb{R}$ dar:

$$z + w, zw\&\bar{z}\bar{w}, z/w, w/z, z^2\&z^20$$

$z + w$:

$$\begin{aligned}
z + w =& 1 - i + 2 + 3i \\
=& 3 + 2i \\
=& a + bi \\
\implies& a = 3, b = 2
\end{aligned}$$

zw:

$$\begin{aligned}
zw =& (1 - i)(2 + 3i) \\
=& 2 + 3i - 2i + 3i^2 \\
=& 2 - 3 + i \\
=& -1 + i \\
=& a + bi \\
\implies& a = -1, b = 1
\end{aligned}$$

$\bar{z}\bar{w}$:

$$\begin{aligned}\bar{z}\bar{w} =&(1+i)(2-3i)\\ =&2-3i+2i-3i^2\\ =&2+3-i\\ =&5-i\\ =&a+bi\\ \implies& a=5, b=-1\end{aligned}$$

z/w:

$$\begin{aligned}\frac{z}{w} =&\frac{1-i}{2+3i}\\ =&(1-i)(2+3i)^{-1}\\ =&(1-i)\left(\frac{2-3i}{2^2+3^2}\right)\\ =&(1-i)\left(\frac{2-3i}{13}\right)\\ =&\frac{2-3i-2i-3i^2}{13}\\ =&\frac{5-5i}{13}\\ =&a+bi\\ \implies& a=\frac{5}{13}, b=\frac{-5}{13}\end{aligned}$$

w/z:

$$\begin{aligned}\frac{w}{z} =&\frac{2+3i}{1-i}\\ =&(2+3i)(1-i)^{-1}\\ =&(2+3i)\left(\frac{1+i}{1^2+(-1)^2}\right)\\ =&(2+3i)\left(\frac{1+i}{2}\right)\\ =&\frac{2+2i+3i+3i^2}{2}\\ =&\frac{-1+5i}{2}\\ =&a+bi\\ \implies& a=\frac{-1}{2}, b=\frac{5}{2}\end{aligned}$$

z^2:

$$\begin{aligned} z^2 =& (1-i)^2 \\ =& 1-2i+i^2 \\ =& 0-2i \\ =& a+bi \\ \implies a=0, b=-2 \end{aligned}$$

z^{20}:

$$\begin{aligned} z^{20} =& (1-i)^{20} \\ =& (0-2i)^{10} \\ =& (-4+0i)^5 \\ =& -1024+0i \\ =& a+bi \\ \implies & a=-1024, b=0 \end{aligned}$$

b) Bestimmen Sie Menge

$$M_1 \left\{ z \in \mathbb{C} : \left| \frac{z+3}{z-3} \right| = 2 \right\},$$

das heißt, leiten Sie eine exakte Beschreibung (auch im Sinne der Schulmathematik) her.

Fall 1 $\frac{z+3}{z-3} = 2$:

$$\begin{aligned} \frac{z+3}{z-3} &= 2 \\ z &= a+bi \\ \frac{a+bi+3}{a+bi-3} &= 2 && |(a+bi-3) \\ a+bi+3 &= 2a+2bi-6 && |-(a+bi) \\ 3 &= a+bi-6 && |+6 \\ 9 &= a+bi \\ 9+0i &= a+bi \\ \implies a &= 9, b=0 \end{aligned}$$

Fall 2 $-\frac{z+3}{z-3} = 2$

$$
\begin{aligned}
-\frac{z+3}{z-3} &= 2 \\
z &= a+bi \\
\frac{-a-bi-3}{a+bi-3} &= 2 && |(a+bi+3) \\
-a-bi-3 &= 2a+2bi-6 && |+(a+bi) \\
-3 &= 3a+3bi-6 && |+6 \\
3 &= 3a+3bi && |:3 \\
1+0i &= a+bi \\
\implies a &= 1, b=0
\end{aligned}
$$

$$M_1 = \{9+0i, 1+0i\}$$

$\square$