Grundlagen Lineare Algebra für Informatiker

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Aufgabe 1 Bestimmen Sie sämtliche Lösungen der folgenden homogenen linearen Gleichungssysteme mit Hilfe des Gaußschen Algorithmus.

a)

$$\begin{aligned} x + 3y - 2z &= 0 \\ 5x + 6y - z &= 0 \\ 2x + y + z &= 0 \end{aligned}$$

b)

$$\begin{aligned} 2x_1 + 3x_2 + x_3 + x_4 &= 0 \\ x_1 - x_3 + 2x_4 &= 0 \\ x_1 + x_2 + x_4 &= 0 \\ x_2 + x_3 - x_4 &= 0 \end{aligned}$$

a)

$$
\begin{aligned}
&\begin{pmatrix} 1 & 3 & -2 \\ 5 & 6 & -1 \\ 2 & 1 & 1 \end{pmatrix}
\begin{array}{ll} \cdot(-5) & \cdot(-2) \\ + & \\ & + \end{array} \\
\sim &\begin{pmatrix} 1 & 3 & -2 \\ 0 & -9 & 9 \\ 0 & -5 & 5 \end{pmatrix}
\begin{array}{ll} + & \\ \cdot\frac{1}{3} & \cdot(-\frac{5}{9}) \\ & + \end{array} \\
\sim &\begin{pmatrix} 1 & 0 & 1 \\ 0 & -9 & 9 \\ 0 & 0 & 0 \end{pmatrix} \quad | \cdot (-\tfrac{1}{9}) \\
\sim &\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \\
\Longleftrightarrow &\begin{array}{ll} x + 0 + z & = 0 \\ 0 + y - z & = 0 \end{array} \Longrightarrow \begin{array}{ll} \text{Sei } z & = \lambda \\ x & = -\lambda \\ y & = \lambda \end{array} \\
\Longrightarrow & L = \left\{ \begin{pmatrix} -\lambda \\ \lambda \\ \lambda \end{pmatrix}, \lambda \in \mathbb{R} \right\}
\end{aligned}
$$

b)

$$
\begin{aligned}
&\begin{pmatrix} 2 & 3 & 1 & 1 \\ 1 & 0 & -1 & 2 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & -1 \end{pmatrix}
\begin{array}{l} \cdot(-\frac{1}{2}) \quad \cdot(-\frac{1}{2}) \\ + \\ + \\ \end{array} \\
\sim &\begin{pmatrix} 2 & 3 & 1 & 1 \\ 0 & -\frac{3}{2} & -\frac{7}{2} & \frac{3}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 1 & -1 \end{pmatrix}
\begin{array}{l} + \\ \cdot 2 \quad \cdot(-\frac{1}{3}) \quad \cdot\frac{2}{3} \\ + \\ + \end{array} \\
\sim &\begin{pmatrix} 2 & 0 & -6 & 4 \\ 0 & -\frac{3}{2} & -\frac{7}{2} & \frac{3}{2} \\ 0 & 0 & \frac{2}{3} & 0 \\ 0 & 0 & -\frac{4}{3} & 0 \end{pmatrix}
\begin{array}{l} + \\ + \\ \cdot 9 \quad \cdot\frac{21}{4} \quad \cdot 2 \\ + \end{array} \\
\sim &\begin{pmatrix} 2 & 0 & 0 & 4 \\ 0 & -\frac{3}{2} & 0 & \frac{3}{2} \\ 0 & 0 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\begin{array}{l} \mid \cdot \frac{1}{2} \\ \mid \cdot (-\frac{2}{3}) \\ \mid \cdot \frac{3}{2} \\ \end{array} \\
\sim &\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}
$$

$$
\iff \begin{aligned} x_1 + 0 + 0 + 2x_4 &= 0 \\ 0 + x_2 + 0 - x_4 &= 0 \\ 0 + 0 + x_3 + 0 &= 0 \end{aligned}
\implies \begin{aligned} \text{Sei } x_4 &= \lambda \\ x_1 &= -2\lambda \\ x_2 &= \lambda \\ x_3 &= 0 \end{aligned}
$$

$$
\implies L = \left\{ \begin{pmatrix} -2\lambda \\ \lambda \\ 0 \\ \lambda \end{pmatrix}, \lambda \in \mathbb{R} \right\}
$$

Aufgabe 2 Bestimmen Sie sämtliche Lösungen des folgenden inhomogenen linearen Gleichungssystems mit Hilfe des Gaußschen Algorithmus.

$$\begin{aligned} 2x + 5y - 3z &= 1 \\ 3x + 4y + z &= 0 \\ 3x + y + 2z &= 8 \end{aligned}$$

$$\left(\begin{array}{ccc|cl} 2 & 5 & -3 & 1 & \mid \cdot \frac{1}{2} \\ 3 & 4 & 1 & 0 & \mid \cdot \frac{1}{3} \\ 3 & 1 & 2 & 8 & \mid \cdot \frac{1}{3} \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|cll} 1 & \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} & \cdot -1 & \cdot -1 \\ 1 & \frac{4}{3} & \frac{1}{3} & 0 & + & \\ 1 & \frac{1}{3} & \frac{2}{3} & \frac{8}{3} & & + \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|cl} 1 & \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} & \\ 0 & -\frac{7}{6} & \frac{11}{6} & -\frac{1}{2} & \mid \cdot \left(-\frac{6}{7}\right) \\ 0 & -\frac{13}{6} & \frac{13}{6} & \frac{13}{6} & \mid \cdot \left(-\frac{6}{13}\right) \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|cll} 1 & \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} & + & \\ 0 & 1 & -\frac{11}{7} & \frac{3}{7} & \cdot\left(-\frac{5}{2}\right) & \cdot(-1) \\ 0 & 1 & -1 & -1 & & + \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|cl} 1 & 0 & \frac{17}{7} & -\frac{4}{7} & \\ 0 & 1 & -\frac{11}{7} & \frac{3}{7} & \\ 0 & 0 & \frac{4}{7} & -\frac{10}{7} & \mid \cdot \frac{7}{4} \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|cll} 1 & 0 & \frac{17}{7} & -\frac{4}{7} & + & \\ 0 & 1 & -\frac{11}{7} & \frac{3}{7} & & + \\ 0 & 0 & 1 & -\frac{5}{2} & \cdot\left(-\frac{17}{7}\right) & \cdot \frac{11}{7} \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{11}{2} \\ 0 & 1 & 0 & -\frac{7}{2} \\ 0 & 0 & 1 & -\frac{5}{2} \end{array}\right)$$

$$\Longleftrightarrow \begin{aligned} x + 0 + 0 &= \tfrac{11}{2} \\ 0 + y + 0 &= -\tfrac{7}{2} \\ 0 + 0 + z &= -\tfrac{5}{2} \end{aligned} \Longrightarrow \begin{aligned} x &= \tfrac{11}{2} \\ y &= -\tfrac{7}{2} \\ z &= -\tfrac{5}{2} \end{aligned}$$

$$\Longrightarrow L = \left\{ \begin{pmatrix} \frac{11}{2} \\ -\frac{7}{2} \\ -\frac{5}{2} \end{pmatrix} \right\}$$

Aufgabe 3 Bestimmen Sie die Inversen der folgenden Matrizen mit Hilfe des Gaußschen Algorithmus.

$$A = \begin{pmatrix} 1 & 3 & -1 \\ 2 & 0 & 2 \\ 1 & 2 & 1 \end{pmatrix} \qquad\qquad B = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & 3 \\ 4 & -2 & 2 \end{pmatrix}$$

a) A^{-1}

$$\left(\begin{array}{ccc|ccc} 1 & 3 & -1 & 1 & 0 & 0 \\ 2 & 0 & 2 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \cdot(-2) \quad \cdot(-1) \\ + \\ \quad\quad + \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 3 & -1 & 1 & 0 & 0 \\ 0 & -6 & 4 & -2 & 1 & 0 \\ 0 & -1 & 2 & -1 & 0 & 1 \end{array}\right) \begin{array}{l} \\ \mid \cdot(-\frac{1}{6}) \\ \mid \cdot(-1) \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 3 & -1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & \frac{1}{3} & -\frac{1}{6} & 0 \\ 0 & 1 & -2 & 1 & 0 & -1 \end{array}\right) \begin{array}{l} \quad\quad + \\ \cdot(-1) \quad \cdot(-3) \\ + \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{2}{3} & \frac{1}{3} & -\frac{1}{6} & 0 \\ 0 & 0 & -\frac{4}{3} & \frac{2}{3} & \frac{1}{6} & -1 \end{array}\right) \begin{array}{l} \\ \\ \mid \cdot(-\frac{3}{4}) \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{2}{3} & \frac{1}{3} & -\frac{1}{6} & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{8} & \frac{3}{4} \end{array}\right) \begin{array}{l} \quad\quad + \\ + \\ \cdot\frac{3}{4} \quad \cdot(-1) \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{5}{8} & -\frac{3}{4} \\ 0 & 1 & 0 & 0 & -\frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{8} & \frac{3}{4} \end{array}\right)$$

$$\begin{pmatrix} \frac{1}{2} & \frac{5}{8} & -\frac{3}{4} \\ 0 & -\frac{1}{4} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{8} & \frac{3}{4} \end{pmatrix}$$

b) B^{-1}

$$
\left(\begin{array}{ccc|ccc}
1 & -1 & 0 & 1 & 0 & 0 \\
0 & 2 & 3 & 0 & 1 & 0 \\
4 & -2 & 2 & 0 & 0 & 1
\end{array}\right)
\begin{array}{l}
\cdot(-4) \\
\\
+
\end{array}
$$

$$
\sim \left(\begin{array}{ccc|cccl}
1 & -1 & 0 & 1 & 0 & 0 & \\
0 & 2 & 3 & 0 & 1 & 0 & \mid \cdot \frac{1}{2} \\
0 & 2 & 2 & -4 & 0 & 1 & \mid \cdot \frac{1}{2}
\end{array}\right)
$$

$$
\sim \left(\begin{array}{ccc|ccc}
1 & -1 & 0 & 1 & 0 & 0 \\
0 & 1 & \frac{3}{2} & 0 & \frac{1}{2} & 0 \\
0 & 1 & 1 & -2 & 0 & \frac{1}{2}
\end{array}\right)
\begin{array}{ll}
 & + \\
\cdot(-1) & \cdot 1 \\
+ &
\end{array}
$$

$$
\sim \left(\begin{array}{ccc|cccl}
1 & 0 & \frac{3}{2} & 1 & \frac{1}{2} & 0 & \\
0 & 1 & \frac{3}{2} & 0 & \frac{1}{2} & 0 & \\
0 & 0 & -\frac{1}{2} & -2 & -\frac{1}{2} & \frac{1}{2} & \mid \cdot (-2)
\end{array}\right)
$$

$$
\sim \left(\begin{array}{ccc|ccc}
1 & 0 & \frac{3}{2} & 1 & \frac{1}{2} & 0 \\
0 & 1 & \frac{3}{2} & 0 & \frac{1}{2} & 0 \\
0 & 0 & 1 & 4 & 1 & -1
\end{array}\right)
\begin{array}{ll}
 & + \\
+ & \\
\cdot(-\frac{3}{2}) & \cdot(-\frac{3}{2})
\end{array}
$$

$$
\sim \left(\begin{array}{ccc|ccc}
1 & 0 & 0 & -5 & -1 & \frac{3}{2} \\
0 & 1 & 0 & -6 & -1 & \frac{3}{2} \\
0 & 0 & 1 & 4 & 1 & -1
\end{array}\right)
$$

$$
\begin{pmatrix}
-5 & -1 & \frac{3}{2} \\
-6 & -1 & \frac{3}{2} \\
4 & 1 & -1
\end{pmatrix}
$$