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This handwriting is for problem 1 and 3.

1. (a) forward algorithm

$O_1 O_2 O_3 = 010$

$\alpha(S_1)$:

$P(O_1=0, S_1=A) = P(O_1=0 \mid S_1=A)\, P(S_1=A) = 0.64$

$P(O_1=0, S_1=B) = P(O_1=0 \mid S_1=B)\, P(S_1=B) = 0.02$

$\alpha(S_2)$:

$P(O_1=0, O_2=1, S_2=A) = P(O_2=1 \mid S_2=A) \sum_{S_1} \alpha(S_1)\, P(S_2 \mid S_1) = 0.1036$

$P(O_1=0, O_2=1, S_2=B) = P(O_2=1 \mid S_2=B) \sum_{S_1} \alpha(S_1)\, P(S_2 \mid S_1) = 0.1278$

$\alpha(S_3)$:

$P(O_1=0, O_2=1, O_3=0, S_3=A) = P(O_3=0 \mid S_3=A) \sum_{S_2} \alpha(S_2)\, P(S_3 \mid S_2) = 0.096976$

$P(O_1=0, O_2=1, O_3=0, S_3=B) = P(O_3=0 \mid S_3=B) \sum_{S_2} \alpha(S_2)\, P(S_3 \mid S_2) = 0.011018$

$\Rightarrow P(O_1=0, O_2=1, O_3=0) = \sum_{S_3} \alpha(S_3) = 0.107994$

$O_1 O_2 O_3 = 101$

$\alpha(S_1)$:

$P(O_1=1, S_1=A) = P(O_1=1 \mid S_1=A)\, P(S_1=A) = 0.16$

$P(O_1=1, S_1=B) = P(O_1=1 \mid S_1=B)\, P(S_1=B) = 0.18$

$\alpha(S_2)$:

$P(O_1=1, O_2=0, S_2=A) = P(O_2=0 \mid S_2=A) \sum_{S_1} \alpha(S_1)\, P(S_2 \mid S_1) = 0.1456$

$P(O_1=1, O_2=0, S_2=B) = P(O_2=0 \mid S_2=B) \sum_{S_1} \alpha(S_1)\, P(S_2 \mid S_1) = 0.0158$

$\alpha(S_3)$:

$P(O_1=1, O_2=0, O_3=1, S_3=A) = P(O_3=1 \mid S_3=A) \sum_{S_2} \alpha(S_2)\, P(S_3 \mid S_2) = 0.024244.$

$P(O_1=1, O_2=0, O_3=1, S_3=B) = P(O_3=1 \mid S_3=B) \sum_{S_2} \alpha(S_2)\, P(S_3 \mid S_2) = 0.036162$

$\Rightarrow P(O_1=1, O_2=0, O_3=1) = \sum_{S_3} \alpha(S_3) = 0.060406.$

$\alpha(S_2)$:

$P(O_1=1, O_2=0, S_2=A) = P(O_2=0 \mid S_2=A) \sum_{S_1} \alpha(S_1)\, P(S_2 \mid S_1) = 0.1456$

$P(O_1=1, O_2=0, S_2=B) = P(O_2=0 \mid S_2=B) \sum_{S_1} \alpha(S_1)\, P(S_2 \mid S_1) = 0.0158$

$\alpha(S_3)$:

$P(O_1=1, O_2=0, O_3=1, S_3=A) = P(O_3=1 \mid S_3=A) \sum_{S_2} \alpha(S_2)\, P(S_3 \mid S_2) = 0.024244.$

$P(O_1=1, O_2=0, O_3=1, S_3=B) = P(O_3=1 \mid S_3=B) \sum_{S_2} \alpha(S_2)\, P(S_3 \mid S_2) = 0.036162$

$\Rightarrow P(O_1=1, O_2=0, O_3=1) = \sum_{S_3} \alpha(S_3) = 0.060406.$

(b) backward algorithm

$O_1 O_2 O_3 = 010$

$\beta(S_2)$:

$P(O_3=0 \mid S_2=A) = \sum_{S_3} \beta(S_3)\, P(O_3=0 \mid S_3)\, P(S_3 \mid S_2=A) = 0.66$

$P(O_3=0 \mid S_2=B) = \sum_{S_3} \beta(S_3)\, P(O_3=0 \mid S_3)\, P(S_3 \mid S_2=B) = 0.31$

$\beta(S_1)$:

$P(O_2=1, O_3=0 \mid S_1=A) = \sum_{S_2} \beta(S_2)\, P(O_2=1 \mid S_2)\, P(S_2 \mid S_1=A) = 0.1614$

$P(O_2=1, O_3=0 \mid S_1=B) = \sum_{S_2} \beta(S_2)\, P(O_2=1 \mid S_2)\, P(S_2 \mid S_1=B) = 0.2349$

$\cancel{\beta(S_0)}$:

$\cancel{P(O_1=0, O_2=1, O_3=0 \mid S_0=A) = \sum_{S_1} \beta(S_1)\, P(O_1=0 \mid S_1)\, P(S_1 \mid S_0=A) =}$

$\cancel{P(O_1=0, O_2=1, O_3=0 \mid S_0=B) = \sum_{S_1} \beta(S_1)\, P(O_1=0 \mid S_1)\, P(S_1 \mid S_0=B) =}$

so $P(O_1=1, O_2=0, O_3=1) = 0.107994.$

$O_1 O_2 O_3 = 101$

$\beta(S_2)$:

$P(O_3=1 \mid S_2=A) = \sum_{S_3} \beta(S_3)\, P(O_3=1 \mid S_3) P(S_3 \mid S_2=A) \approx 0.34$

$P(O_3=1 \mid S_2=B) = \sum_{S_3} \beta(S_3)\, P(O_3=1 \mid S_3) P(S_3 \mid S_2=B) = 0.69$

$\beta(S_1)$:

$P(O_2=0, O_3=1 \mid S_1=A) = \sum_{S_2} \beta(S_2)\, P(O_2=0 \mid S_2) P(S_2 \mid S_1=A) = 0.2314$

$P(O_2=0, O_3=1 \mid S_1=B) = \sum_{S_2} \beta(S_2)\, P(O_2=0 \mid S_2) P(S_2 \mid S_1=B) = 0.1299$

$\cancel{\beta(S_0)}$:

$\cancel{P(O_1=1, O_2=0, O_3=1 \mid S_0=A) = \sum_{S_1} \beta(S_1)\, P(O_1=1 \mid S_1) P(S_1 \mid S_0=A) =}$

$\cancel{P(O_1=1, O_2=0, O_3=1 \mid S_0=B) = \sum_{S_0} \beta(S_1)\, P(O_1=1 \mid S_1) P(S_1 \mid S_0=B) =}$ $P(O_1=1, O_2=0, O_3=1) = 0.060406$

(c) $O_1 O_2 O_3 = 010$

$S_1 = A$ $\quad P(O_1=0, S_1=A) = 0.8 \times 0.8 = 0.64 \leftarrow$ max

path: $A \to B \to A$

$A \to A \to A$ $\quad$ calculate respectively, find that the most likly is

$A \to B \to B$

$A \to A \to B$

$$\begin{array}{ccccc} A & \to & A & \to & A \\ \downarrow & & \downarrow & & \downarrow \\ 0 & & 1 & & 0 \end{array}$$ with $P(O_1=0, O_2=1, O_3=0,\ S_1=A, S_2=A, S_3=A) = 0.065536$

3.(a) $P(D,I,G,L,S) = P(D)P(I)P(G \mid D,I)P(S \mid I)P(L \mid G)$

(b) $P(L=l')$

$= \sum_k P(L=l', g=g^k)$

$= \sum_k P(L=l' \mid g=g^k) P(g=g^k)$

$A \to B \to B$

$A \to A \to B$

$$\begin{array}{ccccc} A & \to & A & \to & A \\ \downarrow & & \downarrow & & \downarrow \\ 0 & & 1 & & 0 \end{array}$$ with $P(O_1=0, O_2=1, O_3=0,\ S_1=A, S_2=A, S_3=A) = 0.065536$

3.(a) $P(D,I,G,L,S) = P(D)P(I)P(G \mid D,I)P(S \mid I)P(L \mid G)$

(b) $P(L=l')$

$= \sum_k P(L=l', g=g^k)$

$= \sum_k P(L=l' \mid g=g^k) P(g=g^k)$

$= \sum_k P(L=l' \mid g=g^k) \sum_{i,d} P(g=g^k \mid I,D) P(I) P(D)$

$= 0.9(0.3\cdot0.7\cdot0.6 + 0.05\cdot0.7\cdot0.4 + 0.9\cdot0.3\cdot0.6 + 0.5\cdot0.4\cdot0.3)$

$+ 0.6(0.4\cdot0.7\cdot0.6 + 0.25\cdot0.7\cdot0.4 + 0.08\cdot0.3\cdot0.6 + 0.3\cdot0.4\cdot0.3)$

$+ 0.01(0.3\cdot0.7\cdot0.6 + 0.7\cdot0.7\cdot0.4 + 0.02\cdot0.3\cdot0.6 + 0.2\cdot0.4\cdot0.3)$

$= 0.3258 + 0.17304 + 3.496 \times 10^{-3} = 0.502336.$

(c) $P(L=l' \mid I=i')$

$= P(L=l', I=i') / P(I=i')$

$= \sum_k P(L=l', g=g^k, I=i') / P(I=i')$

$= \sum_k P(L=l' \mid g=g^k) P(g=g^k \mid I=i')$

$= \sum_k P(L=l' \mid g=g^k) \sum_j (g=g^k \mid I=i', D=d^j) P(D=d^j)$

$= 0.9\cdot(0.9\cdot0.6 + 0.5\cdot0.4) + 0.6\cdot(0.08\cdot0.6 + 0.3\cdot0.4) + 0.01(0.02\cdot0.6 + 0.2\cdot0.4)$

$= 0.666 + 0.1008 + 9.2 \times 10^{-4}$

$= 0.76772.$

d) $P(G=g^1 \mid S=s^1)$

$$
\begin{aligned}
&= P(G=g^1, S=s^1)/P(S=s^1) \\
&= \sum_k P(G=g^1, S=s^1, I=i^k)/P(S=s^1) \\
&= \frac{\sum_k P(G=g^1 \mid I=i^k)\, P(S=s^1 \mid I=i^k)\, P(I=i^k)}{P(S=s^1)}
\end{aligned}
$$

$\sum_k P(S=s^1 \mid I=i^k)\, P(I=i^k)$

$$
\begin{aligned}
&= \sum_k \sum_j P(G=g^1 \mid I=i^k, D=d^j)\, P(D=d^j)\, P(S=s^1 \mid I=i^k)\, P(I=i^k) \;/\; P(S=s^1) \\
&= (0.6\cdot0.7\cdot0.3\cdot0.05 + 0.7\cdot0.4\cdot0.05\cdot0.05 + 0.3\cdot0.6\cdot0.9\cdot0.8 + 0.3\cdot0.4\cdot0.5\cdot0.8)/ \\
&\quad (0.05\cdot0.7 + 0.8\cdot0.3) \approx 0.671
\end{aligned}
$$

e) $P(D=d^1 \mid I=i^1, S=s^1, L=l^1)$

$$
\begin{aligned}
&= \frac{P(D=d^1, I=i^1, S=s^1, L=l^1)}{P(I=i^1, S=s^1, L=l^1)} \\
&= \frac{\sum_k P(D=d^1, I=i^1, S=s^1, L=l^1, G=g^k)}{\sum_k \sum_j P(I=i^1, S=s^1, L=l^1, G=g^k, D=d^j)} \\
&= \frac{\sum_k P(D=d^1)\, P(\cancel{I=i^1})\, P(G=g^k \mid D=d^1, I=i^1)\, P(S\cancel{=s^1 \mid I=i^1})\, P(L=l^1 \mid G=g^k)}{\sum_k \sum_j P(D=d^j)\, P(\cancel{I=i^1})\, P(G=g^k \mid D=d^j, I=i^1)\, P(S\cancel{=s^1 \mid I=i^1})\, P(L=l^1 \mid G=g^k)} \\
&= \frac{0.4\cdot(0.5\cdot0.9 + 0.3\cdot0.6 + 0.2\cdot0.01)}{0.6\cdot(0.9\cdot0.9 + 0.08\cdot0.6 + 0.02\cdot0.01) + 0.4(0.5\cdot0.9 + 0.3\cdot0.6 + 0.2\cdot0.01)} \\
&= \frac{0.2528}{0.51492 + 0.2528} \approx 0.33.
\end{aligned}
$$

$\cancel{P(D=d^1) = 0.4.}$

2.

Iterate 10 times.

K=2

Transition Probability

array([[0.12044337, 0.87955663],
 [0.9910059 , 0.0089941]])

Emission Probability

array([[5.99817846e-01, 2.00209763e-01, 1.99940581e-01,
 3.18095450e-05],
 [4.33452066e-06, 6.66587595e-01, 2.07313187e-16,
 3.33408071e-01]])

K=4

Transition Probability

array([[5.14751585e-05, 7.19341565e-01, 2.69370553e-05,
 2.80580022e-01],
 [4.40058933e-02, 3.45646392e-04, 4.52532364e-01,
 5.03116096e-01],
 [2.09043317e-04, 9.99703756e-01, 2.79228264e-10,
 8.72000617e-05],
 [9.82158118e-01, 1.57920606e-02, 5.65622201e-04,
 1.48419879e-03]])

Emission Probability

array([[8.69938202e-01, 1.30014169e-01, 2.32347424e-08,
4.76056216e-05],
[2.49142908e-05, 9.89941678e-01, 3.16054222e-18,
1.00334079e-02],
[3.05193658e-01, 1.55962763e-01, 5.38843579e-01,
2.29273184e-10],
[3.49642565e-01, 2.32387775e-03, 3.64553001e-06,
6.48029911e-01]])

The most likely sequence of states of length 4 when 1st observation is A:

1) 0101
2) 0130

4.

First:

array([0.24959319, -0.25652131, 0.3468611 , 0.005099 , 0.34297566,
-0.18943673, 0.31385097, -0.32173451, 0.31981745, 0.33853899,
0.20502118, -0.20273245, 0.30984085])

Second:

array([-0.31318631, -0.32130825, 0.11181554, 0.45672596, 0.21985693,
0.15387677, 0.31174761, -0.34918069, -0.2703984 , -0.23885931,
-0.30870354, 0.23495727, -0.07598235])