Measure Theory: Cheatsheet

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1 Measure Spaces

Generation of σ-algebras. Let Ω be a nonempty set. Let $\mathcal{A} \subset 2^{\Omega}$. The smallest σ-algebra containing $\mathcal{A}$ is called the σ-algebra generated by $\mathcal{A}$. We denote it by $\sigma(\mathcal{A})$. Trivially, $\sigma(\mathcal{A}) \subset 2^{\Omega}$.

Similar concepts are defined similarly.

Let X be a metric space. The σ-algbra generated by the open sets (or balls) is called Borel σ-algebra. The sets that belong to Borel σ-algebra are called Borel sets. All open or closed sets are then Borel sets.

It is important to know that sets of the form (a, ∞) generate all Borel sets on $\mathbb{R}$.

Limit superior and limit inferior. Let $\{A_n\}_{n \geq 0}$ be a sequence of sets. Then

$$\limsup A_n = \cap_{n=1}^{\infty} \cup_{k \geq n} A_k, \qquad \liminf A_n = \cup_{n=1}^{\infty} \cap_{k \geq n} A_k \tag{1.1}$$

From the definitions, we derive

$$\liminf A_n \leq \limsup A_n \tag{1.2}$$

The sequence A_n converges to A iff $\liminf A_n = \limsup A_n$.

Monotone class theorem. Let $\mathcal{A}$ be an algebra. Let $\mathcal{F}$ and $\mathcal{M}$ be σ-algebra and monotone class generated by $\mathcal{A}$ respectively. Then $\mathcal{F} = \mathcal{M}$.

Proof is easy. $\mathcal{F}$ is a montone class by definition. This implies that $\mathcal{M} \subset \mathcal{F}$. A simple use of logic will imply $\mathcal{M}$ is also a σ-algebra which means $\mathcal{F} \subset \mathcal{M}$.

Measurable space vs. measure space. Any set Ω bundled with a σ-algebra $\mathcal{F}$ defined on Ω makes a measurable space. If we define a measure $\mu : \mathcal{F} \to [0, \infty]$ and associate it to the measurable space, it becomes a measure space.

In short, $(\Omega, \mathcal{F})$ is a measurable space and $(\Omega, \mathcal{F}, \mu)$ is a measure space.

Lebesgue measure. Consider the measurable space $(\mathbb{R}^n, \mathcal{B}^n)$ where $\mathcal{B}^n$ denote the Borel σ-algebra on $\mathbb{R}^n$. We define Lebesgue measure $l : \mathcal{B}^n \to [0, \infty)$ by

$$l((a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n)) = \prod_{i=1}^{n} (b_i - a_i). \tag{1.3}$$

The value of the measure remains the same if the open intervals are replaced with closed intervals. Lebesgue measure generalizes the sense of length, area, volume.

2 Measurability

Checking measurability Let $(\Omega, \mathcal{F})$ and $(\Omega', \mathcal{F}')$ be two measurable spaces. A function $f : \Omega \to \Omega'$ is measurable if for any $A \in \mathcal{F}'$, we have $f^{-1}(A) \in \mathcal{F}$.

Informally, a measurable function is one whose inverse images of measurable sets are also measurable. That is, it preserves the structure of a measurable space. In that regard, they are like continuous functions which preserve the structure of topological spaces, i.e., the inverse image of an open set is open.

Measurability on a generator. To check if a set function $f : (\Omega, \mathcal{F}) \to (\Omega', \mathcal{F}')$ is $\mathcal{F}$-$\mathcal{F}'$ measurable it is sufficient to check that f is measurable on a set $B \subset \mathcal{F}'$ where B generates $\mathcal{F}'$.

Since intervals of the form $(-\infty, a)$ generate the Borel σ-algebra on $\mathbb{R}$ we sometimes say f is measurable if $\{x : f(x) < a, a \in \mathbb{R}\}$ is measurable.

σ-algebras from measurable functions. $f^{-1}(\mathcal{F}')$ is the smallest σ-algebra for f to be measurable.

We can extend this idea for a collection of functions. Suppose $(\sigma_i, \mathcal{F}_i)$ for some index i are measurable spaces. The smallest σ-algebra so that $f_i : \sigma \to \sigma_i$ are all measurable is then given by

$$\sigma(f_i) = \sigma(\cup f_i^{-1}(\mathcal{F}_i)) \tag{2.1}$$

Operations on measurable functions. Sum, difference, product, scalar product, composition on measurable functions yield measurable functions.

Let f_i be measurable functions. Then $\sup f_i$, $\inf f_i$, $\limsup f_i$ and $\liminf f_i$ are measurable.

The concept of almost everywhere. Informally, we say that a property P holds almost everywhere if the set of elements for which P does not hold has measure zero. In Euclidean spaces, any countable set has measure zero — this implies that any property which holds for all points in $\mathbb{R}^n$ except for countably many points is said to hold almost everywhere.

The mathematical concept of "almost surely" is closely related to this.

Approximation by simple functions. Any nonnegative measurable function f can be written as

$$f(x) = \lim_{n \to \infty} \sum_{i=0}^{n} a_i 1_{E_i}(x) \tag{2.2}$$

where 1_{E_i} is the indicator function of the measurable set E_i. In essence, we are approximating f by an increasing sequence of nonnegative simple functions defined by $\sum_{i=0}^{n} a_i 1_{E_i}(x)$.

3 Lebesgue Integration

Simple functions. From the previous chapter, a simple function is of the form

$$s = \sum_{i=0}^{n} a_i 1_{E_i}(x) \tag{3.1}$$

Then the Lebesgue integral of s with respect to measure μ is

$$\int s d\mu = \sum_{i=0}^{n} a_i \mu(E_i) \tag{3.2}$$

The integral $\int s d\mu$ does not depend on the representation of s.

Measurable functions. From the previous chapter, we know how to approximate a nonnegative measurable function by an increasing sequence of nonnegative simple functions. Let

$$f(x) = \lim_{n\to\infty} \sum_{i=0}^{n} a_i 1_{E_i}(x) \tag{3.3}$$

Then the Lebesgue integral of f with respect to measure μ is

$$\int f d\mu = \sup \left(\sum_{i=0}^{n} a_i (\mu(E_i) \right) \tag{3.4}$$

"Almost everywhere" revisted. From the way how a Lebesgue integral is defined, it follows "almost trivially" that if $f \leq g$ almost everywhere then $\int f d\mu \leq \int g d\mu$. Also if $f = g$ almost everywhere then $\int f d\mu = \int g d\mu$.

Suppose $f = 0$ almost everywhere. Then $\int f d\mu = 0$. The converse is also true.

Well the take-home message is: in the grand scheme of things, irregularities in countably many cases do not make a difference. Well sort of.

Lebesgue integrals are in many ways like the ordinary Riemann integrals. They obey linearity and monotone convergence.

Monotone convergence. Let $\{f_n\}$ be such that $\lim_{n\to\infty} f_n = f$. Then

$$\int f d\mu = \lim_{n\to\infty} \int f_n d\mu. \tag{3.5}$$

This is what helped us to defined Lebesgue integrals for nonnegative measurable functions using a sequence of increasing simple functions. A careful look reveals that $\{f_n\}$ are the simple functions which approximate f.