Functional Analysis

IISER-M

These are lecture notes for the course MTH402: Functional Analysis taught by Chandrakant Aribam during the monsoon session of 2022. I live-TEX-ed them on Emacs. Please report bugs/errors, if any.

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Lectures

Lecture 1

The Course Commences, 22/07/2022

The lecture starts with a short review of elementary ideas of vector spaces. Morphisms are structure preserving maps. An endomorphism on a mathematical structure S is a structure preserving map from S to itself.

Definition 1.1: Linear endomorphism

Let V be a vector space. A linear endomorphism is a linear transformation $T : V \to V$.

Set of all linear endomorphisms on a vector space V over a field F is denoted by $\mathrm{End}_F(V)$. An automorphism on V is an endomorphism on V which is also an isomorphism (one-one and onto).

Recall that every field is also a vector space over itself. Thus, we may define a linear map from a vector space to a field. Indeed, we have a special name for this map.

Definition 1.2: Linear functional

Let V be a vector space. Let F be a field. Then a linear functional is a linear transformation $T : V \to F$.

The following are examples of linear functionals.

(1) A linear map $T : \mathbb{R} \to \mathbb{R}$ defined by $T(x) = \alpha x$ for some $\alpha \in \mathbb{R}$.
(2) A linear map $T : \mathbb{R}^2 \to \mathbb{R}$ defined by $T(x_1, x_2) = \beta x_1$ for some $\beta \in \mathbb{R}$.
(3) A linear map $\pi : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\pi(x_1, x_2) = x_1$.
(4) Let $T_1 : V_1 \to V_1$ be a linear map and $T_2 : V_1 \to F$ be a linear functional. Then $T_2 \circ T_1 : V_1 \to F$ is a linear functional.

Consider the following ODE:

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 = 0. \tag{1}$$

Let $a_0 = 0$. If $y_1(x)$ and $y_2(x)$ are two solutions, then $(y_1 + y_2)(x)$ is also a solution. Also $\alpha y_1(x)$ is also a solution for some $\alpha \in \mathbb{R}$. Let S be the set of all solutions when $a_0 = 0$. Clearly S is a vector space over $\mathbb{R}$.

We now define the operator

$$L := a_n \frac{d^n}{dx^n} + a_{n-1}\frac{d^{n-1}}{dx^{n-1}} + \cdots + a_1 \frac{d}{dx}$$

which operates on the vector space $C^n(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{R} | f \text{ is differentiable on } \mathbb{R} \text{ upto } n \text{ times}\}$ over $\mathbb{R}$. Let X be the set of all functions on $\mathbb{R}$. Then X is a vector space (also called function space) and $L : C^n(\mathbb{R}) \to X$ is a linear map. Clearly $S = \ker(L)$.

Exercise 1

What is the dimension of the vector space L defined above?

Lecture 2

Say Hello to Banach Spaces, 23/07/2022

A vector space is also called a linear space in that it makes sense to form linear combinations. Defining a norm on a linear space makes it a metric space. Then it makes sense to talk about continuous maps between such linear spaces. Banach spaces are linear spaces with a norm such that the metric space generated is also complete.

Let $X \subset \mathbb{R}$. When is X compact? From analysis, we know X is compact if and only if it is closed and bounded. We define

$$C(X) = \{f : X \to \mathbb{R} \mid f \text{ is continuous on } X\}.$$

When is $C(X)$ compact? To answer this question, we need to ask: what is the topology on $C(X)$? Suppose $X = [0, 1]$. What if, the topology is the one induced by the metric space generated by the norm

$$\|h\|_\infty = \sup\{\|h(x)\| \mid x \in X\}.$$

Definition 2.1: Normed linear space

Let N be a vector space over $\mathbb{R}$ (or $\mathbb{C}$). Then N is a normed linear space if $\exists a$ function $\|\cdot\| : N \to \mathbb{R}$ such that

(1) $\|x\| \geq 0$ and $\|x\| = 0 \iff x = 0$.
(2) $\|x + y\| \leq \|x\| + \|y\|$.
(3) $\|\alpha x\| = |\alpha|\|x\|$ for all $\alpha \in \mathbb{R}$ (or $\alpha \in \mathbb{C}$), $x \in N$.

Exercise 2: Norm induces a metric.

(1) Show that $d(x, y) = \|x - y\|$ is a metric on N.
(2) Since N is a metric space, we can talk about Cauchy sequences in N. Let $\{x_n\}$ be a sequence in N. Show that $\{x_n\}$ is Cauchy if and only if $\|x_n - x_m\| \to 0$ as $n, m \to 0$.
(3) Show that if $\{x_n\}$ converges to x, then $\|x_n\|$ converges to $\|x\|$.
(4) Show that $|\|x\| - \|y\|| \leq \|x - y\|$ for all $x, y \in N$.
(5) Show that if $\{x_n\}$ is Cauchy, then $\{\|x_n\|\}$ is also Cauchy.
(6) Suppose $x_n \to x$ and $y_n \to y$. Show that $x_n + y_n \to x + y$.
(7) Suppose $\alpha_n \in \mathbb{R}$, $\alpha_n \to \alpha$ and $x_n \to x$. Show that $\alpha_n x_n = \alpha x$.

Note

See that (3) of Exercise 2 shows the norm on N is continuous.

We recall that a complete metric space is one in which every Cauchy sequence in the space converges to a point in the space itself. Keeping this in mind, we are well-equiped to define a Banach space.

Definition 2.2: Banach space

A complete normed linear space N is called a Banach space.

We look at some examples of Banach spaces:

(1) $\mathbb{R}$, $\mathbb{C}$ are Banach spaces with their usual metric.
(2) $\mathbb{R}^n$ is a Banach space with either of the norms

$$\|(x_1, \ldots, x_n)\| = |x_1| + \cdots + |x_n| \qquad \text{or} \qquad \|(x_1, \ldots, x_n)\| = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2}.$$

(3) l_p^n-spaces are Banach spaces with the p-norm

$$\|(x_1, \ldots, x_n)\| = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}.$$

Exercise 3

Prove the triangle inequality for p-norm.

The scheduled class on 25/08/2022 was dismissed.

Lecture 3

The Misery of Banach Spaces, 26/08/2022

Normed linear spaces can be infinite dimensional over $\mathbb{R}$ or $\mathbb{C}$. Recall that Banach spaces are complete normed linear spaces. The linear space $(\mathbb{Q}, |\cdot|)$ with the usual metric is not complete.

Exercise 4

Can you think of a vector spaces over $\mathbb{R}$ which is not complete with respect to some norm?

Consider l_p^n with $1 \leq p < \infty$. Is this complete? Suppose x_m, $x_{m'} \in l_p^n$. Let $\epsilon > 0$ be any fixed real number. Suppose for sufficiently large m, m' sufficiently large,

$$\|x_m - x_{m'}\| < \epsilon$$

$$\sum |x_{m_i} - x_{m'_i}|^p < \epsilon^p$$

$$|x_{m_i} - x_{m'_i}|^p \leq \sum |x_{m_i} - x_{m'_i}|^p < \epsilon^p$$

$$|x_{m_i} - x_{m'_i}| < \epsilon.$$

That is, $\{x_{m,i}\}_m$ is Cauchy in $\mathbb{R}$. $\{x_{m,1}\}_m$ converges in $\mathbb{R}$, say $\lim\limits_{m\to\infty} x_{m,i} = a_i$.

We put $a = (a_1, \cdots, a_n)$. Now, $|x_{m,i} - a_i| < \epsilon/n^{1/p}$ for $m \geq N_i$. For $N_0 = \max\{N_1, \cdots, N_n\}$, we have,

$$|x_m - a_i| < \epsilon/n^{1/p} \qquad \text{for all } m > N_0 \text{ and for all } i.$$

Then

$$\sum_{i=1}^{m} |x_m - a_i|^p < \epsilon^p$$

$$\|x_m - a\| = \sum_{i=1}^{n} |x_{m,i} - a_i| < \epsilon,$$

for all $m > N_0$. Therefore, $\lim_{m\to\infty} x_m = a$ so that l_p^n is a Banach space.

We represent l_p^∞ as l_p. It is an infinite dimensional linear space such that

$$\sum_{i=1}^{\infty} |x_i|^p < \infty.$$

Taking square root yields its norm, which is

(2) $$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p}.$$

Exercise 5: Triangle inequality

Show that the norm defined above satisfies triangle inequality.

We now introduce L_p space as a space of functions $f : \mathbb{R} \to \mathbb{R}$ equipped with the norm

(3) $$\|f\|_p = \left(\int |f(x)|^p dm(x)\right)^{1/p},$$

where m is a measure on $\mathbb{R}$. We may also write

$$L_p = \{f : \mathbb{R} \to \mathbb{R} \mid \int |f(x)|^p dm(x) < \infty\}.$$

Lecture 4

29/08/2022

Continuous functions on compact sets are always bounded. In fact, continuous functions maps compact sets to compact sets.

Theorem 4.1

Let N be a normed linear space and $M \subset N$ which is a closed linear subspace. Then

(1) Consider the quotient $N/M = \{x + M \mid x \in N \text{ and } x_1 + M = x_2 + M \text{ iff } x_1 - x_2 \in M\}$. Define

$$\|x + M\| = \inf\{\|x + m\| \mid m \in M\},$$

then N/M is a linear subspace over $\mathbb{R}$.

(2) If N is Banach, then N/M is Banach.

Proof. (1) $\|x + M\| = \inf\{\|x + m\| \mid m \in M\} \implies \|x + M\| \geq 0$. Suppose $\|x + M\| = 0$. Then $\exists$ a sequence $\{t_k\} \subset M$ such that $\|x + t_k\| \to 0$. That is, $0 < 1/k \implies \exists \|x + t_k\| < 1/k$. As M is closed, there exists a subsequence $\{t_k\} \subset M$ such that $\{t_k\}$ is convergent and let $t = \lim_{r\to\infty} t_{k_r}$. Then $t \in M$ as M is closed. Since $t_{n_r} \to t$, we have $x + t_{n_r} \to x + t$. Then $\|x + t_{n_r}\| \to \|x + t\|$ so that $\|x + t\| = 0$, from which we get $t = -x$. Therefore, $x + M = 0$.

Now, we want to prove triangle inequality. See that, for $m = u + v \in M$, we have

$$\begin{aligned} \|x+y+m\| &\leq \|x+u\| + \|y+v\| \\ \inf \|x+y+m\| \leq \|x+y+m\| &\leq \|x+u\| + \|y+v\| \end{aligned}$$

Then

$$\begin{aligned} \inf_{u\in M} \|x+u\| &\leq \|x+u\| \\ \left(\inf_{u\in M} \|x+u\|\right) + \|y+v\| &\leq \|x+u\| + \|y+v\| \\ \inf_{m} \|x+y+m\| &\leq \inf_{u\in M} \|x+u\| + \|y+b\|. \end{aligned}$$

Take inf on $v \in M$. Then we are done.

(2) Let $\{x_n + M\}_n$ be a Cauchy sequence in N/M. That is, $\|x_n + M - x_m + M\| \to 0$ as $n, m \to \infty$. We see that

$$\begin{aligned} \|x_{n_1} + M - x_{n_2} + M\|\frac{1}{2} &\implies \|y_1 - y_2\| < \frac{1}{2} \\ \|x_{n_2} + M - x_{n_3} + M\|\frac{1}{2^2} &\implies \|y_2 - y_3\| < \frac{1}{2^2} \\ &\cdots \\ \|x_{n_k} + M - x_{n_{k+1}} + M\| &< \frac{1}{2^k}, \end{aligned}$$

and so on.

□

Note

In the above, defining $\|x + M\| = \|x\|$ does not work. Defining $\|x + M\| = \sup\{\|x + m\| \mid m \in M\}$ will not work either.