

Functional Analysis

IISER-M



These are lecture notes for the course MTH402: Functional Analysis taught by Chandrakant Aribam during the monsoon session of 2022. I live-TEX-ed them on Emacs. Please report bugs/errors, if any.

Ronald Huidrom

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LECTURE 1

THE COURSE COMMENCES, 22/07/2022

The lecture starts with a short review of elementary ideas of vector spaces. Morphisms are structure preserving maps. An endomorphism on a mathematical structure S is a structure preserving map from S to itself.

Definition 1.1: Linear endomorphism

Let V be a vector space. A linear endomorphism is a linear transformation $T : V \rightarrow V$.

Set of all linear endomorphisms on a vector space V over a field F is denoted by $\text{End}_F(V)$. An automorphism on V is an endomorphism on V which is also an isomorphism (one-one and onto).

Recall that every field is also a vector space over itself. Thus, we may define a linear map from a vector space to a field. Indeed, we have a special name for this map.

Definition 1.2: Linear functional

Let V be a vector space. Let F be a field. Then a linear functional is a linear transformation $T : V \rightarrow F$.

The following are examples of linear functionals.

- (1) A linear map $T : \mathbb{R} \rightarrow \mathbb{R}$ defined by $T(x) = \alpha x$ for some $\alpha \in \mathbb{R}$.
- (2) A linear map $T : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by $T(x_1, x_2) = \beta x_1$ for some $\beta \in \mathbb{R}$.
- (3) A linear map $\pi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $\pi(x_1, x_2) = x_1$.
- (4) Let $T_1 : V_1 \rightarrow V_1$ be a linear map and $T_2 : V_1 \rightarrow F$ be a linear functional. Then $T_2 \circ T_1 : V_1 \rightarrow F$ is a linear functional.

Consider the following ODE:

$$(1) \quad a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 = 0.$$

Let $a_0 = 0$. If $y_1(x)$ and $y_2(x)$ are two solutions, then $(y_1 + y_2)(x)$ is also a solution. Also $\alpha y_1(x)$ is also a solution for some $\alpha \in \mathbb{R}$. Let S be the set of all solutions when $a_0 = 0$. Clearly S is a vector space over \mathbb{R} .

We now define the operator

$$L := a_n \frac{d^n}{dx^n} + a_{n-1} \frac{d^{n-1}}{dx^{n-1}} + \cdots + a_1 \frac{d}{dx}$$

which operates on the vector space $C^n(\mathbb{R}) := \{f : \mathbb{R} \rightarrow \mathbb{R} | f \text{ is differentiable on } \mathbb{R} \text{ upto } n \text{ times}\}$ over \mathbb{R} . Let X be the set of all functions on \mathbb{R} . Then X is a vector space (also called function space) and $L : C^n(\mathbb{R}) \rightarrow X$ is a linear map. Clearly $S = \ker(L)$.

Exercise 1

What is the dimension of the vector space L defined above?

LECTURE 2

SAY HELLO TO BANACH SPACES, 23/07/2022

A vector space is also called a linear space in that it makes sense to form linear combinations. Defining a norm on a linear space makes it a metric space. Then it makes sense to talk about continuous maps between such linear spaces. Banach spaces are linear spaces with a norm such that the metric space generated is also complete.

Let $X \subset \mathbb{R}$. When is X compact? From analysis, we know X is compact if and only if it is closed and bounded. We define

$$C(X) = \{f : X \rightarrow \mathbb{R} \mid f \text{ is continuous on } X\}.$$

When is $C(X)$ compact? To answer this question, we need to ask: what is the topology on $C(X)$? Suppose $X = [0, 1]$. What if, the topology is the one induced by the metric space generated by the norm

$$\|h\|_\infty = \sup\{\|h(x)\| \mid x \in X\}.$$

Definition 2.1: Normed linear space

Let N be a vector space over \mathbb{R} (or \mathbb{C}). Then N is a normed linear space if \exists a function $\|\cdot\| : N \rightarrow \mathbb{R}$ such that

- (1) $\|x\| \geq 0$ and $\|x\| = 0 \iff x = 0$.
- (2) $\|x + y\| \leq \|x\| + \|y\|$.
- (3) $\|\alpha x\| = |\alpha| \|x\|$ for all $\alpha \in \mathbb{R}$ (or $\alpha \in \mathbb{C}$), $x \in N$.

Exercise 2: Norm induces a metric.

- (1) Show that $d(x, y) = \|x - y\|$ is a metric on N .
- (2) Since N is a metric space, we can talk about Cauchy sequences in N . Let $\{x_n\}$ be a sequence in N . Show that $\{x_n\}$ is Cauchy if and only if $\|x_n - x_m\| \rightarrow 0$ as $n, m \rightarrow 0$.
- (3) Show that if $\{x_n\}$ converges to x , then $\|x_n\|$ converges to $\|x\|$.
- (4) Show that $\||x| - |y|\| \leq \|x - y\|$ for all $x, y \in N$.
- (5) Show that if $\{x_n\}$ is Cauchy, then $\{\|x_n\|\}$ is also Cauchy.
- (6) Suppose $x_n \rightarrow x$ and $y_n \rightarrow y$. Show that $x_n + y_n \rightarrow x + y$.
- (7) Suppose $\alpha_n \in \mathbb{R}$, $\alpha_n \rightarrow \alpha$ and $x_n \rightarrow x$. Show that $\alpha_n x_n = \alpha x$.

Note

See that (3) of Exercise 2 shows the norm on N is continuous.

We recall that a complete metric space is one in which every Cauchy sequence in the space converges to a point in the space itself. Keeping this in mind, we are well-equipped to define a Banach space.

Definition 2.2: Banach space

A complete normed linear space N is called a Banach space.

We look at some examples of Banach spaces:

- (1) \mathbb{R}, \mathbb{C} are Banach spaces with their usual metric.
- (2) \mathbb{R}^n is a Banach space with either of the norms

$$\|(x_1, \dots, x_n)\| = |x_1| + \dots + |x_n| \quad \text{or} \quad \|(x_1, \dots, x_n)\| = \left(\sum_{i=1}^n |x_i|^2 \right)^{1/2}.$$

- (3) L^p -spaces are Banach spaces with the p -norm

$$\|(x_1, \dots, x_n)\| = \left(\sum_{i=1}^n |x_i|^p \right)^{1/p}.$$

Exercise 3

Prove the triangle inequality for p -norm.

The scheduled class on 25/08/2022 was dismissed.

LECTURE 3

THE MISERY OF BANACH SPACES, 26/08/2022
