Functional Analysis

IISER-M

These are lecture notes for the course MTH402: Functional Analysis taught by Chandrakant Aribam during the monsoon session of 2022. I TEX-ed them on Emacs. Please report bugs/errors, if any.

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Lectures

Lecture 1

The Course Commences, 22/07/2022

The lecture starts with a short review of elementary ideas of vector spaces. Morphisms are structure preserving maps. An endomorphism on a mathematical structure S is a structure preserving map from S to itself.

Definition 1.1: Linear endomorphism

Let V be a vector space. A linear endomorphism is a linear transformation $T : V \to V$.

Set of all linear endomorphisms on a vector space V over a field F is denoted by $\mathrm{End}_F(V)$. An automorphism on V is an endomorphism on V which is also an isomorphism (one-one and onto).

Recall that every field is also a vector space over itself. Thus, we may define a linear map from a vector space to a field. Indeed, we have a special name for this map.

Definition 1.2: Linear functional

Let V be a vector space. Let F be a field. Then a linear functional is a linear transformation $T : V \to F$.

The following are examples of linear functionals.

(1) A linear map $T : \mathbb{R} \to \mathbb{R}$ defined by $T(x) = \alpha x$ for some $\alpha \in \mathbb{R}$.
(2) A linear map $T : \mathbb{R}^2 \to \mathbb{R}$ defined by $T(x_1, x_2) = \beta x_1$ for some $\beta \in \mathbb{R}$.
(3) A linear map $\pi : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\pi(x_1, x_2) = x_1$.
(4) Let $T_1 : V_1 \to V_1$ be a linear map and $T_2 : V_1 \to F$ be a linear functional. Then $T_2 \circ T_1 : V_1 \to F$ is a linear functional.

Consider the following ODE:

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 = 0. \tag{1}$$

Let $a_0 = 0$. If $y_1(x)$ and $y_2(x)$ are two solutions, then $(y_1 + y_2)(x)$ is also a solution. Also $\alpha y_1(x)$ is also a solution for some $\alpha \in \mathbb{R}$. Let S be the set of all solutions when $a_0 = 0$. Clearly S is a vector space over $\mathbb{R}$.

We now define the operator

$$L := a_n \frac{d^n}{dx^n} + a_{n-1} \frac{d^{n-1}}{dx^{n-1}} + \cdots + a_1 \frac{d}{dx}$$

which operates on the vector space $C^n(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{R} | f \text{ is differentiable on } \mathbb{R} \text{ upto } n \text{ times}\}$ over $\mathbb{R}$. Let X be the set of all functions on $\mathbb{R}$. Then X is a vector space (also called function space) and $L : C^n(\mathbb{R}) \to X$ is a linear map. Clearly $S = \ker(L)$.

Exercise 1

What is the dimension of the vector space L defined above?

Lecture 2

The Second Day, 23/07/2022