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Derbyshire's Prime Obsession

Notes, Thoughts & Opinions

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1 Comments

The book presents a good introduction to the prime number theorem (PNT) and Riemann Hypothesis (RH). Further, it accounts the life of Bernhard Riemann and some great mathematicians connected with him—Euler, Gauss, Dirichlet and others. It also gives a historical account of the their times.

Read the notes given at the end of the book. Some of them are really good.

2 Why Read?

Anyone who wants a beginner's introduction to the PNT or RH or the life of Bernhard Riemann should read it.

3 Summary

3.1 Card Trick

3.2 The Soil, the Crop

1. In the aftermath of the revolution, the French were disorganized and ineffective, and disturbed by its republican and antimonarchical ideals. Taking advantage of this situation, in 1792 a huge force of mainly Austrian and Prussian troops, but which included 15,000 emigré French, advanced on Paris. To their surprise, the army of revolutionary France took a stand at the village of Valmy, engaging the invaders in an artillery duel fought in thick fog on September 20 of that year. Edward Creasy, in his classic *Fifteen Decisive Battles of the World*, calls this the Battle of Valmy. Germans call it the Cannonade of Valmy. It marks the beginning of the succession of wars that occupied Europe for the next 23 years—the Napoleonic Wars; it could have been called the First World War, since they included engagements in both the Americas and the Far East. It all ended with a peace treaty at the Congress of Vienna (June 8, 1815) and Europe settled into a long period, almost a century, of relative peace.

2. One consequence of the treaty was a modest tidying up of the German peoples in Europe. Before the French Revolution a German-speaking European might have been a citizen of Hapsburg Austria (in which case he was probably a Catholic) or of the Kingdom of Prussia (making him more likely a Protestant) or of any one of 300-odd petty principalities scattered across the map of what we now call Germany. He might also have been a subject of the king of France, or of the king of Denmark, or a citizen of the Swiss Confederation. Austria still had her empire (which included great numbers of non-Germans: Hungarians, Slavs, Romanians, Czechs, and so on); Switzerland, Denmark, and France still included German speakers. It was a good start, though. The 300-odd entities that comprised 18th Germany were consolidated into 34 sovereign states and 4 free cities, and their cultural unity was recognized by the creation of a German Confederation.

3. The Kingdom of Hanover was something of an oddity in that its king was hardly ever present. Because, for complicated dynastic reasons, he was also king of England.

4. King William IV of England and Hanover died in 1837 and the English throne passed to his niece, Victoria. Hanover was subscribed to the Salic Law of the medieval Franks, according to which only a male could succeed to the throne. England and Hanover thereupon parted company. The new ruler of Hanover was Ernest Augustus whose almost first act was to set aside the liberal constitution granted by William IV four years earlier. Seven eminent professors at Göttingen University refused to swear an oath to uphold the new constitution and were dismissed. Three of them were actually exiled from the kingdom. These dismissed scholars became known as "the Göttingen Seven" and were heroes to social and political reformers all over Europe. Among them were the two brothers Grimm of fairy-tale fame, who were academic philologists. Eventually Hanover got a new liberal constitution. At least one of the Göttingen Seven, the physicist Wilhelm Weber, was reinstated.

5. Riemann was extremely shy. He avoided human contact as far as possible and was ill at ease in company. His only close ties—and they were very close indeed—were with his family, and his only other ties were with other mathematicians. He thought deeply about philosophy. He was a hypochondriac and was subject to spells of very deep unhappiness, especially after the death of his father, whom he worshiped. Riemann dealt with these episodes by losing himself in work. His health was never good.

3.3 The Prime Number Theorem

6. Consequences of the PNT:

   (a) The probability that N is prime $\sim 1/\log N$.

   (b) The N-th prime number is $\sim N \log N$.

   Not only are these consequences of the PNT; it is also a consequence of them. If you could mathematically prove the truth of either, the PNT would follow.

3.4 On the Shoulders of Giants

7. The first person to whom the truth contained in the PNT occurred was Gauss. In his lifetime he was known as Princeps Mathematicorum—the Prince of Mathematics—and at his death the King of Hanover, George V, ordered a commemorative medal in his honor, with that title on it.

8. Gauss came from extremely humble origins. His grandfather was a landless peasant; his father was a jobbing gardener and bricklayer. Gauss attended the poorest kind of local school.

9. Carl Wilhelm Ferdinand, the then Duke of Brunswick, was truly a gentleman. Hearing of the boy Gauss's talent, the Duke asked to see him. Young Gauss cannot have possessed much in the way of social polish at this point. Later in life, after much acquaintance with courts and universities, he is described as mild and affable; but he always had the rough-cut features and stocky physique of his peasant origins. However, the Duke was sufficiently discerning that he took to the boy at once, remained his friend until death parted them, and provided the steady financial support that enabled young Gauss to embark on a long brilliant career.

10. The Duke's ability to support Gauss ended very tragically. In 1806 Napoleon was at the height of his career—he had defeated the combined armies of Russia and Austria at the battle of Austerlitz, having temporarily bought off the Prussians by offering them Hanover. He had then established the Confederation of the Rhine, bringing all the western part of what is now Germany under French rule, and reneged on the Hanover deal, offering it now to Britain. Only Prussia and Saxony held out against him; and their only ally was Russia, gun-shy from the defeat at Austerlitz. To prevent Saxony from becoming a French satellite, the Prussians occupied it, calling the Duke of Brunswick out of retirement—he was 71 years old at this point—to lead their forces. Napoleon declared war and his army struck northwest through Saxony toward Berlin. The Prussians tried to concentrate forces, but the French crushed the main Prussian units. One of Napoleon's flanking corps caught the Duke and routed his troops. Defeated and mortally wounded, the Duke asked Napoleon, via an emissary, for leave to return to his home to die. The Emperor, a thoroughly modern dictator who was not much given to chivalry, laughed in the messenger's face. The unfortunate Duke, blinded and dying, had to be hurried away in a cart to the free territories beyond the Elbe. He had passed through Brunswick on the way, and it is said that Gauss saw the cart from the window of his room opposite the castle gate. The Duchy of Brunswick was then wound up, incorporated into Napoleon's puppet "Kingdom of Westphalia." The Duke's heir, Friedrich Wilhelm, was dispossessed and had to flee to England. He, too, died fighting Napoleon, at the battle of Quatre Bras in 1815, a few days before Waterloo, but not before his duchy had been restored to him.

11. On a later razzia through western Germany, when Gauss was installed at Göttingen, Napoleon spared the city because "the greatest mathematician of all time is living there."

12. Gauss published much less than he wrote. There seem to have been two reasons for this apparent carelessness. One was a lack of ambition. A serene, self-contained, and frugal man, who grew up without material possessions and seems never to have acquired the taste for them, Gauss had little need of anyone's approval and did not seek social advancement. The other factor, much more common among mathematicians in all ages, was perfectionism. Gauss could not bring himself to present any result to the world until it was polished smooth, all in faultless logical order.

13. Published mathematical papers often have irritating assertions of the type: "It now follows that...," or: "It is now obvious that...," when it doesn't follow, and isn't obvious at all, unless you put in the six hours the author did to supply the missing steps and checking them. There is a story about the English mathematician G.H. Hardy. In the middle of delivering a lecture, Hardy arrived at a point in his argument where he said, "It is now obvious that...." Here he stopped, fell silent, and stood motionless with furrowed brow for a few seconds. Then he walked out of the lecture hall. Twenty minutes later he returned, smiling, and began, "Yes, it is obvious that...."

14. Russia entered the modern age somewhat behind the rest of Europe, and her entry was accomplished mainly by the energy and imagination of Peter the Great, a Tsar.

15. The St. Petersburg Academy opened its doors in August, 1725—too late for Tsar Peter to preside over the ceremony; he had died six months earlier. The required personnel were imported. Among the foreign scholars who showed up at the first session of the St. Petersburg Academy were two brothers, Nicholas and Daniel Bernoulli—sons of Johann Bernoulli of Basel in Switzerland. The death of Nicholas created a vacancy and Daniel who had known Euler in Basel recommended him. Euler was happy to take up an academician's post at an early age of 20. The death of Empress Catherine, Peter's wife, who had succeeded Peter and followed through on his plans for the Academy, worsened the academic situation in Russia. Daniel left to return to Basel, and Euler took over the chair of mathematics at the Academy which brought him sufficient income to get married. He chose a Swiss girl, Catherine Gsell, whose father was a painter living in St. Petersburg. Euler left St. Petersburg for Berlin, working in Frederick's court (Frederick was the ruler of Prussia).

16. A second Catherine, Catherine the Great, who was on the throne of Russia was a German princess and it is possible Euler had some acquaintance with her at Frederick's court before she was shipped off to St. Petersburg to marry Peter the Great's grandson. For two-thirds of the 18th century—67 years out of 100—Russia, one of the most difficult nations to govern, was ruled by women, for the most part very successfully. Euler resumed his position in St. Petersburg, spending his last 17 years in Russia, productive to the end, and died in an instant, in full possession of all his powers but sight, at age 76, with a grandchild on his knee.

17. Euler's crystal-clear Latin makes one realize what western civilization lost when scholars ceased writing in that language. Gauss was the last important mathematician to do so; this was one of those changes that came upon us after the Napoleonic wars.

While the Congress of Vienna, which marked the end of those wars, was a gathering of reactionaries intent on restoring the *status quo ante* to Europe, in fact the wars had changed everything, and nothing could be the same after them.

18. Euler lost sight in his right eye when he was barely 30 (the heartless Frederick called him "My Cyclops") and went completely blind in his early 60s. Neither the partial nor the full disability seems to have slowed him down a bit. Of his thirteen children, only five survived into adolescence, and only three outlived him. His wife Catherine died when Euler was 69; a year later he remarried—to another Gsell, Catherine's half-sister.

3.5 Riemann's Zeta Function

19. The Basel Problem: Find a closed form for the infinite series

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

Euler solved it while he was in St. Petersburg. The astonishing number was $\pi^2/6$. In fact, Euler proved more. His solution gave a closed form for

$$1 + \frac{1}{2^N} + \frac{1}{3^N} + \frac{1}{4^N} + \cdots$$

for all even values of N. But what if N is odd? Nobody has been able to find closed forms for these series.

3.6 The Great Fusion

20. Dirichlet married Rebecca Mendelssohn, one of the sisters of the composer Felix Mendelssohn, thereby forming one of the many Mendelssohn-mathematics connections. Dirichlet taught Riemann, who came to revere Dirichlet, considering him to be the second greatest mathematician alive, after Gauss.

3.7 The Golden Key, and an Improved Prime Number Theorem

21. It is true that

$$\zeta(s) = \sum_n n^{-s} = \prod_p (1 - p^{-s})^{-1}.$$

22. The log integral function is defined by

$$Li(x) = \int_0^x \frac{1}{\log t} dt.$$

23. It is found that $\pi(N) \sim Li(N)$. In fact, $Li(N)$ is actually a much better estimate of $\pi(N)$ than $N/\log N$ is.

3.8 Not Altogether Unworthy

24. Gauss examined dissertation papers of Riemann and Dedekind. He had high opinions on Riemann's and a mediocre one on Dedekind's.

25. When Gauss died, Dirichlet succeeded him in his professorship chair. Guass's brain was preserved in the university's physiology department, and it remains to this day.

26. Riemann and Dedekind lectured at Göttingen. Each attended the other's lectures. Riemann wasn't a good lecturer — unable to explain how he arrived at conclusions which seemed obvious to him.

27. Riemann was quite emotionally attached to his family. His family members passed away one after another.

28. Dirichlet had heart attack in 1858 and while he was gravely ill, her wife suddenly died of stroke. Dirichlet followed her soon after. His brain joined Gauss's in the physiology department.

29. In 1850s, Göttingen's fame grew and came to be known as the home of Gauss, Dirichlet and Gauss. (In 1858, Dedekind left for Zurich and his best work was in front of him).

3.9 Domain Stretching

3.10 A Proof and a Turning Point

30. Riemann's 1859 paper was his only publication on number theory and the only one of his productions that contained no geometrical ideas at all.

31. Riemann was a very pure case of the intuitive mathematician. The mathematical personality has two large components, the logical and the intuitive. Weierstrass would be a good example of an extremely logical mathematician.

32. The 1859 paper is revered not for its logical purity and not for its clarity, but for the sheer originality of the methods Riemann used. In terms of rigor, Riemann left many results unproved — including the main result of the paper.

33. Hadamart and de la Vallée Poussin independently proved the PNT in 1896 after being first thought of by Gauss in 1792. Stieltjes claimed to have proved the Riemann Hypothesis. He died and never published it.

34. Many great mathematicians belonged to the absent-minded professor stereotype. There were some exceptions—Descartes was a soldier and courtier, Weierstrass spent his university years drinking and fighting and left without a degree, and von Neumann was quite a *boulevardier*, fond of pretty women and fast cars.

35. Hadamart was a public man but also deeply marked with tragedy such as the great wars of the 20th cen taking all three of his sons. In grief and despair after WWI, he turned to pacifism and the League of Nations. He was a keen naturalist, with museum-grade collections of ferns and fungi. He organized an amateur orchestra at his home; Einstein—a lifelong friend—was a visiting violinist. He was married for 68 years to the same woman and when she died, he was 94 years old. Two years later, his beloved grandson died in a climbing accident robbing him of his spirit and he died a little short of his 98th birthday.

3.11 Nine Zulu Queens Ruled China

36. $\mathbb{R}$ opens the door to analysis as any infinite sequence of real numbers has a limit in $\mathbb{R}$. Irrationals may be either algebraic or transcendental. Both e and π are transcendental.

37. Numbers came in the order: $\mathbb{N}, \mathbb{Q}, \mathbb{R}, \mathbb{Z}, \mathbb{C}$. Natural numbers were known in prehistoric times. Fractions were used by Egyptians in around 3000 B.C.E. Pythagoreans discovered irrational numbers around 600 B.C.E. Negative numbers came in during the Renaissance. Complex numbers appeared in the 17th century.

38. Cantor ended his life in an asylum, which was probably a consequence of depression caused by difficulty in getting his theories accepted.

3.12 Hilbert's Eighth Problem

39. Hilbert was a keen dancer, popular lecturer and something of a skirt-chaser, to the very limited degree. When, during WWI, the university refused to give Noether a regular lecturing position on the grounds that she was a female, Hilbert simply announced a course of lectures to be given by himself, then let Noether deliver them.

40. Hilbert classed a large part of humanity as fools. This was particularly unfortunate because Franz, his only child, was afflicted with serious mental problems. Unable to learn anything much, or to hold down any kind of job, Franz also suffered occasional lapses into paranoia, following which he had to be kept in a mental hospital for a while. Hilbert is recorded as saying, at the time of the first of these incarcerations, "From now on I must consider myself as not having a son."

41. Hilbert had a student who one day presented him with a paper purporting to prove the Riemann Hypothesis. Hilbert studied the paper carefully and was really impressed by the depth of the argument; but unfortunately he found an error in it which even he could not eliminate. The following year the student died. Hilbert asked the grieving parents if he might be permitted to make a funeral oration. While the student's relatives and friends were weeping beside the grave in the rain, Hilbert came forward. He began by saying what a tragedy it was that such a gifted young man had died before he had had an opportunity to show what he could accomplish. But, he continued, in

spite of the fact that this young man's proof of the Riemann Hypothesis contained an error, it was still possible that some day a proof of the famous problem would be obtained along the lines which the deceased had indicated. "In fact," he continued with enthusiasm, standing there in the rain by the dead student's grave, "let us consider a function of a complex variable. . . ."

42. By 1900, it is known that there is an infinite number of zeros of the zeta function, all having real parts between 0 and 1 (the critical strip), they occur in conjugate pairs and their real parts are symmetrical about the critical line ($x = 1/2$).

43. Someone asked Hilbert what he would do if he could be revived after a sleep of several centuries. Hilbert: "I would ask whether anyone had proved the Riemann Hypothesis."

3.13 The Argument Ant and the Value Ant

44. In four dimensions of space, two flat two-dimensional planes can intersect in a single point. Compare the fact, utterly inconceivable to the inhabitants of a two-dimensional universe, that in three dimensions, two non-parallel straight lines need not intersect.

45. It is helpful with complex functions to think of the complex plane as an infinitely stretchable sheet of rubber and ask what a function does to this sheet.

3.14 In the Grip of an Obsession

46. Hardy, more than anyone else, awoke English pure mathematics from its long slumber. Most other mathematicians were engaged in applied mathematics, such as Hamiltonian, Green's theorem, Maxwell's equations, Stokes's formula, Reynolds number, etc.

47. For 12 years, 1919-1931, Hardy held a chair at Oxford, with an exchange year at Princeton, 1928-1929; the rest of his life was spent at Trinity, Cambridge. A handsome and charming man, he never married, nor had any intimate attachments of any kind. His games were cricket and court tennis. It must be remembered that the old Oxford and Cambridge colleges were men-only institutions with a strong flavor of misogyny. Until 1882, Fellows of Trinity were not permitted to marry.

48. Hardy stayed in Denmark with Harald Bohr (younger brother of Niels Bohr) until the very end of the summer vacation, and when he was obliged to return to England to start his lectures there was only a very small boat available. . . . The North Sea can be pretty rough, and the probability that such a small boat would sink was not exactly zero. Still, Hardy took the boat, but sent a postcard to Bohr: "I proved the Riemann Hypothesis. G.H. Hardy." If the boat sinks and Hardy drowns, everybody must believe that he has proved the Riemann Hypothesis. Yet God would not let Hardy have such a great honor and so He will not let the boat sink.

49. Hardy hated having his photograph taken, for example—there are only half a dozen extant photographs of him and when staying in a hotel or guest room, he would cover up all the mirrors. Where Hardy was slender and finely made, Littlewood was stocky and strong, a good all-round sportsman: swimming, rowing, rock climbing, cricket. He took up skiing at age 39 and became very proficient—an unusual thing among Englishmen at that time. He loved music and dancing.

50. Little, a lifelong bachelor, had at least two children. His colleague Béla Bollobás once told that young Littlewood used to go for annual vacations with the family of a doctor, whose children grew up calling him "Uncle John." One of these children was named Ann; Littlewood referred to her as "my niece." However, after becoming close friends with Bollobás and his wife, Littlewood confessed that Ann was, in fact, his daughter. They persuaded him to stop calling her his niece and start saying "my daughter." He accordingly did so, in the faculty common room one evening, and was mortified that none of his colleagues displayed the least surprise. Then, after Littlewood's death in 1977, a middle-aged man showed up at Trinity asking about his effects, explaining that he was Littlewood's son.

51. "Hardy and Littlewood" became such a common byline on mathematical papers in the 1910s and 1920s that jokes were circulating about Littlewood being a fiction, invented by Hardy to take the blame for his mistakes. Edmund Landau was said to have crossed the English Channel solely to confirm his belief that Littlewood did not exist.

52. When people asked for directions to his house, Landau would reply "You can't miss it. It's the finest house in town." Asked if Noether was not an instance of a great female mathematician, Landau replied: "I can testify that Emmy is a great mathematician, but that she is female, I cannot swear."

53. Landau's work ethic was legendary. When one of his junior lecturers was in hospital, recuperating from a serious illness, Landau climbed a ladder and pushed a huge folder of work through the poor man's window. Littlewood: "He simply did not know what it was like to be tired." Hardy says that Landau worked from 7 A.M. until midnight every day. Landau was also a gifted and enthusiastic teacher, and an extraordinarily productive mathematician. He wrote more than 250 papers and 7 books.

54. Landau's book *Handbuch*, considered to be his magnum opus, has probably never been translated into English. Number theorist Hugh Montgomery taught himself German by reading his way through the Handbuch, one finger on the dictionary.

55. Swedish mathematician Helge von Koch's result: If the Riemann Hypothesis is true, then $\pi(x) = Li(x) + O(\sqrt{x}\log x)$. In other words, the difference $Li(x) - \pi(x)$ is trapped between two bounding curves, $C\sqrt{x}\log x$ and its mirror image where C is some constant, as x goes off to infinity.

56. Hardy's 1914 result: Infinitely many of the zeta function's non-trivial zeros satisfy the Riemann Hypothesis—that is, have real part one-half.

57. Littlewood's 1914 result: $Li(x) - \pi(x)$ changes from positive to negative and back infinitely many times.

3.15 Big Oh and Möbius Mu

58. When the great Hungarian number theorist Paul Turán lay dying of cancer in 1976, his wife was at his bedside. She reported that his last murmured words were "Big oh of one. . . ." Mathematicians tell this story with awed admiration. "Doing number theory to the very end! A real mathematician!"

59. Landau did not actually invent big oh. He acknowledges, on page 883 of the *Handbuch*, that he borrowed it from Paul Bachmann's 1894 treatise. It is, therefore, very unfair that it is always referred to as "Landau's big oh," and that most mathematicians probably believe Landau did invent it.

60. Big oh doesn't care about minus signs. To say that some function $f(x)$ is big oh of 1 is to say that $f(x)$ is forever trapped between two horizontal lines $x = 1$ and $x = -1$. Big oh doesn't care about multiples either.

61. Number theory is now densely populated with, results that begin "If the Riemann Hypothesis is true, then. . . ." If it turns out that the Riemann Hypothesis is not true, quite large parts of number theory will have to be rewritten.

62. Using the Möbius function $\mu(n)$, it is true that

$$\frac{1}{\zeta(s)} = \sum_{n} \frac{\mu(n)}{n^s}.$$

3.16 Climbing the Critical Line

63. Hilbert believed in the unbounded power of the human mind to uncover the truths of Nature and mathematics. In his youth, the rather pessimistic theories of the French philosopher duBois-Reymond had been very popular. DuBois-Reymond maintained that certain thing—the nature of matter and of human consciousness, for example—are intrinsically unknowable. He coined the apothegm *ignoramus et ignorabimus*—"we are ignorant and we shall remain ignorant." In one of his most famous speeches, Hilbert said, "We must know, we shall know."

64. When the Nazis had control of Germany in 1933, one of their first decrees (April 7) was intended to bring about the dismissal of all Jews from the civil service but von Hindenburg insisted that there be two categories of exemption to the decree: first, any Jew who had performed military service in WWI, and second, any who had already held a civil service position before August 1914, when that war began. University professors were civil servants and so came under the scope of the decree. Of the five professors teaching mathematics at Göttingen, three—Edmund Landau,

Richard Courant, and Felix Bernstein—were Jewish. A fourth, Hermann Weyl (who had succeeded to Hilbert's chair), had a Jewish wife. As a matter of fact, the decree did not apply to Landau or Courant, since they fell within the Hindenburg exemptions.

65. Between April and November of 1933, Göttingen as a mathematical center was gutted. Not only Jewish faculty were involved; anyone thought to have leftist leanings came under suspicion. The mathematicians fled—most eventually finding their way to the US. Altogether 18 faculty members left or were dismissed from the Mathematics Institute at Göttingen.

66. Landau attempted to resume his calculus classes in November 1933, but the Science Students' Council learned of his intention and organized a boycott. With singular courage, Landau asked the Council leader, a 20-year-old student named Teichmüller, to write out as a letter his reasons for organizing the boycott. Teichmüller did so, and the letter somehow survived. It is clear from his letter that his motivation for the boycott was ideological. He believed, wholeheartedly and sincerely, in the Nazi doctrines, including the racial ones, and felt it improper that German students should be taught by Jews. Landau himself then left Göttingen, brokenhearted and went back to the family home in Berlin. He would not leave his native land to live permanently abroad and died from natural causes at his Berlin home in 1938.

67. Turing was fascinated by the RH. By 1937, he had made up his mind that the RH was false and conceived the idea of constructing a mechanical computing device to generate a counterexample. He applied to the Royal Society for a grant to cover the cost of construction and actually cut some of the gear wheels himself, at the engineering department of King's College, Cambridge, where he was lecturing. His work on the "zeta function machine" stopped abruptly in 1939, when WWII broke out.

68. Carl Siegel discovered, from his researches into Riemann's *Nachlass* at the Göttingen library: Riemann, in the background work for his 1859 paper, had developed a much better method for working out the zeros—and had implemented it and computed the first three zeros for himself! None of this was revealed in the 1859 paper. It was all hidden away in the *Nachlass*. The discovery of Riemann's formula, fine-tuned and published by Siegel to become the Riemann-Siegel formula, made work on the zeros much easier. All significant research depended on it up to the mid-1980s.

3.17 A Little Algebra

69. In 1921 the Austrian mathematician Emil Artin, in his Ph.D. thesis, used field theory to open up a new approach to the RH. If you start from a finite field, there is a way to construct one of these "extension" fields in such a way that a zeta function can be associated with it. "A zeta function" here means a function defined over $\mathbb{C}$, that bears an uncanny resemblance, in its broad properties, to Riemann's zeta function.

70. For the classical Riemann zeta function, the equivalent associated field is $\mathbb{Q}$, the field of ordinary rational numbers. In these past few decades, it has become apparent that

$\mathbb{Q}$ is in some sense deeper and more intractable than the subtle, artificial fields to which the results of Artin, Weil, and Deligne apply. But the techniques developed for the manipulation of those artificial fields have considerable power—Andrew Wiles used them to prove Fermat's Last Theorem!

71. Matrices are important for what they can represent. The characteristic polynomial, the eigenvalues, and the trace are such key concepts. They are properties of the underlying operator, not just of the matrix that represents it. An operator can, infact, be represented by many matrices, all having the same eigenvalues, and so on.

72. All the eigenvalues of a Hermitian matrix are real. This implies that all the coefficients of the characteristic polynomial of a Hermitian matrix are real.

73. The Hilbert-Pólya Conjecture: The non-trivial zeros of the Riemann zeta function correspond to the eigenvalues of some Hermitian operator.

3.18 Number Theory Meets Quantum Mechanics

74. "Splitting the atom" is a misnomer. You split atoms every time you strike a match. What we are really talking about here is the splitting of the atomic nucleus, the heart of the atom. To get a nuclear reaction—controlled or otherwise—going, you must fire a subatomic particle into the nucleus of a very heavy element. If you do that in a certain way, the nucleus splits, firing off new subatomic particles as it does. These particles penetrate the nuclei of neighboring atoms ... and so on, leading to a chain reaction.

75. A random matrix is a Hermitian matrix whose entries (real parts and imaginary parts) are random in a certain special sense. They are plucked at random from a Gaussian-normal distribution. The set of Gaussian-random Hermitian matrices is called the "Gaussian Unitary Ensemble," or GUE (pronounced "goo").

76. The Montgomery-Odlyzko Law: The distribution of the spacings between successive non-trivial zeros of the Riemann zeta function (suitably normalized) is statistically identical with the distribution of eigenvalue spacings in a GUE operator.

77. It seems that the non-trivial zeros of the zeta function and the eigenvalues of random Hermitian matrices are related in some way. The non-trivial zeros of Riemann's zeta function arise from inquiries into the distribution of prime numbers. The eigenvalues of a random Hermitian matrix arise from inquiries into the behavior of systems of subatomic particles under the laws of quantum mechanics. What on earth does the distribution of prime numbers have to do with the behavior of subatomic particles?

3.19 Turning the Golden Key

78. The modified prime counting function $\pi(x)$ can be expressed in terms of the ζ function. This is exactly what Riemann had set out to do, because then all the properties of

the π function will be found encoded somehow in the properties of the ζ function.

3.20 The Riemann Operator and Other Approaches

79. In 1890 Henri Poincaré published a definitive paper on the three-body problem, making it clear not only that the problem has no closed-form solutions, but that it has another, even more disturbing quality: Its solutions are sometimes chaotic. That is, if you vary the initial conditions of the problem—the numbers equivalent to M and a in my two-body example—very slightly, the resulting orbits change drastically, beyond all recognition. Poincaré himself commented that one set of conditions produced "orbits so tangled that I cannot even begin to draw them." Poincaré's paper is generally taken to mark the birth of modern chaos theory.

80. When modern chaos theory first came up, physicists took it to be entirely a classical matter, with no relevance for quantum theory. Chaos arises from issues like the three-body problem because the numbers defining the initial conditions are real numbers, measuring numbers, infinitely divisible; they can be varied by 1 percent, or by 0.1 percent, or by 0.001 percent. . . . Since the conditions are infinitely variable, an infinity of outcomes presents itself. In quantum theory, by contrast, you can vary those initial conditions by 1, 2, or 3 units, but not by fractions. There should be "no room" for chaos in quantum theory. Yet in fact, a certain level of chaos can be observed in quantum-scale dynamical systems. If such quantum-chaotic systems persist for a period, however, the laws of quantum mechanics eventually impose order on them, draining away the chaos. The number of permitted states dwindles; the number of forbidden states swells. The bigger and more complex the system, the longer it takes for the quantum rules to assert order, and the larger the number of permitted states. . . until, on the scale of the everyday world, it would take trillions of years for the quantum order to assert itself, and the number of permitted states is large enough to be taken as infinite. That is why we have chaos in classical physics.

3.21 The Error Term

3.22 Either It's True, or Else It Isn't

81. Riemann's own imagination was very much that of a physical scientist. "Four of the nine papers that he himself managed to publish must be viewed as belonging to physics" (Laugwitz). The distinction between mathematician and physicist was not much made in Riemann's time.

82. Hilbert once ranked three problems in ascending order of difficulty:

    - The RH
    - Fermat's Last Theorem

- "The Seventh"—that is, number 7 in the list of 23 problems Hilbert presented at the 1900 congress. In its more explicit form: If a and b are algebraic numbers, then a^b is transcendental except when it trivially isn't.

Hilbert said that the RH would be resolved in his lifetime, and Fermat's Last Theorem within the lifetime of younger audience members; but "no-one in this room will live to see a proof of the Seventh." In fact the Seventh was proved less than 10 years later.

83. One can decompose the ζ function into different parts, each of which tells us something different about zeta's behavior. One of these parts is the so-called S function. For arguments on the critical line up to a height of around 10^{23}—s mainly hovers between -1 and $+1$. The largest value known is around 3.2. There are strong reasons to think that if S were ever to get up to around 100, then the RH might be in trouble. As a matter of fact, Atle Selberg proved in 1946 that S is unbounded.

Epilogue

84. Riemann died in 1866 a few weeks short of his 40th birthday. He had caught a heavy cold in the fall of 1862, and this had accelerated the tuberculosis from which he had probably suffered since childhood.