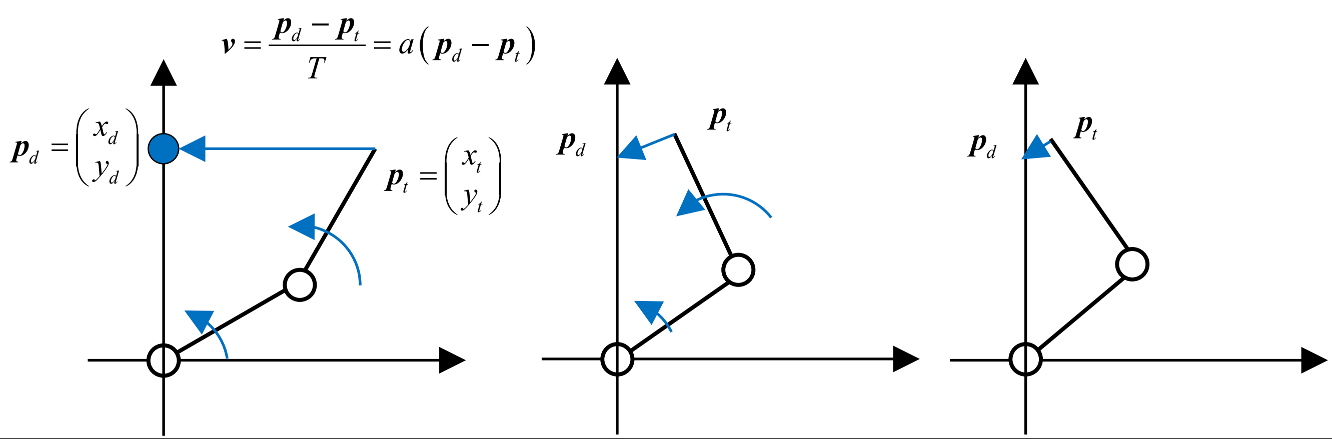
Inverse Kinematics by Jacobin
ヤコビアンによる逆運動学

Inverse Kinematics Solution Based on Velocity
速度に基づく逆運動学解法

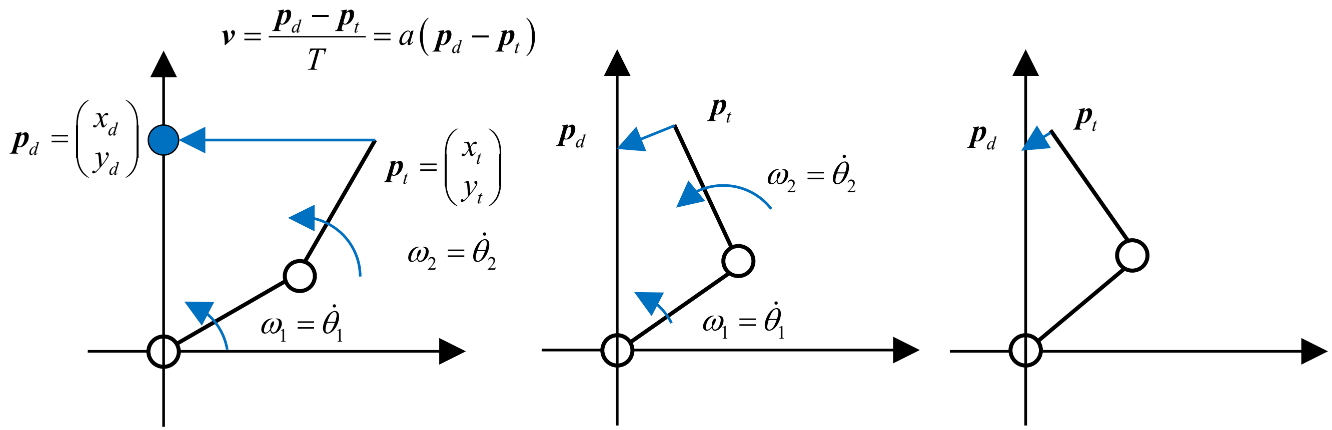


Inverse kinematics solution / 逆運動学解
The hand position should be equal to the target position
手先の位置が目標位置に一致しなければならない

Idea / アイディア
Consider the position vector from the hand position $\boldsymbol{p}_t$ to the target position $\boldsymbol{p}_d$ with a small coefficient as a required velocity of the hand to reach at the target.
手先からの目標位置への位置ベクトルに小さな係数をかけたものを考え，それを手先が発生すべき速度ベクトルと考える

- The hand always moves to the target position.
- If it is far away from the target, the velocity is large.
- If the hand is reached at the target, the velocity is zero

Relation between Hand Velocity and Angular Velocity
手先速度と関節速度の関係



Forward kinematics / 順運動学

$$\boldsymbol{p}_t = fk(\boldsymbol{\theta}_t), \begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} L_1 \cos\theta_{1,t} + L_2 \cos\left(\theta_{1,t} + \theta_{2,t}\right) \\ L_1 \sin\theta_{1,t} + L_2 \sin\left(\theta_{1,t} + \theta_{2,t}\right) \end{pmatrix}$$

Taking the time derivative of the forward kinematics / 順運動学式の時間微分をとる

$$\boldsymbol{v}_t = \dot{\boldsymbol{p}}_t = \frac{d\boldsymbol{p}_t}{dt} \approx \frac{\partial \boldsymbol{p}_t}{\partial \theta_1}\frac{d\theta_1}{dt} + \frac{\partial \boldsymbol{p}_t}{\partial \theta_2}\frac{d\theta_2}{dt} = \frac{\partial \boldsymbol{p}_t}{\partial \theta_1}\dot{\theta}_1 + \frac{\partial \boldsymbol{p}_t}{\partial \theta_2}\dot{\theta}_2$$

$$\dot{x}_t = \left(-L_1 \sin\theta_{1,t} - L_2 \sin\left(\theta_{1,t} + \theta_{2,t}\right)\right)\dot{\theta}_1 - L_2 \sin\left(\theta_{1,t} + \theta_{2,t}\right)\dot{\theta}_2$$

$$\dot{y}_t = \left(L_1 \cos\theta_{1,t} + L_2 \cos\left(\theta_{1,t} + \theta_{2,t}\right)\right)\dot{\theta}_1 + L_2 \cos\left(\theta_{1,t} + \theta_{2,t}\right)\dot{\theta}_2$$

Jacobian of Forward Kinematics of Robot Arm
ロボットアームの順運動学ヤコビアン

We can always have the following relation between the hand translational velocity and the angular velocity. / 手先の並進速度と関節角速度について常に以下の関係が成り立つ

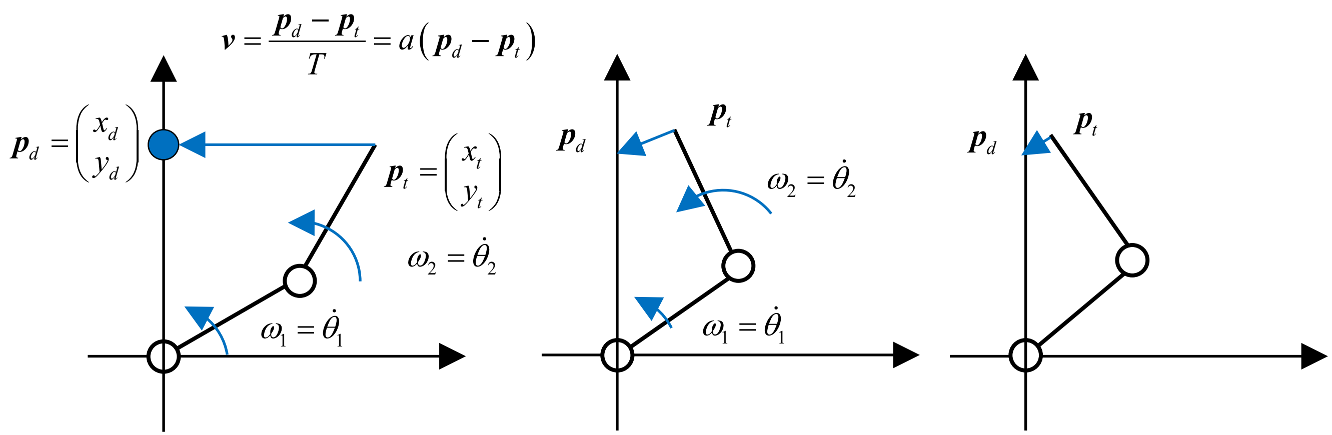
$$\boldsymbol{p}_t = fk(\boldsymbol{\theta}_t), \boldsymbol{p}_t = \begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix}, \boldsymbol{\theta}_t = \begin{pmatrix} \theta_{1,t} \\ \theta_{2,t} \\ \vdots \\ \theta_{n,t} \end{pmatrix}$$

$$\dot{\boldsymbol{p}}_t = \begin{pmatrix} \dot{x}_t \\ \dot{y}_t \\ \dot{z}_t \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial \theta_1}\dot{\theta}_1 + \frac{\partial x}{\partial \theta_2}\dot{\theta}_2 + \cdots + \frac{\partial x}{\partial \theta_n}\dot{\theta}_n \\ \frac{\partial y}{\partial \theta_1}\dot{\theta}_1 + \frac{\partial y}{\partial \theta_2}\dot{\theta}_2 + \cdots + \frac{\partial y}{\partial \theta_n}\dot{\theta} \\ \frac{\partial z}{\partial \theta_1}\dot{\theta}_1 + \frac{\partial z}{\partial \theta_2}\dot{\theta}_2 + \cdots + \frac{\partial z}{\partial \theta_n}\dot{\theta} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial \theta_1} & \frac{\partial x}{\partial \theta_2} & \cdots & \frac{\partial x}{\partial \theta_n} \\ \frac{\partial y}{\partial \theta_1} & \frac{\partial y}{\partial \theta_2} & \cdots & \frac{\partial y}{\partial \theta_n} \\ \frac{\partial z}{\partial \theta_1} & \frac{\partial z}{\partial \theta_2} & \cdots & \frac{\partial z}{\partial \theta_n} \end{pmatrix} \begin{pmatrix} \dot{\theta}_{1,t} \\ \dot{\theta}_{2,t} \\ \vdots \\ \dot{\theta}_{n,t} \end{pmatrix}$$

$$\dot{\boldsymbol{p}} = J\dot{\boldsymbol{\theta}}_t$$

The matrix J, which is called Jacobian, corresponds the angular velocity with the hand translation velocity. ヤコビアンと呼ばれる行列Jが関節角速度と手先の並進速度を関係付ける

Relation between Hand Velocity and Angular Velocity
手先速度と関節速度の関係



If J is regular, because we can have the inverse of J, we can find the angular velocity which yields the hand translational velocity.
もしJが正則なら逆行列を持つので，手先の並進速度を実現する関節角速度見つけることができる

$$\boldsymbol{v} = \dot{\boldsymbol{p}} = J\dot{\boldsymbol{\theta}}_t$$
$$\dot{\boldsymbol{\theta}}_t = J^{-1}\dot{\boldsymbol{p}} = J^{-1}\boldsymbol{v} = aJ^{-1}(\boldsymbol{p}_d - \boldsymbol{p}_t)$$

We can solve the inverse kinematics iteratively as shown in the above diagram.
上の絵の様に繰り返し計算で逆運動学を解くことができる

Two Ways of Finding Jacobian J
ヤコビアンJを見つける二つの方法

(A) Symbolic Jacobian / 記号ヤコビアン
Taking the time derivatives of the forward kinematics function of the robot arm. It requires us many symbolic operations.
順運動学関数を微分することで得られる．たくさんの微分操作が必要

(B) Numerical Jacobian / 数値ヤコビアン
Make the difference of the hand positions as giving a very small change of the joint angle, and divide it by the small change, which is the definition of the derivative.
とても小さな関節角度が違う姿勢に対して手先の位置の差を求め，それを関節角度の差で割る（微係数の定義）

$$J = \begin{pmatrix} \frac{\partial x}{\partial \theta_1} & \frac{\partial x}{\partial \theta_2} & \cdots & \frac{\partial x}{\partial \theta_n} \\ \frac{\partial y}{\partial \theta_1} & \frac{\partial y}{\partial \theta_2} & \cdots & \frac{\partial y}{\partial \theta_n} \\ \frac{\partial z}{\partial \theta_1} & \frac{\partial z}{\partial \theta_2} & \cdots & \frac{\partial z}{\partial \theta_n} \end{pmatrix}$$

$$\begin{aligned} J_i &= \frac{fk\left(\theta_1, \cdots_2, \theta_i + \Delta\theta_i, \cdots, \theta_n\right) - fk\left(\theta_1, \cdots, \theta_i, \cdots, \theta_n\right)}{\left(\theta_i + \Delta\theta_i\right) - \theta_i} \\ &= \frac{fk\left(\theta_1, \cdots_2, \theta_i + \Delta\theta_i, \cdots, \theta_n\right) - fk\left(\theta_1, \cdots, \theta_i, \cdots, \theta_n\right)}{\Delta\theta} \end{aligned}$$

$$J = \left(J_1, J_2, \cdots, J_n\right)$$

Introduction of Pseudo Inverse of J
ヤコビアンJの疑似逆行列の導入

- J is m, the dimension of the workspace, by n, the number of joints. If m = n and J is full rank, we can find the inverse of J. However, we cannot find the inverse in all the other cases.
  Jはm（作業空間の次元）かけるn（関節数）行列である．m=nでJがフルランクならJの逆行列は存在するが，それ以外の場合は逆行列は存在しない
- Therefore, we introduce the pseudo inverse matrix (or generalized inverse matrix), which is the generalized concept of the inverse matrix to any size of matrices.
  そこで逆行列の概念を拡張した，任意のサイズの行列に対する疑似逆行列（または一般化逆行列）を導入する

$$J = \begin{pmatrix} \frac{\partial x}{\partial \theta_1} & \frac{\partial x}{\partial \theta_2} & \cdots & \frac{\partial x}{\partial \theta_n} \\ \frac{\partial y}{\partial \theta_1} & \frac{\partial y}{\partial \theta_2} & \cdots & \frac{\partial y}{\partial \theta_n} \\ \frac{\partial z}{\partial \theta_1} & \frac{\partial z}{\partial \theta_2} & \cdots & \frac{\partial z}{\partial \theta_n} \end{pmatrix}$$

$$JJ^{-1} = J^{-1}J = I$$

$$JJ^{-1} = J^{-1}J = I$$

J^{-1} : the inverse matrix of J

$$JJ^{+}J = J$$

$$J^{+}JJ^{+} = J^{+}$$

$$\left(JJ^{+}\right)^{T} = JJ^{+}$$

$$\left(J^{+}J\right)^{T} = J^{+}J$$

J^{+} : psuedo inverse matrix of J

The Way to Find Pseudo Inverse of J
疑似逆行列の見つけ方

Moore-Penrose pseudo-inverse: A typical instance of Pseudo-inverse matrix
ムーアペンロース型の疑似逆行列：最も代表的な疑似逆行列

$$J^{+} = J^{-1} \quad \text{,if m = n = rank(J)}$$

$$J^{+} = J^{T}\left(JJ^{T}\right)^{-1} \quad \text{, if n > m = rank(J)}$$

$$J^{+} = \left(J^{T}J\right)^{-1}J^{T} \quad \text{, if m > n = rank(J)}$$

$$J = U\Sigma V^{T}, J^{+} = V\Sigma^{+}U^{T}$$

In matlab, you can use pinv(J) instead of inv(J)
matlabではinv(J)の代わりにpinv(J)という関数が使えます