ICT03A: Advanced Robotics
#3 Robot Arm Kinematics

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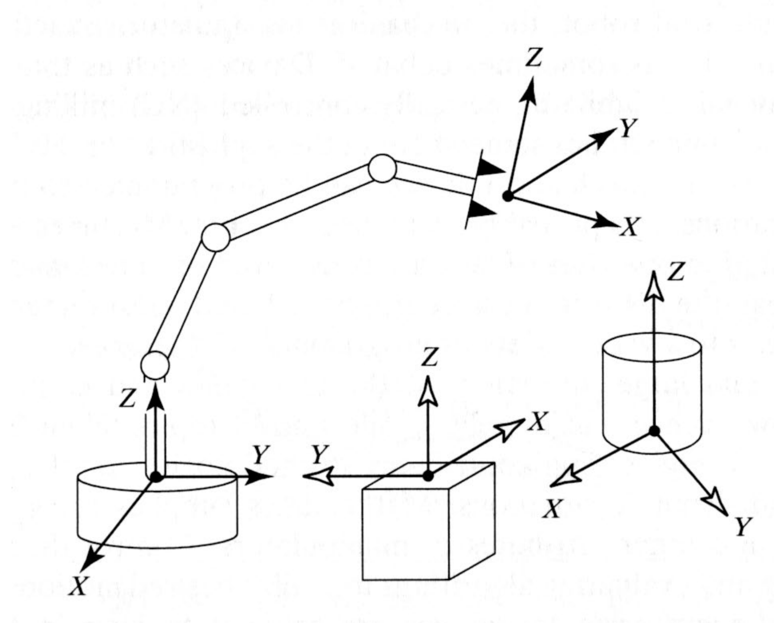
Mechanics
機構学

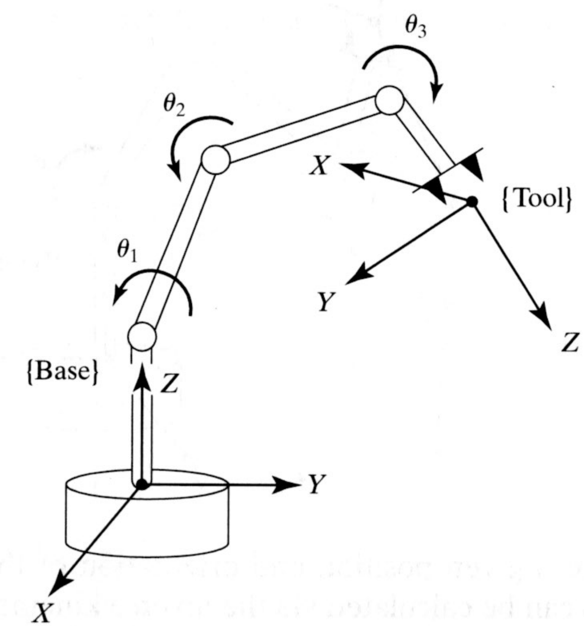
Position and Orientation
位置と向き

- Frame / 座標系・フレーム
  - Attach a coordinate system (frame) rigidly to an object
  - Transforming the description of position and orientation from one frame to another frame



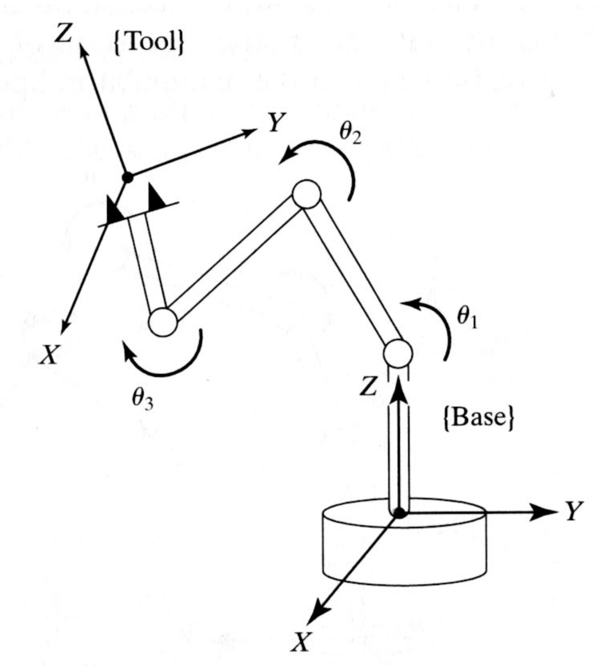
Forward Kinematics
順運動学

- Kinematics / 運動学
  - Theory of motion without forces, mainly position and velocity
- Forward kinematics / 順運動学
  - Relation from joint angles (controllable variables) to joint positions (reference / target)
    関節角度（制御可能変数）から関節位置（参照・目標）の関係
  - We can have a general solution
    一般的な解を求めることができる



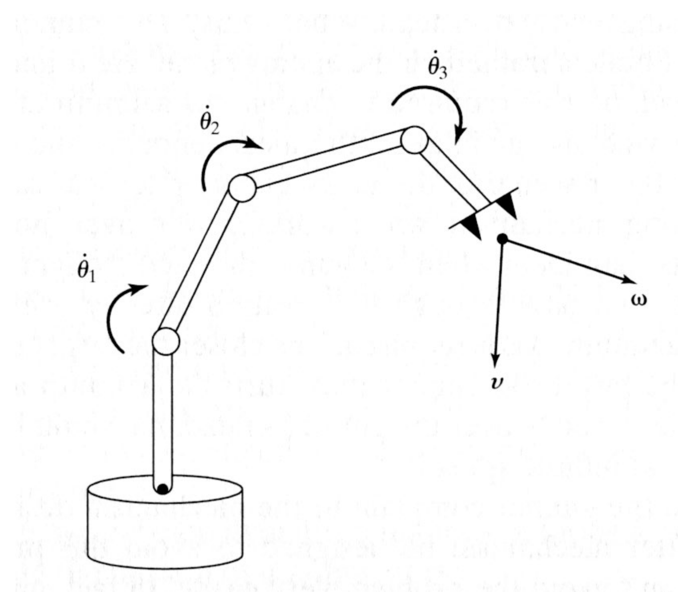
Inverse Kinematics
逆運動学

- Inverse Kinematics / 逆運動学
  - Relation from joint positions (reference / target) to joint angels (controllable variables)
    関節位置（参照・目標）から関節角度（制御変数の関係）
  - We cannot have a general closed-form solution
    閉じた形式の一般解は存在しない
  - Instead, we solve it numerically and iteratively
    数値的な繰り返し計算で求める



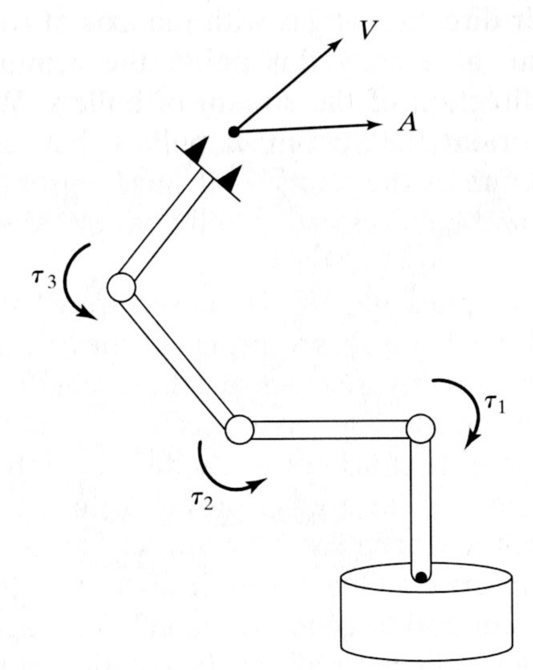
Jacobian
ヤコビアン（ヤコビ行列）

- Jacobian (Matrix)
  - Relation from joint angular velocities to joint translational ones in Cartesian space
    関節速度から直交座標系での関節の並進速度の関係
- Inverse and Singularity
  - Because we use the inverse matrix of Jacobian for the inverse kinematics, singularity causes problems
    逆運動学解ではヤコビアンの逆行列を使うので，特異点が問題となる

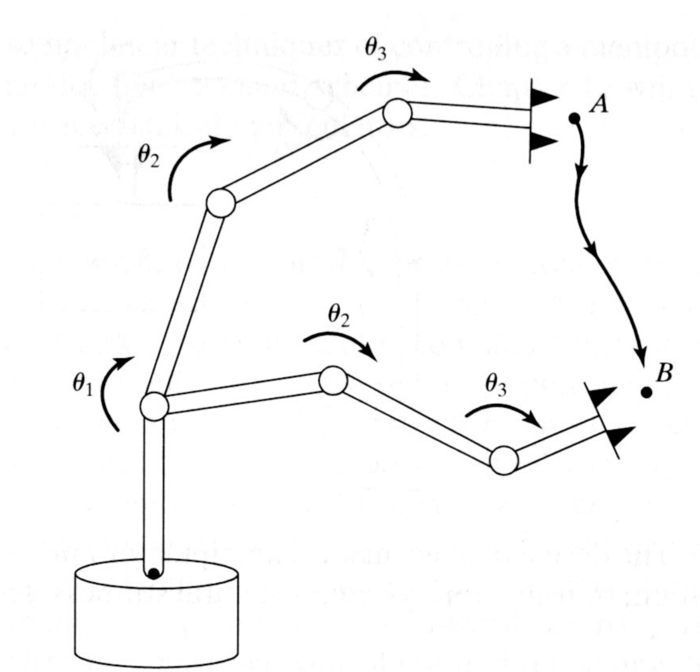


Dynamics
動力学

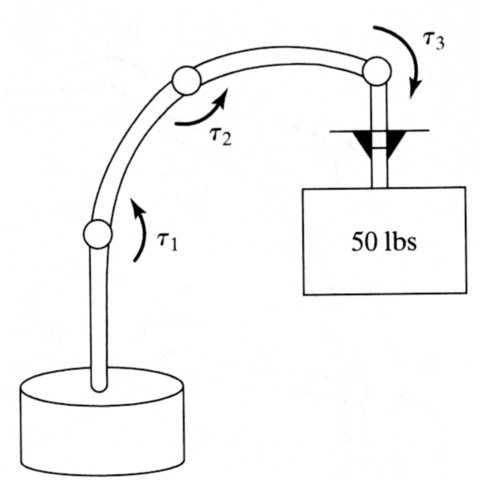
- Dynamics
  - Studying forces and torques required to cause motion
    力・トルクによる運動を考察する
  - Deriving equations of motion
    運動方程式



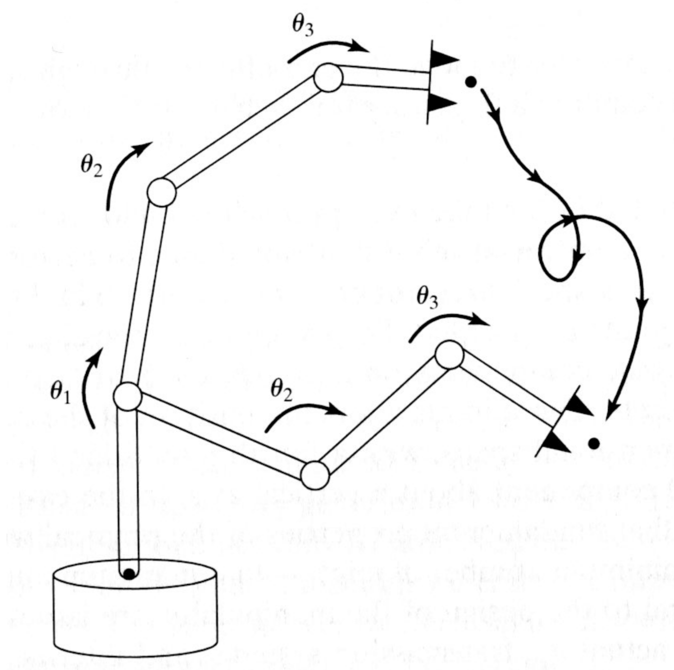
Other Issues
その他

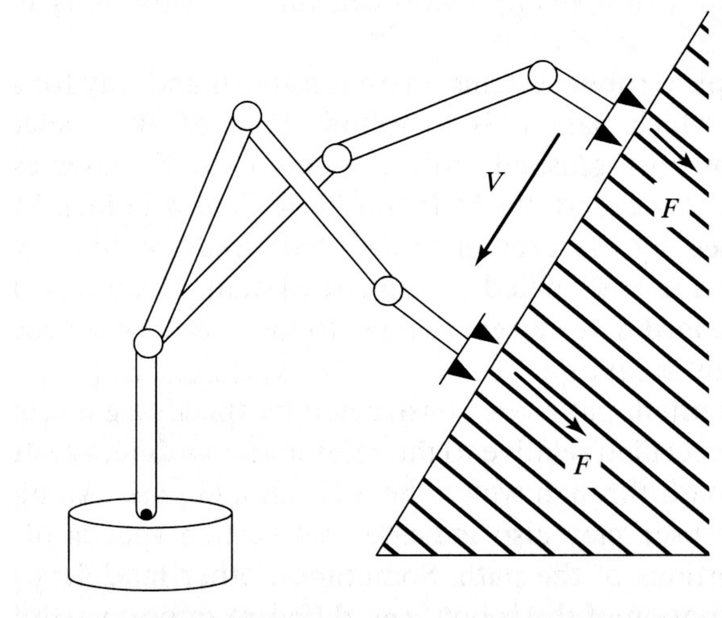


Trajectory generation
軌道生成



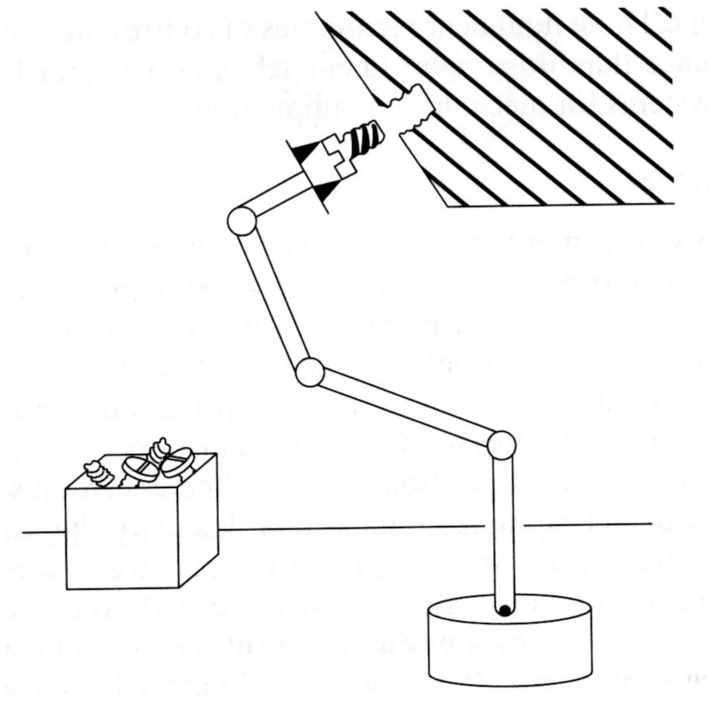
Robot arm design
ロボットアームの設計



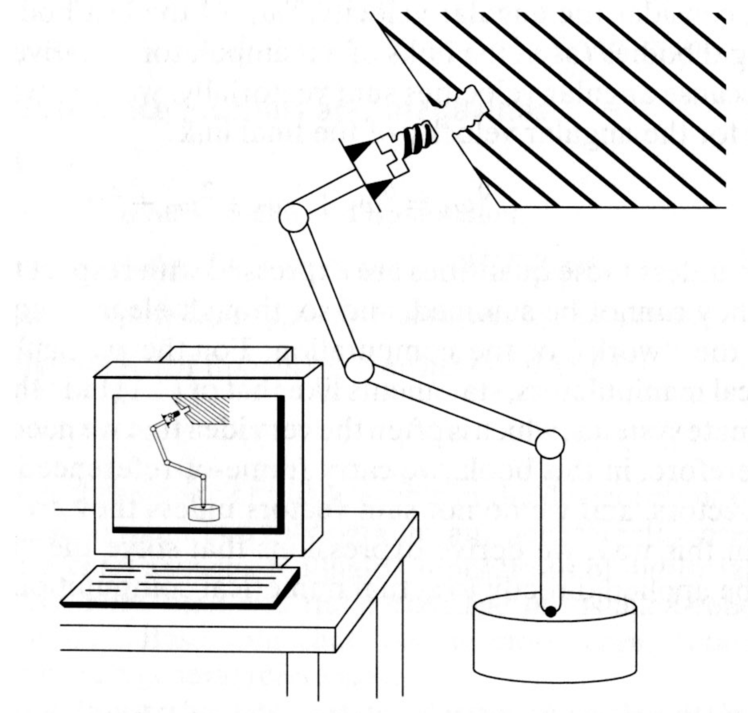


Position control for following a desired trajectory
与えられた手先軌道を実現するような位置制御

Position–force control
力・位置制御



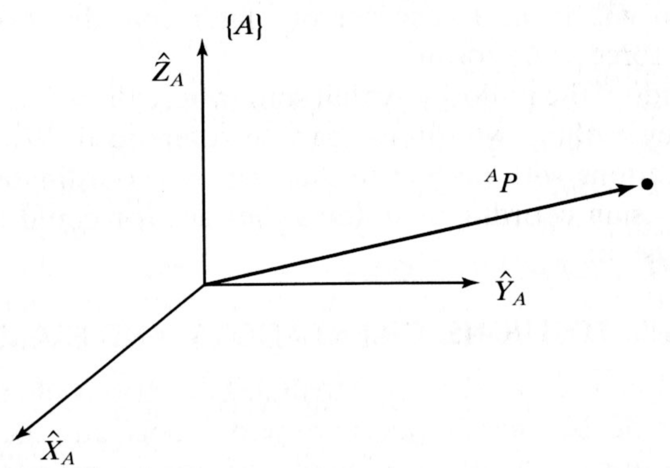
Robot programming language
ロボットのプログラミング言語



Off-line programming system
オフラインプログラミングシステム

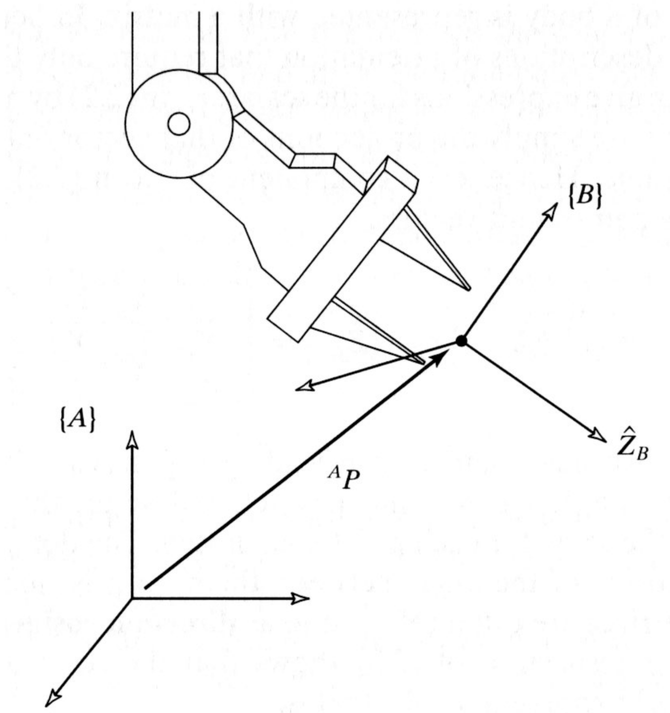
Mathematical preliminary
数学的準備

Description of Position
位置の表現



$$^AP = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$

Description of Orientation
向きの表現



$\hat{X}_B, \hat{Y}_B, \hat{Z}_B$ Unit vectors of frame {B}

${}^A\hat{X}_B, {}^A\hat{Y}_B, {}^A\hat{Z}_B$ Unit vectors of frame {B} written in frame {A}

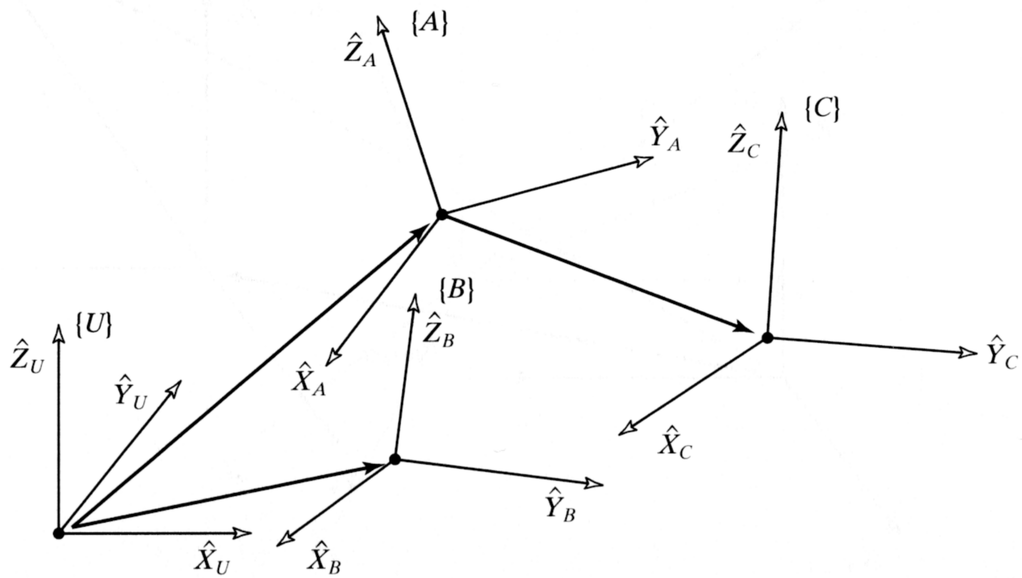
Rotation matrix (from B to A) (B in A)

$$
{}^A_BR = \left[{}^A\hat{X}_B, {}^A\hat{Y}_B, {}^A\hat{Z}_B\right]
= \begin{bmatrix} \hat{X}_B \bullet \hat{X}_A & \hat{Y}_B \bullet \hat{X}_A & \hat{Z}_B \bullet \hat{X}_A \\ \hat{X}_B \bullet \hat{Y}_A & \hat{Y}_B \bullet \hat{Y}_A & \hat{Z}_B \bullet \hat{Y}_A \\ \hat{X}_B \bullet \hat{Z}_A & \hat{Y}_B \bullet \hat{Z}_A & \hat{Z}_B \bullet \hat{Z}_A \end{bmatrix}
$$

$$
{}^A_BR = {}^B_AR^{-1} = {}^B_AR^T
$$

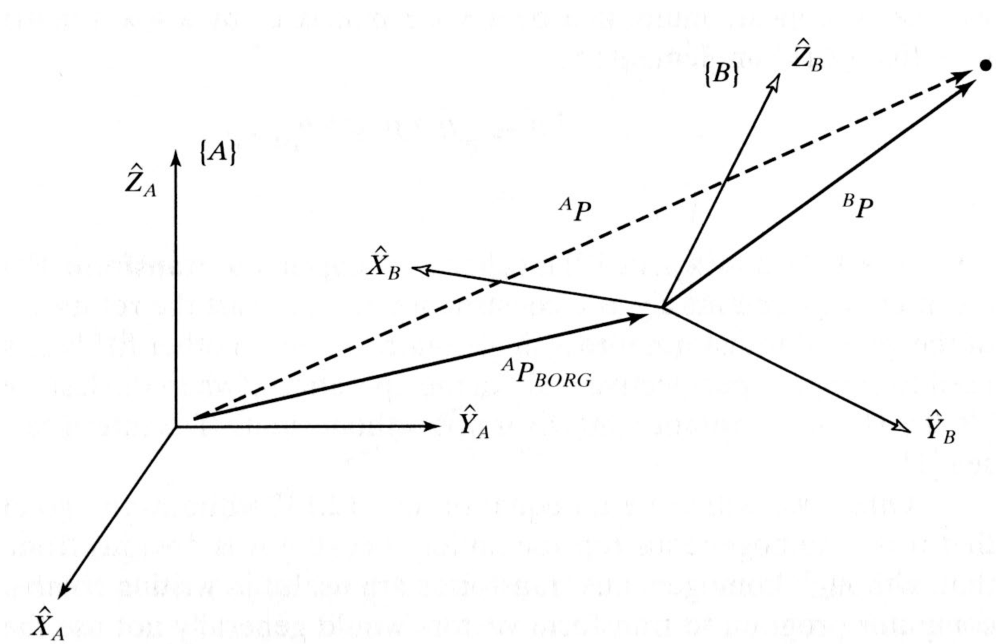
Description of Frame

$$\{B\} = \{{}^{A}_{B}R, {}^{A}P_{B\ ORG}\}$$ Frame {B} is described relative to Frame {A}



$$\{A\} = \{{}^{U}_{A}R, {}^{U}P_{A\ ORG}\}, \{B\} = \{{}^{U}_{B}R, {}^{U}P_{B\ ORG}\}, \{C\} = \{{}^{A}_{C}R, {}^{A}P_{C\ ORG}\}$$

Homogeneous Transformation
同次変換行列



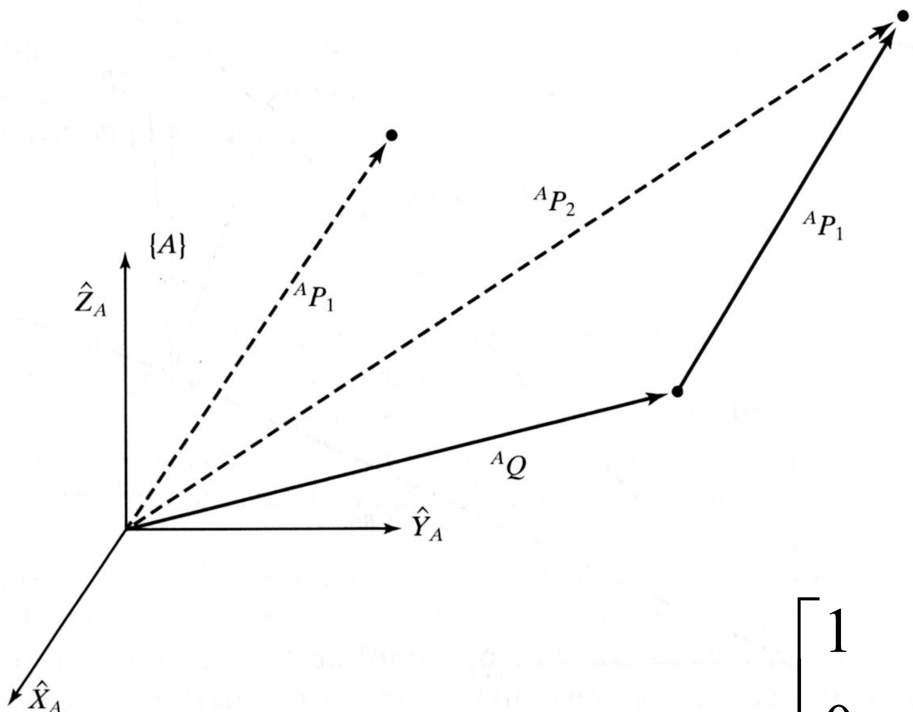
$$ {}^{A}P = {}^{A}_{B}R\,{}^{B}P + {}^{A}P_{B\ ORG} $$

$$\Downarrow$$

$$ {}^{A}P = {}^{A}_{B}T\,{}^{B}P $$

$$ \begin{bmatrix} {}^{A}P \\ 1 \end{bmatrix} = \left[\begin{array}{ccc|c} & {}^{A}_{B}R & & {}^{A}P_{B\ ORG} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} {}^{B}P \\ 1 \end{bmatrix} $$

Translational Operator
平行移動

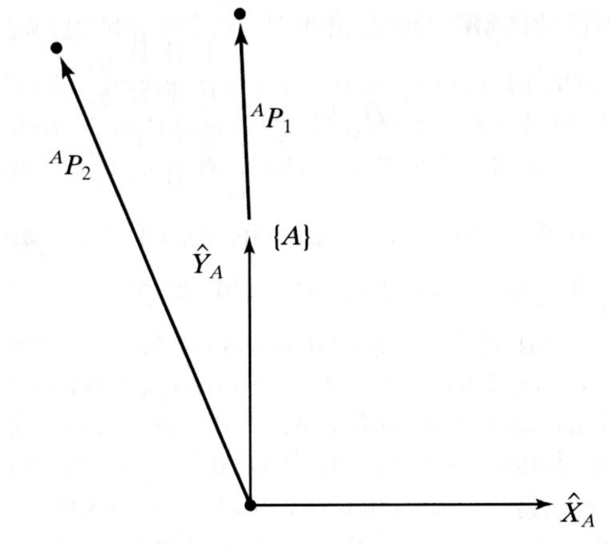


$$^{A}P_2 = {}^{A}P_1 + {}^{A}Q$$

$$^{A}P_2 = D_Q(q)\ {}^{A}P_1$$

$$D_Q(q) = \left[\begin{array}{ccc|c} 1 & 0 & 0 & q_x \\ 0 & 1 & 0 & q_y \\ 0 & 0 & 1 & q_z \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

Rotational Operator
回転移動



Rotation about Z axis

$$R_z(\theta)=\left[\begin{array}{ccc|c}\cos\theta & -\sin\theta & 0 & 0\\ \sin\theta & \cos\theta & 0 & 0\\ 0 & 0 & 1 & 0\\ \hline 0 & 0 & 0 & 1\end{array}\right]$$

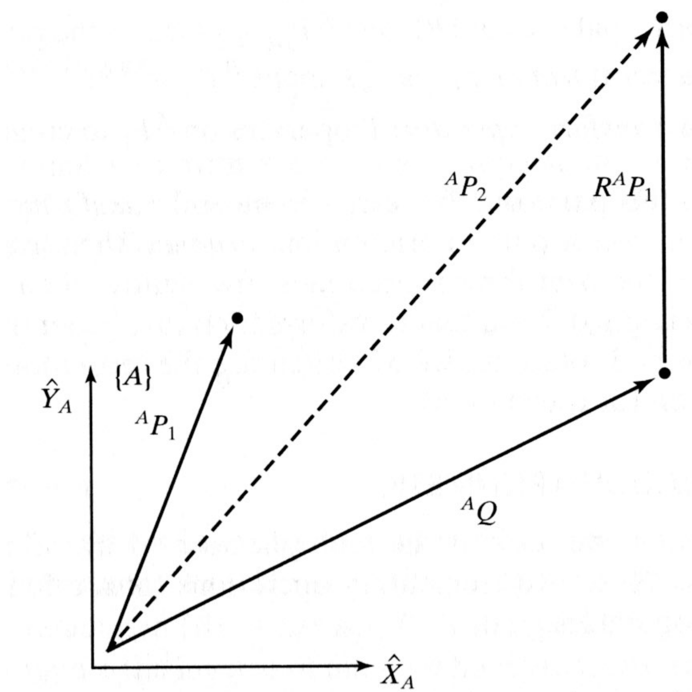
Rotation about Y axis

$$R_Y(\theta)=\left[\begin{array}{ccc|c}\cos\theta & 0 & \sin\theta & 0\\ 0 & 1 & 0 & 0\\ -\sin\theta & 0 & \cos\theta & 0\\ \hline 0 & 0 & 0 & 1\end{array}\right]$$

Rotation about X axis

$$R_X(\theta)=\left[\begin{array}{ccc|c}1 & 0 & 0 & 0\\ 0 & \cos\theta & -\sin\theta & 0\\ 0 & \sin\theta & \cos\theta & 0\\ \hline 0 & 0 & 0 & 1\end{array}\right]$$

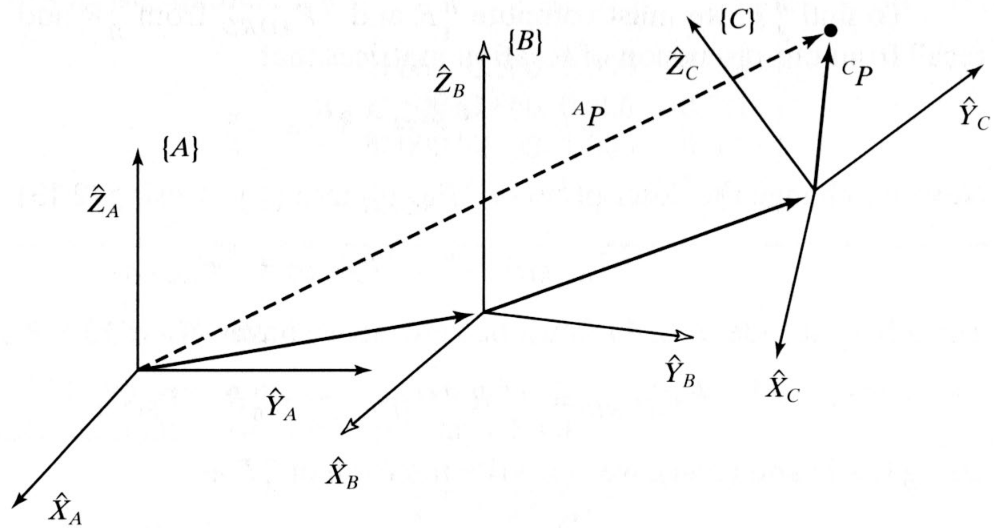
Transformation Operator



Rotate about Z by 30 degree and translate it 10 in X_A, 5 in Y_A

$$T = \left[\begin{array}{ccc|c} 0.866 & -0.500 & 0.000 & 10.0 \\ 0.500 & 0.866 & 0.000 & 5.0 \\ 0.000 & 0.000 & 1.000 & 0.0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

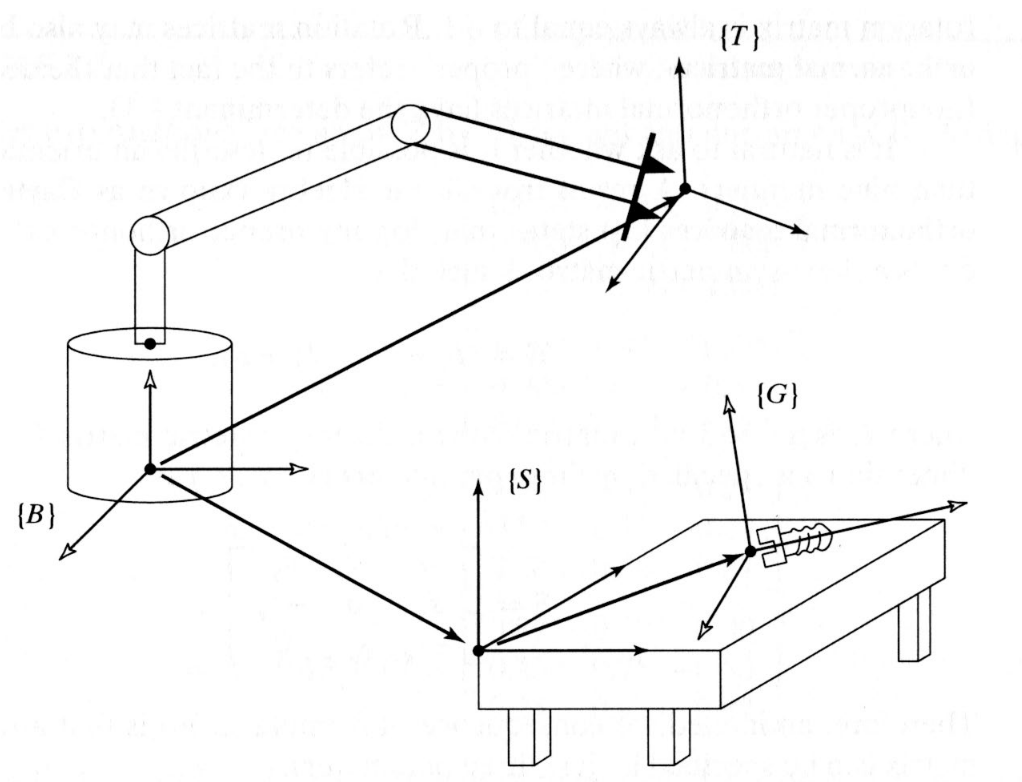
Compound and Invert Transformation
合成変換と逆変換



$$ {}^A_CT = {}^A_BT\ {}^B_CT \qquad {}^A_CT = \left[\begin{array}{ccc|c} & {}^A_BR\ {}^B_CR & & {}^A_BR\ {}^BP_{C\ ORG} + {}^AP_{B\ ORG} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] $$

$$ {}^B_AT = {}^A_BT^{-1} \qquad {}^B_AT = \left[\begin{array}{ccc|c} & {}^A_BR^T & & -{}^A_BR^T\ {}^AP_{B\ ORG} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] $$

Example

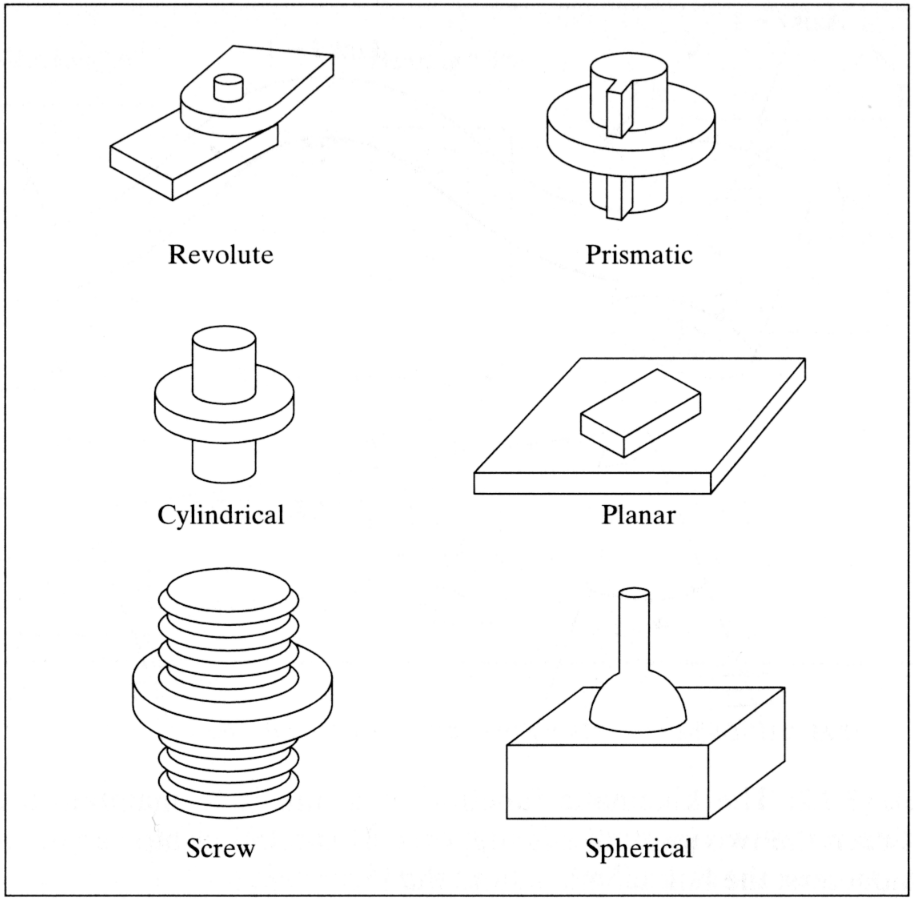


$$ {}^{T}_{G}T = {}^{T}_{B}T \; {}^{B}_{S}T \; {}^{S}_{G}T $$
$$ = {}^{B}_{T}T^{-1} \; {}^{B}_{S}T \; {}^{S}_{G}T $$

B: Base
T: Tool (Gripper)
S: Workspace (Table)
G: Object (Bolt)

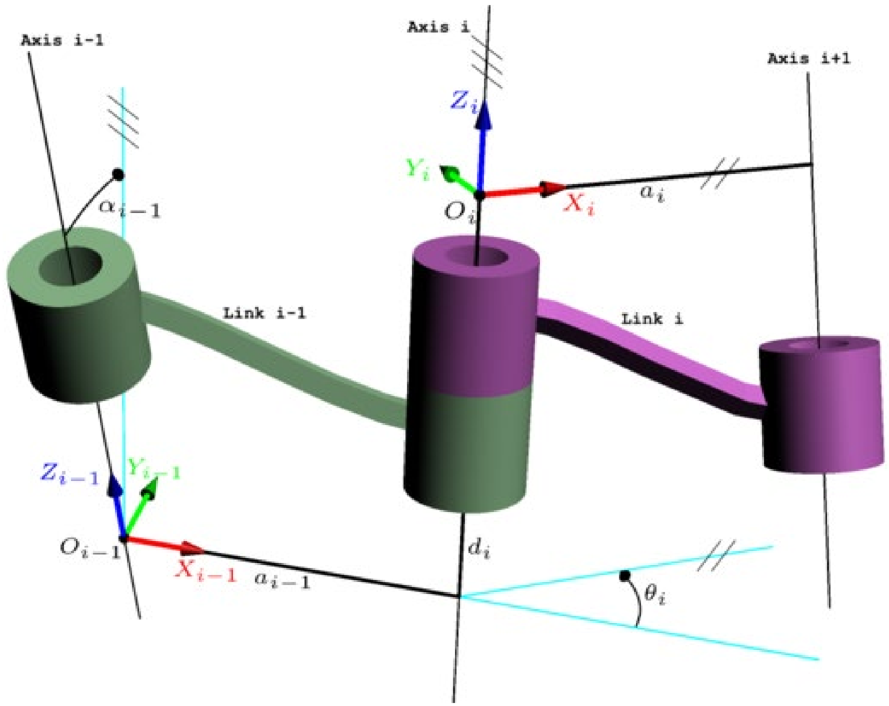
Forward kinematics
順運動学

Lower-pair Joint



Denavit–Hartenberg Parameters

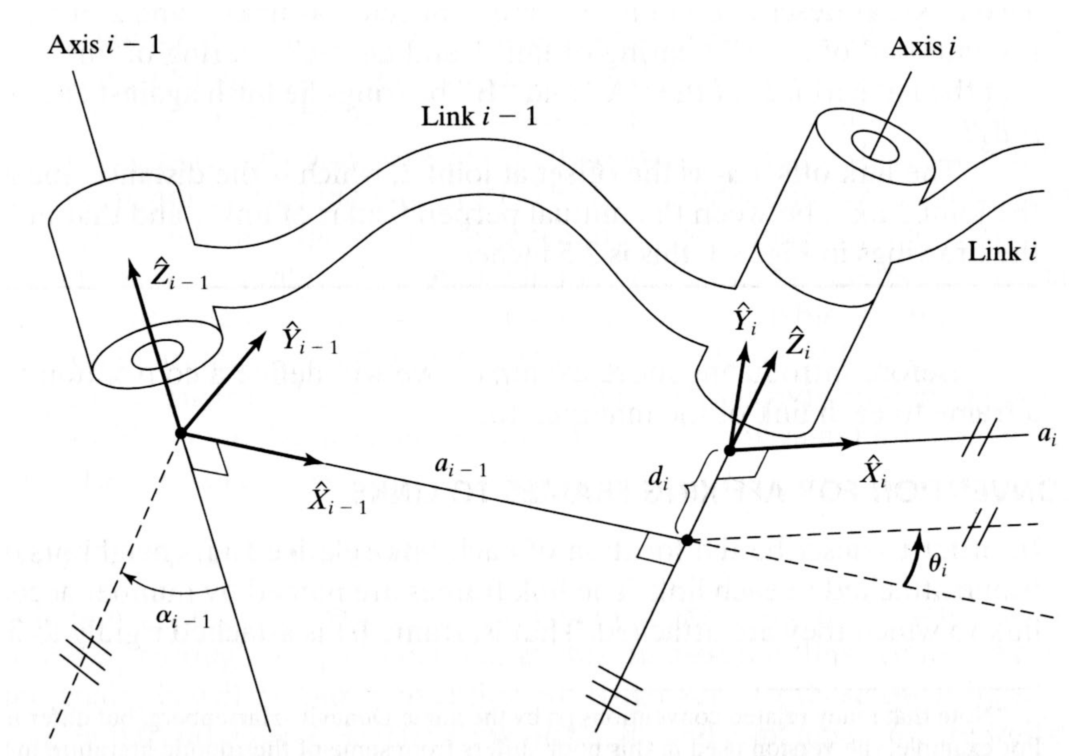
DH法



Standard method to describe robot arm parameters: Assign a frame to each of the joints, and frame relation by homogeneous transformation matrix by four parameters

We study the modified DH parameters developed by Craig in this course

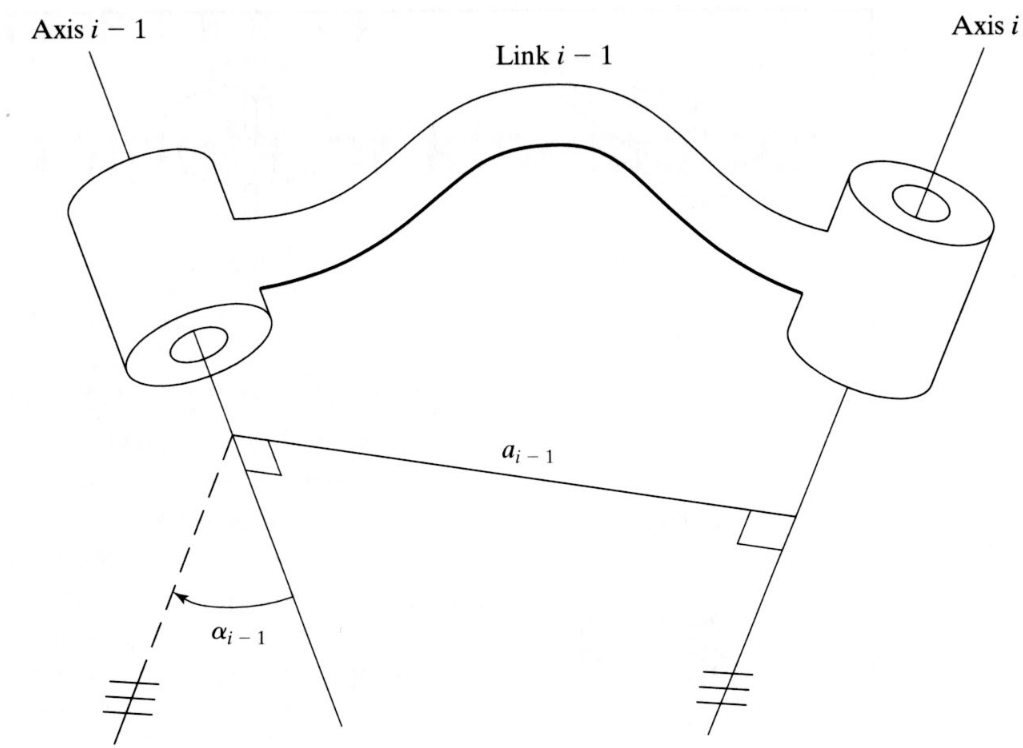
Definition of Link and Frame(Coordinate System)
リンクとフレーム（座標系）の定義



Denaviet-Hartenberg (DH) Mothod:

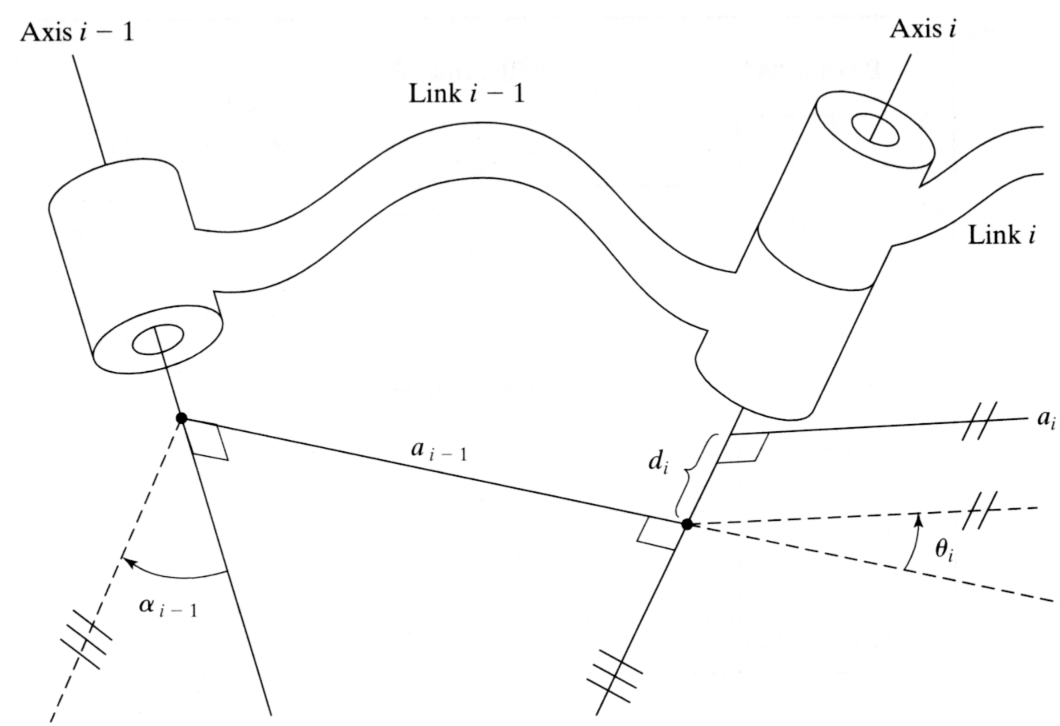
1) Z_{i-1}-axis is set to a joint axis at link i -1
2) X_{i-1} is set as common perpendicular to Z_i and Z_{i-1}, and the direction is from Z_{i-1} to Z_i
3) Y_{i-1} is set as following right hand system

Link Parameters
リンクパラメータ



Link length a_i and link twist α_i
リンク長とねじれ角

Link Parameters
リンクパラメータ



Link offset d_i and joint angle θ_i オフセットと関節角度

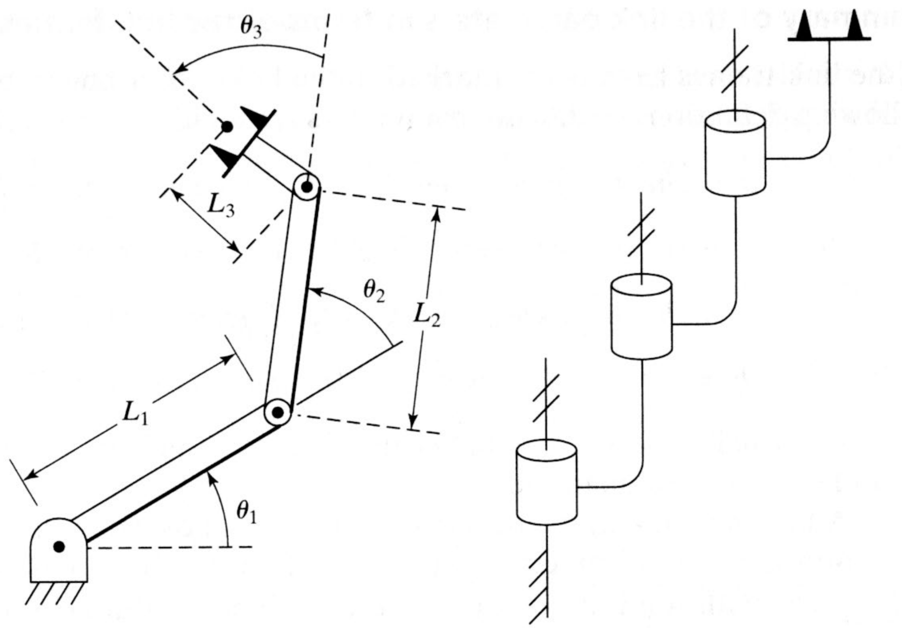
Denavit-Hartenberg notation: we relate two frames by the four parameters

DH法は二つのフレームを以下の４つのパラメータで表現する

(α_i, a_i, d_i, θ_i)

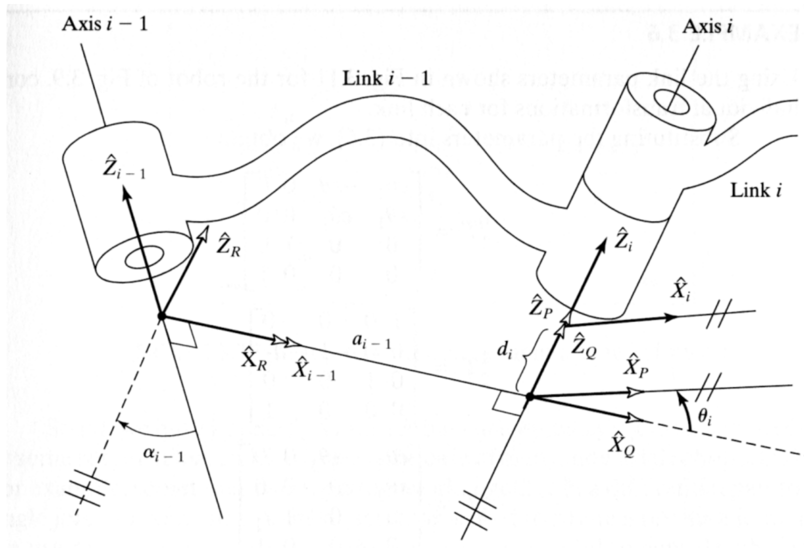
Representation Diagrams of Robot Arm Structure

ロボットアームの構造を表す図



(a) (b)

Frame Transformation Matrix
フレーム変換行列

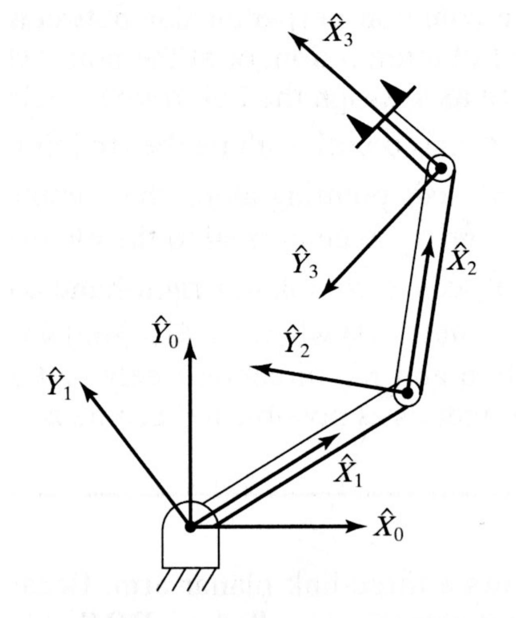


$$
{}^{i-1}_{\ i}T = R_X(\alpha_{i-1})D_X(a_{i-1})R_Z(\theta_i)D_Z(d_i)
$$

$$
= \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1} d_i \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1} d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Example of Robot Arm Representation by DH Method

DH法によるロボットアームの表現



i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	0	L_1	0	θ_2
3	0	L_2	0	θ_3
4	0	L_3	0	0

L_3 does not appear in the link parameter, because our kinematic analysis ends at a frame whose origin lies on the last joint axis.
Final offsets to the end-effecter are dealt with separately later.

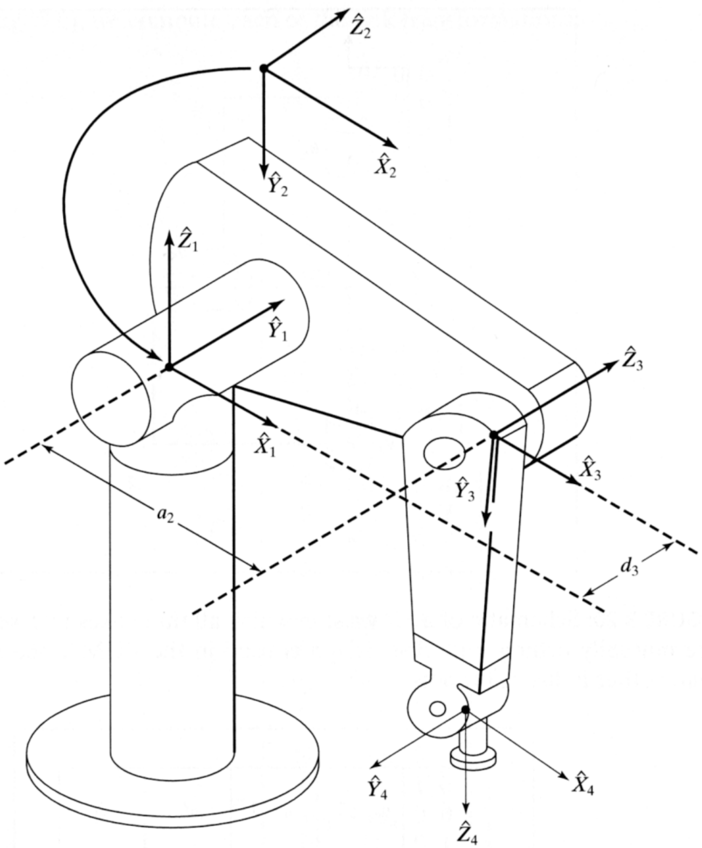
Sample: Jaco Arm

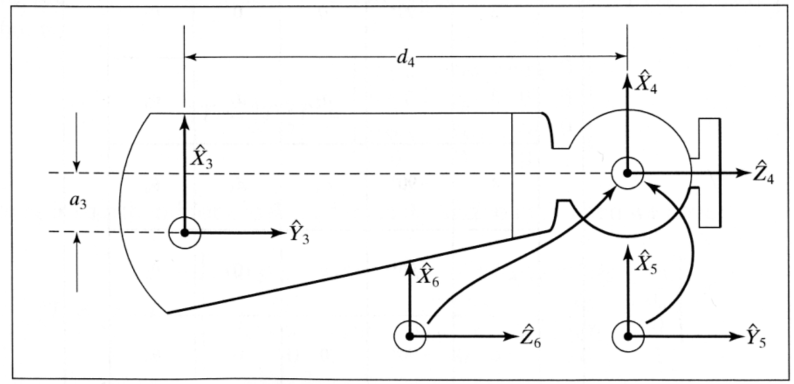
- https://www.kinovarobotics.com/sites/default/files/ULWS-RA-JAC-UG-INT-EN%20201804-1.0%20%28KINOVA%E2%84%A2%20Ultra%20lightweight%20robotic%20arm%20user%20guide%29_0.pdf
- https://github.com/JenniferBuehler/jaco-arm-pkgs/blob/master/jaco_arm/jaco_description/doc/DH%20Parameters%20-%20Kinova%20-%201.1.6.pdf

Example of Actual Manipulator

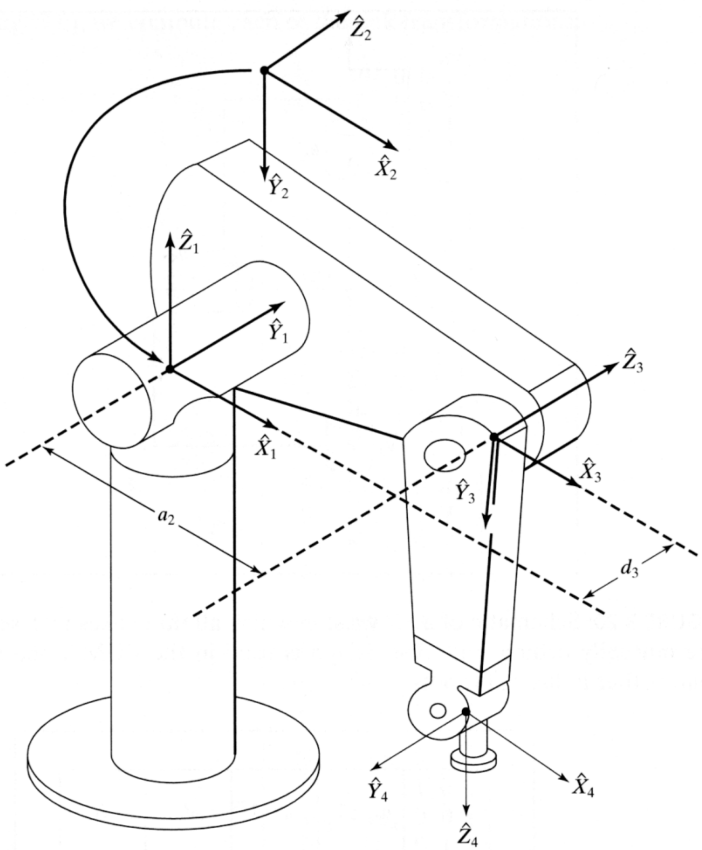
PUMA 560 Manipulator

Structure and Dimensions





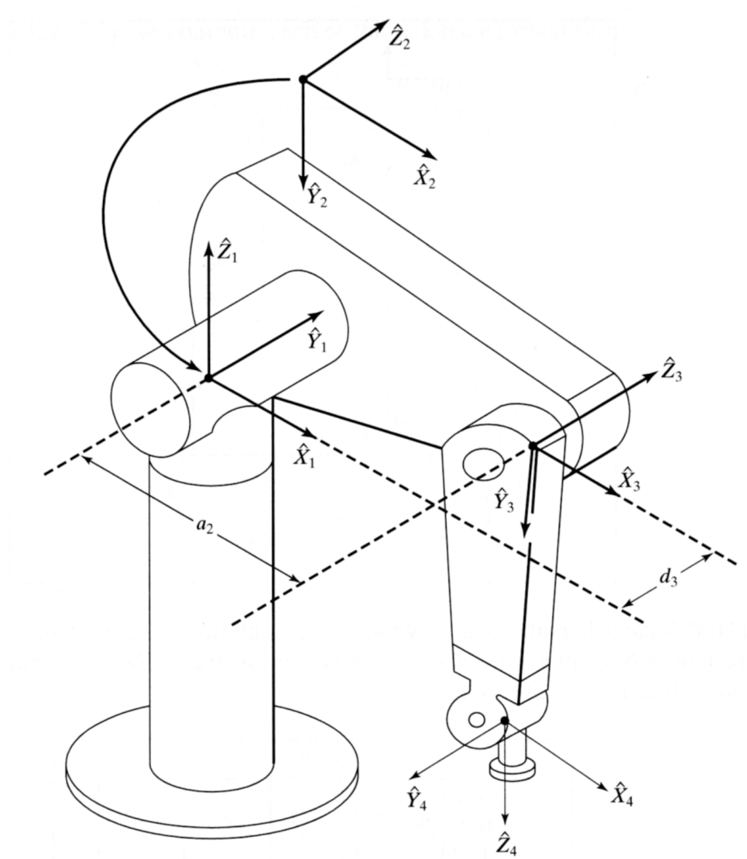
Body	a_i	α_i	d_i	θ_i
Units	m	rad	m	rad
1	0	0	0	z_1
2	0	$-\pi/2$	0	z_2
3	0.4318	0	-0.1491	z_3
4	-0.0203	$\pi/2$	-0.4318	z_4
5	0	$-\pi/2$	0	z_5
6	0	$\pi/2$	0	z_6



$$
{}^0_1T = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ -s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^1_2T = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s\theta_2 & -c\theta_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^2_3T = \begin{bmatrix} c\theta_3 & -s\theta_3 & 0 & a_2 \\ s\theta_3 & c\theta_3 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$



$$
{}^{3}_{4}T = \begin{bmatrix} c\theta_4 & -s\theta_4 & 0 & a_3 \\ 0 & 0 & 1 & d_4 \\ -s\theta_4 & -c\theta_4 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{4}_{5}T = \begin{bmatrix} c\theta_5 & -s\theta_5 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ s\theta_5 & c\theta_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{5}_{6}T = \begin{bmatrix} c\theta_6 & -s\theta_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s\theta_6 & -c\theta_6 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^0_6T = {}^0_1T\ {}^1_2T\ {}^2_3T\ {}^3_4T\ {}^4_5T\ {}^5_6T
= \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$r_{11} = c_1[c_{23}(c_4c_5c_6 - s_4s_5) - s_{23}s_5c_5] + s_1(s_4c_5c_6 + c_4s_6)$$
$$r_{21} = s_1[c_{23}(c_4c_5c_6 - s_4s_5) - s_{23}s_5c_6] - c_1(s_4c_5c_6 + c_4s_6)$$
$$r_{31} = -s_{23}(c_4c_5c_6 - s_4s_5) - c_{23}s_5c_6$$

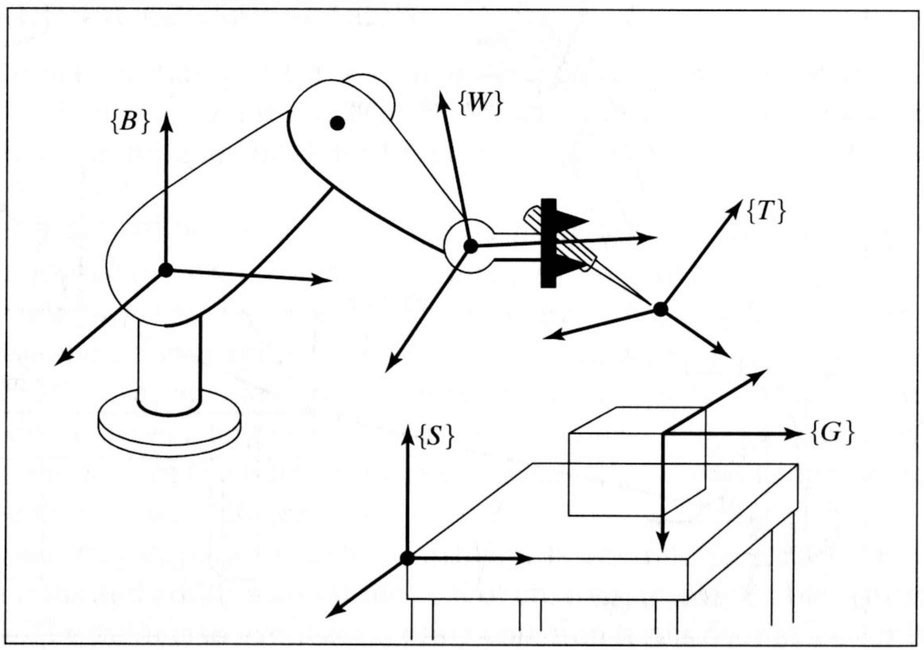
$$r_{12} = c_1[c_{23}(-c_4c_5s_6 - s_4c_6) + s_{23}s_5s_6] + s_1(c_4c_6 - s_4c_5s_6)$$
$$r_{22} = s_1[c_{23}(-c_4c_5s_6 - s_4c_6) + s_{23}s_5s_6] - c_1(c_4c_6 - s_4c_5s_6)$$
$$r_{32} = -s_{23}(-c_4c_5s_6 - s_4c_6) + c_{23}s_5s_6$$

$$r_{13} = -c_1(c_{23}c_4s_5 + s_{23}c_5) - s_1s_4s_5$$
$$r_{23} = -s_1(c_{23}c_4s_5 + s_{23}c_5) + c_1s_4s_5$$
$$r_{33} = s_{23}c_4s_5 - c_{23}c_5$$

$$p_x = c_1(a_2c_2 + a_3c_{23} - d_4s_{23}) - d_3s_1$$
$$p_y = s_1(a_2c_2 + a_3c_{23} - d_4s_{23}) + d_3c_1$$
$$p_z = -a_3s_{23} - a_2s_2 - d_4c_{23}$$

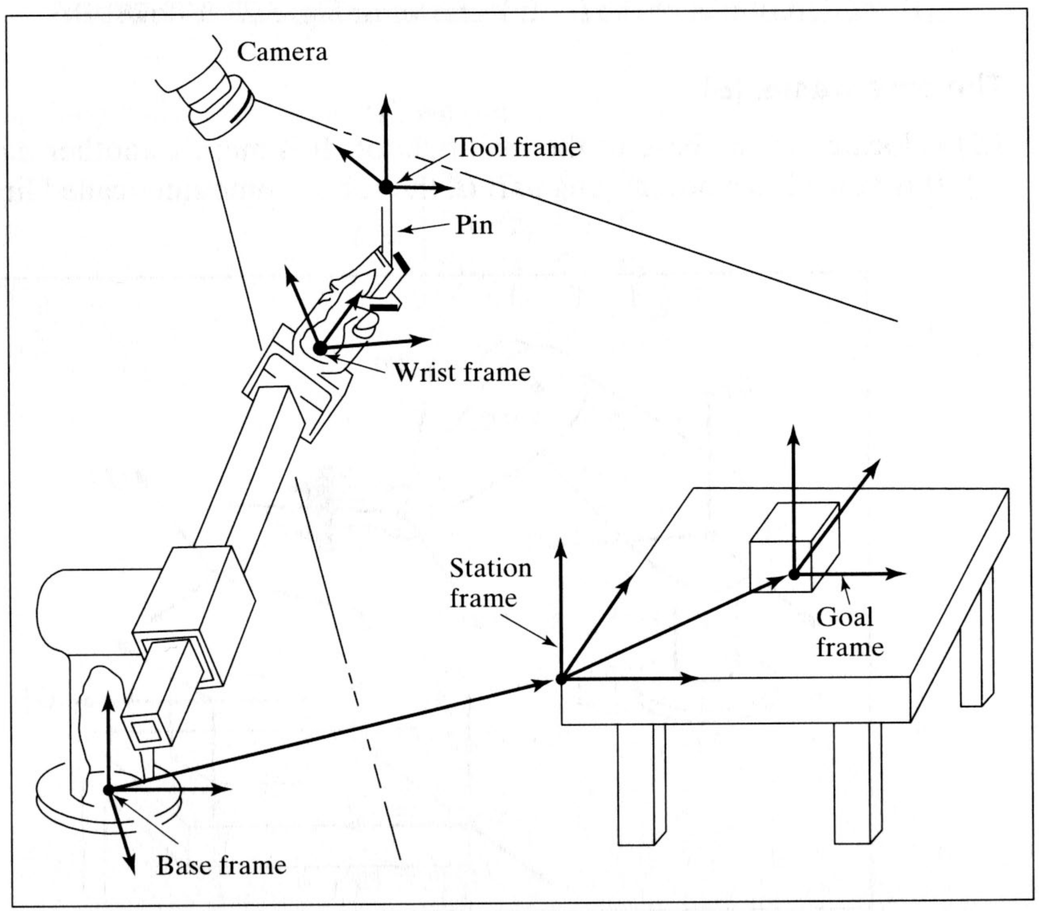
$$c_i = \cos(\theta_i)$$
$$s_i = \sin(\theta_i)$$
$$c_{ij} = \cos(\theta_i + \theta_j)$$
$$s_{ij} = \sin(\theta_i + \theta_j)$$

Standard Frames



Base Frame, {B}
Station Frame, {S}
(Workspace)
Wrist Frame, {W}
Tool Frame, {T}
Goal Frame, {G}
(Workpeace)

Example

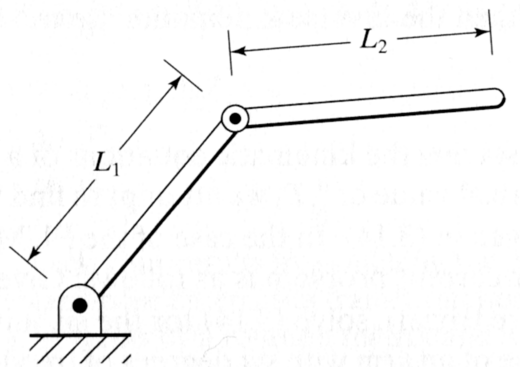

Camera
Tool frame
Pin
Wrist frame
Station
frame
Goal
frame
Base frame

Inverse kinematics
逆運動学

Inverse Kinematics Problem
逆運動学問題

- Given
  - Desired position (and orientation) of the hand relative to the base
    ベースからの手先の目的の位置（と向き）
- Find
  - Set of joint angles which achieve the desired position (and orientation)
    上の手先の位置（と向き）を実現する関節角度

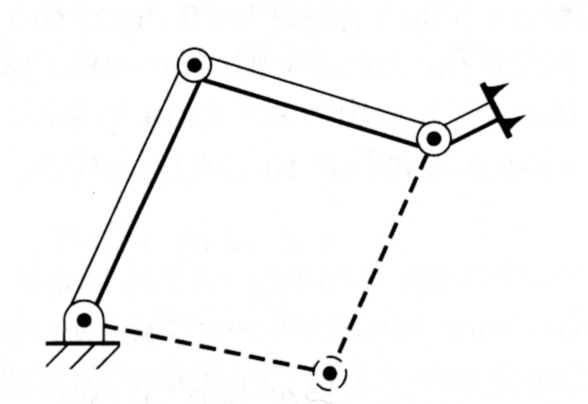
Existence of Solution



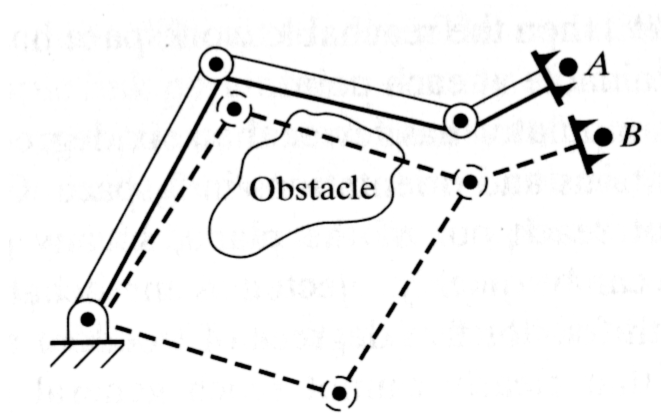
- Solvability
  - Set of nonlinear kinematic equations
- Dexterous workspace
  - Volume that the end-effecter can reach with all orientations
- Reachable workspace
  - Volume that the robot can reach in at least one orientation

If L1 = L2
  Dexterous: Origin
  Reachable workspace:
    Disc of radius 2L1
If L1 != L2
  Dexterous: None
  Reachable: Ring
    outer radius L1 + L2,
    inner radius L1- L2

Multiple Solutions
複数解が存在する

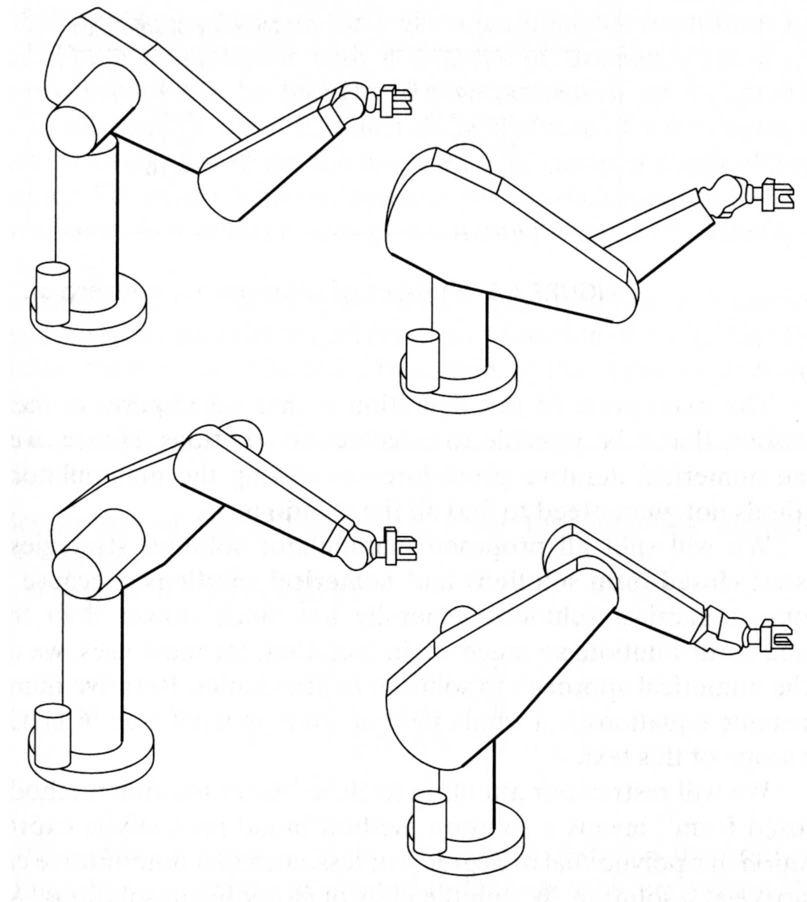


Dashed lines indicate a second solution



One of two possible solutions to reach point B causes a collision

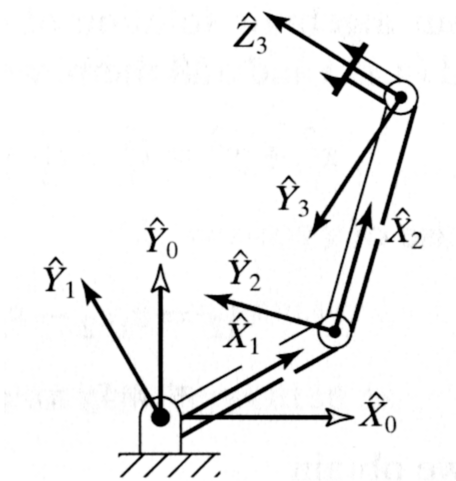
Four Solutions of PUMA 560



Method of Solution

- Closed-form solution
  - Exact solution
  - Algebraic method
  - Geometric method
- Numerical solution
  - Estimated solution
  - Iterative calculation

Algebraic Solution
代数的解法



$$
{}^{B}_{W}T = {}^{0}_{3}T = \begin{bmatrix} c_{12} & -s_{12} & 0 & l_1c_1 + l_2c_{12} \\ s_{12} & c_{12} & 0 & l_1s_1 + l_2s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_\varphi & -s_\varphi & 0 & x \\ s_\varphi & c_\varphi & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	0	L_1	0	θ_2
3	0	L_2	0	0

$$
\begin{aligned}
c_\varphi &= c_{12} \\
s_\varphi &= s_{12} \\
x &= l_1c_1 + l_2c_{12} \\
y &= l_1s_1 + l_2s_{12}
\end{aligned}
$$

Find θ_2

$$x^2 + y^2 = l_1^2 + 2l_1 l_2 c_2 + l_2^2$$

where $c_{12} = c_1 c_2 - s_1 s_2, s_{12} = c_1 s_2 + s_1 c_2$

$$c_2 = \frac{x^2 + y^2 - l_1^2 - l_2^2}{2 l_1 l_2} \qquad s_2 = \pm\sqrt{1 - c_2^2}$$

$$\theta_2 = \mathrm{Atan2}(s_2, c_2)$$

Choice of signs corresponds to the multiple solution:
Elbow-up and elbow-down

Find θ_1

$$x = k_1 c_1 - k_2 s_1, y = k_1 s_1 + k_2 c_1$$

where $k_1 = l_1 + l_2 c_2, k_2 = l_2 s_2$

$$k_1 = r\cos\gamma, k_2 = r\sin\gamma$$

$$\gamma = \mathrm{Atan2}(k_2, k_1), r = \sqrt{k_1^2 + k_2^2}$$

$$\frac{x}{r} = \cos\gamma\cos\theta_1 - \sin\gamma\sin\theta_1$$

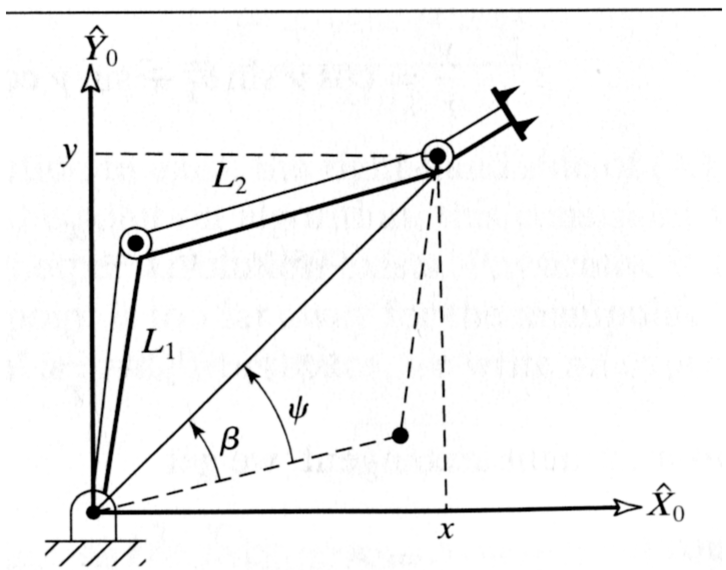
$$\frac{y}{r} = \cos\gamma\sin\theta_1 + \sin\gamma\cos\theta_1$$

$$\cos(\gamma + \theta_1) = \frac{x}{r}, \sin(\gamma + \theta_1) = \frac{y}{r}$$

$$\gamma + \theta_1 = \mathrm{Atan2}(y, x)$$

$$\theta_1 = \mathrm{Atan2}(y, x) - \mathrm{Atan2}(k_2, k_1)$$

Geometric Solution
幾何学的解法



Find θ_2

Law of Cosine

$$x^2+y^2=l_1^2+l_2^2-2l_1l_2\cos(180+\theta_2)$$

$$x^2+y^2=l_1^2+l_2^2-2l_1l_2\cos(180+\theta_2)$$

$$c_2=\frac{x^2+y^2-l_1^2-l_2^2}{2l_1l_2}$$

where $\cos(180+\theta_2)=-\cos(\theta_2)$

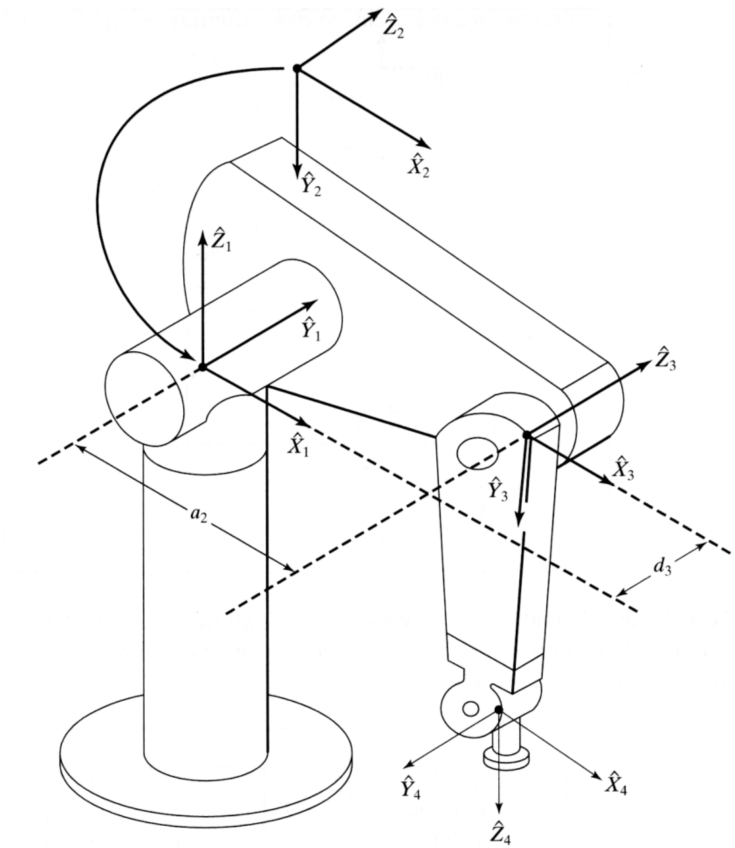
Find θ_1

$$\beta=\mathrm{Atan2}(y,x)$$

$$\cos\psi=\frac{x^2+y^2+l_1^2-l_2^2}{2l_1\sqrt{x^2+y^2}}$$

$$\theta_1=\beta\pm\psi$$

Inverse Kinematics of PUMA 560



$$
{}^0_6T = {}^0_1T\ {}^1_2T\ {}^2_3T\ {}^3_4T\ {}^4_5T\ {}^5_6T
= \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

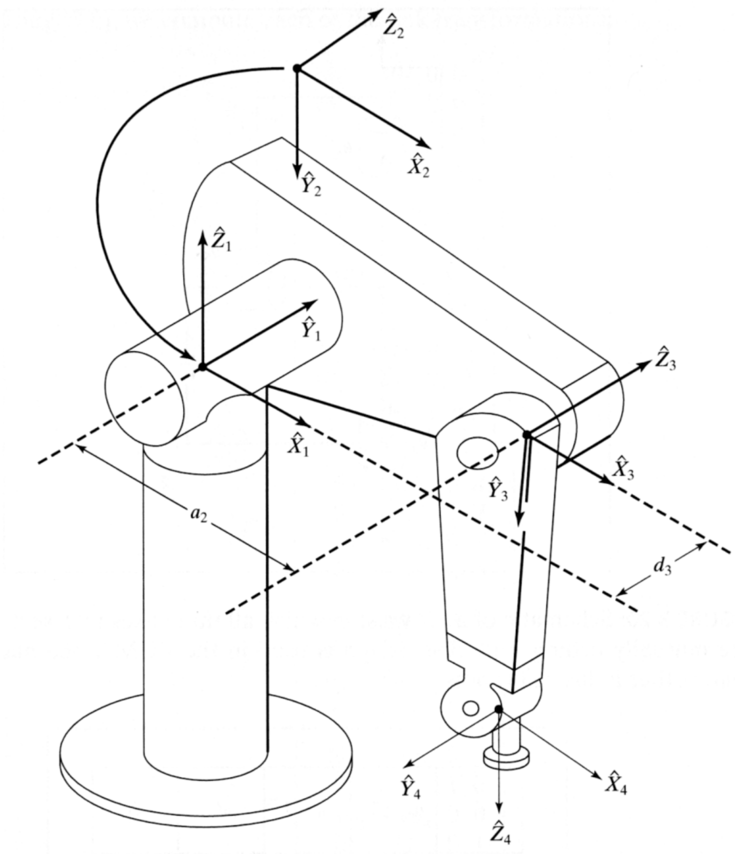
$$
\theta_1 = \mathrm{Atan2}(p_x, p_y) - \mathrm{Atan2}\left(d_3, \pm\sqrt{p_x^2 + p_y^2 - d_3^2}\right)
$$

$$
\theta_3 = \mathrm{Atan2}(a_3, d_4) - \mathrm{Atan2}\left(K, \pm\sqrt{a_3^2 + d_4^2 - K^2}\right)
$$

$$
K = \frac{p_x^2 + p_y^2 + p_z^2 - a_2^2 - a_3^2 - d_3^2 - d_4^2}{2a_2}
$$

$$
\theta_2 = \theta_{23} - \theta_3
$$

$$
\theta_{23} = A\tan 2\begin{pmatrix} (-a_3 - a_2c_3)p_z - (c_1p_x + s_1p_y)(d_4 - a_2s_3), \\ (a_2s_3 - d_4)p_z - (a_3 + a_2c_3)(c_1p_x + s_1p_y) \end{pmatrix}
$$



$$
{}^{0}_{6}T = {}^{0}_{1}T\,{}^{1}_{2}T\,{}^{2}_{3}T\,{}^{3}_{4}T\,{}^{4}_{5}T\,{}^{5}_{6}T
= \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$\theta_4 = \mathrm{Atan2}\left(-r_{13}s_1 + r_{23}c_1, -r_{13}c_1c_{23} - r_{23}s_1c_{23} + r_{33}s_{23}\right)$$

$$\theta_5 = \mathrm{Atan2}\left(s_5, c_5\right)$$

$$\theta_6 = \mathrm{Atan2}\left(s_6, c_6\right)$$

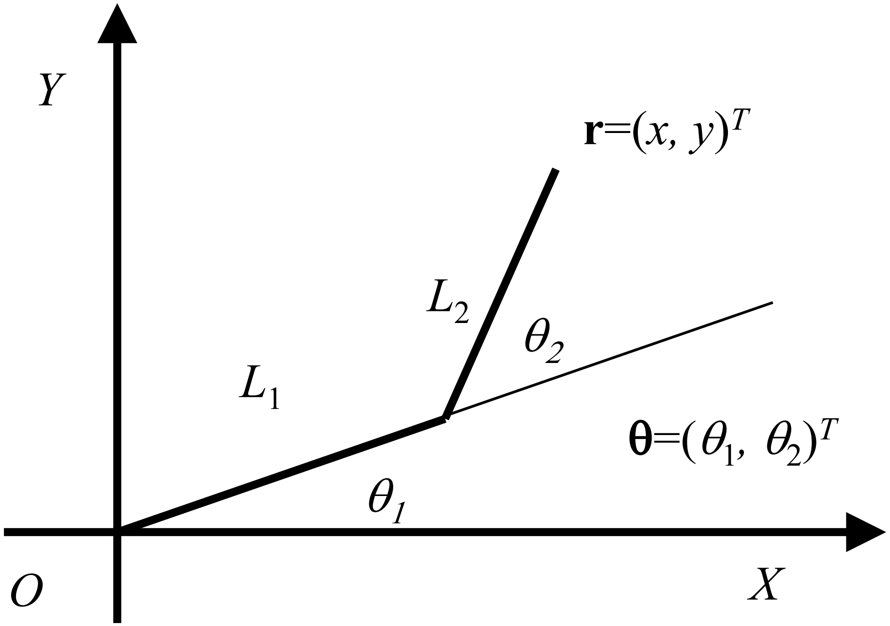
$$\theta_4' = \theta_4 + 180, \theta_5' = -\theta_5, \theta_6' = \theta_6 + 180$$

Jacobian

Objectives

- Velocities and static forces
  - Relation between linear velocities and angular velocities
  - Relation between tip force and joint torque
  - (Relation between linear positions and joint angles)
- Jacobian matrix

Inverse Kinematics



- **r**: Position vector of end effector
- **θ**: Joint angle vector

$$x = L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2)$$
$$y = L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2)$$

Forward Kinematics

$$\mathbf{r} = \mathbf{f}(\boldsymbol{\theta}), \mathbf{r} = \begin{pmatrix} x \\ y \end{pmatrix}, \boldsymbol{\theta} = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$$

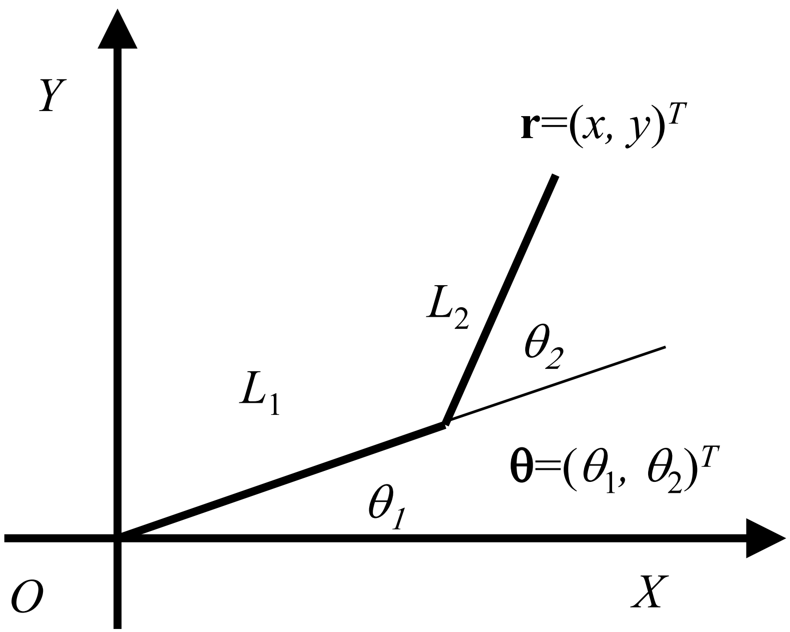
Inverse Kinematics

$$\boldsymbol{\theta} = \mathbf{f}^{-1}(\mathbf{r})$$

How to solve $\mathbf{f}^{-1}$?

Existence of f Inverse

$$\frac{\partial \mathbf{f}}{\partial \boldsymbol{\theta}} = \frac{\partial (x, y)^T}{\partial (\theta_1, \theta_2)^T}$$ called Jacobian, must be a regular matrix



$$\frac{\partial x}{\partial \theta_1} = -L_1 \sin\theta_1 - L_2 \sin\left(\theta_1 + \theta_2\right)$$

$$\frac{\partial y}{\partial \theta_1} = L_1 \cos\theta_1 + L_2 \cos\left(\theta_1 + \theta_2\right)$$

$$\frac{\partial x}{\partial \theta_2} = -L_2 \sin\left(\theta_1 + \theta_2\right)$$

$$\frac{\partial y}{\partial \theta_2} = L_2 \cos\left(\theta_1 + \theta_2\right)$$

Jacobian

$$J = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{\theta}} = \frac{\partial (x, y)^T}{\partial (\theta_1, \theta_2)^T}$$

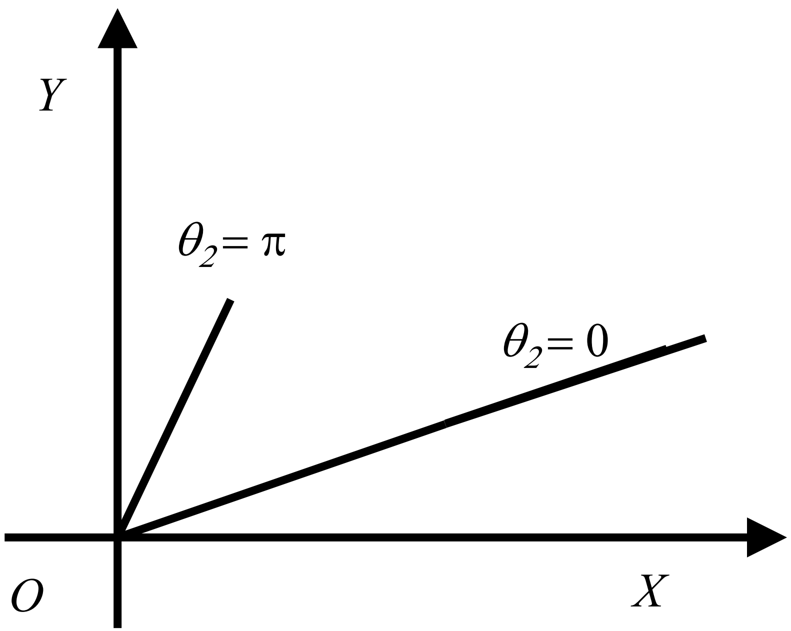
$$= \begin{pmatrix} \dfrac{\partial x}{\partial \theta_1} & \dfrac{\partial x}{\partial \theta_2} \\ \dfrac{\partial y}{\partial \theta_1} & \dfrac{\partial y}{\partial \theta_2} \end{pmatrix} = \begin{pmatrix} -L_1 \sin\theta_1 - L_2 \sin(\theta_1 + \theta_2) & -L_2 \sin(\theta_1 + \theta_2) \\ L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) & L_2 \cos(\theta_1 + \theta_2) \end{pmatrix}$$

For existence of f^{-1}, J must be a regular matrix

$$\begin{aligned} |J| &= \left(-L_1 \sin\theta_1 - L_2 \sin(\theta_1 + \theta_2)\right)\left(L_2 \cos(\theta_1 + \theta_2)\right) \\ &\quad - \left(-L_2 \sin(\theta_1 + \theta_2)\right)\left(L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2)\right) \\ &= L_1 L_2 \sin\theta_2 \end{aligned}$$

What does it mean?

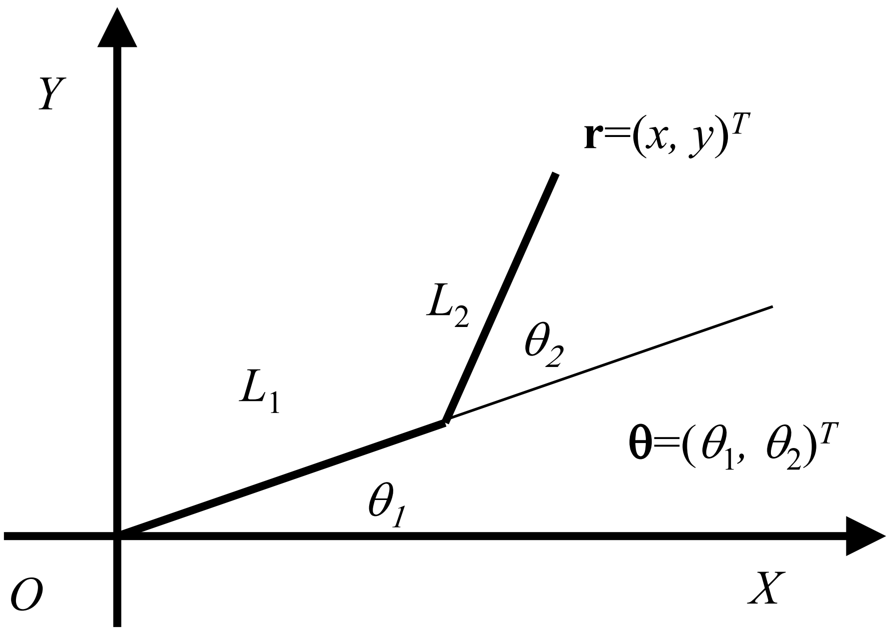
Singularity and Singular Points



$$|J| = L_1 L_2 \sin\theta_2$$

- If |J|=0, which means θ_2 = 0 or π in this case, the end effector can move only in limited direction
- Less motion ability in |J|=0
- These are called singular points

Differential Kinematics



$$\mathbf{r} = \mathbf{f}(\boldsymbol{\theta}), \mathbf{r} = \begin{pmatrix} x \\ y \end{pmatrix}, \boldsymbol{\theta} = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$$

Taking time derivative,

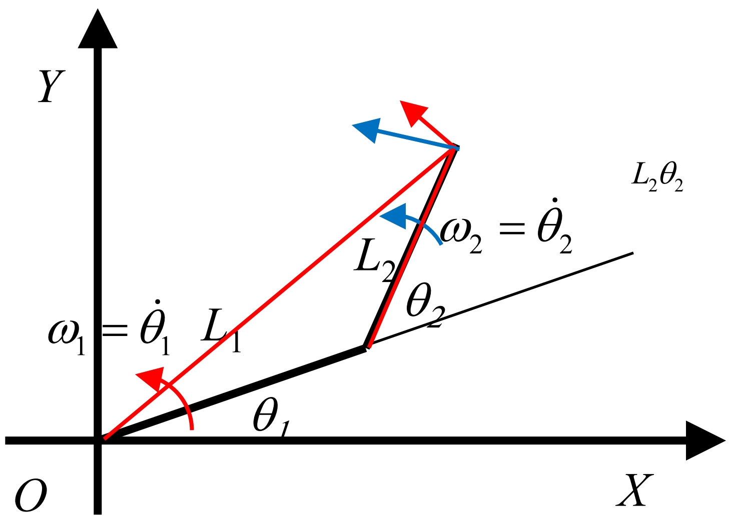
$$\dot{\mathbf{r}} = J(\theta)\dot{\boldsymbol{\theta}}, J(\theta) = \frac{\partial \mathbf{f}}{\partial \boldsymbol{\theta}}$$

If J is regular, $\dot{\boldsymbol{\theta}} = J(\theta)^{-1}\dot{\mathbf{r}}$

- We can associate angle speed with translational speed
- We can solve inverse kinematics problem by numerical method

$$\boldsymbol{\theta}(t+1) = \boldsymbol{\theta}(t) + \dot{\boldsymbol{\theta}}(t)\Delta t$$

What Jacobian Means

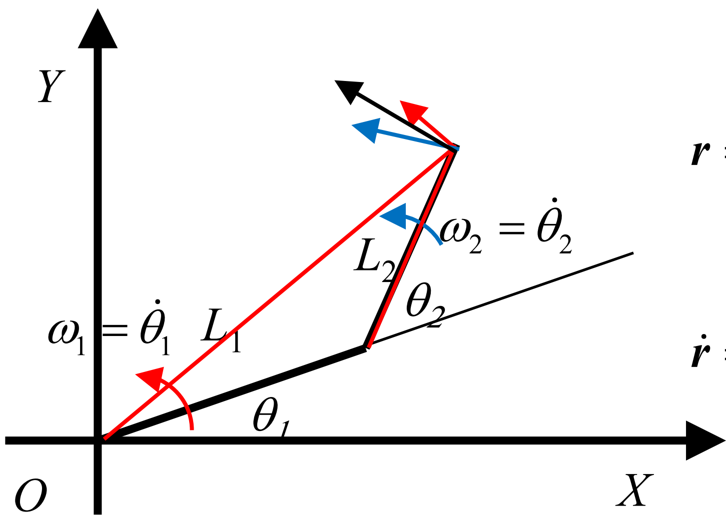


- When each of the joints rotates (or has an angular velocity), a hand gets a translational velocity
- An overall hands velocity is a sum of them

Differential Jacobian
（微分的ヤコビアン）
Find it by taking a time derivative of hand position

Geometrical Jacobian
（幾何学的ヤコビアン）
Find it by geometrical relations

Differential Jacobian
#（微分的ヤコビアン）



$$\boldsymbol{r} = fk(\boldsymbol{\theta}) = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} L_1 \cos(\theta_1) + L_2 \cos(\theta_1 + \theta_2) \\ L_1 \sin(\theta_1) + L_2 \sin(\theta_1 + \theta_2) \end{pmatrix}$$

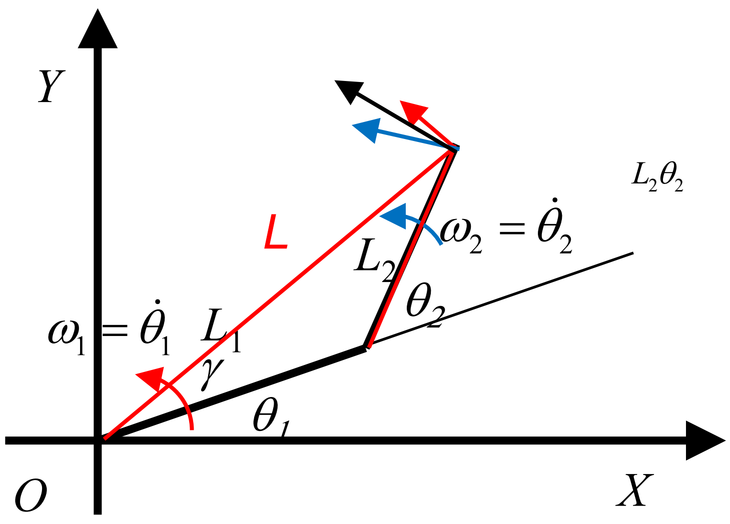
$$\dot{\boldsymbol{r}} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \dfrac{dx}{dt} \\ \dfrac{dy}{dt} \end{pmatrix} = \begin{pmatrix} \dfrac{dx}{d\theta_1}\dfrac{d\theta_1}{dt} + \dfrac{dx}{d\theta_2}\dfrac{d\theta_2}{dt} \\ \dfrac{dy}{d\theta_1}\dfrac{d\theta_1}{dt} + \dfrac{dy}{d\theta_2}\dfrac{d\theta_2}{dt} \end{pmatrix}$$

$$= \begin{pmatrix} -L_1\dot{\theta}_1 \sin(\theta_1) - L_2\left(\dot{\theta}_1 + \dot{\theta}_2\right)\sin(\theta_1 + \theta_2) \\ L_1\dot{\theta}_1 \cos(\theta_1) + L_2\left(\dot{\theta}_1 + \dot{\theta}_2\right)\cos(\theta_1 + \theta_2) \end{pmatrix}$$

$$= \begin{pmatrix} -L_1 \sin(\theta_1) - L_2 \sin(\theta_1 + \theta_2) & -L_2 \sin(\theta_1 + \theta_2) \\ L_1 \cos(\theta_1) + L_2 \cos(\theta_1 + \theta_2) & L_2 \cos(\theta_1 + \theta_2) \end{pmatrix} \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix}$$

$$\dot{\boldsymbol{r}} = J\dot{\boldsymbol{\theta}}$$

Geometrical Jacobian
（幾何学的ヤコビアン）

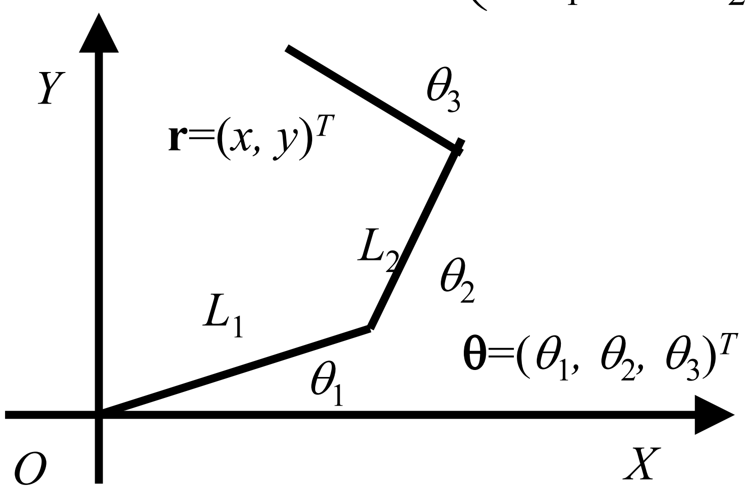


$$
\begin{aligned}
L^2 &= L_1^2 + L_2^2 + 2L_1L_2\cos(\pi - \theta_2) \\
&= L_1^2 + L_2^2 - 2L_1L_2\cos\theta_2 \\
\dot{\boldsymbol{r}}_1 &= \begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \end{pmatrix} = L\dot{\theta}_1 \begin{pmatrix} -\sin(\gamma) \\ \cos(\gamma) \end{pmatrix} \\
\dot{\boldsymbol{r}}_2 &= \begin{pmatrix} \dot{x}_2 \\ \dot{y}_2 \end{pmatrix} = L_2\dot{\theta}_2 \begin{pmatrix} -\sin(\theta_1 + \theta_2) \\ \cos(\theta_1 + \theta_2) \end{pmatrix} \\
\dot{\boldsymbol{r}} &= \dot{\boldsymbol{r}}_1 + \dot{\boldsymbol{r}}_2
\end{aligned}
$$

How to Solve J Inverse

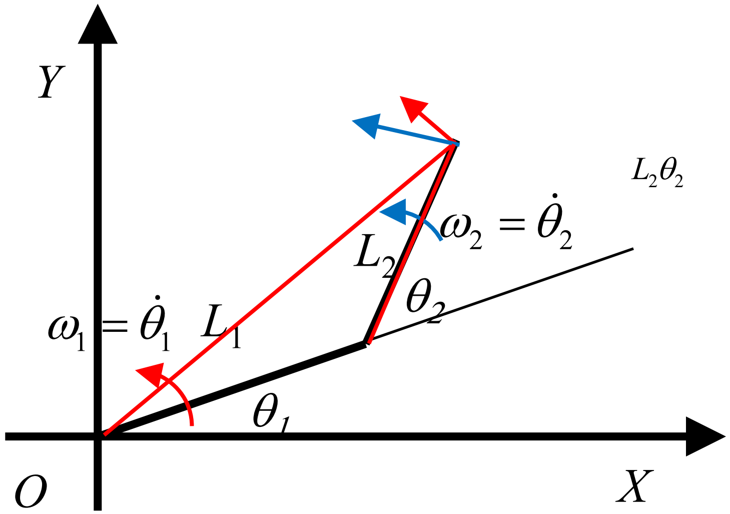
$$r = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}, \boldsymbol{\theta} = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_n \end{pmatrix}, J = \begin{pmatrix} \frac{\partial x_1}{\partial \theta_1} & \frac{\partial x_1}{\partial \theta_2} & \cdots & \frac{\partial x_1}{\partial \theta_n} \\ \frac{\partial x_2}{\partial \theta_1} & \frac{\partial x_2}{\partial \theta_2} & \cdots & \frac{\partial x_2}{\partial \theta_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_m}{\partial \theta_1} & \frac{\partial x_m}{\partial \theta_2} & \cdots & \frac{\partial x_m}{\partial \theta_n} \end{pmatrix}$$

- J is not always a square matrix
- we cannot guarantee J-1 is available all the time



For example, redundant case
m=2 and n=3
J is 2x3 matrix

Two Ways to Find Jacobian



Numerical Jacobian
（数値的ヤコビアン）
Find it by forward kinematics function

$$\frac{\partial x}{\partial \theta_1} = \frac{fk_x(\theta_1 + \Delta\theta_1, \cdots) - fk_x(\theta_1, \cdots)}{(\theta_1 + \Delta\theta_1) - (\theta_1)} = \frac{fk_x(\theta_1 + \Delta\theta_1, \cdots) - fk_x(\theta_1, \cdots)}{\Delta\theta_1}$$

Symbolic Jacobian
（記号的ヤコビアン）
Find it by taking a time derivative of hand position, i.e. differential Jacobian

Pseudo-Inverse Matrix
疑似逆行列

- Suppose J be a m x n matrix
- We introduce $J^{\#}$, n x m matrix, which acts similarly as an inverse matrix
- J# is called pseudo-inverse matrix or Moore–Penrose pseudoinverse matrix

Definition of Pseudo-Inverse Matrix

$$JJ^{\#}J = J$$
$$J^{\#}JJ^{\#} = J^{\#}$$
$$\left(JJ^{\#}\right)^{T} = JJ^{\#}$$
$$\left(J^{\#}J\right)^{T} = J^{\#}J$$

Moore-Penrose inverse:
An instance of Pseudo-inverse matrix

$$J^{\#} = J^{-1} \quad \text{,if m = n = rank(J)}$$
$$J^{\#} = J^{T}\left(JJ^{T}\right)^{-1} \quad \text{, if n > m = rank(J)}$$
$$J^{\#} = \left(J^{T}J\right)^{-1}J^{T} \quad \text{, if m > n = rank(J)}$$

Pseudo-Inverse Matrix and Singular Value Decomposition
疑似逆行列と特異値分解

Suppose J is factorized by singular value decomposition

$$J = U\Sigma V^T, \Sigma = \left(\begin{array}{ccc|c} \sigma_{11} & 0 & 0 & 0 \\ 0 & \sigma_{22} & 0 & 0 \\ 0 & 0 & \sigma_{33} & 0 \end{array}\right), UU^T = U^T U = I_m, VV^T = V^T V = I_n$$

The pseudo-inverse of J is given by

$$J^\# = V\Sigma^\# U^T, \Sigma^\# = \left(\begin{array}{ccc} \frac{1}{\sigma_{11}} & 0 & 0 \\ 0 & \frac{1}{\sigma_{22}} & 0 \\ 0 & 0 & \frac{1}{\sigma_{33}} \\ \hline 0 & 0 & 0 \end{array}\right), \Sigma\Sigma^\# = I_m, \Sigma^\#\Sigma = \left(\begin{array}{c|c} I_m & 0 \\ \hline 0 & 0 \end{array}\right)$$

Numerical solution

Newton-Raphson method

Jacobian matrix

$$\mathbf{J} = \frac{\partial \mathbf{f}}{\partial \mathbf{q}^T} = \begin{bmatrix} \frac{\partial f_1}{\partial q_1} & \frac{\partial f_1}{\partial q_2} & \cdots & \frac{\partial f_1}{\partial q_n} \\ \frac{\partial f_2}{\partial q_1} & \frac{\partial f_2}{\partial q_2} & \cdots & \frac{\partial f_2}{\partial q_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial q_1} & \frac{\partial f_m}{\partial q_2} & \cdots & \frac{\partial f_m}{\partial q_n} \end{bmatrix}$$

If m=n

$$\mathbf{q}_{i+1} = \mathbf{q}_i + k\mathbf{J}^{-1}(\mathbf{r} - \mathbf{r}_i)$$

$$\mathbf{r}_i = \mathbf{f}(\mathbf{q}_i)$$

Calculate the above iteratedly until q_i converges (small change) under a constant k

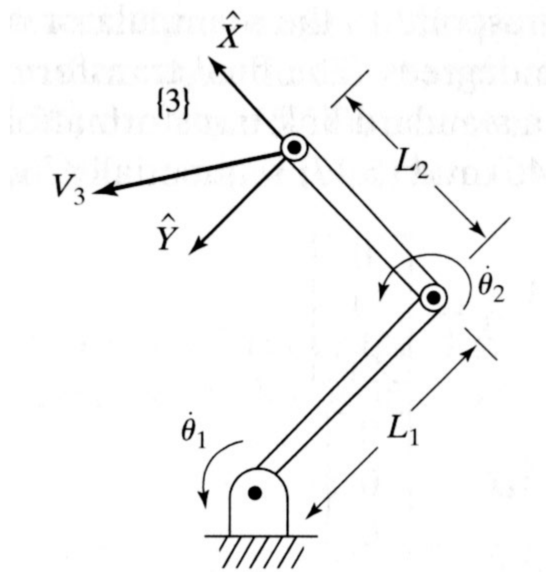
$$\mathbf{r}_n = \mathbf{f}(\mathbf{q}_n)$$

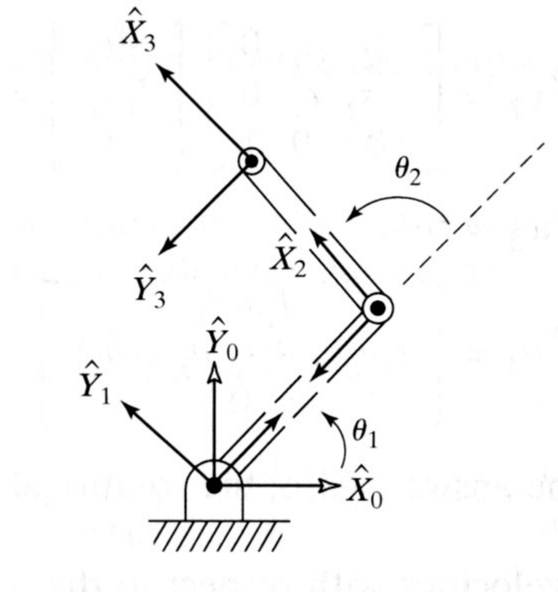
Singularity

$$v = J(\Theta)\dot{\Theta}$$

$$\dot{\Theta} = J^{-1}(\Theta)v$$

- Given joint velocity, we can find linear velocity
- Invertible?
  - Given linear velocity
  - Find joint velocity
- J^{-1} singular?
- Det [$J^{-1}(\Theta)$] = 0?

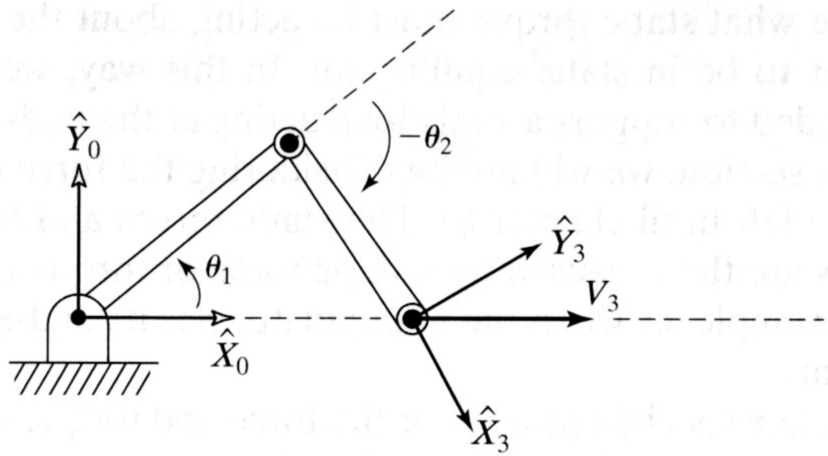




$$ {}^{3}J(\Theta) = \begin{bmatrix} L_1 s_2 & 0 \\ L_1 c_1 + L_2 & L_2 \end{bmatrix} $$

$$ \det\left({}^{3}J(\Theta)\right) = L_1 L_2 s_2 $$

Singular at θ_2 = 0 or 180 deg.
Arm is stretched out or folded back
Less orientation



Moving the tip along X axis at 1.0
Show joint rates

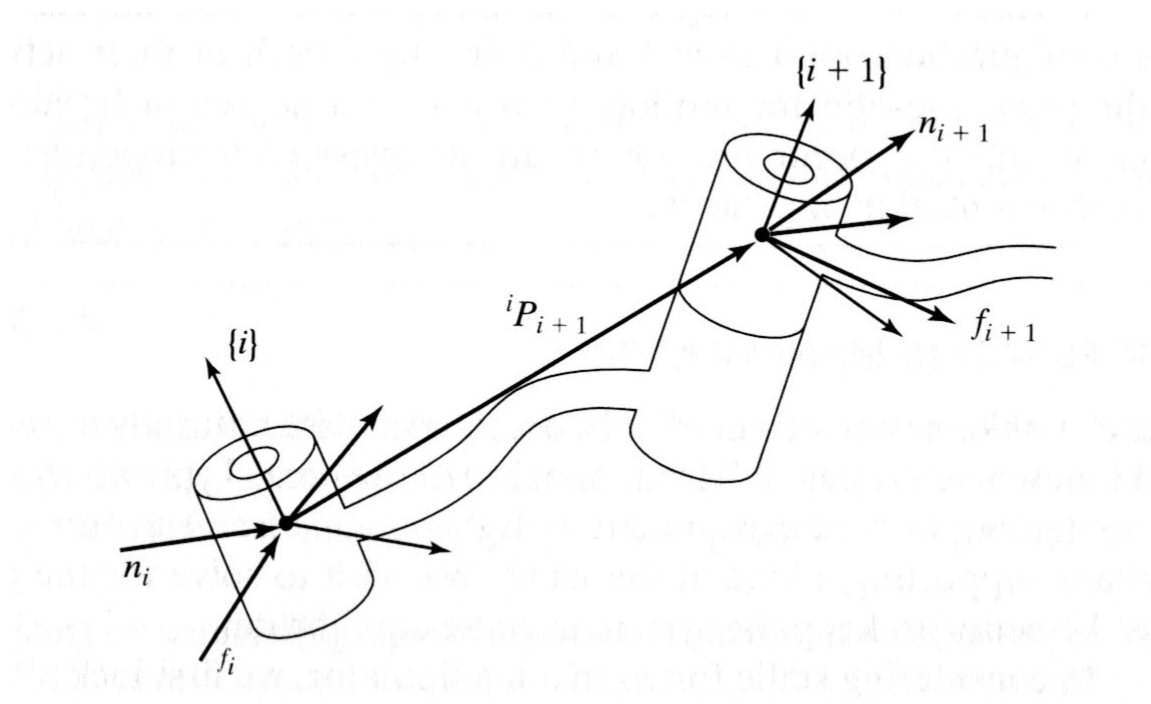
$$\dot{\theta}_1 = \frac{c_{12}}{L_1 s_2}, \dot{\theta}_2 = -\frac{c_1}{L_2 s_2} - \frac{c_{12}}{L_1 s_2}$$

If q2 = 0 or 180, both joint rates go to infinity

$$J(\Theta) = \begin{bmatrix} -L_1 s_1 - L_2 s_{12} & -L_2 s_{12} \\ L_1 c_1 + L_2 c_{12} & L_2 c_{12} \end{bmatrix} \qquad J^{-1}(\Theta) = \frac{1}{L_1 L_2 s_2} \begin{bmatrix} L_2 c_{12} & L_2 s_{12} \\ -L_1 c_1 - L_2 c_{12} & -L_1 s_1 - L_2 s_{12} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

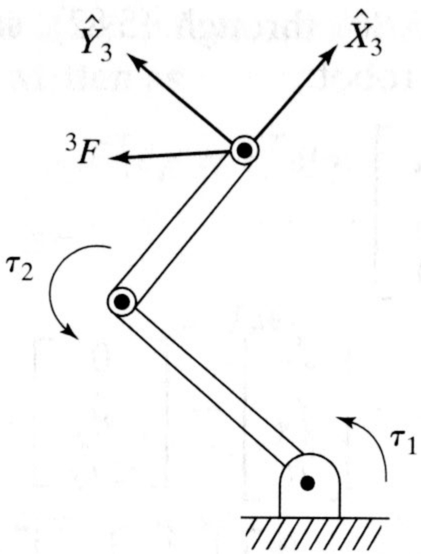
$$\begin{aligned} \det J(\Theta) &= \left| J(\Theta) \right| = \left(-L_1 s_1 - L_2 s_{12} \right) L_2 c_{12} + L_2 s_{12} \left(L_1 c_1 + L_2 c_{12} \right) \\ &= -L_1 L_2 s_1 c_{12} - L_2^2 s_{12} c_{12} + L_1 L_2 c_1 s_{12} + L_2^2 s_{12} c_{12} \\ &= L_1 L_2 \sin\left(\left(\theta_1 + \theta_2 \right) - \theta_1 \right) = L_1 L_2 \sin \theta_2 \end{aligned}$$

Static Force-Moment Balance



$$ {}^{i}f_{i} = {}^{i}f_{i+1} $$

$$ {}^{i}n_{i} = {}^{i}n_{i+1} + {}^{i}P_{i+1} \times {}^{i}f_{i+1} $$



$$ {}^{i}f_i = {}^{i}_{i+1}R \; {}^{i+1}f_{i+1} $$

$$ {}^{i}n_i = {}^{i}_{i+1}R \; {}^{i+1}n_{i+1} + {}^{i}P_{i+1} \times {}^{i}f_i $$

Revolute Joint

$$ \tau_i = {}^{i}n^{T}{}_{i} \; {}^{i}\hat{Z}_i $$

Translational Joint

$$ \tau_i = {}^{i}f^{T}{}_{i} \; {}^{i}\hat{Z}_i $$

Jacobian in force domain

$$ \tau = J^{T} F $$

Jacobian by DH Method

$$\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = {}^0_1T(\theta_1)\,{}^1_2T(\theta_2)\,{}^2_3T(\theta_3)\,{}^3_4T(0)\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$J = \begin{pmatrix} \dfrac{\partial \boldsymbol{r}}{\partial \theta_1} & \dfrac{\partial \boldsymbol{r}}{\partial \theta_2} & \dfrac{\partial \boldsymbol{r}}{\partial \theta_3} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial x}{\partial \theta_1} & \dfrac{\partial x}{\partial \theta_2} & \dfrac{\partial x}{\partial \theta_3} \\ \dfrac{\partial y}{\partial \theta_1} & \dfrac{\partial y}{\partial \theta_2} & \dfrac{\partial y}{\partial \theta_3} \\ \dfrac{\partial z}{\partial \theta_1} & \dfrac{\partial z}{\partial \theta_2} & \dfrac{\partial z}{\partial \theta_3} \end{pmatrix}$$

CC BY

Jacobian by DH Method

$$\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = {}^0_1T(\theta_1)\,{}^1_2T(\theta_2)\,{}^2_3T(\theta_3)\,{}^3_4T(0)\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\frac{\partial \boldsymbol{r}}{\partial \theta_1} = \frac{\partial}{\partial \theta_1}\,{}^0_1T(\theta_1)\,{}^1_2T(\theta_2)\,{}^2_3T(\theta_3)\,{}^3_4T(0)\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \left(\frac{\partial}{\partial \theta_1}\,{}^0_1T(\theta_1)\right){}^1_2T(\theta_2)\,{}^2_3T(\theta_3)\,{}^3_4T(0)\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Jacobian by DH Method

$$\frac{\partial \boldsymbol{r}}{\partial \theta_2} = {}^0_1T(\theta_1)\left(\frac{\partial}{\partial \theta_2}\,{}^1_2T(\theta_2)\right){}^2_3T(\theta_3)\,{}^3_4T(0)\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

$$\frac{\partial \boldsymbol{r}}{\partial \theta_3} = {}^0_1T(\theta_1)\,{}^1_2T(\theta_2)\left(\frac{\partial}{\partial \theta_2}\,{}^2_3T(\theta_3)\right){}^3_4T(0)\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

We only need to know a partial derivative of T by θ_i

Jacobian of Homogeneous Transformation Matrix

$$
{}^{i-1}_{i}T = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1} d_i \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1} d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
\frac{\partial}{\partial \theta_i} {}^{i-1}_{i}T = \begin{bmatrix} -s\theta_i & -c\theta_i & 0 & 0 \\ c\theta_i c\alpha_{i-1} & -s\theta_i c\alpha_{i-1} & 0 & 0 \\ c\theta_i s\alpha_{i-1} & -s\theta_i s\alpha_{i-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
$$