3. Random Variables and Probability Distributions - Distribution functions

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Part 1: Distribution Functions of Common PDFs

Uniform Distribution

For a uniform random variable X on the interval $[a, b]$, the probability density function (pdf) is given by:

$$f_X(x) = \frac{1}{b-a}, \quad a \leq x \leq b.$$

The cumulative distribution function (CDF) $F_X(x)$ is calculated as:

$$F_X(x) = P(X \leq x) = \int_a^x f_X(t)\, dt.$$

Evaluating this integral, we find:

1. For $x < a$, $F_X(x) = 0$.
2. For $a \leq x \leq b$,

$$F_X(x) = \frac{x-a}{b-a}.$$

3. For $x > b$, $F_X(x) = 1$.

Thus, the CDF of a uniform distribution on $[a, b]$ is:

$$F_X(x) = \begin{cases} 0, & x < a, \\ \frac{x-a}{b-a}, & a \leq x \leq b, \\ 1, & x > b. \end{cases}$$

Exponential Distribution

For an exponential random variable X with rate parameter $\lambda > 0$, the pdf is:

$$f_X(x) = \lambda e^{-\lambda x}, \quad x \geq 0.$$

The CDF $F_X(x)$ is calculated as:

$$F_X(x) = \int_0^x \lambda e^{-\lambda t}\, dt.$$

Evaluating this integral,

$$F_X(x) = \left[-e^{-\lambda t}\right]_0^x = 1 - e^{-\lambda x}.$$

Thus, the CDF of an exponential distribution is:

$$F_X(x) = \begin{cases} 0, & x < 0, \\ 1 - e^{-\lambda x}, & x \geq 0. \end{cases}$$

Normal Distribution

For a normal random variable X with mean μ and variance σ^2, the pdf is:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad x \in \mathbb{R}.$$

The CDF $F_X(x)$ is given by:

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(t-\mu)^2}{2\sigma^2}}\,dt.$$

This integral does not have a closed-form expression, so it is typically evaluated using numerical methods or tables.

Part 2: Line through $(1, 0)$ in a Random Direction

A line is drawn through the point $(1, 0)$ in a random direction. Let $(0, Y)$ be the point at which this line intersects the Y-axis. We aim to show that Y has the standard Cauchy distribution with density

$$f_Y(y) = \frac{1}{\pi(1+y^2)}, \quad y \in \mathbb{R}.$$

1. Showing that Y has the Cauchy Distribution

1. Let θ be the angle that the line makes with the positive X-axis, chosen uniformly over $[0, \pi]$.
2. The slope of the line is $\tan\theta$.
3. The line equation through $(1, 0)$ is:

$$y = \tan\theta \cdot (x - 1).$$

4. Setting $x = 0$, we find the y-intercept:

$$Y = -\tan\theta.$$

5. Since θ is uniformly distributed, $\tan\theta$ follows a standard Cauchy distribution.

Thus, Y has the density

$$f_Y(y) = \frac{1}{\pi(1+y^2)}.$$

2. Proving that $1/Y$ has the Same Distribution as Y

1. The reciprocal property of the Cauchy distribution implies that if Y has a standard Cauchy distribution, then so does $1/Y$.
2. Geometrically, rotating the line by 90° around the origin swaps Y with $1/Y$, preserving the Cauchy distribution due to the symmetric nature of the Cauchy density.