3. Random Variables and Probability Distributions - Well-defined: expectation

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Let X be a random variable with density

$$f_X(x) = Cx^{-\alpha}, \quad x \in (0, \infty),$$

where $\alpha > 0$ and C is a normalizing constant to ensure the total probability integrates to 1 over $(0, \infty)$.

Existence of the Expected Value $\mathbb{E}(X)$

The expectation $\mathbb{E}(X)$ is given by

$$\mathbb{E}(X) = \int_0^\infty x \cdot f_X(x)\, dx = \int_0^\infty x \cdot Cx^{-\alpha}\, dx = C \int_0^\infty x^{1-\alpha}\, dx.$$

The integral $\int_0^\infty x^{1-\alpha}\, dx$ converges or diverges depending on the value of α. We analyze this integral over two regions: near $x = 0$ and as $x \to \infty$.

1. Behavior as $x \to 0$

For convergence near $x = 0$, we require that the exponent $1 - \alpha$ does not make the integral diverge. The integral

$$\int_0^1 x^{1-\alpha}\, dx$$

converges if $1 - \alpha > -1$, which simplifies to

$$\alpha < 2.$$

2. Behavior as $x \to \infty$

For convergence as $x \to \infty$, we consider

$$\int_1^\infty x^{1-\alpha}\, dx.$$

This integral converges if $1 - \alpha < -1$, which simplifies to

$$\alpha > 2.$$

Conclusion on $\mathbb{E}(X)$

To satisfy both conditions for convergence, the expected value $\mathbb{E}(X)$ is defined if and only if

$$\alpha > 2.$$

If $\alpha \leq 2$, the integral defining $\mathbb{E}(X)$ diverges, and thus the expected value does not exist.

In summary:

- **If** $\alpha > 2$: The expected value $\mathbb{E}(X)$ exists.
- **If** $\alpha \leq 2$: The expected value $\mathbb{E}(X)$ does not exist due to divergence in the integral.