2. Events and Probabilities - Bernoulli distribution

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Let us consider a Bernoulli distribution;

$$\mathbb{P}(X = 1) = p, \quad \mathbb{P}(X = 0) = 1 - p.$$

1. The expectation $\mathbb{E}(X)$

Ans:

For a Bernoulli random variable X, which can take values 0 or 1, the expectation is computed as:

$$\mathbb{E}(X) = \sum_x x \cdot \mathbb{P}(X = x)$$

Since $X = 1$ with probability p and $X = 0$ with probability $1 - p$, we have:

$$\mathbb{E}(X) = 1 \cdot \mathbb{P}(X = 1) + 0 \cdot \mathbb{P}(X = 0)$$

$$\mathbb{E}(X) = 1 \cdot p + 0 \cdot (1 - p) = p$$

Thus, the expectation of X is:

$$\mathbb{E}(X) = p$$

2. The expectation $\mathbb{E}((X - \mathbb{E}(X))^2)$

Ans:

The quantity $\mathbb{E}((X - \mathbb{E}(X))^2)$ is the **variance** of X. Let's compute it.

First, recall that $\mathbb{E}(X) = p$. The variance is defined as:

$$\mathrm{Var}(X) = \mathbb{E}((X - \mathbb{E}(X))^2)$$

We can expand this expression as:

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$$

For a Bernoulli distribution, since X can only take values 0 or 1, we have $X^2 = X$, so $\mathbb{E}(X^2) = \mathbb{E}(X)$. Therefore:

$$\mathrm{Var}(X) = \mathbb{E}(X) - (\mathbb{E}(X))^2$$

$$\mathrm{Var}(X) = p - p^2$$

We can also factor this expression as:

$$\text{Var}(X) = p(1-p)$$

Thus, the expectation $\mathbb{E}((X - \mathbb{E}(X))^2)$ is:

$$\mathbb{E}((X - \mathbb{E}(X))^2) = p(1-p)$$

3. Denoting $S_n = X_1 + X_2 + \cdots + X_n$ for Bernoulli IID $\{X_n\}$, then S_n becomes a binomial distribution. Prove it.

Ans:

To prove this, let's first define $S_n = X_1 + X_2 + \cdots + X_n$, where each X_i is an independent and identically distributed (IID) Bernoulli random variable with probability p of success ($X_i = 1$) and $1 - p$ of failure ($X_i = 0$).

The random variable S_n represents the sum of n Bernoulli trials, which counts the number of successes (the number of times $X_i = 1$) in n independent trials.

The probability mass function of a binomial distribution is given by:

$$\mathbb{P}(S_n = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Now, let's show that S_n follows this form:

- Each X_i is Bernoulli distributed, so the probability of success in a single trial is p, and the probability of failure is $1 - p$.
- Since the X_i's are independent, the probability of getting exactly k successes (i.e., $S_n = k$) is the probability of choosing k trials to be successes (which can happen in $\binom{n}{k}$ different ways), times the probability of k successes (p^k) and $n - k$ failures $(1-p)^{n-k}$.

Thus, the probability that $S_n = k$ is:

$$\mathbb{P}(S_n = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

This is exactly the probability mass function of a binomial distribution with parameters n and p.

Therefore, we have proven that S_n, the sum of n IID Bernoulli random variables, follows a binomial distribution:

$$S_n \sim \text{Binomial}(n, p)$$