# 3. Random Variables and Probability Distributions - Exponential function

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For a positive $\lambda > 0$, we are given the function

$$f(x) = e^{-\lambda x} 1_{[0,\infty)}(x), \quad x \in \mathbb{R},$$

where $1_{[0,\infty)}(x)$ is the indicator function on the interval $[0, \infty)$.

## 1. Plot of $f(x)$

The function $f(x)$ is defined as:

- $f(x) = e^{-\lambda x}$ for $x \geq 0$.
- $f(x) = 0$ for $x < 0$.

This represents an exponentially decaying function for $x \geq 0$, starting from $f(0) = 1$ and approaching $0$ as $x \to \infty$.

A plot of $f(x)$ would show an exponential decay curve starting from $1$ at $x = 0$ and gradually decreasing towards $0$ as $x \to \infty$.

## 2. Computation of $\int_0^\infty f(x)\,dx$

We aim to compute the integral

$$\int_0^\infty f(x)\,dx = \int_0^\infty e^{-\lambda x}\,dx.$$

To solve this, we calculate the integral:

$$\int_0^\infty e^{-\lambda x}\,dx = \left[\frac{-e^{-\lambda x}}{\lambda}\right]_0^\infty.$$

Evaluating this expression:

1. As $x \to \infty$, $e^{-\lambda x} \to 0$.
2. At $x = 0$, $e^{-\lambda \cdot 0} = 1$.

Thus, we have

$$\int_0^\infty e^{-\lambda x}\,dx = \frac{1}{\lambda}.$$

## 3. Computation of $\int_0^\infty e^{tx} f(x)\,dx$

Next, we compute the integral

$$\int_0^\infty e^{tx} f(x)\,dx = \int_0^\infty e^{tx} e^{-\lambda x}\,dx.$$

Simplifying the integrand:

$$e^{tx} e^{-\lambda x} = e^{(t-\lambda)x}.$$

Thus, the integral becomes

$$\int_0^\infty e^{(t-\lambda)x}\,dx.$$

We need to consider two cases:

1. If $t < \lambda$, the integral converges.
2. If $t \geq \lambda$, the integral diverges.

For $t < \lambda$, we compute the integral:

$$\int_0^\infty e^{(t-\lambda)x}\,dx = \left[\frac{e^{(t-\lambda)x}}{t-\lambda}\right]_0^\infty.$$

Evaluating this:

1. As $x \to \infty$, $e^{(t-\lambda)x} \to 0$ because $t - \lambda < 0$.
2. At $x = 0$, $e^{(t-\lambda)\cdot 0} = 1$.

Thus, we have

$$\int_0^\infty e^{tx} f(x)\,dx = \frac{1}{\lambda - t}, \quad \text{for } t < \lambda.$$

If $t \geq \lambda$, the integral does not converge.

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