3. Random Variables and Probability Distributions - Well-defined: variance

Lai Hui Shan
m5281022

Let Y be a random variable with the standard Cauchy density

$$f_Y(y) = \frac{1}{\pi(1+y^2)}, \quad y \in \mathbb{R}.$$

1. Does the Expectation $\mathbb{E}(Y)$ Exist?

To determine if $\mathbb{E}(Y)$ exists, we consider the integral

$$\mathbb{E}(Y) = \int_{-\infty}^{\infty} y \cdot f_Y(y)\, dy = \int_{-\infty}^{\infty} \frac{y}{\pi(1+y^2)}\, dy.$$

The integrand $\frac{y}{1+y^2}$ does not converge absolutely on $(-\infty, \infty)$ because the tails of the Cauchy distribution decay too slowly for this integral to have a finite value. Specifically, the Cauchy distribution is known to have "heavy tails," meaning that $\int_{-\infty}^{\infty} |y| f_Y(y)\, dy$ diverges.

Therefore, the expectation $\mathbb{E}(Y)$ does not exist.

2. Can We Define $\mathbb{V}(Y)$? If Not, Explain the Reason.

Variance is defined as

$$\mathbb{V}(Y) = \mathbb{E}((Y - \mathbb{E}(Y))^2),$$

but since the expectation $\mathbb{E}(Y)$ does not exist, we cannot define $\mathbb{V}(Y)$ in the usual sense.

Even if we consider the alternative approach of calculating $\mathbb{E}(Y^2)$, the integral

$$\mathbb{E}(Y^2) = \int_{-\infty}^{\infty} y^2 \cdot f_Y(y)\, dy = \int_{-\infty}^{\infty} \frac{y^2}{\pi(1+y^2)}\, dy$$

also diverges due to the heavy tails of the Cauchy distribution.

Therefore, the variance $\mathbb{V}(Y)$ is undefined for the standard Cauchy distribution because $\mathbb{E}(Y^2)$ does not exist.

3. The Median of Y

The median of a Cauchy distribution is the value m such that

$$\mathbb{P}(Y \leq m) = \frac{1}{2}.$$

For the standard Cauchy distribution, which is symmetric about $y = 0$, the median is 0.

Thus, the median of Y is 0.