4. Conditioning and Independent - Conditional Probability

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1. Proof: Conditional Probability is a Probability Measure

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and consider a fixed event $A \in \mathcal{F}$ with $\mathbb{P}(A) > 0$. Define the conditional probability $\mathbb{P}(B \mid A) = \frac{\mathbb{P}(B \cap A)}{\mathbb{P}(A)}$ for $B \in \mathcal{F}$.

We prove that $B \mapsto \mathbb{P}(B \mid A)$ satisfies the axioms of a probability measure:

Step 1: Non-negativity

For any $B \in \mathcal{F}$,

$$\mathbb{P}(B \mid A) = \frac{\mathbb{P}(B \cap A)}{\mathbb{P}(A)} \geq 0$$

since $\mathbb{P}(B \cap A) \geq 0$ and $\mathbb{P}(A) > 0$.

Step 2: Additivity for Disjoint Events

If $B_1, B_2 \in \mathcal{F}$ are disjoint ($B_1 \cap B_2 = \emptyset$),

$$\mathbb{P}((B_1 \cup B_2) \mid A) = \frac{\mathbb{P}((B_1 \cup B_2) \cap A)}{\mathbb{P}(A)}.$$

Using the fact that $B_1 \cap B_2 = \emptyset$,

$$(B_1 \cup B_2) \cap A = (B_1 \cap A) \cup (B_2 \cap A),$$

and since $B_1 \cap A$ and $B_2 \cap A$ are disjoint, we have

$$\mathbb{P}((B_1 \cup B_2) \mid A) = \frac{\mathbb{P}(B_1 \cap A) + \mathbb{P}(B_2 \cap A)}{\mathbb{P}(A)} = \mathbb{P}(B_1 \mid A) + \mathbb{P}(B_2 \mid A).$$

Step 3: Total Probability

For the whole space Ω,

$$\mathbb{P}(\Omega \mid A) = \frac{\mathbb{P}(\Omega \cap A)}{\mathbb{P}(A)} = \frac{\mathbb{P}(A)}{\mathbb{P}(A)} = 1.$$

Thus, $\mathbb{P}(\cdot \mid A)$ is a probability measure on $(\Omega, \mathcal{F})$.

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2. Proof: $\mathbb{P}(B \cup C \mid A) = \mathbb{P}(B \mid A) + \mathbb{P}(C \mid A)$

Given B and C are disjoint ($B \cap C = \emptyset$) and $\mathbb{P}(A) > 0$, we compute:

$$\mathbb{P}(B \cup C \mid A) = \frac{\mathbb{P}((B \cup C) \cap A)}{\mathbb{P}(A)}.$$

Since $B \cap C = \emptyset$,

$$(B \cup C) \cap A = (B \cap A) \cup (C \cap A),$$

and because $(B \cap A)$ and $(C \cap A)$ are disjoint, we have

$$\mathbb{P}((B \cup C) \cap A) = \mathbb{P}(B \cap A) + \mathbb{P}(C \cap A).$$

Therefore,

$$\mathbb{P}(B \cup C \mid A) = \frac{\mathbb{P}(B \cap A) + \mathbb{P}(C \cap A)}{\mathbb{P}(A)} = \mathbb{P}(B \mid A) + \mathbb{P}(C \mid A).$$

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3. Example: Conditional Probabilities with Balls

Scenario: A bag contains red (r) and white (w) balls. Suppose X represents the outcome of interest (e.g., a specific ball being chosen), and Y_1, Y_2, Y_3 represent sequential draws with results r, r, w respectively. We compute:

$$\mathbb{P}(X = b \mid Y_1 = r, Y_2 = r, Y_3 = w).$$

Let us define:

- Total number of balls = $n_r + n_w$,
- Total ways to draw r, r, w.

Step 1: Define Sample Space

Each sequence corresponds to one outcome. Compute probabilities for r, r, w given prior probabilities and update.

Step 2: Compute Conditional Probabilities

By the definition of conditional probability:

$$\mathbb{P}(X = b \mid Y_1 = r, Y_2 = r, Y_3 = w) = \frac{\mathbb{P}(X = b \cap Y_1 = r, Y_2 = r, Y_3 = w)}{\mathbb{P}(Y_1 = r, Y_2 = r, Y_3 = w)}.$$

Explicit calculations depend on specific numbers of r and w.

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4. Proof: Independence of A^c and B^c

Given A and B are independent, $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$.

To prove A^c and B^c are independent, we compute:

$$\mathbb{P}(A^c \cap B^c) = \mathbb{P}((A \cup B)^c) = 1 - \mathbb{P}(A \cup B).$$

By inclusion-exclusion,

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$

Substituting independence,

$$\mathbb{P}(A^c \cap B^c) = 1 - \big(\mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A)\mathbb{P}(B)\big).$$

Simplify:

$$\mathbb{P}(A^c \cap B^c) = (1 - \mathbb{P}(A))(1 - \mathbb{P}(B)) = \mathbb{P}(A^c)\mathbb{P}(B^c).$$

Thus, A^c and B^c are independent.