# 3. Random Variables and Probability Distributions - Random generator

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Let $Z$ be a random variable with a uniform distribution on $[0,1]$, and let $X$ be a continuous-valued random variable with cumulative distribution function $F_X$.

# 1. Why does the Distribution Function $F_X$ Have an Inverse Function $F_X^{-1}$?

For a continuous random variable $X$ with cumulative distribution function $F_X$, the function $F_X(x)$ is monotonic and strictly increasing on its support (the range of possible values for $X$). This monotonicity implies that for each $p \in (0,1)$, there is a unique value $x$ such that $F_X(x) = p$. Therefore, $F_X$ has an inverse function $F_X^{-1}$, which satisfies

$$F_X(F_X^{-1}(p)) = p, \quad p \in (0,1).$$

The inverse $F_X^{-1}$ allows us to "map" probabilities back to the corresponding values of $X$.

# 2. Proving $\mathbb{P}(Z \leq z) = F_X(F_X^{-1}(z)), \quad z \in \mathbb{R}$

Since $Z$ is a uniform random variable on $[0,1]$, we have:

$$\mathbb{P}(Z \leq z) = z, \quad 0 \leq z \leq 1.$$

For $z \in (0,1)$, let $X = F_X^{-1}(Z)$. Then the probability that $Z$ is less than or equal to some value $z$ is the probability that $X \leq F_X^{-1}(z)$:

$$\mathbb{P}(Z \leq z) = \mathbb{P}(X \leq F_X^{-1}(z)).$$

Since $F_X$ is the cumulative distribution function of $X$, we know that

$$\mathbb{P}(X \leq F_X^{-1}(z)) = F_X(F_X^{-1}(z)).$$

Thus, we have shown that:

$$\mathbb{P}(Z \leq z) = F_X(F_X^{-1}(z)), \quad z \in (0,1).$$

# 3. Generating a Value of $X$ Using $U$

To generate a value of $X$ from a uniform random variable $U \sim U[0,1]$, we can use the inverse transform sampling method:

1. Generate a value $U$ from the uniform distribution on $[0,1]$.

2. Compute $X = F_X^{-1}(U)$.

Since $U$ is uniform on $[0, 1]$ and $F_X^{-1}(U)$ maps uniformly distributed values to the distribution of $X$, the random variable $X = F_X^{-1}(U)$ will follow the distribution defined by $F_X$.

This method allows us to transform uniformly distributed values into samples from the distribution of $X$.