Test of Significance for High-dimensional Thresholds with Application to Individualized Minimal Clinically Important Difference

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Abstract

This work is motivated by learning the individualized minimal clinically important difference, a vital concept to assess clinical importance in various biomedical studies. We formulate the scientific question into a high-dimensional statistical problem where the parameter of interest lies in an individualized linear threshold. The goal of this paper is to develop a hypothesis testing procedure for the significance of a single element in this high-dimensional parameter as well as for the significance of a linear combination of this parameter. The difficulty dues to the high-dimensionality of the nuisance component in developing such a testing procedure, and also stems from the fact that this high-dimensional threshold model is nonregular and the limiting distribution of the corresponding estimator is nonstandard. To deal with these challenges, we construct a test statistic via a new bias corrected smoothed decorrelated score approach, and establish its asymptotic distributions under both the null and local alternative hypotheses. In addition, we propose a double-smoothing approach to select the optimal bandwidth parameter in our test statistic and provide theoretical guarantees for the selected bandwidth. We conduct comprehensive simulation studies to demonstrate how our proposed procedure can be applied in empirical studies. Finally, we apply the proposed method to a clinical trial where the scientific goal is to assess the clinical importance of a surgery procedure.

Keyword: Bandwidth selection, High-dimensional statistical inference, Kernel method, Nonstandard asymptotics.

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1 Introduction

1.1 Motivation: Individualized Minimal Clinically Important Difference (iM-CID) under High-dimensionality

In clinical studies, the effect of a treatment or intervention is widely assessed through clinical significance, instead of statistical significance. By leveraging patient-reported outcomes (PRO) that are directly collected from the patients without a third party's interpretation, the aim of assessing clinical significance is to provide clinicians and policy makers the clinical effectiveness of the treatment or intervention. For example, in our motivating study, the ChAMP randomized controlled trial (Bisson et al., 2015), the interest is to identify the smallest WOMAC pain score change such that the corresponding improvement and beyond can be claimed as clinically significant. In Jaeschke et al. (1989), this concept was firstly and formally introduced as the minimal clinically important difference (MCID), "the smallest difference in score in the domain of interest which patients perceive as beneficial and which would mandate a change in the patient's management".

There are roughly three approaches to determine the magnitude of MCID (Lassere et al., 2001; Erdogan et al., 2016; Angst et al., 2017; Jayadevappa et al., 2017): distribution-based, opinionbased, and anchor-based. Although adopted in various studies (Wyrwich et al., 1999a,b; Samsa et al., 1999; Norman et al., 2003; Bellamy et al., 2001), the first two approaches are usually criticized (McGlothlin and Lewis, 2014) due to, for example, "distribution-based methods are not derived from individual patients" and "expert opinion may not be a valid and reliable way to determine what is important to patients". The third approach, anchor-based, conceptually determines the MCID by incorporating both certainty of effective treatment encoded as a continuous variable and the patient's satisfaction collected from the anchor question. It is clinically evident (Wells et al., 2001) that the magnitude of MCID would depend on various factors such as the demographic variables and the patients' baseline status. For example, in a shoulder pain reduction study (Heald et al., 1997), because of the higher expectation for complete recovery, the healthier patients with mild pain at baseline often deemed greater pain reduction as "meaningful" than the ones who suffered from chronic disease. Therefore, it is of scientific interest to generally estimate the individualized MCID (iMCID) based on each individual patient's clinical profile as well as to quantify the uncertainties of those estimates.

Nowadays, there is an increasing use and advancing development of EHR-based (electronic health records) studies in clinical research. The EHR data are complex, diverse and high-dimensional (Abdullah et al., 2020). The rich information contained in the EHR data could facilitate the determination and quantification of iMCID. Therefore, there is a pressing need to develop statistical methods that incorporate the high-dimensional data into both magnitude determination and uncertainty quantification of iMCID.

1.2 Problem Formulation

To facilitate the presentation, we first introduce some notation. Let $X \in \mathbb{R}$ be a continuous variable representing the score change collected from the PRO, e.g., the WOMAC pain score change from baseline to one year after surgery in the ChAMP trial. Let $Y = \pm 1$ be a binary variable derived from the patient's response to the anchor question, where Y = 1 represents an improved health condition and Y = -1 otherwise. We use a d-dimensional vector \mathbf{Z} to denote the patient's clinical profile including demographic variables, clinical biomarkers, disease histories, among many others. Suppose the data we observe are n i.i.d. samples $\{(x_i, y_i, \mathbf{z}_i)\}_{i=1}^n$ of (X, Y, Z). We focus on the high-dimensional setting, i.e., $d \gg n$.

Firstly, if there were no covariate Z, the MCID can be estimated by $\operatorname{argmax}_{\tau}\{\mathbb{P}(X \geq \tau \mid Y = 1) + \mathbb{P}(X < \tau \mid Y = -1)\}$, which is equivalent to

$$\underset{\tau}{\operatorname{argmin}} \ \mathbb{E}[w(Y)L_{01}\{Y(X-\tau)\}], \tag{1.1}$$

where $L_{01}(u) = \frac{1}{2}\{1 - \text{sign}(u)\}$ is the 0-1 loss, sign(u) = 1 if $u \ge 0$ and -1 otherwise, $w(1) = 1/\pi$, $w(-1) = 1/(1-\pi)$ and $\pi = \mathbb{P}(Y=1)$. When the high-dimensional covariate \mathbf{Z} is available, as the focus of this paper, the natural idea is to consider the iMCID with a functional form of \mathbf{Z} , say $\tau(\mathbf{Z})$. In clinical practice, a simple structure, such as linear, is preferred due to its transparency and convenience for interpretation, especially for high-dimensional data. Therefore, we focus on the linear structure $\tau(\mathbf{Z}) = \boldsymbol{\beta}^T \mathbf{Z}$ in this paper. The objective becomes

$$\boldsymbol{\beta}^* = \underset{\boldsymbol{\beta}}{\operatorname{argmin}} R(\boldsymbol{\beta}), \text{ where } R(\boldsymbol{\beta}) = \mathbb{E}[w(Y)L_{01}\{Y(X - \boldsymbol{\beta}^T \boldsymbol{Z})\}].$$
 (1.2)

Throughout this paper we assume that $\boldsymbol{\beta}^*$ exists and is unique. Denote $\boldsymbol{\beta}^* = (\theta^*, \boldsymbol{\gamma}^{*T})^T$, where θ^* is an arbitrary one-dimensional component of $\boldsymbol{\beta}^*$ and $\boldsymbol{\gamma}^*$ represents the rest of the parameter which is high-dimensional. In this paper, we start from considering the hypothesis testing procedure for the parameter θ^* . With a simple reparametrization, the same procedure can be applied to infer the iMCID $\boldsymbol{c}_0^T \boldsymbol{\beta}^*$ for some fixed and known vector $\boldsymbol{c}_0 \in \mathbb{R}^d$.

It is also worthwhile to mention that, although the motivation of this paper is to study iMCID, our formulation of this problem can be similarly applied to other scenarios as well, such as the covariate-adjusted Youden index (Xu et al., 2014), one-bit compressed sensing (Boufounos and Baraniuk, 2008), linear binary response model (Manski, 1975, 1985), and personalized medicine (Wang et al., 2018). Interested readers could refer to Feng et al. (2019) for those examples.

1.3 From Estimation to Inference

Incorporating high-dimensional data in the objective, i.e., moving forward from (1.1) to (1.2), is not trivial, even for the purpose of estimation only. Recently, Mukherjee et al. (2019) established the rate of convergence of the (penalized) maximum score estimator for (1.2) in growing dimension,

where the dimension d of Z is allowed to grow with n. In a related work, Feng et al. (2019) proposed a regularized empirical risk minimization framework with a smoothed surrogate loss for the estimation of the high-dimensional parameter β^* . They showed that the estimation problem is nonregular, in the sense that there do not exist estimators of β^* with root-n convergence rate uniformly over a proper parameter space.

Under (1.2), developing a valid statistical inference procedure is challenging, even for fixed dimensional setting. Manski (1975, 1985) considered the binary response model $Y = \text{sign}(X - \mathbf{Z}^T \boldsymbol{\beta} + \epsilon)$, where ϵ may depend on (X, \mathbf{Z}) but with $\text{Median}(\epsilon | X, \mathbf{Z}) = 0$. It can be shown that the true coefficient $\boldsymbol{\beta}^*$ can be equivalently defined via (1.2) with w(-1) = w(1) = 1/2. The maximum score estimator is proposed to estimate $\boldsymbol{\beta}$, and is later shown to have a non-Gaussian limiting distribution (Kim et al., 1990). To tackle the challenge of nonstandard limiting distribution of the maximum score estimator, Horowitz (1992) proposed the smoothed maximum score estimator which is asymptotically normal in fixed dimension.

On top of the nonregularity of the problem (1.2), the high-dimensionality of the parameter adds an additional layer of complexity for inference. The reason is that the estimator that minimizes the penalized loss function does not have a tractable standard limiting distribution under high dimensionality, due to the bias induced by the penalty term. For regular models (e.g., generalized linear models), there is a growing literature on correcting the bias from the penalty for valid inference, such as Javanmard and Montanari (2014); Zhang and Zhang (2014); Van de Geer et al. (2014); Belloni et al. (2015); Ning et al. (2017); Cai and Guo (2017); Fang et al. (2017); Neykov et al. (2018); Feng and Ning (2019); Fang et al. (2020); Yi and Neykov (2021), among others. Their main idea is to firstly construct a consistent estimator of the high dimensional parameter via proper regularization, and then remove the bias (via debiasing or decorrelation) in order to develop valid inferential statistics. While these methods enjoy great success under regular models, it remains unclear whether they can be applied to conduct valid inference in nonregular models such as the problem we consider in this paper. To the best of our knowledge, our work is the first one that provides valid inferential tools for nonregular models in high dimension.

1.4 Our Contributions

In this paper, we propose a unified hypothesis testing framework for the one-dimensional parameter θ^* as well as for the iMCID encoded as a linear combination of β^* . We start from considering the hypothesis testing problem $H_0: \theta^* = 0$ versus $H_1: \theta^* \neq 0$, where we treat γ^* as a high-dimensional nuisance parameter. Built on the smoothed surrogate estimation framework (Feng et al., 2019), we propose a bias corrected smoothed decorrelated score to form the score test statistic.

There are several new ingredients in the construction of our score statistic. First, the score function is derived based on a smoothed surrogate loss to overcome the nonregularity due to the nonsmoothness of the 0-1 loss. Second, unlike the existing works on high-dimensional inference,

the score function from the smoothed surrogate loss is biased. By explicitly estimating the bias term, we derive the bias corrected score. Third, the decorrelation step, developed by Ning et al. (2017) for regular models, is applied to reduce the uncertainty of estimating high-dimensional nuisance parameters. Theoretically, we show that under some conditions, the proposed score test statistic converges in distribution to a standard Gaussian distribution under the null hypothesis. We further establish the local asymptotic power of the test statistic when θ^* deviates from 0 in a local neighborhood. In particular, we give the conditions under which the test statistic has asymptotic power one.

When constructing the bias corrected smoothed decorrelated score, we need to specify a bandwidth parameter, whose optimal choice depends on the unknown smoothness of the data distribution. We further propose a double-smoothing approach to select the optimal bandwidth parameter by minimizing the mean squared error (MSE) of the score function. To our knowledge, such bandwidth selection procedures have not been studied for high-dimensional models. We show that under some extra smoothness assumptions, the ratio of the data-driven bandwidth to the theoretically optimal bandwidth converges to one in probability. From a practical perspective, with the selected bandwidth, the proposed score statistic is fully data-driven and can be easily applied for empirical research.

1.5 Paper Structure and Notation

The organization of this paper is as follows. In Section 2, we first provide some background on the estimation of iMCID then introduce the bias corrected smoothed decorrelated score and the associated test statistic. In Section 3, we discuss the theoretical properties of the score test. The data-driven bandwidth selection is addressed in Section 4. The corresponding results for $c_0^T \beta^*$, the linear combination of parameters, are briefly summarized in Section 5. Sections 6 and 7 contain simulation studies and a real data example, respectively. All the technical details and proofs are contained in the Appendix.

Throughout the paper, we adopt the following notation. For any set S, we write |S| for its cardinality. For any vector $\mathbf{v} \in \mathbb{R}^d$, we use \mathbf{v}_S to denote the subvector of \mathbf{v} with entries indexed by the set S, and define its ℓ_q norm as $\|\mathbf{v}\|_q = (\sum_{j=1}^d |\mathbf{v}_j|^q)^{1/q}$ for some real number $q \geq 0$. For any matrix $\mathbf{M} \in \mathbb{R}^{d_1 \times d_2}$, we denote $||\mathbf{M}||_{\max} = \max_{i,j} |M_{ij}|$. For any two sequences a_n and b_n , we write $a_n \lesssim b_n$ if there exists some positive constant C such that $a_n \leq Cb_n$ for any n. We let $a_n \times b_n$ stand for $a_n \lesssim b_n$ and $b_n \lesssim a_n$. Denote $a \vee b = \max(a, b)$ and $a \wedge b = \min(a, b)$. For some function $F(\theta, \gamma)$, we use $\nabla_{\theta} F(\theta, \gamma)$ and $\nabla_{\gamma} F(\theta, \gamma)$ to denote the partial derivatives with respect to the corresponding parameters, and similarly $\nabla_{\theta, \theta}^2 F(\theta, \gamma)$ to denote the second order derivative.

2 Methodology

2.1 Review of Penalized Smoothed Surrogate Estimation

Under the condition that $d \gg n$, the estimation of β^* via the empirical risk minimization following (1.2) induces challenges from both statistical and computational perspectives. The non-smoothness of $L_{01}(u)$ would cause the estimator to have a nonstandard convergence rate, which happens even in the fixed low dimensional case (Kim et al., 1990). Moreover, minimizing the empirical risk function based on the 0-1 loss is computationally NP-hard and is often very difficult to implement. To tackle these challenges, Feng et al. (2019) considered the following smoothed surrogate risk

$$R_{\delta}(\boldsymbol{\beta}) = \mathbb{E}\left[w(Y)L_{\delta,K}\left\{Y(X-\boldsymbol{\beta}^{T}\boldsymbol{Z})\right\}\right],$$
(2.1)

where $L_{\delta,K}(u) = \int_{u/\delta}^{\infty} K(t)dt$ is a smoothed approximation of $L_{01}(u)$, K is a kernel function defined in Section 3 and $\delta > 0$ is a bandwidth parameter. As the bandwidth δ shrinks to 0, $L_{\delta,K}(u)$ converges pointwisely to $L_{01}(u)$ (for any $u \neq 0$), from which it can be shown that β^* also minimizes the smoothed risk $R_{\delta}(\beta)$ up to a small approximation error. They further proposed the following penalized smoothed surrogate estimator

$$\widehat{\boldsymbol{\beta}} := \underset{\boldsymbol{\beta}}{\operatorname{argmin}} R_{\delta}^{n}(\boldsymbol{\beta}) + P_{\lambda}(\boldsymbol{\beta}), \tag{2.2}$$

where $P_{\lambda}(\beta)$ is some sparsity inducing penalty (e.g., Lasso) with a tuning parameter λ , and $R_{\delta}^{n}(\beta)$ is the corresponding empirical risk

$$R_{\delta}^{n}(\boldsymbol{\beta}) = \frac{1}{n} \sum_{i=1}^{n} \bar{R}_{\delta}^{i}(\boldsymbol{\beta}) = \frac{1}{n} \sum_{i=1}^{n} w(y_{i}) L_{\delta,K} \Big(y_{i} (x_{i} - \boldsymbol{\beta}^{T} \boldsymbol{z}_{i}) \Big).$$
 (2.3)

Computationally, the empirical surrogate risk $R^n_{\delta}(\beta)$ is a smooth function of β , which renders the optimization more tractable. Statistically, under some conditions, the estimator $\widehat{\beta}$ is shown to be rate-optimal, i.e., the convergence rate of $\widehat{\beta}$ matches the minimax lower bound up to a logarithmic factor. We refer to Feng et al. (2019) for the detailed results.

2.2 Bias Corrected Smoothed Decorrelated Score

While Feng et al. (2019) showed that the penalized smoothed surrogate estimator $\hat{\beta}$ is consistent, it does not automatically equip with a practical inferential procedure for β^* , or even for the parameter θ^* , mainly because of the sparsity inducing penalty. In practice, how to draw valid statistical inference is often the ultimate goal. In our motivating example, it is of critical importance to quantify the uncertainty of $c_0^T \beta^*$ where c_0 represents the realized value of a new patient's clinical profile. In other words, we would like to develop a testing procedure for

$$H_{0L}: \mathbf{c}_0^T \boldsymbol{\beta}^* = 0 \text{ versus } H_{1L}: \mathbf{c}_0^T \boldsymbol{\beta}^* \neq 0.$$
 (2.4)

In this section, we focus on a special case of (2.4), the hypothesis test for θ^* ,

$$H_0: \theta^* = 0 \text{ versus } H_1: \theta^* \neq 0, \tag{2.5}$$

where we treat γ as the nuisance parameter. Below we mainly present the results for (2.5). Once they are clear, we can extend the results to (2.4), which will be presented in Section 5.

In this section, we propose a new bias corrected smoothed decorrelated score to quantify the uncertainty for testing (2.5). It is well known that the classical score test is constructed based on the magnitude of the gradient of the loglikelihood, or more generally, the empirical risk function associated with $R(\beta)$ in (1.2). However, this construction breaks down in our problem due to the following two reasons.

First, to construct the score statistic, one needs to plug in some estimate of the nuisance parameter γ such as $\widehat{\gamma}$ obtained by partitioning $\widehat{\beta} = (\widehat{\theta}, \widehat{\gamma}^T)^T$ in (2.2). However, since γ is a high-dimensional parameter, the estimation error from $\widehat{\gamma}$ may become the leading term in the asymptotic analysis of the score function. To deal with the high-dimensional nuisance parameter, we use the decorrelated score, where the key idea is to project the score of the parameter of interest to a high-dimensional nuisance space (Ning et al., 2017). On the population level, it takes the form

$$\nabla_{\theta} R(\theta, \gamma) - \omega^{*T} \nabla_{\gamma} R(\theta, \gamma), \tag{2.6}$$

where the decorrelation vector is $\boldsymbol{\omega}^* = \left(\nabla_{\boldsymbol{\gamma},\boldsymbol{\gamma}}^2 R(\boldsymbol{\beta}^*)\right)^{-1} \nabla_{\boldsymbol{\gamma},\theta}^2 R(\boldsymbol{\beta}^*)$. The extra term $\boldsymbol{\omega}^{*T} \nabla_{\boldsymbol{\gamma}} R(\theta,\boldsymbol{\gamma})$ is the key to reduce the error from estimating $\boldsymbol{\gamma}$.

Second, even if the above decorrelated score approach can successfully remove the effect of the high-dimensional nuisance parameter, one cannot construct the sample based decorrelated score from (2.6), as the sample version of $R(\theta, \gamma)$ is non-differentiable, leading to the so called non-standard inference. To circumvent this issue, we approximate $R(\theta, \gamma)$ in (2.6) by the smoothed surrogate risk $R_{\delta}(\theta, \gamma)$ in (2.1), that is

$$\nabla_{\theta} R(\theta, \gamma) - \boldsymbol{\omega}^{*T} \nabla_{\gamma} R(\theta, \gamma) = \left\{ \nabla_{\theta} R_{\delta}(\theta, \gamma) - \boldsymbol{\omega}^{*T} \nabla_{\gamma} R_{\delta}(\theta, \gamma) \right\} - \text{approximation bias.}$$
 (2.7)

Since the empirical version of $R_{\delta}(\theta, \gamma)$ is smooth, we define the (empirical) smoothed decorrelated score function as

$$S_{\delta}(\theta, \gamma) = \nabla_{\theta} R_{\delta}^{n}(\theta, \gamma) - \omega^{*T} \nabla_{\gamma} R_{\delta}^{n}(\theta, \gamma).$$

With γ estimated by $\hat{\gamma}$, the estimated score function is then naturally defined as

$$\widehat{S}_{\delta}(\theta,\widehat{\gamma}) = \nabla_{\theta} R_{\delta}^{n}(\theta,\widehat{\gamma}) - \widehat{\omega}^{T} \nabla_{\gamma} R_{\delta}^{n}(\theta,\widehat{\gamma}), \tag{2.8}$$

where $\hat{\omega}$ is an estimator of ω^* defined more precisely in Section 2.3.

In view of (2.7) and (2.8), the sample version of $\nabla_{\theta} R_{\delta}(\theta, \gamma) - \omega^{*T} \nabla_{\gamma} R_{\delta}(\theta, \gamma)$ is given by $\widehat{S}_{\delta}(\theta, \widehat{\gamma})$ and therefore, to construct a valid score function, it remains to estimate the approximation bias in (2.7). To proceed, we first analyze this term, which is simply $v^{*T} \nabla R_{\delta}(\beta^{*})$ at $\beta = \beta^{*}$, where

 $v^* = (1, -\omega^{*T})^T$. After some analysis, we can show that the magnitude of the approximation bias depends on the smoothness of f(x|y, z), the conditional density of X given Y and Z. To obtain an explicit form of the approximation bias, we assume that f(x|y, z) is ℓ th order differentiable for some $\ell \geq 2$, which is defined more precisely in Section 3. Under this assumption, we can show that as the bandwidth parameter $\delta \to 0$,

$$\mathbf{v}^{*T} \nabla R_{\delta}(\boldsymbol{\beta}^*) = \delta^{\ell} \mu^* (1 + o(1)), \tag{2.9}$$

where

$$\mu^* := \boldsymbol{v}^{*T} \boldsymbol{b}^* = \boldsymbol{v}^{*T} \left(\int K(u) \frac{u^{\ell}}{\ell!} du \right) \sum_{y \in \{-1,1\}} w(y) \int y \boldsymbol{z} f^{(\ell)}(\boldsymbol{\beta}^{*T} \boldsymbol{z} | y, \boldsymbol{z}) f(y, \boldsymbol{z}) d\boldsymbol{z},$$

$$= \underbrace{\left(\int K(u) \frac{u^{\ell}}{\ell!} du \right)}_{\gamma_{K,\ell}} \boldsymbol{v}^{*T} \underbrace{\mathbb{E} \left[w(Y) Y \boldsymbol{Z} f^{(\ell)}(\boldsymbol{\beta}^{*T} \boldsymbol{Z} | Y, \boldsymbol{Z}) \right]}_{T^{(\ell)}(\boldsymbol{\beta}^{*})}, \tag{2.10}$$

and $f^{(\ell)}(x|y,z)$ denotes the ℓ th order derivative of f(x|y,z) with respect to x.

To estimate the approximation bias $\mathbf{v}^{*T}\nabla R_{\delta}(\boldsymbol{\beta}^{*})$, it suffices to estimate μ^{*} . From (2.10), we can see that, once $f^{(\ell)}(x|y,\mathbf{z})$ at $x = \boldsymbol{\beta}^{*T}\mathbf{z}$ is estimated, we can construct a plug-in estimator for μ^{*} . To be specific, assume that a pilot kernel estimator with some kernel function U and bandwidth h is available to estimate $f^{(\ell)}(\boldsymbol{\beta}^{*T}\mathbf{z}|y,\mathbf{z})$. Then we can estimate μ^{*} by

$$\widehat{\mu} = \gamma_{K,\ell} \widehat{\boldsymbol{v}}^T \widehat{T}_{h,U}^{(\ell),n}(\widehat{\boldsymbol{\beta}}), \tag{2.11}$$

where
$$\widehat{T}_{h,U}^{(\ell),n}(\widehat{\boldsymbol{\beta}}) := \frac{1}{n} \sum_{i=1}^n w(y_i) y_i \frac{\boldsymbol{z}_i}{h^{1+\ell}} U^{(\ell)}(\widehat{\boldsymbol{\beta}}^T \boldsymbol{z}_i - x_i)$$
 and $\widehat{\boldsymbol{v}} = (1, -\widehat{\boldsymbol{\omega}}^T)^T$.

The last step to construct a valid score test is to find the asymptotic variance of the smoothed decorrelated score $S_{\delta}(\boldsymbol{\beta}^*)$. Lemma 1 in the next section shows that the asymptotic variance of the standardized decorrelated score $(n\delta)^{1/2}S_{\delta}(\boldsymbol{\beta}^*)$ is $\sigma^{*2} = \boldsymbol{v}^{*T}\boldsymbol{\Sigma}^*\boldsymbol{v}^*$, where

$$\Sigma^* := \sum_{y \in \{-1,1\}} w(y)^2 \int z z^T \int K(u)^2 du f(\beta^{*T} z | y, z) f(y, z) dz,$$

$$= \underbrace{\left(\int K(u)^2 du\right)}_{\widetilde{\mu}_K} \underbrace{\mathbb{E}\left[w(Y)^2 Z Z^T f(\beta^{*T} Z | Y, Z)\right]}_{H(\beta^*)}, \qquad (2.12)$$

and thus σ^* can be estimated by

$$\widehat{\sigma} = \sqrt{\widetilde{\mu}_K \widehat{\mathbf{v}}^T \widehat{H}_{g,L}^n(\widehat{\boldsymbol{\beta}}) \widehat{\mathbf{v}}}, \tag{2.13}$$

where $\widehat{H}_{g,L}^n(\widehat{\boldsymbol{\beta}}) = \frac{1}{n} \sum_{i=1}^n w^2(y_i) \boldsymbol{z}_i \boldsymbol{z}_i^T \frac{1}{g} L(\frac{x_i - \widehat{\boldsymbol{\beta}}^T \boldsymbol{z}_i}{g})$ with some kernel function L and bandwidth g.

Equipped with the smoothed decorrelated score $\widehat{S}_{\delta}(\theta, \widehat{\gamma})$ in (2.8), the estimate of the approximation bias $\delta^{\ell}\widehat{\mu}$ in (2.11) and the estimate of the asymptotic variance $\widehat{\sigma}^2$ in (2.13), we define the bias corrected smoothed decorrelated score statistic as

$$\widehat{U}_n = \sqrt{n\delta} \left(\frac{\widehat{S}_\delta(0, \widehat{\gamma}) - \delta^\ell \widehat{\mu}}{\widehat{\sigma}} \right). \tag{2.14}$$

Remark 1. Compared to the existing decorrelated score approach (Ning et al., 2017), our methodological innovation is to develop an explicit bias correction step to remove the approximation bias in (2.7) induced by the smoothed surrogate risk. From the theoretical aspect, our test statistic \hat{U}_n is rescaled by $(n\delta)^{1/2}$ rather than the classical $n^{1/2}$ factor, which leads to the non-standard rate of the decorrelated score not only under the null but also under local alternatives; see Section 3.

2.3 Detailed Implementation

For numerical implementation, we follow the path-following algorithm presented in Feng et al. (2019) to compute the initial estimator $\widehat{\beta}$. For the estimator $\widehat{\omega}$, recall that ω^* satisfies $\nabla^2_{\gamma,\gamma}R(\beta^*)\omega^* = \nabla^2_{\gamma,\theta}R(\beta^*)$. Since $\nabla^2R(\beta^*)$ can be approximated by the Hessian of the smoothed surrogate loss $\nabla^2R^n_\delta(\beta^*)$, we consider the following Dantzig type estimator $\widehat{\omega}$, where

$$\widehat{\boldsymbol{\omega}} = \underset{\boldsymbol{\omega}}{\operatorname{argmin}} ||\boldsymbol{\omega}||_{1} \qquad s.t. \quad ||\nabla_{\boldsymbol{\gamma}, \boldsymbol{\theta}}^{2} R_{\delta}(\widehat{\boldsymbol{\beta}}) - \nabla_{\boldsymbol{\gamma}, \boldsymbol{\gamma}}^{2} R_{\delta}(\widehat{\boldsymbol{\beta}}) \boldsymbol{\omega}||_{\infty} \leq \lambda', \tag{2.15}$$

for some tuning parameter $\lambda' > 0$. Lemma 10 presented in the Appendix shows the convergence rate of this estimator $\hat{\omega}$ as required in Assumption 6.

For implementing \widehat{U}_n , we note that the analysis of the asymptotic distribution of \widehat{U}_n is complicated by the dependence between the estimator $\widehat{\beta}$ and $S_{\delta}(\theta, \gamma)$. To decouple the dependence and ease theoretical development, we apply the cross-fitting technique to construct the bias corrected smoothed decorrelated score. Specifically, instead of utilizing the same set of samples for estimating $\widehat{\beta}$, $\widehat{\omega}$ and constructing the score function $S_{\delta}(\theta, \gamma)$, we will firstly estimate $\widehat{\beta}$ using one set of samples, and then use the rest of samples for estimating $\widehat{\omega}$ and constructing $S_{\delta}(\theta, \gamma)$. We can further switch the samples and aggregate the decorrelated score. Without loss of generality, assume the sample size n is even and we divide the samples into two halves with equal size for this purpose. Formally, denote $\widehat{\beta}^{(i)}$, $\widehat{\omega}^{(i)}$, i = 1, 2 as the estimator based on the ith fold of the samples, \mathcal{N}_i , and similarly $\nabla R_{\delta}^{n_{(i)}}(\beta)$, $\nabla^2 R_{\delta}^{n_{(i)}}(\beta)$ as the corresponding gradient and Hessian. Define

$$\widehat{S}_{\delta}^{(1)}(\boldsymbol{\theta},\widehat{\boldsymbol{\gamma}}^{(2)}) = \nabla_{\boldsymbol{\theta}} R_{\delta}^{n_{(1)}}(\boldsymbol{\theta},\widehat{\boldsymbol{\gamma}}^{(2)}) - \widehat{\boldsymbol{\omega}}^{(1)T} \nabla_{\boldsymbol{\gamma}} R_{\delta}^{n_{(1)}}(\boldsymbol{\theta},\widehat{\boldsymbol{\gamma}}^{(2)}),$$

and $\widehat{S}^{(2)}_{\delta}(\theta,\widehat{\gamma}^{(1)})$ in a similar way. The estimated decorrelated score via cross-fitting is

$$\widehat{S}_{\delta}(\theta,\widehat{\gamma}) = \frac{1}{2} \left(\widehat{S}_{\delta}^{(1)}(\theta,\widehat{\gamma}^{(2)}) + \widehat{S}_{\delta}^{(2)}(\theta,\widehat{\gamma}^{(1)}) \right). \tag{2.16}$$

Similarly, we define the cross-fitted estimators $\hat{\mu}$ and $\hat{\sigma}$ as

$$\widehat{\mu} = \frac{1}{2} \gamma_{K,\ell} (\widehat{\boldsymbol{v}}^{(1)T} \widehat{T}_{h,U}^{(\ell),n_{(1)}} (\widehat{\boldsymbol{\beta}}^{(2)}) + \widehat{\boldsymbol{v}}^{(2)T} \widehat{T}_{h,U}^{(\ell),n_{(2)}} (\widehat{\boldsymbol{\beta}}^{(1)})),$$

$$\widehat{\sigma}^2 = \frac{\widetilde{\mu}_K}{2} \left[\widehat{\boldsymbol{v}}^{(1)T} \widehat{H}_{g,K}^{n_{(1)}} (\widehat{\boldsymbol{\beta}}^{(2)}) \widehat{\boldsymbol{v}}^{(1)} + \widehat{\boldsymbol{v}}^{(2)T} \widehat{H}_{g,K}^{n_{(2)}} (\widehat{\boldsymbol{\beta}}^{(1)}) \widehat{\boldsymbol{v}}^{(2)} \right],$$
(2.17)

where

$$\widehat{T}_{h,U}^{(\ell),n_{(1)}}(\widehat{\boldsymbol{\beta}}^{(2)}) = \frac{1}{|\mathcal{N}_{1}|} \sum_{i \in \mathcal{N}_{1}} w(y_{i}) y_{i} \frac{\boldsymbol{z}_{i}}{h^{1+\ell}} U^{(\ell)} \left(\frac{\widehat{\boldsymbol{\beta}}^{(2)T} \boldsymbol{z}_{i} - x_{i}}{h} \right),$$

$$\widehat{H}_{g,L}^{n_{(1)}}(\widehat{\boldsymbol{\beta}}^{(2)}) = \frac{1}{|\mathcal{N}_{1}|} \sum_{i \in \mathcal{N}_{1}} w^{2}(y_{i}) \boldsymbol{z}_{i} \boldsymbol{z}_{i}^{T} \frac{1}{g} L(\frac{x_{i} - \widehat{\boldsymbol{\beta}}^{(2)T} \boldsymbol{z}_{i}}{g}), \tag{2.18}$$

and similarly for $\widehat{T}_{h,U}^{(\ell),n_{(2)}}(\widehat{\boldsymbol{\beta}}^{(1)}), \widehat{H}_{g,L}^{n_{(2)}}(\widehat{\boldsymbol{\beta}}^{(1)})$. Given $\widehat{S}_{\delta}(\theta,\widehat{\boldsymbol{\gamma}})$ in (2.16) and the above estimators $\widehat{\mu}$ and $\widehat{\sigma}$, we can form the score test statistic \widehat{U}_n in the same way as in (2.14).

3 Theory

3.1 Assumptions

In this paper, we consider the following definition of function smoothness.

Definition 1. We say the conditional density f(x|y, z) of X given Y, Z is ℓ th order smooth, if for any z and $y \in \{-1, 1\}$, the conditional density f(x|y, z) is ℓ -times continuously differentiable in x with derivatives $f^{(i)}(x|y, z)$ bounded by a constant C, $|f^{(i)}(x|y, z)| \leq C$ for $i = 1, \ldots, \ell$, and $f^{(\ell)}(x|y, z)$ is Hölder continuous with some exponent $0 < \zeta \leq 1$, that is, for any z, Δ and $y \in \{-1, 1\}$,

$$|f^{(\ell)}(x + \Delta|y, \mathbf{z}) - f^{(\ell)}(x|y, \mathbf{z})| \le L\Delta^{\zeta}, \tag{3.1}$$

where L > 0 is some constant.

Assumption 1. We assume f(x|y,z) is ℓ th order smooth with some integer $\ell \geq 2$.

Assumption 1 concerns the smoothness of f(x|y, z). To see why the smoothness condition is important, notice that the gradient functions of (2.1) and (1.2) are

$$\nabla R_{\delta}(\boldsymbol{\beta}) = \sum_{y \in \{-1,1\}} w(y) \int yz \left[\int \frac{1}{\delta} K(\frac{y(x - \boldsymbol{\beta}^T \boldsymbol{z})}{\delta}) f(x|y, \boldsymbol{z}) dx \right] f(y, \boldsymbol{z}) d\boldsymbol{z}$$

$$\nabla R(\boldsymbol{\beta}) = \sum_{y \in \{-1,1\}} w(y) \int yz f(\boldsymbol{\beta}^T \boldsymbol{z}|y, \boldsymbol{z}) f(y, \boldsymbol{z}) d\boldsymbol{z}, \tag{3.2}$$

from which we can see that $f(\boldsymbol{\beta}^T \boldsymbol{z}|y, \boldsymbol{z})$ in $\nabla R(\boldsymbol{\beta})$ is substituted by its kernel approximation $\int \frac{1}{\delta} K(\frac{y(x-\boldsymbol{\beta}^T \boldsymbol{z})}{\delta}) f(x|y, \boldsymbol{z}) dx$, and thus the difference between $\nabla R_{\delta}(\boldsymbol{\beta})$ and $\nabla R(\boldsymbol{\beta})$ naturally depends on the smoothness of $f(x|y, \boldsymbol{z})$.

Notice that our smoothness condition in Definition 1 is slightly stronger than the standard Hölder smoothness condition in the nonparametric literature (Tsybakov, 2009). In particular, we require that $f^{(\ell)}(x|y,z)$ is Hölder continuous with some exponent $0 < \zeta \le 1$ in (3.1). This additional assumption is essential to show the rate of the bias estimator $\hat{\mu}$ in (2.11).

Our next assumption is about the kernel function K(t) that we first introduced in the surrogate risk $R_{\delta}(\beta)$ in (2.1).

Assumption 2. We assume K(t) is a kernel function with support [-1,1] that satisfies: K(t) = K(-t), $|K(t)| \le K_{\max} < \infty \ \forall \ t \in \mathbb{R}$, $\int K(t)dt = 1$, $\int K^2(t)dt < \infty$, and $|K'| < \infty$. We also assume that K degenerates at the boundaries. A kernel is said to be of order $\ell \ge 1$ if it satisfies $\int t^j K(t)dt = 0$, $\forall \ j = 1, \ldots, \ell - 1$, $\int t^\ell K(t)dt \ne 0$, and $\int |t|^q |K(t)|dt$ are bounded by a constant for any $q \in [\ell, \ell + 1]$.

We now impose regularity conditions on (X, Y, \mathbf{Z}) .

Assumption 3. There exists a constant c > 0 such that $c \leq \mathbb{P}(Y = 1) \leq 1 - c$ and the weight function $w(\cdot)$ is positive and upper bounded by a constant.

Assumption 4. We assume $\forall j = 1, ..., d, |Z_j| \leq M_n$ for some M_n that possibly depends on n, where $M_n^2 \leq C\sqrt{n\delta/\log(d)}$ for some constant C > 0. We also assume that $\mathbb{E}[Z_j^4|Y=y]$ is bounded by a constant for $y \in \{1, -1\}$.

Assumption 5. We assume $\sigma^* = \sqrt{\boldsymbol{v}^{*T}\boldsymbol{\Sigma}^*\boldsymbol{v}^*}$ and $|\mu^*| = |\boldsymbol{v}^{*T}\boldsymbol{b}^*|$ are bounded away from 0 and infinity by some constants.

Assumption 4 requires the boundedness of Z and the fourth order moment. Notice that if each component of Z is sub-Gaussian with bounded sub-Gaussian norm, Assumption 4 is satisfied with high probability providing $(\log d)^3/(n\delta) = \mathcal{O}(1)$, which is a mild assumption. Assumption 5 ensures that the asymptotic variance of the smoothed decorrelated score σ^* and the approximation bias μ^* are nonzero and bounded.

We impose the following assumption on the estimators of β^* and ω^* .

Assumption 6. Assume there are estimators $\widehat{\beta}$ and $\widehat{v} = (1, -\widehat{\omega}^T)^T$ with

$$||\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}^*||_1 \lesssim \eta_1(n) \quad \text{ and } \quad ||\widehat{\boldsymbol{v}} - \boldsymbol{v}^*||_1/||\boldsymbol{v}^*||_1 \lesssim \eta_2(n),$$

for some non-random sequences $\eta_1(n), \eta_2(n)$ converging to 0 as $n \to \infty$.

It is shown by Feng et al. (2019) that, under some conditions, the estimator $\widehat{\boldsymbol{\beta}}$ in (2.2) achieves the (near) minimax-optimal rate $\eta_1(n) = \sqrt{s}(\frac{s\log(d)}{n})^{\ell/(2\ell+1)}$ by choosing $\lambda \asymp \sqrt{\log d/(n\delta)}$ and $\delta \asymp (s\log d/n)^{1/(2\ell+1)}$, where $s = \|\boldsymbol{\beta}^*\|_0$. For $\widehat{\boldsymbol{v}} = (1, -\widehat{\boldsymbol{\omega}}^T)^T$, we assume $\|\widehat{\boldsymbol{v}} - \boldsymbol{v}^*\|_1 \lesssim \|\boldsymbol{v}^*\|_1 \eta_2(n)$. Notice that the term $\|\boldsymbol{v}^*\|_1$ is not absorbed into $\eta_2(n)$ only for notational simplicity. In Lemma 10 in the Appendix, we show that a Dantzig type estimator $\widehat{\boldsymbol{v}}$ attains a fast rate $\eta_2(n)$.

3.2 Theoretical Results

We start from the following lemma which characterizes the asymptotic distribution of the decorrelated score function evaluated at the true parameter β^* .

Lemma 1. Under Assumptions 1 - 5, if $(||v^*||_1 M_n)^3/(n\delta)^{1/2} = o(1)$ and $\delta = o(1)$, then

$$\sqrt{n\delta} \frac{\boldsymbol{v}^{*T}(\nabla R_{\delta}^{n}(\boldsymbol{\beta}^{*}) - \nabla R_{\delta}(\boldsymbol{\beta}^{*}))}{\sqrt{\boldsymbol{v}^{*T}\boldsymbol{\Sigma}^{*}\boldsymbol{v}^{*}}} \stackrel{d}{\to} N(0,1), \tag{3.3}$$

where

$$\boldsymbol{v}^{*T} \nabla R_{\delta}(\boldsymbol{\beta}^{*}) = \delta^{\ell} \boldsymbol{v}^{*T} \boldsymbol{b}^{*} (1 + o(1)). \tag{3.4}$$

This lemma illustrates the asymptotic bias and variance of the smoothed decorrelated score. The former is essentially the approximation bias term in (2.7). Since $\mu^* = \mathbf{v}^{*T}\mathbf{b}^*$ and $\sigma^* = \sqrt{\mathbf{v}^{*T}\mathbf{\Sigma}^*\mathbf{v}^*}$ are bounded by constants, the asymptotic bias and standard deviation of $\mathbf{v}^{*T}\nabla R^n_{\delta}(\boldsymbol{\beta}^*)$ are of order δ^ℓ and $(n\delta)^{-1/2}$, respectively. Thus, choosing $\delta = cn^{-1/(2\ell+1)}$ for any constant c>0 attains the optimal bias and variance trade-off in terms of the rate. Indeed, we show in Section 4 that the bandwidth δ that minimizes an estimate of the MSE of the decorrelated score not only attains the optimal order $n^{-1/(2\ell+1)}$ but also finds the optimal constant c. Note that in this lemma we require $(||\mathbf{v}^*||_1 M_n)^3/(n\delta)^{1/2} = o(1)$ to verify the Lindeberg condition in the central limit theorem, which holds as long as δ does not shrink to 0 too fast.

Our first main theorem characterizes the asymptotic normality of the decorrelated score under the null hypothesis with nuisance parameters γ^* and ω^* estimated by those in Assumption 6.

Theorem 1. Under Assumptions 1 - 6, if $(||\boldsymbol{v}^*||_1 M_n)^3/(n\delta)^{1/2} = o(1)$, $\frac{\log(d)}{n\delta^3} = o(1)$, $n\delta^{2\ell+1} = O(1)$, and

$$(n\delta)^{1/2}||\boldsymbol{v}^*||_1\left(\frac{\eta_1(n)}{\delta}\vee\eta_2(n)\right)\left(\sqrt{\frac{\log(d)}{n\delta}}\vee\delta^\ell\vee M_n^2\eta_1(n)\right)=o(1),\tag{3.5}$$

then under $H_0: \theta^* = 0$, it holds that

$$\frac{\sqrt{n\delta}\widehat{S}_{\delta}(0,\widehat{\gamma}) - \sqrt{n\delta^{2\ell+1}}\mu^*}{\sigma^*} \stackrel{d}{\to} N(0,1). \tag{3.6}$$

Theorem 1 implies that the decorrelated score with some high-dimensional plug-in estimators $\widehat{\gamma}$ and $\widehat{\omega}$ has the same asymptotic distribution as in Lemma 1. Several conditions are needed to show this result. The first condition $(||v^*||_1 M_n)^3/(n\delta)^{1/2} = o(1)$ is from Lemma 1, and the second condition $\frac{\log(d)}{n\delta^3} = o(1)$ is also mild as long as δ does not go to zero too fast. The third condition $n\delta^{2\ell+1} = O(1)$ guarantees that the higher order bias of the decorrelated score can be ignored and therefore it suffices to only correct for the leading bias term in (3.4). In particular, if the rate optimal bandwidth $\delta = cn^{-1/(2\ell+1)}$ is chosen, from (3.6), we can see that the asymptotic bias of $\sqrt{n\delta}\widehat{S}_{\delta}(0,\widehat{\gamma})$ is a constant $\sqrt{n\delta^{2\ell+1}}\mu^* = c^{\ell+1/2}\mu^*$, which is further removed to attain the mean zero Gaussian limiting distribution.

We now elaborate the condition (3.5). Roughly speaking, the term $\sqrt{\frac{\log(d)}{n\delta}} \vee \delta^{\ell} \vee M_n^2 \eta_1(n)$ comes from the bound for $\|\nabla R_{\delta}^{n_{(1)}}(\theta, \widehat{\gamma}^{(2)}) - \nabla R(\theta, \gamma)\|_{\infty}$, in which the three terms correspond to the stochastic and approximation error of $\nabla R_{\delta}^{n_{(1)}}(\theta, \widehat{\gamma}^{(2)})$ and the plug-in error from $\widehat{\gamma}^{(2)}$, respectively. Indeed, the cross-fitting technique guarantees the independence between $\widehat{\gamma}^{(2)}$ and $\nabla R_{\delta}^{n_{(1)}}(\theta, \gamma)$,

which is the key to obtain a sharp bound for the stochastic and approximation error. Thus, condition (3.5) simply means that these error terms interacting with the estimation error of $\widehat{\gamma}$ and $\widehat{\omega}$ are sufficiently small. We can further simplify the condition (3.5) by plugging the order of $\eta_1(n)$ and $\eta_2(n)$. Recall that Feng et al. (2019) derived $\eta_1(n) = \sqrt{s} (\frac{s \log(d)}{n})^{\ell/(2\ell+1)}$, where $s = \|\beta^*\|_0$. The Lemma 10 in the Appendix implies that $\eta_2(n) = s'(\log(d)/n)^{(\ell-1)/(2\ell+1)}$, where $s' = \|\omega^*\|_0$. Assuming $\|v^*\|_1$, $M_n = \mathcal{O}(1)$ for now, condition (3.5) becomes

$$s^{(4\ell+1)/(4\ell+2)}(s^{(4\ell+1)/(4\ell+2)} \vee s')n^{-(\ell-1)/(2\ell+1)}(\log d)^{(4\ell-1)/(4\ell+2)} = o(1)$$
(3.7)

when taking $\delta \approx n^{-1/(2\ell+1)}$. If we consider the extreme case with $\ell \to \infty$, (3.7) reduces to $s(s \vee s') \log d = o(n^{1/2})$.

Recall that in our score statistic \widehat{U}_n in (2.14), we plug in the estimators $\widehat{\mu}$ and $\widehat{\sigma}$ for μ^* and σ^* . In Lemmas 11 and 12 in the Appendix, we establish the rate of convergence of $\widehat{\mu}$ and $\widehat{\sigma}$. Under the assumption that $|\widehat{\mu} - \mu^*| = o_p(1)$ and $|\widehat{\sigma} - \sigma^*| = o_p(1)$, the Slutsky's theorem implies that the bias corrected decorrelated score statistic $\widehat{U}_n \stackrel{d}{\to} N(0,1)$ under the null hypothesis.

Accordingly, given the desired significance level α , we define the test function as

$$T_{DS} = I(|\widehat{U}_n| > \Phi^{-1}(1 - \alpha/2)),$$

where $\Phi^{-1}(\cdot)$ is the inverse function of the cdf of the standard normal distribution. Thus, our result shows that the Type I error of the test T_{DS} converges to α asymptotically, i.e., $\mathbb{P}(T_{DS}=1|H_0) \to \alpha$. Now denote

$$\nabla^2_{\theta|\boldsymbol{\gamma}}R(\boldsymbol{\beta}^*) = \nabla^2_{\theta\theta}R(\boldsymbol{\beta}^*) - \nabla^2_{\theta\boldsymbol{\gamma}}R(\boldsymbol{\beta}^*)(\nabla^2_{\boldsymbol{\gamma}\boldsymbol{\gamma}}R(\boldsymbol{\beta}^*))^{-1}\nabla^2_{\boldsymbol{\gamma}\boldsymbol{\theta}}R(\boldsymbol{\beta}^*).$$

Our second main theorem characterizes the limiting behavior of \widehat{U}_n under the local alternative hypothesis $H_1: \theta^* = \widetilde{C} n^{-\phi}$ for some constants $\widetilde{C} \neq 0$ and $\phi > 0$.

Theorem 2. Under the conditions in Theorem 1 and Lemmas 11 and 12 in the Appendix, we further assume

$$||\boldsymbol{v}^*||_1^2 M_n^4 n^{1-4\phi}/\delta = o(1), \quad (n\delta)^{1/2} ||\boldsymbol{v}^*||_1 (\eta_1(n) \vee \eta_2(n)) M_n n^{-\phi} = o(1), \tag{3.8}$$

and $\widehat{\mu}, \widehat{\sigma}$ are consistent estimators of μ^*, σ^* . Then the following results hold under the local alternative hypothesis $H_1: \theta^* = \widetilde{C} n^{-\phi}$.

• If $\widetilde{C}(n\delta)^{1/2}n^{-\phi}\nabla^2_{\theta|\gamma}R(\boldsymbol{\beta}^*)\sigma^{*-1}\to \xi$ for some constant ξ , then it holds that

$$\widehat{U}_n \stackrel{d}{\to} N(-\xi, 1). \tag{3.9}$$

• If $\widetilde{C}(n\delta)^{1/2}n^{-\phi}\nabla^2_{\theta|\gamma}R(\boldsymbol{\beta}^*)\sigma^{*-1}\to\infty$, then for any fixed t, it holds that

$$\lim_{n \to \infty} \mathbb{P}(|\widehat{U}_n| > t) = 1. \tag{3.10}$$

In addition to the conditions imposed in Theorem 1 and Lemmas 11 and 12 in the Appendix, we further require two additional conditions involving the magnitude of θ^* in (3.8). The first condition $||\boldsymbol{v}^*||_1^2 M_n^4 n^{1-4\phi}/\delta = o(1)$ is imposed to ensure the local asymptotic normality (LAN) in terms of the parameter θ^* . Assuming $||\boldsymbol{v}^*||_1$ and $M_n = O(1)$, this condition requires $\phi > \frac{\ell+1}{2(2\ell+1)}$ with the rate optimal bandwidth $\delta \approx n^{-1/(2\ell+1)}$. The second condition in (3.8) is similar to (3.5), that controls the magnitude of $||\nabla R(0, \gamma^*)||_{\infty}$ and $||\nabla^2 R_{\delta}(\theta^*, \gamma^*) - \nabla^2 R_{\delta}(0, \gamma^*)||_{\max}$ under the alternative hypothesis.

The conclusions in Theorem 2 are shown in two settings (for now assume $\nabla^2_{\theta|\gamma}R(\boldsymbol{\beta}^*)$ is a constant and $\delta \simeq n^{-1/(2\ell+1)}$). Firstly, if $\widetilde{C}(n\delta)^{1/2}n^{-\phi}\nabla^2_{\theta|\gamma}R(\boldsymbol{\beta}^*)\sigma^{*-1} \to \xi$, which holds when $\phi = \ell/(2\ell+1)$, then it implies that the asymptotic power of the test T_{DS} at level α is

$$\mathbb{P}(T_{DS} = 1|H_1) \to A_{\alpha} := 1 - \Phi(\xi + \Phi^{-1}(1 - \alpha/2)) + \Phi(\xi - \Phi^{-1}(1 - \alpha/2)).$$

Notice that A_{α} is greater than the level α as long as $\xi \neq 0$. This indicates the asymptotic unbiasedness of the test T_{DS} . Second, the power of the test tends to 1 when $(n\delta)^{1/2}n^{-\phi}\nabla^2_{\theta|\gamma}R(\beta^*)\sigma^{*-1} \to \infty$, which holds when $\phi < \ell/(2\ell+1)$. This implies that the proposed test can successfully detect the nonzero θ^* whose magnitude exceeds the order of $n^{-\ell/(2\ell+1)}$. In contrast, for regular models, the local alternative that is detectable is of the standard parametric rate $n^{-1/2}$. This is a major theoretical difference between our work and the existing ones on debiased and decorrelated methods.

4 Data-Driven Bandwidth Selection

In the previous section, we establish the theoretical property of the bias corrected smoothed decorrelated score when the underlying conditional density f(x|y,z) is ℓ th order smooth. However, this smoothness parameter ℓ is typically unknown in practice, leading to the following two complications. First, in Assumption 2, a kernel function K of the same order is applied, which implicitly requires the knowledge on the smoothness parameter ℓ . In practice, the choice of kernel functions is often determined by the user's preference rather than the theory. Since high order kernels may exacerbate the problem of variability, choosing low order kernels of 2 or 4 is often recommended (even if the density is more smooth); see Härdle et al. (1992). Second, the optimal bandwidth $\delta \approx n^{-1/(2\ell+1)}$ that balances the asymptotic bias and variance of the decorrelated score in Lemma 1 also depends on the unknown ℓ . It is well known from the nonparametric literature that the choice of bandwidth is an extremely important problem of both theoretical and practical values (Silverman, 1986; Bowman, 1984; Sheather and Jones, 1991; Hall et al., 1992; Jones et al., 1996). In this section, we focus on how to choose the bandwidth δ in a data-driven manner. In view of the above discussion on the kernels, we assume that a low order kernel K is chosen (for simplicity, we still denote its order by ℓ) and meanwhile the underlying conditional density has a higher order smoothness parameter.

Assumption 7. We assume that the kernel K is of order ℓ and f(x|y, z) is $(\ell + r)$ th order smooth for some $\ell \geq 2$ and r > 0.

We define the optimal bandwidth δ^* as the one that minimizes the MSE of the smoothed decorrelated score:

$$\delta^* = \operatorname*{argmin}_{\delta} M(\delta), \quad \text{where} \quad M(\delta) = \mathbb{E}[(\boldsymbol{v}^{*T} \nabla R_{\delta}^n(\boldsymbol{\beta}^*))^2]. \tag{4.1}$$

A direct bias-variance decomposition of $M(\delta)$ gives

$$M(\delta) = \frac{1}{n} \mathbb{E}[(\boldsymbol{v}^{*T} \nabla \bar{R}_{\delta}^{1}(\boldsymbol{\beta}^{*}))^{2}] + \frac{n-1}{n} (\boldsymbol{v}^{*T} \nabla R_{\delta}(\boldsymbol{\beta}^{*}))^{2} := \frac{1}{n} V(\delta) + \frac{n-1}{n} SB(\delta), \tag{4.2}$$

where $\nabla \bar{R}_{\delta}^{1}(\boldsymbol{\beta}^{*})$ is defined in (2.3). To simplify the estimation, we define $V(\delta) = \mathbb{E}[(\boldsymbol{v}^{*T}\nabla \bar{R}_{\delta}^{1}(\boldsymbol{\beta}^{*}))^{2}]$ and use it as a proxy for the variance. We use $SB(\delta) = (\boldsymbol{v}^{*T}\nabla R_{\delta}(\boldsymbol{\beta}^{*}))^{2}$ to denote the squared bias.

To estimate δ^* , the main idea is to construct estimators $\widehat{V}(\delta)$ and $\widehat{SB}(\delta)$ for $V(\delta)$ and $SB(\delta)$ and then estimate δ^* by

$$\widehat{\delta} = \underset{\delta}{\operatorname{argmin}} \widehat{M}(\delta) \text{ where } \widehat{M}(\delta) = \frac{1}{n}\widehat{V}(\delta) + \frac{n-1}{n}\widehat{SB}(\delta).$$

Remark 2. Before we proceed to develop the estimators of $V(\delta)$ and $SB(\delta)$, we first clarify the difference between the variance and squared bias terms $V(\delta)$ and $SB(\delta)$ and the asymptotic variance and bias σ^{*2} and $\delta^{\ell}\mu^{*}$ in Lemma 1. From the proof of Lemma 1, we can show that $SB(\delta) = (\delta^{\ell}\mu^{*})^{2}(1+o(1))$ and $V(\delta) = \delta^{-1}\sigma^{*2}(1+o(1))$ as $\delta \to 0$, and thus σ^{*2}/δ and $(\delta^{\ell}\mu^{*})^{2}$ are the asymptotic versions of $V(\delta)$ and $SB(\delta)$, respectively. As a result, one may attempt to estimate the optimal bandwidth by minimizing the asymptotic MSE $\hat{\sigma}^{2}/\delta + (\delta^{\ell}\hat{\mu})^{2}$ with the plug-in estimators $\hat{\sigma}$ and $\hat{\mu}$ developed in the previous section. However, the asymptotic MSE depends on the unknown smoothness ℓ and therefore is not appropriate for bandwidth selection in practice. Thus, we have to use a different strategy to estimate $V(\delta)$ and $SB(\delta)$.

To estimate $V(\delta)$, we consider the following moment estimator

$$\widehat{V}(\delta) = \frac{1}{2} (\widehat{\boldsymbol{v}}^{(1)T} \widehat{\boldsymbol{\Gamma}}^{(1)}(\delta) \widehat{\boldsymbol{v}}^{(1)} + \widehat{\boldsymbol{v}}^{(2)T} \widehat{\boldsymbol{\Gamma}}^{(2)}(\delta) \widehat{\boldsymbol{v}}^{(2)}), \tag{4.3}$$

where $\widehat{\mathbf{\Gamma}}^{(1)}(\delta) = \frac{1}{|\mathcal{N}_1|} \sum_{i \in \mathcal{N}_1} \nabla \bar{R}^i_{\delta}(\widehat{\boldsymbol{\beta}}^{(2)}) \nabla \bar{R}^i_{\delta}(\widehat{\boldsymbol{\beta}}^{(2)})^T$ and similarly for $\widehat{\mathbf{\Gamma}}^{(2)}$.

Next, we focus on the problem of estimating the squared bias $SB(\delta)$. After some algebra, the bias term $B(\delta)$ can be written as

$$B(\delta) = \mathbf{v}^{*T} \underbrace{\left(\nabla R_{\delta}(\boldsymbol{\beta}^{*}) - \nabla R(\boldsymbol{\beta}^{*})\right)}_{A(\boldsymbol{\beta}^{*}, \delta)} = \int_{u} K(u) \left\{ \mathbf{v}^{*T} (\nabla R(u\delta, \boldsymbol{\beta}^{*}) - \nabla R(\boldsymbol{\beta}^{*})) \right\} du, \tag{4.4}$$

where with a bit abuse of notation, we use $\nabla R(u\delta, \boldsymbol{\beta}^*) = \sum_y w(y) \int_{\boldsymbol{z}} \boldsymbol{z} y f(u\delta + \boldsymbol{\beta}^{*T} \boldsymbol{z} | y, \boldsymbol{z}) f(y, \boldsymbol{z}) d\boldsymbol{z}$ to denote the population gradient with a bias induced by $u\delta$. Consider the following estimator with

cross-fitting

$$\widehat{B}(\delta) = \frac{1}{2} \left(\widehat{\boldsymbol{v}}^{(1)T} \frac{1}{|\mathcal{N}_1|} \sum_{i \in \mathcal{N}_1} A_i(\widehat{\boldsymbol{\beta}}^{(2)}, \delta) + \widehat{\boldsymbol{v}}^{(2)T} \frac{1}{|\mathcal{N}_2|} \sum_{i \in \mathcal{N}_2} A_i(\widehat{\boldsymbol{\beta}}^{(1)}, \delta) \right), \tag{4.5}$$

where

$$A_{i}(\widehat{\boldsymbol{\beta}}, \delta) = \int_{\mathcal{U}} K(u)w(y_{i}) \frac{\boldsymbol{z}_{i}y_{i}}{b} \left[J(\frac{x_{i} - \widehat{\boldsymbol{\beta}}^{T}\boldsymbol{z}_{i} - u\delta}{b}) - J(\frac{x_{i} - \widehat{\boldsymbol{\beta}}^{T}\boldsymbol{z}_{i}}{b}) \right] du, \tag{4.6}$$

and J is another pilot kernel function of order r with bandwidth b. Essentially, we substitute $\nabla R(u\delta, \boldsymbol{\beta}^*)$ and $\nabla R(\boldsymbol{\beta}^*)$ in (4.4) with their corresponding kernel estimators. This approach is motivated by the "double-smoothing" technique in nonparametric statistics (Härdle et al., 1992; Neumann et al., 1995), and is also related to the "smoothed cross validation" approach (Hall et al., 1992). In these works, when the target function has higher order smoothness than the kernel function applied for estimation, a second kernel smoothing procedure is applied for estimating the bias. Using a similar strategy, in our case, we apply the second smoother with kernel J in the estimators of $\nabla R(u\delta, \boldsymbol{\beta}^*)$ and $\nabla R(\boldsymbol{\beta}^*)$. We now estimate the squared bias by $\widehat{SB}(\delta) = \widehat{B}(\delta)^2$.

To analyze theoretical properties of the estimates, let's define $\Delta = [q_1 n^{-1+\epsilon_1}, q_2 n^{-1+\epsilon_2}]$ as the range of bandwidth δ for some constants $0 < q_1 \le q_2$ and $0 < \epsilon_1 < \epsilon_2 < 1$. Since the optimal bandwidth δ^* is of order $n^{-1/(2\ell+1)}$, we can guarantee $\delta^* \in \Delta$ for some suitable ϵ_1 and ϵ_2 . Under some conditions, the uniform convergence rates of $\widehat{V}(\delta)$ and $\widehat{SB}(\delta)$ are given by

$$|\widehat{V}(\delta) - V(\delta)| \lesssim \frac{\psi_1(n,\delta)}{\delta}, \quad |\widehat{SB}(\delta) - SB(\delta)| \lesssim \delta^{2\ell} \psi_2(n,\delta),$$

uniformly over all $\delta \in \Delta$, where

$$\psi_1(n,\delta) = ||\boldsymbol{v}^*||_1^2 \bigg(\eta_2(n) \vee \sqrt{\frac{\log(n \vee d)}{n\delta}} \vee M_n \eta_1(n) \bigg),$$

and

$$\psi_2(n,\delta) = ||\boldsymbol{v}^*||_1^2 \left(\sqrt{\frac{\log(n \vee d)}{nb^{2\ell+1}}} \vee (\delta \vee b)^r \vee M_n \eta_1(n) (1 \vee \frac{M_n \eta_1(n)}{\delta^\ell}) \vee \eta_2(n) \right).$$

We refer to Lemmas 2 and 3 in Appendix A for the formal statement of the results and further interpretations of the rates. Now we are ready to present the following theorem.

Theorem 3. Under the conditions in Lemmas 2 and 3 in Appendix A, if $\psi_1(n, \delta), \psi_2(n, \delta) = o(1)$, then with probability tending to 1,

$$\frac{\widehat{\delta} - \delta^*}{\delta^*} \lesssim \psi_1(n, \delta^*) \vee \psi_2(n, \delta^*).$$

Thus, $\widehat{\delta}/\delta^* \to 1$ in probability.

Notice that in Theorem 3, the MSE-optimal bandwidth satisfies $\delta^* \simeq n^{-1/(2\ell+1)}$. To examine the order of $\psi_1(n, \delta^*) \vee \psi_2(n, \delta^*)$, we consider $d \gtrsim n$ (high-dimensional case) and optimize the bandwidth

b in $\psi_2(n, \delta^*)$, leading to $b \approx (\log d/n)^{1/(2\ell+2r+1)}$. Furthermore, take $\eta_1(n), \eta_2(n)$ as the rate from Feng et al. (2019) and Lemma 10, and assume that $||\boldsymbol{v}^*||_1, M_n = \mathcal{O}(1)$, then $\psi_1(n, \delta^*) \vee \psi_2(n, \delta^*)$ can be simplified to

$$\left(\frac{\log d}{n}\right)^{r/(2\ell+2r+1)} \vee s^{(4\ell+1)/(2\ell+1)} \left(\frac{\log^2 d}{n}\right)^{\ell/(2\ell+1)} \vee s' \left(\frac{\log d}{n}\right)^{(\ell-1)/(2\ell+1)}. \tag{4.7}$$

There are in general two scenarios: firstly, if r is small relative to ℓ while s, s' do not grow too fast, then $(\log d/n)^{r/(2\ell+2r+1)}$ from the extra smoothing step for estimating $SB(\delta)$ will be the dominant term. However, if r is relatively large such that $r/(2\ell+2r+1) \geq (\ell-1)/(2\ell+1)$, then the last two terms in (4.7) due to $\widehat{\beta}$ and $\widehat{\omega}$ dominate. In this case, even if f(x|y,z) has a large amount of extra smoothness r, the convergence rate of $\widehat{\delta}$ cannot be further improved, as the rate is dominated by the error from $\widehat{\beta}$ and $\widehat{\omega}$.

5 Hypothesis Testing for the Linear Combination

Our results presented thus far are for the hypothesis testing (2.5), where θ is simply a single element in β . In applications, researchers may also be interested in the hypothesis testing (2.4), a linear combination of the parameter β . For example, as we mentioned at the beginning of Section 2.2, in the study of inferring iMCID, it is of interest to test $\mathbf{c}_0^T \beta^* = 0$ where \mathbf{c}_0 represents the realized value of a new patient's clinical profile and we assume $c_{01} \neq 0$ without loss of generality. Below, we show that all the methods and results developed for (2.5) can be extended to (2.4) in a parallel manner.

To test the hypothesis (2.4), consider the one to one reparametrization $(\theta, \gamma) \to (\xi, \gamma)$, where $\xi = \mathbf{c}_0^T \boldsymbol{\beta}$. Under this new set of parameters, the null hypothesis can be written as $H_{0L}: \xi^* = 0$, and the smoothed surrogate loss reduces to $R_{\delta}^n(\frac{\xi - \mathbf{c}_{02}^T \boldsymbol{\gamma}}{c_{01}}, \boldsymbol{\gamma})$, where we write $\mathbf{c}_0 = (c_{01}, \mathbf{c}_{02}^T)^T$ with

$$c_{02} \in \mathbb{R}^{d-1}$$
. Define $C = \begin{bmatrix} \frac{1}{c_{01}} & 0 \\ \frac{-c_{02}}{c_{01}} & I_{d-1} \end{bmatrix} \in \mathbb{R}^{d \times d}$. From the chain rule, we can show that

$$\nabla_{(\xi,\gamma)} R_{\delta}^{n}(\frac{\xi - \boldsymbol{c}_{02}^{T} \boldsymbol{\gamma}}{c_{01}}, \boldsymbol{\gamma}) = \boldsymbol{C} \nabla R_{\delta}^{n}(\theta, \boldsymbol{\gamma}), \quad \nabla_{(\xi,\gamma),(\xi,\gamma)}^{2} R_{\delta}^{n}(\frac{\xi - \boldsymbol{c}_{02}^{T} \boldsymbol{\gamma}}{c_{01}}, \boldsymbol{\gamma}) = \boldsymbol{C} \nabla^{2} R_{\delta}^{n}(\theta, \boldsymbol{\gamma}) \boldsymbol{C}^{T}, \quad (5.1)$$

and similarly for $R_{\delta}(\frac{\xi - c_{02}^{T} \gamma}{c_{01}}, \gamma)$ and $R(\frac{\xi - c_{02}^{T} \gamma}{c_{01}}, \gamma)$. Therefore, following the same idea as in Section 2.2, we define the smoothed decorrelated score as

$$S_{\delta}^{L}(\xi, \gamma, \omega_{L}^{*}) = \nabla_{\xi} R_{\delta}^{n}(\frac{\xi - \boldsymbol{c}_{02}^{T} \gamma}{c_{01}}, \gamma) - \omega_{L}^{*T} \nabla_{\gamma} R_{\delta}^{n}(\frac{\xi - \boldsymbol{c}_{02}^{T} \gamma}{c_{01}}, \gamma), \tag{5.2}$$

where $\boldsymbol{\omega}_L^* = \left[\nabla_{\boldsymbol{\gamma},\boldsymbol{\gamma}}^2 R(\frac{\xi - \boldsymbol{c}_{02}^T \boldsymbol{\gamma}}{c_{01}}, \boldsymbol{\gamma})\right]^{-1} \nabla_{\boldsymbol{\gamma},\xi}^2 R(\frac{\xi - \boldsymbol{c}_{02}^T \boldsymbol{\gamma}}{c_{01}}, \boldsymbol{\gamma}).$

Write $\boldsymbol{v}_L^* = (1, \boldsymbol{\omega}_L^{*T})^T$ and denote μ_L^*, σ_L^* as the (scaled) asymptotic bias and standard deviation of the score function $S_{\delta}^L(\xi, \boldsymbol{\gamma}, \boldsymbol{\omega}_L^*)$, i.e.,

$$\mu_L^* = \boldsymbol{v}_L^{*T} \boldsymbol{C} \boldsymbol{b}^*, \qquad \sigma_L^* = \sqrt{\boldsymbol{v}_L^{*T} \boldsymbol{C} \boldsymbol{\Sigma}^* \boldsymbol{C}^T \boldsymbol{v}_L^*},$$
 (5.3)

where b^* and Σ^* are defined in (2.10) and (2.12), respectively. From above we can see that the estimation methods for ω^* , μ^* and σ^* proposed in Section 2.2 can be easily extended to obtain corresponding estimators for ω_L^* , μ_L^* and σ_L^* . Given these estimators, we define the test statistics for H_{0L} as

$$\widehat{U}_n^L = (n\delta)^{1/2} \frac{S_\delta^L(0,\widehat{\gamma},\widehat{\omega}_L) - \delta^\ell \widehat{\mu}_L}{\widehat{\sigma}_L}.$$

Accordingly, all the parallel results presented in Section 3 and Section 4 can be developed. In the interest of space, we only present the following result that characterizes the asymptotic distribution of \hat{U}_n^L under the null hypothesis H_{0L} . All other parallel results are omitted.

Theorem 4. If Assumptions 1 - 6 hold with $\mu^*, \sigma^*, v^*, \hat{v}$ substituted by $\mu_L^*, \sigma_L^*, v_L^*, \hat{v}_L$, and in addition $\hat{\mu}_L$ and $\hat{\sigma}_L$ are consistent estimators of μ_L^* and σ_L^* , respectively, then under the same conditions as in Theorem 1 and the null hypothesis $H_{0L}: \xi^* = 0$, it holds that

$$\widehat{U}_n^L \stackrel{d}{\to} N(0,1).$$

6 Simulation Studies

In this section, we evaluate the empirical performance of the proposed methods. Although many models can be formulated as special cases of our problem (1.2), here we mainly consider the following binary response model

$$Y = \operatorname{sign}(X - \boldsymbol{\beta}^{*T} \boldsymbol{Z} + \epsilon), \tag{6.1}$$

where ϵ possibly depends on X and Z but the median of ϵ given X and Z is 0.

6.1 Experiments with pre-specified bandwidth

In the first set of experiments, we evaluate the performance of the proposed test statistic with pre-specified bandwidth. We use Gaussian kernel K of order 2 with bandwidth pre-specified at $\delta = n^{-1/5}$. The choices of other tuning parameters are detailed in Appendix D. Throughout this subsection, we consider sample size n = 800, dimension d = 100, 500, 1000 and generate $\beta_2^*, \ldots, \beta_s^*$ by sampling from a uniform distribution within [1,2] for s = 3,10. The first coordinate β_1^* would vary depending on the purpose of the experiment, and the rest coordinates of $\boldsymbol{\beta}^*$ are all set to 0. After that, the coefficient vector is then normalized such that $||\boldsymbol{\beta}^*||_2 = 1$. We generate $X \sim N(0,1)$ and $\mathbf{Z} \sim N(0, \mathbf{\Sigma}_{\rho})$, where $(\mathbf{\Sigma}_{\rho})_{jk} = \rho^{|j-k|}$ with $\rho = 0.2, 0.5, 0.7$. For all cases, the simulations are repeated 250 times.

In the first scenario, we let $\epsilon \sim N(0, 0.2^2(1 + 2(X - \boldsymbol{\beta}^{*T}\boldsymbol{Z})^2))$, which is referred to as Heteroskedastic Gaussian scenario later on. We compare the proposed smoothed decorrelated score test (SDS) with the decorrelated score test method (DS) (Ning et al., 2017) and Honest confidence region method (Honest) (Belloni et al., 2016) from the "hdm" package. We fix the significance level

at 0.05 and firstly evaluate the performance of the tests under the null hypothesis $H_0: \beta_1^* = 0$. In this case, we set $\beta_1^* = 0$. Note that the R code for the DS and Honest approaches is tailored for the high-dimensional logistic regression, which differs from the above data generating process.

Table 6.1 reports the empirical Type I error rate under the first scenario. The error rate from the SDS method is generally close to the nominal significance level 0.05, which empirically verifies the theoretical results in Theorem 1. For both Honest and DS methods, the empirical Type I error rate seems to be consistently higher or lower than the nominal level. This is expected as these two methods only work for the logistic regression. By taking a closer look at the Normal Q-Q plot of the test statistics, we observe that the distribution of the test statistics from the Honest and DS methods deviate substantially from Gaussian, as opposed to those yield by the proposed SDS method. Please see Appendix E.2 for more details.

Table 6.1: The empirical Type I error rate of the tests under the Heteroskedastic Gaussian scenario from SDS, DS and Honest methods.

| | | | s = 3 | | | s = 10 | |
|------|--------|--------------|--------------|--------------|--------------|--------------|--------------|
| d | method | $\rho = 0.2$ | $\rho = 0.5$ | $\rho = 0.7$ | $\rho = 0.2$ | $\rho = 0.5$ | $\rho = 0.7$ |
| 100 | SDS | 5.6% | 5.0% | 6.4% | 4.8% | 4.8% | 5.2% |
| | DS | 1.2% | 2.0% | 2.0% | 2.0% | 1.8% | 1.8% |
| | Honest | 5.2% | 5.6% | 7.6% | 5.4% | 5.2% | 6.8% |
| 500 | SDS | 4.8% | 4.4% | 5.6% | 5.6% | 5.0% | 4.8% |
| | DS | 0.2% | 0.4% | 0.4% | 0.2% | 0.0% | 0.4% |
| | Honest | 7.0% | 10.8% | 7.6% | 8.2% | 6.8% | 7.2% |
| 1000 | SDS | 4.4% | 6.0% | 5.6% | 5.0% | 5.4% | 5.0% |
| | DS | 0.0% | 0.4% | 0.2% | 0.0% | 0.0% | 0.4% |
| | Honest | 10.0% | 10.4% | 12.6% | 12.4% | 6.4% | 15.2% |

In the second scenario, we let $\epsilon \sim 0.2 \cdot \text{Unif}(-G(X, \mathbf{Z}), G(X, \mathbf{Z}))$, where $G(x, \mathbf{z}) = \sqrt{1 + 2(x - \boldsymbol{\beta}^{*T} \mathbf{z})^2}$. In other words, the error ϵ follows a uniform distribution such that its range depends on the covariates X, \mathbf{Z} (we will call it Heteroskedastic Uniform scenario). Similar to the Heteroskedastic Gaussian case, we compare SDS method with DS and Honest methods and study the empirical Type I error rate. From Table 6.2 we can see that the proposed method yields Type I error close to the nominal level as opposed to the other two. The Normal QQ-plots in Figure E.1 in Appendix E.2 further confirm the asymptotic normality of our SDS test statistics. The above results suggest that in practice, if the underlying data generating process is the binary response model, our proposed approach provides valid inferential results while the existing approaches fail.

Next, we investigate the empirical power of the SDS method. We use the same data gen-

Table 6.2: The empirical Type I error rate of the tests under the Heteroskedastic Uniform scenario from SDS, DS and Honest methods.

| | | | s = 3 | | | s = 10 | |
|------|--------|--------------|--------------|--------------|--------------|--------------|--------------|
| d | method | $\rho = 0.2$ | $\rho = 0.5$ | $\rho = 0.7$ | $\rho = 0.2$ | $\rho = 0.5$ | $\rho = 0.7$ |
| 100 | SDS | 6.0% | 5.6% | 6.8% | 7.2% | 6.8% | 7.2% |
| | DS | 2.0% | 0.4% | 0.8% | 1.6% | 0.8% | 1.2% |
| | Honest | 9.6% | 17.6% | 19.2% | 9.6% | 18.4% | 21.6% |
| 500 | SDS | 6.4% | 6.8% | 6.4% | 6.0% | 7.6% | 7.2% |
| | DS | 1.6% | 0.8% | 0.0% | 0.8% | 1.2% | 0.4% |
| | Honest | 11.2% | 15.6% | 20.0% | 13.6% | 17.2% | 22.8% |
| 1000 | SDS | 8.4% | 7.6% | 8.8% | 9.2% | 7.6% | 8.0% |
| | DS | 0.0% | 0.4% | 0.4% | 1.2% | 0.8% | 0.0% |
| | Honest | 13.6% | 15.2% | 22.4% | 16.0% | 19.2% | 20.8% |

erating processes as in the above two scenarios, but instead of setting $\beta_1^* = 0$, we vary β_1^* in the grid $\{0.02, 0.05, 0.075, 0.10, 0.15, 0.20, 0.25, 0.30\}$ for the Heteroskedastic Gaussian case, and $\{0.025, 0.05, 0.075, 0.10, 0.125, 0.15, 0.175\}$ for the Heteroskedastic Uniform case. Similarly, we consider s = 3, 10, d = 100, 500, 1000 and $\rho = 0.2, 0.5, 0.7$. Figure 6.1 shows the empirical rejection rate of the SDS method when s = 10 (see Appendix E.3 for the results when s = 3). Note that we do not compare with the DS and Honest methods for the empirical power, because these two tests do not maintain the desired Type I error in our scenarios. We can see that for all considered cases, the empirical power converges to 1 as the magnitude of the signal β_1^* becomes larger, which agrees with Theorem 2. In addition, we find that the dimension d has minor effects on the empirical power, which is reasonable as Theorem 2 only depends on log d via the condition (3.8). Finally, we note that the power of the test deteriorates as the correlation of the design increases.

6.2 Experiments with data-driven bandwidth

In the next set of experiments, we study the empirical performance of the data-driven bandwidth selection approach. We firstly study the type I error and power of our SDS method for testing $H_0: \beta_1^* = 0$ versus $H_1: \beta_1^* \neq 0$ with data-driven bandwidth. We consider the same data generating processes as in Section 6.1 with n = 800, d = 100, s = 3, 10 and $\rho = 0.2, 0.5, 0.7$. We seek for the minimizer of the estimated MSE over $\delta \in [0.1, 1.2]$ and each experiment is repeated 250 times. After $\hat{\delta}$ is obtained, we plug-in it into the test statistic and estimate the bias and variance as discussed in Appendix D. With the same implementations, we also evaluate the empirical power of the test

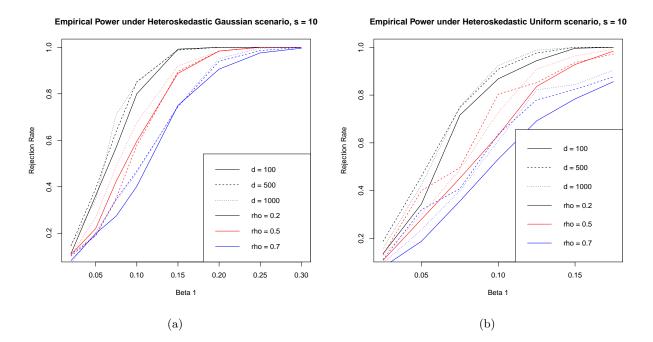


Figure 6.1: Empirical rejection rate of the proposed test under both scenarios with s = 10, d = 100, 500, 1000 and $\rho = 0.2, 0.5, 0.7$.

by varying β_1^* in the same grid as in Section 6.1.

Table 6.3: The empirical Type I error rate of the tests under the Heteroskedastic Gaussian and Uniform scenarios with data-driven bandwidth $\hat{\delta}$.

| | s=3 | | | s = 10 | | |
|--------------------------|--------------|--------------|--------------|--------------|--------------|--------------|
| Data generating process | $\rho = 0.2$ | $\rho = 0.5$ | $\rho = 0.7$ | $\rho = 0.2$ | $\rho = 0.5$ | $\rho = 0.7$ |
| Heteroskedastic Gaussian | 6.8% | 7.2% | 5.6% | 8.4% | 7.2% | 6.4% |
| Heteroskedastic Uniform | 8.8% | 8.0% | 7.6% | 8.4% | 6.8% | 5.2% |

Table 6.3 shows the empirical Type I error rate over 250 repetitions when $\beta_1^* = 0$, and Figure 6.2 shows the empirical power for different $\beta_1^* \neq 0$ in these scenarios. Similar to the case when the bandwidth δ is pre-specified, the empirical Type I errors are generally close to the nominal level 0.05, and the empirical power converges to 1 as β_1^* becomes larger.

In the Appendix E.4, we show that the data-driven bandwidth is indeed close to the theoretically optimal bandwidth that minimizes the MSE. From the above results, we recommend using the data-driven bandwidth selection approach in practice.

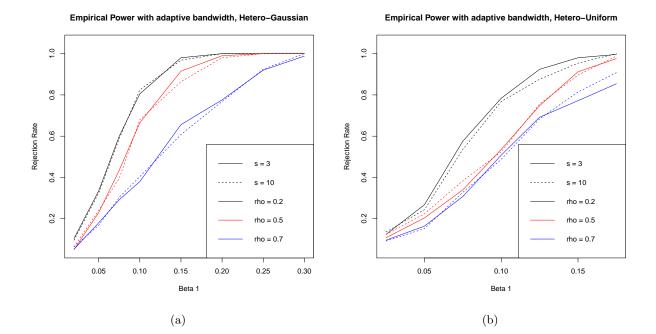


Figure 6.2: Empirical power of the tests under the Heteroskedastic Gaussian and Uniform scenarios with data-driven bandwidth $\hat{\delta}$.

7 Analysis of ChAMP Trial

In this section we analyze the ChAMP (Chondral Lesions And Meniscus Procedures) trial (Bisson et al., 2017), which contains clinical information about n=138 patients undergoing arthroscopic partial meniscectomy (APM), a knee surgery for meniscal tears. The response variable is Y=1 if the patient is healthy/satisfactory and -1 otherwise, obtained from the SF-36 survey. The continuous measurement X encodes the WOMAC pain score change from the baseline to one-year after the surgery. The dataset also contains d=160 additional variables from the patient's clinical profile, denoted by Z. The scientific question is to determine the iMCID, defined as a linear combination of the covariates $\beta^T Z$, such that the treatment of debriding chondral lesions can be claimed as clinically significant by comparing the WOMAC pain score change with this individualized threshold. As we can see, this application naturally fits into our formulation (1.2) with weight function $w(y) = 1/\mathbb{P}(Y=y)$. The goal of the analysis is to address this question by providing valid inferential results for each component of β .

We apply the proposed SDS test for $H_{0j}: \beta_j = 0$ versus $H_{1j}: \beta_j \neq 0$, where $1 \leq j \leq d$. We use the same tuning parameter setting for estimating β^*, ω^* and the asymptotic bias and variance of the score function following Appendix D. For comparison, we also apply the DS and Honest methods discussed in Section 6.1.

Table 7.1 lists the three significant variables (i.e., those with p-value smaller than 0.05/d = 3.125e-

Table 7.1: The three significant variables (with p-value < 0.05/d) identified by the proposed SDS method, and their corresponding p-values obtained from the DS and Honest methods.

| p-value | KQOL_6wk | flex_inj_pre | KSymp_3mo |
|---------|-----------|--------------|------------|
| SDS | 3.583e-07 | 5.575 e-05 | 4.482 e-05 |
| DS | 0.0168 | 0.0340 | 0.0213 |
| Honest | 0.0591 | 0.0241 | 0.4383 |

04) from the proposed SDS approach. Interestingly, all of them are clinically relevant and can provide meaningful implications for iMCID. The significance of the variable KQOL_6wk, which represents the KOOS score for quality of life at 6-week, definitely indicates how the patients recover at a relatively early stage after the surgery. The variable flex_inj_pre means the degree of flexion right before the surgery. Its significance recommends that the baseline disease severity would affect the magnitude of iMCID—this similar phenomenon was also discovered in the clinical literature for other types of diseases, such as the shoulder pain reduction study (Heald et al., 1997). The third variable KSymp_3mo is the KOOS score for other symptoms at 3-month. In some previous analysis of ChAMP trial where only estimate is available but without inference results, this variable has the second largest coefficient (Feng et al., 2019).

The results from the DS and the Honest methods are different. Firstly, the DS method only yields 1 significant variable and the Honest method yields 13. From Table 7.1, the three significant variables identified by the proposed SDS method cannot be identified by either DS or Honest. In general, compared to the proposed SDS method, DS identifies fewer significant variables while Honest identifies more. This phenomenon is also evident from the simulation results in Section 6.1. On the other hand, the results from the three methods do not completely contradict with each other. For instance, the two significant variables, KQOL_6wk and KSymp_3mo, identified from the proposed SDS method, has the fourth and fifth smallest p-values in the DS method. Please refer to Appendix E.5 for more results of this analysis.

In general, recall that DS and Honest methods are devised for the logistic regression, while our proposed SDS method can produce valid inference results under the binary response model (6.1), which is more flexible since the distribution of ϵ is left unspecified. Therefore, we expect that the significant variables identified by SDS are potentially more reliable and clinically more relevant. Our results presented in this session echo this rationale.

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Supplementary Appendix

A Discussions and Lemmas for the bandwidth selection

Lemma 2. Under Assumptions 1 - 6, if $\sqrt{\frac{\log(n\vee d)}{n\delta}}\vee M_n\eta_1(n)=o(1)$, and $M_n^2\sqrt{\log(n\vee d)/n^{\epsilon_1}}=\mathcal{O}(1)$, it holds that

$$|\widehat{V}(\delta) - V(\delta)| \lesssim \frac{\psi_1(n,\delta)}{\delta},$$
 (A.1)

uniformly over all $\delta \in \Delta$, where

$$\psi_1(n,\delta) = ||\boldsymbol{v}^*||_1^2 \left(\eta_2(n) \vee \sqrt{\frac{\log(n \vee d)}{n\delta}} \vee M_n \eta_1(n)\right). \tag{A.2}$$

Proof. See Section B.4 for a detailed proof.

Lemma 3. Suppose Assumptions 1- 7 hold. Choose J as a proper kernel function of order r satisfying the conditions of K in Assumption 2. In addition, assume J is ℓ times continuously differentiable and $J^{(i)}$ degenerates at the boundary for $i = 1, \ldots, \ell - 1$. If $M_n \leq C\sqrt{nb/\log(nd)}$ for some constant C and $\frac{\log(n \vee d)}{nb^{2\ell+1}} = o(1)$, then it holds that

$$|\widehat{SB}(\delta) - SB(\delta)| \lesssim \delta^{2\ell} \psi_2(n, \delta)$$
 (A.3)

uniformly over all $\delta \in \Delta$, where

$$\psi_2(n,\delta) = ||\boldsymbol{v}^*||_1^2 \left(\sqrt{\frac{\log(n \vee d)}{nb^{2\ell+1}}} \vee (\delta \vee b)^r \vee M_n \eta_1(n) (1 \vee \frac{M_n \eta_1(n)}{\delta^\ell}) \vee \eta_2(n) \right). \tag{A.4}$$

Proof. See Section B.5 for a detailed proof.

In view of (A.3) and (A.4), the error terms $\sqrt{\frac{\log(n\vee d)}{nb^{2\ell+1}}}$ and $(\delta\vee b)^r$ correspond to the variance and bias when estimating $A(\beta^*,\delta)$ in (4.4) with the extra smoothing step. This rate is similar to the case of estimating the ℓ th derivative of a density function using a kernel of order r in the context of kernel density estimation under the smoothness Assumption 7. The terms $M_n\eta_1(n)(1\vee\frac{M_n\eta_1(n)}{\delta^\ell})$ and $\eta_2(n)$ come from the plug-in error of $\widehat{\beta}$ and \widehat{v} , respectively.

B Proofs of Main Results

B.1 Proof of Lemma 1

Recall that

$$\nabla \bar{R}_{\delta}^{i}(\boldsymbol{\beta}^{*}) = w(y_{i}) \frac{y_{i} \boldsymbol{z}_{i}}{\delta} K\left(\frac{y_{i}(x_{i} - \boldsymbol{\beta}^{*T} \boldsymbol{z}_{i})}{\delta}\right). \tag{B.1}$$

Let $T_i = \frac{\sqrt{\delta} v^{*T} (\nabla \bar{R}_{\delta}^i(\boldsymbol{\beta}^*) - \nabla R_{\delta}(\boldsymbol{\beta}^*))}{\sqrt{v^{*T} \boldsymbol{\Sigma}^* v^*}}$, we know by definition $\mathbb{E} T_i = 0$. Consider $Var[\boldsymbol{v}^{*T} \nabla \bar{R}_{\delta}^i(\boldsymbol{\beta}^*)] = \mathbb{E}[(\boldsymbol{v}^{*T} \nabla \bar{R}_{\delta}^i(\boldsymbol{\beta}^*))^2] - (\boldsymbol{v}^{*T} \nabla R_{\delta}(\boldsymbol{\beta}^*))^2. \tag{B.2}$

Here

$$\mathbb{E}[(\boldsymbol{v}^{*T}\nabla\bar{R}_{\delta}^{i}(\boldsymbol{\beta}^{*}))^{2}] = \sum_{y\in\{-1,1\}} w(y)^{2} \int \frac{(\boldsymbol{v}^{*T}\boldsymbol{z})^{2}}{\delta^{2}} \int K^{2}(\frac{\boldsymbol{x}-\boldsymbol{\beta}^{*T}\boldsymbol{z}}{\delta})f(\boldsymbol{x}|\boldsymbol{y},\boldsymbol{z})d\boldsymbol{x}f(\boldsymbol{y},\boldsymbol{z})d\boldsymbol{z}$$

$$= \sum_{y\in\{-1,1\}} w(y)^{2} \int \frac{(\boldsymbol{v}^{*T}\boldsymbol{z})^{2}}{\delta} \int K^{2}(\boldsymbol{u})f(\boldsymbol{u}\delta + \boldsymbol{\beta}^{*T}\boldsymbol{z}|\boldsymbol{y},\boldsymbol{z})d\boldsymbol{u}f(\boldsymbol{y},\boldsymbol{z})d\boldsymbol{z}$$

$$= \frac{1}{\delta} \sum_{y\in\{-1,1\}} w(y)^{2} \int (\boldsymbol{v}^{*T}\boldsymbol{z})^{2} \int K^{2}(\boldsymbol{u})(f(\boldsymbol{\beta}^{*T}\boldsymbol{z}|\boldsymbol{y},\boldsymbol{z}) + \boldsymbol{u}\delta f'(\boldsymbol{\beta}^{*T}\boldsymbol{z}|\boldsymbol{y},\boldsymbol{z}) + o(\delta))d\boldsymbol{u}f(\boldsymbol{y},\boldsymbol{z})d\boldsymbol{z}$$

$$= \frac{1}{\delta}(\boldsymbol{v}^{*T}\boldsymbol{\Sigma}^{*}\boldsymbol{v}^{*}(1+o(1))),$$
(B.3)

where the second equality is due to a change of variable, and the third equality is due to Assumption 1. Meanwhile, we know

$$\boldsymbol{v}^{*T} \nabla R_{\delta}(\boldsymbol{\beta}^{*}) = \sum_{y \in \{-1,1\}} w(y)y \int \frac{(\boldsymbol{v}^{*T}\boldsymbol{z})}{\delta} \int K(\frac{x - \boldsymbol{\beta}^{*T}\boldsymbol{z}}{\delta}) f(x|y, \boldsymbol{z}) dx f(y, \boldsymbol{z}) d\boldsymbol{z}$$

$$= \sum_{y \in \{-1,1\}} w(y)y \int (\boldsymbol{v}^{*T}\boldsymbol{z}) \int K(u) f(u\delta + \boldsymbol{\beta}^{*T}\boldsymbol{z}|y, \boldsymbol{z}) du f(y, \boldsymbol{z}) d\boldsymbol{z}$$

$$= \sum_{y \in \{-1,1\}} w(y)y \int (\boldsymbol{v}^{*T}\boldsymbol{z}) \int K(u) \frac{(u\delta)^{\ell}}{\ell!} \left(f^{(\ell)}(\boldsymbol{\beta}^{*T}\boldsymbol{z}|y, \boldsymbol{z}) + \mathcal{O}((u\delta)^{\zeta}) \right) du f(y, \boldsymbol{z}) d\boldsymbol{z}$$

$$= \delta^{\ell} \boldsymbol{v}^{*T} \boldsymbol{b}^{*} (1 + o(1)). \tag{B.4}$$

This together with Assumption 5 implies that $Var[\boldsymbol{v}^{*T}\nabla R_{\delta}^{i}(\boldsymbol{\beta}^{*})] = \frac{1}{\delta}\boldsymbol{v}^{*T}\boldsymbol{\Sigma}^{*}\boldsymbol{v}^{*}(1+o(1))$ and therefore $Var(T_{i}) = 1 + o(1)$. Now we verify the Lyapunov condition

$$\frac{1}{n^{3/2}} \sum_{i}^{n} \mathbb{E}|T_{i}|^{3} = \frac{1}{n^{3/2}} \sum_{i}^{n} \mathbb{E}\left|\frac{\sqrt{\delta} \boldsymbol{v}^{*T} (\nabla \bar{R}_{\delta}^{i}(\boldsymbol{\beta}^{*}) - \nabla R_{\delta}(\boldsymbol{\beta}^{*}))}{\sqrt{\boldsymbol{v}^{*T} \boldsymbol{\Sigma}^{*} \boldsymbol{v}^{*}}}\right|^{3}.$$
 (B.5)

By Assumption 5,

$$\mathbb{E} \left| \frac{\sqrt{\delta} \boldsymbol{v}^{*T} (\nabla \bar{R}_{\delta}^{i}(\boldsymbol{\beta}^{*}) - \nabla R_{\delta}(\boldsymbol{\beta}^{*}))}{\sqrt{\boldsymbol{v}^{*T} \boldsymbol{\Sigma}^{*} \boldsymbol{v}^{*}}} \right|^{3} \lesssim \delta^{3/2} \mathbb{E} \left| \boldsymbol{v}^{*T} (\nabla \bar{R}_{\delta}^{i}(\boldsymbol{\beta}^{*}) - \nabla R_{\delta}(\boldsymbol{\beta}^{*})) \right|^{3} \\
\lesssim \delta^{3/2} (\mathbb{E} |\boldsymbol{v}^{*T} \nabla \bar{R}_{\delta}^{i}(\boldsymbol{\beta}^{*})|^{3} + |\boldsymbol{v}^{*T} \nabla R_{\delta}(\boldsymbol{\beta}^{*})|^{3}), \tag{B.6}$$

where we can show that

$$\mathbb{E}|\boldsymbol{v}^{*T}\nabla\bar{R}_{\delta}^{i}(\boldsymbol{\beta}^{*})|^{3} \leq ||\boldsymbol{v}^{*}||_{1}^{3} \sum_{y\in\{-1,1\}} \int \delta \max_{j} \left|\frac{y\boldsymbol{z}_{j}}{\delta}K(u)\right|^{3} f(u\delta + \boldsymbol{\beta}^{*T}\boldsymbol{z}, y, \boldsymbol{z}) dud\boldsymbol{z}$$

$$\lesssim ||\boldsymbol{v}^{*}||_{1}^{3} M_{n}^{3}/\delta^{2}, \tag{B.7}$$

and from Lemma 4 it's easy to see that the first term on the RHS of (B.6) is dominant. This implies that

$$\frac{1}{n^{3/2}} \sum_{i=1}^{n} \mathbb{E}|T_{i}|^{3} = \mathcal{O}((||\boldsymbol{v}^{*}||_{1}M_{n})^{3}/(n\delta)^{1/2}).$$
 (B.8)

Therefore, under the condition of this Lemma the Lyapunov condition holds, which completes the proof by applying Lindeberg Feller Central Limit Theorem.

B.2 Proof of Theorem 1

It suffices to show that $(n\delta)^{1/2}|\widehat{S}_{\delta}(\widehat{\beta}_0) - S_{\delta}(\beta^*)| = o_{\mathbb{P}}(1)$ where $\widehat{\beta}_0 = (0, \widehat{\gamma})$. Here we only show that $(n\delta)^{1/2}|\widehat{S}_{\delta}^{(1)}(\widehat{\beta}_0^{(2)}) - S_{\delta}(\beta^*)| = o_{\mathbb{P}}(1)$ and the desired result shall follow naturally from Lemma 1. By definition, we have

$$(n\delta)^{1/2} |\widehat{S}_{\delta}^{(1)}(\widehat{\boldsymbol{\beta}}_{0}^{(2)}) - S_{\delta}(\boldsymbol{\beta}^{*})|$$

$$= (n\delta)^{1/2} |\widehat{\boldsymbol{v}}^{(1)T} \nabla R_{\delta}^{n_{(1)}}(\widehat{\boldsymbol{\beta}}_{0}^{(2)}) - \boldsymbol{v}^{*T} \nabla R_{\delta}^{n_{(1)}}(\boldsymbol{\beta}^{*})|$$

$$\leq (n\delta)^{1/2} |\boldsymbol{v}^{*T}(\nabla R_{\delta}^{n_{(1)}}(\widehat{\boldsymbol{\beta}}_{0}^{(2)}) - \nabla R_{\delta}^{n_{(1)}}(\boldsymbol{\beta}^{*}))| + (n\delta)^{1/2} |(\widehat{\boldsymbol{v}}^{(1)} - \boldsymbol{v}^{*}) \nabla R_{\delta}^{n_{(1)}}(\widehat{\boldsymbol{\beta}}_{0}^{(2)})|$$

$$:= I_{1} + I_{2}. \tag{B.9}$$

The fact that $\mathbf{v}^{*T}\nabla^2_{\boldsymbol{\cdot}\boldsymbol{\gamma}}R(\boldsymbol{\beta}^*)=0$ and Lemma 7 implies that

$$I_{1} = (n\delta)^{1/2} |\mathbf{v}^{*T} \nabla_{\cdot \gamma}^{2} R_{\delta}^{n_{(1)}}(\beta^{*}) (\widehat{\boldsymbol{\beta}}^{(2)} - \boldsymbol{\beta}^{*})| + o_{\mathbb{P}}(1)$$

$$\leq (n\delta)^{1/2} ||\mathbf{v}^{*}||_{1} \left(||\nabla_{\cdot \gamma}^{2} R_{\delta}^{n_{(1)}}(\beta^{*}) - \nabla_{\cdot \gamma}^{2} R_{\delta}(\beta^{*})||_{\max} + ||\nabla_{\cdot \gamma}^{2} R_{\delta}(\beta^{*}) - \nabla_{\cdot \gamma}^{2} R(\beta^{*})||_{\max} \right) ||\widehat{\boldsymbol{\beta}}^{(2)} - \boldsymbol{\beta}^{*}||_{1} + o_{\mathbb{P}}(1)$$

$$\lesssim (n\delta)^{1/2} ||\mathbf{v}^{*}||_{1} \frac{\eta_{1}(n)}{\delta} \left(\sqrt{\frac{\log(d)}{n\delta}} + \delta^{\ell} \right) + o_{\mathbb{P}}(1)$$

$$= o_{\mathbb{P}}(1). \tag{B.10}$$

Similarly, since $\nabla R(\boldsymbol{\beta}^*) = 0$, imply that

$$I_{2} = (n\delta)^{1/2} |(\widehat{\boldsymbol{v}}^{(1)} - \boldsymbol{v}^{*})(\nabla R_{\delta}^{n_{(1)}}(\widehat{\boldsymbol{\beta}}_{0}^{(2)}) - \nabla R(\boldsymbol{\beta}^{*}))|$$

$$\leq (n\delta)^{1/2} ||\widehat{\boldsymbol{v}}^{(1)} - \boldsymbol{v}^{*}||_{1} ||\nabla R_{\delta}^{n_{(1)}}(\widehat{\boldsymbol{\beta}}_{0}^{(2)}) - \nabla R(\boldsymbol{\beta}^{*})||_{\infty}$$

$$\leq (n\delta)^{1/2} ||\widehat{\boldsymbol{v}}^{(1)} - \boldsymbol{v}^{*}||_{1} \left(||\nabla R_{\delta}^{n_{(1)}}(\widehat{\boldsymbol{\beta}}_{0}^{(2)}) - \nabla R_{\delta}(\widehat{\boldsymbol{\beta}}_{0}^{(2)})||_{\infty} + ||\nabla R_{\delta}(\widehat{\boldsymbol{\beta}}_{0}^{(2)}) - \nabla R(\widehat{\boldsymbol{\beta}}_{0}^{(2)})||_{\infty} + ||\nabla R(\widehat{\boldsymbol{\beta}}_{0}^{(2)}) - \nabla R(\boldsymbol{\beta}^{*})||_{\infty} \right). \tag{B.11}$$

Since $\widehat{\beta}_0^{(2)}$ depends on the set of samples that is disjoint with \mathcal{N}_1 , Lemma 5 together with Lemma 13 implies that

$$||\nabla R_{\delta}^{n_{(1)}}(\widehat{\beta}_0^{(2)}) - \nabla R_{\delta}(\widehat{\beta}_0^{(2)})||_{\infty} = \mathcal{O}_{\mathbb{P}}(\sqrt{\frac{\log(d)}{n\delta}}).$$

This in combine with Lemma 4 and 6 further implies that

$$I_2 \lesssim (n\delta)^{1/2} ||\boldsymbol{v}^*||_1 \eta_2(n) \left(M_n \eta_1(n) + \sqrt{\frac{\log(d)}{n\delta}} + \delta^{\ell} \right) = o_{\mathbb{P}}(1).$$

Putting all above together with the results from Lemma 11 and Lemma 12 as well as the Slutsky's theorem, we obtain that the bias corrected decorrelated score statistic $\hat{U}_n \stackrel{d}{\to} N(0,1)$ under the null hypothesis.

B.3 Proof of Theorem 2

Recall that
$$\widehat{S}_{\delta}(0,\widehat{\gamma}) = \frac{1}{2} (\widehat{S}_{\delta}^{(1)}(0,\widehat{\gamma}^{(2)}) + \widehat{S}_{\delta}^{(2)}(0,\widehat{\gamma}^{(1)}))$$
. Let's focus on $\widehat{S}_{\delta}^{(1)}(0,\widehat{\gamma}^{(2)})$. By definition
$$\widehat{S}^{(1)}(0,\widehat{\gamma}^{(2)}) = \widehat{\mathbf{v}}^{(1)T} \nabla R_{\delta}^{n_{(1)}}(0,\widehat{\gamma}^{(2)}) = \underbrace{(\widehat{\mathbf{v}}^{(1)} - \mathbf{v}^*)^T \nabla R_{\delta}^{n_{(1)}}(0,\widehat{\gamma}^{(2)})}_{I_1} + \mathbf{v}^{*T} \nabla R_{\delta}^{n_{(1)}}(0,\widehat{\gamma}^{(2)}) + \mathbf{v}^{*T} \nabla R_{\delta}^{n_{(1)}}(0,\widehat{\gamma}^{(2)}) = \underbrace{(\mathbf{v}^{(1)} - \mathbf{v}^*)^T \nabla R_{\delta}^{n_{(1)}}(0,\widehat{\gamma}^{(2)})}_{I_2} + \mathbf{v}^{*T} \nabla R_{\delta}^{n_{(1)}}(0,\widehat{\gamma}^{(2)}) + \mathbf{v}^{*T} \nabla R_{\delta}^{n_{(1)}}(0,\widehat{\gamma}^*) = \underbrace{I_1 + \underbrace{\mathbf{v}^{*T} (\nabla R_{\delta}^{n_{(1)}}(0,\widehat{\gamma}^{(2)}) - \nabla R_{\delta}^{n_{(1)}}(0,\widehat{\gamma}^*))}_{I_2} + \mathbf{v}^{*T} \nabla R_{\delta}^{n_{(1)}}(0,\widehat{\gamma}^*) + \mathbf{v}^{*T} \nabla R_{\delta}^{n_{(1)}}(0,\widehat{\gamma}^*) - \underbrace{\nabla R_{\delta}^{n_{(1)}}(\beta^*) + \theta^* \mathbf{v}^{*T} \nabla_{\cdot \theta}^2 R(\beta^*)}_{I_3} = S^{(1)}(\beta^*) - \theta^* \mathbf{v}^{*T} \nabla_{\cdot \theta}^2 R(\beta^*) + I_1 + I_2 + I_3.$$
(B.12)

For I_1 , we know

$$I_1 \leq ||\widehat{\boldsymbol{v}}^{(1)} - \boldsymbol{v}^*||_1 ||\nabla R_{\delta}^n(0, \widehat{\boldsymbol{\gamma}})||_{\infty} = \mathcal{O}_{\mathbb{P}}(||\boldsymbol{v}^*||_1 \eta_2(n)(\sqrt{\log(d)/(n\delta)} + \delta^{\ell} + M_n(\theta^* + \eta_1(n)))) = o_{\mathbb{P}}((n\delta)^{-1/2}).$$

Following a similar proof of Lemma 7, we can show that

$$I_2 = \mathcal{O}_{\mathbb{P}}(||v^*||_1\eta_1(n)[\delta^{\ell-1} + \sqrt{\log(d)/(n\delta^3)} + M_n\theta^*]) = o_{\mathbb{P}}((n\delta)^{-1/2}).$$

Finally, for I_3 , since by definition

$$\mathbf{v}^{*T} \nabla^{2}_{\cdot \theta} R(\boldsymbol{\beta}^{*}) = \nabla^{2}_{\theta | \boldsymbol{\gamma}} R(\boldsymbol{\beta}^{*}), \tag{B.13}$$

Lemma 9 implies that $I_3 = o_{\mathbb{P}}((n\delta)^{-1/2})$. Put all pieces together, we have shown that

$$|\widehat{S}^{(1)}(0,\widehat{\gamma}^{(2)}) - (S^{(1)}(\beta^*) - \theta^* \nabla^2_{\theta|\gamma} R(\beta^*))| = o_{\mathbb{P}}((n\delta)^{-1/2}).$$

A similar result will also hold for $\widehat{S}^{(2)}(0,\widehat{\gamma}^{(1)})$, and thus we conclude

$$|\widehat{S}(0,\widehat{\gamma}) - (S(\beta^*) - \theta^* \nabla^2_{\theta|\gamma} R(\beta^*))| = o_{\mathbb{P}}((n\delta)^{-1/2}).$$
(B.14)

At this point the conclusion of (3.9) is shown. To show (3.10), the above formula also implies that

$$\mathbb{P}(|(n\delta)^{1/2}\widehat{S}(0,\widehat{\gamma})/\sigma^*| \le t) \le \mathbb{P}(L(n) \le (n\delta)^{1/2}S(\beta^*)/\sigma^* \le U(n)) + o(1), \tag{B.15}$$

where $L(n) = -t - q(n) + \widetilde{C}\sigma^{*-1}\nabla^2_{\theta|\gamma}R(\boldsymbol{\beta}^*)n^{-\phi}(n\delta)^{1/2}$, $U(n) = t + q(n) + \widetilde{C}\sigma^{*-1}\nabla^2_{\theta|\gamma}R(\boldsymbol{\beta}^*)n^{-\phi}(n\delta)^{1/2}$, and q(n) = o(1) is some deterministic sequence. Since $n^{-\phi}(n\delta)^{1/2}\nabla^2_{\theta|\gamma}R(\boldsymbol{\beta}^*)\sigma^{*-1} \to \infty$, it is easily seen that $\mathbb{P}(|(n\delta)^{1/2}\widehat{S}(0,\widehat{\gamma})/\sigma^*| \leq t) \to 0$. Since $\widehat{\sigma}$ is consistent, for n large enough we will get $|\widehat{\sigma}/\sigma - 1| \leq 3$, which finally implies the desired result. This completes the proof.

B.4 Proof of Lemma 2

It suffices to show the rate for $\widehat{\boldsymbol{v}}^{(1)T}\widehat{\boldsymbol{\Gamma}}^{(1)}(\delta)\widehat{\boldsymbol{v}}^{(1)}$. Let $\widetilde{\boldsymbol{\Gamma}}(\delta) = \mathbb{E}\left[\nabla \bar{R}_{\delta}^{1}(\widehat{\boldsymbol{\beta}}^{(2)})\nabla \bar{R}_{\delta}^{1}(\widehat{\boldsymbol{\beta}}^{(2)})^{T}\right]$, and $\boldsymbol{\Gamma}(\delta) = \mathbb{E}\left[\nabla \bar{R}_{\delta}^{1}(\boldsymbol{\beta}^{*})\nabla \bar{R}_{\delta}^{1}(\boldsymbol{\beta}^{*})^{T}\right]$. We firstly look at

$$||\widehat{\mathbf{\Gamma}}^{(1)}(\delta) - \mathbf{\Gamma}(\delta)||_{\max} \le ||\widehat{\mathbf{\Gamma}}^{(1)}(\delta) - \widetilde{\mathbf{\Gamma}}(\delta)||_{\max} + ||\widetilde{\mathbf{\Gamma}}(\delta) - \mathbf{\Gamma}(\delta)||_{\max}.$$
(B.16)

Now we bound the first term. For any $\delta \in \Delta$, we can always find a $\widetilde{\delta} \in \widetilde{\Delta}$ such that $|\delta - \widetilde{\delta}| \lesssim n^{-\rho}$, where ρ is some constant that can be arbitrarily large and the set $\widetilde{\Delta}$ that is a subset of Δ has cardinality of order $\mathcal{O}(n^{\rho})$. For such pair $(\delta, \widetilde{\delta})$, we can show that

$$\sqrt{n\delta^{3}}||\widehat{\mathbf{\Gamma}}^{(1)}(\delta) - \widetilde{\mathbf{\Gamma}}(\delta)||_{\max}
\lesssim \sqrt{n\delta^{3}} \left(||\widehat{\mathbf{\Gamma}}^{(1)}(\widetilde{\delta}) - \widetilde{\mathbf{\Gamma}}(\widetilde{\delta})||_{\max} + ||\widehat{\mathbf{\Gamma}}^{(1)}(\delta) - \widehat{\mathbf{\Gamma}}^{(1)}(\widetilde{\delta})||_{\max} + ||\widetilde{\mathbf{\Gamma}}(\widetilde{\delta}) - \widetilde{\mathbf{\Gamma}}(\delta)||_{\max}\right)
\lesssim \sqrt{n\delta^{3}}||\widehat{\mathbf{\Gamma}}^{(1)}(\widetilde{\delta}) - \widetilde{\mathbf{\Gamma}}(\widetilde{\delta})||_{\max} + \mathcal{O}(n^{-1/2}),$$
(B.17)

where the first inequality is by taking ρ large enough. To see last step of above, recall that under Assumptions 2, 3 and 4

$$||\widehat{\boldsymbol{\Gamma}}^{(1)}(\delta) - \widehat{\boldsymbol{\Gamma}}^{(1)}(\widetilde{\delta})||_{\max} = \max_{j,k} \left| \frac{1}{|\mathcal{N}_1|} \sum_{i \in \mathcal{N}_1} w(y_i)^2 \boldsymbol{z}_{ij} \boldsymbol{z}_{ik} \left[\frac{K^2((x_i - \widehat{\boldsymbol{\beta}}^{(2)T} \boldsymbol{z}_i)/\delta)}{\delta^2} - \frac{K^2((x_i - \widehat{\boldsymbol{\beta}}^{(2)T} \boldsymbol{z}_i)/\widetilde{\delta})}{\widetilde{\delta}^2} \right] \right| \lesssim M_n^2 (\frac{1}{\delta^2} - \frac{1}{\widetilde{\delta}^2}).$$
(P.18)

Thus, taking ρ large enough will ensure $\sqrt{n\widetilde{\delta}^3}(1+o(1))||\widehat{\boldsymbol{\Gamma}}^{(1)}(\delta)-\widehat{\boldsymbol{\Gamma}}^{(1)}(\widetilde{\delta})||_{\max}=\mathcal{O}(n^{-1/2})$. With a similar derivation we will also obtain $\sqrt{n\widetilde{\delta}^3}(1+o(1))||\widetilde{\boldsymbol{\Gamma}}(\delta)-\widetilde{\boldsymbol{\Gamma}}(\widetilde{\delta})||_{\max}=\mathcal{O}(n^{-1/2})$, and thus (B.17) holds.

Now we start to bound $\max_{\widetilde{\delta} \in \widetilde{\Delta}} \sqrt{n\widetilde{\delta}^3} ||\widehat{\Gamma}^{(1)}(\widetilde{\delta}) - \widetilde{\Gamma}(\widetilde{\delta})||_{\max}$. Since

$$|\widehat{\mathbf{\Gamma}}_{ijk}^{(1)}(\widetilde{\delta})| = |(\nabla \bar{R}_{\widetilde{\delta}}^{i}(\widehat{\boldsymbol{\beta}}^{(2)})\nabla \bar{R}_{\widetilde{\delta}}^{i}(\widehat{\boldsymbol{\beta}}^{(2)})^{T})_{jk}| \lesssim \frac{M_{n}^{2}}{\widetilde{\delta}^{2}}$$

and

$$Var[(\nabla \bar{R}^{i}_{\widetilde{\delta}}(\widehat{\boldsymbol{\beta}}^{(2)})\nabla \bar{R}^{i}_{\widetilde{\delta}}(\widehat{\boldsymbol{\beta}}^{(2)})^{T})_{jk}] \lesssim \frac{1}{\widetilde{\delta^{3}}},$$

by applying Bernstein inequality similar to the proof of Lemma 5 , conditioned on $\widehat{m{\beta}}^{(2)}$ we can show that

$$\mathbb{P}_{\widehat{\boldsymbol{\beta}}^{(2)}}(\max_{\widetilde{\delta} \in \widetilde{\Delta}} \sqrt{n\widetilde{\delta}^{3}} || \widehat{\boldsymbol{\Gamma}}^{(1)}(\widetilde{\delta}) - \widetilde{\boldsymbol{\Gamma}}(\widetilde{\delta}) ||_{\max} > t)$$

$$\leq \sum_{\widetilde{\delta} \in \widetilde{\Delta}} \sum_{j=1}^{d} \sum_{k=1}^{d} \mathbb{P}_{\widehat{\boldsymbol{\beta}}^{(2)}}(\sqrt{n\widetilde{\delta}^{3}} |\widehat{\boldsymbol{\Gamma}}^{(1)}(\widetilde{\delta}) - \widetilde{\boldsymbol{\Gamma}}(\widetilde{\delta}) |_{jk} > t)$$

$$\lesssim n^{\rho} d^{2} \exp\left(-\frac{\frac{1}{2}t^{2}/\widetilde{\delta}^{3}}{\frac{c_{0}}{\widetilde{\delta}^{3}} + \frac{c_{1}M_{n}^{2}t}{3\widetilde{\delta}^{3}\sqrt{n\widetilde{\delta}}}}\right), \tag{B.19}$$

where $c_0, c_1 > 0$ are some constants. By taking $t = c_2 \sqrt{\log(nd)}$ for some constant c_2 large enough, we obtain that conditioned on $\widehat{\beta}^{(2)}$, with probability greater than $1 - \mathcal{O}((nd)^{-1})$

$$||\widehat{\mathbf{\Gamma}}^{(1)}(\widetilde{\delta}) - \widetilde{\mathbf{\Gamma}}(\widetilde{\delta})||_{\max} \lesssim \sqrt{\frac{\log(nd)}{n\widetilde{\delta}^3}}$$
(B.20)

uniformly over $\widetilde{\delta} \in \widetilde{\Delta}$. This combines with Lemma 13 further implies that with probability approaching to 1

$$||\widehat{\mathbf{\Gamma}}^{(1)}(\delta) - \widetilde{\mathbf{\Gamma}}(\delta)||_{\max} \lesssim \sqrt{\frac{\log(nd)}{n\delta^3}}$$
 (B.21)

uniformly over $\delta \in \Delta$.

For the second term on the RHS of (B.16), similar to Lemma 6, we can show that

$$||\widetilde{\Gamma}(\delta) - \Gamma(\delta)||_{\max} \lesssim \frac{M_n \eta_1(n)}{\delta}.$$
 (B.22)

This implies that

$$||\widehat{\Gamma}(\delta) - \Gamma(\delta)||_{\max} = \frac{1}{\delta} \mathcal{O}_{\mathbb{P}}(\sqrt{\frac{\log(nd)}{n\delta}} \vee M_n \eta_1(n)).$$
(B.23)

Meanwhile, since

$$||\delta \cdot \mathbf{\Gamma}(\delta)||_{\max} = \max_{i,j} \left| \sum_{y} w(y)^{2} \int \frac{z_{i}z_{j}}{\delta} \int K^{2} \left(\frac{y(x - \boldsymbol{\beta}^{*T} \boldsymbol{z})}{\delta} \right) f(x|y, \boldsymbol{z}) dx f(y, \boldsymbol{z}) d\boldsymbol{z} \right|$$

$$= \mathcal{O}(1), \tag{B.24}$$

under the condition that $\sqrt{\frac{\log(nd)}{n\delta}} \vee M_n \eta_1(n) = o(1)$, we know $||\delta \widehat{\Gamma}(\delta)||_{\infty} = \mathcal{O}_{\mathbb{P}}(1)$. Finally, by triangle inequality

$$|\widehat{\boldsymbol{v}}^{(1)T}\widehat{\boldsymbol{\Gamma}}^{(1)}(\delta)\widehat{\boldsymbol{v}}^{(1)} - \boldsymbol{v}^{*T}\boldsymbol{\Gamma}(\delta)\boldsymbol{v}^{*}|$$

$$\leq ||\widehat{\boldsymbol{v}}^{(1)} - \boldsymbol{v}^{*}||_{1}||\widehat{\boldsymbol{\Gamma}}^{(1)}(\delta)(\widehat{\boldsymbol{v}}^{(1)} - \boldsymbol{v})||_{\infty} + 2||\boldsymbol{v}^{*T}\widehat{\boldsymbol{\Gamma}}^{(1)}(\delta)||_{\infty}||\widehat{\boldsymbol{v}}^{(1)} - \boldsymbol{v}^{*}||_{1} + |\boldsymbol{v}^{*T}(\widehat{\boldsymbol{\Gamma}}^{(1)}(\delta) - \boldsymbol{\Gamma}(\delta))\boldsymbol{v}^{*}|,$$
(B.25)

which further implies that with probability approaching 1

$$|\widehat{\boldsymbol{v}}^{(1)T}\widehat{\boldsymbol{\Gamma}}^{(1)}(\delta)\widehat{\boldsymbol{v}}^{(1)} - \boldsymbol{v}^{*T}\boldsymbol{\Gamma}(\delta)\boldsymbol{v}^*| \lesssim \frac{||\boldsymbol{v}^*||_1^2}{\delta} \left(\eta_2(n) + \sqrt{\frac{\log(nd)}{n\delta}} + M_n\eta_1(n)\right)$$

uniformly over $\delta \in \Delta$. This completes the proof.

B.5 Proof of Lemma 3

It suffices to bound $\left| \left(\widehat{v}^{(1)T} \frac{1}{|\mathcal{N}_1|} \sum_{i \in \mathcal{N}_1} A_i(\widehat{\beta}^{(2)}, \delta) \right)^2 - SB(\delta) \right|$. Define

$$\widetilde{A}(\boldsymbol{\beta}, \delta) = \mathbb{E}_{\boldsymbol{\beta}}(A_1(\boldsymbol{\beta}, \delta))
= \sum_{y} w(y)y \int_{\boldsymbol{z}} \boldsymbol{z} \int_{t} J(t) \int_{u} K(u)[f(u\delta + tb + \boldsymbol{\beta}^T \boldsymbol{z}|y, \boldsymbol{z}) - f(tb + \boldsymbol{\beta}^T \boldsymbol{z}|y, \boldsymbol{z})] dudt f(y, \boldsymbol{z}) d\boldsymbol{z}.$$
(B.26)

By definition,

$$\left| \left(\widehat{\boldsymbol{v}}^{(1)T} \frac{1}{|\mathcal{N}_{1}|} \sum_{i \in \mathcal{N}_{1}} A_{i}(\widehat{\boldsymbol{\beta}}^{(2)}, \delta) \right)^{2} - SB(\delta) \right| \\
= \left| \left(\widehat{\boldsymbol{v}}^{(1)T} \frac{1}{|\mathcal{N}_{1}|} \sum_{i \in \mathcal{N}_{1}} A_{i}(\widehat{\boldsymbol{\beta}}^{(2)}, \delta) \right)^{2} - \left(\boldsymbol{v}^{*T} A(\boldsymbol{\beta}^{*}, \delta) \right)^{2} \right| \\
\leq \left| \left(\widehat{\boldsymbol{v}}^{(1)T} \frac{1}{|\mathcal{N}_{1}|} \sum_{i \in \mathcal{N}_{1}} A_{i}(\widehat{\boldsymbol{\beta}}^{(2)}, \delta) \right)^{2} - \left(\widehat{\boldsymbol{v}}^{(1)T} A(\widehat{\boldsymbol{\beta}}^{(2)}, \delta) \right)^{2} \right| + \left| \left(\widehat{\boldsymbol{v}}^{(1)T} A(\widehat{\boldsymbol{\beta}}^{(2)}, \delta) \right)^{2} - \left(\boldsymbol{v}^{*T} A(\boldsymbol{\beta}^{*}, \delta) \right)^{2} \right| \\
\leq \underbrace{\left| \left(\widehat{\boldsymbol{v}}^{(1)T} \left(\frac{1}{|\mathcal{N}_{1}|} \sum_{i \in \mathcal{N}_{1}} A_{i}(\widehat{\boldsymbol{\beta}}^{(2)}, \delta) - \widetilde{\boldsymbol{A}}(\widehat{\boldsymbol{\beta}}^{(2)}, \delta) \right) \right)^{2} \right|}_{I_{1}} \\
+ 2 \underbrace{\left| \left(\widehat{\boldsymbol{v}}^{(1)T} \widetilde{\boldsymbol{A}}(\widehat{\boldsymbol{\beta}}^{(2)}, \delta) \cdot \widehat{\boldsymbol{v}}^{(1)T} \left(\frac{1}{|\mathcal{N}_{1}|} \sum_{i \in \mathcal{N}_{1}} A_{i}(\widehat{\boldsymbol{\beta}}^{(2)}, \delta) - \widetilde{\boldsymbol{A}}(\widehat{\boldsymbol{\beta}}^{(2)}, \delta) \right) \right|}_{I_{2}} \\
+ \underbrace{\left| \left(\widehat{\boldsymbol{v}}^{(1)T} \widetilde{\boldsymbol{A}}(\widehat{\boldsymbol{\beta}}^{(2)}, \delta) \right)^{2} - \left(\widehat{\boldsymbol{v}}^{(1)T} A(\widehat{\boldsymbol{\beta}}^{(2)}, \delta) \right)^{2} \right|}_{I_{3}} + \underbrace{\left| \left(\widehat{\boldsymbol{v}}^{(1)T} A(\widehat{\boldsymbol{\beta}}^{(2)}, \delta) \right)^{2} - \left(\boldsymbol{v}^{*T} A(\boldsymbol{\beta}^{*}, \delta) \right)^{2} \right|}_{I_{4}} + \underbrace{\left| \left(\widehat{\boldsymbol{v}}^{(1)T} A(\widehat{\boldsymbol{\beta}}^{(2)}, \delta) \right)^{2} - \left(\boldsymbol{v}^{*T} A(\boldsymbol{\beta}^{*}, \delta) \right)^{2} \right|}_{I_{4}} + \underbrace{\left| \left(\widehat{\boldsymbol{v}}^{(1)T} A(\widehat{\boldsymbol{\beta}}^{(2)}, \delta) \right)^{2} - \left(\boldsymbol{v}^{*T} A(\boldsymbol{\beta}^{*}, \delta) \right)^{2} \right|}_{I_{4}} + \underbrace{\left| \left(\widehat{\boldsymbol{v}}^{(1)T} A(\widehat{\boldsymbol{\beta}}^{(2)}, \delta) \right)^{2} - \left(\boldsymbol{v}^{*T} A(\boldsymbol{\beta}^{*}, \delta) \right)^{2} \right|}_{I_{4}} + \underbrace{\left| \left(\widehat{\boldsymbol{v}}^{(1)T} A(\widehat{\boldsymbol{\beta}}^{(2)}, \delta) \right)^{2} - \left(\boldsymbol{v}^{*T} A(\boldsymbol{\beta}^{*T}, \delta) \right)^{2} \right|}_{I_{4}} + \underbrace{\left| \left(\widehat{\boldsymbol{v}}^{(1)T} A(\widehat{\boldsymbol{\beta}}^{(2)}, \delta) \right)^{2} - \left(\boldsymbol{v}^{*T} A(\boldsymbol{\beta}^{*T}, \delta) \right)^{2} \right|}_{I_{4}} + \underbrace{\left| \left(\widehat{\boldsymbol{v}}^{(1)T} A(\widehat{\boldsymbol{\beta}}^{(2)}, \delta) \right)^{2} - \left(\boldsymbol{v}^{*T} A(\boldsymbol{\beta}^{*T}, \delta) \right)^{2} \right|}_{I_{4}} + \underbrace{\left| \left(\widehat{\boldsymbol{v}}^{(1)T} A(\widehat{\boldsymbol{\beta}}^{(2)}, \delta) \right)^{2} - \left(\boldsymbol{v}^{*T} A(\boldsymbol{\beta}^{*T}, \delta) \right)^{2} \right|}_{I_{4}} + \underbrace{\left| \left(\widehat{\boldsymbol{v}}^{(1)T} A(\widehat{\boldsymbol{\beta}}^{(2)}, \delta) \right)^{2} - \left(\boldsymbol{v}^{*T} A(\boldsymbol{\beta}^{(2)}, \delta) \right)^{2} \right|}_{I_{4}} + \underbrace{\left| \left(\widehat{\boldsymbol{v}}^{(1)T} A(\widehat{\boldsymbol{\beta}}^{(2)}, \delta) \right)^{2} - \left(\boldsymbol{v}^{*T} A(\boldsymbol{\beta}^{(2)}, \delta) \right)^{2} \right|}_{I_{4}} + \underbrace{\left| \left(\widehat{\boldsymbol{v}}^{(1)T} A(\widehat{\boldsymbol{\beta}}^{(2)}, \delta) \right)^{2} - \left(\boldsymbol{v}^{*T} A(\boldsymbol{\beta}^{(2)}, \delta) \right)^{2} \right|}_{I_{4}} + \underbrace{\left| \left(\widehat{\boldsymbol{v}}^{(1)T} A(\widehat{\boldsymbol{\beta$$

Now we start to bound each term.

• For I_1 , we firstly study $\delta^{-\ell}||\frac{1}{|\mathcal{N}_1|}\sum_{i\in\mathcal{N}_1}A_i(\widehat{\boldsymbol{\beta}}^{(2)},\delta)-\widetilde{A}(\widehat{\boldsymbol{\beta}}^{(2)},\delta)||_{\infty}$ condition on $\widehat{\boldsymbol{\beta}}^{(2)}$. Similar to the proof of Lemma 12, it suffices to bound

$$\widetilde{\delta}^{-\ell} || \frac{1}{|\mathcal{N}_1|} \sum_{i \in \mathcal{N}_1} A_i(\widehat{\boldsymbol{\beta}}^{(2)}, \widetilde{\delta}) - \widetilde{A}(\widehat{\boldsymbol{\beta}}^{(2)}, \widetilde{\delta}) ||_{\infty}$$
(B.28)

uniformly over $\widetilde{\delta} \in \widetilde{\Delta}$, where $\widetilde{\Delta}$ is a subset of Δ with cardinality of order $\mathcal{O}(n^{\rho})$ for some constant $\rho > 0$, such that for each $\delta \in \Delta$, there exist $\widetilde{\delta} \in \widetilde{\Delta}$ with $|\delta - \widetilde{\delta}| \lesssim n^{-\rho}$.

Let
$$T_{ij}(\widetilde{\delta}) = \widetilde{\delta}^{-\ell} (A_i(\widehat{\boldsymbol{\beta}}^{(2)}, \widetilde{\delta}) - \widetilde{A}(\widehat{\boldsymbol{\beta}}^{(2)}, \widetilde{\delta}))_j$$
, we know $\mathbb{E}_{\widehat{\boldsymbol{\beta}}^{(2)}} T_{ij}(\widetilde{\delta}) = 0$. Moreover for all j ,

$$\begin{split} \widetilde{\delta}^{-\ell} | (A_{i}(\widehat{\boldsymbol{\beta}}^{(2)}, \widetilde{\delta}))_{j} | = & \widetilde{\delta}^{-\ell} \bigg| w(y_{i}) \frac{z_{ij}y_{i}}{b} \int K(u) \Big[J(\frac{x_{i} - \widehat{\boldsymbol{\beta}}^{(2)T} \boldsymbol{z}_{i} - u\widetilde{\delta}}{b}) - J(\frac{x_{i} - \widehat{\boldsymbol{\beta}}^{(2)T} \boldsymbol{z}_{i}}{b}) \Big] \bigg\} du \bigg| \\ = & \widetilde{\delta}^{-\ell} \bigg| w(y_{i}) \frac{z_{ij}y_{i}}{b} \int K(u) \frac{\widetilde{\delta}^{\ell}}{b^{\ell}} \frac{u^{\ell}}{\ell!} J^{\ell} (\frac{x_{i} - \widehat{\boldsymbol{\beta}}^{(2)T} \boldsymbol{z}_{i} - \tau u\widetilde{\delta}}{b}) \Big] \bigg\} du \bigg| \\ \lesssim & \frac{M_{n}}{b^{\ell+1}}, \end{split} \tag{B.29}$$

where $\tau \in [0,1]$ and the second equality is because K is of order ℓ and mean value theorem. This implies that $|T_{ij}(\widetilde{\delta})| \lesssim \frac{M_n}{b^{\ell+1}}$. Now we look at $\mathbb{E}_{\widehat{\boldsymbol{\beta}}^{(2)}}[T_{ij}^2(\widetilde{\delta})]$.

Notice that for all j

$$\widetilde{A}(\widehat{\boldsymbol{\beta}}^{(2)}, \widetilde{\delta})_{j} = \sum_{y} w(y) \int_{\boldsymbol{z}} \boldsymbol{z}_{j} y \int_{t} J(t) \int_{u} K(u) [f(u\widetilde{\delta} + tb + \widehat{\boldsymbol{\beta}}^{(2)T} \boldsymbol{z} | y, \boldsymbol{z}) - f(tb + \widehat{\boldsymbol{\beta}}^{(2)T} \boldsymbol{z} | y, \boldsymbol{z})] du dt f(y, \boldsymbol{z}) d\boldsymbol{z}$$

$$= \sum_{y} w(y) \int_{\boldsymbol{z}} \boldsymbol{z}_{j} y \int_{t} J(t) \int_{u} K(u) \frac{(u\widetilde{\delta})^{\ell}}{\ell!} f^{(\ell)}(\tau u\widetilde{\delta} + tb + \widehat{\boldsymbol{\beta}}^{(2)T} \boldsymbol{z} | y, \boldsymbol{z}) du dt f(y, \boldsymbol{z}) d\boldsymbol{z}$$

$$\lesssim \widetilde{\delta}^{\ell}, \tag{B.30}$$

where the second equality is because K is of order ℓ and $\tau \in [0,1]$. For the second moment, we have

$$\begin{split} &\mathbb{E}_{\widehat{\boldsymbol{\beta}}^{(2)}}\big[(\widetilde{\delta}^{-\ell}A_{1}(\widehat{\boldsymbol{\beta}}^{(2)},\widetilde{\delta}))_{j}^{2}\big] \\ =&\widetilde{\delta}^{-2\ell}\mathbb{E}_{\widehat{\boldsymbol{\beta}}^{(2)}}\left[w(y)^{2}\frac{z_{j}^{2}}{b^{2}}\bigg(\int K(u)\big[J(\frac{x-\widehat{\boldsymbol{\beta}}^{(2)T}\boldsymbol{z}-u\widetilde{\delta}}{b})-J(\frac{x-\widehat{\boldsymbol{\beta}}^{(2)T}\boldsymbol{z}}{b})\big]\bigg\}du\bigg)^{2}\right] \\ =&\widetilde{\delta}^{-2\ell}\mathbb{E}_{\widehat{\boldsymbol{\beta}}^{(2)}}\bigg[w(y)^{2}\frac{z_{j}^{2}}{b^{2}}\bigg(\int K(u)\frac{\widetilde{\delta}^{\ell}}{b^{\ell}}\frac{u^{\ell}}{\ell!}J^{\ell}(\frac{x-\widehat{\boldsymbol{\beta}}^{(2)T}\boldsymbol{z}-\tau u\widetilde{\delta}}{b})du\bigg)^{2}\bigg] \\ =&\frac{1}{b^{2\ell+1}}\bigg|\sum_{y}\int_{z}w(y)^{2}z_{j}^{2}\int_{t}\bigg(\int K(u)\frac{u^{\ell}}{\ell!}J^{\ell}(t-\frac{\tau u\widetilde{\delta}}{b})du\bigg)^{2}f(tb+\widehat{\boldsymbol{\beta}}^{(2)T}\boldsymbol{z}|\boldsymbol{y},\boldsymbol{z})dtf(\boldsymbol{y},\boldsymbol{z})d\boldsymbol{z}\bigg| \\ \lesssim&\frac{1}{b^{2\ell+1}}, \end{split} \tag{B.31}$$

where the last equality is due to a change of variable $t = \frac{x - \widehat{\boldsymbol{\beta}}^{(2)T} \boldsymbol{z}}{b}$. Since $b \to 0$, we conclude that $\mathbb{E}_{\widehat{\boldsymbol{\beta}}^{(2)}}[T_{ij}^2(\widetilde{\delta})] \lesssim \frac{1}{b^{2\ell+1}}$. Now applying Bernstein inequality similar to Lemma 12 with $M_n \sqrt{\frac{\log(d)}{nb}} = \mathcal{O}(1)$, second moment of order $\frac{1}{b^{2\ell+1}}$ and each term is bounded by $\frac{M_n}{b^{\ell+1}}$, we will obtain that with probability greater than $1 - \mathcal{O}((nd)^{-1})$, conditioned on $\widehat{\boldsymbol{\beta}}^{(2)}$

$$||\frac{1}{|\mathcal{N}_1|} \sum_{i \in \mathcal{N}_1} A_i(\widehat{\boldsymbol{\beta}}^{(2)}, \widetilde{\boldsymbol{\delta}}) - \mathbb{E}A_i(\widehat{\boldsymbol{\beta}}^{(2)}, \widetilde{\boldsymbol{\delta}})||_{\infty} \lesssim \frac{\widetilde{\boldsymbol{\delta}}^{\ell}}{b^{\ell}} \sqrt{\frac{\log(nd)}{nb}}, \tag{B.32}$$

which together with Lemma 13 implies that with probability approaching 1

$$||\frac{1}{|\mathcal{N}_1|} \sum_{i \in \mathcal{N}_1} A_i(\widehat{\boldsymbol{\beta}}^{(2)}, \delta) - \mathbb{E}_{\widehat{\boldsymbol{\beta}}^{(2)}} A_i(\widehat{\boldsymbol{\beta}}^{(2)}, \delta)||_{\infty} \lesssim \frac{\delta^{\ell}}{b^{\ell}} \sqrt{\frac{\log(nd)}{nb}}$$
(B.33)

holds uniformly for all $\delta \in \Delta$.

Therefore we conclude that

$$I_1 = \mathcal{O}_{\mathbb{P}}(||\boldsymbol{v}^*||_1^2 \frac{\delta^{2\ell}}{b^{2\ell}} \frac{\log(nd)}{nb}). \tag{B.34}$$

• For I_2 , (B.30) implies that $||\mathbb{E}_{\widehat{\boldsymbol{\beta}}^{(2)}}A_1(\widehat{\boldsymbol{\beta}}^{(2)},\delta)||_{\infty} \lesssim \delta^{\ell}$ regardless the choice of $\widehat{\boldsymbol{\beta}}^{(2)}$. This combined with (B.33) further implies that

$$I_2 = \mathcal{O}_{\mathbb{P}}(||\boldsymbol{v}^*||_1^2 \delta^{\ell} \sqrt{\frac{\delta^{2\ell} \log(nd)}{nb}}) = \mathcal{O}_{\mathbb{P}}(||\boldsymbol{v}^*||_1^2 \frac{\delta^{2\ell}}{b^{\ell}} \sqrt{\frac{\log(nd)}{nb}}). \tag{B.35}$$

• For I_3 , by definition for all j we can write

$$\begin{split} & \left(\widetilde{A}(\widehat{\boldsymbol{\beta}}^{(2)}, \boldsymbol{\delta}) - A(\widehat{\boldsymbol{\beta}}^{(2)}, \boldsymbol{\delta})\right)_{j} \\ &= \sum_{y} w(y) \int_{\boldsymbol{z}} \boldsymbol{z}_{j} y \int_{t} J(t) \int_{u} K(u) [(f(u\boldsymbol{\delta} + t\boldsymbol{b} + \widehat{\boldsymbol{\beta}}^{(2)T} \boldsymbol{z} | \boldsymbol{y}, \boldsymbol{z}) - f(t\boldsymbol{b} + \widehat{\boldsymbol{\beta}}^{(2)T} \boldsymbol{z} | \boldsymbol{y}, \boldsymbol{z})) \\ & - (f(u\boldsymbol{\delta} + \widehat{\boldsymbol{\beta}}^{(2)T} \boldsymbol{z} | \boldsymbol{y}, \boldsymbol{z}) - f(\widehat{\boldsymbol{\beta}}^{(2)T} \boldsymbol{z} | \boldsymbol{y}, \boldsymbol{z}))] du dt f(\boldsymbol{y}, \boldsymbol{z}) d\boldsymbol{z} \\ &= \sum_{y} w(y) \int_{\boldsymbol{z}} \boldsymbol{z}_{j} y \int_{u} \sum_{j=\ell}^{\ell+r-1} \frac{(u\boldsymbol{\delta})^{j}}{j!} K(u) \int_{t} J(t) [f^{(j)}(t\boldsymbol{b} + \widehat{\boldsymbol{\beta}}^{(2)T} \boldsymbol{z} | \boldsymbol{y}, \boldsymbol{z}) - f^{(j)}(\widehat{\boldsymbol{\beta}}^{(2)T} \boldsymbol{z} | \boldsymbol{y}, \boldsymbol{z})] dt du f(\boldsymbol{y}, \boldsymbol{z}) d\boldsymbol{z} \\ & + \mathcal{O}(\boldsymbol{\delta}^{\ell+r}) \\ &= \sum_{y} w(y) \int_{\boldsymbol{z}} \boldsymbol{z}_{j} y \int_{u} \sum_{j=\ell}^{\ell+r-1} \frac{(u\boldsymbol{\delta})^{j}}{j!} K(u) \int_{t} J(t) \frac{(t\boldsymbol{b})^{\ell+r-j}}{(\ell+r-j)!} f^{(\ell+r)}(\tau t\boldsymbol{b} + \widehat{\boldsymbol{\beta}}^{(2)T} \boldsymbol{z} | \boldsymbol{y}, \boldsymbol{z}) dt du f(\boldsymbol{y}, \boldsymbol{z}) d\boldsymbol{z} + \mathcal{O}(\boldsymbol{\delta}^{\ell+r}) \\ &= \mathcal{O}(\boldsymbol{\delta}^{\ell}(\boldsymbol{\delta} \vee \boldsymbol{b})^{r}), \end{split}$$

where $\tau \in [0, 1]$. Here the first equality is by definition, the second equality uses the property that $\int K(u)u^i du = 0 \,\,\forall \,\, i < \ell$, and the third equality uses the property that $\int J(u)u^i du = 0 \,\,\forall \,\, i < r$ and the mean value theorem. This implies that

$$||\widetilde{A}(\widehat{\beta}^{(2)}, \delta) - A(\widehat{\beta}^{(2)}, \delta)||_{\infty} = \mathcal{O}(\delta^{\ell}(\delta \vee b)^{r}).$$
(B.37)

Meanwhile, by (B.30) and a similar derivation, we know both $||\widetilde{A}(\widehat{\beta}^{(2)}, \delta)||_{\infty}$ and $||A(\widehat{\beta}^{(2)}, \delta)||_{\infty}$ are of order δ^{ℓ} . These together imply that

$$I_3 = \mathcal{O}(||\boldsymbol{v}^*||_1^2 \delta^{2\ell} (\delta \vee b)^r). \tag{B.38}$$

(B.36)

• For I_4 , again $||A(\widehat{\beta}^{(2)}, \delta)||_{\infty} \approx ||A(\beta^*, \delta)||_{\infty} = \mathcal{O}(\delta^{\ell})$. For the difference, we have

$$||A(\widehat{\boldsymbol{\beta}}^{(2)}, \delta) - A(\boldsymbol{\beta}^*, \delta)||_{\infty}$$

$$= \max_{j} \left| \int_{u} K(u) \left[\sum_{y} w(y) \int_{z} z_{j} y \left(f(u\delta + \widehat{\boldsymbol{\beta}}^{(2)T} \boldsymbol{z} | \boldsymbol{y}, \boldsymbol{z}) - f(\widehat{\boldsymbol{\beta}}^{(2)T} \boldsymbol{z} | \boldsymbol{y}, \boldsymbol{z}) \right) - f(u\delta + \boldsymbol{\beta}^{*T} \boldsymbol{z} | \boldsymbol{y}, \boldsymbol{z}) + f(\boldsymbol{\beta}^{*T} \boldsymbol{z} | \boldsymbol{y}, \boldsymbol{z}) \right) f(\boldsymbol{y}, \boldsymbol{z}) d\boldsymbol{z} \right] d\boldsymbol{u} \right|$$

$$= \max_{j} \left| \sum_{y} w(y) \int_{z} z_{j} y \int_{u} K(u) \left[\left(f(u\delta + \widehat{\boldsymbol{\beta}}^{(2)T} \boldsymbol{z} | \boldsymbol{y}, \boldsymbol{z}) - f(u\delta + \boldsymbol{\beta}^{*T} \boldsymbol{z} | \boldsymbol{y}, \boldsymbol{z}) \right) - \left(f(\widehat{\boldsymbol{\beta}}^{(2)T} \boldsymbol{z} | \boldsymbol{y}, \boldsymbol{z}) - f(\boldsymbol{\beta}^{*T} \boldsymbol{z} | \boldsymbol{y}, \boldsymbol{z}) \right) \right] f(\boldsymbol{y}, \boldsymbol{z}) d\boldsymbol{z} d\boldsymbol{u} \right|$$

$$= \max_{j} \left| \sum_{y} w(y) \int_{z} z_{j} y \int_{u} K(u) \left[(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}^{*})^{T} \boldsymbol{z} \left(f'(u\delta + \boldsymbol{\beta}^{*T} \boldsymbol{z} | \boldsymbol{y}, \boldsymbol{z}) - f'(\boldsymbol{\beta}^{*T} \boldsymbol{z} | \boldsymbol{y}, \boldsymbol{z}) \right) + ((\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}^{*})^{T} \boldsymbol{z})^{2} \left(f''(u\delta + \widehat{\boldsymbol{\beta}}^{T} \boldsymbol{z}) - f''(\check{\boldsymbol{\beta}}^{T} \boldsymbol{z}) \right) \right] d\boldsymbol{u} f(\boldsymbol{y}, \boldsymbol{z}) d\boldsymbol{z} \right|$$

$$\lesssim M_{n} ||\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}^{*}||_{1} \delta^{\ell} + M_{n}^{2} ||\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}^{*}||_{1}^{2}, \tag{B.39}$$

where the last step follows from an ℓ order Taylor expansion of $f'(u\delta + \boldsymbol{\beta}^{*T}\boldsymbol{z}|y,\boldsymbol{z}) - f'(\boldsymbol{\beta}^{*T}\boldsymbol{z}|y,\boldsymbol{z})$ for the first term and boundedness of f'' for the second term. This implies that

$$||A(\widehat{\beta}^{(2)}, \delta) - A(\beta^*, \delta)||_{\infty} \lesssim M_n \eta_1(n) (\delta^{\ell} + M_n \eta_1(n))$$
(B.40)

with probability approaching 1 by Assumption 6. Therefore, we can obtain

$$I_{4} \leq \left| (\boldsymbol{v}^{(1)T} A(\widehat{\boldsymbol{\beta}}^{(2)}, \delta) - \boldsymbol{v}^{*T} A(\boldsymbol{\beta}^{*}, \delta)) (\boldsymbol{v}^{(1)T} A(\widehat{\boldsymbol{\beta}}^{(2)}, \delta) + \boldsymbol{v}^{*T} A(\boldsymbol{\beta}^{*}, \delta)) \right|$$

$$\leq \left| \boldsymbol{v}^{(1)T} (A(\widehat{\boldsymbol{\beta}}^{(2)}, \delta) - A(\boldsymbol{\beta}^{*}, \delta)) + (\boldsymbol{v}^{(1)T} - \boldsymbol{v}^{*T}) A(\boldsymbol{\beta}^{*}, \delta) \right| \left| \boldsymbol{v}^{(1)T} A(\widehat{\boldsymbol{\beta}}^{(2)}, \delta) + \boldsymbol{v}^{*T} A(\boldsymbol{\beta}^{*}, \delta) \right|$$

$$= ||\boldsymbol{v}^{*}||_{1}^{2} \delta^{2\ell} \mathcal{O}_{\mathbb{P}}(M_{n} \eta_{1}(n)(1 \vee M_{n} \eta_{1}(n)/\delta^{\ell}) + \eta_{2}(n)).$$
(B.41)

Combining all bounds for I_1, \ldots, I_4 , we know that uniformly over all $\delta \in \Delta$

$$\left| \left(\widehat{\boldsymbol{v}}^{(1)T} \frac{1}{|\mathcal{N}_{1}|} \sum_{i \in \mathcal{N}_{1}} A_{i}(\widehat{\boldsymbol{\beta}}^{(2)}, \delta) \right)^{2} - SB(\delta) \right| \\
\lesssim ||\boldsymbol{v}^{*}||_{1}^{2} \delta^{2\ell} \left(\frac{\log(nd)}{nb^{2\ell+1}} + \sqrt{\frac{\log(nd)}{nb^{2\ell+1}}} + (\delta \vee b)^{r} + M_{n}\eta_{1}(n)(1 \vee \frac{M_{n}\eta_{1}(n)}{\delta^{\ell}}) + \eta_{2}(n) \right). \tag{B.42}$$

This completes the proof.

B.6 Proof of Theorem 3

By Lemma 2 and Lemma 3, uniformly over all $\delta \in \Delta$

$$|\widehat{M}(\delta) - M(\delta)|$$

$$= ||\boldsymbol{v}^*||_1^2 \frac{1}{n\delta} \mathcal{O}_{\mathbb{P}} \left(\eta_2(n) + \sqrt{\frac{\log(nd)}{n\delta}} + M_n \eta_1(n) \right)$$

$$+ ||\boldsymbol{v}^*||_1^2 \delta^{2\ell} \mathcal{O}_{\mathbb{P}} \left(\frac{\log(nd)}{nb^{2\ell+1}} + \sqrt{\frac{\log(nd)}{nb^{2\ell+1}}} + (\delta \vee b)^r + M_n \eta_1(n) (1 \vee \frac{M_n \eta_1(n)}{\delta^\ell}) + \eta_2(n) \right)$$

$$\lesssim \frac{1}{n\delta} \psi_1(n) + \delta^{2\ell} \psi_2(n). \tag{B.43}$$

Recall that $M(\delta) = \frac{1}{n}V(\delta) + \frac{n-1}{n}SB(\delta)$, where

$$V(\delta) = \frac{1}{\delta} (\boldsymbol{v}^{*T} \boldsymbol{\Sigma}^* \boldsymbol{v}^*) (1 + o(1)) \quad \text{and} \quad SB(\delta) = \delta^{2\ell} (\boldsymbol{v}^{*T} \boldsymbol{b}^*)^2 (1 + o(1)).$$
 (B.44)

This implies that $\delta^* = (\frac{1}{n} \frac{\boldsymbol{v}^{*T} \boldsymbol{\Sigma}^* \boldsymbol{v}^*}{2\ell(\boldsymbol{v}^{*T} \boldsymbol{b}^*)^2})^{1/(2\ell+1)} (1 + o(1))$. By Assumption 5 we know $M(\delta) \gtrsim \frac{1}{n\delta} + \delta^{2\ell}$, which implies

$$\frac{|\widehat{M}(\delta) - M(\delta)|}{M(\delta)} \lesssim \psi_1(n) \vee \psi_2(n) = o_{\mathbb{P}}(1),$$

uniformly over all $\delta \in \Delta$. That means for any $\epsilon > 0$, uniformly over all $\delta \in \Delta$, the event $(1 - \epsilon)M(\delta) \le \widehat{M}(\delta) \le (1 + \epsilon)M(\delta)$ holds with probability tending to 1. Under this event, we have that

$$M(\widehat{\delta}) = \widehat{M}(\widehat{\delta}) + [M(\widehat{\delta}) - \widehat{M}(\widehat{\delta})]$$

$$\leq \widehat{M}(\delta^*) + [M(\widehat{\delta}) - \widehat{M}(\widehat{\delta})]$$

$$\leq (1 + \epsilon)M(\delta^*) + \epsilon M(\widehat{\delta}), \tag{B.45}$$

where the first inequality holds because $\hat{\delta}$ is the minimizer of $\widehat{M}(\delta)$ and the second inequality follows from $\widehat{M}(\delta^*) \leq (1+\epsilon)M(\delta^*)$ and $\widehat{M}(\widehat{\delta}) \geq (1-\epsilon)M(\widehat{\delta})$. As a result, we obtain $M(\widehat{\delta}) \leq \frac{1+\epsilon}{1-\epsilon}M(\delta^*)$, from which we can claim that $\widehat{\delta} \leq \frac{3}{2}\delta^*$. To see this, let us consider the complement case $\widehat{\delta} > \frac{3}{2}\delta^*$. When it holds, we know from (B.44) that

$$M(\widehat{\delta}) \ge (\frac{3}{2})^{2\ell} SB(\delta^*)(1+o(1)) = (\frac{3}{2})^{2\ell} \frac{1}{2\ell+1} M(\delta^*)(1+o(1)) \ge 1.01 M(\delta^*)(1+o(1)),$$

where the second step follows from $M(\delta^*) = (2\ell+1)SB(\delta^*)(1+o(1))$ and last step holds by the condition $\ell \geq 2$. However, the above inequality contradicts with $M(\widehat{\delta}) \leq \frac{1+\epsilon}{1-\epsilon}M(\delta^*)$ for some sufficiently small ϵ . This justifies the statement $\widehat{\delta} \leq \frac{3}{2}\delta^*$. Following a similar argument, we can show that $\widehat{\delta} \geq \frac{1}{2}\delta^*$. Thus, with probability tending to 1, we have $|\widehat{\delta} - \delta^*|/\delta^* \leq 1/2$.

By definition, we have $M'(\delta^*) = 0$ and $\widehat{M}'(\widehat{\delta}) = 0$. This implies that

$$\widehat{M}'(\widehat{\delta}) - M'(\widehat{\delta}) = -M''(\widetilde{\delta})(\widehat{\delta} - \delta^*), \tag{B.46}$$

for some intermediate value $\widetilde{\delta}$, which gives

$$(\widehat{\delta} - \delta^*) = \frac{\widehat{M}'(\widehat{\delta}) - M'(\widehat{\delta})}{-M''(\widehat{\delta})}.$$
(B.47)

For any $\delta \simeq \delta^*$, after some algebra, we can show that

$$M''(\delta) \approx \frac{1}{\delta^3 n}.$$
 (B.48)

Similar to the proof of (B.43), we can show that $(\widehat{M}'-M')(\delta)$ has the same order as $(\widehat{M}-M)(\delta)$, except for an additional factor $\frac{1}{\delta}$. Since $|\widehat{\delta}-\delta^*|/\delta^* \leq 1/2$ holds in probability, it implies $\widehat{\delta}$ and $\widehat{\delta}$ are of the same order of δ^* . Thus, (B.47) implies

$$\frac{\widehat{\delta} - \delta^*}{\delta^*} \lesssim \psi_1(n) \vee \psi_2(n). \tag{B.49}$$

This completes the proof.

B.7 Proof of Theorem 4

The proof is very similar to the proof of Theorem 1 so we only give a sketch here. Following the same derivation of Lemma 1, we can show that

$$(n\delta)^{1/2} \frac{S_{\delta}^{L}(0, \boldsymbol{\gamma}^*, \boldsymbol{\omega}_{L}^*) - \delta^{\ell} \mu_{L}^*}{\sigma_{L}^*} \stackrel{d}{\to} N(0, 1).$$
(B.50)

Next, following the derivation of Theorem 1, we can show that $I_1 = (n\delta)^{1/2} | \boldsymbol{v}_L^{*T} \boldsymbol{C}(\nabla R_{\delta}^{n_{(1)}}(\widehat{\boldsymbol{\beta}}_0^{(2)}) - \nabla R_{\delta}^{n_{(1)}}(\boldsymbol{\beta}^*))|$ and $I_2 = (n\delta)^{1/2} | (\widehat{\boldsymbol{v}}_L^{(1)} - \boldsymbol{v}_L^*) \boldsymbol{C} \nabla R_{\delta}^{n_{(1)}}(\widehat{\boldsymbol{\beta}}_0^{(2)})|$ are $o_{\mathbb{P}}(1)$, which further implies that $(n\delta)^{1/2} | S_{\delta}^{L(1)}(0, \widehat{\boldsymbol{\gamma}}^{(2)}, \widehat{\boldsymbol{\omega}}_L^{(1)}) - S_{\delta}^{L}(0, \boldsymbol{\gamma}^*, \boldsymbol{\omega}_L^*)| = o_{\mathbb{P}}(1)$. This shall hold similarly for $S_{\delta}^{L(2)}(0, \widehat{\boldsymbol{\gamma}}^{(1)}, \widehat{\boldsymbol{\omega}}_L^{(2)})$. Invoking Slutsky's theorem completes the proof.

C Proofs of additional technical lemmas

Lemma 4. Under Assumptions 1 - 4, for any fixed β , we have

$$||\nabla R_{\delta}(\boldsymbol{\beta}) - \nabla R(\boldsymbol{\beta})||_{\infty} \lesssim \delta^{\ell},$$

$$||\nabla^{2} R_{\delta}(\boldsymbol{\beta}) - \nabla R(\boldsymbol{\beta})||_{\max} \lesssim \delta^{\ell-1}.$$
 (C.1)

Proof of Lemma 4. We focus on proving the result for $||\nabla^2 R_{\delta}(\beta) - \nabla^2 R(\beta)||_{\text{max}}$ and the proof for

the other one is very similar. By definition

$$||\nabla^{2}R_{\delta}(\boldsymbol{\beta}) - \nabla^{2}R(\boldsymbol{\beta})||_{\text{max}}$$

$$= \max_{j,k} |\sum_{y} w(y) \int_{z} yz_{j}z_{k} \left(\int \frac{-1}{\delta^{2}} K'(\frac{x - \boldsymbol{\beta}^{T}\boldsymbol{z}}{\delta}) f(x|y,\boldsymbol{z}) dx - f'(\boldsymbol{\beta}^{T}\boldsymbol{z}|y,\boldsymbol{z}) \right) f(y,\boldsymbol{z}) d\boldsymbol{z}|$$

$$= \max_{j,k} |\sum_{y} w(y) \int_{z} yz_{j}z_{k} \left(\int \frac{-1}{\delta} K'(u) f(u\delta + \boldsymbol{\beta}^{T}\boldsymbol{z}|y,\boldsymbol{z}) du - f'(\boldsymbol{\beta}^{T}\boldsymbol{z}|y,\boldsymbol{z}) \right) f(y,\boldsymbol{z}) d\boldsymbol{z}|$$

$$= \max_{j,k} |\sum_{y} w(y) \int_{z} yz_{j}z_{k} \int K(u) (f'(u\delta + \boldsymbol{\beta}^{T}\boldsymbol{z}|y,\boldsymbol{z}) - f'(\boldsymbol{\beta}^{T}\boldsymbol{z}|y,\boldsymbol{z})) du f(y,\boldsymbol{z}) d\boldsymbol{z}|$$

$$= \max_{j,k} |\sum_{y} w(y) \int_{z} yz_{j}z_{k} \int K(u) \frac{(u\delta)^{\ell-1}}{(\ell-1)!} f^{(\ell)} (\tau u\delta + \boldsymbol{\beta}^{T}\boldsymbol{z}|y,\boldsymbol{z}) du f(y,\boldsymbol{z}) d\boldsymbol{z}|$$

$$\lesssim \mathcal{O}(\delta^{\ell-1}), \tag{C.2}$$

where the first equality is by definition, the second equality follows from a change of variable, the third equality follows from an integration by parts, the last equality follows from Assumptions 1, 2 and the last inequality follows from Assumptions 3, 4. The proof is complete.

Lemma 5. Under Assumptions 1 - 4, for any fixed β , we have with probability greater than $1 - \mathcal{O}(d^{-1})$

$$||\nabla R_{\delta}^{n}(\boldsymbol{\beta}) - \nabla R_{\delta}(\boldsymbol{\beta})||_{\infty} \lesssim \sqrt{\frac{\log(d)}{n\delta}},$$

$$||\nabla^{2} R_{\delta}^{n}(\boldsymbol{\beta}) - \nabla^{2} R_{\delta}(\boldsymbol{\beta})||_{\max} \lesssim \sqrt{\frac{\log(d)}{n\delta^{3}}}.$$
(C.3)

Proof of Lemma 5. We focus on proving the result for $||\nabla^2 R_{\delta}^n(\beta) - \nabla^2 R_{\delta}(\beta)||_{\text{max}}$ and the proof for the other one is very similar. Denote

$$T_{ijk} = (\nabla^2 \bar{R}_{\delta}^i(\boldsymbol{\beta}) - \nabla^2 R_{\delta}(\boldsymbol{\beta}))_{jk}$$

$$= -w(y_i)y_i \frac{z_{ij}z_{ik}}{\delta^2} K'(\frac{x_i - \boldsymbol{\beta}^T \boldsymbol{z}_i}{\delta}) - (\nabla^2 R_{\delta}(\boldsymbol{\beta}))_{jk}.$$
(C.4)

By definition, we know $\mathbb{E}[T_{ijk}] = 0$. Meanwhile for $Var[T_{ijk}]$, we know

$$\mathbb{E}[(\nabla^2 \bar{R}_{\delta}^i(\boldsymbol{\beta}))_{jk}^2] = \sum_{y} \int w(y)^2 \frac{z_j^2 z_k^2}{\delta^4} K'^2 (\frac{x - \boldsymbol{\beta}^T \boldsymbol{z}}{\delta}) f(x|y, \boldsymbol{z}) dx f(y, \boldsymbol{z}) d\boldsymbol{z}$$

$$= \sum_{y} \int w(y)^2 \frac{z_j^2 z_k^2}{\delta^3} K'^2 (u) f(u\delta + \boldsymbol{\beta}^T \boldsymbol{z}|y, \boldsymbol{z}) du f(y, \boldsymbol{z}) d\boldsymbol{z}$$

$$= \mathcal{O}(\frac{1}{\delta^3}), \tag{C.5}$$

and with a similar derivation $(\nabla^2 R_{\delta}(\beta))_{jk}^2 = \mathcal{O}(\frac{1}{\delta^2})$. This shows that $Var[T_{ijk}] = \mathcal{O}(\frac{1}{\delta^3})$. Since $|T_{ijk}| \lesssim \frac{M_n^2}{\delta^2}$, by Bernstein inequality we can show that with probability greater than $1 - \mathcal{O}(d^{-1})$

$$||\nabla^2 R_{\delta}^n(\boldsymbol{\beta}) - \nabla^2 R_{\delta}(\boldsymbol{\beta})||_{\max} \lesssim \sqrt{\frac{\log(d)}{n\delta^3}}.$$
 (C.6)

This completes the proof.

Lemma 6. Under Assumptions 1 - 4, it holds that

$$||\nabla R(\widehat{\boldsymbol{\beta}}) - \nabla R(\boldsymbol{\beta}^*)||_{\infty} \lesssim M_n ||\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}^*||_1,$$

$$||\nabla^2 R(\widehat{\boldsymbol{\beta}}) - \nabla^2 R(\boldsymbol{\beta}^*)||_{\max} \lesssim M_n ||\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}^*||_1.$$
 (C.7)

Proof of Lemma 6. Here we prove the second inequality and the first one should follow similarly. By definition,

$$||\nabla^{2}R(\widehat{\boldsymbol{\beta}}) - \nabla^{2}R(\boldsymbol{\beta}^{*})||_{\max}$$

$$= \max_{j,k} \left| \sum_{y} w(y) \int_{\boldsymbol{z}} z_{j} z_{k} \left[f'(\widehat{\boldsymbol{\beta}}^{T} \boldsymbol{z}|y, \boldsymbol{z}) - f'(\boldsymbol{\beta}^{*T} \boldsymbol{z}|y, \boldsymbol{z}) \right] f(y, \boldsymbol{z}) d\boldsymbol{z} \right|$$

$$\leq ||\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}^{*}||_{1} M_{n} |f''|_{\infty} \max_{j,k} \sum_{y} \mathbb{E}[|Z_{j}Z_{k}||Y = y]$$

$$\lesssim M_{n} ||\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}^{*}||_{1}. \tag{C.8}$$

Lemma 7. Under the conditions in Theorem 1, we have

$$(n\delta)^{1/2} \left| \boldsymbol{v}^{*T} \bigg(\nabla R_{\delta}^{n_{(j)}}(\widehat{\boldsymbol{\beta}}^{(k)}) - \nabla R_{\delta}^{n_{(j)}}(\boldsymbol{\beta}^*) - \nabla^2 R_{\delta}^{n_{(j)}}(\boldsymbol{\beta}^*)(\widehat{\boldsymbol{\beta}}^{(k)} - \boldsymbol{\beta}^*) \bigg) \right| = o_{\mathbb{P}}(1),$$

for $(j,k) \in \{(1,2),(2,1)\}.$

Proof of Lemma 7. With some algebra we obtain

$$\left| \boldsymbol{v}^{*T} \left(\nabla R_{\delta}^{n_{(j)}}(\widehat{\boldsymbol{\beta}}^{(k)}) - \nabla R_{\delta}^{n_{(j)}}(\boldsymbol{\beta}^{*}) - \nabla^{2} R_{\delta}^{n_{(j)}}(\boldsymbol{\beta}^{*})(\widehat{\boldsymbol{\beta}}^{(k)} - \boldsymbol{\beta}^{*}) \right) \right|$$

$$= \left| \frac{1}{\mathcal{N}_{j}} \sum_{i \in \mathcal{N}_{j}} w(y_{i}) \frac{\boldsymbol{z}_{i}^{T} \boldsymbol{v}^{*} y_{i}}{\delta} \int_{\frac{x_{i} - \widehat{\boldsymbol{\beta}}^{(k)T} \boldsymbol{z}_{i}}{\delta}}^{\frac{x_{i} - \widehat{\boldsymbol{\beta}}^{(k)T} \boldsymbol{z}_{i}}{\delta}} K''(t) \left(\frac{x_{i} - \widehat{\boldsymbol{\beta}}^{(k)T} \boldsymbol{z}_{i}}{\delta} - t \right) dt \right|$$

$$\leq ||\boldsymbol{v}^{*}||_{1} \left| \left| \frac{1}{\mathcal{N}_{j}} \sum_{i \in \mathcal{N}_{j}} w(y_{i}) \frac{\boldsymbol{z}_{i} y_{i}}{\delta} \int_{\frac{x_{i} - \widehat{\boldsymbol{\beta}}^{(k)T} \boldsymbol{z}_{i}}{\delta}}^{\frac{x_{i} - \widehat{\boldsymbol{\beta}}^{(k)T} \boldsymbol{z}_{i}}{\delta}} K''(t) \left(\frac{x_{i} - \widehat{\boldsymbol{\beta}}^{(k)T} \boldsymbol{z}_{i}}{\delta} - t \right) dt \right| \right|_{\infty}.$$
(C.9)

Now we start to analyze $\left\| \frac{1}{\mathcal{N}_j} \sum_{i \in \mathcal{N}_j} w(y_i) \frac{\mathbf{z}_i y_i}{\delta} \int_{\frac{x_i - \hat{\boldsymbol{\beta}}^{(k)T} \mathbf{z}_i}{\delta}}^{\frac{x_i - \hat{\boldsymbol{\beta}}^{(k)T} \mathbf{z}_i}{\delta}} K''(t) \left(\frac{x_i - \hat{\boldsymbol{\beta}}^{(k)T} \mathbf{z}_i}{\delta} - t \right) dt \right\|_{\infty}$. Denote $G_i = w(y_i) \frac{\mathbf{z}_i y_i}{\delta} \int_{\frac{x_i - \hat{\boldsymbol{\beta}}^{(k)T} \mathbf{z}_i}{\delta}}^{\frac{x_i - \hat{\boldsymbol{\beta}}^{(k)T} \mathbf{z}_i}{\delta}} K''(t) \left(\frac{x_i - \hat{\boldsymbol{\beta}}^{(k)T} \mathbf{z}_i}{\delta} - t \right) dt$ and $G_{im}, 1 \leq m \leq d$ as its coordinates. Consider the

event $A := \{||\widehat{\boldsymbol{\beta}}^{(k)} - \boldsymbol{\beta}^*||_1 \lesssim C\eta_1(n)\}$ for some constant C. Notice that for each m

$$\sum_{y} w(y) \int_{\mathbf{z}} \frac{z_{m}y}{\delta} \int_{x} \int_{\frac{x-\hat{\boldsymbol{\beta}}^{(k)T}\mathbf{z}}{\delta}}^{\frac{x-\hat{\boldsymbol{\beta}}^{(k)T}\mathbf{z}}{\delta}} K''(t) (\frac{x-\hat{\boldsymbol{\beta}}^{(k)T}\mathbf{z}}{\delta} - t) dt f(x|y, \mathbf{z}) dx f(y, \mathbf{z}) dz$$

$$(u = (x - \underline{\boldsymbol{\beta}}^{*T}\mathbf{z})/\delta) \sum_{y} w(y) \int_{\mathbf{z}} z_{m}y \int_{u} \int_{u}^{u+\triangle} K''(t) (u + \triangle - t) dt f(u\delta + \boldsymbol{\beta}^{*T}\mathbf{z}|y, \mathbf{z}) du f(y, \mathbf{z}) dz$$

$$= \sum_{y} w(y) \int_{\mathbf{z}} z_{m}y \int_{u} \int_{u}^{u+\triangle} K''(t) (u + \triangle - t) dt f(\boldsymbol{\beta}^{*T}\mathbf{z}|y, \mathbf{z}) du f(y, \mathbf{z}) dz$$

$$+ \sum_{y} w(y) \int_{\mathbf{z}} z_{m}y \int_{u} u\delta \int_{u}^{u+\triangle} K''(t) (u + \triangle - t) dt f'(\tau u\delta + \boldsymbol{\beta}^{*T}\mathbf{z}|y, \mathbf{z}) du f(y, \mathbf{z}) d\mathbf{z},$$
(C.10)

where $\triangle = \frac{(\widehat{\boldsymbol{\beta}}^{(k)} - \boldsymbol{\beta})^T \boldsymbol{z}}{\delta}$ and $\tau \in [0, 1]$. Here the first step is by definition and the last step is from the mean value theorem. Since

$$\int_{u} \int_{u}^{u+\Delta} K''(t)(u+\Delta-t)dtdu = 0,$$
(C.11)

we can show that the first term on the RHS of the last step is 0. The second term on the RHS can be bounded by $C'\delta|\Delta|^2 \lesssim M_n^2\eta_1(n)^2/\delta$ for some constant C' on event A. This implies that $\mathbb{E}[G_{im}|A] \lesssim M_n^2\eta_1(n)^2/\delta$.

Now we look at its variance. For the second moment, for each $i \in \mathcal{N}_j, m = 1, \dots, d$, we have

$$\mathbb{E}\left[\left(w(y_{i})\frac{z_{im}y_{i}}{\delta}\int_{\frac{x_{i}-\boldsymbol{\beta}^{*T}\boldsymbol{z}_{i}}{\delta}}^{\frac{x_{i}-\boldsymbol{\beta}^{(k)}\boldsymbol{z}_{i}}{\delta}}K''(t)\left(\frac{x_{i}-\boldsymbol{\hat{\beta}}^{(k)}\boldsymbol{z}_{i}}{\delta}-t\right)dt\right)^{2}\middle|A\right]$$

$$=\mathbb{E}\left[\sum_{y}w(y)^{2}\int_{\boldsymbol{z}}\frac{z_{m}^{2}}{\delta}\int_{u}\left(\int_{u}^{u+\Delta}K''(t)(u+\Delta-t)dt\right)^{2}f(u\delta+\boldsymbol{\beta}^{*T}\boldsymbol{z}|\boldsymbol{y},\boldsymbol{z})duf(\boldsymbol{y},\boldsymbol{z})d\boldsymbol{z}|A\right]$$

$$\leq\mathbb{E}\left[\sum_{y}w(y)^{2}\int_{\boldsymbol{z}}\frac{z_{m}^{2}}{\delta}2|f|_{\infty}|K''|_{\infty}^{2}\Delta^{4}f(\boldsymbol{y},\boldsymbol{z})d\boldsymbol{z}|A\right]$$

$$\lesssim\frac{M_{n}^{4}\eta_{1}(n)^{4}}{\delta^{5}}.$$
(C.12)

Also we know for each $i \in \mathcal{N}_j, m = 1, \ldots, d$, $\left| \frac{z_{im}y_i}{\delta} \int_{\frac{x_i - \widehat{\boldsymbol{\beta}}^{(k)} z_i}{\delta}}^{\frac{x_i - \widehat{\boldsymbol{\beta}}^{(k)} z_i}{\delta}} K''(t) \left(\frac{x_i - \widehat{\boldsymbol{\beta}}^{(k)T} z_i}{\delta} - t \right) dt \right| \lesssim \frac{M_n}{\delta} \left(\frac{M_n \eta_1(n)}{\delta} \right)^2$ on A, and therefore

$$Var[G_{im}|A] \lesssim \frac{M_n^4 \eta_1(n)^4}{\delta^5}.$$
 (C.13)

Therefore, applying Bernstein inequality with $M_n \sqrt{\frac{\log(d)}{n\delta}} = \mathcal{O}(1)$, we can obtain that

$$\mathbb{P}\left(\max_{m} \left| \frac{1}{\mathcal{N}_{j}} \sum_{i \in \mathcal{N}_{j}} G_{im} - \mathbb{E}G_{m} \right| > \frac{M_{n}^{2} \eta_{1}(n)^{2}}{\delta} \sqrt{\frac{\log(d)}{n\delta^{3}}} \right| A\right) \leq \mathcal{O}(d^{-1}), \tag{C.14}$$

which further implies that

$$\mathbb{P}\left(\max_{m} \left| \frac{1}{\mathcal{N}_{j}} \sum_{i \in \mathcal{N}_{j}} G_{im} - \mathbb{E}G_{m} \right| > \frac{M_{n}^{2} \eta_{1}(n)^{2}}{\delta} \sqrt{\frac{\log(d)}{n\delta^{3}}} \right) \\
\leq \mathbb{P}\left(\max_{m} \left| \frac{1}{\mathcal{N}_{j}} \sum_{i \in \mathcal{N}_{j}} G_{im} - \mathbb{E}G_{m} \right| > \frac{M_{n}^{2} \eta_{1}(n)^{2}}{\delta} \sqrt{\frac{\log(d)}{n\delta^{3}}} \right| A \right) + \mathbb{P}(A^{C}) \\
= o(1), \tag{C.15}$$

where the last step follows from Assumption 6.

$$\left\| \frac{1}{\mathcal{N}_j} \sum_{i \in \mathcal{N}_j} w(Y_i) \frac{\mathbf{Z}_i Y_i}{\delta} \int_{\frac{X_i - \widehat{\boldsymbol{\beta}}^{(k)} \mathbf{Z}_i}{\delta}}^{\frac{X_i - \widehat{\boldsymbol{\beta}}^{(k)} \mathbf{Z}_i}{\delta}} K''(t) \left(\frac{X_i - \widehat{\boldsymbol{\beta}}^{(k)} \mathbf{Z}_i}{\delta} - t \right) dt \right\|_{\infty} = \mathcal{O}_{\mathbb{P}} \left(\frac{M_n^2 \eta_1(n)^2}{\delta} \right). \tag{C.16}$$

Under the conditions of Theorem 1, the desired result holds. This completes the proof. \Box

Lemma 8. Under the conditions of Theorem 1, let $s' = ||\omega^*||_1$ and $\xi > 0$ is some constant. If $s'(M_n\eta_1(n) \vee \sqrt{\frac{\log(d)}{n\delta^3}} \vee \delta^{\ell-1}) = o(1)$, then with probability tending to one it holds that $\kappa_D(s') \geq \kappa/\sqrt{2}$, where

$$\kappa_D(s') = \min \left\{ rac{s'^{1/2} (oldsymbol{v}^T
abla_{\gamma \gamma}^2 R_\delta^n(\widehat{oldsymbol{eta}}) oldsymbol{v})^{1/2}}{||oldsymbol{v}_{s'}||_1} : oldsymbol{v} \in \mathbb{R}^{d-1} ackslash \{0\}, ||oldsymbol{v}_{s'^c}||_1 \le \xi ||oldsymbol{v}_{s'}||_1
ight\}.$$

Proof. This proof is similar to the proof of Lemma J.3 in Ning et al. (2017) so we only give a sketch here. Firstly we have

$$\kappa_D(s')^2 \ge \min\left\{\frac{\boldsymbol{v}^T \nabla_{\boldsymbol{\gamma}\boldsymbol{\gamma}}^2 R_{\delta}^n(\widehat{\boldsymbol{\beta}}) \boldsymbol{v}}{||\boldsymbol{v}||_2^2} : \boldsymbol{v} \in \mathbb{R}^{d-1} \setminus \{0\}, ||\boldsymbol{v}_{s'^c}||_1 \le \xi ||\boldsymbol{v}_{s'}||_1\right\}. \tag{C.17}$$

Similar to the proof of Theorem 1, Lemmas 4, 5, 6 and 13 together imply that

$$\left| \frac{\mathbf{v}^{T}(\nabla_{\gamma\gamma}^{2}R_{\delta}^{n}(\widehat{\boldsymbol{\beta}}) - \nabla_{\gamma\gamma}^{2}R(\boldsymbol{\beta}^{*}))\mathbf{v}}{||\mathbf{v}||_{2}^{2}} \right|
= \left| \frac{\mathbf{v}^{T}\left[\frac{1}{2}\left(\nabla_{\gamma\gamma}^{2}R_{\delta}^{n_{(1)}}(\widehat{\boldsymbol{\beta}}^{(2)}) + \nabla_{\gamma\gamma}^{2}R_{\delta}^{n_{(2)}}(\widehat{\boldsymbol{\beta}}^{(1)})\right) - \nabla_{\gamma\gamma}^{2}R(\boldsymbol{\beta}^{*})\right]\mathbf{v}}{||\mathbf{v}||_{2}^{2}} \right|
\lesssim s'(\xi+1)^{2}\left[M_{n}\eta_{1}(n) \vee \sqrt{\frac{\log(d)}{n\delta^{3}}} \vee \delta^{\ell-1}\right] = o_{\mathbb{P}}(1),$$
(C.18)

where the inequality is because $||\boldsymbol{v}||_1^2 \leq s'(\xi+1)^2||\boldsymbol{v}||_2^2$. Therefore for n large enough, we have $|\frac{\boldsymbol{v}^T(\nabla^2_{\gamma\gamma}R^n_\delta(\tilde{\boldsymbol{\beta}})-\nabla^2_{\gamma\gamma}R(\boldsymbol{\beta}^*))\boldsymbol{v}}{||\boldsymbol{v}||_2^2}|\leq \frac{1}{2}\kappa^2$. This implies $\kappa_D(\tilde{\boldsymbol{s}}^*)^2\geq \frac{1}{2}\kappa^2$ with probability tending to 1. This completes the proof.

Lemma 9. Under the same conditions as in Theorem 2, for j = 1, 2, it holds that

$$(n\delta)^{1/2}|\boldsymbol{v}^{*T}(\nabla R^{n_{(j)}}_{\delta}(0,\boldsymbol{\gamma}^*)-\nabla R^{n_{(j)}}_{\delta}(\boldsymbol{\beta}^*))+\theta^*\boldsymbol{v}^{*T}\nabla^2_{\cdot\theta}R(\boldsymbol{\beta}^*)|=o_{\mathbb{P}}(1).$$

Proof. By definition

$$|\boldsymbol{v}^{*T}(\nabla R_{\delta}^{n_{(j)}}(0,\boldsymbol{\gamma}^{*}) - \nabla R_{\delta}^{n_{(j)}}(\boldsymbol{\beta}^{*})) + \boldsymbol{\theta}^{*}\boldsymbol{v}^{*T}\nabla_{\cdot\boldsymbol{\theta}}^{2}R(\boldsymbol{\beta}^{*})|$$

$$\leq ||\boldsymbol{v}^{*}||_{1} \left[\underbrace{||\nabla R_{\delta}^{n_{(j)}}(0,\boldsymbol{\gamma}^{*}) - \nabla R_{\delta}^{n_{(j)}}(\boldsymbol{\beta}^{*}) + \boldsymbol{\theta}^{*}\nabla_{\cdot\boldsymbol{\theta}}^{2}R_{\delta}^{n_{(j)}}(\boldsymbol{\beta}^{*})||_{\infty}}_{I_{1}} + \underbrace{||\boldsymbol{\theta}^{*}(\nabla_{\cdot\boldsymbol{\theta}}^{2}R_{\delta}(\boldsymbol{\beta}^{*}) - \nabla_{\cdot\boldsymbol{\theta}}^{2}R(\boldsymbol{\beta}^{*}))||_{\infty}}_{I_{2}} + \underbrace{||\boldsymbol{\theta}^{*}(\nabla_{\cdot\boldsymbol{\theta}}^{2}R_{\delta}(\boldsymbol{\beta}^{*}) - \nabla_{\cdot\boldsymbol{\theta}}^{2}R(\boldsymbol{\beta}^{*}))||_{\infty}}_{I_{3}} \right]. \tag{C.19}$$

For I_1 , similar to Lemma 7, we can write

$$|\nabla R_{\delta}^{n_{(j)}}(0, \boldsymbol{\gamma}^*) - \nabla R_{\delta}^{n_{(j)}}(\boldsymbol{\beta}^*) + \theta^* \nabla_{\cdot \theta}^2 R_{\delta}^{n_{(j)}}(\boldsymbol{\beta}^*)|$$

$$= \left| \frac{1}{\mathcal{N}_j} \sum_{i \in \mathcal{N}_j} w(y_i) \frac{y_i \boldsymbol{z}_i}{\delta} \int_{\frac{\boldsymbol{x}_i - \boldsymbol{\beta}_0^{*T} \boldsymbol{z}_i}{\delta}}^{\frac{\boldsymbol{x}_i - \boldsymbol{\beta}_0^{*T} \boldsymbol{z}_i}{\delta}} K''(t) (\frac{\boldsymbol{x}_i - \boldsymbol{\beta}_0^{*T} \boldsymbol{z}_i}{\delta} - t) dt \right|$$

$$:= \left| \frac{1}{\mathcal{N}_j} \sum_{i \in \mathcal{N}_i} G_i \right|, \tag{C.20}$$

where $G_i = w(y_i) \frac{y_i z_i}{\delta} \int_{\frac{x_i - \beta_0^{*T} z_i}{\delta}}^{\frac{x_i - \beta_0^{*T} z_i}{\delta}} K''(t) (\frac{x_i - \beta_0^{*T} z_i}{\delta} - t) dt \in \mathbb{R}^d$. Thus, similar to the derivation of Lemma 7, for each $1 \leq m \leq d$, we can show that $\mathbb{E}[G_{im}] \lesssim M_n^2 \theta^{*2} / \delta$, $|G_{im}| \lesssim M_n^3 \theta^{*2} / \delta^3$ and $Var[G_{im}] \lesssim M_n^4 \theta^{*4} / \delta^5$, and thus applying Bernstein inequality yields

$$I_1 = \mathcal{O}_{\mathbb{P}}(\frac{M_n^2 \theta^{*2}}{\delta}).$$

Meanwhile, Lemma 5 and Lemma 4 imply that $I_2 = \mathcal{O}_{\mathbb{P}}(\theta^* \sqrt{\frac{\log(d)}{n\delta^3}})$ and $I_3 = \mathcal{O}(\theta^* \delta^{\ell-1})$. Combing the above results, we obtain

$$|\boldsymbol{v}^{*T}(\nabla R_{\delta}^{n_{(j)}}(0,\boldsymbol{\gamma}^{*}) - \nabla R_{\delta}^{n_{(j)}}(\boldsymbol{\beta}^{*})) + \theta^{*}\boldsymbol{v}^{*T}\nabla_{\cdot\theta}^{2}R(\boldsymbol{\beta}^{*})| = \mathcal{O}_{\mathbb{P}}(\frac{||\boldsymbol{v}^{*}||_{1}\theta^{*}}{\delta}(M_{n}^{2}\theta^{*}\vee\sqrt{\frac{\log(d)}{n\delta}}\vee\delta^{\ell})). \tag{C.21}$$

This together with the condition in Theorem 2 completes the proof.

Lemma 10. Let $s' = ||\omega^*||_0$. Suppose Assumptions 1-5 hold, and $\lambda_{\min}(\nabla^2_{\gamma,\gamma}R(\boldsymbol{\beta}^*)) \geq c$ for some constant c > 0. If we choose $\delta \simeq (\log(d)/n)^{1/(2\ell+1)}$ and tuning parameter $\lambda' \simeq ||\omega^*||_1(M_n\eta_1(n) + (\log(d)/n)^{(\ell-1)/(2\ell+1)})$ and $s'(M_n\eta_1(n) + (\log(d)/n)^{(\ell-1)/(2\ell+1)}) = o(1)$, then it holds that

$$||\widehat{\boldsymbol{\omega}} - {\boldsymbol{\omega}}^*||_1 \lesssim ||{\boldsymbol{v}}^*||_1 s'(M_n \eta_1(n) + (\log(d)/n)^{(\ell-1)/(2\ell+1)}).$$
 (C.22)

Proof. Here we focus on the rate for $\widehat{\boldsymbol{\omega}}^{(1)}$ and the result will follow accordingly. Denote $\widehat{\triangle} = \widehat{\boldsymbol{\omega}}^{(1)} - \boldsymbol{\omega}^*$. By definition, we can show that $||\widehat{\triangle}_{\widetilde{s}^{*c}}||_1 \leq ||\widehat{\triangle}_{\widetilde{s}^*}||_1$.

With some algebra, we will get

$$\widehat{\triangle}^{T} \nabla_{\gamma \gamma}^{2} R_{\delta}^{n_{(1)}}(\widehat{\boldsymbol{\beta}}^{(2)}) \widehat{\triangle} = \widehat{\triangle}^{T} (\nabla_{\gamma}^{2} R_{\delta}^{n_{(1)}}(\widehat{\boldsymbol{\beta}}^{(2)}) \boldsymbol{v}^{*}) - \widehat{\triangle}^{T} (\nabla_{\gamma}^{2} R_{\delta}^{n_{(1)}}(\widehat{\boldsymbol{\beta}}^{(2)}) \widehat{\boldsymbol{v}}^{(1)}) \\
\leq ||\widehat{\triangle}||_{1} ||\nabla_{\gamma}^{2} R_{\delta}^{n_{(1)}}(\widehat{\boldsymbol{\beta}}^{(2)}) \boldsymbol{v}^{*}||_{\infty} + ||\widehat{\triangle}||_{1} ||\nabla_{\gamma \theta}^{2} R_{\delta}^{n_{(1)}}(\widehat{\boldsymbol{\beta}}^{(2)}) - \nabla_{\gamma \gamma}^{2} R_{\delta}^{n_{(1)}}(\widehat{\boldsymbol{\beta}}^{(2)}) \widehat{\boldsymbol{\omega}}^{(1)}||_{\infty}.$$
(C.23)

By definition, the second term is bounded by $\lambda'||\widehat{\triangle}||_1$. For the first term, recall by definition $\nabla^2_{\gamma} R(\beta^*) v^* = 0$, and thus similar to the proof of Theorem 1, Lemma 4, 5 and 6 imply that with probability approaching to 1

$$||\nabla_{\boldsymbol{\gamma}}^2 R_{\delta}^{n_{(1)}}(\widehat{\boldsymbol{\beta}}^{(2)}) \boldsymbol{v}^*||_{\infty} \lesssim ||\boldsymbol{v}^*||_1 \bigg[M_n \eta_1(n) \vee \sqrt{\frac{\log(d)}{n\delta^3}} \vee \delta^{\ell-1} \bigg],$$

and thus with the choice of λ' it holds that $\widehat{\triangle}^T \nabla^2_{\gamma \gamma} R^{n_{(1)}}_{\delta}(\widehat{\boldsymbol{\beta}}^{(2)}) \widehat{\triangle} \lesssim \lambda' ||\widehat{\triangle}||_1$. This together with Lemma 8 implies that $||\widehat{\triangle}||_1 \lesssim s' \lambda'$ with high probability. This completes the proof.

Lemma 11. Under the same conditions of Theorem 1, if U is a proper kernel of order ℓ satisfying the same condition as K in Assumption 2, and in addition U is ℓ times continuously differentiable and $U^{(i)}$ degenerates at the boundary for $i = 0, \ldots, \ell - 1$, then when $\sqrt{\frac{\log(d)}{nh^{2\ell+1}}} + (M_n\eta_1(n) \vee h)^{\zeta} = o(1)$, it holds that

$$|\widehat{\mu} - v^{*T}b^*| \lesssim ||v^*||_1 \left(\eta_2(n) + \sqrt{\frac{\log(d)}{nh^{2\ell+1}}} + (M_n\eta_1(n) \vee h)^{\zeta}\right).$$

Proof. It suffices to proof the results for $\widehat{v}^{(1)T}\widehat{T}_{h,U}^{(\ell),n_{(1)}}(\widehat{\boldsymbol{\beta}}^{(2)})$. By definition,

$$|\boldsymbol{v}^{*T}T^{(\ell)}(\boldsymbol{\beta}^{*}) - \widehat{\boldsymbol{v}}^{(1)T}\widehat{T}_{h,U}^{(\ell),n_{(1)}}(\widehat{\boldsymbol{\beta}}^{(2)})| \leq |(\boldsymbol{v}^{*} - \widehat{\boldsymbol{v}})^{T}\widehat{T}_{h,U}^{(\ell),n_{(1)}}(\widehat{\boldsymbol{\beta}}^{(2)})| + |\boldsymbol{v}^{*T}(T^{(\ell)}(\boldsymbol{\beta}^{*}) - \widehat{T}_{h,U}^{(\ell),n_{(1)}}(\widehat{\boldsymbol{\beta}}^{(2)}))|.$$
(C.24)

We firstly look at $||T^{(\ell)}(\boldsymbol{\beta}^*) - \widehat{T}_{h,U}^{(\ell),n_{(1)}}(\widehat{\boldsymbol{\beta}}^{(2)})||_{\infty}$. Direct calculation gives that

$$||T^{(\ell)}(\boldsymbol{\beta}^*) - \widehat{T}_{h,U}^{(\ell),n_{(1)}}(\widehat{\boldsymbol{\beta}}^{(2)})||_{\infty} \\ \leq ||T^{(\ell)}(\boldsymbol{\beta}^*) - T^{(\ell)}(\widehat{\boldsymbol{\beta}}^{(2)})||_{\infty} + ||T^{(\ell)}(\widehat{\boldsymbol{\beta}}^{(2)}) - \widetilde{T}_{h,U}^{(\ell)}(\widehat{\boldsymbol{\beta}}^{(2)})||_{\infty} + ||\widetilde{T}_{h,U}^{(\ell)}(\widehat{\boldsymbol{\beta}}^{(2)}) - \widehat{T}_{h,U}^{(\ell),n_{(1)}}(\widehat{\boldsymbol{\beta}}^{(2)})||_{\infty},$$
(C.25)

where $\widetilde{T}_{h,U}^{(\ell)}(\widehat{\boldsymbol{\beta}}^{(2)}) := \sum_{y \in \{-1,1\}} w(y) y \int \frac{\boldsymbol{z}}{h^{1+\ell}} \int U^{(\ell)} \left(\frac{\widehat{\boldsymbol{\beta}}^{(2)T} \boldsymbol{z} - \boldsymbol{x}}{h}\right) f(\boldsymbol{x}|\boldsymbol{y}, \boldsymbol{z}) d\boldsymbol{x} f(\boldsymbol{y}, \boldsymbol{z}) d\boldsymbol{z}$. For the first term, we have

$$||T^{(\ell)}(\boldsymbol{\beta}^*) - T^{(\ell)}(\widehat{\boldsymbol{\beta}}^{(2)})||_{\infty}$$

$$= \left| \left| \sum_{y \in \{-1,1\}} w(y) \int y \boldsymbol{z} (f^{(\ell)}(\boldsymbol{\beta}^{*T} \boldsymbol{z} | y, \boldsymbol{z}) - f^{(\ell)}(\widehat{\boldsymbol{\beta}}^{(2)T} \boldsymbol{z} | y, \boldsymbol{z})) f(y, \boldsymbol{z}) d\boldsymbol{z} \right| \right|_{\infty}$$

$$= \mathcal{O}_{\mathbb{P}}((M_n \eta_1(n))^{\zeta}), \tag{C.26}$$

where the last step follows from Assumption 1. For the second term, notice that we can rewrite

$$\widetilde{T}_{h,U}^{(\ell)}(\widehat{\boldsymbol{\beta}}^{(2)}) = \sum_{y \in \{-1,1\}} w(y)y \int \frac{\boldsymbol{z}}{h} \int U\left(\frac{\widehat{\boldsymbol{\beta}}^{(2)T}\boldsymbol{z} - \boldsymbol{x}}{h}\right) f^{(\ell)}(\boldsymbol{x}|\boldsymbol{y}, \boldsymbol{z}) d\boldsymbol{x} f(\boldsymbol{y}, \boldsymbol{z}) d\boldsymbol{z}
= \sum_{y \in \{-1,1\}} w(y)y \int \boldsymbol{z} \int U(u) f^{(\ell)}(\widehat{\boldsymbol{\beta}}^{(2)T}\boldsymbol{z} - hu|\boldsymbol{y}, \boldsymbol{z}) du f(\boldsymbol{y}, \boldsymbol{z}) d\boldsymbol{z},$$
(C.27)

where the first step is by repeated integration by parts and the second step is by a change of variable. This implies that

$$||T^{(\ell)}(\widehat{\boldsymbol{\beta}}^{(2)}) - \widetilde{T}_{h,U}^{(\ell)}(\widehat{\boldsymbol{\beta}}^{(2)})||_{\infty}$$

$$= \left| \left| \sum_{y \in \{-1,1\}} w(y)y \int \boldsymbol{z} \int U(u) \left(f^{(\ell)}(\widehat{\boldsymbol{\beta}}^{(2)T}\boldsymbol{z}|y,\boldsymbol{z}) - f^{(\ell)}(\widehat{\boldsymbol{\beta}}^{(2)T}\boldsymbol{z} - hu|y,\boldsymbol{z}) \right) du f(y,\boldsymbol{z}) d\boldsymbol{z} \right| \right|_{\infty}$$

$$= \mathcal{O}(h^{\zeta}). \tag{C.28}$$

Finally, similar to the proof of Theorem 1, we can show that the third term on the RHS of (C.25) is $\mathcal{O}_{\mathbb{P}}(\sqrt{\frac{\log(d)}{nh^{2\ell+1}}})$ based on Lemmas 5 and 13. Putting all three terms together we have

$$||T^{(\ell)}(\boldsymbol{\beta}^*) - \widehat{T}_{h,U}^{(\ell),n_{(1)}}(\widehat{\boldsymbol{\beta}}^{(2)})||_{\infty} = \mathcal{O}_{\mathbb{P}}\left(\sqrt{\frac{\log(d)}{nh^{2\ell+1}}} + (M_n\eta_1(n) \vee h)^{\zeta}\right), \tag{C.29}$$

and thus $||\widehat{T}_{h,U}^{(\ell),n_{(1)}}(\widehat{\boldsymbol{\beta}}^{(2)})|| = \mathcal{O}_{\mathbb{P}}(1)$ by the condition of this lemma. Plugging this back to (C.24) gives the desired result. This completes the proof.

Lemma 12. Under the same conditions of Theorem 1, if L is a proper kernel of order ℓ satisfying Assumption 2 and $g^{\ell} + \sqrt{\frac{\log(d)}{ng}} + M_n \eta_1(n) = o(1)$, then we have

$$|\widehat{\sigma}^2 - \boldsymbol{v}^{*T} \boldsymbol{\Sigma}^* \boldsymbol{v}^*| \lesssim ||\boldsymbol{v}^*||_1^2 \left(\eta_2(n) + g^{\ell} + \sqrt{\frac{\log(d)}{ng}} + M_n \eta_1(n) \right).$$
 (C.30)

Proof. It suffices to show the convergence rate of $\widetilde{\mu}_K \widehat{\boldsymbol{v}}^{(1)T} \widehat{H}_{g,K}^{n_{(1)}}(\widehat{\boldsymbol{\beta}}^{(2)}) \widehat{\boldsymbol{v}}^{(1)}$. Similar to the proof of Theorem 1, following Lemmas 4, 5, 6 and 13, we can show that

$$||\widehat{H}_{g,K}^{(1)}(\widehat{\boldsymbol{\beta}}^{(2)}) - H(\boldsymbol{\beta}^*)||_{\max} = \mathcal{O}_{\mathbb{P}}\left(g^{\ell} + \sqrt{\frac{\log(d)}{ng}} + M_n\eta_1(n)\right).$$
 (C.31)

Since $||H(\boldsymbol{\beta}^*)||_{\max} = \mathcal{O}(1)$, we know $||\widehat{H}_{g,K}^{(1)}(\widehat{\boldsymbol{\beta}}^{(2)})||_{\max} = \mathcal{O}_{\mathbb{P}}(1)$ when $g^{\ell} + \sqrt{\frac{\log(d)}{ng}} + M_n \eta_1(n) = o(1)$. Now by triangle inequality, we obtain

$$|\widehat{\boldsymbol{v}}^{(1)T}\widehat{H}_{g,K}^{(1)}(\widehat{\boldsymbol{\beta}}^{(2)})\widehat{\boldsymbol{v}}^{(1)} - \boldsymbol{v}^{*T}H(\boldsymbol{\beta}^{*})\boldsymbol{v}^{*}| \\ \leq ||\widehat{\boldsymbol{v}}^{(1)} - \boldsymbol{v}^{*}||_{1}^{2}||\widehat{H}_{g,K}^{(1)}(\widehat{\boldsymbol{\beta}}^{(2)})||_{\max} + 2||\boldsymbol{v}^{*T}\widehat{H}_{g,K}^{(1)}(\widehat{\boldsymbol{\beta}}^{(2)})||_{\infty}||\widehat{\boldsymbol{v}}^{(1)} - \boldsymbol{v}^{*}||_{1} + |\boldsymbol{v}^{*T}(\widehat{H}_{g,K}^{(1)}(\widehat{\boldsymbol{\beta}}^{(2)}) - H(\boldsymbol{\beta}^{*}))\boldsymbol{v}^{*}|.$$
(C.32)

This implies that

$$|\widehat{\sigma}^2 - \boldsymbol{v}^{*T} \boldsymbol{\Sigma}^* \boldsymbol{v}^*| = ||\boldsymbol{v}^*||_1^2 \mathcal{O}_{\mathbb{P}} \left(\eta_2(n) + g^{\ell} + \sqrt{\frac{\log(d)}{ng}} + M_n \eta_1(n) \right).$$
 (C.33)

This completes the proof.

Lemma 13 (Lemma 6.1 of Chernozhukov et al. (2018)). Let $\{X_m\}$ and $\{Y_m\}$ be sequences of random vectors. If $||X_m|| = \mathcal{O}_{\mathbb{P}}(A_m)$ conditional on $\{Y_m\}$ for a sequence of positive constants $\{A_m\}$, then $||X_m|| = \mathcal{O}_{\mathbb{P}}(A_m)$ unconditionally.

D Practical Considerations

In practice, in order to apply the smoothed decorrelated score test presented in Section 2.2, we need to select the bandwidth and regularization parameter in the initial estimators $\hat{\omega}$ and $\hat{\beta}$, two bandwidths for the plug-in asymptotic bias and variance estimators in (2.11) and (2.13), and the final bandwidth δ for the score test statistic. In this section, we discuss practical considerations for the proposed test as well as the data-driven selection procedure for δ .

We use a path-following algorithm to compute the initial estimator $\widehat{\beta}$ and apply a two way cross-validation approach to choose the tuning parameters (δ, λ) in the optimization (2.2); see Feng et al. (2019) for details. The initial values of the path-following algorithm used in the simulation are shown in the appendix Section E. To compute the Dantzig estimator $\widehat{\omega}$, we use the "flare" package in R. The same cross-validation method can be used to select the tuning parameters (δ, λ') in (2.15). Meanwhile, we find the empirical performance of the Dantzig estimator $\widehat{\omega}$ is not very sensitive to the choice of (δ, λ') . In the simulation and real data analysis, to ease the computation, we set $\lambda' = 2(\log d/n)^{1/5}$ and $\delta = 1$ in (2.15). Notice that in rare cases, the Dantzig solver may become unstable numerically when the Hessian $\nabla^2 R_{\delta}^n(\widehat{\beta})$ is ill-conditioned. In this case, we will firstly project it onto the cone of positive definite matrices and then plug-in the projected matrix into (2.15) to solve $\widehat{\omega}$.

For the bandwidth parameters h and g in the plug-in bias and variance estimators, Lemma 11 and Lemma 12 imply that the theoretical optimal order for h and g are $(\log(d)/n)^{2\ell+2\zeta+1}$ (if $M_n\eta_1(n)=o(h)$ which holds under mild conditions) and $(\log(d)/n)^{2\ell+1}$, respectively. In practice, we recommend choosing $h=c_1(\log(d)/n)^{2\ell'+3}$ and $g=c_2(\log(d)/n)^{2\ell'+1}$ for some $c_1,c_2>0$ (e.g., $c_1=c_2=2$ in our simulation), where ℓ' is the order of the kernel function K.

Finally, for the main bandwidth parameter δ appearing in the score test statistic, if a prespecified value is preferred, we recommend choosing $\delta = cn^{-1/(2\ell'+1)}$ for some constant c>0 (e.g., c=1). When using our proposed data-driven bandwidth selector, notice that another bandwidth parameter b, whose optimal order is $(\log(d)/n)^{1/(2\ell+2r+1)}$ (see the discussion after Theorem 3), is required for estimating the squared bias $SB(\delta)$. Similarly, we suggest choosing $b=c_0(\log(d)/n)^{1/(2\ell'+2r'+1)}$ for some c_0 (e.g., $c_0=0.5$), where c_0 is the order of kernel c_0 .

To further automate and robustify the whole procedure with data-driven bandwidth $\hat{\delta}$, we provide an alternative variance estimator for the score function which does not require an additional bandwidth parameter. Given $\hat{\delta}$, the numerator of the score statistic \hat{U}_n can be written as

$$\widehat{S}_{\widehat{\delta}}(0,\widehat{\gamma}) - \widehat{\delta}^{\ell}\widehat{\mu} = \widehat{\boldsymbol{v}}^T \nabla R_{\widehat{\delta}}^n(\widehat{\boldsymbol{\beta}}_0) - \widehat{\delta}^{\ell} \gamma_{K,\ell} \widehat{\boldsymbol{v}}^T \widehat{T}_{h,U}^{\ell}(\widehat{\boldsymbol{\beta}}) = D_{\widehat{\delta},h}^n(\widehat{\boldsymbol{\beta}},\widehat{\boldsymbol{v}}), \tag{D.1}$$

where

$$D_{\delta,h}^{n}(\boldsymbol{\beta},\boldsymbol{v}) = \frac{1}{n} \sum_{i=1}^{n} w(y_i) y_i \frac{\boldsymbol{v}^T \boldsymbol{z}_i}{\delta} M(x_i, \boldsymbol{z}_i; \boldsymbol{\beta}),$$

and $M(x_i, \mathbf{z}_i; \boldsymbol{\beta}) = K(\frac{\boldsymbol{\beta}_0^T \mathbf{z}_i - x_i}{h}) - \frac{\delta^{\ell+1}}{h^{\ell+1}} U^{(\ell)}(\frac{\boldsymbol{\beta}^T \mathbf{z}_i - x_i}{h})$. To this end, we may alternatively seek to estimate the variance

$$\operatorname{Var}[D_{\delta,h}^{n}(\boldsymbol{\beta},\boldsymbol{v})] = \frac{1}{n} \left\{ \mathbb{E}\left[(w(Y)Y \frac{\boldsymbol{v}^{T}\boldsymbol{Z}}{\delta} M(X,\boldsymbol{Z};\boldsymbol{\beta}))^{2} \right] - \mathbb{E}\left[w(Y)Y \frac{\boldsymbol{v}^{T}\boldsymbol{Z}}{\delta} M(X,\boldsymbol{Z};\boldsymbol{\beta}) \right]^{2} \right\}$$
(D.2)

by replacing the expectations above with sample averages under $\hat{\delta}, \hat{\beta}$ and \hat{v} . We can see that this variance estimator does not require additional bandwidth parameters as in the pilot estimator (2.13) and thus makes the procedure more convenient in practice. In addition, we find this estimator empirically robust when the bandwidth parameter δ is selected adaptively using the data.

E Additional Numerical Results

E.1 Computational details of the path-following algorithm

In Section 6, we apply the path-following algorithm proposed in Feng et al. (2019) to estimate the true coefficient β . We fix the number of stages N=25, $\nu=0.25$, $\eta=0.25$, $\epsilon_{tgt}=0.0001$ and set $\Omega=\{\beta:||\beta||_2\leq 10^3\}$ for each parameter in Algorithm 2.1 and 2.2 in Feng et al. (2019).

E.2 Normal Q-Q plots in Section 6.1

Figure E.1 contains the Normal Q-Q plots for the test statistics obtained from the experiments in Section 6.1 under both scenarios. We can see that the Q-Q plots from the proposed SDS method appear close to Gaussian, while the Q-Q plots for DS and Honest method deviate from standard Gaussian.

E.3 Empirical power in Section 6.1

Figure E.2 shows the empirical power under the Heteroskedastic Uniform scenario in Section 6.1. Similar to the figures for the Heteroskedastic Gaussian scenario, the empirical power converges to 1 as the magnitude of β^* becomes larger. In addition, as the correlation between Z becomes larger, the power also decreases.

E.4 Bandwidth selection in Section 6.2

Finally, we investigate whether the data-driven bandwidth is close to the theoretically optimal bandwidth that minimizes the MSE. We focus on the Heteroskedastic Gaussian scenario and consider the setting with $n=800, d=50, s=10, \rho=0.2$ and set $\beta_1^*=\ldots=\beta_{10}^*=1/\sqrt{10}$. We compare $M(\delta)$ with $\widehat{M}(\delta)$, where $\widehat{M}(\delta)$ is obtained using the implementation discussed in Section D, but with different bandwidth b=0.15, 0.2, 0.25, 0.3 for squared bias estimation. Notice that since directly obtaining $M(\delta)$ is hard, we apply Monte Carlo integration to approximate this quantity.

Figure E.3 shows the true $M(\delta)$ as well as $\widehat{M}(\delta)$ in each case. The red line is the true MSE $M(\delta)$ approximated by Monte Carlo method, and the black solid line is the average estimated MSE $\widehat{M}(\delta)$ with different bandwidths b over 100 repetitions. The black dashed line is the one standard deviation band around $\widehat{M}(\delta)$ over these repetitions. As we can see, for all cases, $M(\delta)$ lies within the one standard deviation band around $\widehat{M}(\delta)$ and the minimizer of $M(\delta)$ is very close to the minimizer of the average $\widehat{M}(\delta)$. This experiment confirms that the data-driven bandwidth $\widehat{\delta}$ is close to the optimal δ^* , provided that the bandwidth b for squared-bias estimation lies in a suitable range.

E.5 Additional results from ChAMP analysis

Table E.1 lists the five most significant variables using the DS and Honest approach. Notice that the significant variables identified by the proposed approach are also in the list for the DS approach. Honest yields very different set of variables compared to the other two.

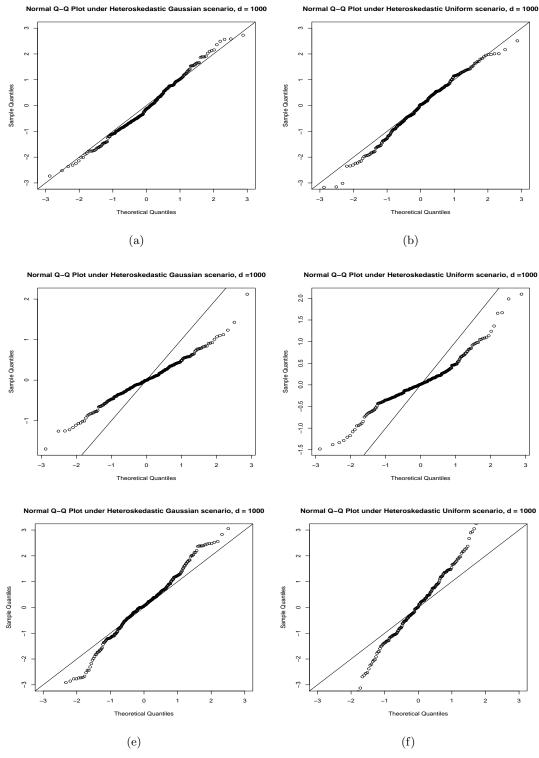


Figure E.1: Gaussian Q-Q plot of the test statistics under the setting $d=1000, s=10, \rho=0.5$ from: (a, b) SDS method, (c, d) DS method, and (e, f) Honest method.

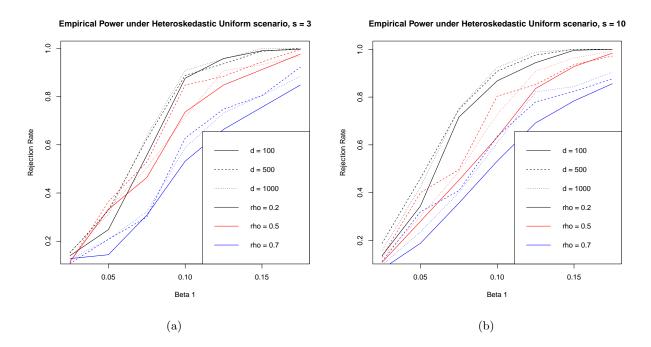


Figure E.2: Empirical rejection rate of the proposed test under both scenarios with $s=3,\ d=100,500,1000$ and $\rho=0.2,0.5,0.7.$

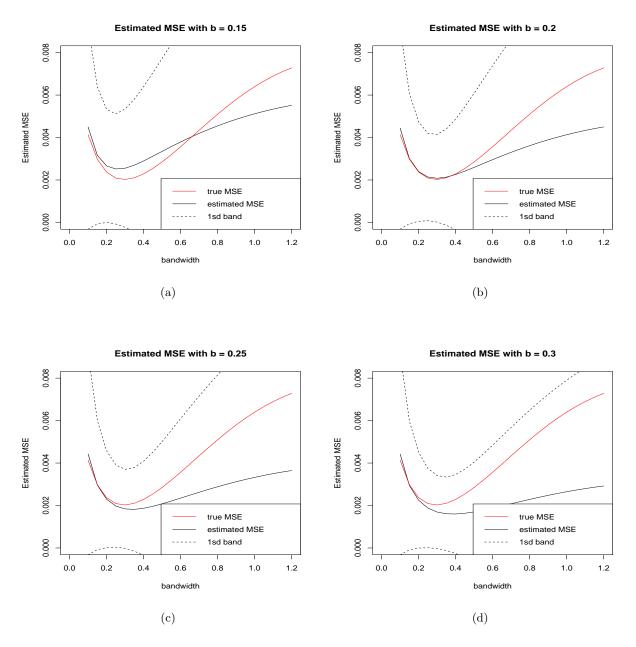


Figure E.3: The true $M(\delta)$ approximated by Monte Carlo Method (red solid line) and the average estimated $\widehat{M}(\delta)$ (black solid line) with bandwidth b=0.15,0.20,0.25,0.30 for squared-bias estimation over 100 repetitions. The black dashed line is one standard deviation bands around $\widehat{M}(\delta)$.

Table E.1: Five most significant variables using DS and Honest approach, sorted from most significant (left) to less significant (right), with their corresponding p-values

| Method | | | Significant variables | | |
|--------|-----------|---------------------|---------------------------|---------------------------------|-----------|
| DS | MMenB | X(WOMAC Pain Score) | $WFunc_6mo$ | $\mathrm{KQOL}_{-6}\mathrm{wk}$ | KSymp_3mo |
| | 9.754e-05 | 3.402 e-03 | 1.649 e-02 | 1.685 e-02 | 2.136e-02 |
| Honest | ActualGrp | MMenTear | $\operatorname{TrochDam}$ | ${\bf TrochCenLes}$ | MMenMgt |
| | <1e-16 | <1e-16 | <1e-16 | <1e-16 | <1e-16 |