① US facts

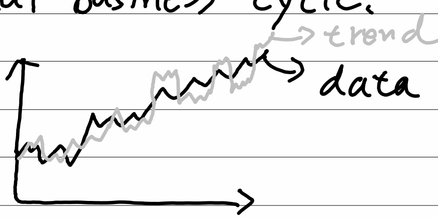
gov spending : high in GDP share but less velotile and not correlated with GDP.

Net export : in corr, IM > EX ⇒ corr < 0.

Net inventory invest : miniscule but velotile!

GDP is 2~3 times more velotile than investment.

② Real business cycle.



The difference between real GDP and trend is the real business cycle.

HP filter : $\{y_t\}_{t=1}^T$ data, $\{\tau_t\}_{t=1}^T$ trend

$$\min_{\{\tau_t\}_{t=1}^T} \underbrace{\sum_{t=1}^{T} (y_t - \tau_t)^2}_{\text{squared penalty on deviation from trend}} + \underbrace{\lambda}_{\text{parameter for smooth data.}} \sum_{t=2}^{T-1} \underbrace{\left[(\tau_{t+1} - \tau_t) - (\tau_t - \tau_{t-1})\right]^2}_{\text{penalty on peaks (too high frequency)}}$$

③ Real business cycle model.

1. Household problem:

$$\max_{\{n_t, C_t, K_{t+1}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t u(C_t, 1-n_t)$$

$$\text{s.t.} \quad \begin{aligned} &C_t \le W_t n_t + r_{k,t} K_t + \Pi_t - I_t \\ &K_{t+1} \le (1-\delta) K_t + I_t \\ &n_t \in [0, 1]. \end{aligned}$$

2. Firm max profit:

$$\max_{\{K_t, n_t\}_{t=0}^{\infty}} [Y_t - W_t n_t - r_{k,t} K_t],$$

where $Y_t = z_t F(K_t, n_t X_t)$

X_t: "effective labor" measurement of the productivity of labor growth, growth determistically.

F: HD1 in (K, nX), strictly $\nearrow$ in K, nX
strictly $\frown$.

3. Market clear:

$$C_t + K_{t+1} \le (1-\delta) K_t + Y_t.$$

④ Planner's problem:

$$\max_{\{n_t, C_t, K_{t+1}\}} E_0 \sum_{t=0}^{\infty} \beta^t u(C_t, 1-n_t)$$

s.t. $C_t + K_{t+1} \leq Y_t + (1-\delta)K_t$.

$Y_t = z_t F(K_t, n_t X_t)$

use market-clearing condition as budget

⑤ Define growth rate as $\frac{X_{t+1}}{X_t} = \gamma_x$

Transform each constraint into growth rate version:

$$Y_t = C_t + I_t \Rightarrow \gamma_y^t Y_0 = \gamma_c^t C_0 + \gamma_I^t I_0$$

$$\Rightarrow Y_0 = \left(\frac{\gamma_c}{\gamma_y}\right)^t C_0 + \left(\frac{\gamma_I}{\gamma_y}\right)^t I_0$$

$\because Y_0 = C_0 + I_0$, $\therefore \gamma_c = \gamma_y$, $\gamma_I = \gamma_y$.

$$K_{t+1} = (1-\delta)K_t + I_t \Rightarrow \gamma_k^t K_1 = (1-\delta)\gamma_k^t K_0 + \gamma_I^t I_0$$

$$\Rightarrow K_1 = (1-\delta)K_0 + \left(\frac{\gamma_I}{\gamma_k}\right)^t I_0 \Rightarrow \gamma_I = \gamma_k$$

$$Y_t = z_t F(K_t, n_t X_t) \Rightarrow \gamma_y^t Y_0 = z_t F(\gamma_k^t K_0, \gamma_x^t n_0 X_0), \ \gamma_n = 1$$

$$\Rightarrow Y_0 = z_t F\left(\left(\frac{\gamma_k}{\gamma_y}\right)^t K_0, \left(\frac{\gamma_x}{\gamma_y}\right)^t n_0 X_0\right)$$

$\because Y_0 = z_t F(K_0, n_0 X_0) \Rightarrow \gamma_x = \gamma_y = \gamma_k$.

For price,

$$Y_t = Z_t F(K_t, n_t X_t) = Z_t n_t X_t F\left(\frac{K_t}{X_t n_t}, 1\right)$$

$$\equiv Z_t X_t n_t f(k_t), \quad k_t \equiv \frac{K_t}{X_t n_t}$$

$$\because r_{k,t} = \frac{\partial Y_t}{\partial K_t} \Rightarrow Z_t X_t n_t f'(k_t) \cdot \frac{\partial k_t}{\partial K_t}$$

$$= Z_t f'(k_t) \quad k_t = \frac{\gamma_k^t K_0}{\gamma_x^t X_0 n_0} = k_0$$

$$\gamma_{r_k} r_{k,0} = Z_t f'(k_0) \Rightarrow \gamma_{r_k} = 1$$

$\because$ real interest rate = return on capital

$$\Rightarrow r_t + 1 = r_{k,t} + 1 - \delta \Rightarrow \gamma_r r_0 + 1 = \gamma_{r_k} r_{k,0} + 1 - \delta$$

$$\Rightarrow \gamma_r = 1$$

$$W_t = \frac{\partial Y_t}{\partial n_t X_t} = Z_t f(k_t) + Z_t n_t X_t f'(k_t) \cdot \left(-\frac{K_t}{(X_t n_t)^2}\right)$$

$$= Z_t f(k_t) - Z_t k_t f'(k_t)$$

$$\gamma_W^t W_0 = Z_t f(k_0) - Z_t k_0 f'(k_0)$$

$$\Rightarrow \gamma_W = 1$$

$$\because W_t = w_t X_t \Rightarrow \gamma_x = \gamma_w$$

$\Rightarrow$ all growth rate $= 1$

Conclusion: In balanced growth path, all growth rate $= 1$

⑥ Existence of balance growth path:

$u(c, 1-n)$ need to satisfy

$$\beta^t u(c_t, 1-n_t) = (\underbrace{\beta \gamma_x^{1-\sigma}}_{\beta^*})^t u\left(\frac{c_t}{\gamma_x^t}, 1-n_t\right)$$

So if

① $\gamma_x = 1$, or

② $u(c_t, 1-n_t) = \frac{c^{1-\sigma}}{1-\sigma} v(1-n)$

$= \log c + v(1-n)$

then we have BGP.

⑦ Growth-deflated model:

Define $y_t = Y_t / X_t$; $c_t = C_t / X_t$; $i_t = I_t / X_t$; $k_t = K_t / X_t$

the planner's problem become:

$$\max_{\{c_t, n_t, y_t, i_t, k_{t+1}\}} \sum_{t=0}^{\infty} (\beta^*)^t u(c_t, 1-n_t)$$

s.t. $y_t = z_t F(k_t, n_t)$

$y_t \geq c_t + i_t$

$k_{t+1} \leq (1-\delta) k_t + i_t$

$0 \leq n_t \leq 1$.

⑧ RBC model w/ growth-deflated:

$$V(z_i; k) = \max_{\{c, n, k'\}} \left[u(c, 1-n) + \sum_{j=1}^{N_z} \pi_{ij} V(k', z_j) \right]$$

s.t. $c + \gamma_x k' \leq (1-\delta) k + z F(k, n)$.

z, k given.

$$\mathcal{L} = u(c, 1-n) + \sum_{j=1}^{N_z} \pi_{ij} V(k', z_j) + \lambda \left[(1-\delta) k + z F(k, n) - c - \gamma_x k' \right]$$

$[c]$: $D_1 u(c, 1-n) = \lambda$

$[k']$: $\beta \sum_{j=1}^{N_z} \pi_{ij} \lambda_j' D_1 V(k', z_j) = \gamma_x \lambda$

$[n]$: $+D_2 u(c, 1-n) + \lambda z D_2 F(k, n) = 0$

$[\lambda]$: $c + \gamma_x k' = (1-\delta) k + z F(k, n)$

By Benvensite - Scheinkman Thm,

$$D_1 V(k, z_i) = z D_1 F(k, n) + (1-\delta) \Rightarrow D_1 V(k', z_j') = z_j D_1 F(k', n') + (1-\delta)$$

$$[k'] : \gamma \lambda = \beta \sum_{j=1}^{N_z} \pi_{ij} \lambda_j' [z_j D_1 F(k', n') + (1-\delta)]$$

Apply certainty equivalence, we can take the expected value of $[k']$ and get

$$\begin{aligned}
&\gamma \lambda = \beta \lambda_j' [z_j D_1 F(k', n') + (1-\delta)] \\
&D_1 U(c, 1-n) = \lambda \\
&D_2 U(c, 1-n) + \lambda z D_2 F(k, n) = 0 \\
&c + \gamma_x k' = (1-\delta) k + z F(k, n).
\end{aligned} \quad (\star)$$

9 elasticity of labor supply.

Let $U(c_t, 1-n_t) = \frac{c^{1-\sigma}}{1-\sigma} - \chi \frac{n^{1+\eta}}{1+\eta}$

$F(k_t, n_t) = k_t^{\alpha} n_t^{1-\alpha}$

1. Frisch elasticity of labor supply:

$$\max_n \frac{\bar{c}^{1-\sigma}}{1-\sigma} - \chi \frac{n^{1+\eta}}{1+\eta} + \bar{\lambda} [\text{Wealth} + wn - c]$$

$$[n] : \chi n^{\eta} + \bar{\lambda} w = 0 \Rightarrow n = \left(\frac{\bar{\lambda} w}{\chi}\right)^{\frac{1}{\eta}}$$

$$\frac{\partial n}{\partial w} = \frac{1}{\eta} \left(\frac{\bar{\lambda} w}{\chi}\right)^{\frac{1}{\eta} - 1} \cdot \frac{\bar{\lambda}}{\chi} = \frac{1}{\eta} \cdot n \cdot \frac{\chi}{w \bar{\lambda}} \cdot \frac{\bar{\lambda}}{\chi} = \frac{n}{\eta w}$$

$$\eta^F_{n,w} = \frac{\partial n}{\partial w} \frac{w}{n} = \frac{1}{\eta}$$

$\hookrightarrow$ elasticity w/o income effect, only substitution effect.

2. Actually elasticity of supply

$$\max_{c,n} \frac{c^{1-\sigma}}{1-\sigma} - \chi \frac{n^{1+\eta}}{1+\eta} + \lambda[\text{Wealth} + wn - c]$$

$[c]: c^{-\sigma} = \lambda$

$[n]: \chi n^{\eta} = \lambda w = c^{-\sigma} w$

$$\Rightarrow n = \left(\frac{c^{-\sigma} w}{\chi}\right)^{\frac{1}{\eta}} = (c^{-\sigma} w)^{\frac{1}{\eta}} \chi^{-\frac{1}{\eta}}$$

Now c is also a function of w. We don't hold c as constant anymore.

$$\frac{\partial n}{\partial w} = \chi^{-\frac{1}{\eta}} \cdot \frac{1}{\eta} (c^{-\sigma} w)^{\frac{1}{\eta}-1}$$
$$\cdot [-\sigma c^{-\sigma-1} \frac{\partial c}{\partial w} \cdot w + c^{-\sigma}]$$

Define $\frac{\partial c}{\partial w} \frac{w}{c} = \eta_{c,w}$

$$\eta_{n,w} = \frac{\partial n}{\partial w} \frac{w}{n} = \left\{ \frac{1}{\eta} \cdot \cancel{n} \cdot \cancel{w^{-1}}\right.$$
$$\left. \cdot [-\sigma c^{-\sigma-1} \frac{\partial c}{\partial w} \frac{w}{c} \cdot \cancel{c} + c^{-\sigma}] \right\} \cancel{\frac{w}{n}}$$

$$= \frac{1}{\eta} \cdot c^{-\sigma} [1 - \sigma \eta_{c,w}] < \frac{1}{\eta}$$

Actuall elasticity of labor supply is including income and substitution effects.

$\because \eta^F_{n,w} > \eta_{n,w}$, $\therefore$ income effect plays opposite rule on substitution effect.

⑩ RBC model & steady state.

1. eliminate uncertainty $\Rightarrow z$ constant.

2. $u(c,n) = \log c + B(1-n)$

$F(k,n) = k^{\alpha} n^{1-\alpha}$

In steady, $k' = k$, $\lambda_j' = \lambda$, $n' = n$

(※) become

$$\frac{1}{c^*} = \lambda^* \ ; \ B = \lambda^*(1-\alpha)\frac{y^*}{n^*}$$

$$r = \beta\left(\alpha\frac{y^*}{k^*} + (1-\delta)\right) \ ; \ c^* + r_x k^* = y^* + (1-\delta)k^*$$

$$\frac{y^*}{k^*} = \left[\frac{r}{\beta} - (1-\delta)\right]\frac{1}{\alpha} = \frac{r-\beta(1-\delta)}{\alpha\beta} \Rightarrow \frac{k^*}{y^*} = \frac{\alpha\beta}{r-\beta(1-\delta)}$$

$$\frac{y^*}{n^*} = \frac{c^* B}{1-\alpha} \Rightarrow \frac{n^*}{y^*} = \frac{1-\alpha}{c^* B}.$$

$$i^* = y^* - c^* = r_x k^* - (1-\delta)k^* \Rightarrow \frac{i^*}{k^*} = r_x - 1 + \delta$$

$$c^* = y^* - i^* \Rightarrow \frac{c^*}{y^*} = 1 - \frac{i^*}{y^*} = 1 - \frac{i^*}{k^*}\cdot\frac{k^*}{y^*}.$$

$$w^* = (1-\alpha)\frac{y^*}{n^*} \ ; \ r_k^* = \alpha\frac{y^*}{k^*}$$

So we can retrive all steady state variables by getting.

$$k^* = \left[\frac{k^*}{y^*}\cdot(n^*)^{1-\alpha}\right]^{\frac{1}{1-\alpha}}$$

⑪ Linear approximation on RBC model.

Assume shocks are AR(1) process:

$$\log(Z_{t+1}) = \rho \log(Z_t) + \varepsilon_{t+1},$$

where $\varepsilon_t \overset{iid}{\sim} N(0, \sigma_\varepsilon^2)$,

ρ as persistence of tech shocks.

Put the time subscripts back to (✻),

1. $C_t^{-1} = \lambda_t$
2. $B = \lambda_t Z_t D_2 F(k_t, n_t)$
3. $\gamma \lambda_t = \beta \lambda_{t+1}[Z_{t+1} D_1 F(k_{t+1}, n_{t+1}) + (1-\delta)]$
4. $C_t + \gamma_x k_{t+1} = Z_t F(k_t, n_t) + (1-\delta) k_t$.

Define $\hat{x}_t = \dfrac{x_t - x^*}{x^*} \Rightarrow \hat{x}_t x^* = x_t - x^*$

Apply 1st-order Taylor expansion on all eqs,

1. $-(\cancel{C^*})^{-2} \cdot \hat{C}_t \cancel{C^*} = \hat{\lambda}_t \cancel{\lambda^*} \Rightarrow -\hat{C}_t = \hat{\lambda}_t$

2. $0 = Z^* D_2 F(k^*, n^*) \lambda^* \hat{\lambda}_t$
$\quad + \lambda^* Z^* D_2 F(k^*, n^*) \hat{Z}_t$
$\quad + \lambda^* Z^* D_{21} F(k^*, n^*) \cdot k^* \hat{k}_t$
$\quad + \lambda^* Z^* D_{22} F(k^*, n^*) \cdot n^* \hat{n}_t$

$\Rightarrow 0 = \cancel{y^*} \hat{\lambda}_t + \cancel{y^*} \hat{Z}_t + \cancel{y^*} \alpha \hat{k}_t + \cancel{y^*} (-\alpha) \hat{n}_t$

$\Rightarrow 0 = \hat{\lambda}_t + \hat{Z}_t + \alpha \hat{k}_t + (-\alpha) \hat{n}_t$

3. $\gamma \cancel{\lambda^*} \hat{\lambda}_t = \beta \cancel{\lambda^*} [Z^* D_1 F(k^*, n^*) + (1-\delta)] \hat{\lambda}_t$
$\quad + \beta \cancel{\lambda^*} Z^* D_1 F(k^*, n^*) \hat{Z}_t$
$\quad + \beta \cancel{\lambda^*} Z^* D_{11} F(k^*, n^*) k^* \hat{k}_t$
$\quad + \beta \cancel{\lambda^*} Z^* D_{12} F(k^*, n^*) n^* \hat{n}_t$

$\because \gamma \cancel{\lambda^*} = \beta \cancel{\lambda^*} [Z^* D_1 F(k^*, n^*) + (1-\delta)]$

$\Rightarrow \beta Z^* D_1 F(k^*, n^*) = \gamma - \beta(1-\delta)$

$$\hat{\lambda}_t = \hat{\lambda}_{t+1} + \frac{\gamma - \beta(1-\delta)}{\gamma} \left[\hat{Z}_{t+1} + (\alpha - 1)\hat{k}_{t+1} + (1-\alpha)\hat{n}_{t+1} \right]$$

4. $\hat{c}_t c^* + \gamma_x k^* \hat{k}_{t+1} = F(k^*, n^*) \cdot Z^* \hat{Z}_t$
$+ [Z^* D_1 F(k^*, n^*) + (1-\delta)] k^* \hat{k}_t$
$+ Z^* D_2 F(k^*, n^*) n^* \hat{n}_t$

$$\Rightarrow \frac{c^*}{y^*} \hat{c}_t + \frac{\gamma_x k^*}{y^*} \hat{k}_{t+1}$$
$$= \hat{Z}_t + \left[\frac{Z^* D_1 F(k^*, n^*) k^*}{y^*} + (1-\delta) \frac{k^*}{y^*} \right] \hat{k}_t$$
$$+ \frac{Z^* D_2 F(k^*, n^*) n^*}{F(k^*, n^*)} \hat{n}_t$$

Define $s_c = \frac{c^*}{y^*}$, $s_k = \frac{\gamma_x k^*}{y^*}$,

$$\Rightarrow s_c \hat{c}_t + s_k \hat{k}_{t+1} = \hat{Z}_t + (\alpha + (1-\delta) s_k \gamma^{-1}) \hat{k}_t$$
$$+ (1-\alpha) \hat{n}_t.$$

☆ Note on linearization:

① linearize everything w/ t subscript.

② take first order derivative at steady state
ex. $F(k_t, n_t) \Rightarrow D_1 F(k^*, n^*)$

③ times $x^* \hat{x}_t$
ex. $F(k_t, n_t) \Rightarrow D_1 F(k^*, n^*) \cdot k^* \hat{k}_t$

Do more linearization on everything

5. $y_t = z_t k_t^{\alpha} n_t^{1-\alpha}$

$$\hat{y}_t \cancel{y^*} = k^{*\alpha} n^{*1-\alpha} z^* \hat{z}_t + \alpha z^* k^{*\alpha-1} n^{*1-\alpha} k^* \hat{k}_t$$
$$+ (1-\alpha) z^* k^{*\alpha} n^{*-\alpha} n^* \hat{n}_t$$
$$= \cancel{y^*} \hat{z}_t + \cancel{y^*} \hat{k}_t (\alpha) + \cancel{y^*} \hat{n}_t (1-\alpha)$$
$$\hat{y}_t = \hat{z}_t + \alpha \hat{k}_t + (1-\alpha) \hat{n}_t$$

6. $i_t = y_t - c_t$

$$i^* \hat{i}_t = y^* \hat{y}_t - c^* \hat{c}_t$$
$$\hat{i}_t \left(\frac{i^*}{y^*}\right) = \hat{y}_t - \hat{c}_t \left(\frac{c^*}{y^*}\right)$$

7. $w_t = (1-\alpha) y_t (n_t)^{-1}$

$$w^* \hat{w}_t = (1-\alpha)(n^*)^{-1} y^* \hat{y}_t + (1-\alpha) y^* (-n^*)^{-2} \cdot n^* \hat{n}_t$$
$$\hat{w}_t = \hat{y}_t - \hat{n}_t$$

8. $r_{k,t} = \alpha y_t (k_t)^{-1} - \delta \Rightarrow \alpha y^* (k^*)^{-1} = r_k^* + \delta$

$$\hat{r}_{k,t} r_k^* = \alpha (k^*)^{-1} y^* \hat{y}_t$$
$$+ \alpha y^* (-k^*)^{-2} \cdot k^* \hat{k}_t$$
$$\hat{r}_{k,t} = \frac{r_k^* + \delta}{r_k^*} \left[\hat{y}_t - \hat{k}_t \right]$$

9. $\log Z_{t+1} = \rho \log Z_t + \varepsilon_{t+1}$

$$\Rightarrow \frac{1}{\cancel{Z^*}} \cdot \cancel{Z^*} \hat{Z}_{t+1} = \rho \cdot \frac{1}{\cancel{Z^*}} \cancel{Z^*} \hat{Z}_t + \underbrace{\varepsilon^*}_{\varepsilon_s = 0} \hat{\varepsilon}_{t+1}$$

$$\Rightarrow \hat{Z}_{t+1} = \rho \hat{Z}_t .$$

(12) State space system reduction

$$-\hat{c}_t = \hat{\lambda}_t$$

$$\alpha \hat{n}_t = \hat{\lambda}_t + \alpha \hat{k}_t + \hat{z}_t$$

$$\therefore \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & \alpha \end{bmatrix}}_{F_1} \begin{bmatrix} \hat{c}_t \\ \hat{n}_t \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & \alpha & 1 \end{bmatrix}}_{F_2} \begin{bmatrix} \hat{\lambda}_t \\ \hat{k}_t \\ \hat{z}_t \end{bmatrix}$$

Goal: represent $\hat{\lambda}_t$ as a function of $\hat{k}_t$ and $\hat{z}_t$, and next step is to represent everything in terms of $\hat{k}_t$ and $\hat{z}_t$

$$\Rightarrow \begin{bmatrix} \hat{c}_t \\ \hat{n}_t \end{bmatrix} = \underbrace{F_1 \backslash F_2}_{F} \begin{bmatrix} \hat{\lambda}_t \\ \hat{k}_t \\ \hat{z}_t \end{bmatrix}$$

$$\Rightarrow \hat{c}_t = F(1,1)\hat{\lambda}_t + F(1,2)\hat{k}_t + F(1,3)\hat{z}_t$$

$$\hat{n}_t = F(2,1)\hat{\lambda}_t + F(2,2)\hat{k}_t + F(2,3)\hat{z}_t$$

Let $\theta = \gamma(\beta(1-\delta))/\gamma$.

$$\Rightarrow \hat{\lambda}_t = \hat{\lambda}_{t+1} + \theta[\hat{z}_{t+1} + (\alpha-1)\hat{k}_{t+1} + (1-\alpha)\hat{n}_{t+1}]$$

$$= \hat{\lambda}_{t+1} + \theta\hat{z}_{t+1} + \theta(\alpha-1)\hat{k}_{t+1} + \theta(1-\alpha)F(2,1)\hat{\lambda}_{t+1} + \theta(1-\alpha)F(2,2)\hat{k}_{t+1} + \theta(1-\alpha)F(2,3)\hat{z}_{t+1}$$

$$= \overbrace{[1+\theta(1-\alpha)F(2,1)]}^{M_L(2,2)}\hat{\lambda}_{t+1} + \overbrace{\theta(1-\alpha)(1-F(2,2))}^{M_L(2,1)}\hat{k}_{t+1} + \overbrace{\theta(1+(1-\alpha)F(2,3))}^{C(2,1)}\hat{z}_{t+1}$$

$$\Rightarrow \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix} \begin{bmatrix} \hat{\lambda}_{t+1} \\ \hat{k}_{t+1} \end{bmatrix} = \begin{bmatrix} \hat{\lambda}_t \\ \hat{k}_t \end{bmatrix} + \begin{bmatrix} \square \\ \square \end{bmatrix} \begin{bmatrix} \hat{z}_{t+1} \\ \hat{z}_t \end{bmatrix}$$

$$s_k \hat{k}_{t+1} = -s_c \hat{c}_t + (1-\alpha)\hat{n}_t + \hat{z}_t + [\alpha + (1-\delta) s_k \gamma_x^{-1}] \hat{k}_t$$

$$\begin{aligned}
&= -s_c \left[F(1,1)\hat{\lambda}_t + F(1,2)\hat{k}_t + F(1,3)\hat{z}_t \right] \\
&\quad + (1-\alpha)\left[F(2,1)\hat{\lambda}_t + F(2,2)\hat{k}_t + F(2,3)\hat{z}_t \right] \\
&\quad + \hat{z}_t + [\alpha + (1-\delta) s_k \gamma_x^{-1}] \hat{k}_t \\
&= \Big[\overbrace{-s_c F(1,1) + (1-\alpha) F(2,1)}^{M_R(1,2)} \Big] \hat{\lambda}_t \\
&\quad + \Big[\overbrace{\alpha + (1-\delta) s_k \gamma_x^{-1} - s_c F(1,2) + (1-\alpha) F(2,2)}^{M_R(1,1)} \Big] \hat{k}_t \\
&\quad + \Big[\overbrace{1 - s_c F(1,3) + (1-\alpha) F(2,3)}^{C(1,2)} \Big] \hat{z}_t
\end{aligned}$$

$$\Downarrow$$

$$M_L \begin{bmatrix} \hat{k}_{t+1} \\ \hat{\lambda}_{t+1} \end{bmatrix} = M_R \begin{bmatrix} \hat{k}_t \\ \hat{\lambda}_t \end{bmatrix} + C \begin{bmatrix} \hat{z}_{t+1} \\ \hat{z}_t \end{bmatrix}$$

$$M_L = \begin{bmatrix} s_k & 0 \\ M_L(2,1) & M_L(2,2) \end{bmatrix}$$

$$M_R = \begin{bmatrix} M_R(1,1) & M_R(1,2) \\ 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & C(1,2) \\ C(2,1) & 0 \end{bmatrix}$$

Denote $S_t = \begin{bmatrix}\hat{k}_t \\ \hat{\lambda}_t\end{bmatrix}$ as the vector of state variables.

λ is not state variable. But we can eventually transform λ as a function of k and z, so that $S_t = \begin{bmatrix}\hat{k}_t \\ \hat{z}_t\end{bmatrix}$.

Therefore, we call λ as "co-state" variables.

$$\Rightarrow M_L S_{t+1} = M_R S_t + C U_t, \quad U_t = \begin{bmatrix}\hat{z}_{t+1} \\ \hat{z}_t\end{bmatrix}$$

$$\Rightarrow S_{t+1} = M_L^{-1} M_R S_t + M_L^{-1} C U_t.$$

Let $W = M_L^{-1} M_R$, $\tilde{R} = M_L^{-1} C$.

$$\Rightarrow S_{t+1} = W S_t + \tilde{R} U_t.$$

$\because W$ is full rank, $\therefore \exists P$ and Λ s.t. $W = \overset{\text{eigenvector}}{P}\,\overset{\text{eigenvalue}}{\Lambda}\,P^{-1}$.

$$S_{t+1} = P \Lambda P^{-1} S_t + \tilde{R} U_t$$

Now we are going to do "de-coupling". We want to form a formula that only the eigenvalue is multiplied the S_t term, and all other messy terms goes with U_t.
$\Rightarrow$ To let only k_t affect k_{t+1}, and only λ_t affect λ_{t+1}.

Multiple both side with P^{-1},

$$P^{-1} S_{t+1} = \Lambda P^{-1} S_t + P^{-1} \tilde{R} U_t.$$

Define canonical variable $\tilde{S}_t = P^{-1} S_t$,

$$\mathring{S}_{t+1} = \Lambda \mathring{S}_t + P^{-1}\tilde{R} U_t.$$

Define $P^{-1}\tilde{R} = \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} \mathring{k}_{t+1} \\ \mathring{\lambda}_{t+1} \end{bmatrix} = \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix} \begin{bmatrix} \mathring{k}_t \\ \mathring{\lambda}_t \end{bmatrix} + \begin{bmatrix} \gamma_{1,1} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix} \begin{bmatrix} \hat{Z}_{t+1} \\ \hat{Z}_t \end{bmatrix}$$

$$\Rightarrow \begin{cases} \mathring{k}_{t+1} = \mu_1 \mathring{k}_t + \gamma_{11} \hat{Z}_{t+1} + \gamma_{12} \hat{Z}_t & -(a) \\ \mathring{\lambda}_{t+1} = \mu_2 \mathring{\lambda}_t + \gamma_{21} \hat{Z}_{t+1} + \gamma_{22} \hat{Z}_t & -(b) \end{cases}$$

where $\mu_1 < 1$ is stable eigenvector, and $\mu_2 > \frac{1}{\beta} > 1$ is not stable.

(a) is not important, because $\hat{k}_t$ is already our state variable. We want to transform (b) into a function of k and z.

$$(b) \Rightarrow \mathring{\lambda}_t = \frac{1}{\mu_2} \mathring{\lambda}_{t+1} - \frac{1}{\mu_2}(\gamma_{21} \hat{Z}_{t+1} + \gamma_{22} \hat{Z}_t)$$

$$= \left(\frac{1}{\mu_2}\right)\left[\frac{1}{\mu_2} \mathring{\lambda}_{t+2} - \frac{1}{\mu_2}(\gamma_{21} \hat{Z}_{t+2} + \gamma_{22} \hat{Z}_{t+1})\right] - \frac{1}{\mu_2}(\gamma_{21} \hat{Z}_{t+1} + \gamma_{22} \hat{Z}_t)$$

$$= \left(\frac{1}{\mu_2}\right)^2 \mathring{\lambda}_{t+2} - \frac{1}{\mu_2}(\gamma_{21} \hat{Z}_{t+1} + \gamma_{22} \hat{Z}_t) - \left(\frac{1}{\mu_2}\right)^2 [\gamma_{21} \hat{Z}_{t+2} + \gamma_{22} \hat{Z}_{t+1}]$$

Iterated N times, we get

$$\mathring{\lambda}_t = \left(\frac{1}{\mu_2}\right)^N \mathring{\lambda}_{t+N} - \sum_{k=1}^{N} \left(\frac{1}{\mu_2}\right)^k [r_{21} \hat{z}_{t+k} + r_{22} \hat{z}_{t+k-1}]$$

By transversality condition,

$$\lim_{t \to \infty} \mathbb{E}_0 (\beta^*)^t \lambda_t k_{t+1} = 0$$

i.e., the growth rate of $\lambda_t k_{t+1}$ is slower than the decreasing rate of $(\beta^*)^t$.

$\because \mu_2 > \frac{1}{\beta^*} \Rightarrow \beta^* > \frac{1}{\mu_2}$, so $\frac{1}{\mu_2}$ decreases even faster than β^*,

$\Rightarrow \left(\frac{1}{\mu_2}\right)^N$ decreases faster than the increase of $\mathring{\lambda}_{t+N}$

As $N \to \infty$,

$$\mathring{\lambda}_t = - \sum_{k=1}^{\infty} \left(\frac{1}{\mu_2}\right)^k [r_{21} \hat{z}_{t+k} + r_{22} \hat{z}_{t+k-1}]$$

$$= - \sum_{k=0}^{\infty} \left(\frac{1}{\mu_2}\right)^{k+1} [r_{21} \hat{z}_{t+k+1} + r_{22} \hat{z}_{t+k}]$$

By 9., $\hat{z}_{t+1} = \rho \hat{z}_t$,

$$\mathring{\lambda}_t = -\sum_{k=0}^{\infty}\left(\frac{1}{\mu_2}\right)^{k+1}\left[\gamma_{21}\cdot\rho\,\hat{Z}_{t+k} + \gamma_{22}\,\hat{Z}_{t+k}\right]$$

$$= -\sum_{k=0}^{\infty}\left(\frac{1}{\mu_2}\right)^{k+1}\left[\gamma_{21}\,\rho^{k+1} + \gamma_{22}\,\rho^{k}\right]\hat{Z}_t$$

$$= -\sum_{k=0}^{\infty}\left(\frac{\rho}{\mu_2}\right)^{k+1}\underbrace{\left[\gamma_{21} + \frac{\gamma_{22}}{\rho}\right]\hat{Z}_t}_{\text{not related to } k}$$

$\Rightarrow$ put outside of sum

$$= -\left[\frac{\rho\gamma_{21}+\gamma_{22}}{\rho}\right]\hat{Z}_t\sum_{k=0}^{\infty}\left(\frac{\rho}{\mu_2}\right)^{k+1}$$

$$= -\left[\frac{\rho\gamma_{21}+\gamma_{22}}{\rho}\right]\hat{Z}_t\cdot\frac{\frac{\rho}{\mu_2}}{1-\frac{\rho}{\mu_2}}$$

$$= -\left[\frac{\rho\gamma_{21}+\gamma_{22}}{\cancel{\rho}}\right]\left[\frac{\frac{\cancel{\rho}}{\cancel{\mu_2}}}{\frac{\mu_2-\rho}{\cancel{\mu_2}}}\right]\hat{Z}_t$$

$$= \underbrace{-\left[\frac{\rho\gamma_{21}+\gamma_{22}}{\mu_2-\rho}\right]}_{g}\hat{Z}_t$$

$$\because \begin{bmatrix}\mathring{k}_t\\ \mathring{\lambda}_t\end{bmatrix} = \mathring{S}_t = P^{-1}S_t = \begin{bmatrix}P^{-1}_{11}\hat{k}_t + P^{-1}_{12}\hat{\lambda}_t\\ P^{-1}_{21}\hat{k}_t + P^{-1}_{22}\hat{\lambda}_t\end{bmatrix}.$$

$$\therefore \mathring{\lambda}_t = P^{-1}_{21}\hat{k}_t + P^{-1}_{22}\hat{\lambda}_t$$

$$\Rightarrow \hat{\lambda}_t = \frac{1}{P^{-1}_{22}}\left[\mathring{\lambda}_t - P^{-1}_{21}\hat{k}_t\right] = \frac{1}{P^{-1}_{22}}\left[g\hat{Z}_t - P^{-1}_{21}\hat{k}_t\right]$$

Successfully express λ as a function of Z and k!

Denote $\hat{\lambda}_t = f_{\lambda k}\hat{k}_t + f_{\lambda z}\hat{z}_t$.

Recall $S_{t+1} = W S_t + \tilde{R} U_t$

$$\Rightarrow \begin{bmatrix} \hat{k}_{t+1} \\ \hat{\lambda}_{t+1} \end{bmatrix} = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \begin{bmatrix} \hat{k}_t \\ \hat{\lambda}_t \end{bmatrix} + \begin{bmatrix} 0 & \tilde{R}_{12} \\ \tilde{R}_{21} & 0 \end{bmatrix} \begin{bmatrix} \rho\hat{z}_t \\ \hat{z}_t \end{bmatrix}$$

$$\hat{k}_{t+1} = W_{11}\hat{k}_t + W_{12}(f_{\lambda k}\hat{k}_t + f_{\lambda z}\hat{z}_t) + \tilde{R}_{12}\hat{z}_t$$

$$= \underbrace{(W_{11} + W_{12} f_{\lambda k})}_{m_{11}}\hat{k}_t + \underbrace{(\tilde{R}_{12} + W_{12} f_{\lambda z})}_{m_{12}}\hat{z}_t$$

Since $\hat{z}_{t+1} = \rho\hat{z}_t$,

$$\begin{bmatrix} \hat{k}_{t+1} \\ \hat{z}_{t+1} \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} \\ 0 & \rho \end{bmatrix} \begin{bmatrix} \hat{k}_t \\ \hat{z}_t \end{bmatrix}$$

$$\because \begin{bmatrix} \hat{c}_t \\ \hat{n}_t \end{bmatrix} = F \begin{bmatrix} \hat{\lambda}_t \\ \hat{k}_t \\ \hat{z}_t \end{bmatrix} \Rightarrow \begin{bmatrix} \hat{\lambda}_t \\ \hat{c}_t \\ \hat{n}_t \end{bmatrix} = \begin{bmatrix} f_{\lambda k} & f_{\lambda z} \\ \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix} \begin{bmatrix} \hat{k}_t \\ \hat{z}_t \end{bmatrix}$$

we can also recover linearized $\hat{y}_t, \hat{i}_t, \hat{w}_t, \hat{n}_t$ as a function of $\hat{k}_t, \hat{z}_t$,

$$\begin{bmatrix} \hat{c}_t \\ \hat{n}_t \\ \hat{y}_t \\ \hat{i}_t \\ \hat{w}_t \\ \hat{r}_t \end{bmatrix} = \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \\ \pi_{31} & \pi_{32} \\ \pi_{41} & \pi_{42} \\ \pi_{51} & \pi_{52} \\ \pi_{61} & \pi_{62} \end{bmatrix} \begin{bmatrix} \hat{k}_t \\ \hat{z}_t \end{bmatrix}$$

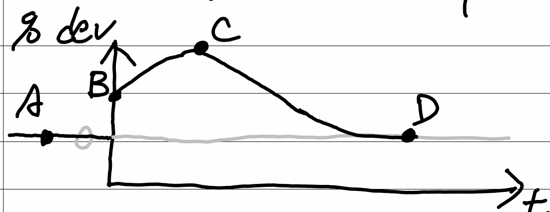
⑬ Impulse response and intepretation

1. Shocks.

Temporary shocks: shocks only last for one period.
Persistent shocks: shocks persists for at least a period or forever.

2. Phases:

we define each phase during the shocks:

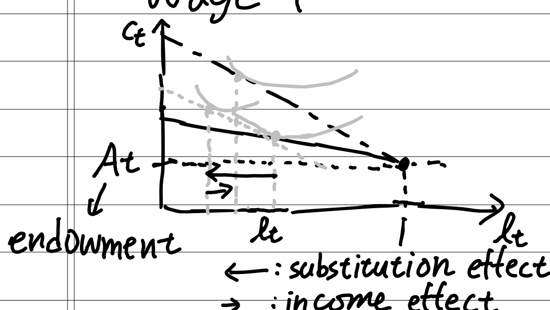


A: before shock
B: 1st period of shock
C: the highest deviation
D: back to steady state.

Phase 1: A ⇒ B
Phase 2: B ⇒ C
Phase 3: C ⇒ D

3. Income effect & Substitution effect

Wage ↑



SE: relative price for leisure ↑
⇒ l_t↓ ⇒ n_t↑, C_t↑

IE: income ↑
⇒ C_t↑ ⇒ l_t↑, n_t↓

4. Impulse response

4.1 Firms:

$w = zD_2F(K,N)$

$r_k + \delta = zD_1F(K,N)$.

$\therefore z\uparrow \Rightarrow w\uparrow \Rightarrow n\uparrow \Rightarrow y$

$\Rightarrow r_k+\delta\uparrow \Rightarrow k\uparrow \nearrow$

when $w\uparrow \Rightarrow$ income effect: $n\downarrow, \ell\uparrow, C\uparrow$

substitution effect: $n\uparrow, \ell\downarrow, C\uparrow$

4.2 Household $u(c,L) = \log(c) + BL$.

[n]: $W_t = BC_t$, from labor-leisure condition

[k']: $r_x D_1U(C_t, L_t) = \beta R_{t+1} D_1 u(C_{t+1}, L_t)$

$$\Rightarrow \frac{r_x C_{t+1}}{C_t} = \beta R_{t+1}.$$

We wish to derive the saving from B.C.:
(B.C. from Aubhik's sequential equilibrium)

$$C_t + K_{t+1} = R_t K_t + W_t N_t$$
$$\Rightarrow C_t + S_t = R_t S_{t-1} + W_t N_t$$
$$\Rightarrow C_t = R_t S_{t-1} - S_t + W_t N_t.$$

Let $A_t = R_t S_{t-1} - S_t$ be the endowment

$$C_t = W_t N_t + R_t S_{t-1} - S_t$$

$$\Rightarrow S_t = W_t N_t + R_t S_{t-1} - C_t$$

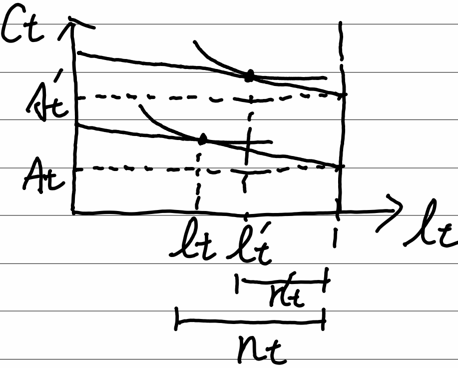
$$\begin{aligned}\Rightarrow C_{t+1} &= W_{t+1} N_{t+1} + R_{t+1} S_t - S_{t+1} \\ &= W_{t+1} N_{t+1} + R_{t+1} [W_t N_t + R_t S_{t-1} - C_t] - S_{t+1} \\ &= R_{t+1} [\underbrace{W_t N_t + R_t S_{t-1}}_{e_t^1} - C_t] + \underbrace{W_{t+1} N_{t+1} - S_{t+1}}_{e_{t+1}^2} \\ &= R_{t+1} (e_t^1 - C_t) + e_{t+1}^2\end{aligned}$$

e_t^1 : income this period from rental and labor.
e_{t+1}^2 : labor income - saving.

OK, so Let's begin.

(a) effect from labor supply :
(i) from $W \uparrow \Rightarrow$ substitution effect : $n\uparrow, l\downarrow, C\uparrow$
income effect : $n\downarrow, l\uparrow, C\uparrow$.
(ii) from $A_t \uparrow$: "wealth effect"

two channels of income effects

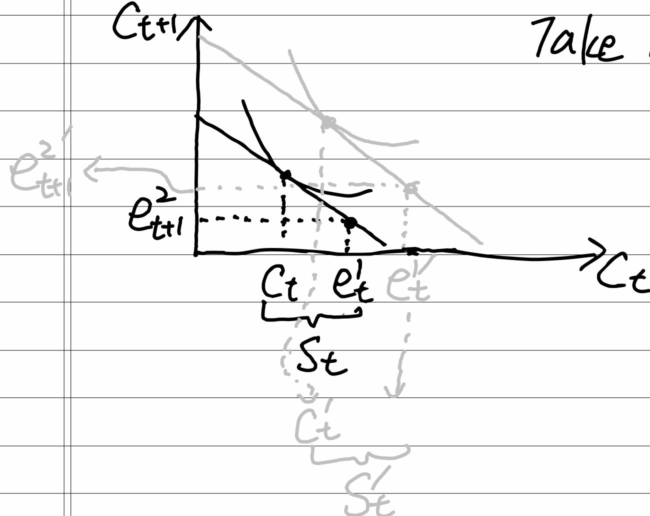


$A_t \uparrow \Rightarrow C_t \uparrow, n_t \downarrow$

(b) effect from saving:

(i) At the period when shock happened.

$C_{t+1} = R_{t+1}(e^1_t - C_t) + e^2_{t+1}$



Take e^1_t as endowment.

$\because W_t$ and $R_t \uparrow$ as $z \uparrow \Rightarrow e^1_t \uparrow$,
and we also expect $W_{t+1} \uparrow \Rightarrow e^2_{t+1} \uparrow$.

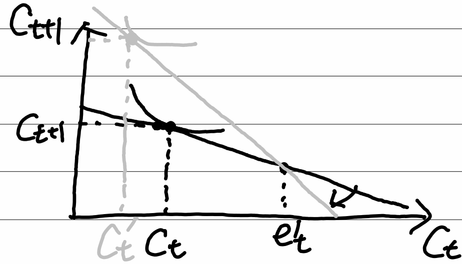
Since $\Delta(e^{1\prime}_t - e^1_t) > \Delta(C'_t - C_t)$

$\Rightarrow$ Saving increase, $S'_t - S_t > 0$

$\Rightarrow$ investment increases, $I'_t > I_t$.

(ii) After 1st period, can adjust capital:

HH anticipate higher R_{t+1} in future



Substitution effect: $C_t \downarrow$, $C_{t+1} \uparrow$, $(S_t \uparrow)$

$\because$ $y_t = C_t + I_t$, and $K_{t+1} = (1-\delta) K_t + I_t$

So as $K_{t+1} \uparrow \Rightarrow I_t \uparrow \Rightarrow$ given y_t, $C_t \downarrow$.

So $R_{t+1} \uparrow$ increases the relative price of C_t

$R_{t+1} \uparrow \Rightarrow K_{t+1} \uparrow \Rightarrow I_t \uparrow \Rightarrow y_t$ fixed, $C_t \downarrow$

Income effect: $C_t \uparrow$, $C_{t+1} \uparrow$, $(S_t \downarrow)$

Since as the $Z \uparrow \Rightarrow R_{t+1} \uparrow$, and the area of the budget set expanded, leads to higher consumption today and tomorrow.

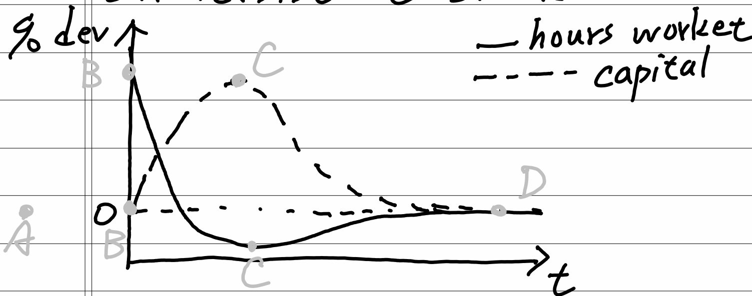
σ controls for the size of substitution effect

① Intertemporal substitution of consumption

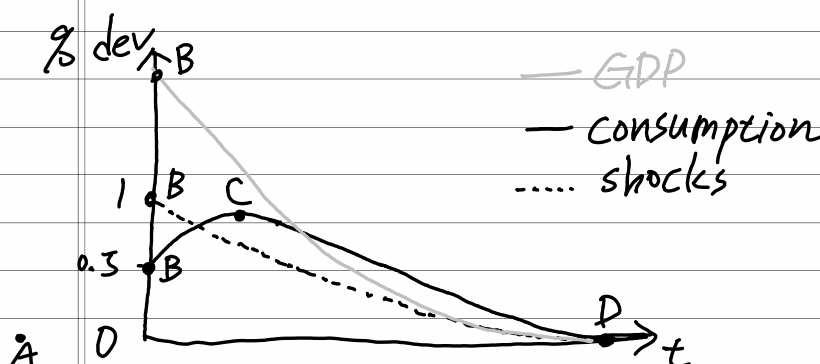
② degree of risk aversion

5. Interpretation of Impulse response.

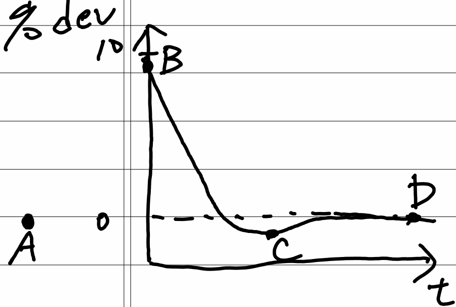
5.1 Persistent Shock



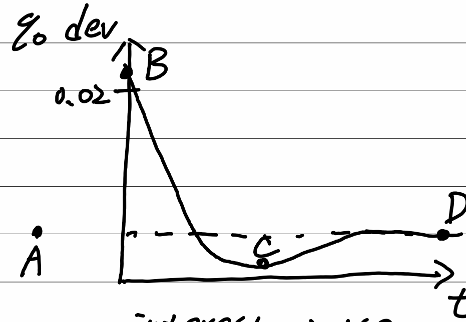
Phase 1 (A→B): $w\uparrow \Rightarrow N\uparrow$ ($\Rightarrow Y\uparrow$)
Phase 2 (B→C): ($I\uparrow \Rightarrow$) $K'\uparrow$ ($\Rightarrow Y$ keep $\uparrow$)
even when $Z\downarrow$, $\hat{i} > \delta$, $\Rightarrow K\uparrow$
Phase 3 (C→D): $K\downarrow \Rightarrow Y\downarrow \Rightarrow C\downarrow$
$\hookrightarrow w = z D_2 F(K,N)$, $w\downarrow$



Phase 1 (A→B): $\because$ consumption smoothing & high expected MPK_{t+1}
$\Rightarrow \Delta C < \Delta Y$, $\Delta I > \Delta Y$
Phase 2 (B→C): ($I\uparrow \Rightarrow$) $K'\uparrow \Rightarrow Y$ keep $\uparrow$
$\because \exists \rho$, $\Rightarrow C$ lasting
Phase 3 (C→D): $K\downarrow \Rightarrow Y\downarrow \Rightarrow C\downarrow$
$\Rightarrow w\downarrow$



investment



interest rate

Phase 1 $(A \to B)$: $\Delta I > \Delta Y$

Phase 2 $(B \to C)$: $I\uparrow \Rightarrow K'\uparrow \Rightarrow Y\uparrow$

Phase 3 $(C \to D)$: To maintain C that higher than steady state $\Rightarrow I < I^*$, $r < r^*$

5.2 Tempory Shock

No expected $R_{t+1}\uparrow$, only consumption smoothing.

$\therefore$ Only $Z_0 \uparrow \Rightarrow W_0 \uparrow \Rightarrow n_0 \uparrow \Rightarrow y_0 \uparrow \Rightarrow C_0 \uparrow$
$\Rightarrow r_0 \uparrow \Rightarrow i_0 \uparrow \Rightarrow k_1 \uparrow$

Z_1 back to 0 % dev,
$\Rightarrow C_1 > C^*$, since consumption smoothing
$\Rightarrow W_1$ also follow consumption smoothing
$\because C_1 > C^*$ and $y_1 \to y^*$
$\Rightarrow r_1 < r^*$, $i_1 < i^*$ to maintain C_t
Since the increase in MPN just disappear
$\Rightarrow n_1 \to n^*$, n is not intertemporal element
$\Rightarrow k_1 > k^*$, since capital is intertemporal, so we can smooth our capital accumulation.