Lecture 7
Representative Firm

Hui-Jun Chen

National Tsing Hua University

September 24, 2025

Overview: Lecture 4 - 7

Provide micro-foundation for the macro implication (Lucas critique)

- **Representative Consumer:**
  - Lecture 4: preference, constraints
  - Lecture 5: optimization, application
  - Lecture 6: Numerical Examples
- **Representative Firm:**
  - Lecture 7: production, optimization, application

Outline

Production Function

Production function describes the technology possibility for converting inputs into outputs.
Representative firm produces output Y with production function

$$Y = zF(K, N^d) \tag{1}$$

- Y: output (consumption goods)
- z: total factor productivity (TFP) (productivity for the economy)
- K: capital (fixed for now, $\because$ 1-period model)
- N^d: labor demand (chose by firm, d represents demand)

Properties of Production Function: Marginal Product

- **Marginal product**: how much $Y \uparrow$ by one unit of $K \uparrow$ or $N^d \uparrow$.
  - **Marginal product of capital** (MPK): $zD_K F(K, N^d)$
  - **Marginal product of labor** (MPN): $zD_N F(K, N^d)$
- Marginal product is positive and diminishing:
  - **Positive MP**: $Y \uparrow$ if either $K \uparrow$ or $N^d \uparrow$
    - more inputs result in more output
  - **Diminishing MP**: MPK $\downarrow$ as $K \uparrow$; MPN $\downarrow$ as $N^d \uparrow$
    - the rate/speed of output increasing is decreasing
- **Increasing marginal cross-products**:
  - e.g. MPK $\uparrow$ as $N \uparrow$; MPN $\uparrow$ as $K \uparrow$

Properties of Production Function: Return to Scale

- **Return to scale**: how Y will change when both K and N increase

- **Constant return to scale** (CRS): $xzF(K, N^d) = zF(xK, xN^d)$
  - small firms are as efficient as large firms

- **Increasing return to scale** (IRS): $xzF(K, N^d) > zF(xK, xN^d)$
  - small firms are less efficient than large firms

- **Decreasing return to scale** (DRS): $xzF(K, N^d) < zF(xK, xN^d)$
  - small firms are more efficient than large firms

Example: Cobb-Douglas Production Function

- **Cobb-Douglas**: $zF(K,N) = zK^{\alpha}N^{1-\alpha}$, α is the share of capital contribution to output
- **Positive MPK & MPN**:
  - $\text{MPK} = D_K zF(K,N) = z\alpha K^{\alpha-1}N^{1-\alpha} = z\alpha\left(\frac{K}{N}\right)^{\alpha-1} > 0$
  - $\text{MPN} = D_N zF(K,N) = z(1-\alpha)K^{\alpha}N^{-\alpha} = z(1-\alpha)\left(\frac{K}{N}\right)^{\alpha} > 0$
- **Diminishing MP**:
  - For K, $D_K\left(z\alpha K^{\alpha-1}N^{1-\alpha}\right) = z\alpha(\alpha-1)K^{\alpha-2}N^{1-\alpha} < 0$
  - For N, $D_N\left(z(1-\alpha)K^{\alpha}N^{-\alpha}\right) = z(1-\alpha)(-\alpha)K^{\alpha}N^{-\alpha-1} < 0$
- **Increasing marginal cross-product**:
  - For MPK, $D_N\left(z\alpha K^{\alpha-1}N^{1-\alpha}\right) = z\alpha(1-\alpha)K^{\alpha-1}N^{-\alpha} > 0$
  - For MPN, $D_K\left(z(1-\alpha)K^{\alpha}N^{-\alpha}\right) = z(1-\alpha)\alpha K^{\alpha-1}N^{-\alpha} > 0$

Example: Cobb-Douglas and Return to Scale

Let's assume that Cobb-Douglas production is $zF(K,N) = zK^{\alpha}N^{\beta}$
So if both inputs are increasing by twice, then

$$\begin{aligned} zF(2K,2N) &= z(2K)^{\alpha}(2N)^{\beta} = 2^{\alpha} \times 2^{\beta} zK^{\alpha}N^{\beta} \\ &= 2^{\alpha+\beta} zK^{\alpha}N^{\beta} = 2^{\alpha+\beta} Y \end{aligned}$$

1. If $\alpha + \beta = 1$, then $zF(2K,2N) = 2Y$, constant return to scale
2. If $\alpha + \beta < 1$, then $zF(2K,2N) = 2^{\alpha+\beta}Y < 2Y$, decreasing return to scale
3. If $\alpha + \beta > 1$, then $zF(2K,2N) = 2^{\alpha+\beta}Y > 2Y$, increasing return to scale

Visualization

Figure: Diminishing Marginal Product

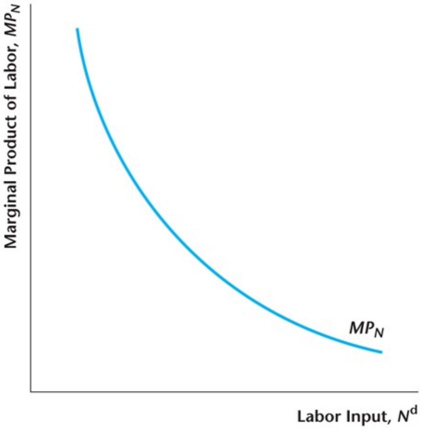
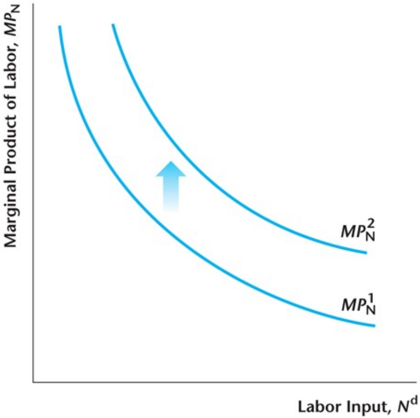


Figure: Increasing Marginal Cross-product



Visualization: Changes in TFP

Figure: TFP shifts up the Production Function

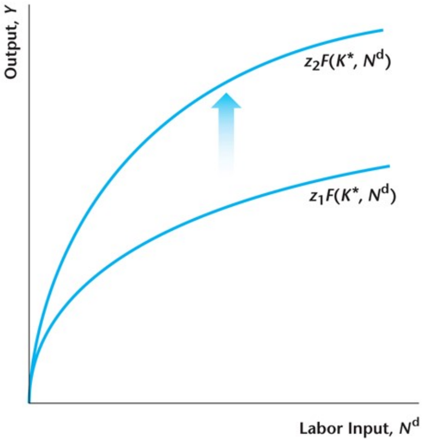
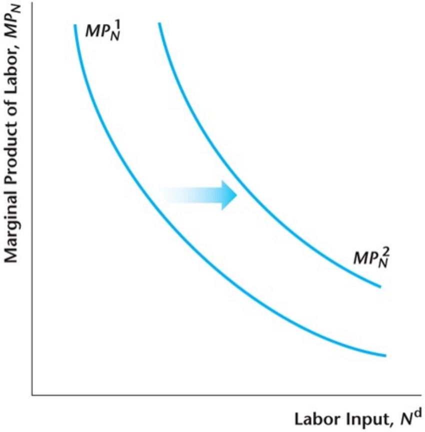
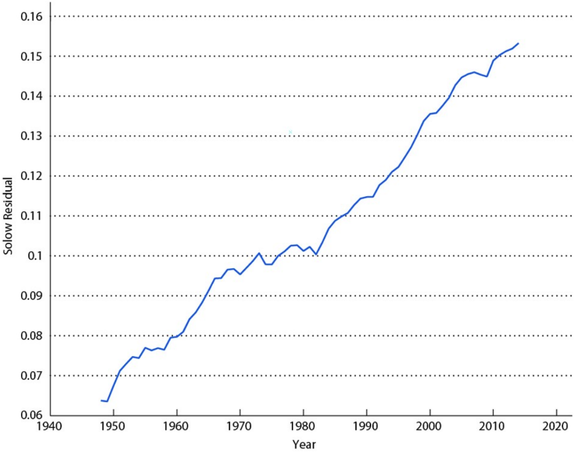


Figure: TFP increases MPN



TFP in Data

Figure: Solow Residual for US



We cannot see TFP, how to measure it?

- Assume Cobb-Douglas production function: $Y = zK^{\alpha}N^{1-\alpha}$
- By data, $K/Y = 0.3 \Rightarrow \alpha = 0.3$
- Can observe K, Y, N in data:

$$z = \frac{Y}{K^{0.3}N^{0.7}}$$

Outline

Firm's Problem: Profit Maximization

Firm maximizes profit (π), which is the revenue minus the wage bill:

$$\pi = \max_{N^d} zF(K, N^d) - wN^d \tag{2}$$

- **Constraints**: $N^d > 0$, relatively simple!

$$\text{Cobb-Douglas:} \quad zF(K, N^d) = zK^{\alpha}(N^d)^{1-\alpha} \tag{3}$$

$$\text{FOC:} \quad w = z(1-\alpha)K^{\alpha}(N^d)^{-\alpha} \tag{4}$$

$$(N^d)^{\alpha} = \frac{z(1-\alpha)K^{\alpha}}{w} \tag{5}$$

$$\text{Labor demand:} \quad N^d = \left(\frac{z(1-\alpha)K^{\alpha}}{w}\right)^{\frac{1}{\alpha}} = \left(\frac{z(1-\alpha)}{w}\right)^{\frac{1}{\alpha}} K \tag{6}$$

As $w \uparrow, N^d \downarrow \Rightarrow$ downward-sloping demand.

Outline

Experiment 1: Payroll Tax

Payroll tax: suppose firms have to pay additional per-unit tax $t > 0$ on the wage bill, then

$$\text{Firm Problem:} \quad \max_{N^d} zK^\alpha (N^d)^{1-\alpha} - w(1+t)N^d \tag{7}$$

$$\text{FOC:} \quad w(1+t) = z(1-\alpha)K^\alpha (N^d)^{-\alpha} \tag{8}$$

$$N^d = K\left(\frac{z(1-\alpha)}{w(1+t)}\right)^{\frac{1}{\alpha}} \tag{9}$$

- **wage** $\uparrow$: $w \uparrow \Rightarrow N^d \downarrow$ (same as benchmark)
- **tax** $\uparrow$: $t \uparrow \Rightarrow N^d \downarrow$
- **capital** $\uparrow$: $K \uparrow \Rightarrow N^d \uparrow \Rightarrow$ what if firm can also choose K?

Experiment 2: Choice of Capital

Capital rent: suppose that firm can choose capital level but have to pay r of per-unit rent.

$$\text{Firm Problem:} \quad \max_{K, N^d} zK^{\alpha}(N^d)^{1-\alpha} - rK - wN^d \tag{10}$$

$$\text{FOC on N:} \quad w = z(1-\alpha)K^{\alpha}(N^d)^{-\alpha} \tag{11}$$

$$\text{FOC on K:} \quad r = z\alpha K^{\alpha-1}(N^d)^{1-\alpha} \tag{12}$$

$$\text{Divide (11) with (12)}: \quad \frac{w}{r} = \frac{(1-\alpha)}{\alpha}\frac{K}{N^d} \tag{13}$$

$$\text{Capital-Labor ratio:} \quad \frac{K}{N^d} = \frac{w}{r}\frac{\alpha}{1-\alpha} \tag{14}$$

When firm can choose K, they choose both capital and labor such that (14) satisfied!