Lecture 6
Numerical Examples

Hui-Jun Chen

National Tsing Hua University

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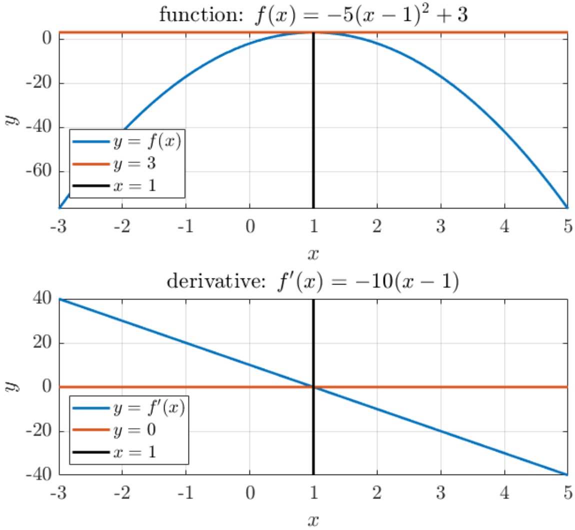
Overview: Lecture 4 - 7

Provide micro-foundation for the macro implication (Lucas critique)

- **Representative Consumer:**
  - Lecture 4: preference, constraints
  - Lecture 5: optimization, application
  - Lecture 6: Numerical Examples
- **Representative Firm:**
  - Lecture 7: production, optimization, application

Outline

1 Variable



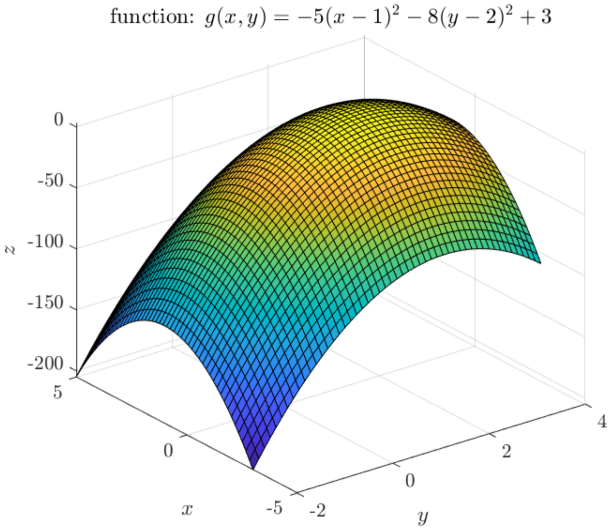
In general, want to solve $\max_x f(x)$

- find "peak" of function
- at peak, slope is 0
- **First order condition** (FOC) is when the 1st order derivative, i.e., the slope is 0:

$$f'(x^*) = 0,$$

where x^* is the peak

2 Variables



In general, want to solve $\max_{x,y} g(x,y)$

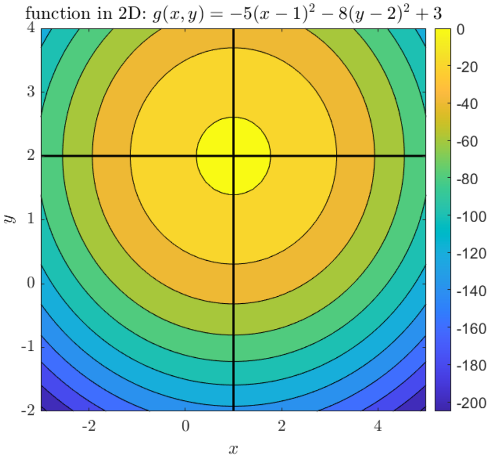
- at peak, slope is 0 in both directions, i.e., the FOCs are

$$D_x g(x^*, y^*) = 0$$
$$D_y g(x^*, y^*) = 0'$$

  where the bundle (x^*, y^*) is the peak

- Hard for my brain to process 3-D graph...resolution?

Visualizing 3-D function on 2-D plane

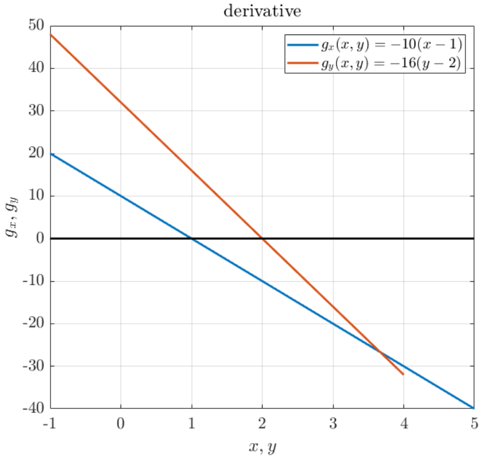


- **Contours:**"standing" at the peak and look down
  - e.g. map on Alltrails
- Fix the level of $g = -20$ (a horizontal slice of 3-D figure)
- Find x and y such that

$$-20 = -5(x-1)^2 - 8(y-2)^2 + 3$$

- repeat for any value of g
- **Exactly where indifference curve came from!**

Solving 2 Variables Optimizations



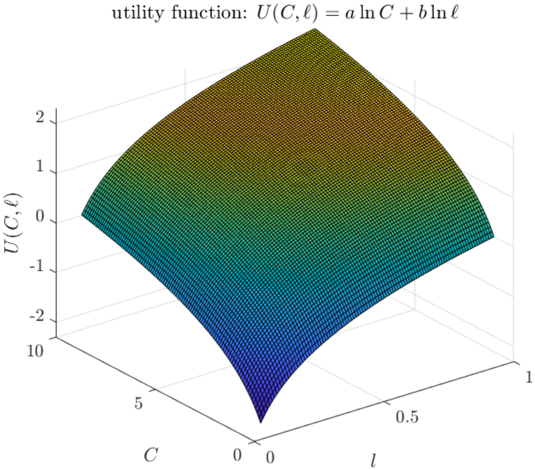
$$D_x g(x^*, y^*) = -10(x-1) = 0$$
$$D_y g(x^*, y^*) = -16(y-2) = 0$$

- Intersection between 0 and line is the solution.
- For other functional form, $D_x g(x,y)$ can depend on y, and $D_y g(x,y)$ can depend on x
- May have constraints on the relationship between x and y

Outline

Utility Function in 3-D

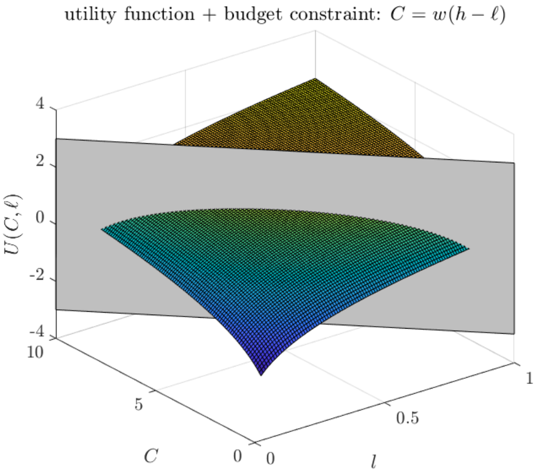
Here $a = b = 1$, where is the peak?



- Seems like to be at $C^* = 10$ and $l^* = 1$
- Recall monotonicity: more is better!
- What stops the consumer from choose $(C, l) = (10, 1)$?

Utility Function + Budget Set in 3-D

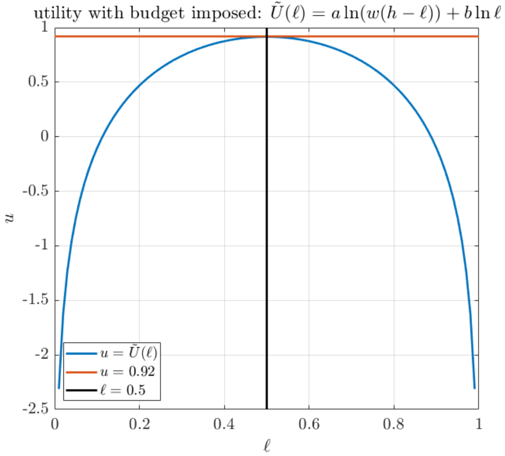
Let $w = 10$ and $h = 1$, and the gray surface represents the border of the budget set.



- Consumers have to choose (C, l) bundles inside the budget set
  - $(C, l) = (10, 1)$ is outside of the budget set $\Rightarrow$ not feasible
- **Binding budget constraint:** candidates for optimal are points in gray
- Which one?

Collapsing 3-D Problem into 2-D: Slice

How? Binding budget constraint!

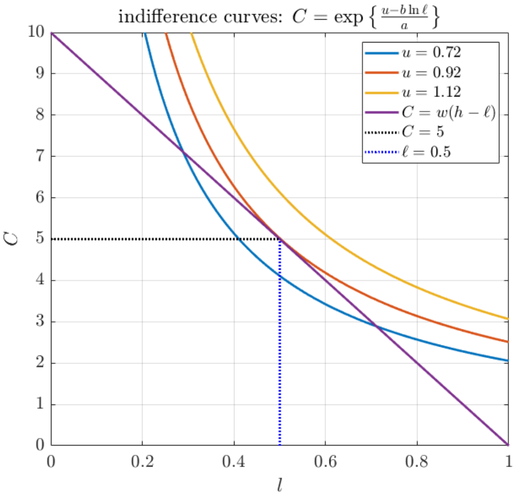


$$
\begin{aligned}
\text{Binding:} \quad & C = w(h-l) \\
& U(C,l) = a\ln C + b\ln l \\
\text{Plug in:} \quad & \tilde{U}(l) = a\ln(w(h-l)) + b\ln l \\
\text{FOC:} \quad & D_l\tilde{U}(l) = 0 \\
& a\frac{-w}{w(h-l)} + b\frac{1}{l} = 0 \\
& \frac{a}{h-l} = \frac{b}{l} \\
& l = \frac{b}{a+b}h
\end{aligned}
$$

$l = 0.5$, let $C = 5$, $u = 0.91629\ldots$

Collapsing 3-D Problem into 2-D: Contours

Recall contours, for any utility level u, $u = a\ln C + b\ln l \Rightarrow C = e^{\frac{u-b\ln l}{a}}$



- What is the highest u feasible given budget constraint?
- Or push up IC (increase u) such that IC is tangent to budget line:

$$-MRS_{l,C} = -w$$

$$\frac{bC}{al} = \frac{bw(h-l)}{al} = w$$

$$l = \frac{b}{a+b}h$$

2-D versions: Pros and Cons

Both 2-D formulations are delivering the same answer.

1. Slice: 1 variable optimization problem, x-axis: l, y-axis: u
   - **Straightforward**: operate on (l, u) plane, good for problem solving
   - **General**: can collapse higher dimension problem
   - **Cons**: lack of trade off between C and $l \Rightarrow$ economcis intuition
2. Contours: 2 variable optimization problem, x-axis: l, y-axis: C
   - **Intuitive**: direct trade off between C and l through $MRS_{l,C}$
   - **Cons:** harder to solve and to generalize to higher dimension

Outline

Review: Models from Last Lecture

1. Utility function: $U(C, l) = a \ln C + b \ln l$

2. Budget constraint: $C \leq w(h - l) + \pi - T$

3. After-tax dividend: $x = \pi - T$

4. wage rate: w

- **Benchmark**: in section Consumer Example

- **Experiment 1**: increase in after-tax dividend: $x_1 > x_0$

- **Experiment 2**: increase wage rate: $w_2 > w_0$

Solve for Benchmark Case

- **Marginal utilities:** $D_C U(C,l) = \frac{a}{C}; D_l U(C,l) = \frac{b}{l}$.
- **Binding budget constraint:** $C = w(h-l) + \pi - T$
- **Optimality:** $MRS_{l,C} = w \Rightarrow \frac{D_l U(C,l)}{D_C U(C,l)} = w \Rightarrow w = \frac{bC}{al}$

Plug binding budget constraints into optimality and solve for l:

$$w = \frac{b(w(h-l)+x)}{al} \tag{1}$$

$$\Rightarrow \quad wal = b(w(h-l)+x) \tag{2}$$

$$\Rightarrow \quad wal = bwh - bwl + bx \tag{3}$$

$$\Rightarrow \quad (a+b)wl = bwh + bx \tag{4}$$

$$\Rightarrow \quad l = \frac{b}{a+b}\left(h + \frac{x}{w}\right) \tag{5}$$

Solve for Benchmark Case (Cont.)

Solve for C, we get

$$
\begin{aligned}
& l = \frac{b}{a+b}\left(h + \frac{x}{w}\right) \Rightarrow wl = \frac{b}{a+b}(wh+x) && (6)\\
& C = w(h-l) + \pi - T = w(h-l) + x && (7)\\
\Rightarrow \quad & C = w\left[h - \frac{b}{a+b}\left(h+\frac{x}{w}\right)\right] + x && (8)\\
\Rightarrow \quad & C = wh - \frac{b}{a+b}(wh+x) + x && (9)\\
\Rightarrow \quad & C = \frac{a}{a+b}wh + \frac{a}{a+b}x && (10)\\
\Rightarrow \quad & C = \frac{a}{a+b}(wh+x) && (11)
\end{aligned}
$$

Property for this utility function: consumer "split" fixed share of "wealth": $wl = s(wh+x)$, and $C = (1-s)(wh+x)$.

Solve for Experiment 1: $x \uparrow$

(l_0, C_0, x_0): benchmark value; (l_1, C_1, x_1): experiment 1 value.
With pure income effect, no change in real wage: $w_1 = w_0 = w$
The difference between experiment 1 and benchmark case is

$$
\begin{aligned}
l_1 - l_0 &= \frac{b}{a+b}\left(h + \frac{x_1}{w}\right) - \frac{b}{a+b}\left(h + \frac{x_0}{w}\right) && (12)\\
&= \frac{b}{a+b}\left(\frac{x_1}{w} - \frac{x_0}{w}\right) && (13)\\
&= \frac{b}{(a+b)w}(x_1 - x_0) > 0 && (14)\\
C_1 - C_0 &= \frac{a}{a+b}(wh + x_1) - \frac{a}{a+b}(wh + x_0) && (15)\\
&= \frac{a}{a+b}(x_1 - x_0) > 0 && (16)
\end{aligned}
$$

Namely, with pure income effect, both leisure and consumption increases.

Solve for Experiment 1: Graphical Intuition

$w_1 = w_0 = 10; x_1 = 1 > x_0 = 0$

Figure: Both leisure and consumption are higher

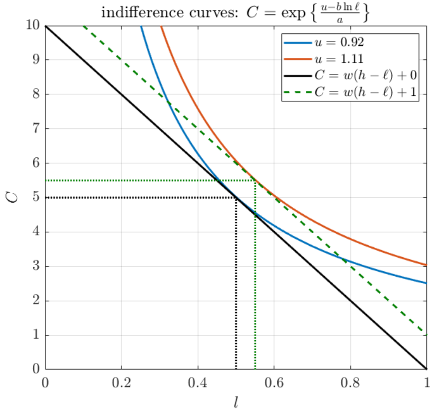
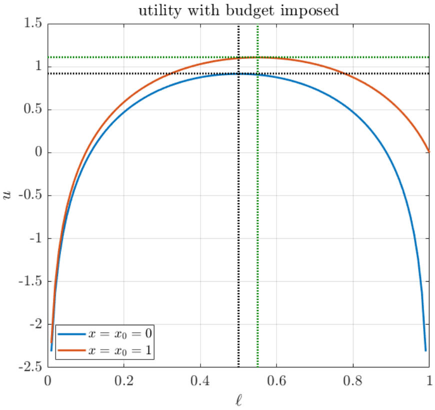


Figure: Budget constraint is "eased"



Solve for Experiment 2: $w \uparrow$

(l_0, C_0, x_0): benchmark value; (l_2, C_2, x_2): experiment 2 value.
With both income and substitution effects, analysis is complicated:

$$
\begin{aligned}
l_2 - l_0 &= \frac{b}{a+b}\left(h + \frac{x_2}{w_2}\right) - \frac{b}{a+b}\left(h + \frac{x_0}{w_0}\right) && (17)\\
&= \frac{b}{a+b}\left(\frac{x_2}{w_2} - \frac{x_0}{w_0}\right) \gtreqless 0 && (18)\\
C_2 - C_0 &= \frac{a}{a+b}(w_2 h + x_2) - \frac{a}{a+b}(w_0 h + x_0) && (19)\\
&= \frac{a}{a+b}(h(w_2 - w_0) + (x_2 - x_0)) > 0 && (20)
\end{aligned}
$$

Although the consumption is certainly increasing, the change in leisure is uncertain $\Rightarrow$ need numerical solution (put numbers in).

Solve for Experiment 2: $w \uparrow$ (Cont.)

Let $w_2 = 15 > w_0 = 10; x_2 = x_0 = 0$.

$$l_2 - l_0 = \frac{b}{a+b}\left(\frac{x_2}{w_2} - \frac{x_0}{w_0}\right) = \frac{b}{a+b}\left(\frac{0}{15} - \frac{0}{10}\right) = 0 \tag{21}$$

Leisure remain the same.
Compare with experiment 1, $w_2 = 15 > w_1 = 10; x_2 = 0 < x_1 = 1; h = 1$:

$$l_2 - l_1 = \frac{b}{a+b}\left(\frac{x_2}{w_2} - \frac{x_1}{w_1}\right) = \frac{b}{a+b}\left(\frac{0}{15} - \frac{1}{10}\right) < 0 \tag{22}$$

$$C_2 - C_1 = \frac{a}{a+b}\left(h(w_2 - w_1) + (x_2 - x_1)\right) \tag{23}$$

$$= \frac{a}{a+b}(1(15-10) + (0-1)) > 0 \tag{24}$$

Experiment 2 v.s. Benchmark: Graphical Intuition

Figure: Total Effect

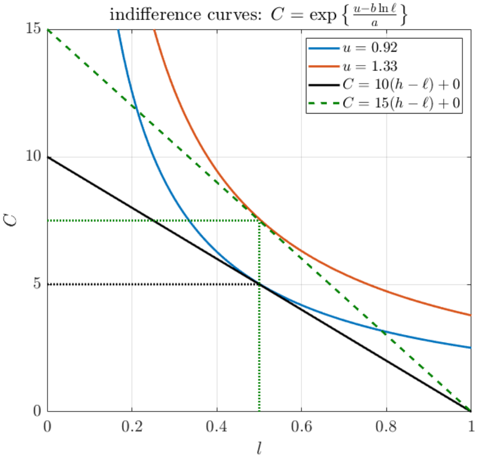


Figure: Income and Substitution Effect