utility function: $u(C)+u(C')$

Time endowment: 1

Human capital accumulation requires "time": $1-\phi$ of time goes to education, only ϕ of time goes to work.

Household's human capital endowment: H

$\phi \neq 1$

human capital accumulation process: $H' = H + (1-\phi) H$

Household's owns the capital and doing investment to accumulate capital.

physical capital endowment: K

physical capital accumulation process: $K' = (1-\delta) K + I$

Production function: $Y = K^{\alpha}(\phi H)^{1-\alpha}; Y' = K'^{\alpha}(\phi' H')^{1-\alpha}$

Consumer owns the firm, i.e., claim the whole π

No government

labor \$

Consumer's current budget constraint: $C \leqslant w\phi H + rK - I + \pi$

capital rental \$

where $\pi = Y - w\phi H - rK$

$\Rightarrow C \leqslant w\phi H + rK - I + (Y - w\phi H - rK) = Y - I$

$C = Y - I$

$\Rightarrow C' = Y'$, no I since this is the last period.

$C' = Y'$

Social planner's problem:

$\phi' = 1$

$\max_{C, C', \phi, K', H'} u(C) + u(C')$

s.t. $C \leqslant Y - I$

$\boxed{\phi H \Rightarrow N}$

$C' = Y'$

$H' = H + (1-\phi) H$

$H' = H + (1-\phi) H$
$= (2-\phi) H$

$K' = (1-\delta) K + I$

$$C \leqslant Y - I = K^{\alpha}(\phi H)^{1-\alpha} - \underbrace{(K' - (1-\delta) K)}_{I}$$

$$C' = Y' = K'^{\alpha}(\phi' H')^{1-\alpha} = K'^{\alpha}(\phi' (2-\phi) H)^{1-\alpha}$$

$K' = \underbrace{(1-\delta) K}_{\text{undepreciated capital}} + I$

$H' = H + (1-\phi) H \Rightarrow H' = (2-\phi) H$

$$\left.\begin{aligned} C &= Y - I \\ C' &= Y' \end{aligned}\right\} \Rightarrow \max_{\phi, K'} u(C(\phi, K')) + u(C'(\phi, K'))$$

$$K' = (1-\delta)\,K + I \Rightarrow I = K' - (1-\delta)\,K$$

$$\Rightarrow \max_{\phi,K'} u(\underbrace{K^\alpha(\phi H)^{1-\alpha} - (K' - (1-\delta)\,K)}_{C}) + u(\underbrace{K'^\alpha(\overset{1}{\phi'}(2-\phi)\,H)^{1-\alpha}}_{C'})$$

$\phi' = 1$ since no third period

$$\Rightarrow \max_{\phi,K'} u(K^\alpha(\phi H)^{1-\alpha} - (K' - (1-\delta)\,K)) + u(K'^\alpha((2-\phi)\,H)^{1-\alpha})$$

$$[K']: u'(C) = u'(C')\,\alpha K'^{\alpha-1}((2-\phi)\,H)^{1-\alpha}$$

$$\Rightarrow \frac{u'(C)}{u'(C')} = \alpha K'^{\alpha-1}((2-\phi)\,H)^{1-\alpha} \Leftrightarrow MRS_{C,C'} = MPK$$

$$[\phi]: \overbrace{u'(C)\,\underbrace{(1-\alpha)K^\alpha(\phi H)^{-\alpha}\,H}_{\text{MP of } \phi \text{ fraction of time}}}^{\text{MC of time}} = \overbrace{u'(C')\,K'^\alpha\,\underbrace{(1-\alpha)\,((2-\phi)\,H)^{-\alpha}\,H}_{\text{MP of } H'}}^{\text{MB of time}}$$

$$\Rightarrow u'(C)K^\alpha(\phi H)^{-\alpha}\,H = u'(C')\,K'^\alpha\,((2-\phi)\,H)^{-\alpha}\,H$$

$$\Rightarrow u'(C)K^\alpha(\phi)^{-\alpha} = u'(C')\,K'^\alpha\,((2-\phi))^{-\alpha}$$

$$\Rightarrow \frac{u'(C)}{u'(C')} = \left(\frac{K'}{K}\right)^\alpha\left(\frac{(2-\phi)}{\phi}\right)^{-\alpha} \Rightarrow MRS_{C,C'} = \frac{MPH'}{MP\phi}$$

$$\Rightarrow \overbrace{\left(\frac{K'}{K}\right)^\alpha\left(\frac{(2-\phi)}{\phi}\right)^{-\alpha}}^{MP\phi} = \overbrace{\alpha K'^{\alpha-1}((2-\phi)\,H)^{1-\alpha}}^{MPK}$$

$$\Rightarrow K'\left(\frac{1}{K}\right)^\alpha\left(\frac{1}{\phi}\right)^{-\alpha} = \alpha\,(2-\phi)\,H^{1-\alpha} \checkmark$$

$$\Rightarrow K'\,K^{-\alpha}\,\phi^\alpha = \alpha\,(2-\phi)\,H^{1-\alpha} \checkmark$$

$$\Rightarrow \boxed{\frac{\phi^\alpha}{2-\phi}\,K' = \underbrace{\alpha K^\alpha\,H^{1-\alpha}}_{\text{constant}}}$$

$\downarrow \quad \uparrow$