Lecture 10
Examples on Competitive Equilibrium and Social Planner's Problem

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Overview

After constructing both consumers' and firms' problem, we start to bring them together in one-period model:

- Lecture 8: competitive equilibrium (CE)
  - each agent solve their problems individually
  - aggregate decision determines "prices" (wage, rent, etc.)
- Lecture 9: social planer's problem (SPP) → most effecient
  - imaginary and benevolent social planner determines the allocation
  - should be the most efficient outcome
- Lecture 10: CE and SPP examples

Two Dimensional Chain Rule

consumer make decision given w

Suppose we have a utility function $U(C, l)$, where C is the consumption, and l is the leisure, and both $C = C(w)$ and $l = l(w)$ are the function of equilibrium wage w, then

how much utility change because of consumption change

how much C change because of w change

$$\frac{d}{dw}[U(C(w), l(w))] = D_C U(C(w), l(w)) \times \boxed{\frac{dC(w)}{dw}} + D_l U(C(w), l(w)) \times \frac{dl(w)}{dw} \quad (1)$$

affect

"Taken as Given"

Here is a good rule of thumb:

When you solve the problem of an agent who chooses y taking x as given,
the answer should take the form of $y(x)$.

Example: the consumer maximizes utility by choosing consumption, leisure, and labor supply, taking the wage and profits as given. $(G = 0)$

$$\max_{C,l,N^s} U(C,l) \quad \text{subject to} \quad C = wN^s + \pi \quad \text{and} \quad l + N^s = h \qquad (2)$$

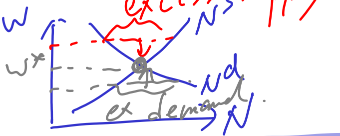
$c(w)$ $\quad l(w)$ $\quad \pi(w)$

- solution takes the form: $C(w,\pi), l(w,\pi), N^s(w,\pi) \Rightarrow N^s(w)$

  exogenous to the model.

- why not h, or utility parameters? Not **endogenous to the model!**
- can repeat this idea for the firm to get $N^d(w), Y(w), \pi(w)$

"Endogenous to the Model"



What does equilibrium do? Figures out what level of "taken as given" but endogenous variables has to occur:

- consumer: $\pi = \pi(w)$ from firm's problem
- labor supply can be rewrite as: $N^s(w, \pi) = N^s(w, \pi(w)) = \boxed{N^s(w)}$
- labor market clearing: $N^d(w^*) = N^s(w^*)$, where w^* is eqm wage

Question: any of the "taken as given variables" show up in the SPP?

- Ans: NO! Social planner is benevolent dictator!

Model Environment

constant rate of risk aversion

CRRA

- Consumer: $U(C,l) = \frac{C^{1-b}}{1-b} + \frac{l^{1-d}}{1-d}$, where $b = 2$ and $d = \frac{3}{2}$.
  - b, d are parameters
  - $h = 1$ is time endowment to allocate between leisure and labor supply
  - owns the firm, subject to lump-sum tax $T \geq 0$

share of capital

- Firm: $zF(K,N) = zK^{\alpha}N^{1-\alpha}$, where $K = 1$ and $\alpha = \frac{1}{2}$ (param)

K endowment

- Government: $T = G$
- Labor market: both consumer and firm take wage rate w as given

Experiments

1. **Benchmark**: $z = 1$ and $G = 0$

   $\downarrow z \uparrow$

2. **Experiment 1**: $z = 1.2$ and $G = 0$

   $G \uparrow$

3. **Experiment 2**: $z = 1$ and $G = 0.5$

Directly solve social planner's problem

$\Rightarrow \because$ all underlying assumptions hold

$\Rightarrow \boxed{CE = SPP.}$

Solve Benchmark in Social Planner's Problem

$Y = C + G \qquad Y = zF(K, N) = zK^{\alpha} N^{1-\alpha} = zN^{1-\alpha}.$

- PPF: $C + G = \underbrace{zN^{1-\alpha}}_{Y}$, where $\alpha = \frac{1}{2}$ $=1$
- Time: $N = h - l$, where $h = 1$
- Social Planner's Problem:

$$\begin{aligned} \max_{l} \quad & U(C(l), l) = \frac{\boxed{C(l)}^{1-b}}{1-b} + \frac{l^{1-d}}{1-d} \\ \text{s.t.} \quad & C = Y - G \Rightarrow C = zN^{1-\alpha} - G \\ & Y = zN^{1-\alpha} \qquad = \boxed{z(1-l)^{1-\alpha} - G} \\ & N = \boxed{1 - l} \\ \Rightarrow \quad \max_{l} \quad & \frac{(\overbrace{z(1-l)^{1-\alpha} - G}^{C})^{1-b}}{1-b} + \frac{l^{1-d}}{1-d} \end{aligned} \tag{3}$$

Solve Benchmark in Social Planner's Problem (Cont.)

$\frac{1}{1-b} C^{1-b} \Rightarrow \cancel{(1-b)} \cdot \frac{1}{\cancel{1-b}} \cdot C^{1-b-1} \Rightarrow C^{-b}$

$z(1-l)^{1-\alpha} \Rightarrow z \cdot (1-\alpha) \cdot (1-l)^{-\alpha}$

$$\max_{l} \boxed{\frac{(z(1-l)^{1-\alpha} - G)^{1-b}}{1-b}} + \boxed{\frac{l^{1-d}}{1-d}} \tag{4}$$

FOC: $$\underbrace{(z(1-l)^{1-\alpha} - \cancel{G})^{-b}}_{\frac{(\cdot)^{1-b}}{1-b}} \times \underbrace{(1-\alpha)z(1-l)^{-\alpha}}_{z(1-l)^{1-\alpha}} \times \underbrace{(-1)}_{-l} + l^{-d} = 0 \tag{5}$$

$G = 0:$ $$z^{-b}(1-l)^{-b(1-\alpha)} \times (1-\alpha)z(1-l)^{-\alpha} = l^{-d} \tag{6}$$

$$(1-\alpha)z^{1-b}(1-l)^{-\alpha-b+\alpha b} = l^{-d} \tag{7}$$

$1-b = 1-2 = -1$

$-\alpha-b+\alpha b = -\frac{1}{2} - 2 + \frac{1}{2}\cdot 2 = -\frac{3}{2}$

$$\alpha = 1/2; \quad b = 2; \quad d = 3/2 \tag{8}$$

Apply: $$\frac{1}{2} z^{-1}(1-l)^{-\frac{3}{2}} = l^{-\frac{3}{2}} \Rightarrow \frac{1}{2z} = (\frac{1-l}{l})^{\frac{3}{2}} \tag{9}$$

$$\Rightarrow \frac{1-l}{l} = (\frac{1}{2z})^{\frac{2}{3}} \Rightarrow l(z,0) = \frac{1}{1+(2z)^{-\frac{2}{3}}} \tag{10}$$

$$z = 1 \quad \Rightarrow l \approx 0.61, N \approx 0.39, Y = C \approx 0.62, w = \frac{z}{2} N^{-\frac{1}{2}} \approx 0.8 \tag{11}$$

$$\frac{1}{2} \cdot \overbrace{Z^{-1}}^{\frac{1}{2}} \cdot \underline{(1-\ell)}^{-\frac{3}{2}} = \underline{\ell}^{-\frac{3}{2}}$$

$$\frac{1}{2Z} = \left(\frac{\ell}{1-\ell}\right)^{-\frac{3}{2}}$$

$$= \left(\frac{1-\ell}{\ell}\right)^{\frac{3}{2}}$$

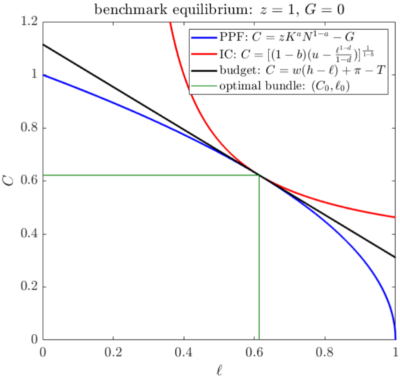
$$\Rightarrow \left(\frac{1}{2Z}\right)^{\frac{2}{3}} = \frac{1-\ell}{\ell}$$

$$\Rightarrow \left(\frac{1}{2Z}\right)^{\frac{2}{3}} \cdot \ell = 1-\ell$$

$$\Rightarrow \left(\left(\frac{1}{2Z}\right)^{\frac{2}{3}} + 1\right)\ell = 1$$

$$\Rightarrow \ell = \frac{1}{\left(\left(\frac{1}{2Z}\right)^{\frac{2}{3}} + 1\right)}$$

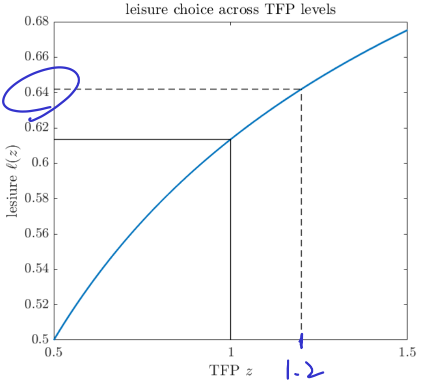
Visualization: Benchmark in SPP



Indifference curve and PPF are tangent at optimal bundle

$$
\begin{aligned}
& \text{slope at tangency } (C_0, l_0) \\
= \quad & \text{slope of IC}(-MRS_{l,C}) \\
= \quad & \text{slope of budget line}(-w) \\
= \quad & \text{slope of PPF}(-MRT_{l,C}) \\
= \quad & \text{slope of production fcn}(-MPN)
\end{aligned}
$$

Solving with New TFP



Recall that we solved for the equilibrium quantity of leisure as a function of TFP:

$$l(z) = \frac{1}{1 + (2z)^{-\frac{2}{3}}} \qquad (12)$$

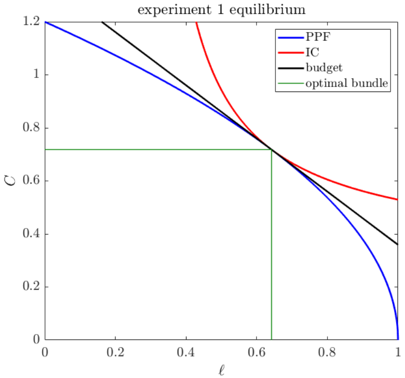
So now we've solved for all possible "experiment 1's"! Just plug in $z = 1.2$ to get $l \approx 0.642$, and plug in to get all the rest as well.

$C(\ell) \rightarrow Y = C + G$

$\downarrow$

$MPN = w \qquad N = 1 - \ell$

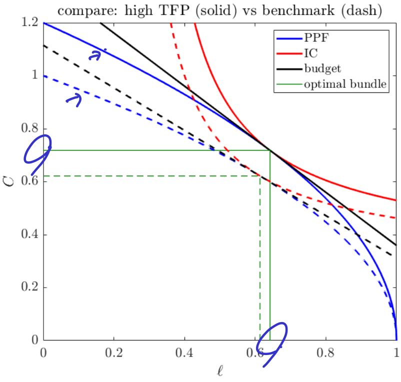
Visualization: Experiment 1



Tangency preserved, just shifted

$$\begin{aligned} & \text{slope at tangency } (C_1, l_1) \\ = \quad & \text{slope of IC}(-MRS_{l,C}) \\ = \quad & \text{slope of budget line}(-w) \\ = \quad & \text{slope of PPF}(-MRT_{l,C}) \\ = \quad & \text{slope of production fcn}(-MPN) \end{aligned}$$

Comparison: Experiment 1 and Benchmark

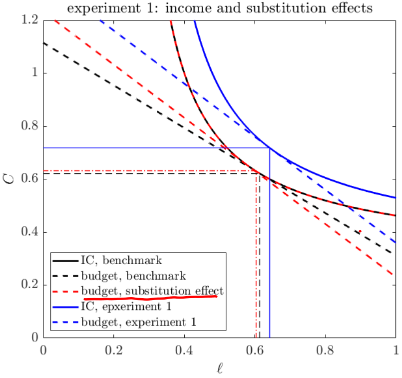


What's different?

- higher productivity means PPF shifts outward
- outward shift of PPF makes higher utility level (IC) attainable
- tangency is steeper: wage increases
- both consumption and leisure increase!

Experiment 1: Income and Substitution Effect

$$\uparrow w = MPN = \frac{\partial Y}{\partial N} = \frac{\partial (Z \cdot N^{\frac{1}{2}})}{\partial N} = \frac{1}{2} Z \uparrow N^{-\frac{1}{2}}$$

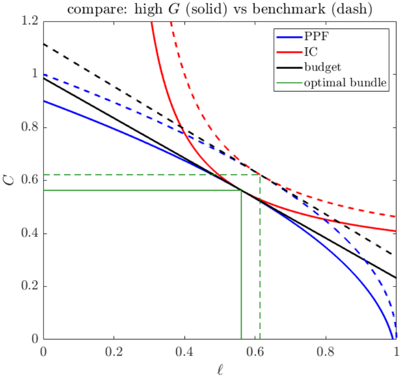


Recall wage increase case from the consumer problem:

- **substitution effect:** move along IC but reflect new wage (i,e, new budget or new PPF)
  - C increases, l decreases
- **income effect:** move up to new budget line / PPF
  - C and l both increase
- here, income effect wins and leisure increases

Comparison: Experiment 2 and Benchmark

dashed : benchmark
solid : experiment 2.



$Y = \downarrow C + G \uparrow$

Note: SPP harder to solve by hand with $G \neq 0$ details. But, can still analyze with graphs! $\downarrow C = Y - G \uparrow$

- higher government spending shifts PPF inward
- inward shift of PPF lowers utility level (IC) attainable
- budget shallower: wage falls
- consumption, leisure fall (recall normal goods assumption)
- can show output increases

MPC

Response to Data

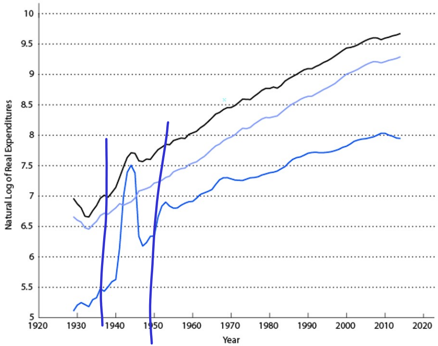
Effect of ↑ in	TFP	G
Output	Increase	Increase
Consumption	Increase	Decrease
Employment	Ambiguous	Increase
Wage	Increase	Decrease

TFP is a overall better match! Real Business Cycle theory

- recall key business cycle facts: employment, consumption, real wage are all procyclical
- recall key trend: output has grown steadily for last century
- question: which model is more consistent with these facts?

Data: Government Spending from WWII

Figure 5.7 GDP, Consumption, and Government Expenditures



- large increase in G to finance war effort
- modest increase in Y
- slight decline in C
- consistent with our model!

Data: Solow Residual, $z = \frac{Y}{K^{\alpha} N^{1-\alpha}}$

Figure 4.18 The Solow Residual for the United States

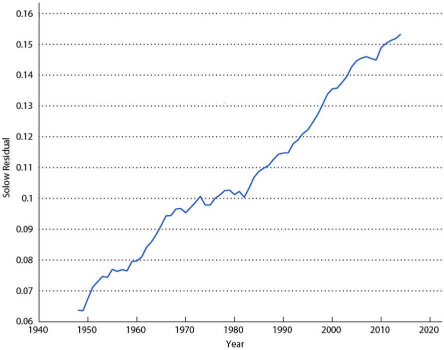
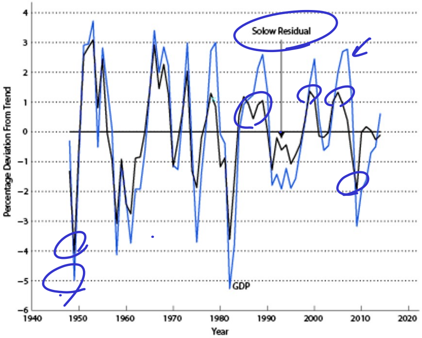


Figure 5.11 Deviations from Trend in GDP and the Solow Residual



Appendix

How to solve $G \neq 0$

Back

$$\max_{l} \quad \frac{(z(1-l)^{1-\alpha} - G)^{1-b}}{1-b} + \frac{l^{1-d}}{1-d} \tag{13}$$

$$\text{FOC:} \quad z(1-l)^{1-\alpha} - G)^{-b} \times (1-\alpha)z(1-l)^{-\alpha} = l^{-d} \tag{14}$$

$$\text{Divide :} \quad (z(1-l)^{1-\alpha} - G)^{-b} = \frac{l^{-d}}{(1-\alpha)z(1-l)^{-\alpha}} \tag{15}$$

$$\text{power of } -\frac{1}{b} : \quad z(1-l)^{1-\alpha} - G = \left[\frac{l^{-d}}{(1-\alpha)z(1-l)^{-\alpha}}\right]^{-\frac{1}{b}} \tag{16}$$

$$\text{Solve } G : \quad G = F(l) = z(1-l)^{1-\alpha} - \left[\frac{l^{-d}}{(1-\alpha)z(1-l)^{-\alpha}}\right]^{-\frac{1}{b}} \tag{17}$$

$$\iff l = F^{-1}(G) \tag{18}$$