

Lecture 6

Numerical Example

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Overview: Lecture 4 - 7

Provide **micro-foundation** for the **macro implication** (**Lucas critique**)

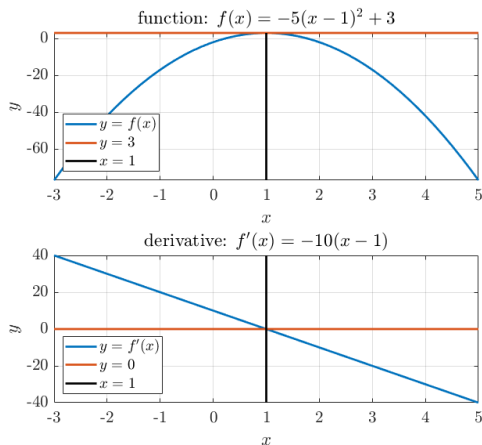
■ Representative Consumer:

- Lecture 4: **preference**, **constraints**
- Lecture 5: **optimization**, **application**
- Lecture 6: Numerical Examples

■ Representative Firm:

- Lecture 7: **production**, **optimization**, **application**

1 Variable



In general, want to solve

$$\max_x f(x)$$

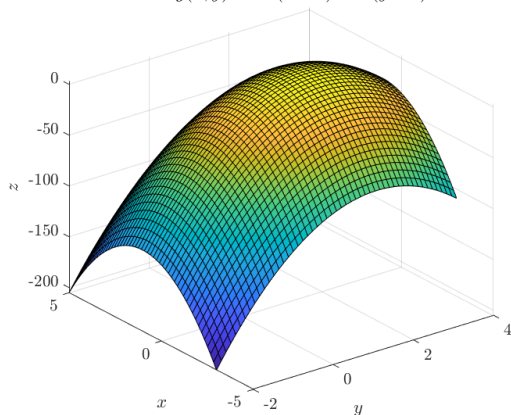
- find “peak” of function
- at peak, slope is 0
- **First order condition** (FOC) is when the 1st order derivative, i.e., the slope is 0:

$$f'(x^*) = 0,$$

where x^* is the peak

2 Variables

function: $g(x, y) = -5(x - 1)^2 - 8(y - 2)^2 + 3$



In general, want to solve

$$\max_{x,y} g(x, y)$$

- at peak, slope is 0 in both directions, i.e., the FOCs are

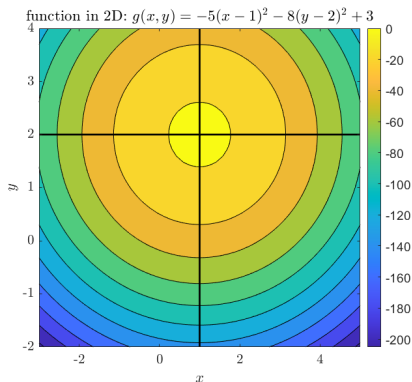
$$D_x g(x^*, y^*) = 0$$

$$D_y g(x^*, y^*) = 0'$$

where the bundle (x^*, y^*) is the peak

- Hard for my brain to process 3-D graph...resolution?

Visualizing 3-D function on 2-D plane



- **Contours:** “standing” at the peak and look down

- e.g. [map on Alltrails](#)

- Fix the level of $g = -20$ (a [horizontal slice](#) of 3-D figure)

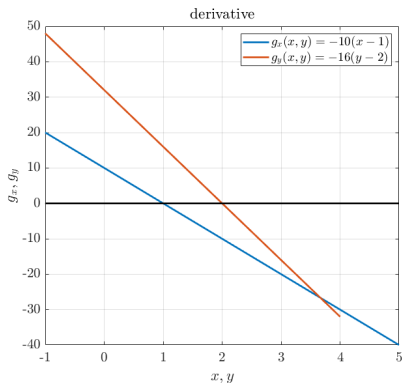
- Find x and y such that

$$-20 = -5(x - 1)^2 - 8(y - 2)^2 + 3$$

- repeat for any value of g

- **Exactly where indifference curve came from!**

Solving 2 Variables Optimizations



$$D_x g(x^*, y^*) = -10(x - 1) = 0$$

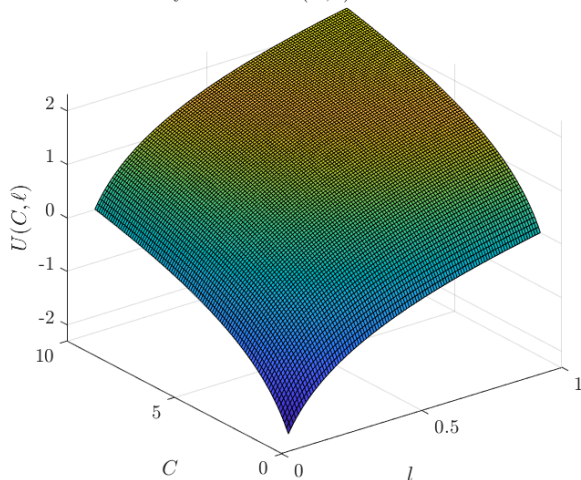
$$D_y g(x^*, y^*) = -16(y - 2) = 0$$

- **Intersection** between 0 and line is the solution.
- For other functional form,
 $D_x g(x, y)$ can depend on y , and
 $D_y g(x, y)$ can depend on x
- May have constraints on the relationship between x and y

Utility Function in 3-D

Here $a = b = 1$, where is the peak?

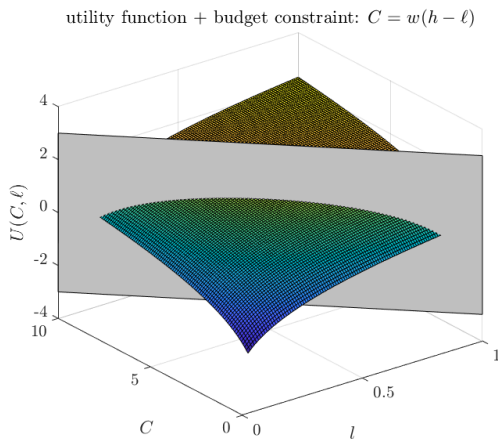
utility function: $U(C, \ell) = a \ln C + b \ln \ell$



- Seems like to be at $C^* = 10$ and $\ell^* = 1$
- Recall **monotonicity**: more is better!
- What **stops** the consumer from choose $(C, \ell) = (10, 1)$?

Utility Function + Budget Set in 3-D

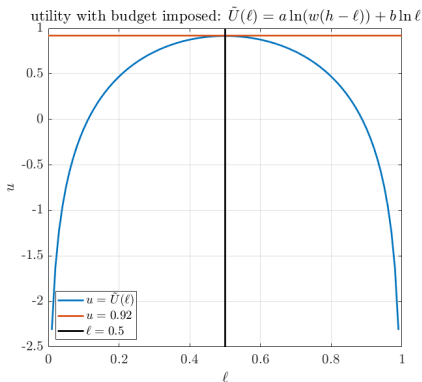
Here $w = 10$ and $h = 1$, and the gray surface represents the **border** of the budget set.



- Consumers have to choose (C, ℓ) bundles **inside the budget set**
 - $(C, \ell) = (10, 1)$ is **outside** of the budget set \Rightarrow not feasible
- **Binding budget constraint:** candidates for optimal are points in gray
- Which one?

Collapsing 3-D Problem into 2-D: Slice

How? Binding budget constraint!



Binding: $C = w(h - l)$

$$U(C, l) = a \ln C + b \ln l$$

Plug in: $\tilde{U}(l) = a \ln(w(h - l)) + b \ln l$

FOC: $D_l \tilde{U}(l) = 0$

$$a \frac{-w}{w(h - l)} + b \frac{1}{l} = 0$$

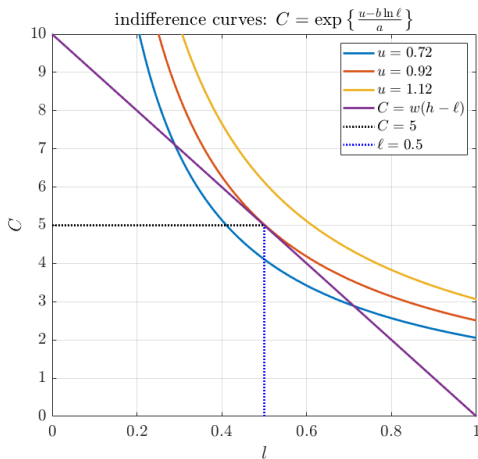
$$\frac{a}{h - l} = \frac{b}{l}$$

$$l = \frac{b}{a + b} h$$

$$l = 0.5, C = 5, u = 0.91629\dots$$

Collapsing 3-D Problem into 2-D: Contours

Recall **contours**, for any utility level u , $u = a \ln C + b \ln l \Rightarrow C = e^{\frac{u-b \ln l}{a}}$



■ What is the highest u feasible given budget constraint?

■ Or push up IC (increase u) such that IC is tangent to budget line:

$$-MRS_{l,C} = -w$$

$$\frac{bC}{al} = \frac{bw(h-l)}{al} = w$$

$$l = \frac{b}{a+b}h$$

2-D versions: Pros and Cons

Both 2-D formulations are delivering the same answer.

- ① Slice: 1 variable optimization problem, x -axis: l , y -axis: u
 - **Straightforward**: operate on (l, u) plane, good for problem solving
 - **General**: can collapse higher dimension problem
 - **Cons**: lack of trade off between C and $l \Rightarrow$ [economcis intuition](#)
- ② Contours: 2 variable optimization problem, x -axis: l , y -axis: C
 - **Intuitive**: direct trade off between C and l through $MRS_{l,C}$
 - **Cons**: harder to solve and to generalize to higher dimension

Review: Models from Last Lecture

- ① Utility function: $U(C, l) = a \ln C + b \ln l$
 - ② Budget constraint: $C \leq w(h - l) + \pi - T$
 - ③ After-tax dividend: $x = \pi - T$
 - ④ wage rate: w
- **Benchmark:** in section Consumer Example
 - **Experiment 1:** increase in after-tax dividend: $x_1 > x_0$
 - **Experiment 2:** increase wage rate: $w_2 > w_0$

Solve for Benchmark Case

■ **Marginal utilities:** $D_C U(C, l) = \frac{a}{C}$; $D_l U(C, l) = \frac{b}{l}$.

■ **Binding budget constraint:** $C = w(h - l) + \pi - T$

■ **Optimality:** $MRS_{l,C} = w \Rightarrow \frac{D_l U(C, l)}{D_C U(C, l)} = w \Rightarrow w = \frac{bC}{al}$

Plug binding budget constraints into optimality and solve for l :

$$w = \frac{b(w(h - l) + x)}{al} \quad (1)$$

$$\Rightarrow wal = b(w(h - l) + x) \quad (2)$$

$$\Rightarrow wal = bwh - bwl + bx \quad (3)$$

$$\Rightarrow (a + b)wl = bwh + bx \quad (4)$$

$$\Rightarrow l = \frac{b}{a + b} \left(h + \frac{x}{w} \right) \quad (5)$$

Solve for Benchmark Case (Cont.)

Solve for C , we get

$$l = \frac{b}{a+b} \left(h + \frac{x}{w} \right) \Rightarrow wl = \frac{b}{a+b} (wh + x) \quad (6)$$

$$C = w(h - l) + \pi - T = w(h - l) + x \quad (7)$$

$$\Rightarrow C = w \left[h - \frac{b}{a+b} \left(h + \frac{x}{w} \right) \right] + x \quad (8)$$

$$\Rightarrow C = wh - \frac{b}{a+b} (wh + x) + x \quad (9)$$

$$\Rightarrow C = \frac{a}{a+b} wh + \frac{a}{a+b} x \quad (10)$$

$$\Rightarrow C = \frac{a}{a+b} (wh + x) \quad (11)$$

Property for this utility function: consumer “split” fixed share of “wealth”: $wl = s(wh + x)$, and $C = (1 - s)(wh + x)$.

Solve for Experiment 1: $x \uparrow$

(l_0, C_0, x_0) : benchmark value; (l_1, C_1, x_1) : experiment 1 value.

With pure income effect, **no change in real wage**: $w_1 = w_0 = w$

The difference between experiment 1 and benchmark case is

$$l_1 - l_0 = \frac{b}{a+b} \left(h + \frac{x_1}{w} \right) - \frac{b}{a+b} \left(h + \frac{x_0}{w} \right) \quad (12)$$

$$= \frac{b}{a+b} \left(\frac{x_1}{w} - \frac{x_0}{w} \right) \quad (13)$$

$$= \frac{b}{(a+b)w} (x_1 - x_0) > 0 \quad (14)$$

$$C_1 - C_0 = \frac{a}{a+b} (wh + x_1) - \frac{a}{a+b} (wh + x_0) \quad (15)$$

$$= \frac{a}{a+b} (x_1 - x_0) > 0 \quad (16)$$

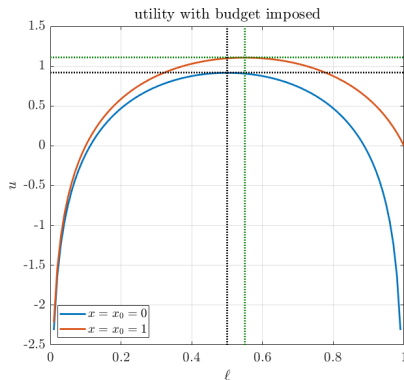
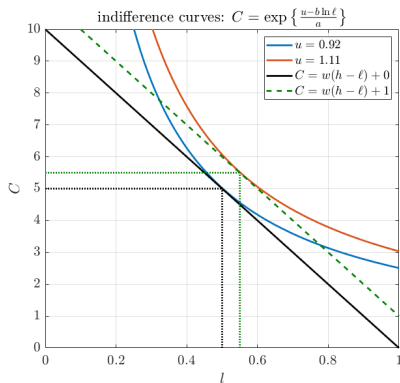
Namely, with pure income effect, **both leisure and consumption increases**.

Solve for Experiment 1: Graphical Intuition

$$w_1 = w_0 = 10; x_1 = 1 > x_0 = 0$$

Both leisure and consumption are higher

Budget constraint is “eased”



Solve for Experiment 2: $w \uparrow$

(l_0, C_0, x_0) : benchmark value; (l_2, C_2, x_2) : experiment 2 value.

With both income and substitution effects, analysis is complicated:

$$l_2 - l_0 = \frac{b}{a+b} \left(h + \frac{x_2}{w_2} \right) - \frac{b}{a+b} \left(h + \frac{x_0}{w_0} \right) \quad (17)$$

$$= \frac{b}{a+b} \left(\frac{x_2}{w_2} - \frac{x_0}{w_0} \right) \begin{matrix} \geq 0 \\ \leq 0 \end{matrix} \quad (18)$$

$$C_2 - C_0 = \frac{a}{a+b} (w_2 h + x_2) - \frac{a}{a+b} (w_0 h + x_0) \quad (19)$$

$$= \frac{a}{a+b} (h(w_2 - w_0) + (x_2 - x_0)) > 0 \quad (20)$$

Although the consumption is certainly increasing, the change in leisure is uncertain \Rightarrow need numerical solution (put numbers in).

Solve for Experiment 2: $w \uparrow$ (Cont.)

Let $w_2 = 15 > w_0 = 10$; $x_2 = x_0 = 0$.

$$l_2 - l_0 = \frac{b}{a+b} \left(\frac{x_2}{w_2} - \frac{x_0}{w_0} \right) = \frac{b}{a+b} \left(\frac{0}{15} - \frac{0}{10} \right) = 0 \quad (21)$$

Leisure remain the same.

Compare with experiment 1, $w_2 = 15 > w_1 = 10$; $x_2 = 0 < x_1 = 1$; $h = 1$:

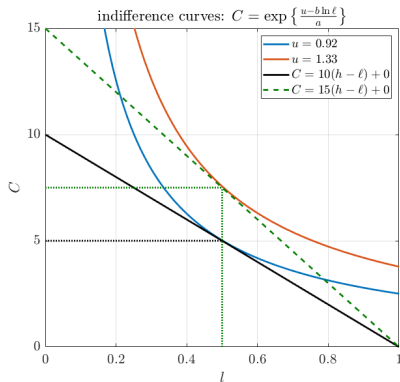
$$l_2 - l_1 = \frac{b}{a+b} \left(\frac{x_2}{w_2} - \frac{x_1}{w_1} \right) = \frac{b}{a+b} \left(\frac{0}{15} - \frac{1}{10} \right) < 0 \quad (22)$$

$$C_2 - C_1 = \frac{a}{a+b} (h(w_2 - w_1) + (x_2 - x_1)) \quad (23)$$

$$= \frac{a}{a+b} (1(15 - 10) + (0 - 1)) > 0 \quad (24)$$

Experiment 2 v.s. Benchmark: Graphical Intuition

Total Effect



Income and Substitution Effect

