Lecture 17
The Real Business Cycle Model
Part 4: Formal Examples

Hui-Jun Chen

National Tsing Hua University

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Overview

- Recall that in Lecture 13, there is no production in dynamic model.
- The following 5 lectures is for **Real Business Cycle** (RBC) model:
  - Lecture 14: consumer
  - Lecture 15: firm
  - Lecture 16: competitive equilibrium
  - Lecture 17: formal example
  - Lecture 18: application to bring RBC to data

Outline

1. Setup
2. Labor Market
3. Output Market

Assumptions

- **consumer**: assume discounting factor $\beta \in (0, 1)$ and utility function is
$$\tilde{U}(C, N, C') = \ln C + \beta \ln C' + \gamma \ln(1 - N),$$
where $\gamma > 0$, and consumer endowed with 1 unit of time.
  - we assume no dis-utility in date 1 labor supply to simplify analysis

- **firm**: assume production is Cobb-Douglas in both periods:
$$Y = zK^{\alpha}N^{1-\alpha} \text{ and } Y' = z'K'^{\alpha}N'^{1-\alpha},$$
where K is initial capital, TFP $z = 1$, and depreciation $\delta \in (0, 1)$

- **government**: spend G and G', which is financed by lump-sum taxes T, T' and deficit B

Competitive Equilibrium

Given exogenous quantities $\{G, G', z, z', K\}$, a competitive equilibrium is a set of (1) consumer choices $\{C, C', N_S, N_S', l, l', S\}$; (2) firm choices $\{Y, Y', \pi, \pi', N_D, N_D', I, K'\}$; (3) government choices $\{T, T', B\}$, and (4) prices $\{w, w', r\}$ such that

1. Taken $\{w, w', r, \pi, \pi'\}$ as given, consumer chooses $\{C', N_S, N_S'\}$ to solve

$$\max_{C', N_S, N_S'} \ln\left(wN_S + \pi - T + \frac{w'N_S' + \pi' - T' - C'}{1+r}\right) + \beta \ln C' + \gamma \ln(1 - N_S),$$

   where we can back out $\{C, S, l, l'\}$.

2. Taken $\{w, w', r\}$ as given, firm chooses $\{N_D, N_D', K'\}$ to solve

$$\max_{N_D, N_D', K'} zK^{\alpha}N_D^{1-\alpha} - wN_D - [K' - (1-\delta)K] + \frac{z'(K')^{\alpha}(N_D')^{1-\alpha} - w'N_D' + (1-\delta)K'}{1+r},$$

   where we can back out $\{Y, Y', \pi, \pi', I\}$.

3. Taxes and deficit satisfy $T + \dfrac{T'}{1+r} = G + \dfrac{G'}{1+r}$ and $G - T = B$.

4. All markets clear: (i) labor, $N_S = N_D$ & $N_S' = N_D'$; (ii) goods, $Y = C + G$ & $Y' = C' + G'$; (iii) bonds at date 0, $S = B$.

Outline

Step 0: Result Implied by Assumptions

1. $N'_S = 1$, since consumer don't value leisure at date 1.
   - If consumer don't value leisure, then choose the highest possible N'_S can expand the budget set without decreasing the utility.

2. $N'_D = N'_S = 1$, by future labor market clearing.

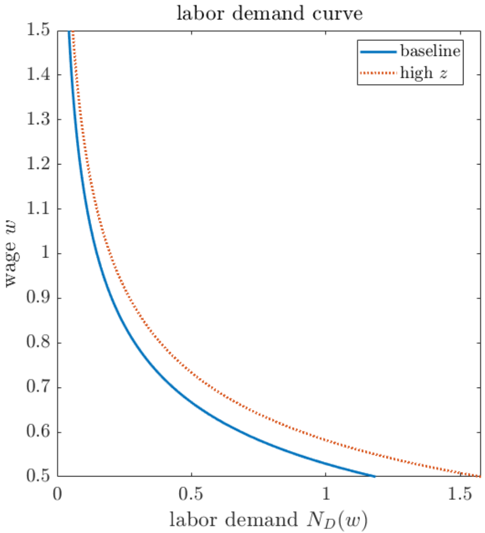
3. The future wage w' is determined by MPN':

$$MPN' = z'(1-\alpha)\left(\frac{K'}{N'_D}\right)^{\alpha},$$

   where $N'_D = 1$ leads to

$$w' = z'(1-\alpha)(K')^{\alpha}.$$

Step 1: Firm's Current Labor Demand



For date 0 labor demand,

$$MPN = z(1-\alpha)\left(\frac{K}{N_D}\right)^{\alpha} = w$$

$$\Rightarrow N_D = \left(\frac{z(1-\alpha)}{w}\right)^{\frac{1}{\alpha}} K$$

- $N_D \downarrow$ in current wage w
- $N_D \uparrow$ in current TFP z (dotted line)
- N_D invariant to interest rate

Step 2: Consumer & Current Labor Supply

- labor supply at date 0:

$$
\begin{aligned}
MRS_{l,C} = -MRS_{N,C} &= -\frac{D_N \tilde{U}(\cdot)}{D_C \tilde{U}(\cdot)} \\
&= -\frac{-\gamma/(1-N_S)}{1/C} = \frac{\gamma C}{1-N_S} = w
\end{aligned}
$$

- Saving at date 0:

$$
MRS_{C,C'} = \frac{1/C}{\beta/C'} = \frac{C'}{\beta C} = 1 + r \Rightarrow C' = \beta(1+r)C
$$

- Recall $N_S' = 1$, we can denote the x notation to be the part of the income that is NOT directly affected by consumer choice:

$$
x = \pi - T \quad \text{and} \quad x' = w' + \pi' - T'
$$

Step 2: Consumer & Current Labor Supply (Cont.)

Recall consumer budget constraint,

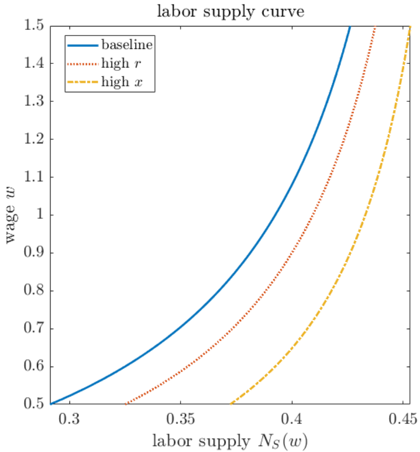
$$\begin{aligned} C + \frac{C'}{1+r} &= wN_S + \pi - T + \frac{w'N_S' + \pi' - T'}{1+r} \\ C + \frac{\beta(1+r)C}{1+r} &= wN_S + x + \frac{x'}{1+r} \\ C &= \frac{1}{1+\beta}\left(wN_S + x + \frac{x'}{1+r}\right) \end{aligned}$$

plug back to labor supply condition:

$$\begin{aligned} w(1-N_S) &= \gamma C \\ w(1-N_S) &= \frac{\gamma}{1+\beta}\left(wN_S + x + \frac{x'}{1+r}\right) \\ wN_S\left(\frac{\gamma}{1+\beta} + 1\right) &= w - \frac{\gamma}{1+\beta}\left(x + \frac{x'}{1+r}\right) \\ N_S &= \frac{1+\beta}{1+\beta+\gamma} - \frac{1}{w}\frac{\gamma}{1+\beta+\gamma}\left(x + \frac{x'}{1+r}\right) \end{aligned}$$

Check: Labor Supply Assumptions

Figure: yellow dotted line is supposed to label as ”low x”

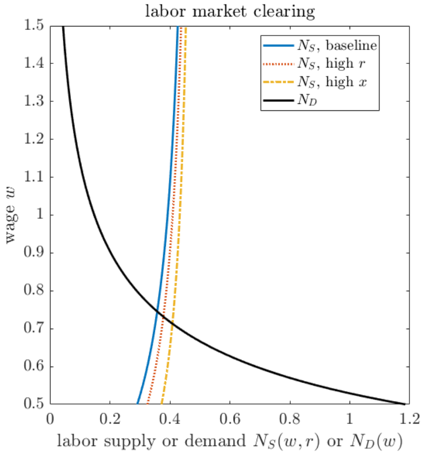


Recall **N1-N3** assumptions,

- **N1**: labor supply ↑ in wage, $dN_S/dw > 0$ (all lines)
- **N2**: labor supply ↑ in real interest rate, $dN_S/dr > 0$ (red v.s. blue)
- **N3**: labor supply ↓ in lifetime wealth, $dN_S/d(x + x') < 0$ (yellow v.s. blue)

Check: Labor Market Clearing

Figure: yellow dotted line is supposed to label as ”low x”



higher interest rate (**N2**), lower lifetime wealth (**N3**) both shifts out labor supply curve:

- wage $w^*(r)$ decreases
- equilibrium quantity of labor $N^*(r)$ increases

Next: construct output supply curve

Outline

Step 3: Output Supply Curve

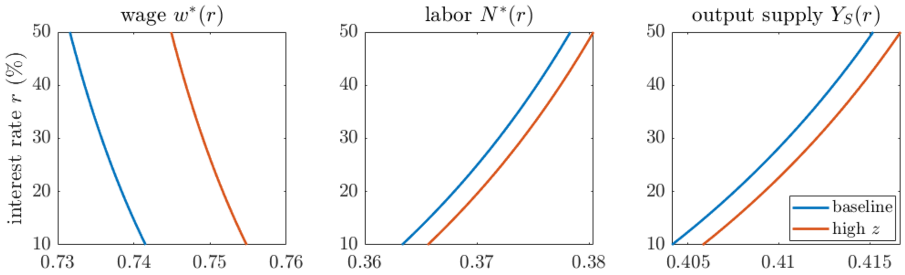
Labor market clearing requires:

$$N_S = \frac{1+\beta}{1+\beta+\gamma} - \frac{1}{w}\frac{\gamma}{1+\beta+\gamma}\left(x + \frac{x'}{1+r}\right) = \left(\frac{z(1-\alpha)}{w}\right)^{\frac{1}{\alpha}} K = N_D.$$

...Yeah, it is very difficult to solve it by hand (actually cannot), but notice

- most of the terms are parameters: $\alpha, \beta, \gamma, z, K$,
- or lifetime wealth that needs gov: x and x'.
- Out main goal is to solve for $w^*(r)$!
  - solve real wage w as a function of real interest rate r
  - then, back out $N^*(r)$ and $Y_S(r)$
    - get $N^*(r)$ by plug $w^*(r)$ into either N_D or N_S
    - get $Y_S(r)$ by plug $N^*(r)$ into $zK^\alpha(N^*)^{1-\alpha}$

Check: Output Supply Curve



Confirm our intuition:

- $r \uparrow$ leads to $w \downarrow$ and $N^*(r) \uparrow$
- given positive *MPN* and fixed K, more labor means more production, so output supply shifts up.

Step 4: Output Demand Curve

Recall that the date 0 output demand curve are composite of

- government spending G and G': exogenous (easy!)
- firm's investment demand $I_D(r)$ (next slide)
- consumer's consumption demand $C_D(r, Y)$:
  - recall **income-expenditure identity**, total income $=$ total demand,

$$C + \frac{C'}{1+r} = wN + \pi - T + \frac{w'N' + \pi' - T'}{1+r}$$

$$\because \pi = Y - wN - I; \pi' = Y' - w'N' + (1-\delta)K'$$

$$(1+\beta)C = Y + \frac{Y'}{1+r} - I + \frac{(1-\delta)K'}{1+r} - \left(T + \frac{T'}{1+r}\right)$$

  - given r, we can solve consumption-saving problem.

Firm's Optimal Investment

Recall

- labor market clearing at date 1: $N_D' = N_S' = N' = 1$, and
- MPK at date 1: $MPK' = z'\alpha(K')^{\alpha-1}$.

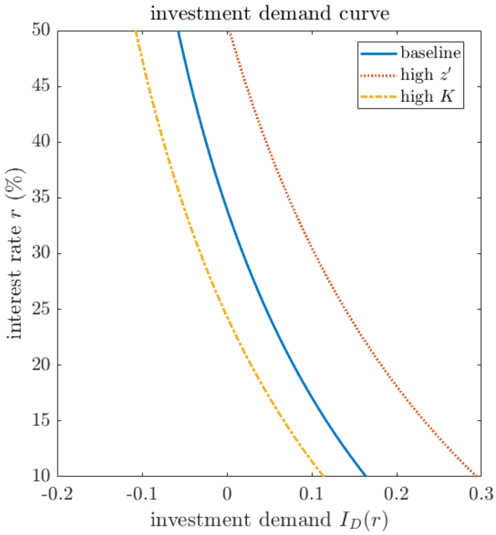
Thus, according to optimal investment schedule,

$$\begin{aligned} MPK' - \delta &= r \\ z'\alpha(K')^{\alpha-1} &= r + \delta \\ K' &= \left(\frac{z'\alpha}{r+\delta}\right)^{\frac{1}{1-\alpha}} \end{aligned}$$

and we can also determine investment by capital accumulation process:

$$I_D = K' - (1-\delta)K = \left(\frac{z'\alpha}{r+\delta}\right)^{\frac{1}{1-\alpha}} - (1-\delta)K$$

Check: Investment Demand Assumption

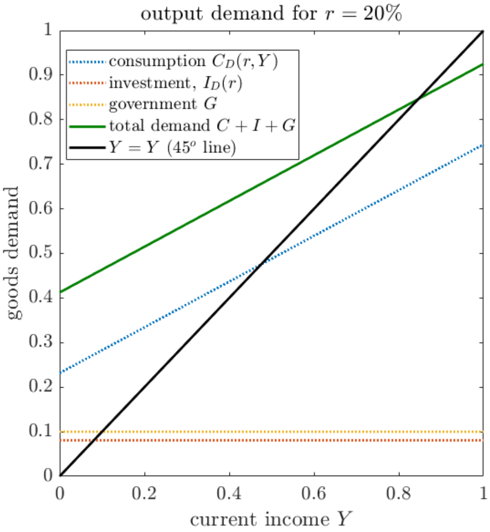


$$I_D = \left(\frac{z'\alpha}{r+\delta}\right)^{\frac{1}{1-\alpha}} - (1-\delta)K$$

Recall assumptions from Lecture 15:

- $I_D(r) \downarrow$ in r (✓)
- $I_D(r)$ shifts in when $K \uparrow$: yellow v.s. blue
- $I_D(r)$ shifts out when $z' \uparrow$: red v.s. blue

Constructing the Output Demand Curve



Aggregate all three components:

- investment (red) and government (yellow) are horizontal
- consumption (blue) increase in income with slope $\approx \frac{1}{1+\beta}$
- total output demand (green) gain the slope from consumption, and is the sum of all three

Constructing the Output Demand Curve (Cont.)

$$r \uparrow \Rightarrow I_D(r) \downarrow \Rightarrow \text{total demand} \downarrow$$