

Lecture 7

Representative Firm

Hui-Jun Chen

The Ohio State University

January 13, 2023

Overview: Lecture 4 - 7

Provide **micro-foundation** for the **macro implication** (**Lucas critique**)

■ Representative Consumer:

- Lecture 4: **preference**, **constraints**
- Lecture 5: **optimization**, **application**
- Lecture 6: Numerical Examples

■ Representative Firm:

- Lecture 7: **production**, **optimization**, **application**

Production Function

Production function describes the technology possibility for converting inputs into outputs.

Representative firm produces output Y with production function

$$Y = zF(K, N^d) \quad (1)$$

- Y : output (consumption goods)
- z : total factor productivity (TFP) (productivity for the economy)
- K : capital (fixed for now, \therefore 1-period model)
- N^d : labor demand (chose by firm, d represents demand)

Properties of Production Function: Marginal Product

- **Marginal product:** how much $Y \uparrow$ by one unit of $K \uparrow$ or $N^d \uparrow$.
 - **Marginal product of capital (MPK):** $zD_K F(K, N^d)$
 - **Marginal product of labor (MPN):** $zD_N F(K, N^d)$
- Marginal product is **positive** and **diminishing**:
 - **Positive MP:** $Y \uparrow$ if either $K \uparrow$ or $N^d \uparrow$
 - more inputs result in more output
 - **Diminishing MP:** MPK \downarrow as $K \uparrow$; MPN \downarrow as $N^d \uparrow$
 - the **rate/speed** of output increasing is decreasing
- **Increasing marginal cross-products:**
 - e.g. MPK \uparrow as $N \uparrow$; MPN \uparrow as $K \uparrow$

Properties of Production Function: Return to Scale

- **Return to scale:** how Y will change when both K and N increase
- **Constant return to scale (CRS):** $xzF(K, N^d) = zF(xK, xN^d)$
 - small firms are **as efficient as** large firms
- **Increasing return to scale (IRS):** $xzF(K, N^d) > zF(xK, xN^d)$
 - small firms are **less efficient than** large firms
- **Decreasing return to scale (DRS):** $xzF(K, N^d) < zF(xK, xN^d)$
 - small firms are **more efficient than** large firms

Example: Cobb-Douglas Production Function

- **Cobb-Douglas:** $zF(K, N) = zK^\alpha N^{1-\alpha}$, α is the share of capital contribution to output

- **Positive MPK & MPN:**

- $MPK = D_K zF(K, N) = z\alpha K^{\alpha-1} N^{1-\alpha} = z\alpha \left(\frac{K}{N}\right)^{\alpha-1} > 0$
- $MPN = D_N zF(K, N) = z(1-\alpha)K^\alpha N^{-\alpha} = z(1-\alpha) \left(\frac{K}{N}\right)^\alpha > 0$

- **Diminishing MP:**

- For K , $D_K (z\alpha K^{\alpha-1} N^{1-\alpha}) = z\alpha(\alpha-1)K^{\alpha-2} N^{1-\alpha} < 0$
- For N , $D_N (z(1-\alpha)K^\alpha N^{-\alpha}) = z(1-\alpha)(-\alpha)K^\alpha N^{-\alpha-1} < 0$

- **Increasing marginal cross-product:**

- For MPK, $D_N (z\alpha K^{\alpha-1} N^{1-\alpha}) = z\alpha(1-\alpha)K^{\alpha-1} N^{-\alpha} > 0$
- For MPN, $D_K (z(1-\alpha)K^\alpha N^{-\alpha}) = z(1-\alpha)\alpha K^{\alpha-1} N^{-\alpha} > 0$

Example: Cobb-Douglas and Return to Scale

Let's assume that Cobb-Douglas production is $zF(K, N) = zK^\alpha N^\beta$

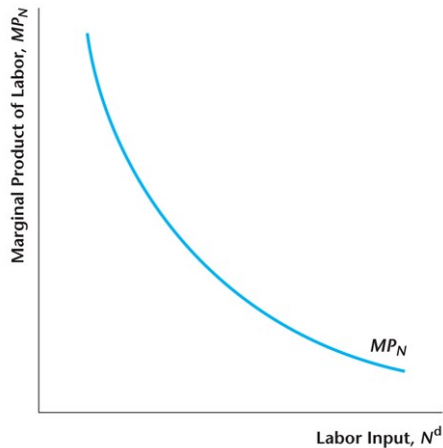
So if both inputs are increasing by twice, then

$$\begin{aligned} zF(2K, 2N) &= z(2K)^\alpha (2N)^\beta = 2^\alpha \times 2^\beta zK^\alpha N^\beta \\ &= 2^{\alpha+\beta} zK^\alpha N^\beta = 2^{\alpha+\beta} Y \end{aligned}$$

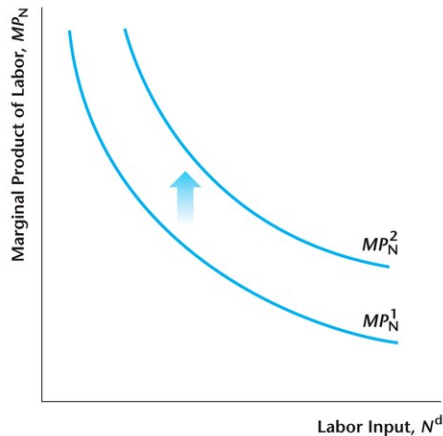
- 1 If $\alpha + \beta = 1$, then $zF(2K, 2N) = 2Y$, constant return to scale
- 2 If $\alpha + \beta < 1$, then $zF(2K, 2N) = 2^{\alpha+\beta} Y < 2Y$, decreasing return to scale
- 3 If $\alpha + \beta > 1$, then $zF(2K, 2N) = 2^{\alpha+\beta} Y > 2Y$, increasing return to scale

Visualization

Diminishing Marginal Product

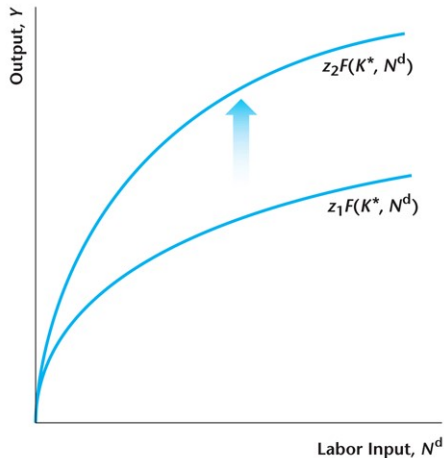


Increasing Marginal Cross-product

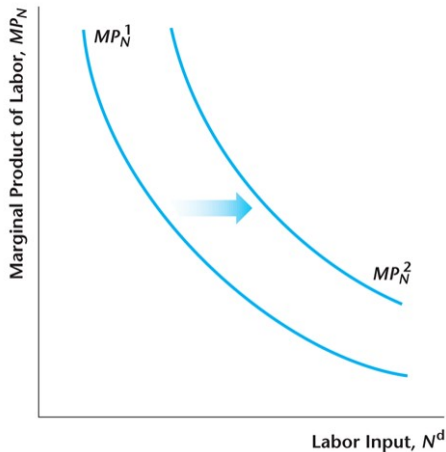


Visualization: Changes in TFP

TFP shifts up the Production Function

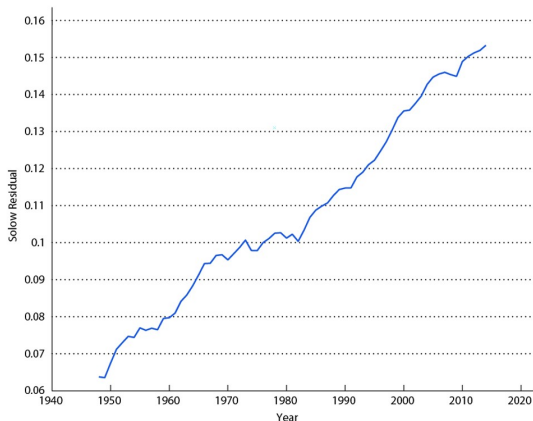


TFP increases MPN



TFP in Data

Solow Residual for US



We cannot see TFP, **how to measure it?**

- Assume Cobb-Douglas production function:
$$Y = zK^{\alpha}N^{1-\alpha}$$
- By data, $K/Y = 0.3 \Rightarrow \alpha = 0.3$
- Can observe K , Y , N in data:

$$z = \frac{Y}{K^{0.3}N^{0.7}}$$

Firm's Problem: Profit Maximization

Firm maximizes profit (π), which is the revenue minus the wage bill:

$$\pi = \max_{N^d} zF(K, N^d) - wN^d \quad (2)$$

■ **Constraints:** $N^d > 0$, relatively simple!

$$\text{Cobb-Douglas: } zF(K, N^d) = zK^\alpha (N^d)^{1-\alpha} \quad (3)$$

$$\text{FOC: } w = z(1 - \alpha)K^\alpha (N^d)^{-\alpha} \quad (4)$$

$$(N^d)^\alpha = \frac{z(1 - \alpha)K^\alpha}{w} \quad (5)$$

$$\text{Labor demand: } N^d = \left(\frac{z(1 - \alpha)K^\alpha}{w} \right)^{\frac{1}{\alpha}} = \left(\frac{z(1 - \alpha)}{w} \right)^{\frac{1}{\alpha}} K \quad (6)$$

As $w \uparrow$, $N^d \downarrow \Rightarrow$ **downward-sloping** demand.

Experiment 1: Payroll Tax

Payroll tax: suppose firms have to pay additional per-unit tax $t > 0$ on the wage bill, then

$$\text{Firm Problem: } \max_{N^d} zK^\alpha (N^d)^{1-\alpha} - w(1+t)N^d \quad (7)$$

$$\text{FOC: } w(1+t) = z(1-\alpha)K^\alpha (N^d)^{-\alpha} \quad (8)$$

$$N^d = K \left(\frac{z(1-\alpha)}{w(1+t)} \right)^{\frac{1}{\alpha}} \quad (9)$$

- **wage** \uparrow : $w \uparrow \Rightarrow N^d \downarrow$ (same as benchmark)
- **tax** \uparrow : $t \uparrow \Rightarrow N^d \downarrow$
- **capital** \uparrow : $K \uparrow \Rightarrow N^d \uparrow \Rightarrow$ what if firm can also choose K ?

Experiment 2: Choice of Capital

Capital rent: suppose that firm can choose capital level but have to pay r of per-unit rent.

$$\text{Firm Problem: } \max_{\mathbf{K}, N^d} zK^\alpha (N^d)^{1-\alpha} - rK - wN^d \quad (10)$$

$$\text{FOC on N: } w = z(1 - \alpha)K^\alpha (N^d)^{-\alpha} \quad (11)$$

$$\text{FOC on K: } r = z\alpha K^{\alpha-1} (N^d)^{1-\alpha} \quad (12)$$

$$\text{Divide (11) with (12): } \frac{w}{r} = \frac{(1 - \alpha)}{\alpha} \frac{K}{N^d} \quad (13)$$

$$\text{Capital-Labor ratio: } \frac{K}{N^d} = \frac{w}{r} \frac{\alpha}{1 - \alpha} \quad (14)$$

When firm can choose K , they choose both capital and labor such that (14) satisfied!