Lecture 6
Numerical Example

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September 9, 2023

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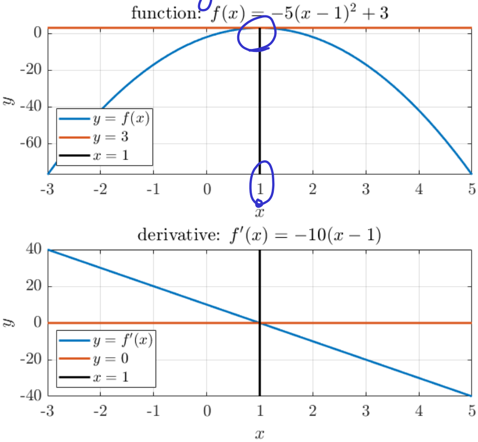
Overview: Lecture 4 - 7

Provide micro-foundation for the macro implication (Lucas critique)

- **Representative Consumer**:
  - Lecture 4: preference, constraints
  - Lecture 5: optimization, application
  - Lecture 6: Numerical Examples
- **Representative Firm**:
  - Lecture 7: production, optimization, application

1 Variable

$y = f(x) = 3$



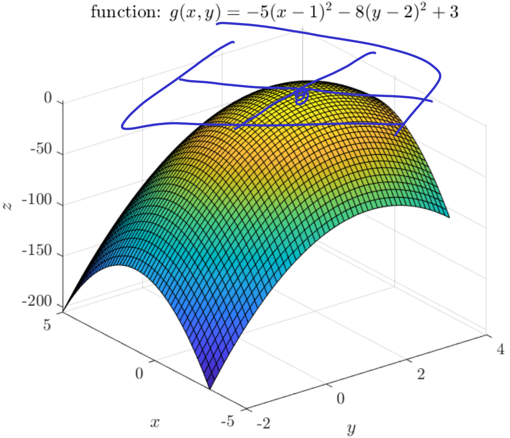
In general, want to solve $\max_x f(x)$

- find "peak" of function
- at peak, slope is 0
- **First order condition** (FOC) is when the 1st order derivative, i.e., the slope is 0:

$$f'(x^*) = 0,$$

where x^* is the peak

2 Variables



In general, want to solve $\max_{x,y} g(x,y)$

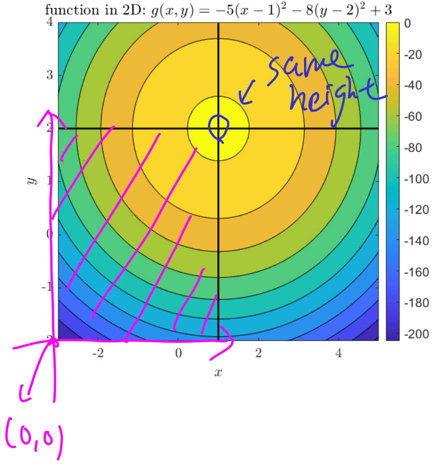
- at peak, slope is 0 in both directions, i.e., the FOCs are

$$\begin{cases} D_x g(x^*,y^*) = 0 \\ D_y g(x^*,y^*) = 0 \end{cases},$$

where the bundle (x^*,y^*) is the peak

- Hard for my brain to process 3-D graph...resolution?

Visualizing 3-D function on 2-D plane



- **Contours**:"standing" at the peak and look down
    - e.g. map on Alltrails
- Fix the level of $g = -20$ (a horizontal slice of 3-D figure)
- Find x and y such that

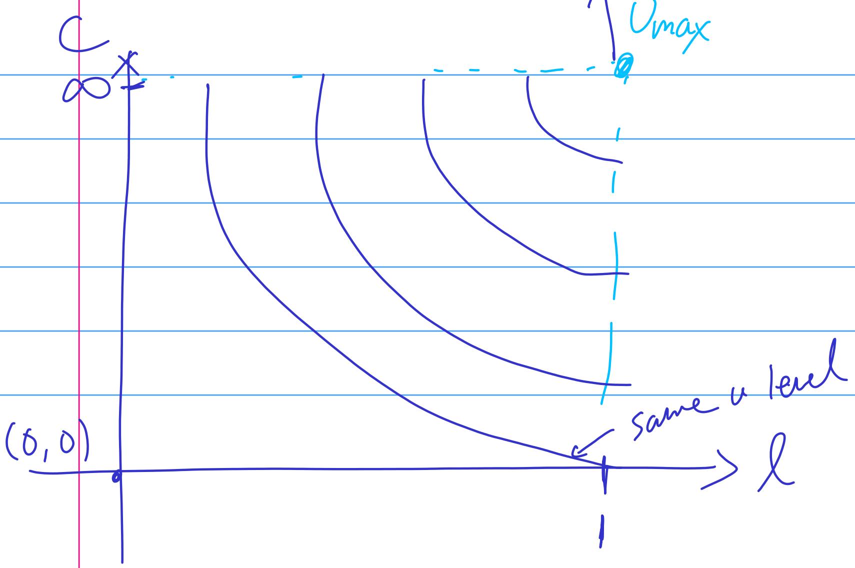
$$-20 = -5(x-1)^2 - 8(y-2)^2 + 3$$

- repeat for any value of g
- **Exactly where indifference curve came from!**

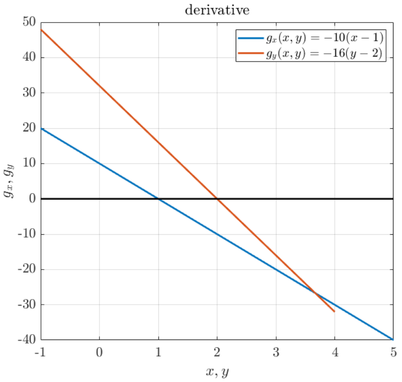
$U(l, c)$

max: $l \to 1$, $c \to \infty$

peak of the mountain



Solving 2 Variables Optimizations

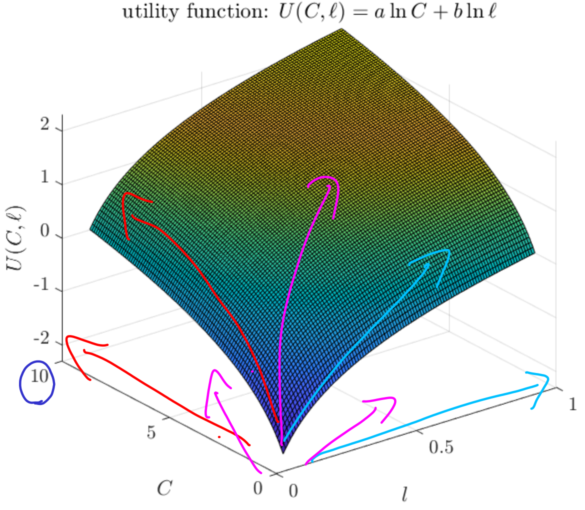


$$\begin{cases} D_x g(x^*, y^*) = -10(x-1) = 0 \\ D_y g(x^*, y^*) = -16(y-2) = 0 \end{cases}$$

- Intersection between 0 and line is the solution.
- For other functional form, $D_x g(x, y)$ can depend on y, and $D_y g(x, y)$ can depend on x
- May have constraints on the relationship between x and y

Utility Function in 3-D

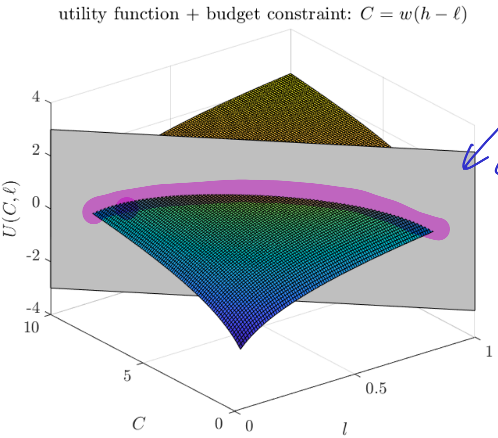
Here $a = b = 1$, where is the peak?



- Seems like to be at $C^* = 10$ and $l^* = 1$
- Recall monotonicity: more is better!
- What stops the consumer from choose $(C, l) = (10, 1)$? $(\infty, 1)$

Utility Function + Budget Set in 3-D

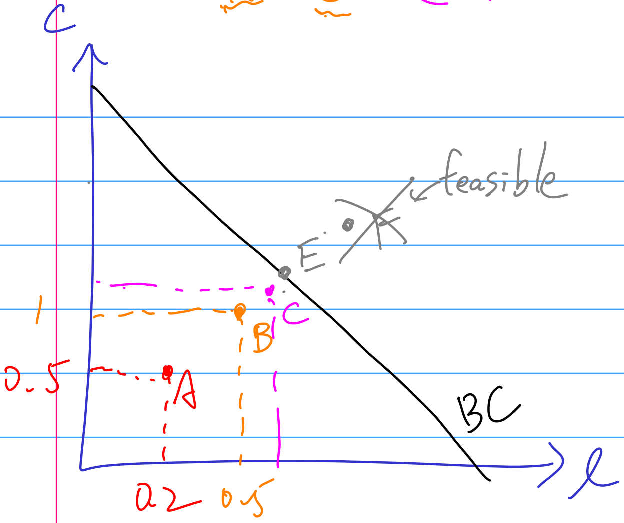
Let $w = 10$ and $h = 1$, and the gray surface represents the border of the budget set.



budget constraint

- Consumers have to choose (C, l) bundles inside the budget set
  - $(C, l) = (10, 1)$ is outside of the budget set $\Rightarrow$ not feasible
- **Binding budget constraint**: candidates for optimal are points in gray
- Which one?

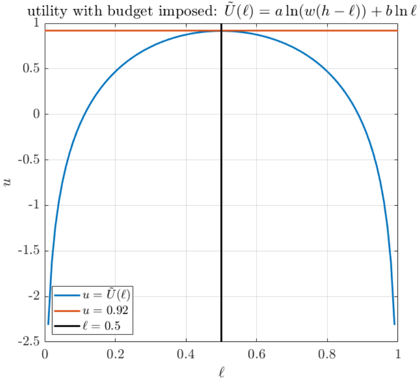
$A < B < C < \cdots\cdots < E$



Collapsing 3-D Problem into 2-D: Slice

How? Binding budget constraint!

pink



$\to C = 10(1-\ell)$

Binding: $C = w(h-l)$

$U(C,l) = a\ln C + b\ln l$

Plug in: $\tilde{U}(l) = a\ln(w(h-l)) + b\ln l$

FOC: $D_l\tilde{U}(l) = 0$

$$a\frac{-w}{w(h-l)} + b\frac{1}{l} = 0$$

$$\frac{a}{h-l} = \frac{b}{l}$$

$$l = \frac{b}{a+b}h$$

$l = 0.5$, let $C = 5$, $u = 0.91629...$

$$\max_{c,\ell}\; U(\underset{(w)}{c}, \underset{(h)}{\ell}) = \ln c + \ln \ell$$

$$\text{s.t.}\quad c = 10(1-\ell)$$

$$\Rightarrow \max_{\ell}\; \ln\underbrace{\left[10(1-\ell)\right]}_{c} + \ln \ell$$

FOC.

$$[\ell]:\; \underbrace{\frac{\partial U}{\partial c}\frac{\partial c}{\partial \ell}}_{\text{indirect effect}} + \underbrace{\frac{\partial U}{\partial \ell}}_{\text{direct effect}} = 0$$

$$\frac{\partial U}{\partial c} = \frac{1}{c}\;;\quad \frac{\partial c}{\partial \ell} = \frac{\partial\big(\overbrace{10(1-\ell)}^{c}\big)}{\partial \ell}$$

$$= \frac{\partial\left[\cancel{10} - 10\ell\right]}{\partial \ell} = -10$$

$$\frac{\partial U}{\partial c}\frac{\partial c}{\partial l} = \frac{-10}{c} = \frac{-10}{10(1-l)}$$

$$\frac{\partial U}{\partial l} = \frac{1}{l}$$

$$-\frac{10}{10(1-l)} + \frac{1}{l} = 0$$

$$\frac{1}{l} = \frac{10}{10(1-l)} \Rightarrow 10l = 10(1-l)$$

$$= 10 - 10l$$

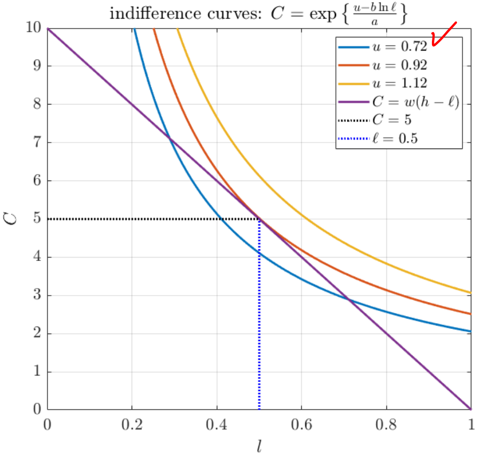
$$\Rightarrow 20l = 10 \Rightarrow l = \frac{1}{2} = 0.5$$

$$c = 10(1-l) = 10 \cdot 0.5 = 5$$

Collapsing 3-D Problem into 2-D: Contours

Recall contours, for any utility level u, $u = a \ln C + b \ln l \Rightarrow C = e^{\frac{u - b \ln l}{a}}$

0.72



- What is the highest u feasible given budget constraint?

- Or push up IC (increase u) such that IC is tangent to budget line:

bind BC

$$-MRS_{l,C} = -w \quad 10$$

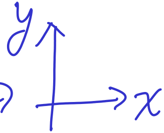
$$\frac{bC}{al} = \frac{bw(h-l)}{al} = w$$

$$l = \frac{b}{a+b}h$$

$$e^{\ln x} = x$$

$$u = 0.72$$

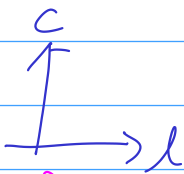
$$\Rightarrow 0.72 = a \ln C + b \ln l$$



$$y = \boxed{}\, x^2 + \boxed{}\, x + \boxed{}$$

$$a \ln C = 0.72 - b \ln l$$

$$\ln C = \frac{0.72 - b \ln l}{a}$$



$$\cancel{e}^{\cancel{\ln} C} = e^{\frac{0.72 - b \ln l}{a}}$$

indifference curve

$$C = e^{\frac{0.72 - b \ln l}{a}}$$

$$C = g(l)$$

2-D versions: Pros and Cons

Both 2-D formulations are delivering the same answer.

1. Slice: 1 variable optimization problem, x-axis: l, y-axis: u
   - **Straightforward**: operate on (l, u) plane, good for problem solving
   - **General**: can collapse higher dimension problem
   - **Cons**: lack of trade off between C and $l \Rightarrow$ economcis intuition
2. Contours: 2 variable optimization problem, x-axis: l, y-axis: C
   - **Intuitive**: direct trade off between C and l through $MRS_{l,C}$
   - **Cons:** harder to solve and to generalize to higher dimension

Review: Models from Last Lecture

1. Utility function: $U(C, l) = a \ln C + b \ln l$.
2. Budget constraint: $C \leq \underbrace{w(h-l)}_{\text{labor income}} + \underbrace{\pi}_{\text{dividend}} - T$ lump-sum tax
3. After-tax dividend: $x = \pi - T$
   "non-labor" income ⇒ not related to C-l trade off.
4. wage rate: w

- **Benchmark**: in section Consumer Example
- **Experiment 1**: increase in after-tax dividend: $x_1 > x_0$
- **Experiment 2**: increase wage rate: $w_2 > w_0$

Solve for Benchmark Case

$a \ln C \Rightarrow \frac{a}{C}$

- **Marginal utilities**: $D_C U(C,l) = \frac{a}{C}$; $D_l U(C,l) = \frac{b}{l}$.

labor non-labor

- **Binding budget constraint**: $\boxed{C = w(h-l) + \pi - T}$

$MRS_{l,C}$

- **Optimality**: $MRS_{l,C} = w \Rightarrow \frac{D_l U(C,l)}{D_C U(C,l)} = w \Rightarrow \boxed{w = \frac{bC}{al}}$

Plug binding budget constraints into optimality and solve for l:

$$w = \frac{b\overbrace{(w(h-l)+x)}^{C}}{al} \tag{1}$$

$$\Rightarrow \quad wal = b(w(h-l)+x) \tag{2}$$

$$\Rightarrow \quad wal = bwh - bwl + bx \tag{3}$$

$$\Rightarrow \quad \boxed{(a+b)w}l = bwh + bx \tag{4}$$

$$\Rightarrow \quad l = \frac{b}{a+b}\left(h + \frac{x}{w}\right) \tag{5}$$

Solve for Benchmark Case (Cont.)

$w\cdot(h-l)$

Solve for C, we get

mkt value of leisure

wh: mkt value of your time

$$l = \frac{b}{a+b}\left(h + \frac{x}{w}\right) \Rightarrow wl = \boxed{\frac{b}{a+b}(wh+x)} \quad (6)$$

$$C = w(h-l) + \pi - T = w(h-l) + x \quad (7)$$

$$\Rightarrow \quad C = w\left[h - \frac{b}{a+b}\left(h + \frac{x}{w}\right)\right] + x \quad (8)$$

$$\Rightarrow \quad C = wh - \frac{b}{a+b}(wh + x) + x \quad (9)$$

$$\left(1 - \frac{b}{a+b}\right)wh$$

Denote $s = \frac{b}{a+b}$

$$\Rightarrow \quad C = \boxed{\frac{a}{a+b}}wh + \frac{a}{a+b}x \quad (10)$$

$$\Rightarrow \quad C = \frac{a}{a+b}(wh+x) \quad (11)$$

Property for this utility function: consumer "split" fixed share of "wealth": $wl = s(wh+x)$, and $C = (1-s)(wh+x)$.

Solve for Experiment 1: $x \uparrow$ ⇒ income effect ⇒ $C \uparrow$, $l \uparrow$

(l_0, C_0, x_0): benchmark value; (l_1, C_1, x_1): experiment 1 value.

With pure income effect, no change in real wage: $w_1 = w_0 = w$

The difference between experiment 1 and benchmark case is

$$l_1 - l_0 = \frac{b}{a+b}\left(h + \frac{x_1}{w}\right) - \frac{b}{a+b}\left(h + \frac{x_0}{w}\right) \tag{12}$$

$$= \frac{b}{a+b}\left(\frac{x_1}{w} - \frac{x_0}{w}\right) \tag{13}$$

$$= \boxed{\frac{b}{(a+b)w}}(x_1 - x_0) \geq 0 \tag{14}$$

$x_1 > x_2$ ⇒ $l_1 > l_0$

$$C_1 - C_0 = \frac{a}{a+b}(wh + x_1) - \frac{a}{a+b}(wh + x_0) \tag{15}$$

$$= \frac{a}{a+b}(x_1 - x_0) > 0 \tag{16}$$

$x_1 > x_0$ ⇒ $C_1 > C_0$

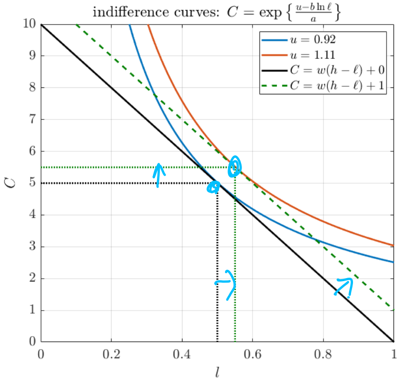
Namely, with pure income effect, both leisure and consumption increases.

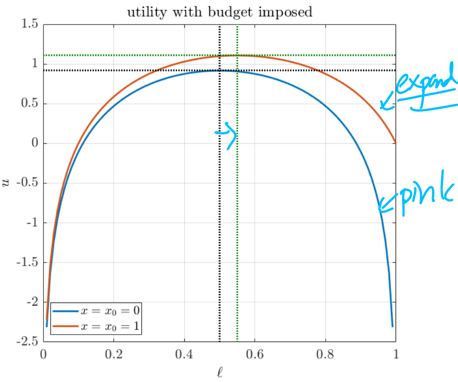
Solve for Experiment 1: Graphical Intuition

$w_1 = w_0 = 10; x_1 = 1 > x_0 = 0$

Both leisure and consumption are higher

Budget constraint is "eased"





Solve for Experiment 2: $w \uparrow$

$w \uparrow$ → C ☺ ; → Firm ☹ → $\pi \downarrow \Rightarrow x = \pi - T \downarrow$ (General eq)

(l_0, C_0, x_0): benchmark value; (l_2, C_2, x_2): experiment 2 value.

With both income and substitution effects, analysis is complicated:

→ $x_2 \neq x_0$

$W_2 > W_0$

$$l_2 - l_0 = \frac{b}{a+b}\left(h + \frac{x_2}{w_2}\right) - \frac{b}{a+b}\left(h + \frac{x_0}{w_0}\right) \tag{17}$$

$$= \frac{b}{a+b}\boxed{\left(\frac{x_2}{w_2} - \frac{x_0}{w_0}\right)} \gtreqless 0 \tag{18}$$

$$C_2 - C_0 = \frac{a}{a+b}(w_2 h + x_2) - \frac{a}{a+b}(w_0 h + x_0) \tag{19}$$

$$= \frac{a}{a+b}\left(h(w_2 - w_0) + (x_2 - x_0)\right) > 0 \tag{20}$$

Although the consumption is certainly increasing, the change in leisure is uncertain ⇒ need numerical solution (put numbers in).

$$\max_{N^D} \overbrace{f(N^D)}^{\text{production fcn}} - \overbrace{wN^D}^{\text{wage bill}}$$

$$\underbrace{f(N^D) - w\uparrow N^D}_{\pi \downarrow}$$

Solve for Experiment 2: $w \uparrow$ (Cont.)

Let $w_2 = 15 > w_0 = 10$; $x_2 = x_0 = 0$.

$$l_2 - l_0 = \frac{b}{a+b}\left(\frac{x_2}{w_2} - \frac{x_0}{w_0}\right) = \frac{b}{a+b}\left(\frac{0}{15} - \frac{0}{10}\right) = 0 \tag{21}$$

Leisure remain the same. v.s. 2

Compare with experiment 1, $w_2 = 15 > w_1 = 10$; $x_2 = 0 < x_1 = 1$; $h = 1$:

$$l_2 - l_1 = \frac{b}{a+b}\left(\frac{x_2}{w_2} - \frac{x_1}{w_1}\right) = \frac{b}{a+b}\left(\frac{0}{15} - \frac{1}{10}\right) < 0 \tag{22}$$

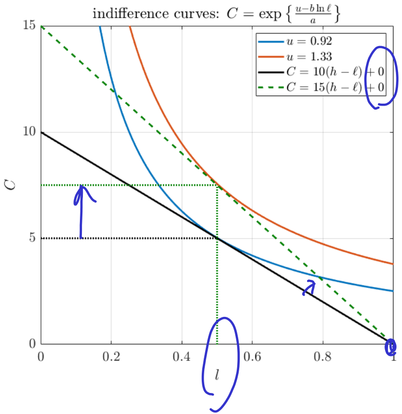
$$C_2 - C_1 = \frac{a}{a+b}\left(h(w_2 - w_1) + (x_2 - x_1)\right) \tag{23}$$

$$= \frac{a}{a+b}(1(15 - 10) + (0 - 1)) > 0 \tag{24}$$

Experiment 2 v.s. Benchmark: Graphical Intuition

Equivalent valuation

Total Effect



Income and Substitution Effect