Debt Financing, Used Capital Market and Capital Reallocation

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March 31, 2023

Introduction

Motivation and Question

How much do financial frictions and endogenous capital irreversibility explain the slow recovery of the Great Recession?

- My conjecture: frictions disproportionally impact small firms
- Young firm holding more old capital (Ma, Murfin and Pratt (2022))
- High demand for cheap old capital push up price (Lanteri and Rampini (2023))
- Willing to exchange future cost for current growth (Eisfeldt and Rampini (2007)

What I do

- This paper: small firms invest, expose them to volatile used K price.
  - endogenous tightening of collateral constraints harms small firms more.
- What I do: collateral constraint + Lanteri (2018) (RBC & used K)
- Contribution: evaluate the joint effect of both frictions
  - Khan and Thomas (2013): predicts countercyclical capital reallocation yet the data is procyclical.
  - Lanteri (2018): explains only 30% of the cyclical volatility of total capital reallocation in data.

Empirical Evidence

- Matched by Lanteri (2018)
  - $> 20\%$ share of used capital in four industries in US. table
  - Price of used investment is $2 \sim 4$ times volatile than new one. figure
- My paper is going to match:
  - Firms holding $10 \sim 30\%$ used capital based on firm size/age. table
    - need some modification.
  - Small firms are buyers in used capital market. table
  - Debt financing is significantly and positively correlated to capital reallocation. table

Model

Overview

I consider a heterogeneous firm model with real and financial friction:

- **Used investment market**: trade price q is determined by the supply (downward-adjust) and the demand (upward-adjust) Def
- **Households:** own firms $\Rightarrow$ firms discount as HH. HH Problem
- **Firms:** idio.: ϵ_i; TFP: z_f; exogenous exit prob π_d.
  - Upward-adjusting firms: purchase effective capital at cost Q.
    - LoM: $CES(i_{used}, i_{new}) \Longrightarrow K$
  - Downward-adjusting firms: sells used investment goods at price q.
  - Collateral constraint: $b' \leq q\zeta k$.

Technology CES cost minimization problem

Following Lanteri (2018), used & new inv. are imperfect substitution:

- capital process for upward-adjusting firms:

$$k' = (1-\delta)k + I(i_{new}, i_{used})$$
$$I(i_{new}, i_{used}) = \left[\eta^{\frac{1}{s}} (i_{new})^{\frac{s-1}{s}} + (1-\eta)^{\frac{1}{s}} (i_{used})^{\frac{s-1}{s}}\right]^{\frac{s}{s-1}}, \quad (1)$$

$\eta \in [0, 1]$: average ratio; $s > 0$: elasticity of substitution Estimation.

- Agg. price index $Q = \left[\eta + (1-\eta)(q+\gamma)^{1-s}\right]^{\frac{1}{1-s}}$, $q + \gamma < 1$, $Q < 1$.
- $\frac{i_{used}}{i_{new}} = \frac{1-\eta}{\eta}(q+\gamma)^{-s}$.
  - needs modification to match share of used capital ↓ with firm size.
- capital process for downward-adjusting firms: $k' = (1-\delta)k - d$.

Production and Value Function

- Following Khan and Thomas (2013),

$$v_0(k,b,\varepsilon;z_f,\mu) = \begin{array}{l} \pi_d \max_n [x^d(k,b,\varepsilon;z_f)] \\ \qquad + (1-\pi_d) v(k,b,\varepsilon;z_f,\mu) \end{array}, \tag{2}$$

where $x^d(\cdot)$ is the cash-on-hand for downward-adjusting firms. Def

- Conditional on survival, firm chooses upward- or downward-adjusting:

$$v(k,b,\varepsilon;z_f,\mu) = \max\{v^u(k,b,\varepsilon;z_f,\mu), v^d(k,b,\varepsilon;z_f,\mu)\}. \tag{3}$$

Upward-adjusting Firm

$$v^u(k,b,\varepsilon;z_f;\mu) = \max_{k',b',D} D + \sum_{g=1}^{N_z} \pi^z_{fg} d_g(z_f;\mu) \sum_{j=1}^{N_\varepsilon} \pi^\varepsilon_{ij} v_0(k',b',\varepsilon'_j;z'_g;\mu'), \tag{4}$$

subject to

$$0 \leq D \leq x^u(k,b,\varepsilon_i;z_f) + q_b b' - Qk', \qquad \text{(Budget: Up)}$$

$$x^u(k,b,\varepsilon_i;z_f) = z_f \epsilon_i F(k,n) - w(z_f,\mu)n - b + Q(1-\delta)k \qquad \text{(Cash: Up)}$$

$$b' \leq q\zeta k, \qquad \text{(Collateral)}$$

$$k' \geq (1-\delta)k, \qquad \text{(K range)}$$

$$\mu' = \Gamma(z_f;\mu), \qquad \text{(Distribution)}$$

q_b: bond price; $d_g(z_f,\mu)$: SDF; ζ: efficiency of financial sector.

Downward-adjusting Firm Back

$$v^d(k,b,\varepsilon_i;z_f,\mu) = \max_{k',b',D} D + \sum_{g=1}^{N_z} \pi^z_{fg} d_g(z_f;\mu) \sum_{j=1}^{N_\varepsilon} \pi^\varepsilon_{ij} v_0(k',b',\varepsilon'_j;z'_g,\mu'), \tag{5}$$

subject to

$$0 \le D \le x^d(k,b,\varepsilon;z_f) + q_b b' - qk', \qquad \text{(Budget: Down)}$$

$$x^d(k,b,\varepsilon;z_f) = z_f \epsilon_i F(k,n) - w(z_f,\mu)n - b + q(1-\delta)k \qquad \text{(Cash: Down)}$$

$$b' \le q\zeta k, \qquad \text{(Collateral)}$$

$$k' \le (1-\delta)k, \qquad \text{(K range)}$$

$$\mu' = \Gamma(z_f;\mu), \qquad \text{(Distribution)}$$

Definition of *recursive equilibrium*, Rewrite (2), (3), (4), (5) in terms of $p(z_f;\mu)$

Steady State Calibration

Frequency and Functional Form

- Model frequency: annual
- HH utility function: $u(c, l) = \log c + \varphi l$
- Production function: $z\varepsilon F(k, n) = z\varepsilon k^{\alpha} n^{\nu}$
- Initial capital for normal entrant: $k_0 = \chi \int k\widetilde{\mu}(d[k \times b \times \varepsilon])$
- Initial bond holding for normal entrant: $b_0 = 0$
- Idiosyncratic productivity shock: $\log \varepsilon' = \rho_{\varepsilon} \log \varepsilon + \eta'_{\varepsilon}$
  - 30-state Markov chain discretized from Tauchen algorithm

Aggregate Data

parameter	target		model	KT13 Rep
$\beta = 0.96$	real rate	$= 0.04$	0.04	0.04
$\nu = 0.6$	labor share	$= 0.6$	0.600	0.599
$\delta = 0.065$	investment/capital	$= 0.069$	0.069	0.067
$\alpha = 0.27$	capital/output	$= 2.39$	2.343	2.322
$\varphi = 2.15$	hours worked	$= 0.33$	0.333	0.331
$\pi_d = 0.1$	exit & entry rate of firms		0.10	0.10
$\chi = 0.1$	new / typical firm size		0.10	0.10
$\omega_e = 0.291$				
$\alpha_e = 0.140$	debt-to-capital ratio	$= 0.37$	0.3739	0.358
$\zeta = 1.38$				

Firm-Level Data KT13

LRD Cooper and Haltiwanger (2006)		model	parameters
$\sigma(i/k)$	$= 0.337$	0.3938	$\gamma = 0.018$
$\rho(i/k)$	$= 0.058$	0.0607	$\rho_{\eta_\varepsilon} = 0.681$
lumpy investment ($> 20\%$)	$= 0.186$	0.2234	$\sigma_{\eta_\varepsilon} = 0.115$
Compustat Eisfeldt and Rampini (2006)			
reallocation / investment	$= 0.2389$	0.2119	$\eta = 0.80$
			$s = 5.0$
Untargeted moments (LRD CH(2006))			
mean(i/k)	$= 0.122$	0.1236	
inaction freq ($abs(i/k) < 1\%$)	$= 0.081$	0.4720	
disinvestment freq ($i/k < -1\%$)	$= 0.104$	0.1940	
lumpy disinvestment ($i/k < -20\%$)	$= 0.018$	0.0989	

[1] reallocation: SPPE & Acquisition

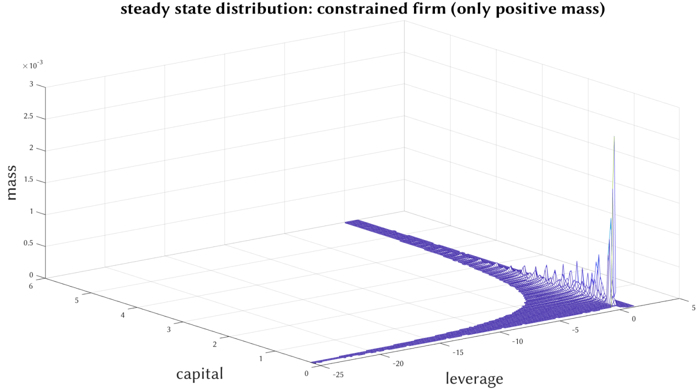
[2] investment: SPPE & new investment & Acquisition

Steady State Results

Steady State Aggregates

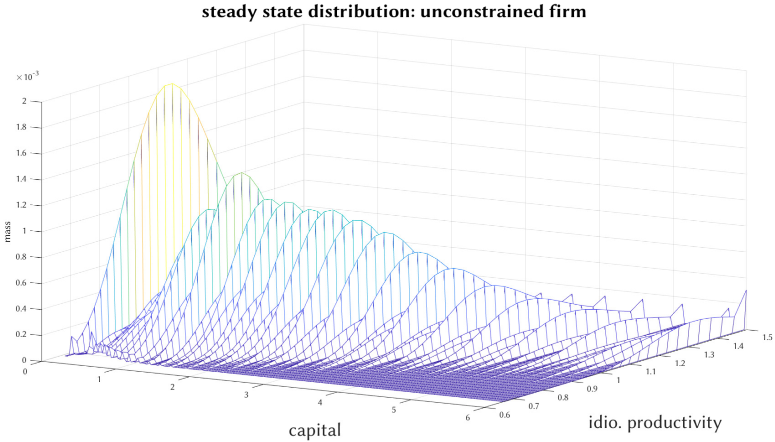
Aggregates	description	model	KT13 Rep
q	used investment price	0.9580	0.9540
Q	effective capital price	0.9949	1.0000
q/Q	capital reversibility	0.9628	0.9540
K	aggregate capital	1.3712	1.3429
$B > 0$	aggregate debt	0.5128	0.4808
Y	aggregate output	0.5850	0.5782
$\hat{z}$	measured TFP	1.0381	1.0353

Steady State distribution: median productivity KT13

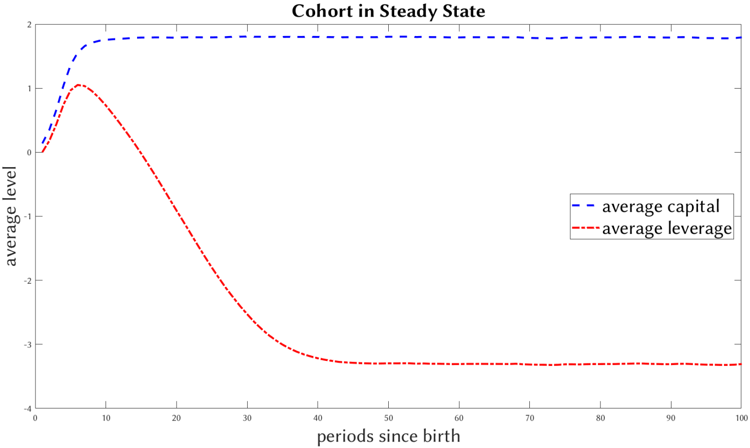


- new firm k: 0.1371
- avg constrained k: 1.2449
- avg unconstrained k: 1.6263
- # constrained: 66%
- firms w/ *currently* binding collateral: 13.5%

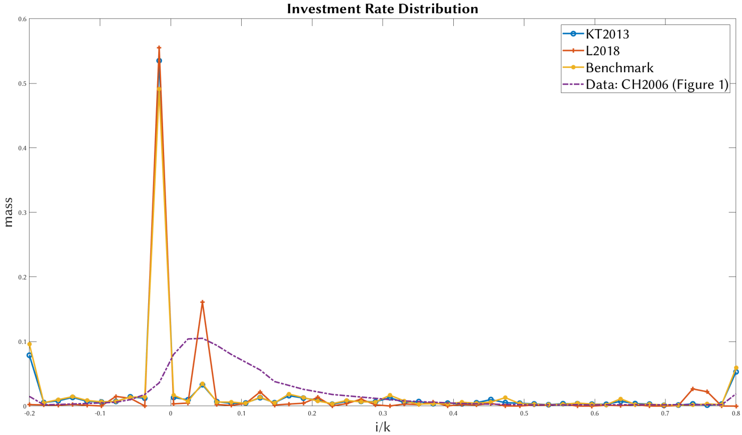
Steady State distribution for unconstrained firm KT13



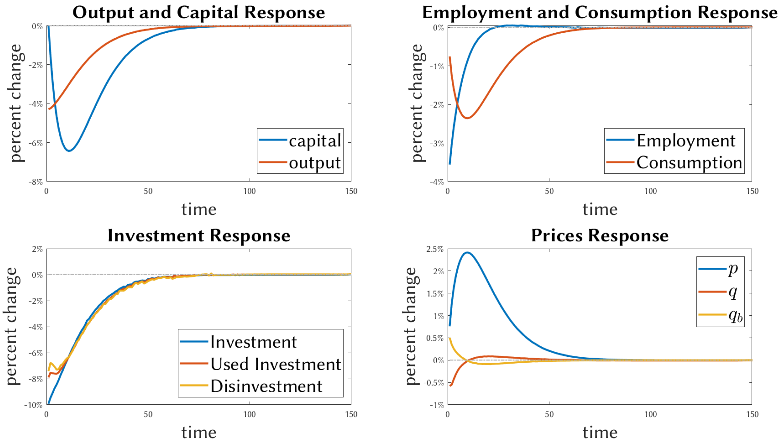
Life Cycle Aspect: Investment & Saving KT13



Investment Rate Distribution



Preliminary Result on Perfect Foresight



Response to 2.18% decrease in productivity shock with persistence $\rho_z = 0.909$, simulated for 150 periods

Brief Discussion

Compared with KT13, with endogenous capital irreversibility, firms are

- enjoying higher aggregates
- accumulating more capital
- willing to be more risky in investing capital even constrained
- having more incentive to disinvest in used investment market

The resulting capital irreversibility, $1 - q/Q$, is smaller than KT13, and way smaller than Lanteri (2018) (0.933).

In PF, capital and employment drops tremendously, reflecting effect of endogenous tightening of collateral constraints on small firms

Next Steps

1. Compute perfect foresight to examine the impulse response
    1. disentangle two channels: hold collateral constraint as $b' \leq q_{SS}\zeta k$
    2. credit shock: effect of ζ drop on q
2. Having a "real" old capital? What's the diff between new and old?
3. Introduce agg. uncertainty, response from used investment mkt?
    - credit shock v.s. TFP shock on this benefit?
4. Evaluate the effect of endogenous collateral constraint
    - Size matters? (Un)constrained matters?

Appendix

References I

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Empirical Evidence

Table: Lanteri (2018)

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TABLE 1—SHARES OF ASSET TYPES IN US EQUIPMENT STOCK

Type	Aircraft	Ships	Autos and trucks	Construction	Total
Share of equipment (%)	6.11	1.33	11.86	3.51	22.81

Source: Bureau of Economic Analysis Asset Tables 2015, author's calculations

Figure: Lanteri (2018)

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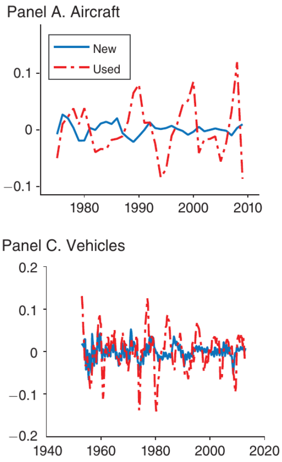

FIGURE 2. PRICES OF NEW AND USED CAPITAL (*Cyclical Components*)

Notes: Log-deviations from trend of price index of new capital and price index of used capital for the following types of capital: Aircraft, Ships, Vehicles, Construction equipment. Data definitions and elaboration are explained under Table 2. More details on data sources and construction are in online Appendix A.

Table: Eisfeldt and Shi (2018)

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Table 1 Cyclical properties of reallocation and productivity dispersion; deviations from trend are computed using an annual HP filter (Hodrick & Prescott 1997)

	Correlation with GDP	Unconditional mean	Boom mean	Recession mean
Panel *a*: Capital reallocation turnover rate				
Total reallocation turnover	0.5752*** (0.1454)	1.96%	2.30%***	1.61%
Sales of PP&E turnover	0.3455* (0.1680)	0.40%	0.43%**	0.36%
Acquisition turnover	0.5861*** (0.1413)	1.56%	1.87%***	1.25%
Panel *b*: Benefits to reallocation				
Standard deviation of Tobin's q (firm level, $0 \leq q \leq 5$)	−0.0580 (0.2250)	0.77	0.77	0.77
Standard deviation of TFP growth rates (3-digit NAICS level)	−0.1463 (0.3003)	3.79	3.56	3.99
Standard deviation of capacity utilization (3-digit NAICS level)	−0.4948*** (0.1650)	5.20	4.69	5.64
Panel *c*: Labor reallocation				
Job creation rate	0.6180*** (0.1540)	16.69%	17.65%	15.68%
Job destruction rate	−0.3760 (0.2391)	14.71%	14.51%	14.93%
Excess job reallocation rate	−0.1030 (0.3153)	14.42%	14.51%	14.32%

Data: Compustat

Table: Eisfeldt and Rampini (2007)

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Table 1
Ratio of used capital expenditures to total capital expenditures across asset, employment, and sales deciles

Decile	By assets				By employment		By sales	
	Decile cutoff (millions)	Used capital (%)	Used structures (%)	Used equipment (%)	Decile cutoff (thousands)	Used capital (%)	Decile cutoff (millions)	Used capital (%)
1st	0	27.79	28.77	26.21	0	30.27	0	20.38
2nd	0.10	20.17	21.69	17.32	0.01	17.86	0.53	23.28
3rd	0.36	18.51	21.43	15.36	0.03	16.31	2.05	18.93
4th	1.04	17.13	20.20	14.46	0.07	13.54	5.97	16.79
5th	2.94	16.14	20.08	12.97	0.18	11.69	13.65	16.40
6th	7.55	15.07	19.04	12.44	0.52	11.92	27.40	14.86
7th	16.89	12.69	16.15	10.64	0.67	10.52	51.15	13.21
8th	34.46	12.16	15.80	9.72	0.92	10.85	94.93	12.67
9th	69.24	11.22	15.33	9.18	1.45	10.33	186.51	11.81
10th	186.55	10.10	13.04	8.34	3.09	9.23	490.25	9.94

Data: Vehicle Inventory and Use Survey (VIUS) and Annual Capital Expenditures Survey (ACES)

Table: Eisfeldt and Shi (2018)

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Table 2 Reallocation versus productivity dispersion and financial flows; deviations from trend are computed using an annual HP filter (Hodrick & Prescott 1997)

	Total reallocation turnover	Sales of PP&E turnover	Acquisition turnover
Panel *a*: Correlation with benefit of reallocation			
Standard deviation of Tobin's q (F) ($0 \le q \le 5$)	−0.0732 (0.2454)	0.1464 (0.2951)	−0.0922 (0.2363)
Standard deviation of TFP growth rates (I)	0.1437 (0.3416)	0.0261 (0.3047)	0.1488 (0.3490)
Standard deviation of capacity utilization (I)	−0.5646*** (0.1218)	−0.2920 (0.1647)	−0.5778*** (0.1207)
Panel *b*: Correlation with financial variables			
Debt financing	0.6590*** (0.1530)	0.4507* (0.2205)	0.6581*** (0.1526)
Equity financing	−0.1661 (0.4199)	0.0766 (0.3439)	−0.1876 (0.4180)
Total financing	0.5261** (0.2114)	0.4768** (0.2029)	0.5122** (0.2144)
VIX	−0.0691 (0.3377)	0.2176 (0.2913)	−0.1082 (0.3287)
Uncertainty shock	0.1744 (0.3183)	0.3433 (0.2194)	0.1518 (0.3247)

Edgerton (2011): Estimation I

Back

- Study the impact and incidence of tax incentives for investment.
- Estimation model using used & new capital in production function.
  - $F(K_{new}, K_{used})$, and two types of LoM.
- Estimation of elasticity of substitution between used & new:
  - Farm machinery: 1.7 to 2.0
  - Aircraft: 1.8 to 10.5
  - Construction machinery: 1.9 to 2.4

Edgerton (2011): Estimation II

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Table 4: Regressions of Log Used/New Price Ratio on ITC and I/K

Panel A: Farm Machinery	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
ITC	-.089	-.149	-.164	-.164	-.159	-.177	-.177	-.199	-.174
	(.044)**	(.037)***	(.035)***	(.028)***	(.049)***	(.045)***	(.033)***	(.090)**	(.133)
Log I/K		.501	.539	.539	.528	.581	.581	.583	.588
		(.134)***	(.136)***	(.060)***	(.177)***	(.176)***	(.088)***	(.191)***	(.198)***
Observations	21	21	24	24	14	17	17	21	21
R^2	.179	.538	.577	.577	.519	.551	.551	.548	.55
Start Year	1984	1984	1984	1984	1984	1984	1984	1984	1984
End Year	1990	1990	1990	1990	1988	1988	1988	1990	1990
Exclude Q1-Q3 1986	Yes	Yes	No	No	Yes	No	No	Yes	Yes
Time Trend	None	None	None	None	None	None	None	Linear	Quadr.

Panel B: Aircraft	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
ITC	-.489	-.465	-.423	-.423	-.202	-.161	-.161	-.165	-.070
	(.056)***	(.067)***	(.067)***	(.120)***	(.107)*	(.095)*	(.122)	(.112)	(.094)
Log I/K		.095	.124	.124	.492	.543	.543	.104	.146
		(.148)	(.152)	(.143)	(.246)**	(.228)**	(.268)**	(.130)	(.105)
Observations	33	33	36	36	17	20	20	33	33
R^2	.712	.716	.665	.665	.732	.697	.697	.788	.867
Start Year	1982	1982	1982	1982	1984	1984	1984	1982	1982
End Year	1990	1990	1990	1990	1988	1988	1988	1990	1990
Exclude Q1-Q3 1986	Yes	Yes	No	No	Yes	No	No	Yes	Yes
Time Trend	None	None	None	None	None	None	None	Linear	Quadr.

This table presents regressions of the form: Reciprocal of coefficient is elasticity of substitution

$$\ln \frac{p_t^U}{p_t^N} = \eta_0 \mathrm{ITC}_t + \eta_1 \ln \frac{I_t^N}{K_{t-1}^U} + \epsilon_t,$$

where ITC is a dummy variable indicating the presence of a 10% investment tax credit. Standard errors in Columns 4 and 7 are Newey-West with a lag length of 4.

*** indicates statistical significance at the 1% level, ** at 5%, and * at 10%.

Edgerton (2011): Estimation III

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Table 5: Regressions of Log Used/New Price Ratio on BONUS and I/K

	Construction Machinery				
	(1)	(2)	(3)	(4)	(5)
BONUS	-.088	.034	.034	.010	-.012
	(.038)**	(.021)	(.029)	(.020)	(.019)
Log I/K		.524	.524	.501	.415
		(.046)***	(.054)***	(.042)***	(.043)***
Observations	39	39	39	39	39
R^2	.129	.811	.811	.852	.892
Time Trend	None	None	None	Linear	Quadr.

This table presents regressions of the form:

$$\ln \frac{p_t^U}{p_t^N} = \eta_0 \text{BONUS}_t + \eta_1 \ln \frac{I_t^N}{K_{t-1}^U} + \epsilon_t,$$

where bonus is a dummy variable indicating the presence of 50% bonus depreciation. Standard error in Column 3 is Newey-West with a lag length of 4.

Model Appendix

(S, s) threshold in Lanteri (2018)

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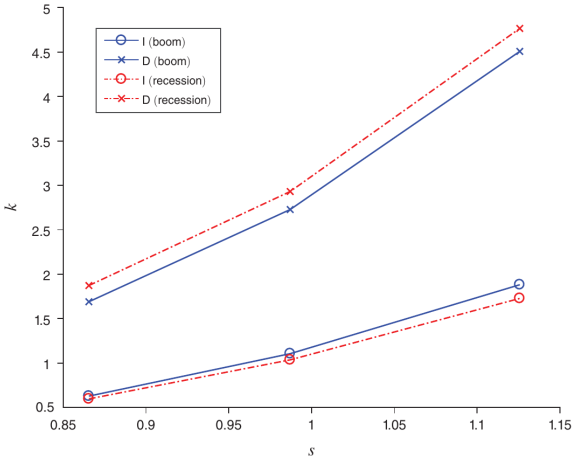

FIGURE 7. THRESHOLDS FOR INVESTMENT AND DISINVESTMENT

Notes: x-axis: idiosyncratic productivity s. y-axis: capital level k. Blue solid lines represent investment (I) and disinvestment (D) thresholds before the aggregate negative shock, while red dashed-dotted lines represent the thresholds after the aggregate negative shock hits.

Calibration Result in Lanteri (2018)

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TABLE 5—BUSINESS-CYCLE STATISTICS: BASELINE MODEL (*HP-Filter* $\lambda = 6.25$)

Statistic	Y	C	I	K	N	r	q	q/Q	reall
mean	0.613	0.509	0.103	1.574	0.336	0.041	0.918	0.933	0.042
$\sigma(\cdot)/\sigma(Y)$	(1.51)	0.482	3.679	0.247	0.534	0.074	0.187	0.133	2.972
corr$(\cdot, Y)$	1	0.983	0.99	−0.335	0.986	0.866	0.986	0.987	0.986
autocorr	0.085	0.144	0.062	0.504	0.061	−0.045	0.184	0.184	0.033

Notes: Rows: mean, standard deviation relative to standard deviation of output, autocorrelation. Columns: output, consumption, investment, capital, hours, real interest rate, price of used capital, degree of irreversibility (price of used capital relative to marginal cost of expanding a firm), capital reallocation (value in terms of output good).

TABLE 7—BUSINESS-CYCLE STATISTICS: US ANNUAL DATA (*HP-Filter with* $\lambda = 6.25$)

Statistic	Y	C	I	K	N	w	r	*TFP*	reall	SPPE only
$\sigma(\cdot)/\sigma(Y)$	(1.44)	0.529	2.86	0.977	1.209	0.568	0.828	0.498	11.022	5.208
corr$(\cdot, Y)$	1	0.81	0.792	0.573	0.894	0.184	0.049	0.402	0.712	0.305
autocorr	0.177	0.27	0.265	0.393	0.276	0.172	0.044	0.177	0.199	0.192

Notes: US business-cycle statistics 1947–2015. Rows: standard deviation relative to standard deviation of GDP, correlation with GDP, autocorrelation. Columns: real GDP, consumption (personal consumption expenditures on nondurables and services, deflated with GDP deflator), investment (fixed private investment and personal consumption expenditures on durables, deflated with GDP deflator), capital (fixed private assets and stock of consumer durables, deflated with GDP deflator), hours (all persons, nonfarm business sector), real wage (real compensation per hour, nonfarm business sector), real interest rate (three-month T-bill, net of ex post GDP-deflator inflation), aggregate TFP (constructed as in the model, i.e., $\log(\text{GDP}) - \alpha \log(K) - \nu \log(N)$), capital reallocation (SPPE + Acquisitions) and SPPE (1971–2011), deflated with GDP deflator.
Sources: BEA, BLS, Board of Governors of the Federal Reserve System, Compustat, author's calculations.

CES Cost Minimization Problem I

Back: What I do, Back: Tech

The CES cost minimization problem to at least achieve $\bar{I}$ level of investment is given by

$$\begin{aligned} \min_{i_{new}, i_{used}} \quad & c_{new} i_{new} + c_{used} i_{used} \\ \text{s.t.} \quad & \left[\eta^{\frac{1}{s}} (i_{new})^{\frac{s-1}{s}} + (1-\eta)^{\frac{1}{s}} (i_{used})^{\frac{s-1}{s}}\right]^{\frac{s}{s-1}} \geq \bar{I} \end{aligned} \quad (6)$$

Note that constraint must bind, so we can denote

$$\bar{I}^{\frac{s-1}{s}} = \left[\eta^{\frac{1}{s}} (i_{new})^{\frac{s-1}{s}} + (1-\eta)^{\frac{1}{s}} (i_{used})^{\frac{s-1}{s}}\right]. \quad (7)$$

CES Cost Minimization Problem II

Back: What I do, Back: Tech

Let the Lagrangian multiplier be λ, the FOC w.r.t. i_{new} and i_{used} are

$$\begin{aligned} [i_{new}]: \quad & c_{new} = \lambda \eta^{\frac{1}{s}} i_{new}^{-\frac{1}{s}} \bar{I}^{\frac{1}{s}} \\ [i_{used}]: \quad & c_{used} = \lambda (1-\eta)^{\frac{1}{s}} i_{used}^{-\frac{1}{s}} \bar{I}^{\frac{1}{s}} \end{aligned}, \tag{8}$$

Rearrange (8) w.r.t. investment,

$$\begin{aligned} i_{new} &= \eta \bar{I} (\frac{c_{new}}{\lambda})^{-s} \\ i_{used} &= (1-\eta) \bar{I} (\frac{c_{used}}{\lambda})^{-s} \end{aligned}. \tag{9}$$

CES Cost Minimization Problem III

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Divide, and let $c_{new} = 1$, $c_{used} = q + \gamma$, we get

$$\frac{i_{used}}{i_{new}} = \frac{1-\eta}{\eta}(q+\gamma)^{-s}. \tag{10}$$

Substitute (9) back to binding constraint and solve for Lagrangian multiplier λ, we get the CES price index as

$$Q = \left[\eta + (1-\eta)(q+\gamma)^{1-s}\right]^{\frac{1}{1-s}}. \tag{11}$$

Model: Household Problem

Back

Representative households maximize their lifetime utility by choosing consumption (c), labor supply (n^h), future firm share holding (λ') and future bond holding (ϕ'):

$$\begin{aligned} V^h(\lambda, \phi; z_f, \mu) = \max_{c, n^h, \phi', \lambda'} & \left\{ u(c, 1-n^h) + \beta \sum_{g=1}^{N_z} \pi^z_{fg} V^h(\lambda', \phi'; z'_g, \mu') \right\} \\ \text{s.t.} \quad & c + q(z_f;\mu)\phi' + \int \rho_1(k', b', \varepsilon'_j, z'_g; \mu')\lambda'(d[k' \times b' \times \epsilon']) \; , \\ & \leq w(z_f;\mu)n^h + \phi + \int \rho_0(k, b, \varepsilon_i, z_f; \mu)\lambda(d[k \times b \times \epsilon]) \end{aligned} \tag{12}$$

where $\rho_0(\cdot)$ is the dividend-inclusive price of the current share, and $\rho_1(\cdot)$ is the ex-dividend price of the future share.

Recursive Equilibrium I

Back: Overview, Back: Downward adjusting

A *recursive competitive equilibrium* is a set of function,

$$w, q, q_b, \{d_g\}_{g=1}^{N_z}, \rho_0, \rho_1, v_0, N, K, B, D, I, I_{new}, I_{used}, d, V^h, C^h, N^h, \Phi^h, \Lambda^h \tag{13}$$

such that

1. v_0 solves (2)-(5), and N is the corresponding policy functions for exiting firms, and (N, K, B, D) are the corresponding policy functions for continuing firms.

2. V^h solves (12), and (C^h, N^h, Λ^h) are the corresponding policy functions for households.

3. $\Lambda^h(k', b', \varepsilon_j', \lambda, \phi; z_f, \mu) = \mu'(k', b', \varepsilon_j'; z_f, \mu)$ for all $(k', b', \epsilon_j) \in \mathbf{S}$.

Recursive Equilibrium II

Back: Overview, Back: Downward adjusting

4. Labor market clears:

$$N^h(\lambda, \phi; z_f, \mu) = \int_{\mathbf{S}} [N(k, \epsilon_i; z_f, \mu)]\mu(d[k \times b \times \epsilon]), \tag{14}$$

5. For upward-adjusting firms, i.e., firms such that $v^u(k, b, \varepsilon_i, z_f, \mu) \geq v^d(k, b, \varepsilon_i, z_f, \mu)$, the policy function $K(k, b, \varepsilon_i, z_f, \mu)$ solves (4), and the investment $I(k, b, \varepsilon_i, z_f, \mu) = K(k, b, \varepsilon_i, z_f, \mu) - (1-\delta)k$. Furthermore, the allocation of $I_{used}(k, b, \varepsilon_i, z_f, \mu)$ and $I_{new}(k, b, \varepsilon_i, z_f, \mu)$ is (10) and the corresponding aggregate price index is (11).

Recursive Equilibrium III

Back: Overview, Back: Downward adjusting

6. For downward-adjusting firms, i.e., $v^u(k, b, \varepsilon_i, z_f, \mu) < v^d(k, b, \varepsilon_i, z_f, \mu)$, the policy function $K(k, b, \varepsilon_i, z_f, \mu)$ solves (5), and $d(k, b, \varepsilon_i, z_f, \mu) = (1-\delta)k - K(k, b, \varepsilon_i, z_f, \mu)$.

7. Good markets clear:

$$C(z_f, \mu) = \int_{\mathbf{S}} \Big\{ z_f \epsilon_i F(k, N(k, \epsilon_i; z_f, \mu)) \\ - (1-\pi_d) Q(z_f, \mu) I(k, b, \varepsilon_i, z_f, \mu) \\ + (1-\pi_d) q(z_f, \mu) d(k, b, \varepsilon_i, z_f, \mu) \\ + \pi_d [q(z_f, \mu)(1-\delta)k - k_0] \Big\} \mu(d[k \times b \times \epsilon]), \qquad (15)$$

Recursive Equilibrium IV

Back: Overview, Back: Downward adjusting

where k_0 is the initial capital stock. We assume k_0 for each entering firm is a fixed χ fraction of the long-run aggregate capital stock, i.e.,

$$k_0 = \chi \int k\tilde{\mu}(d[k \times b \times \epsilon]). \tag{16}$$

8. The used investment price $q(z_f, \mu)$ clears the market of used capital:

$$\int_{\mathbf{S}} d(k, b, \varepsilon_i, z_f, \mu)\mu(d[k \times b \times \epsilon]) = \int_{\mathbf{S}} i_{used}(k, b, \varepsilon_i, z_f, \mu)\mu(d[k \times b \times \epsilon]). \tag{17}$$

Recursive Equilibrium V

Back: Overview, Back: Downward adjusting

9. Evolution of distribution $\Gamma(\mathbf{S}, \mu)$ is defined by

$$\mu'(A, \epsilon_i) = (1 - \pi_d) \int_{\{(k,b,\epsilon_i)|K(k,b,\varepsilon_i,z_f;\mu),B(k,b,\varepsilon_i,z_f;\mu)\in A\}} \mu(d[k \times b \times \epsilon]) \\ + \pi_d \chi(k_0) H(\epsilon_j) , \tag{18}$$

where $\chi(k_0) = 1$ if $(k_0, 0) \in A$, and 0 otherwise.

Recursive Equilibrium VI

Back: Overview, Back: Downward adjusting

10. Bond market clear condition

$$\Phi^h(z_f;\mu) = \int_{\mathbf{s}} B(k,b,\varepsilon,z_f,\mu)\mu(d[k\times b\times\epsilon]) \tag{19}$$

is satisfying Walras's law, where Φ^h is household's policy functions for bond.

Analysis I

Back

Let $u(c,l) = \log c + \psi l$, and $F(k,n) = k^{\alpha} n^{\nu}$, $\alpha + \nu < 1$.
In households' problem, the following three conditions ensure that good market, labor market and bond market clear in this economy:

$$p(z_f;\mu) = D_1 u(c, 1-n^h) = \frac{1}{c} \tag{20}$$

$$w(z_f;\mu) = \frac{D_2 u(c, 1-n^h)}{D_1 u(c, 1-n^h)} = \frac{\psi}{p(z_f;\mu)} \tag{21}$$

$$q_b(z_f;\mu) \equiv \beta \sum_{g=1}^{N_z} \pi^z_{fg} \frac{D_1 u(c_g, 1-n^h_g)}{D_1 u(c, 1-n^h)} = \beta \sum_{g=1}^{N_z} \pi^z_{fg} \frac{p(z_g;\mu')}{p(z_f;\mu)}, \tag{22}$$

where $p(z_f;\mu)$ is the output price when firms current dividends is discounted using households' subjective discount factor.

Analysis II

Back

Following Khan and Thomas (2013), we can rewrite equations (2)-(5) as

$$\begin{aligned} V_0(k, b, \varepsilon_i; z_f, \mu) = \pi_d \max_n [p(z_f, \mu) x^d(k, b, \varepsilon_i; z_f)] \\ + (1 - \pi_d) V(k, b, \varepsilon_i; z_f, \mu) \end{aligned}, \tag{23}$$

where

$$V(k, b, \varepsilon_i; z_f, \mu) = \max\{V^u(k, b, \varepsilon_i; z_f, \mu), V^d(k, b, \varepsilon_i; z_f, \mu)\}. \tag{24}$$

Analysis III

Back

The dynamic problem for upward-adjusting firms is

$$
\begin{aligned}
V^u(k,b,\varepsilon_i;z_f,\mu) = \max_{k',b',D}\, & p(z_f,\mu)D + \beta \sum_{g=1}^{N_z}\sum_{j=1}^{N_\varepsilon} \pi^z_{fg}\pi^s_{ij} V_0(k',b',\varepsilon'_j;z'_g,\mu') \\
\text{s.t. } & 0 \le D \le x^u(k,b,\varepsilon_i;z_f) + q_b b' - Qk' \\
& x^u(k,b,\varepsilon_i;z_f) = z_f\varepsilon_i F(k,n) - w(z_f,\mu)n - b + Q(1-\delta)k \\
& k' \ge (1-\delta)k; \quad b' \le q\zeta k; \quad \mu' = \Gamma(z_f;\mu)
\end{aligned}
,
\tag{25}
$$

Analysis IV

Back

and the dynamic problem for downward-adjusting firms is

$$
\begin{aligned}
V^d(k,b,\varepsilon_i;z_f;\mu) = \max_{k',b',D}\ & p(z_f,\mu)D + \beta \sum_{g=1}^{N_z}\sum_{j=1}^{N_s} \pi^z_{fg}\pi^\varepsilon_{ij} V_0(k',b',\varepsilon'_j;z'_g,\mu') \\
\text{s.t. }\ & 0 \le D \le x^d(k,b,\varepsilon_i;z_f) + q_b b' - q k' \\
& x^d(k,b,\varepsilon_i;z_f) = z_f \varepsilon_i F(k,n) - w(z_f,\mu)n - b + q(1-\delta)k \\
& k' \le (1-\delta)k; \quad b' \le q\zeta k; \quad \mu' = \Gamma(z_f;\mu)
\end{aligned}
\quad . \tag{26}
$$

Khan and Thomas (2013) Replication

Khan and Thomas (2013) Replication Firm-Level Data

Back

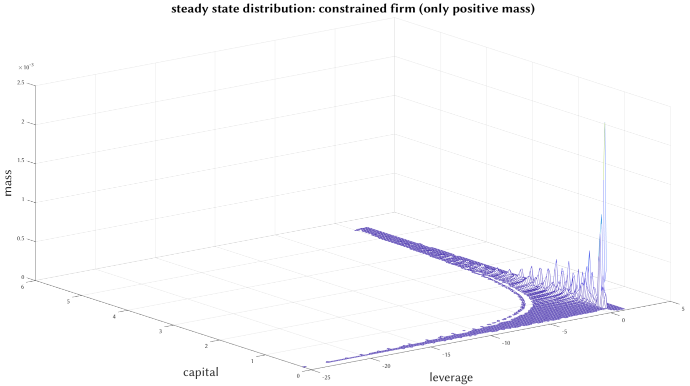
LRD Cooper and Haltiwanger (2006)	model	parameters
$\sigma(i/k) = 0.337$	0.338	$\theta_k = 0.954$
$\rho(i/k) = 0.058$	0.062	$\rho_{\eta_\varepsilon} = 0.659$
lumpy investment ($> 20\%$) $= 0.186$	0.193	$\sigma_{\eta_\varepsilon} = 0.118$

Compustat Eisfeldt and Rampini (2006)	
reallocation / investment $= 0.2389$	0.1716

Untargeted moments (LRD CH(2006))	
$mean(i/k) = 0.122$	0.105
inaction freq ($< 1\%$) $= 0.081$	0.544
disinvestment freq ($< -1.5\%$) $= 0.104$	0.148
lumpy disinvestment ($< -20\%$) $= 0.018$	0.065

KT13 Rep SS distribution: median productivity

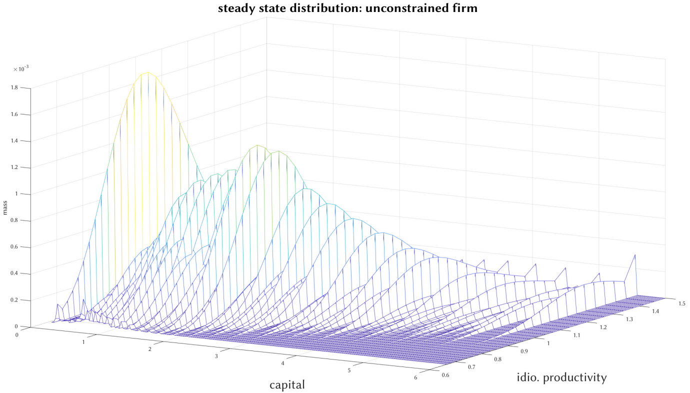
Back



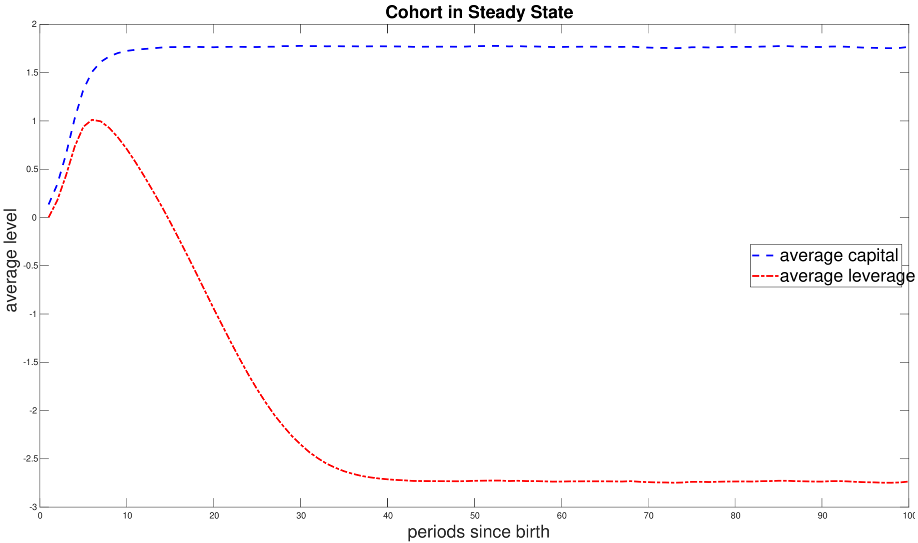
- new firm k: 0.1342
- avg constrained k: 1.202
- avg unconstrained k: 1.603
- # constrained: 65%
- firms w/ *currently* binding collateral: 18.7%

KT13 Steady State distribution for unconstrained firm

Back



KT13 Rep Life Cycle: investment & Saving Back



Algorithm Appendix

Bisection on two prices

- Harvey and Stenger (1976) extends bisection method to two dimensions.
- Instead of bisecting on sections on the line, this method bisects on area of triangles.
- The YouTube video by Oscar Veliz provides a great video explaining the simplified Harvey-Stenger bisection and visualizing the whole process with high aesthetic value. His implementation also hosted on GitHub.
- I solve this model using my own implementation of simplified Harvery-Stenger bisection.

Simplified Harvey-Stenger Bisection: Overview

Harvey and Stenger (1976) algorithm separate into two parts:

1. generate an polygon that contains the roots, and
2. bisect on polygon and find triangles containing roots & continue.

My implementation

- simplified 1 by **checking whether the initial triangular area contains roots**. If not, then exit.
- If contains roots, then following 2 and continue bisecting triangles.

Harvey and Stenger (1976) provides a **L test** to detect whether $(0, 0)$ is inside the functional evaluated triangle.

Simplified Harvey-Stenger Bisection: Algorithm I

We find $(x, y) \in \mathbb{R}^2$ such that $f(x, y) = 0$ and $g(x, y) = 0$ for both f and g are continuous function of two variables,

1. Take three points $A = (x_1, y_1), B = (x_2, y_2), C = (x_3, y_3)$ to form a triangle $\triangle ABC$ such that line $\overline{AB}$ is the longest.
2. Evaluate three points with f and g and form triangle $\triangle A'B'C'$ such that $A' = (f(A), g(A))$ and so on.
3. Use **L test** to check whether $(0, 0)$ is inside $\triangle A'B'C'$. If not, back to 1 and start with new $\triangle ABC$.
4. Otherwise, find the mid-point D on $\overline{AB}$ and evaluate $D' = (f(D), g(D))$.

Simplified Harvey-Stenger Bisection: Algorithm II

5. Find the centeroid $E = \frac{A+B+C}{3}$ and linearly interpolate E' with weight $\omega \equiv \frac{\|E-C\|}{\|D-C\|}$ such that $E' = \omega C' + (1-\omega) D'$, and $\|\cdot\|$ is Euclidean norm.

6. Starting iteration on bisecting triangles with stopping criteria $\max\{\|E'\|, \|\overline{AB}\|\} < \varepsilon$.

7. Inside loop, use **L test** to check which of the following is true:
   - $(0,0) \in \triangle A'D'C' \Rightarrow \triangle ADC$ become $\triangle ABC$
   - $(0,0) \in \triangle B'D'C' \Rightarrow \triangle BDC$ become $\triangle ABC$
   - Neither contains $(0,0) \Rightarrow$ exit iteration and report failure.

Simplified Harvey-Stenger Bisection: Algorithm III

8. Rotate $\triangle ABC$ such that $\overline{AB}$ is the longest. Repeat 4 and 5 to get D' and E'.

9. If $\max\{\|E'\|, \|\overline{AB}\|\} < \varepsilon$, then stop and report $E = (x_E, y_E)$ as solution. Otherwise, repeat 6, 7 and 8.

Simplified Harvey-Stenger Bisection: L function

Let A, B, and V be three points $(x_i, y_i), i \in \{A, B, V\}$. Define

$$L(A, B, V) = (y_B - y_A)(x_V - x_A) - (x_B - x_A)(y_V - y_A). \tag{27}$$

If $L(A, B, V) = 0$, then it means V is on the line $\overline{AB}$:

$$\begin{aligned} L(A, B, V) &= 0 \\ (y_B - y_A)(x_V - x_A) &= (x_B - x_A)(y_V - y_A) \\ \frac{y_V - y_A}{x_V - x_A} &= \frac{y_B - y_A}{x_B - x_A} \end{aligned}$$

If $L(A, B, V)$ is nonzero, then V is either on the right-hand side or left-hand side of $\overline{AB}$, depends on whether V is in between $\overline{AB}$ or outside.

Simplified Harvey-Stenger Bisection: L test

The sufficient condition to detect whether $V = (0,0)$ is inside $\triangle ABC$ is

$$
\begin{aligned}
& L(A,B,V)L(A,B,C) \geq 0 \\
\&\& \quad & L(B,C,V)L(B,C,A) \geq 0 \\
\&\& \quad & L(C,A,V)L(C,A,B) \geq 0
\end{aligned}
$$

where $L(A,B,V)L(A,B,C)$ means that point V and the other point C are on the same side of line $\overline{AB}$.

The requirement for all three conditions to hold ensures that V always on the same side as the third point, which means V is inside $\triangle ABC$.