Lecture 6
Numerical Example

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Overview: Lecture 4 - 7

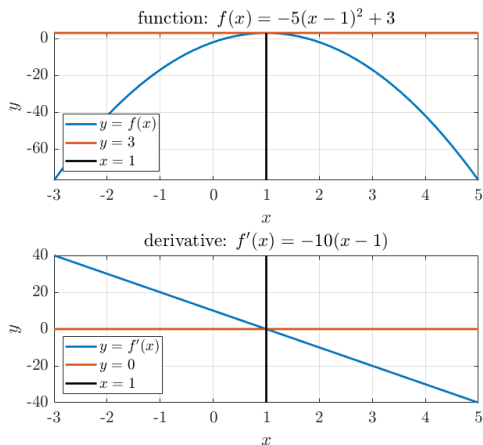Provide micro-foundation for the macro implication (Lucas critique)

- **Representative Consumer**:
  - Lecture 4: preference, constraints
  - Lecture 5: optimization, application
  - Lecture 6: Numerical Examples

- **Representative Firm**:
  - Lecture 7: production, optimization, application

1 Variable

function: $f(x) = -5(x-1)^2 + 3$

derivative: $f'(x) = -10(x-1)$
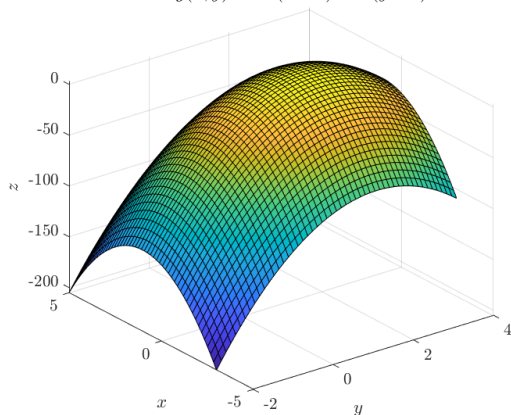
In general, want to solve
$\max_x f(x)$

- find "peak" of function

- at peak, slope is 0

- **First order condition**
  (FOC) is when the 1st
  order derivative, i.e., the
  slope is 0:

  $$f'(x^*) = 0,$$

  where x^* is the peak

2 Variables

function: $g(x,y) = -5(x-1)^2 - 8(y-2)^2 + 3$

In general, want to solve
$\max_{x,y} g(x,y)$

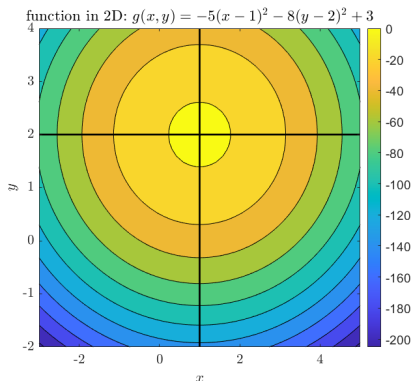- at peak, slope is 0 in both directions, i.e., the FOCs are

$$D_x g(x^*, y^*) = 0$$
$$D_y g(x^*, y^*) = 0,$$

where the bundle (x^*, y^*) is the peak

- Hard for my brain to process 3-D graph...resolution?

Visualizing 3-D function on 2-D plane

function in 2D: $g(x,y) = -5(x-1)^2 - 8(y-2)^2 + 3$

- **Contours**: "standing" at the peak and look down
  - e.g. map on Alltrails

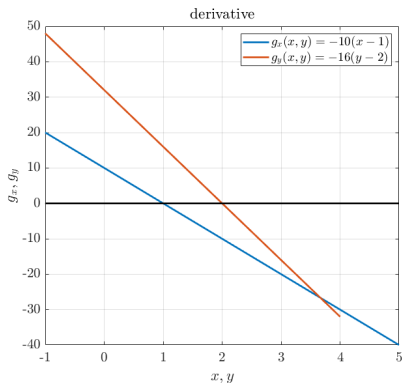- Fix the level of $g = -20$ (a horizontal slice of 3-D figure)

- Find x and y such that

$$-20 = -5(x-1)^2 - 8(y-2)^2 + 3$$

- repeat for any value of g

- **Exactly where indifference curve came from!**

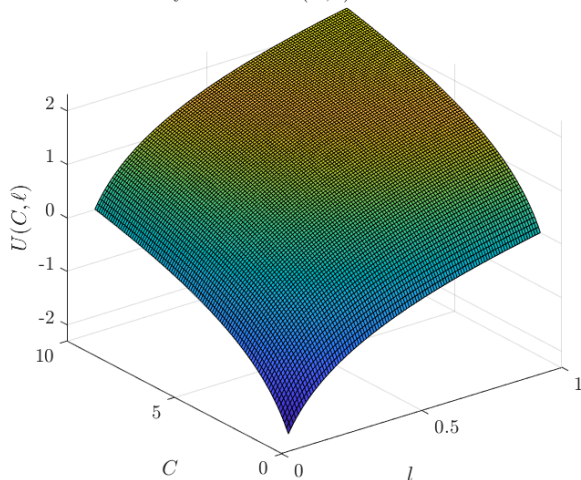## Solving 2 Variables Optimizations

$$D_x g(x^*, y^*) = -10(x-1) = 0$$
$$D_y g(x^*, y^*) = -16(y-2) = 0$$

- Intersection between 0 and line is the solution.

- For other functional form, $D_x g(x, y)$ can depend on y, and $D_y g(x, y)$ can depend on x

- May have constraints on the relationship between x and y

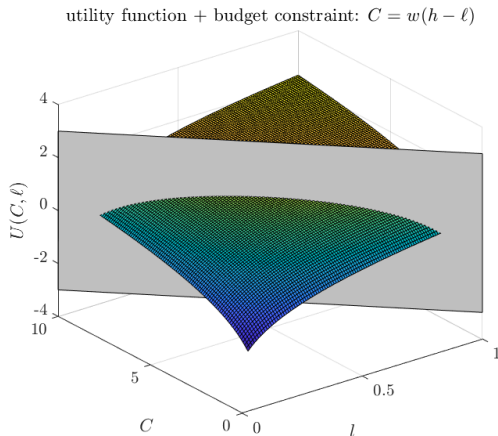# Utility Function in 3-D

Here $a = b = 1$, where is the peak?

utility function: $U(C, \ell) = a \ln C + b \ln \ell$

- Seems like to be at $C^* = 10$ and $l^* = 1$

- Recall monotonicity: more is better!

- What stops the consumer from choose $(C, l) = (10, 1)$?

Utility Function + Budget Set in 3-D

Here $w = 10$ and $h = 1$, and the gray surface represents the border of the budget set.

utility function + budget constraint: $C = w(h - \ell)$

- Consumers have to choose (C, l) bundles inside the budget set

  - $(C, l) = (10, 1)$ is outside of the budget set $\Rightarrow$ not feasible

- **Binding budget constraint**: candidates for optimal are points in gray

- Which one?

Collapsing 3-D Problem into 2-D: Slice

How? Binding budget constraint!

utility with budget imposed: $\tilde{U}(\ell) = a\ln(w(h-\ell)) + b\ln\ell$
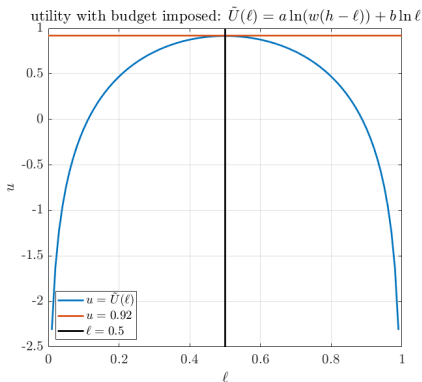
Binding: $C = w(h - l)$

$$U(C, l) = a \ln C + b \ln l$$

Plug in: $\tilde{U}(l) = a \ln(w(h - l)) + b \ln l$

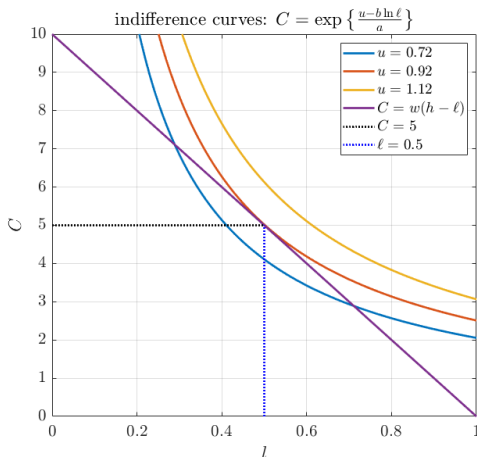FOC: $D_l \tilde{U}(l) = 0$

$$a\frac{-w}{w(h - l)} + b\frac{1}{l} = 0$$

$$\frac{a}{h - l} = \frac{b}{l}$$

$$l = \frac{b}{a + b}h$$

$l = 0.5,\ C = 5,\ u = 0.91629...$

Collapsing 3-D Problem into 2-D: Contours

Recall contours, for any utility level u, $u = a \ln C + b \ln l \Rightarrow C = e^{\frac{u - b \ln l}{a}}$

indifference curves: $C = \exp\left\{\frac{u - b \ln \ell}{a}\right\}$

- $u = 0.72$
- $u = 0.92$
- $u = 1.12$
- $C = w(h - \ell)$
- $C = 5$
- $\ell = 0.5$

- What is the highest u feasible given budget constraint?

- Or push up IC (increase u) such that IC is tangent to budget line:

$$-MRS_{l,C} = -w$$

$$\frac{bC}{al} = \frac{bw(h - l)}{al} = w$$

$$l = \frac{b}{a + b} h$$

2-D versions: Pros and Cons

Both 2-D formulations are delivering the same answer.

1. Slice: 1 variable optimization problem, x-axis: l, y-axis: u

   - **Straightforward**: operate on (l, u) plane, good for problem solving

   - **General**: can collapse higher dimension problem

   - **Cons**: lack of trade off between C and $l \Rightarrow$ economcis intuition

2. Contours: 2 variable optimization problem, x-axis: l, y-axis: C

   - **Intuitive**: direct trade off between C and l through $MRS_{l,C}$

   - **Cons:** harder to solve and to generalize to higher dimension

Review: Models from Last Lecture

1. Utility function: $U(C, l) = a \ln C + b \ln l$

2. Budget constraint: $C \leq w(h - l) + \pi - T$

3. After-tax dividend: $x = \pi - T$

4. wage rate: w

- **Benchmark**: in section Consumer Example

- **Experiment 1**: increase in after-tax dividend: $x_1 > x_0$

- **Experiment 2**: increase wage rate: $w_2 > w_0$

Solve for Benchmark Case

- **Marginal utilities**: $D_C U(C, l) = \frac{a}{C}$; $D_l U(C, l) = \frac{b}{l}$.

- **Binding budget constraint**: $C = w(h - l) + \pi - T$

- **Optimality**: $MRS_{l,C} = w \Rightarrow \frac{D_l U(C,l)}{D_C U(C,l)} = w \Rightarrow w = \frac{bC}{al}$

Plug binding budget constraints into optimality and solve for l:

$$w = \frac{b(w(h - l) + x)}{al} \tag{1}$$

$$\Rightarrow \quad wal = b(w(h - l) + x) \tag{2}$$

$$\Rightarrow \quad wal = bwh - bwl + bx \tag{3}$$

$$\Rightarrow \quad (a + b)wl = bwh + bx \tag{4}$$

$$\Rightarrow \quad l = \frac{b}{a + b}\left(h + \frac{x}{w}\right) \tag{5}$$

Solve for Benchmark Case (Cont.)

Solve for C, we get

$$l = \frac{b}{a+b}\left(h + \frac{x}{w}\right) \Rightarrow wl = \frac{b}{a+b}\left(wh + x\right) \tag{6}$$

$$C = w(h - l) + \pi - T = w(h - l) + x \tag{7}$$

$$\Rightarrow \quad C = w\left[h - \frac{b}{a+b}\left(h + \frac{x}{w}\right)\right] + x \tag{8}$$

$$\Rightarrow \quad C = wh - \frac{b}{a+b}\left(wh + x\right) + x \tag{9}$$

$$\Rightarrow \quad C = \frac{a}{a+b}wh + \frac{a}{a+b}x \tag{10}$$

$$\Rightarrow \quad C = \frac{a}{a+b}\left(wh + x\right) \tag{11}$$

Property for this utility function: consumer "split" fixed share of "wealth": $wl = s(wh + x)$, and $C = (1 - s)(wh + x)$.

Solve for Experiment 1: $x \uparrow$

(l_0, C_0, x_0): benchmark value; (l_1, C_1, x_1): experiment 1 value.

With pure income effect, no change in real wage: $w_1 = w_0 = w$

The difference between experiment 1 and benchmark case is

$$l_1 - l_0 = \frac{b}{a+b}\left(h + \frac{x_1}{w}\right) - \frac{b}{a+b}\left(h + \frac{x_0}{w}\right) \tag{12}$$

$$= \frac{b}{a+b}\left(\frac{x_1}{w} - \frac{x_0}{w}\right) \tag{13}$$

$$= \frac{b}{(a+b)w}\left(x_1 - x_0\right) > 0 \tag{14}$$

$$C_1 - C_0 = \frac{a}{a+b}\left(wh + x_1\right) - \frac{a}{a+b}\left(wh + x_0\right) \tag{15}$$
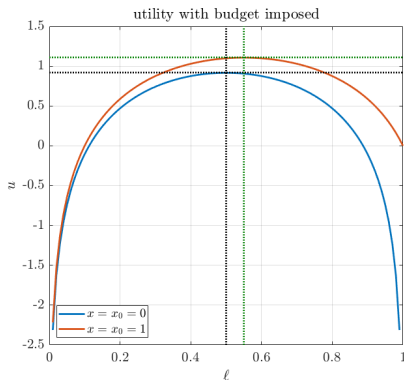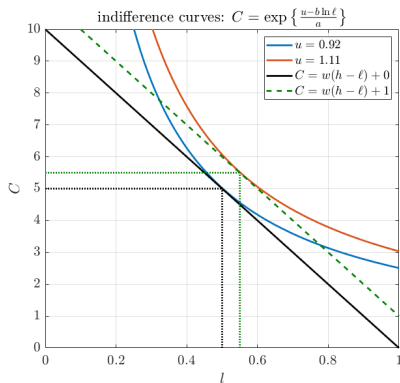
$$= \frac{a}{a+b}\left(x_1 - x_0\right) > 0 \tag{16}$$

Namely, with pure income effect, both leisure and consumption increases.

Solve for Experiment 1: Graphical Intuition

$w_1 = w_0 = 10;\ x_1 = 1 > x_0 = 0$

Both leisure and consumption are higher

Budget constraint is "eased"

indifference curves: $C = \exp\left\{\frac{u - b \ln \ell}{a}\right\}$

——	$u = 0.92$
——	$u = 1.11$
——	$C = w(h - \ell) + 0$
- - -	$C = w(h - \ell) + 1$

utility with budget imposed

——	$x = x_0 = 0$
——	$x = x_0 = 1$

Solve for Experiment 2: $w \uparrow$

(l_0, C_0, x_0): benchmark value; (l_2, C_2, x_2): experiment 2 value.

With both income and substitution effects, analysis is complicated:

$$l_2 - l_0 = \frac{b}{a+b}\left(h + \frac{x_2}{w_2}\right) - \frac{b}{a+b}\left(h + \frac{x_0}{w_0}\right) \tag{17}$$

$$= \frac{b}{a+b}\left(\frac{x_2}{w_2} - \frac{x_0}{w_0}\right) \gtreqless 0 \tag{18}$$

$$C_2 - C_0 = \frac{a}{a+b}(w_2 h + x_2) - \frac{a}{a+b}(w_0 h + x_0) \tag{19}$$

$$= \frac{a}{a+b}(h(w_2 - w_0) + (x_2 - x_0)) > 0 \tag{20}$$

Although the consumption is certainly increasing, the change in leisure is uncertain $\Rightarrow$ need numerical solution (put numbers in).

Solve for Experiment 2: $w \uparrow$ (Cont.)

Let $w_2 = 15 > w_0 = 10$; $x_2 = x_0 = 0$.

$$l_2 - l_0 = \frac{b}{a+b}\left(\frac{x_2}{w_2} - \frac{x_0}{w_0}\right) = \frac{b}{a+b}\left(\frac{0}{15} - \frac{0}{10}\right) = 0 \qquad (21)$$

Leisure remain the same.

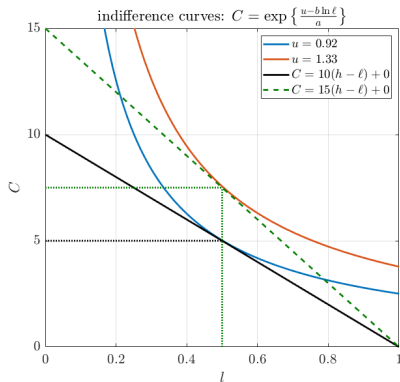Compare with experiment 1, $w_2 = 15 > w_1 = 10$; $x_2 = 0 < x_1 = 1$; $h = 1$:

$$l_2 - l_1 = \frac{b}{a+b}\left(\frac{x_2}{w_2} - \frac{x_1}{w_1}\right) = \frac{b}{a+b}\left(\frac{0}{15} - \frac{1}{10}\right) < 0 \qquad (22)$$

$$C_2 - C_1 = \frac{a}{a+b}\left(h(w_2 - w_1) + (x_2 - x_1)\right) \qquad (23)$$

$$= \frac{a}{a+b}(1(15 - 10) + (0 - 1)) > 0 \qquad (24)$$

Experiment 2 v.s. Benchmark: Graphical Intuition