Math 380 **Autumn 2020**

Problem Set #1 **Due Friday, October 9th, 12:30 PM**

First off, here are some homework guidelines.

Detail. If a problem asks you to compute something, you do not need to show all of your computations. Just give me a summary of the results. If a problem asks a yes-or-no question, you *must* justify your answer with a proof. Your proof does not need to be particularly formal, but it should be clear and convincing.

Format. You should submit your work as a PDF. Your solutions can be handwritten (and scanned) or typed. Answer in complete sentences; do not turn in scratch work.

Group work. You may work in groups of at most *four*. Group work should be collaborative: I expect you all to be carefully reading through the submission and talking together about the writeup. Your work should be much better than it would be if each of you submitted a few problems on your own.

Group work cover page. If you work as a group, you *must* include a cover page answering the following questions:

- How did your group work together, and how often did you meet? Did you talk over Zoom? Discuss the problems over chat? Collaborate in Google Drive?
- What role did each group member play on each problem? Who did what?

Assignment submission. Submit your assignment on Gradescope (linked from Canvas) by uploading a PDF of your work. When you upload your work, you will be asked to select which pages each problem appears on. Please fill this out carefully, or I may not see all of your work when I grade! If you work in a group, only one member should upload the assignment on Gradescope. Once the assignment is uploaded and pages have been assigned, click "view or edit group" to add group members.

Scoring. Each problem will be scored on the following scale:

10 points: Basically perfect.

9 points: Small but nontrivial errors.

5 points: Partly correct, but significant errors or gaps.

1 point: Mostly incorrect.

0 points: Blank.

Whew. Here are the problems:

1. **Anonymity in the Senate**

   Forgetting for the moment about filibusters and vetoes, consider the Senate as a straightforward voting system between two alternatives (namely, whether a bill should pass or not pass). It has 101 voters: 100 senators who vote yes or no under a simple majority system, and the vice president who casts a tie-breaking vote whenever the senators are split 50-50. Is this system anonymous?

2. **Nobody's Favorite**

Let's say a voting method is *favorite-friendly* if the winner must have gotten at least one first-place vote. Determine, with proof, which of the following methods are favorite-friendly.

(a) Plurality voting.

(b) The Hare method.

(c) The Borda count.

(d) The Coombs method.

(e) A dictatorship.

3. **Anti-majority Candidates**

Let's say a candidate is the *anti-majority candidate* if the majority (not just a plurality) of voters rank that candidate in last place. Is it possible for the anti-majority candidate to win an election...

(a) under plurality voting?

(b) under the Coombs method?

(c) under the Borda count?

(d) under sequential pairwise voting?

4. **Condorcet Losers**

A *Condorcet loser* is a candidate who would lose to every other candidate in a one-on-one simple majority race. Is it possible for a Condorcet loser to win an election...

(a) under plurality voting?

(b) under the Hare system?

(c) under sequential pairwise voting?

(d) under a dictatorship?

5. **Wow, Remember January?**

In this problem, I'd like you to consider how various social choice procedures would have affected a real-life race with a large number of candidates.

(a) Come up with a plausible national profile for the 2020 Democratic primary, circa January or February. You don't need to include all 20+ candidates; just pick four or five. There aren't a lot of polls that asked voters about their full rankings of the candidates, so just make a reasonable guess.

(b) Suppose that the primary were held all at once, using your profile from part (a). Who would win using each of the following methods?

   i. Plurality voting.
   ii. The Hare method.
   iii. The Borda count.
   iv. The Coombs method.

(c) Is there a Condorcet candidate in your profile?

6. **Condorcet and Changing Agendas**

(a) Prove that if a sequential pairwise voting election has a Condorcet candidate, then they will always win regardless of the agenda.

(b) Prove that if a three-candidate election does *not* have a Condorcet candidate, then every sequential pairwise ordering will result in a different winner.

7. **Constructing a Profile from Pairwise Results**

In an election with three candidates, suppose you know how every possible *pair* of candidates would perform in a one-on-one race. (In other words, you know what the final results would be if A ran against B, if B ran against C, or if A ran against C.)

(a) Is this always enough information to determine the tabulated profile for the voters (that is, the number of voters who would choose each of the six possible ballots in a three-way race)? If so, explain how you would use this information to find the profile. If not, give an example of two different profiles which would lead to the same result in any one-on-one matchup.

(b) Generalize your results to an election with four or more candidates. Is your answer the same?

8. **Variations on Borda**

For this problem, consider the following voter profiles:

Voter 1	Voter 2	Voter 3	Voter 4
B	C	C	A
A	A	D	D
D	B	B	B
C	D	A	C

(a) Find the winner using the Borda count.

(b) Suppose we made a new version of the Borda count, where each first place vote gets 8 points, each second place vote gets 4 points, each third place vote gets -4 points, and each fourth place vote gets -8 points. Redo the election using this new method.

(c) Redo it again, this time using the points -1, -5, -9, -13 for (respectively) first, second, third, and fourth place.

(d) And again, this time using the points 9, 4, 1, and 0.

(e) Once more now, with values 5, 4, 3, and 2.

(f) When did the modified Borda count give the same results as the usual Borda count? Propose a condition on the new set of point values that will guarantee the results are the same as the original method. Prove it.