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CSC3001 Discrete Mathematics
Practice **Exercise** 3 Sep **2020**

Direct proof

1. Show that the product of an even number and an odd number is even.
2. Let $a, b, k \in \mathbb{Z}$. Show that if a and b are both multiples of k, then $a+b$ is also a multiple of k.
3. Let $a, b, c \in \mathbb{N}$, where $c \leq b \leq a$. Show that $\binom{a}{b}\binom{b}{c} = \binom{a}{b-c}\binom{a-b+c}{c}$.
4. Show that the product of any five consecutive integers is divisible by 120. (For example, the product of 3, 4, 5, 6, and 7 is 2520, and $2520 = 120 \times 21$.)
5. If n is odd, then $(n^2 - 1)$ is a multiple of 8.

Proof by contrapositive

1. Let $p \in \mathbb{Z}$. Show that if p^2 is a multiple of 3, then p is a multiple of 3.
2. Suppose $x \in \mathbb{R}$. If $x^2 + 5x < 0$ then $x < 0$.
3. Let $p \in \mathbb{Z}$. Show that if p^k is even, then p is even.
4. Let $a, b \in \mathbb{Z}$. Show that if $a^2(b^2 - 2b)$ is odd, then a and b are odd.
5. If $n \in \mathbb{N}$ and $2^n - 1$ is prime, then n is prime.

Proof by contradiction

1. Show that $\sqrt[3]{2}$ is not rational.
2. Show that $\sqrt{3}$ is not rational. (**Hint:** use Q1 in the **Proof by contrapositive**.)
3. Let $a, b \in \mathbb{R}$. Show that if a is rational and ab is not rational, then b cannot be rational.
4. Show that there exist no integers a and b such that $18a + 6b = 1$.
5. Suppose n students took a quiz and the average score is 80 (out of 100). Show that at least $n/2$ students score greater than 60.
6. Let $a, b \in \mathbb{Z}$. Show that $a^2 - 4b - 3 \neq 0$.

Proof by cases

1. Show that $|x||y| = |xy|$ for all real numbers x, y.
2. Show that $\max(x, y) + \min(x, y) = x + y$.
3. Show that $\max(x, y) = (|x + y| + |x - y|)/2$, for all positive real numbers x, y.
4. (Triangle inequality) Show that $|a + b|$ is less than or equal to $|a| + |b|$.
5. Let n is a positive integer. Show that $n^7 - n$ is divisible by 7.

Direct Proof.

(4.) Suppose five integers are $n-2, n-1, n, n+1, n+2$. $(n \geq 3)$.

$\Rightarrow P = (n-2)(n-1)n(n+1)(n+2)$

$= (n^2-4)(n^2-1)n = (n^4-5n+4)n = n^5 - 5n^2 + 4n$.

① W.T.S P is divisable by 8.

$n-2, n, n+2$ are even. $\Rightarrow 2^3 = 8$ is divisor of P.

$n-1, n+1$ are even. $n-1 = 2, n+1 = 4 \Rightarrow 8$ is divisor of P.

② W.T.S. P is divisable by 3.

$3k, 3k+1, 3k+2 \Rightarrow$ At least one of $n-2, n-1, n, n+1, n+2$ is the multiple of 3 $\Rightarrow$ 3 is divisor of P.

③ W.T.S. P is divisable by 5.

$5k, 5k+1, 5k+2, 5k+3, 5k+4. \Rightarrow$ At least one of $n-2, n-1, n, n+1, n+2$ is the multiple of $5 \Rightarrow 5$ is divisor of P.

$\Rightarrow P = 8 \cdot 3 \cdot 5k = 120k$. $\forall k \in \mathbb{N}$. P is divisible by 120.

Proof by Contrapositive.

(5). W.T.S. If n is not prime, then 2^n-1 is not prime. $(n \in \mathbb{N})$.

n is not prime $\Rightarrow n = a \cdot b$. $a, b \neq 1$ and $a, b \neq n$.

$\Rightarrow 2^n - 1 = 2^{a \cdot b} - 1 = (2^a)^b - 1 = (2^a - 1)(2^{a(b-1)} + 2^{a(b-2)} + \cdots + 2^a + 1)$

$2^a - 1 > 1$, $2^{a(b-1)} + \cdots + 2^a + 1 > 1 \Rightarrow 2^n - 1$ has factor except 1.

$\Rightarrow 2^n - 1$ is not prime.

Thus, if 2^n-1 is prime, then n is prime.

Proof by contradiction

(6) Let $a, b \in \mathbb{Z}$. Suppose $a^2-4b-3=0 \Rightarrow a^2=4b+3$

① If a is even, $\Rightarrow a^2$ is even. $4b+3$ is odd for $\forall b \in \mathbb{Z}$.

$\Rightarrow$ Form a contradiction.

② If a is odd. $a=2k+1$, $k \in \mathbb{Z}$. $\Rightarrow a^2=(2k+1)^2=4(k^2+k)+1$

$\Rightarrow a^2$ can not equal to $4b+3$

$\Rightarrow$ Form a contradiction.

Thus, if $a, b \in \mathbb{Z}$, then $a^2-4b-3 \neq 0$.

Proof by Cases.

(4). ① If $a>0, b>0$, $\Rightarrow |a+b| = a+b = |a|+|b|$, obviously.

② If $a>0, b>0$ $\Rightarrow |a+b| = -a-b = |a|+|b|$, obviously.

③ If $a>0, b<0$,
$$\begin{cases} |a| \geq |b| \Rightarrow |a+b| = a+b < a-b = |a|+|b| \\ |a| \leq |b| \Rightarrow |a+b| = -a-b < a-b = |a|+|b| \end{cases}$$

④ If $a<0, b>0$,
$$\begin{cases} |a| \geq |b| \Rightarrow |a+b| = -a-b < -a+b = |a|+|b| \\ |a| \leq |b| \Rightarrow |a+b| = a+b < -a+b = |a|+|b|. \end{cases}$$

$\Rightarrow |a+b| \leq |a|+|b|$

(5). Let $S=n^7-n$.

① If $n=7k$, $k \in \mathbb{N}$. $\Rightarrow S = n(n^6-1) = 7k(n^6-1)$

$\Rightarrow S$ is divisible by 7.

② If $n=7k+1$, $k \in \mathbb{N}$. $\Rightarrow S = n(n^6-1) = n(n-1)(n^5+n^4+n^3+n^2+n+1)$
$= n \cdot 7k(n^5+\cdots+n+1)$

$\Rightarrow S$ is divisible by 7

③ If $n=7k+2$, $k \in \mathbb{N}$ $\Rightarrow S = n(n^6-1) = n[(n-2)(n^5+2n^4+4n^3+8n^2+16n+32)+63]$
$= n[7k(n^5+\cdots+16n+32)+63]$

$\Rightarrow S$ is divisible by 7. $(63=7\times 9)$

④ If $n=7k+3$, $k\in N$ $\Rightarrow$ $S=n(n^6-1)=n[(n-3)(n^5+3n^4+9n^3+27n^2+81n+243)+728]$

$=n[7k(n^5+\cdots+81n+243)+728]$

$\Rightarrow$ S is divisible by 7 $(728=7\times104)$

⑤ If $n=7k+4$, $k\in N$ $\Rightarrow$ $S=n(n^6-1)=n[(n-4)(n^5+\cdots+256n+1024)+4095]$

$(4095=7\times585)$

⑥ If $n=7k+5$, $k\in N$ $\Rightarrow$ $S=n(n^6-1)=n[(n-5)(n^5+\cdots+625n+3125)+15624]$

$(15624=7\times2232)$

⑦ If $n=7k+6$, $k\in N$ $\Rightarrow$ $S=n(n^6-1)=n[(n-6)(n^5+\cdots+1296n+7776)+46655]$

$(46655=7\times6665)$

$\Rightarrow$ ⑤, ⑥, ⑦, S is divisible by 7

Thus, n^7+n is divisible by 7.