MAT 3007 – Optimization

Exercise Sheet 3

Exercise E3.1 (Multiple Choice – Duality and the Simplex Method):
We consider an LP in its standard form:

$$\min_x \; c^\top x \qquad \text{s.t.} \qquad Ax = b, \quad x \geq 0, \tag{1}$$

where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$ are given. We assume that A has full row rank. Decide whether the following statements are true or false and explain your answer (i.e., give a short proof or a counterexample).

a) The dual of the auxiliary primal problem considered in phase I of the simplex method is always feasible.

☐ True. ☐ False.

b) If the unboundedness criterion in the simplex algorithm is satisfied, then the dual problem of (1) is infeasible.

☐ True. ☐ False.

Exercise E3.2 (The Dual of a Linear Program):
Consider the linear programming problem:

$$\begin{array}{rll} \text{minimize} & x_1 - x_2 & \\ \text{subject to} & 2x_1 + 3x_2 - x_3 + x_4 & \leq 0 \\ & 3x_1 + x_2 + 4x_3 - 2x_4 & \geq 3 \\ & -x_1 - x_2 + 2x_3 + x_4 & = 6 \\ & x_1 & \leq 0 \\ & x_2, x_3 & \geq 0. \end{array}$$

Write down the corresponding dual problem.

Assignment A3.1 (Duality and Complementarity): (approx. *20* points)
Consider the following linear program:

$$\begin{array}{rll} \text{maximize} & 5x_1 + 2x_2 + 5x_3 & \\ \text{subject to} & 2x_1 + 3x_2 + x_3 & \leq 4 \\ & x_1 + 2x_2 + 3x_3 & \leq 7 \\ & x_1, x_2, x_3 & \geq 0. \end{array}$$

a) Derive the corresponding dual problem.

b) Solve the dual problem graphically.

c) Use the complementarity conditions to solve the primal problem.

Assignment A3.2 (The Dual of the Dual): (approx. *10* points)
We consider the general linear optimization problem:

$$\min_x \; c^\top x \qquad \text{s.t.} \qquad Ax \le b, \quad Cx = d, \tag{2}$$

where $A \in \mathbb{R}^{m\times n}, C \in \mathbb{R}^{p\times n}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, and $d \in \mathbb{R}^p$ are given. Derive the dual of problem (2) and show that the dual of the dual is equivalent to problem (2).

Assignment A3.3 (Examples – Duality): (approx. *20* points)
In this exercise, we want to derive and construct different examples:

a) Give an example in which there is a duality gap between the primal and dual problem, i.e., the optimal objective function values of the primal and dual problem do not coincide.

b) Give an example of a pair (primal and dual) of linear optimization problems such that both the primal and the dual problem have multiple optimal solutions.

c) Give an example of a pair (primal and dual) of linear programs such that both the primal and the dual problem have a unique optimal solution.

d) Give an example in which the primal problem is in standard form and has a degenerate optimal basic feasible solution, but the dual has a unique optimal solution.

Assignment A3.4 (Game Theory and Duality): (approx. *25* points)
Consider the two-player game: Players I and II simultaneously call out one of the numbers one, two, three, or four. Player I wins if the sum of the numbers is odd. Player II wins if the sum of the numbers is even. The amount paid to the winner by the loser is always the sum of the called numbers in RMB.

		Player II			
	Number	1	2	3	4
Player I	1	-2	3	-4	5
	2	3	-4	5	-6
	3	-4	5	-6	7
	4	5	-6	7	-8

Table 1: Player's I winnings = Player's II losses

In this exercise, we want to investigate optimal gaming strategies $x, y \in \mathbb{R}^4$ for player I and II, respectively. We are interested in average winnings or losses and in a probabilistic strategy, i.e., player I decides to call out number i with probability x_i, $i = 1, ..., 4$ (similarly for player II). In the following, let $A \in \mathbb{R}^{4\times 4}$ denote the pay-off or winning matrix shown in Table 1.

a) Player I wants to determine his/her strategy via solving the linear program:

$$\max_{x,t} \; t \qquad \text{s.t.} \qquad \left[\begin{array}{ll} Ax & \ge t\mathbb{1}, \\ \mathbb{1}^\top x & = 1, \\ x & \ge 0, \end{array}\right. \tag{3}$$

where $\mathbb{1} \in \mathbb{R}^4$ is the vector of all ones. Interpret the linear optimization problem (3) and discuss the meaning of its solution in the context of the described situation. Calculate the solution and the optimal value p^* of (3) (e.g., by using MATLAB).

b) Compute the dual of problem (3) and its optimal objective function value d^*. Discuss and interpret the meaning of the dual problem.

c) Let us define $\mathcal{P} := \{x \in \mathbb{R}^4 : x \geq 0,\, \mathbb{1}^\top x = 1\}$ (this set is called standard simplex or probability simplex). Then, the optimization problem (3) can be written as

$$\max_{x\in\mathcal{P}} \left[\max_{t}\ t \quad \text{s.t.} \quad Ax \geq t\mathbb{1}\right].$$

Apply duality in an appropriate way to verify $p^* = \max_{x\in\mathcal{P}} \min_{y\in\mathcal{P}} y^\top Ax = d^*$.

d) Discuss whether the game is fair. What is your answer when player I and II play the same game but only use the numbers one and two?

Assignment A3.5 (Duality in Linear Regression): (approx. *25* points)

In this exercise, we consider the optimization problem

$$\text{minimize}_{x\in\mathbb{R}^n}\ \|Ax - b\|_\infty, \tag{4}$$

where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$ are given and $\|y\|_\infty := \max_{1\leq i\leq m} |y_i|$, $y \in \mathbb{R}^m$, denotes the standard maximum norm.

a) Reformulate problem (4) as a linear optimization problem.

b) Derive the corresponding dual of the linear formulation of problem (4).

c) Show that the dual problem can be equivalently expressed as the following maximization problem:

$$\max_{y\in\mathbb{R}^m}\ b^\top y \qquad \text{s.t.} \qquad A^\top y = 0, \quad \|y\|_1 \leq 1,$$

where $\|y\|_1 = \sum_{i=1}^m |y_i|$ is the ℓ_1-norm of the vector y.

d) Let $\mathcal{Y} := \{y \in \mathbb{R}^m : A^\top y = 0,\ \|y\|_1 \leq 1\}$ denote the feasible set of the dual problem. Verify that:

$$\min_x\ \|Ax - b\|_\infty = \max_{y\in\mathcal{Y}}\ b^\top y.$$

e) Write a MATLAB program to solve the optimization problem (4) and its associated dual problem for general input data $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. Test your implementation with

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A = [ones(m),ones(m)],  b = (1:m)',  m = 100
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and report the optimal objective function values. Compare the runtime of your code for the primal and dual problem; what is your observation?

Sheet 3 is due on **Jul, 4th**. Submit your solutions before **Jul, 4th**, **11:00 am**.