Stochastic Processes

Lecture 18: Stationary distributions for CTMC

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Initial distribution

- Assume $X = \{X(t), t \geq 0\}$ is a CTMC on state space $S = \{1, 2, 3\}$ with generator

$$G = \begin{pmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

- Given

$$\mathbb{P}(X(0) = 1) = 1/4, \quad \mathbb{P}(X(0) = 2) = 1/2, \quad \mathbb{P}(X(0) = 3) = 1/4.$$

  Find $\mathbb{E}(X(1))$.

- First find

$$\mathbb{P}(X(1) = 1) =?, \quad \mathbb{P}(X(1) = 2) =?, \quad \mathbb{P}(X(1) = 3) =?.$$

Distribution at time 1

- Using Python,

$$P(1) = \texttt{expm}(G) = \begin{pmatrix} 0.1703 & 0.3974 & 0.4323 \\ 0.1520 & 0.4157 & 0.4323 \\ 0.0935 & 0.3389 & 0.5677 \end{pmatrix}$$

- The distribution of $X(1)$ is

$$(1/4, 1/2, 1/4)P(1) = (0.1419 \quad 0.3919 \quad 0.4662).$$

- Expectation

$$\mathbb{E}(X(1)) = 1(0.1419) + 2(0.3919) + 3(0.4662).$$

Distribution at time 10

- Using `Python`,

$$P(10) = \texttt{expm}(10 * G) = \begin{pmatrix} 0.1250 & 0.3750 & 0.5000 \\ 0.1250 & 0.3750 & 0.5000 \\ 0.1250 & 0.3750 & 0.5000 \end{pmatrix}$$

- The distribution of $X(10)$ is

$$(1/4, 1/2, 1/4)P(10) = (0.1250 \quad 0.3750 \quad 0.5000).$$

- Given

$$\mathbb{P}(X(0) = 1) = 0.125,\ \mathbb{P}(X(0) = 2) = 0.375,\ \mathbb{P}(X(0) = 3) = 0.5.$$

the distribution of $X(1)$ is

$$(\qquad , \qquad , \qquad)$$

Stationary distribution

Definition

A row vector $\pi = (\pi_i, i \in S)$ is said to be a *stationary distribution* if

$$\pi = \pi P(t) \text{ for all } t \geq 0,$$
$$\pi_i \geq 0 \text{ and } \sum_{i \in S} \pi_i = 1.$$

Theorem

A distribution π is a stationary distribution of a CTMC with generator G if and only if

$$\pi G = 0.$$

Proof (when S is finite)

- Kolmogorov backward equation

$$P'(t) = GP(t) \quad t \geq 0$$

- If $\pi G = 0$, then

$$\pi P'(t) = \pi G P(t) = 0.$$

- Thus

$$\frac{d}{dt}(\pi P(t)) = 0 \text{ and } \pi P(t) = \pi P(0) = \pi.$$

- On the other hand, if $\pi = \pi P(t)$, then

$$\frac{d}{dt}(\pi P(t)) = 0 \text{ and } \pi P'(0+) = 0,$$

where $G = P'(0+)$.

Computing a stationary distribution

- For example, solving

$$(\pi_1, \pi_2, \pi_3) \begin{pmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix} = 0$$

gives $\pi = (1/8, 3/8, 4/8)$.

Computing the Stationary Distribution

π is a *stationary distribution* of the CTMC if

$$\pi G = 0 \tag{1}$$

Since $G_{ii} = -\lambda(i)$, and $G_{ij} = \lambda_{ij}$ when $i \neq j$, equation (1) means

$$\pi_i \lambda(i) = \sum_{j \neq i} \pi_j \lambda_{ji}, \quad i \in S \tag{2}$$

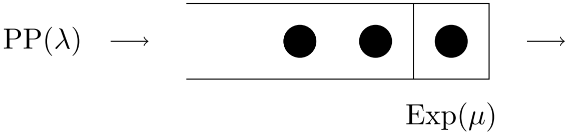
Interpretation of equation (2): For every state of the CTMC,

$$\boxed{\textit{Rate Out} = \textit{Rate In}}$$

Example: M/M/1 Queue

Customers *arrive* according to a Poisson process with rate λ.

The *service* times are iid exponential with rate μ.



$X(t)$ = number of customers in the system at time t.

$\{X(t), t \geq 0\}$ is a CTMC with state space $S = \{0, 1, \dots\}$

Example: M/M/1 Queue

Generator Matrix:

$$G_{0,0} = -\lambda, \quad G_{0,1} = \lambda$$

For $i = 1, 2, \ldots,$

$$G_{i,i-1} = \mu, \quad G_{i,i} = -(\lambda+\mu), \quad G_{i,i+1} = \lambda$$

$$G = \begin{bmatrix} -\lambda & \lambda & 0 & 0 & 0 & \cdots \\ \mu & -(\lambda+\mu) & \lambda & 0 & 0 & \cdots \\ 0 & \mu & -(\lambda+\mu) & \lambda & 0 & \cdots \\ 0 & 0 & \mu & -(\lambda+\mu) & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

Cut method

Lemma

$\pi G = 0$ is equivalent to that for any "cut" (partition) (A, A^c):

$$\sum_{i\in A}\sum_{j\in A^c}\pi(i)\lambda_{ij} = \sum_{i\in A^c}\sum_{j\in A}\pi(i)\lambda_{ij} \tag{3}$$

- For $M/M/1$ queue, solving $\pi G = 0$ (or using "Rate Out = Rate In"),

$$\boxed{\pi_i = \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^i, \quad i = 0, 1, 2, \ldots}$$

- rate diagram

Irreducibility and positive recurrence

Definition

A CTMC is said to be irreducible if its jump matrix, as a transition probability matrix of a DTMC, is irreducible.

Definition

A state $i \in S$ is said to be positive recurrent for a CTMC if

$$\mathbb{E}[T_{i,i}] < \infty,$$

where $T_{i,i}$ is the first return time to state i, starting from state i at time 0.

Two big theorems

Theorem

Assume a CTMC is irreducible. (a) There is at most one stationary distribution. (b) When S is finite, the CTMC has a unique stationary distribution. (c) The CTMC has a stationary distribution if and only if it is positive recurrent.

Theorem (SLLN)

Assume that a CTMC is irreducible and positive recurrent. Assume that $f : S \to \mathbb{R}_+$ is a nonnegative function. Then

$$\mathbb{P}\Big\{ \lim_{T\to\infty} \frac{1}{T} \int_0^T f(X(t))dt = \sum_{i\in S} \pi_i f(i) \Big\} = 1,$$

where π is the unique stationary distribution.

Performance Measures

$f(i)$ = "cost" or "reward" for being in state i

What's the long-run average cost/reward?

Example: M/M/1 Queue

Some Performance Measures:

- $f(i) = i \overset{\textbf{SLLN}}{\Longrightarrow}$ with probability 1,

  $$\text{long-run } \textit{average number of customers in sys.} = \sum_{i=0}^{\infty} i\pi_i = \frac{\lambda}{\mu - \lambda}$$

- $f(i) = \mathbf{1}\{i > 0\} \overset{\textbf{SLLN}}{\Longrightarrow}$ with probability 1,

  $$\text{long-run } \textit{fraction of time the server is busy} = \sum_{i=1}^{\infty} \pi_i = \frac{\lambda}{\mu}$$

- $f(i) = \mathbf{1}\{i = j\} \overset{\textbf{SLLN}}{\Longrightarrow}$ with probability 1,

  long-run *fraction of time there're j customers in the system* $= \pi_j$