1. Since $a_k = Re((5+i)^k) = \dfrac{(5+i)^k + (5-i)^k}{2}$,

$b_k = Im((5+i)^k) = \dfrac{(5+i)^k - (5-i)^k}{2i}$,

then $a_k \cdot b_k = \dfrac{[(5+i)^k + (5-i)^k]\cdot[(5+i)^k - (5-i)^k]}{4i}$

$$= \frac{(5+i)^{2k} - (5-i)^{2k}}{4i}$$

Thus, $\displaystyle\sum_{k=1}^{\infty} \frac{a_k b_k}{28^k} = \frac{1}{4i} \cdot \sum_{k=1}^{\infty} \left[\frac{(5+i)^{2k}}{28^k} - \frac{(5-i)^{2k}}{28^k} \right]$

① Consider $\displaystyle\sum_{k=1}^{\infty} \frac{(5+i)^{2k}}{28^k} = \sum_{k=1}^{\infty} \left(\frac{6}{7} + \frac{5}{14} i\right)^k$

Let $\frac{6}{7} + \frac{5}{14} i = r(\cos\theta + i\sin\theta)$, $r = \frac{13}{14}$,

then $\left(\frac{6}{7} + \frac{5}{14} i\right)^k = r^k(\cos k\theta + i\sin k\theta)$

$$= r^k \cos k\theta + i \cdot r^k \sin k\theta.$$

$$\left|\left(\frac{6}{7} + \frac{5}{14} i\right)^k\right| = \sqrt{(r^k \cos k\theta)^2 + (r^k \sin k\theta)^2}$$

$$= \sqrt{r^{2k}(\cos^2 k\theta + \sin^2 k\theta)}$$

$$= r^k, \text{ where } r = \frac{13}{14}$$

Since $\left|\left(\frac{6}{7} + \frac{5}{14} i\right)^k - 0\right| \to 0$ as $k \to \infty$,

then $\left(\frac{6}{7} + \frac{5}{14} i\right)^k \to 0$ as $k \to \infty$.

Thus, $\displaystyle\sum_{k=1}^{\infty} \left(\frac{6}{7} + \frac{5}{14} i\right)^k = \frac{\left(\frac{6}{7} + \frac{5}{14} i\right)}{1 - \left(\frac{6}{7} + \frac{5}{14} i\right)} = \frac{12+5i}{2-5i}$

② Consider $\displaystyle\sum_{k=1}^{\infty} \frac{(5-i)^{2k}}{28^k} = \sum_{k=1}^{\infty} \left(\frac{6}{7} - \frac{5}{14} i\right)^k$

Let $\frac{6}{7} - \frac{5}{14} i = r(\cos\theta + i\sin\theta)$, $r = \frac{13}{14}$.

Similarly, $\left|\left(\frac{6}{7} - \frac{5}{14} i\right)^k\right| = r^k$. $r = \frac{3}{14}$.

then, $\left(\frac{6}{7} - \frac{5}{14} i\right)^k \to 0$ as $k \to \infty$

Thus, $\displaystyle\sum_{k=1}^{\infty} \left(\frac{6}{7} - \frac{5}{14} i\right)^k = \frac{\frac{6}{7} - \frac{5}{14} i}{1 - \left(\frac{6}{7} - \frac{5}{14} i\right)} = \frac{12-5i}{2+5i}$

By ① and ②, $\displaystyle\sum_{k=1}^{\infty} \frac{a_k b_k}{28^k} = \frac{1}{4i}\cdot\left(\frac{12+5i}{2-5i} - \frac{12-5i}{2+5i}\right)$

$$= \frac{1}{4i}\cdot\frac{140i}{29}$$

$$= \frac{35}{29}$$

2. $f(z) = u(r,\theta) + i v(r,\theta)$, where $z = \cos\theta + i\sin\theta$.

① Let $\Delta z = \Delta r$, then

$$\lim_{\Delta r\to 0} \frac{u(r+\Delta r,\theta) + i v(r+\Delta r,\theta) - u(r,\theta) - i v(r,\theta)}{(\Delta r)\cdot(\cos\theta + i\sin\theta)}$$

$$= \lim_{\Delta r\to 0} \frac{u(r+\Delta r,\theta) - u(r,\theta)}{(\Delta r)\cdot(\cos\theta+i\sin\theta)} + i\cdot\lim_{\Delta r\to 0}\frac{v(r+\Delta r,\theta) - v(r,\theta)}{(\Delta r)(\cos\theta+i\sin\theta)}$$

$$= \frac{1}{\cos\theta + i\sin\theta}\cdot\left[\frac{\partial u}{\partial r}(r,\theta) + i\frac{\partial v}{\partial r}(r,\theta)\right]$$

$$= (\cos\theta - i\sin\theta)\cdot\left[\frac{\partial u}{\partial r}(r,\theta) + i\frac{\partial v}{\partial r}(r,\theta)\right]$$

② Let $\Delta z = \Delta\theta$, then

$$\lim_{\theta\to 0} \frac{u(r,\theta+\Delta\theta) + i v(r,\theta+\Delta\theta) - u(r,\theta) - i v(r,\theta)}{r(\cos(\theta+\Delta\theta) + i\sin(\theta+\Delta\theta)) - r(\cos\theta + i\sin\theta)}$$

$$= \frac{1}{r}\cdot\lim_{\Delta\theta\to 0}\left[\frac{u(r,\theta+\Delta\theta) - u(r,\theta)}{\Delta\theta} + i\cdot\frac{v(r,\theta+\Delta\theta) - v(r,\theta)}{\Delta\theta}\right]$$

$$\cdot\left[\frac{\Delta\theta}{\cos(\theta+\Delta\theta) + i\sin(\theta+\Delta\theta) - (\cos\theta + i\sin\theta)}\right]$$

$$= \frac{1}{r}\cdot\frac{1}{-\sin\theta + i\cos\theta}\cdot\left[\frac{\partial u}{\partial\theta}(r,\theta) + i\frac{\partial v}{\partial\theta}(r,\theta)\right]$$

$$= \frac{1}{r}\cdot(-\sin\theta - i\cos\theta)\left[\frac{\partial u}{\partial\theta}(r,\theta) + i\frac{\partial v}{\partial\theta}(r,\theta)\right]$$

By ① and ②, then the equations are given by

"Re" part / "Im" part

$$\begin{cases} \cos\theta\cdot u_r + \sin\theta\cdot v_r = \cos\theta\cdot\frac{1}{r}\cdot v_\theta + \sin\theta\cdot\left(-\frac{1}{r}\right)\cdot u_\theta \\ \cos\theta\, v_r - \sin\theta\, u_r = \cos\theta\cdot\left(-\frac{1}{r}\right)u_\theta - \sin\theta\cdot\left(\frac{1}{r}\right)\cdot v_\theta \end{cases}$$

Thus, $\begin{cases} u_r = \frac{1}{r} v_\theta \\ v_r = -\frac{1}{r} u_\theta \end{cases}$

3. Proof. ① Since $f(z) = \sqrt{r}\,(\cos(\frac{\theta}{2}) + i\sin(\frac{\theta}{2}))$.

$= \sqrt{r}\cos(\frac{\theta}{2}) + i\cdot\sqrt{r}\sin(\frac{\theta}{2})$, $r>0$, $-\pi<\theta<\pi$.

then $u(r,\theta) = \sqrt{r}\cdot\cos(\frac{\theta}{2})$, $v(r,\theta) = \sqrt{r}\cdot\sin(\frac{\theta}{2})$.

$u_r = \frac{1}{2\sqrt{r}}\cdot\cos(\frac{\theta}{2})$. $\quad v_r = \frac{1}{2\sqrt{r}}\cdot\sin(\frac{\theta}{2})$

$u_\theta = -\frac{1}{2}\sqrt{r}\sin(\frac{\theta}{2})$. $\quad v_\theta = \frac{1}{2}\sqrt{r}\cdot\cos(\frac{\theta}{2})$.

For any $r_0>0$, and $-\pi<\theta_0<\pi$, consider $z_0 = r_0(\cos\theta_0 + i\sin\theta_0)$

Since ① partial derivatives exists in a neighborhood of z_0

② Cauchy-Riemann satisfied at z_0

③ partial derivatives are continuous at z_0

then f is complex differentiable at z_0, and z_0 is arbitrary for $r_0>0$, $-\pi<\theta_0<\pi$, thus f is analytic in the domain $r>0$, $-\pi<\theta<\pi$.

② Since $z = r(\cos\theta + i\sin\theta)$, $f(z) = \sqrt{r}(\cos(\frac{\theta}{2}) + i\sin(\frac{\theta}{2}))$

then $f^2(z) = \left[\sqrt{r}(\cos(\frac{\theta}{2}) + i\sin(\frac{\theta}{2}))\right]^2$

$= r(\cos\theta + i\sin\theta) = z$.

Thus, $f(z) = \sqrt{z}$, and $f(z) = -\sqrt{z}$ is not satisfied.

$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

$$= \lim_{h\to 0} \frac{f^2(z+h) - f^2(z)}{h(f(z+h) + f(z))}$$

$$= \lim_{h\to 0} \frac{z+h-z}{h(f(z+h)+f(z))}$$

$$= \lim_{h\to 0} \frac{1}{f(z+h)+f(z)}$$

$$= \frac{1}{2f(z)}$$

$$= \frac{1}{2\sqrt{z}}$$