MAT 3253 Lecture 20

Example $f(z) = z^3 - 4z + 5z - 2$

$z = 1$ is a zero.

$$z-1 \,\big)\overline{\, z^3 - 4z + 5z - 2 \,}$$

quotient $= z^2 - 3z + 2 = (z-1)(z-2)$

$$f(z) = (z-1)^2 (z-2) .$$

$z = 1$ is a double root

$z = 2$ is a simple root

$$f(z) = \quad + \underset{=0}{a_1}(z-1) + \underset{\neq 0}{a_2}(z-1)^2 + a_3(z-1)^3 + \ldots$$

$$\frac{f(z)}{z-1} = a_1 + a_2(z-1) + \ldots$$

$$\lim_{z \to 1} \frac{f(z)}{z-1} = \lim_{z \to 1} \frac{z^3 - 4z^2 + 5z - 2}{z-1}$$

$$= \lim_{z \to 1} \frac{3z^2 - 8z + 5}{-1} = 0$$

$$\lim_{z \to 1} \frac{f(z)}{(z-1)^2} = \ldots \neq 0 \Rightarrow \text{order of zero at } z=1 \text{ is } 2$$

Example : $f(z) = \dfrac{z}{(z+2)^3}$

$z = 0$ is a zero of order 1

$z = -2$ is a pole of order 3 (triple pole)

$$g(z) = \frac{1}{f(z)} = \frac{(z+2)^3}{z}$$

$g(-2) = 0 \Rightarrow -2$ is a zero

$$\lim_{z \to -2} \frac{g(z)}{z+2} = \lim_{z \to -2} \frac{(z+2)^2}{z} = 0$$

$$\lim_{z \to -2} \frac{g(z)}{(z+2)^2} = \lim_{z \to -2} \frac{z+2}{z} = 0$$

$$\lim_{z \to 2} \frac{g(z)}{(z+2)^3} = \lim_{z \to -2} \frac{1}{z} \neq 0$$

Laurent series $\displaystyle\sum_{k=-\infty}^{\infty} a_k z^k$

$$\sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

$$\underbrace{\sum_{k=0}^{\infty} a_k (z - z_0)^k}_{\text{analytic part}} + \underbrace{\sum_{k=1}^{\infty} \frac{b_k}{(z - z_0)^k}}_{\text{principal part}}$$

Example

$$f(z) = \frac{1}{z(1-z)}$$

(1) Expand at $z=0$

$a_0 + a_1 z + a_2 z^2 + \dots$
$+ b_1/z + \frac{b_2}{z^2} + \dots$

$$\frac{1}{z} \cdot (1 + z + z^2 + z^3 + \dots)$$

$$= \underbrace{\frac{1}{z}}_{\text{principal part}} + 1 + z^2 + z^3 + z^4 + \dots$$

$|z| < 1$

(2) Expand at $z=1$

$a_0 + a_2(z-1) + a_3(z-1)^2$
$+ \frac{b_1}{z-1} + \dots$

$$\frac{1}{z(1-z)} = -\frac{1}{z(z-1)}$$

$$= -\frac{1}{z-1} \cdot \frac{1}{(1+z-1)}$$

$$= -\frac{1}{z-1}\left(1 - (z-1) + (z-1)^2 - (z-1)^3 + \dots\right)$$

$|z-1| < 1$

$$= \underbrace{-\frac{1}{z-1}}_{\text{principal part}} + 1 + (z-1) - (z-1)^2 + \dots$$

(3) Expand at $z=2$

$$\frac{1}{z(1-z)} = \frac{1}{z} + \frac{1}{1-z}$$

$$= \frac{1}{2+z-2} - \frac{1}{1+z-2}$$

$|z-2| < 1$

$$= \frac{1}{2}\,\frac{1}{1+\frac{z-2}{2}} - \frac{1}{1+(z-2)}$$ expand. . . . .

Essential singularity

Example. Compute the Laurent series of $e^{\frac{1}{z}}$

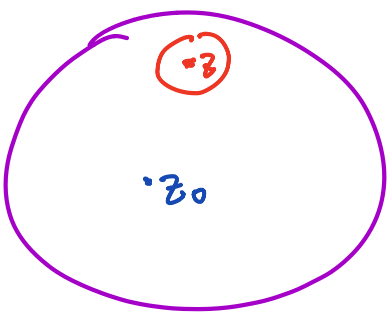
$$e^{z} = 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \frac{z^4}{4!} + \dots$$

$$e^{\frac{1}{z}} = \underbrace{1}_{\text{analytic part}} + \underbrace{\frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3!\,z^3} + \frac{1}{4!\,z^4} + \dots}_{\text{principal part}}$$

Casorati - Weierstrass theorem for essential singularity

If z_0 is an essential singular point of $f(z)$ then the range of f in any disc centered at z_0

$R = \{ f(z) : 0 < |z - z_0| < \varepsilon \}$ is <u>dense</u> in $\mathbb{C}$.

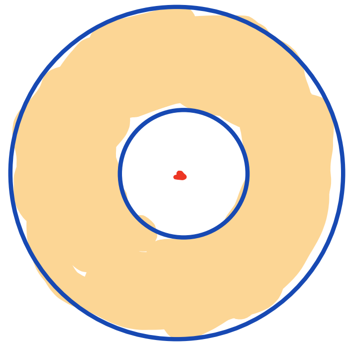


$w \in \mathbb{C}$, $\delta \in \mathbb{R}$

$|f(z) - w| < \delta$

Theorem A Laurent series $\sum_{k=-\infty}^{\infty} a_k z^k$ in general converges in an annulus

$$R_1 < |z| < R_2$$



Proof :

$\sum_{k=0}^{\infty} a_k z^k$ converges

if $|z| < \dfrac{1}{\limsup_k |a_k|^{1/k}}$ R_2

$\sum_{k=1}^{\infty} b_k z^{-k}$ $\boxed{u = \frac{1}{z}}$

$b_k = a_{-k}$

converges if $|u| < \dfrac{1}{\limsup_k |b_k|^{1/k}}$

$|z| > \limsup_k |a_{-k}|^{1/k}$ R_1

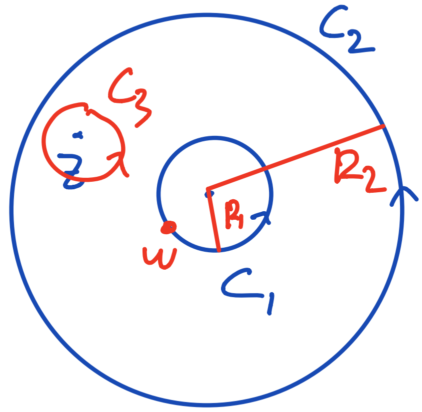
∎

Theorem A function f analytic in an annulus $R_1 < |z| < R_2$ can be expanded as a Laurent series.

$$a_k \neq \frac{f^{(k)}(0)}{k!}$$

$f(0)$, $f^{(1)}(0)$, ... need not be defined

Proof.



$$f(z)\cdot 2\pi i = \int_{C_3} \frac{f(w)}{w-z}\,dw$$

$$= \underbrace{\int_{C_2} \frac{f(w)}{w-z}\,dw}_{\text{represented by Taylor series}} - \int_{C_1} \frac{f(w)}{w-z}\,dw$$

The first integral can be expanded as a Taylor series, as we have shown in Lecture 16.

$$\frac{1}{2\pi i}\int_{C_2} \frac{f(w)}{w-z}\,dw = \sum_{k=0}^{\infty} a_k z^k \quad \text{where} \quad a_k = \frac{1}{2\pi i}\int_{C_2} \frac{f(w)}{w^{k+1}}\,dw$$

For the second integral, write

$$-\frac{1}{w-z} = \frac{1}{z-w} = \frac{1}{z(1-\frac{w}{z})} = \frac{1}{z}\left(1+\frac{w}{z}+\left(\frac{w}{z}\right)^2+\dots\right)$$

converges for $|w| < |z|$

If we can interchange infinite sum and integration, we can expand the second integral as

$$-\int_{C_1} \frac{f(w)}{w-z}\,dw = \sum_{k=1}^{\infty} \underbrace{\left(\int_{C_1} f(w)\,w^{k-1}\,dw\right)}_{\text{coefficient } b_k} \frac{1}{z^k}$$

To make the argument more rigorous, we avoid infinite series by lumping the tail terms as a remainder

$$1 + \frac{w}{z} + \frac{w^2}{z^2} + \dots + \frac{w^{n-1}}{z^{n-1}} + \underbrace{\frac{w^n}{z^n} + \frac{w^{n+1}}{z^{n+1}} + \dots}_{\Downarrow}$$

converges for $|z| > |w|$

$$= 1 + \frac{w}{z} + \frac{w^2}{z^2} + \dots + \frac{w^{n-1}}{z^{n-1}} + \frac{w^n}{z^{n-1}(z-w)}$$

$$\therefore -\int_{C_1} \frac{f(w)}{w-z}\,dw = \sum_{k=1}^{\infty} \left(\int_{C_1} f(w)\, w^{k-1}\, dw \right) z^{-k}$$

$$+ \underbrace{\int_{C_1} \frac{f(w)}{z-w} \frac{w^n}{z^n}\, dw}_{\text{remainder term. Let's call it } A_n}$$

For $w \in C_1$ and z outside C_1, we have

$$|w| \leq r_1 \quad , \quad |z-w| \geq |z-r_1|$$

Furthermore $|f(w)| \leq M$ for all $w \in C_1$ for some M

because $f(w)$ is continuous and C_1 is compact.

ML inequality $\Rightarrow$ $|A_n| \leq \dfrac{M}{|z-r_1|} \dfrac{R_1^n}{|z|^n} \cdot 2\pi r_1$

Since $\dfrac{R_1}{|z|} < 1$, $\dfrac{R_1^n}{|z|^n} \to 0$ as $n \to \infty$.

Therefore $A_n \to 0$ as $n \to \infty$.

$$\therefore -\frac{1}{2\pi i}\int \frac{f(w)}{w-z}\,dw = \sum_{k=1}^{\infty} b_k z^{-k} \text{ where } b_k = \frac{1}{2\pi i}\int_{C_1} f(w)\, w^{k-1}\, dw.$$

$\blacksquare$