Exercise I

Exercise 1. Find an uncountable antichain of $P(\mathbb{N})$.

Let $a = \{a_1, a_2, a_3, \cdots\}$ be a binary sequence, and A the set of all binary sequences. Define a subset N_a of $\mathbb{N}$ by

$$\begin{aligned} 2n &\in N_a \qquad \text{iff} \quad a_n = 1, \\ 2n+1 &\in N_a \qquad \text{iff} \quad a_n = 0. \end{aligned}$$

If two binary sequences $a \neq b$, then there exists $n \in \mathbb{N}$ such that $a_n \neq b_n$. Assume, without loss of generality, that $a_n = 0$ and $b_n = 1$. Then

$$2n \notin N_a, \quad 2n \in N_b, \quad 2n+1 \in N_a, \quad 2n+1 \notin N_b.$$

Hence, $N_a \not\subset N_b$ and $N_b \not\subset N_a$ as long as $a \neq b$. And the set $\{N_a\}_{a\in A}$ is the desired antichain.

Exercise 2. The Monotone Convergence Theorem implies the Archimedean Property.

Proof. Suppose AP is not true, meaning there exits $x \in \mathbb{R}$ such that $n \leq x$ for all $n \in \mathbb{N}$, thus $\mathbb{N}$ is bounded above. Consider the sequences $\{s_n\}$ and $\{t_n\}$

$$s_n = n \qquad \text{and} \qquad t_n = n+1, \qquad n \in \mathbb{N}.$$

Then MCT says that s_n and t_n both converge, and clearly they converge to the same limit say $a \in \mathbb{R}$. The algebraic property of limits – which based only on the definition of limits, not on any completeness property of $\mathbb{R}$ – yields that

$$1 = t_n - s_n \to a - a = 0,$$

which is a contradiction, and it hence completes the proof of MCT implies AP. □

Exercise 3. The Monotone Convergence Theorem implies the Nest Interval Property.

Proof. Let $I_n = [a_n, b_n]$ form a sequence of nested closed intervals, that is

$$I_1 \supset I_2 \supset I_3 \supset \cdots.$$

Then $\{a_n\}_{n=1}^{\infty}$ is a increasing bounded above sequence, since $a_n \le a_{n+1} < b_1$ for each $n \in \mathbb{N}$. Then MCT implies that $\{a_n\}_{n=1}^{\infty}$ converges to a real number x. For any $m \in \mathbb{N}$, we shall show that $a_m \le x \le b_m$.

Let $m \in \mathbb{N}$ be fixed. Note that $a_n \le b_m$ for all $n \in \mathbb{N}$. The order property of limits – which based on the order of real numbers and the definition of limits, not on any completeness property of real numbers – show that $x \le b_m$.

For the other part $x \ge a_m$, we prove it by contradiction. Suppose $x < a_m$, then $x+\epsilon < a_m$, where $\epsilon = (a_m - x)/2$. Moreover, $x+\epsilon < a_n$ for all $n \ge m$ since $\{a_n\}_{n=1}^{\infty}$ is increasing. Now the sequence $\{a_n\}_{n=m}^{\infty}$ converges to x and the order property of limits implies that $x+\epsilon \le x$, which is a contradiction.

Hence $x \in I_m$ for each $m \in \mathbb{N}$, which means that $\bigcap_{m=1}^{\infty} I_m$ contains x and is therefore not empty.

□

Note. For the last part, why the following seemingly obvious "proof" is not valid:

Since $x = \sup\{x_n \mid n \ge 1\}$ from the MCT (an implication in proving MCT), then $x_m \le x$.

Exercise 4. NIP+AP implies AoC (LUBP).

Proof. Assume A is a nonempty bounded above set in $\mathbb{R}$. Let $a \in A$ and b be an upper bound of A. We define a sequence of nested closed intervals $I_n = [a_n, b_n]$ as follows.

Set $a_1 = a$ and $b_1 = b$.

Take $c_1 = \frac{a_1+b_1}{2}$. Then, if $c_1 \in A$, we set $a_2 = c_1$ and $b_2 = b_1$. Otherwise, we set $a_2 = a_1$ and $b_2 = c_1$.

In general, after choosing a_n and b_n, set $c_n = \frac{a_n+b_n}{2}$. Then, if $c_n \in A$, we set $a_{n+1} = c_n$ and $b_{n+1} = b_n$. Otherwise, we set $a_{n+1} = a_n$ and $b_{n+1} = c_n$.

It is readily seen that

$$I_1 \supset I_2 \supset I_3 \supset I_4 \supset \cdots,$$

and NIP says that there is $x \in \mathbb{R}$ such that

$$x \in \bigcap_{n=1}^{\infty} I_n.$$

We shall show that $x = \sup A$.

We first show that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = x.$$

Note that $b_n - a_n = \frac{b-a}{2^n}$. For any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$b_n - a_n = \frac{b-a}{2^n} < \epsilon, \qquad \forall n \ge N.$$

Thus

$$|x - a_n| \leq b_n - a_n < \epsilon, \qquad \forall n \geq N.$$

and

$$|x - b_n| \leq b_n - a_n < \epsilon, \qquad \forall n \geq N.$$

Thus,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = x.$$

Note each b_n is an upper bounds for A. Given any $\alpha \in A$, we have $\alpha < b_n$. Taking $n \to \infty$, the order property of limits yields $\alpha \leq x$, and hence x is an upper bound of A. On the other hand, given any upper bound β of A, we have $a_n \leq \beta$ and the order property of limits yields $x \leq \beta$. Hence $x = \sup A$.

□

Note. Where do we use the AP in the above argument?

Exercise 5. Assume $\{x_n\}$ is a bounded sequence. Show that

$$\limsup_{n\to\infty} x_n = \sup E,$$

where

$$E = \{x \in \mathbb{R} \,|\, \text{there exists a subsequence } \{x_{n_k}\} \text{ of } \{x_n\} \text{ converges to } x\}$$

Proof. (i) Given $x \in E$ and assume that $\{x_{n_k}\} \to x$. Note that $n_k \geq k$, thus

$$x_{n_k} \leq \sup\{x_n \,|\, n \geq k\}$$

Taking $k \to \infty$ and by the order property of limits, we have

$$x \leq \lim_{k\to\infty} \Big(\sup\{x_n \,|\, n \geq k\} \Big) = \limsup_{n\to\infty} x_n,$$

which means that $\limsup_{n\to\infty} x_n$ is an upper bound of E.

(ii) Assume $b \in \mathbb{R}$ is an upper bound of E. We claim that for every $\epsilon > 0$ there is $N \in \mathbb{N}$ such that $x_n \leq b + \epsilon$ for all $n \geq N$. Suppose not true, there are infinitely many $n \in \mathbb{N}$ such that $x_n > b + \epsilon$. By the Bolzano–Weierstrass theorem and the order property, there is a subsequence of $\{x_n\}$ converges to a limit x with $x \geq b + \epsilon$, Thus $x \in E$ and b is not an upper bound of E, which is a contraction with the hypothesis that b is an upper bound of E.

Hence, whenever $m \geq N$, we have$\{x_n\}_{n=m}^{\infty} \leq b + \epsilon$, and so that

$$\sup\{x_n \,|\, n \geq m\} \leq b + \epsilon \qquad \forall m \geq N.$$

Taking $m \to \infty$ and by the order property of limits, we have

$$\limsup_{n\to\infty} x_n = \lim_{m\to\infty} \Big(\sup\{x_n \,|\, n \geq m\} \Big) \leq b + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we have

$$\limsup_{n\to\infty} x_n \leq b.$$

Therefore,

$$\limsup_{n\to\infty} x_n = \sup E.$$

$\square$

Exercise 6 (Construction of Real Numbers)**.** In this course, we assume all the properties about $\mathbb{N}$ and $\mathbb{Q}$ are known, and the set of real numbers $\mathbb{R}$ is the completion of $\mathbb{Q}$: The real numbers were defined simply as an extension of the rational numbers in which bounded sets have least upper bounds (LUBP serves as the Axiom of Completeness), but no attempt was made to demonstrate that such an extension is actually possible.

There are several ways for constructing $\mathbb{R}$ from $\mathbb{Q}$. In Section 8.6 of the textbook, the Dedekind's cut approach is introduced. Read this section and finish all the exercises there.