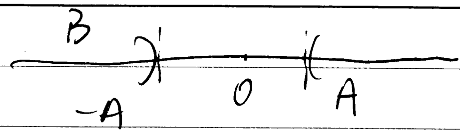
MAT 2006 Tutorial 2. 9.21.

1. (⟸) LUBP. Proof.



Let $B := -A := \{-x \mid x \in A\}$, $A \neq \emptyset \Rightarrow B \neq \emptyset$.

A is bdd below $\Rightarrow \exists\, l \in \mathbb{R}$ s.t. $\forall x \in A$, $x \geq l$.

$\Rightarrow \forall y \in B$, $y = -x$ for some $x \in A$.

$\Rightarrow y = -x \leq -l$.

$\Rightarrow -l$ is an U.B of B $\Rightarrow$ B is bdd above.

LUBP $\Rightarrow$ $\underset{=:s}{\sup B}$ exists.

W.T.S. $-s = \inf A$.

① $\forall x \in A$, $-x \in B$ $\Rightarrow -x \leq s \Rightarrow x \geq -s$.

$\Rightarrow -s$ is a LB of A.

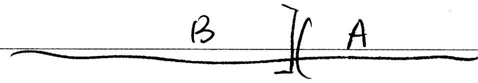
② $\forall$ LB b of A.

$(-b)$ is an U.B of B. $\Rightarrow -b \geq s$. $\Rightarrow -s \geq b$

$\Rightarrow -s$ is the greatest L.B of A.

✓ Alternative Method:

Take $B := \{x \in \mathbb{R} \mid x \text{ is a L.B of } A\}$.



prove $\sup B = \inf A$.

①

2. Proof (a) ((☺) ⇒ A.P) $\forall x > 0$ take $\varepsilon = 1$, by (☺), $\exists M \in \mathbb{N}$ s.t. $M\varepsilon = M > x$.

$\forall x \leq 0$, take $n = 1$, then $n > x$.

② (A.P ⇒ (☺)) $\forall x > 0$, $\forall \varepsilon > 0$. by A.P.

$k-1 \leq \frac{x}{h} < k$. $\exists N \in \mathbb{N}$ s.t. $N > \frac{x}{\varepsilon}$ so $N\varepsilon > x$.

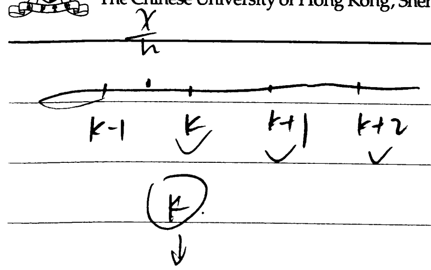
(b) ((☹) ⇒ A.P). $\forall x \in \mathbb{R}$, by (☹), take $h = 1$.

$\exists$ unique $k \in \mathbb{Z}$ s.t. $kh = k > x \geq (k-1)h = k-1$

Take $N = \max\{1, k\}$ $\Rightarrow N > x$

(A.P ⇒ (☹)). W.T.S. $\exists$ unique k s.t. $k-1 \leq \frac{x}{h} < k$.

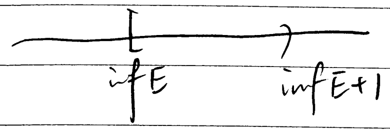
$\{n \in \mathbb{Z} \mid n > \frac{x}{h}\}$



By A.P. $\exists n \in \mathbb{N}$ s.t. $N > \frac{x}{h}$. So $h \in E$, $E \neq \emptyset$

E is bdd below by $\frac{x}{h}$. $\Rightarrow$ $\inf E$ exists!

By def of inf. $\exists k \in E$. s.t. $\inf E \leq k < \inf E + 1$.



$\Rightarrow k-1 < \inf E$. $\Rightarrow k-1 \notin E$.

$\forall m \leq k-1$. $m \in \mathbb{Z}$. $m < \inf E$. $\Rightarrow m \notin E$.

So if $n \in E$. then $n \geq k \in E$. $\Rightarrow k = \min E$.

$k \in E \Rightarrow k > \frac{x}{h} \Rightarrow k-1 \in k-1 \notin E$

Uniqueness: If $k_1 > k$, and $k_1 \in \mathbb{Z}$. then $k_1 \geq k+1$

then $k_1 - 1 \geq k > \frac{x}{h}$.

✓ k_1 does not satisfy condition!

If $k_2 < k$, and $k_2 \in \mathbb{Z}$. then $k_2 \leq k-1$

then $k_2 \leq k-1 \leq \frac{x}{h}$.

✓ k_2 does not — !

3. proof. $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n \in \mathbb{N}} A_n = \{x \in \mathbb{R} \mid x \in A_n \text{ for some } n \in \mathbb{N}\}$

$\bigcap_{n=1}^{\infty} A_n = \bigcap_{n \in \mathbb{N}} A_n = \{x \in \mathbb{R} \mid x \in A_n \text{ for } \forall n \in \mathbb{N}\}$.

W.T.S $\forall x \in \mathbb{R}$ $x \notin \bigcap_{n=1}^{\infty} (0, \frac{1}{n})$.

① $\forall x \leq 0$. $x \notin (0,1)$. $\Rightarrow x \notin \bigcap_{n=1}^{\infty} (0, \frac{1}{n})$

② $\forall x > 0$. by A.P $\exists n \in \mathbb{N}$. s.t. $\frac{1}{N} < x$.

So $x \notin (0, \frac{1}{N})$

$\Rightarrow x \notin \bigcap_{n=1}^{\infty} (0, \frac{1}{n})$

4. proof. we say $A \sim B$ if A & B have the same card..

(i) Take $f: A \to A$. where $f(x) = x$. $\Rightarrow A \sim A$.

(ii) $A \sim B \Rightarrow \exists f: A \to B$ bijective.

$\exists f^{-1}: B \to A$. $\Rightarrow B \sim A$.

$f(g(y)) = f(x) = y.$

$g(f(x)) = g(y) = x \quad \Rightarrow g = f^{-1}.$

(iii). $A \sim B \Rightarrow \exists f: A \to B$. bijective.

$B \sim C \Rightarrow \exists f: B \to C$. bijective.

$g \circ f: A \to C$. is bijective.

① 1-1: $x_1, x_2 \in A$. $x_1 \neq x_2$.

f is 1-1, $\Rightarrow f(x_1) \neq f(x_2)$

g is 1-1 $\Rightarrow g(f(x_1)) \neq g(f(x_2))$

$\Rightarrow g \circ f$ is 1-1.

② onto: $\forall z \in C$.

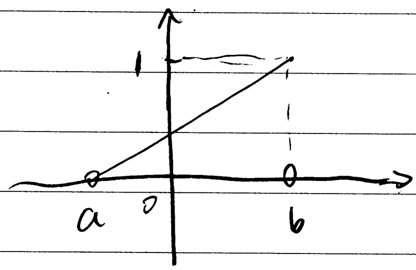
g is onto $\Rightarrow \exists y \in B$. s.t. $g(y) = z$.

f is onto $\Rightarrow \exists x \in A$. s.t. $f(x) = y$.

$\Rightarrow g(f(x)) = z$. $\forall z \in C$.

$\Rightarrow g \circ f$ is onto.

5. proof. (i).



$f(x) = \frac{1}{b-a}(x-a)$ is bijective.
from (a,b) to $(0,1)$.

(ii). $f(x) = \tan x$. is bijective from $(-\frac{\pi}{2}, \frac{\pi}{2})$ to $\mathbb{R}$.

$\Rightarrow (-\frac{\pi}{2}, \frac{\pi}{2}) \sim \mathbb{R}$

(i) $\Rightarrow (-\frac{\pi}{2}, \frac{\pi}{2}) \sim (0,1)$

(i) $\Rightarrow (a,b) \sim (0,1)$.

ctb. $\Rightarrow \{a,b\} \cup A$ ctb.

$\Rightarrow (a,b) \sim (-\frac{\pi}{2}, \frac{\pi}{2}) \sim \mathbb{R}$

$\Rightarrow (a,b) \sim \mathbb{R}$.

$a,b\} \cup A \sim A$.

(iii). $\mathbb{Q} \cap (0,1)$ ctb. $\Rightarrow \mathbb{Q} \cap (0,1) = \{r_1, r_2, \cdots\}$.

$g: \mathbb{Q} \cap (0,1) \to \mathbb{Q} \cap [0,1]$ by

$$g: \begin{array}{cccc} r_1 & r_2 & r_3 & r_4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 0 & 1 & r_1 & r_2 \end{array} \cdots$$

$h: (0,1) \to [0,1]$ by $h(x) = \begin{cases} g(x) & \text{if } x \in \mathbb{Q} \cap (0,1) \\ x, & \text{if } x \in \mathbb{Q}^c \cap (0,1). \end{cases}$

$\Rightarrow [0,1] \sim (0,1)$.

✓ More generally: take $\forall$ ctb set $\{a_1, a_2, \dots\}$.

$$\begin{array}{cccccc} a_1 & a_2 & a_3 & a_4 & \cdots \\ \Downarrow & \Downarrow & \Downarrow & \Downarrow & \\ 0 & 1 & a_1 & a_2 & \end{array}$$

✓ Schroder Bernstein Th'm. $\nearrow$ card $A \le$ card B

$\left.\begin{array}{l} \exists\ 1\text{-}1: f: A \to B \\ \exists\ 1\text{-}1: g: B \to A \end{array}\right\} \Rightarrow A \sim B$

$\searrow$ card $A \ge$ card B.

Take $f: [0,1] \to (0,1)$ $\quad f(x) = \frac{1}{2}x + \frac{1}{4}$.

$g: (0,1) \to [0,1]$ $\quad g(x) = x$.