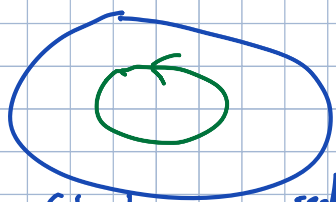
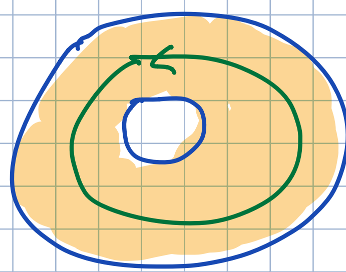
MAT 3253 Lecture 16

Simply connected region



simply connected

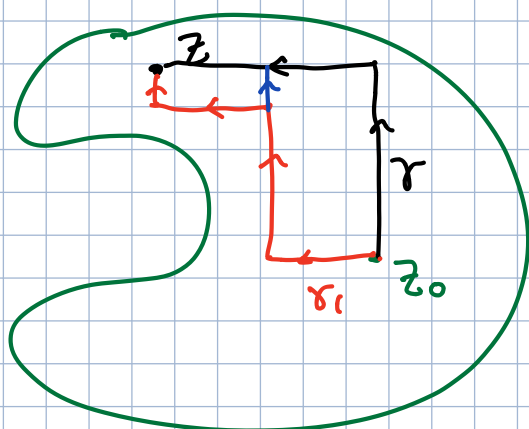


multiply connected

<u>Theorem</u> Suppose D is simply connected region and f is analytic in D. Then

$$\int_C f(z)\,dz = 0$$

for all closed and smooth curve C.

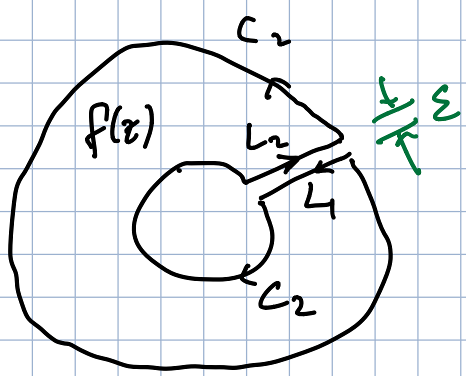


* $F(z) \hat{=} \int_\gamma f(z)\,dz$

* $F'(z) = f(z)$

* $\int_C f(z)\,dz = 0$

Multiply connected region.



$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

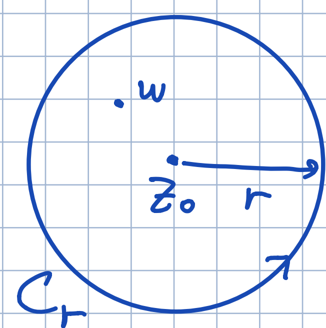
$$\int_{C_1 + (-C_2)} f(z)\,dz = 0$$

$$\int_{C_1} + \int_{L_1} + \int_{-C_2} + \int_{L_2} = 0$$

$\varepsilon \to 0$ $\qquad \int_{L_1} + \int_{L_2} \to 0$

Cauchy integral formula

f is analytic inside C.



$$f(w) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-w}\,dz$$

$\forall\, w \in \operatorname{int}(C)$

Proof

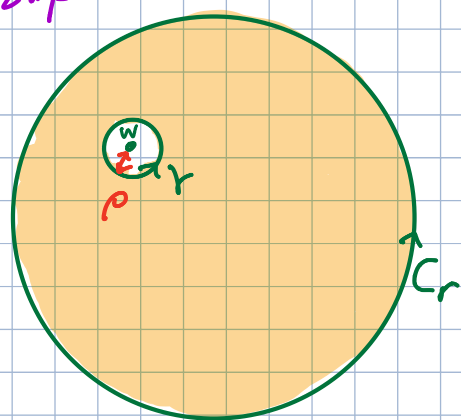
Want to show $\displaystyle\int_{C_r} \frac{f(z)}{z-w}\,dz = 2\pi i\, f(w)$

Write $f(z) = \underbrace{f(w)} + \underbrace{f(z) - f(w)}_{\triangleq\, g(z)}$ $\quad g(w) = 0$

$$\int_{C_r} \frac{f(w)}{z-w}\,dz = f(w)\int_{C_r} \frac{1}{z-w}\,dz = f(w)\int_{\gamma} \frac{1}{z-w}\,dz$$
$$= f(w)\cdot 2\pi i$$

$$\boxed{\int_{|z|=r} \frac{1}{z}\,dz = 2\pi i}$$

$$\left|\int_{C_r} \frac{g(z)}{z-w}\,dz\right| = \left|\int_{\gamma} \frac{g(z)}{z-w}\,dz\right| \leq \frac{m}{\rho}\cdot 2\pi\rho$$
$$= 2\pi m$$



$\because$ g is continuous and $g(w)=0$

Let $m = \sup_{z\in\gamma} |g(z)|$

$= \max_{z\in\gamma} |g(z)|$

m can be arbitrarily small

$$\therefore \int_{C_r} \frac{g(z)}{z-w}\,dz = 0$$

$$\int_{C_r} \frac{f(z)}{z-w}\,dw = \int_{C_r} \frac{f(w)}{z-w}\,dw + \int_{C_r} \frac{f(z)-f(w)}{z-w}\,dz$$
$$= 2\pi i\, f(w) + 0 \qquad \boxtimes$$

Example Calculate $\int_C \frac{1}{z^2-1}\,dz$

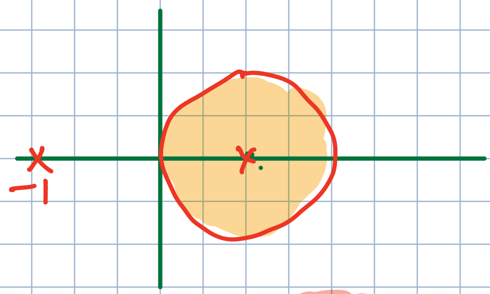
for C (i) $|z-1| = 1$

(ii) $|z+1| = 1$

(iii) $|z| = 2$

$$\frac{1}{z^2-1} = \frac{1}{(z-1)(z+1)} \qquad \frac{f(z)}{z-w}$$

(i)

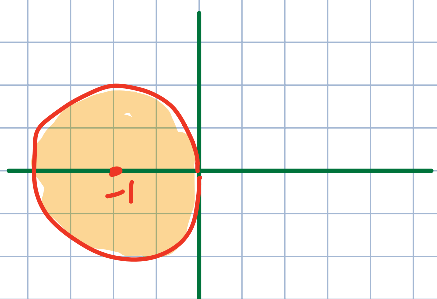


$$\frac{1}{(z-1)(z+1)} = \frac{\frac{1}{z+1}}{z-1}$$

$$\int_{|z-1|=1} \frac{\frac{1}{z+1}}{z-1}\, dz = 2\pi i \left(\frac{1}{z+1}\right)\Bigg|_{z=1} = \pi i$$

$$w = 1$$

(ii)



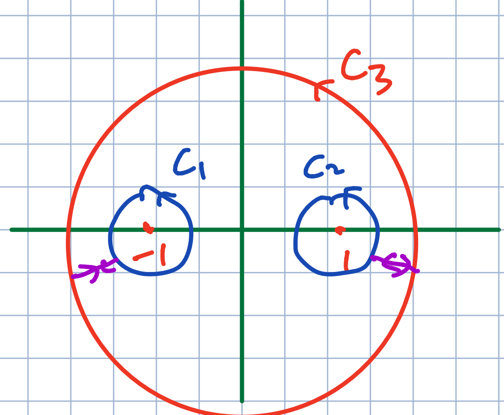
$$\int_{|z+1|=1} \frac{\frac{1}{z-1}}{z+1}\, dz$$

$(w=-1)$

$$= 2\pi i \left(\frac{1}{z-1}\right)\Bigg|_{z=-1}$$

$$= -\pi i$$

(iii)



$$\int_{C_3} f(z)\, dz$$

$$= \int_{C_1} f(z)\, dz + \int_{C_2} f(z)\, dz$$

$$= \pi i + (-\pi i)$$

$$= 0$$

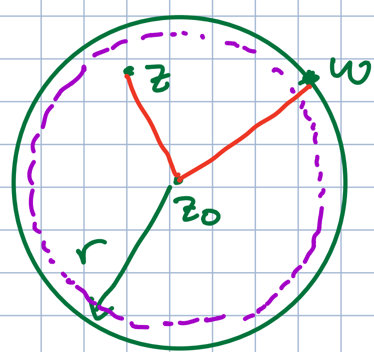
Theorem Suppose f is analytic in an open disk $D(z_0 : r)$. Then

$f(z)$ has Taylor series expansion $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$

where $$a_n = \frac{1}{2\pi i} \oint_C \frac{f(w)}{(w-z_0)^{n+1}} dw$$

$$C : |z - z_0| = r.$$

Proof $$f(z) = \frac{1}{2\pi i} \int \frac{f(w)}{w-z} dw$$



$$\frac{1}{w-z} = \frac{1}{w-z_0 - (z-z_0)}$$

$$= \frac{1}{w-z_0}\left(\frac{1}{1 - \frac{z-z_0}{w-z_0}}\right)$$

$$= \frac{1}{w-z_0} + \frac{z-z_0}{(w-z_0)^2} + \frac{(z-z_0)^2}{(w-z_0)^3} + \dots$$

converges uniformly when $|z - z_0| < \rho < r$

Proof idea

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z_0} dw + \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z_0)^2} dw\,(z-z_0)$$

$$+ \vdots + \frac{1}{2\pi i}\int_C \frac{f(w)\,dw}{(w-z_0)^{n+1}} (z-z_0)^n + \dots$$

$$\frac{1}{w-z} = \frac{1}{w-z_0} + \frac{z-z_0}{(w-z_0)^2} + \dots + \frac{(z-z_0)^n}{(w-z_0)^{n+1}} + \text{remainder term}$$

remainder term $= \dfrac{1}{w-z}\,\dfrac{(z-z_0)^{n+1}}{(w-z_0)^{n+1}}$

$$\frac{1}{2\pi i}\int_C \frac{f(w)}{w-z_0}\,dw + \cdots + \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z_0)^{n+1}}\,dw\,(z-z_0)^{n+1}$$

$$(*)\quad \boxed{+\frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}\cdot\frac{(z-z_0)^{n+1}}{(w-z_0)^{n+1}}\,dw}$$

f is continuous

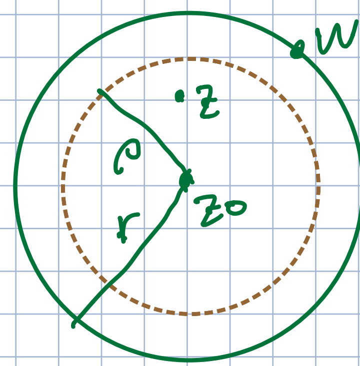
$\exists M \quad f(z) \le M$

$\forall z$ in C

$|z-z_0| < \rho$

$|w-z| \geq \rho - |z-z_0|$

$|w-z_0| = r$



$$|*| \leq \frac{1}{2\pi}\,\frac{M}{\rho-|z-z_0|}\cdot\frac{\rho^{n+1}}{r^{n+1}}\cdot 2\pi r$$

$$= \frac{M\rho}{\rho-|z-z_0|}\cdot\left(\frac{\rho}{r}\right)^n \qquad \rho < r$$

$n \to \infty \quad |*| \to 0$

$\blacksquare$