STA 4001 Homework 6 118010350

1. (a) Since $X_{n+1} = \begin{cases} (3-X_n)^+ + 15 - D_{n+1}), & \text{for } X_n = 0,1,2,3,4 \\ X_n - D_{n+1}, & \text{for } X_n \geq 5. \end{cases}$

for $X_n = 0$, $D_{n+1} = 0$, then $X_{n+1} = 8$, which is the max inventory.

Thus, the state space is $\{0,1,2,3,4,5,6,7,8\}$.

The transition matrix is given by

$$P = \begin{bmatrix} 0 & 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 \\ 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 & 0 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 & 0 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 & 0 \\ 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 \\ 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 \\ 0 & 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \end{bmatrix}$$

Let the stationary distribution be $\pi = [\pi_0, \pi_1, \cdots \pi_8]$.

Solve for the equation $\pi = \pi P$. then

$\pi_0 = 1/6 \cdot \pi_3 + 1/6 \cdot \pi_4 + 1/6 \cdot \pi_5$

$\pi_1 = 1/6 \cdot \pi_2 + 1/6 \cdot \pi_3 + 1/6 \cdot \pi_4 + 1/6 \cdot \pi_5 + 1/6 \pi_6$

$\pi_2 = 1/6 \cdot \pi_1 + 1/6 \cdot \pi_2 + 1/6 \cdot \pi_3 + 1/6 \cdot \pi_4 + 1/6 \cdot \pi_5 + 1/6 \cdot \pi_6 + 1/6 \cdot \pi_7$

$\pi_3 = 1/6 \cdot \pi_0 + 1/6 \cdot \pi_2 + \cdots + 1/6 \cdot \pi_7 + 1/6 \cdot \pi_8$.

$\pi_4 = 1/6 \cdot \pi_0 + 1/6 \cdot \pi_2 + \cdots + 1/6 \cdot \pi_7 + 1/6 \cdot \pi_8$.

$\pi_5 = 1/6 \cdot \pi_0 + 1/6 \cdot \pi_2 + \cdots + 1/6 \cdot \pi_7 + 1/6 \cdot \pi_8$.

$\pi_6 = 1/6 \pi_0 + 1/6 \cdot \pi_1 + 1/6 \pi_2 + 1/6 \pi_6 + 1/6 \cdot \pi_7 + 1/6 \cdot \pi_8$.

$\pi_7 = 1/6 \cdot \pi_0 + 1/6 \cdot \pi_1 + 1/6 \cdot \pi_7 + 1/6 \cdot \pi_8$.

$\pi_8 = 1/6\,\pi_0 + 1/6\,\pi_8$.

and $\pi_0 + \pi_1 + \cdots + \pi_8 = 1$. Thus, the stationary distribution is given by $\pi = [0.0833, 0.1222, 0.1500, 0.1667, 0.1667, 0.1667, 0.0833, 0.0444, 0.0167]$

(b) Let $f(x)$ be the expected profit for the next week, given $X_n = x$.

Since $100 \cdot E(D_{n+1}) = 100 \cdot \frac{1}{6} \cdot (0+1+2+3+4+5) = 250$

and $E(\max(5,x) - D_{n+1}) = \max(5,x) - 2.5$.

then $f(x) = -15(3-x)^+ - 35(5-x)^+ - 10\cdot(\max(5,x) - 2.5) + 250$.

That is,

$$f(x) = \begin{cases} 50x + 5, & \text{for } x = 0,1,2,3. \\ 35x + 50, & \text{for } x = 4,5. \\ -10x + 275, & \text{for } x = 6,7,8. \end{cases}$$

Then the long-run average profit is given by $\sum_{k=0}^{8} \pi_k f(k)$.

where $f(x=0) = 5$, $f(x=1) = 55$, $f(x=2) = 150$, $f(x=3) = 200$.

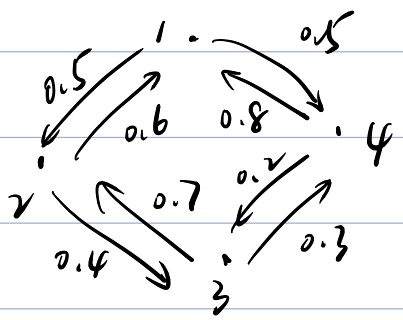
$f(x=4) = 190$, $f(x=5) = 225$.

$f(x=6) = 215$, $f(x=7) = 205$, $f(x=8) = 195$.

Thus, $\sum_{k=0}^{8} \pi_k f(k) = 148.17$, that is the long-run weekly profit of the store is 148.17.

2. (a) Yes, the Markov chain is periodic.

Suppose the state space is $S = \{1,2,3,4\}$, then the state diagram is given by



Then $d(1) = d(2) = d(3) = d(4) = 2$.

(b) Yes. Since $\pi_1+\pi_2+\pi_3+\pi_4 = 33/96 + 27/96 + 15/96 + 21/96 = 1$.

and

$$P = \begin{bmatrix} 0 & 0.5 & 0 & 0.5 \\ 0.6 & 0 & 0.4 & 0 \\ 0 & 0.7 & 0 & 0.3 \\ 0.8 & 0 & 0.2 & 0 \end{bmatrix}$$

then $\pi_1 = 0.6\cdot\pi_2 + 0.8\cdot\pi_4 = 0.6\cdot(27/96) + 0.8\cdot(21/96) = 33/96$.

$\pi_2 = 0.5\pi_1 + 0.7\pi_3 = 0.5\cdot(33/96) + 0.7\cdot(15/96) = 27/96$.

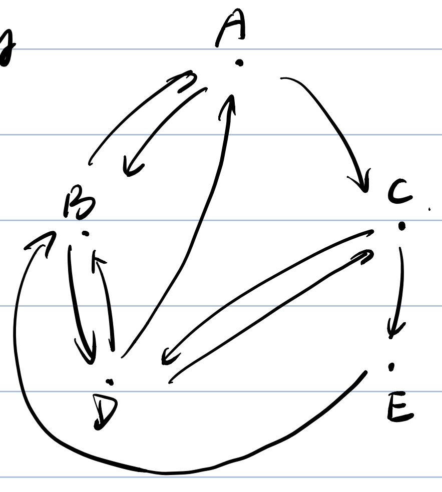
$\pi_3 = 0.4\pi_2 + 0.2\pi_4 = 0.4\cdot(27/96) + 0.2(21/96) = 15/96$.

$\pi_4 = 0.5\pi_1 + 0.3\pi_3 = 0.5(33/96) + 0.3(27/96) = 21/96$.

Thus, $\pi = \pi P$. the π is a valid stationary distribution.

(c). $P_{11}^{100} \neq \pi_1$, $P_{11}^{101} \neq \pi_1$. Since the Markov chain is periodic with period $d=2$, then $\pi_1 = \frac{1}{2}\cdot(P_{11}^{100} + P_{11}^{101})$

3. The state space is $S = \{A, B, C, D, E\}$, then the state diagram is given by



Then the transition matrix is given by

$$P = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Suppose the stationary distribution is $\pi = [\pi_A, \pi_B, \pi_C, \pi_D, \pi_E]$.

then solve for $\pi = \pi P$.

$$\pi_A = 1/2 \cdot \pi_B + 1/3 \cdot \pi_D$$

$$\pi_B = 1/2 \cdot \pi_A + 1/3 \cdot \pi_D + \pi_E$$

$$\pi_C = 1/2 \cdot \pi_A + 1/3 \cdot \pi_D$$

$$\pi_D = 1/2 \cdot \pi_B + 1/2 \cdot \pi_C$$

$$\pi_E = 1/2 \cdot \pi_C.$$

and $\pi_A + \pi_B + \pi_C + \pi_D + \pi_E = 1$, thus the stationary distribution is $\pi = [0.215385, 0.276923, 0.184615, 0.230769, 0.092308]$

Therefore, the PageRank of these five pages is B, D, A, C, E.