Stochastic Processes

Lecture 10: Positive recurrence, Decomposition of state space, Limiting behavior, Period

Hailun Zhang@SDS of CUHK-Shenzhen

March 3, 2021

Two Examples

One dimensional symmetric random walk

Reflected random walks

Positive recurrence criterion

- Let $N_i(n) = \sum_{k=1}^{n} 1_{\{X_k=i\}}$ be the number of times visiting state i in $[1, n]$. Then

$$\mathbb{E}_i(N_i(n)) = \sum_{k=1}^{n} \mathbb{E}_i 1_{\{X_k=i\}} = \sum_{k=1}^{n} \mathbb{P}_i\{X_k = i\} = \sum_{k=1}^{n} P_{ii}^k.$$

THEOREM

State i is positive recurrent if and only if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} P_{ii}^k > 0.$$

- Proof.

Comparison with recurrence criterion

- Recall that state i is recurrent iff

$$\sum_{k=1}^{\infty} P_{ii}^{k} = \infty.$$

- State i is positive recurrent iff

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} P_{ii}^{k} > 0.$$

Solidarity of positive recurrence

Lemma 1

Assume states i and j communicate. State i is p.r. iff state j is p.r.

- Proof: there exist k_1 and k_2 such that $P_{ij}^{k_1} > 0$ and $P_{ji}^{k_2} > 0$.
- Assume j is p.r. Then $\lim_{n\to\infty}(1/n)\sum_{k=1}^{n} P_{jj}^{k} > 0$. Lemma follows from

$$P_{ii}^{k_1+k+k_2} \geq P_{ij}^{k_1} P_{jj}^{k} P_{ji}^{k_2},$$

$$\frac{1}{n}\sum_{k=1}^{n+k_1+k_2} P_{ii}^{k} = \frac{1}{n}\sum_{k=1}^{n} P_{ii}^{k_1+k+k_2} + \frac{1}{n}\sum_{k=1}^{k_1+k_2} P_{ii}^{k} > 0$$

  when n is large enough.
- The proof for solidarity of recurrence is left as exercise.

Limiting behavior of transition matrix P

- Assume that the DTMC is irreducible.
- If it is positive recurrent, for every pair of states $i, j \in S$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (P^k)_{ji} = \frac{1}{\mathbb{E}_i(T_i)} > 0.$$

  Namely,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} P^k = P^{(\infty)},$$

  where $P^{(\infty)}_{ij} = \pi_j = 1/\mathbb{E}_j(T_j)$.
- If it is not positive recurrent, for every pair of states

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (P^k)_{ji} = 0.$$

Communicating classes

Definition

(a) A set $C \subset S$ is said to be a communicating class if i, j communicate for any $i, j \in C$ and i, j does not communicate if $i \in C$ and $j \notin C$.

(b) A communicating class is said to be *closed* if $i \in C$ and $i \to j$ imply $j \in C$.

Theorem

Let C be a communicating class. Then either all states in C are transient or all are recurrent.

Theorem

Every recurrent class is closed.

Decomposition of states

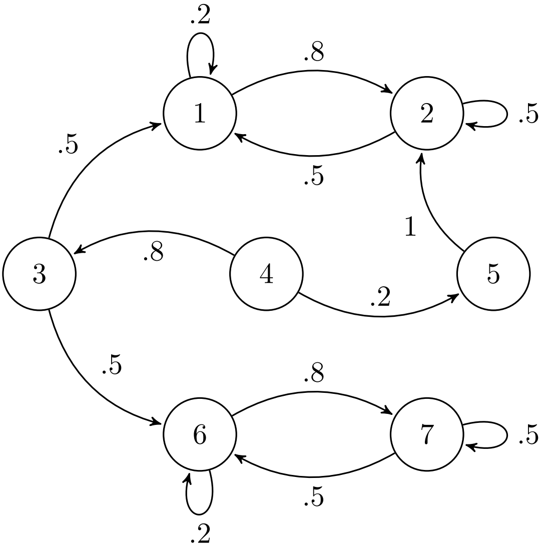
- The state space

$$S = T \cup C_1 \cup C_2 \cup \ldots,$$

where C_i is a closed, communicating recurrent class, and T the set of transient states.
- For a finite state DTMC, there exists at least one (closed) recurrent class.
- Counter example when S is infinite.

A reducible DTMC

Consider the following DTMC.



Limiting behavior

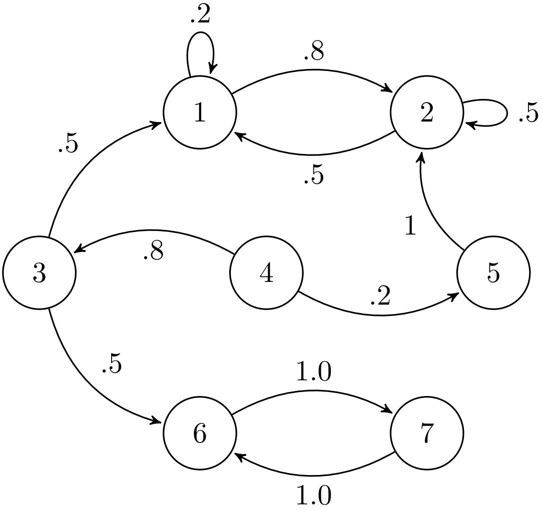
- compute $\lim_{n\to\infty} P^n$.

$$\begin{pmatrix} 5/13 & 8/13 & 0 & 0 & 0 & 0 & 0 \\ 5/13 & 8/13 & 0 & 0 & 0 & 0 & 0 \\ (1/2)(5/13) & (1/2)(8/13) & 0 & 0 & 0 & (1/2)(5/13) & (1/2)(8/13) \\ (.6)(5/13) & (.6)(8/13) & 0 & 0 & 0 & (.4)(5/13) & (.4)(8/13) \\ 5/13 & 8/13 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5/13 & 8/13 \\ 0 & 0 & 0 & 0 & 0 & 5/13 & 8/13 \end{pmatrix}$$

- $S = T \cup C_1 \cup C_2 = \{3,4,5\} \cup \{1,2\} \cup \{6,7\}$
- When computing rows 1, 2, you can just forget about states except for 1 and 2 because there is no arrow going out. Same for rows 6, 7.

Another reducible DTMC

Consider the following DTMC.



Limiting distribution?

- $\lim_{n\to\infty} P^n$ does not exist. $\lim_{n\to\infty}(P^n + P^{n+1})/2$ exists.

$$\begin{pmatrix} 5/13 & 8/13 & 0 & 0 & 0 & 0 & 0 \\ 5/13 & 8/13 & 0 & 0 & 0 & 0 & 0 \\ (1/2)(5/13) & (1/2)(8/13) & 0 & 0 & 0 & (1/2)(.5) & (1/2)(.5) \\ (.6)(5/13) & (.6)(8/13) & 0 & 0 & 0 & (.4)(.5) & (.4)(.5) \\ 5/13 & 8/13 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & .5 & .5 \\ 0 & 0 & 0 & 0 & 0 & .5 & .5 \end{pmatrix}$$

Periodicity

Definition

The *period* of state i of a DTMC is $d(i) = \gcd\{n : P_{ii}^n > 0\}$.

Theorem (Solidarity property)

If state i and j communicate, then $d(i) = d(j)$.

- Assume $P_{ij}^{k_1} > 0$ and $P_{ji}^{k_2} > 0$. For $k \geq 0$,

$$P_{ii}^{k+k_1+k_2} \geq P_{ij}^{k_1} P_{jj}^{k} P_{ji}^{k_2}$$

- Take $k = 0$, $P_{ii}^{k_1+k_2} > 0$, which implies $d(i) \mid k_1 + k_2$.
- Whenever $P_{jj}^k > 0$, $P_{ii}^{k+k_1+k_2} > 0$, thus, $d(i) \mid k + k_1 + k_2$, which implies $d(i) \mid k$. Thus, $d(i) \leq d(j)$.

Periodicity and limit

Definition

An irreducible DTMC is *aperiodic* if $d = 1$. Otherwise, it's *periodic*.

Theorem

If an irreducible DTMC is aperiodic, then

$$\lim_{n\to\infty} P^n = P^{(\infty)}$$

exists, where $P_{ij}^{(\infty)} = 1/\mathbb{E}_j(T_j)$. *Therefore, when the DTMC is positive recurrent, every row of the limiting matrix* $P^{(\infty)}$ *is equal to the DTMC's stationary distribution* π.

The Theorem is false if the DTMC is periodic!

Random walks on circles

- R.w. on a circle of three points.
- R.w. on a circle of four points.