Chapter 6. Differentiation *

1 Discussion: Are Derivatives Continuous?

The myriad applications of the derivative function are the topic of much of the calculus sequence, as well as several other upper-level courses in mathematics. None of these applied questions are pursued here in any length, but it should be pointed out that the rigorous underpinnings for differentiation worked out in this chapter are an essential foundation for any applied study. Eventually, as the derivative is subjected to more and more complex manipulations, it becomes crucial to know precisely how differentiation is defined and how it interacts with other mathematical operations.

Although physical applications are not explicitly discussed, we will encounter several questions of a more abstract quality as we develop the theory. Many of these are concerned with the relationship between differentiation and continuity. Are continuous functions always differentiable? If not, how nondifferentiable can a continuous function be? Are differentiable functions continuous? Given that a function f has a derivative at every point in its domain, what can we say about the function f'? Is f' continuous? How accurately can we describe the set of all possible derivatives, or are there no restrictions? Put another way, if we are given an arbitrary function g, is it always possible to find a differentiable function f such that $f' = g$, or are there some properties that g must possess for this to occur? In our study of continuity, we saw that restricting our attention to monotone functions had a significant impact on the answers to questions about sets of discontinuity. What effect, if any, does this same restriction have on our questions about potential sets of nondifferentiable points? Some of these issues are harder to resolve than others, and some remain unanswered in any satisfactory way.

A particularly useful class of examples for this discussion are functions of the form

$$g_n(x) = \begin{cases} x^n \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

*Lecture notes for CUHKSZ course MAT2006: Elementary Real Analysis.

When $n = 0$, we have seen that the oscillations of $\sin(1/x)$ prevent $g_0(x)$ from being continuous at $x = 0$. When $n = 1$, these oscillations are squeezed between $|x|$ and $-|x|$, the result being that g_1 is continuous at $x = 0$. Is $g_1'(0)$ defined? By the definition

$$g_1'(0) = \lim_{x \to 0} \frac{g_1(x)}{x} = \lim_{x \to 0} \sin \frac{1}{x},$$

which, as we now know, does not exist. Thus, g_1 is not differentiable at $x = 0$. On the other hand, the same calculation shows that g_2 is differentiable at zero. In fact, we have

$$g_2'(0) = \lim_{x \to 0} \frac{g_2(x)}{x} = \lim_{x \to 0} x \sin \frac{1}{x} = 0.$$

At points different from zero, we can use the familiar rules of differentiation (soon to be justified) to conclude that g_2 is differentiable everywhere in $\mathbb{R}$ with

$$g_2'(x) = \begin{cases} -\cos(1/x) + 2x\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

But now consider

$$\lim_{x \to 0} g_2'(x)$$

Because the $\cos(1/x)$ term is not preceded by a factor of x, we must conclude that this limit does not exist and that, consequently, the derivative function is not continuous. To summarize, the function $g_2(x)$ is continuous and differentiable everywhere on $\mathbb{R}$, the derivative function g_2' is thus defined everywhere on $\mathbb{R}$, but g_2' has a discontinuity at zero. The conclusion is that derivatives need not, in general, be continuous!

The discontinuity in g_2' is essential, meaning $\lim_{x \to 0} g_2'(x)$ does not exist as a one-sided limit. But, what about a function with a simple jump discontinuity? For example, does there exist a function h such that

$$h'(x) = \begin{cases} -1 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0. \end{cases}$$

A first impression may bring to mind the absolute value function, which has slopes of -1 at points to the left of zero and slopes of 1 to the right. However, the absolute value function is not differentiable at zero. We are seeking a function that is differentiable everywhere, including the point zero, where we are insisting that the slope of the graph be -1. The degree of difficulty of this request should start to become apparent. Without sacrificing differentiability at any point, we are demanding that the slopes jump from -1 to 1 and not attain any value in between.

Although we have seen that continuity is not a required property of derivatives, the intermediate value property will prove a more stubborn quality to ignore.

2 Derivatives and the Intermediate Value Property

2.1 Differentiability

Although the definition would technically make sense for more complicated domains, all of the interesting results about the relationship between a function and its derivative require that the domain of the given function be an interval. Thinking geometrically of the derivative as a rate of change, it should not be too surprising that we would want to confine the independent variable to move about a connected domain.

Definition 1 (Differentiability)**.** Let $g : A \to \mathbb{R}$ be a function defined on an interval A. Given $c \in A$, the *derivative* of g at c is defined by

$$g'(x) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c},$$

provided this limit exists. In this case we say g is *differentiable* at c. If g' exists for all points $c \in A$, we say that g is *differentiable on* A.

Example 2.1. (i) Consider $f(x) = x^n$ where $n \in \mathbb{N}$. For any $c \in \mathbb{R}$, we have

$$f'(x) = \lim_{x \to c} \frac{x^n - c^n}{x - c} = nc^{n-1}.$$

(ii) If $g(x) = |x|$, then attempting to compute the derivative at $c = 0$ produces the limit

$$g'(0) = \lim_{x \to 0} \frac{|x|}{x}$$

which is $+1$ or -1 depending on whether x approaches zero from the right or left. Consequently, this limit does not exist, and we conclude that g is not differentiable at zero.

This is a reminder that continuity of g does not imply that g is necessarily differentiable. On the other hand, if g is differentiable at a point, then it is true that g must be continuous at this point.

Theorem 1. *If $g : A \to \mathbb{R}$ is differentiable at a point $c \in A$, then g is continuous at c as well.*

Proof. We are assuming that

$$g'(c) = \lim_{x \to c} \frac{g(x) - g(c)}{x - c}$$

exists, and we want to prove that $\lim_{x \to c} g(x) = g(c)$. But notice that the Algebraic Limit Theorem for functional limits allows us to write

$$\lim_{x \to c}(g(x) - g(c)) = \lim_{x \to c} \frac{g(x) - g(c)}{x - c} \cdot \lim_{x \to c}(x - c) = g'(c) \cdot 0 = 0.$$

It follows that $\lim_{x \to c} g(x) = g(c)$. □

2.2 Differentiation Rules

The Algebraic Limit Theorem of functional limits led easily to the conclusion that algebraic combinations of continuous functions are continuous. With only slightly more work, we arrive at a similar conclusion for sums, products, and quotients of differentiable functions.

Theorem 2 (Algebraic Differentiability Theorem)**.** *Let f and g be functions defined on an interval A, and assume both are differentiable at some point $c \in A$. Then,*

(i) $(f+g)'(c) = f'(c) + g'(c)$,

(ii) $(kf)'(c) = kf'(c)$, *for all* $k \in \mathbb{R}$,

(iii) $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$,

(iv) $(f/g)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{[g(c)]^2}$ *provided that* $g(c) \neq 0$.

Theorem 3 (Chain Rule)**.** *Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ satisfy $f(A) \subset B$ so that the composition $g \circ f$ is defined. If f is differentiable at $c \in A$ and if g is differentiable at $f(c) \in B$, then $g \circ f$ is differentiable at c with $(g \circ f)'(c) = g'(f(c))f'(c)$.*

2.3 Darboux's Theorem

One conclusion from this chapter's introduction is that although continuity is necessary for the derivative to exist, it is not the case that the derivative function itself will always be continuous. Our specific example was $g_2(x) = x^2 \sin(1/x)$, where we set $g_2(0) = 0$. By tinkering with the exponent of the leading x^2 factor, it is possible to construct examples of differentiable functions with derivatives that are unbounded, or twice-differentiable functions that have discontinuous second derivatives. The underlying principle in all of these examples is that by controlling the size of the oscillations of the original function, we can make the corresponding oscillations of the slopes volatile enough to prevent the existence of the relevant limits.

It is significant that for this class of examples, the discontinuities that arise are never simple jump discontinuities. We are now ready to confirm our earlier suspicions that although derivatives do not in general have to be continuous, they do possess the intermediate value property. This surprising observation is a fairly straightforward corollary to the more obvious observation that differentiable functions attain maximums and minimums only at points where the derivative is equal to zero.

Theorem 4 (Interior Extremum Theorem)**.** *Let f be differentiable on an open interval (a, b). If f attains a maximum value at some point $c \in (a, b)$ (i.e., $f(x) \leq f(c)$ for all $x \in (a, b)$), then $f'(c) = 0$. The same is true if $f(c)$ is a minimum value.*

Proof. Because c is in the open interval (a, b), we can construct two sequences $\{x_n\}$ and $\{y_n\}$, which converge to c and satisfy $x_n < c < y_n$ for all $n \in \mathbb{N}$. The fact that $f(c)$ is a maximum implies that $f(y_n) - f(c) \leq 0$ for all n, and thus

$$f'(c) = \lim_{n\to\infty} \frac{f(y_n) - f(x)}{y_n - c} \leq 0$$

by the Order Limit Theorem. In a similar way,

$$f'(c) = \lim_{n\to\infty} \frac{f(x_n) - f(x)}{x_n - c} \geq 0,$$

and therefore $f'(c) = 0$, as desired. □

The Interior Extremum Theorem is the fundamental fact behind the use of the derivative as a tool for solving applied optimization problems. This idea, discovered and exploited by Pierre de Fermat, is as old as the derivative itself. In a sense, finding maximums and minimums is arguably why Fermat invented his method of finding slopes of tangent lines. It was 200 years later that the French mathematician Gaston Darboux (1842–1917) pointed out that Fermat's method of finding maximums and minimums carries with it the implication that if a derivative function attains two distinct values $f'(a)$ and $f'(b)$, then it must also attain every value in between.

Theorem 5 (Darboux's Theorem)**.** *If f is differentiable on an interval $[a, b]$, and if α satisfies $f'(a) < \alpha < f'(b)$ (or $f'(a) > \alpha > f'(b)$), then there exists a point $c \in (a, b)$ where $f'(c) = \alpha$.*

Proof. We first simplify matters by defining a new function $g(x) = f(x) - \alpha x$ on $[a, b]$. Notice that g is differentiable on $[a, b]$ with $g'(x) = f'(x) - \alpha$. In terms of g, our hypothesis states that $g'(a) < 0 < g'(b)$, and we hope to show that $g'(c) = 0$ for some $c \in (a, b)$.

Now we claim there exists $x \in (a, b)$ such that $g(a) > g(x)$. For otherwise, if $g(a) \leq g(x)$ for all $x \in (a, b)$, then

$$g'(a) = \lim_{x\to a} \frac{g(x) - g(a)}{x - a} \geq 0$$

by the Order Limit Theorem, which is a contradiction. In a similar manner, we can show that there exists $y \in (a, b)$ such that $g(b) > g(y)$. Therefore $g(x)$ attains its minimum value at some interior point $c \in (a, b)$. By the Interior Extremum Theorem, $g'(c) = 0$, and thus $f'(c) = \alpha$ as desired. □

Exercise 1. Exactly one of the following requests is impossible. Decide which it is, and provide examples for the other three. In each case, let's assume the functions are defined on all of $\mathbb{R}$.

(a) Functions f and g not differentiable at zero but where fg is differentiable at zero.

(b) A function f not differentiable at zero and a function g differentiable at zero where fg is differentiable at zero.

(c) A function f not differentiable at zero and a function g differentiable at zero where $f+g$ is differentiable at zero.

(d) A function f differentiable at zero but not differentiable at any other point.

Exercise 2. Let $f_a(x) = \begin{cases} x^a & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases}$

(a) For which values of a is f continuous at zero?

(b) For which values of a is f differentiable at zero? In this case, is the derivative function continuous?

(c) For which values of a is f twice-differentiable?

Exercise 3. Let

$$g_a = \begin{cases} x^a \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Find a particular (potentially noninteger) value for a so that

(a) g_a is differentiable on $\mathbb{R}$ but such that $g'(a)$ is unbounded on $[0,1]$.

(b) g_a is differentiable on $\mathbb{R}$ with g'_a continuous but not differentiable at zero.

(c) g_a is differentiable on $\mathbb{R}$ and g'_a is differentiable on $\mathbb{R}$, but such that g''_a is not continuous at zero.

Exercise 4. Decide whether each conjecture is true or false. Provide an argument for those that are true and a counterexample for each one that is false.

(a) If f' exists on an interval and is not constant, then f' must take on some irrational values.

(b) If f' exists on an open interval and there is some point c where $f'(c) > 0$, then there exists a δ-neighborhood $V_\delta(c)$ around c in which $f'(x) > 0$ for all $x \in V_\delta(c)$.

(c) If f is differentiable on an interval containing zero and if $\lim_{x\to 0} f'(x) = L$, then it must be that $L = f'(0)$.

Exercise 5. Recall that a function $f : (a,b) \to \mathbb{R}$ is increasing on (a,b) if $f(x) \leq f(y)$ whenever $x < y$ in (a,b). A familiar mantra from calculus is that a differentiable function is increasing if its derivative is positive, but this statement requires some sharpening in order to be completely accurate.

Show that the function

$$g(x) = \begin{cases} x/2 + x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is differentiable on $\mathbb{R}$ and satisfies $g'(0) > 0$. Now, prove that g is not increasing over any open interval containing 0.

In the next section we will see that f is indeed increasing on (a, b) if and only if $f'(x) \geq 0$ for all $x \in (a, b)$.

Exercise 6 (Inverse functions)**.** If $f : [a, b] \to \mathbb{R}$ is one-to-one, then there exists an inverse function f^{-1} defined on the range of f given by $f^{-1}(y) = x$ where $y = f(x)$. In the previous chapter, we saw that if f is continuous on $[a, b]$, then f^{-1} is continuous on its domain. Let's add the assumption that f is differentiable on $[a, b]$ with $f'(x) \neq 0$ for all $x \in [a, b]$. Show f^{-1} is differentiable with

$$(f^{-1})'(y) = \frac{1}{f(x)}, \qquad \text{where } y = f(x).$$

3 The Mean Value Theorems

The Mean Value Theorem makes the geometrically plausible assertion that a differentiable function f on an interval [a, b] will, at some point, attain a slope equal to the slope of the line through the endpoints (a, f(a)) and (b, f(b)). More tersely put,

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

for at least one point $c \in (a, b)$.

On the surface, there does not appear to be anything especially remarkable about this observation. Its validity appears undeniable—much like the Intermediate Value Theorem for continuous functions—and its proof is rather short. The ease of the proof, however, is misleading, as it is built on top of some hard-fought accomplishments from the study of limits and continuity. In this regard, the Mean Value Theorem is a kind of reward for a job well done. As we will see, it is a prize of exceptional value. Although the result itself is geometrically unsurprising, the Mean Value Theorem is the cornerstone of the proof for almost every major theorem pertaining to differentiation. We will use it to prove L'Hospital's rules regarding limits of quotients of differentiable functions. A rigorous analysis of how infinite series of functions behave when differentiated requires the Mean Value Theorem, and it is the crucial step in the proof of the Fundamental Theorem of Calculus. It is also the fundamental concept underlying Lagrange's Remainder Theorem which approximates the error between a Taylor polynomial and the function that generates it.

The Mean Value Theorem can be stated in various degrees of generality, each one important enough to be given its own special designation. Recall that the Extreme Value Theorem states that continuous functions on compact sets always attain maximum and minimum values. Combining this observation with the Interior Extremum Theorem for differentiable functions yields a special case of the Mean Value Theorem first noted by the mathematician Michel Rolle (1652–1719).

Theorem 6 (Rolle's Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there exists a point $c \in (a, b)$ where $f'(c) = 0$.*

Proof. Because f is continuous on a compact set, f attains a maximum and a minimum. If both the maximum and minimum occur at the endpoints, then f is necessarily a constant function and $f'(x) = 0$ on all of (a, b). In this case, we can choose c to be any point we like. On the other hand, if either the maximum or minimum occurs at some point c in the interior (a, b), then it follows from the Interior Extremum Theorem that $f'(c) = 0$. □

Theorem 7 (Mean Value Theorem, Lagrange)**.** *If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b), then there exists a point $c \in (a, b)$ where*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof. Notice that the Mean Value Theorem reduces to Rolle's Theorem in the case where $f(a) = f(b)$. The strategy of the proof is to reduce the more general statement to this special case.

The equation of the line through $(a, f(a))$ and $(b, f(b))$ is

$$y = \left(\frac{f(b) - f(a)}{b - a}\right)(x - a) + f(a).$$

We want to consider the difference between this line and the function $f(x)$. To this end, let

$$d(x) = f(x) - \left(\frac{f(b) - f(a)}{b - a}\right)(x - a) - f(a),$$

and observe that d is continuous on $[a, b]$, differentiable on (a, b), and satisfies $d(a) = d(b) = 0$. Thus, by Rolle's Theorem, there exists a point $c \in (a, b)$ where $d'(c) = 0$. Because

$$d'(x) = f'(x) - \frac{f(b) - f(a)}{b - a},$$

we get

$$0 = d'(c) = f'(c) - \frac{f(b) - f(a)}{b - a},$$

which completes the proof. □

The point has been made that the Mean Value Theorem manages to find its way into nearly every proof of any statement related to the geometrical nature of the derivative.

Corollary 8. *If $g : A \to \mathbb{R}$ is differentiable on an interval A and satisfies $g'(x) = 0$ for all $x \in A$, then $g(x) = k$ for some constant $k \in \mathbb{R}$.*

Proof. Take $x, y \in A$ and assume $x < y$. Applying the Mean Value Theorem to g on the interval $[x, y]$, we see that

$$\frac{g(y) - g(x)}{y - x} = g'(c) = 0$$

for some $c \in (x, y)$, so we conclude that $g(y) = g(x)$. Set k equal to this common value. Because x and y are arbitrary, it follows that $g(x) = k$ for all $x \in A$. □

Corollary 9. *If f and g are differentiable functions on an interval A and satisfy $f'(x) = g'(x)$ for all $x \in A$, then $f(x) = g(x) + k$ for some constant $k \in \mathbb{R}$.*

The Mean Value Theorem has a more general form due to Cauchy. It is this generalized version of the theorem that is needed to analyze L'Hospital's rules and Lagrange's Remainder Theorem.

Theorem 10 (Generalized Mean Value Theorem, Cauchy)**.** *If f and g are continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a point $c \in (a, b)$ where*

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c).$$

If g' is never zero on (a, b), then the conclusion can be stated as

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Proof. This result follows by applying the Mean Value Theorem to the function $h(x) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x)$. □

L'Hospital's Rules

Theorem 11 (L'Hospital's Rule, 0/0 case)**.** *Let f and g be continuous on an interval containing a, and assume f and g are differentiable on this interval with the possible exception of the point a. If $f(a) = g(a) = 0$ and $g'(x) \neq 0$ for all $x \neq a$, then*

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = L \qquad \textit{implies} \qquad \lim_{x \to a} \frac{f(x)}{g(x)} = L.$$

Proof. Let $\epsilon > 0$. Because $L = \lim_{x\to a} f'(x)/g'(x)$, there exists a $\delta > 0$ such that

$$\left|\frac{f'(y)}{g'(y)} - L\right| < \epsilon, \qquad 0 < |y-a| < \delta.$$

Given any x with $a-\delta < x < a$ (the case when $a < x < a+\delta$ is similar), it follows from the GMVT that

$$\frac{f'(c)}{g'(c)} = \frac{f(x)-f(a)}{g(x)-g(a)} = \frac{f(x)}{g(x)}$$

where $c \in (x,a)$. Now $|x-a| < \delta$ implies that $0 < |c-a| < \delta$, and it then follows that

$$\left|\frac{f(x)}{g(x)} - L\right| = \left|\frac{f'(c)}{g'(c)} - L\right| < \epsilon,$$

which completes the proof. □

Definition 2. Given $g : A \to \mathbb{R}$ and a limit point c of A, we say that $\lim_{x\to c} g(x) = \infty$if, for every $M > 0$, there exists a $\delta > 0$ such that whenever $0 < |x-c| < \delta$ it follows that $g(x) \geq M$.

We can define $\lim_{x\to c} g(x) = -\infty$ in a similar way.

Theorem 12 (L'Hospital's Rule, ∞/∞ case)**.** *Assume f and g are differentiable in a deleted neighborhood of a and that $g'(x) \neq 0$ there. If $\lim_{x\to a} g(x) = \infty$ (or $-\infty$,) then*

$$\lim_{x\to a}\frac{f'(x)}{g'(x)} = L \qquad \textit{implies} \qquad \lim_{x\to a}\frac{f(x)}{g(x)} = L.$$

Proof. Let $\epsilon > 0$. Because $\lim_{x\to a}\frac{f'(x)}{g'(x)} = L$, there exists a $\delta_1 > 0$ such that

$$L - \frac{\epsilon}{2} < \frac{f'(x)}{g'(x)} < L + \frac{\epsilon}{2}$$

for all $x \in V^0_{\delta_1}(a)$. We shall concentrate on the case when $a < x < a+\delta_1$ (the other case is similar). Set $y = a+\delta_1$

Note that $f(x)$ and $g(x)$ might be not defined at a. But we may apply the GMVT to the interval $[x,y] \subset (a,y]$,

$$\frac{f(x)-f(y)}{g(x)-g(y)} = \frac{f'(c)}{g'(c)}$$

for some $c \in (x,y)$. Thus, the choice of y implies that

$$L - \frac{\epsilon}{2} < \frac{f(x)-f(y)}{g(x)-g(y)} < L + \frac{\epsilon}{2} \tag{$*$}$$

for all $x \in (a, y)$.

Since $\lim_{x\to a} g(x) = \infty$, there exists δ_2 such that (keeping in mind that y is fixed),

$$g(x) > g(y), \qquad 0 < |y - x| < \delta_2,$$

which implies that

$$1 - \frac{g(y)}{g(x)} > 0, \qquad 0 < |y - x| < \delta_2.$$

It then follows from (*) that

$$\left(L - \frac{\epsilon}{2}\right)\left(1 - \frac{g(y)}{g(x)}\right) < \frac{f(x) - f(y)}{g(x)} < \left(L + \frac{\epsilon}{2}\right)\left(1 - \frac{g(y)}{g(x)}\right)$$

which is

$$L - \frac{\epsilon}{2} - \frac{(L - \frac{\epsilon}{2})g(y) - f(y)}{g(x)} < \frac{f(x)}{g(x)} < L + \frac{\epsilon}{2} - \frac{(L + \frac{\epsilon}{2})g(y) - f(y)}{g(x)}.$$

Again, it follow from $\lim_{x\to a} g(x) = \infty$ that there exists $\delta_3 > 0$ such that

$$\left|\frac{(L - \frac{\epsilon}{2})g(y) - f(y)}{g(x)}\right| < \frac{\epsilon}{2} \quad \text{and} \quad \left|\frac{(L + \frac{\epsilon}{2})g(y) - f(y)}{g(x)}\right| < \frac{\epsilon}{2}$$

whenever $0 < |x - a| < \delta$.

Choose $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Then, we have

$$\left|\frac{f(x)}{g(x)} - L\right| < \epsilon$$

whenever $0 < |x - a| < \delta$, as desired. □

Exercise 7. A fixed point of a function f is a value x where $f(x) = x$. Show that if f is differentiable on an interval with $f'(x) \neq 1$, then f can have at most one fixed point.

Exercise 8. Assume f is continuous on an interval containing zero and differentiable for all $x \neq 0$. If $\lim_{x\to 0} f'(x) = L$, show $f'(0)$ exists and equals L.

Exercise 9. Let $f(x) = x \sin(1/x^4)e^{-1/x^2}$ and $g(x) = e^{1/x^2}$. Using the familiar properties of these functions, compute the limit as x approaches zero of $f(x)$, $g(x)$, $f(x)/g(x)$, and $f'(x)/g'(x)$. Explain why the results are surprising but not in conflict with the content of L'Hosipital's Rule.

Exercise 10. If f is twice differentiable on an open interval containing a and f'' is continuous at a, show

$$f''(a) = \lim_{h\to 0} \frac{f(a + h) - 2f(a) + f(a - h)}{h^2}$$

4 A Continuous Nowhere-Differentiable Function

Exploring the relationship between continuity and differentiability has led to both fruitful results and pathological counterexamples. The bulk of discussion to this point has focused on the continuity of derivatives, but historically a significant amount of debate revolved around the question of whether continuous functions were necessarily differentiable. Early in the chapter, we saw that continuity was a requirement for differentiability, but, as the absolute value function demonstrates, the converse of this proposition is not true. A function can be continuous but not differentiable at some point. But just how nondifferentiable can a continuous function be? Given a finite set of points, it is not difficult to imagine how to construct a graph with corners at each of these points, so that the corresponding function fails to be differentiable on this finite set. The trick gets more difficult, however, when the set becomes infinite. For instance, is it possible to construct a function that is continuous on all of R but fails to be differentiable at every rational point? Not only is this possible, but the situation is even more disconcerting. In 1872, Karl Weierstrass presented an example of a continuous function that was not differentiable at any point. (It seems to be the case that Bernhard Bolzano had his own example of such a beast as early as 1830, but it was not published until much later.)

Weierstrass actually discovered a class of nowhere-differentiable functions of the form

$$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n x)$$

where the values of a and b are carefully chosen. Such functions are specific examples of Fourier series. The details of Weierstrass' argument are simplified if we replace the cosine function with a piecewise linear function that has oscillations qualitatively like $\cos(x)$.

Define

$$h(x) = |x|.$$

on the interval $[-1, 1]$ and extend the definition of h to all of $\mathbb{R}$ by requiring that $h(x+2) = h(x)$. The result is a periodic function.

Now, define a function as a series of functions,

$$g(x) = \sum_{n=0}^{\infty} h_n(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} h(2^n x).$$

The claim is that $g(x)$ is continuous on all of $\mathbb{R}$ but fails to be differentiable at any point.

Fix $x \in \mathbb{R}$. By the Comparison Test we may show that the series

$$\sum_{n=0}^{\infty} \frac{1}{2^n} h(2^n x)$$

converges absolutely and thus $g(x)$ is properly defined. Note that each term of the above series is a continuous function. We shall show in the next chapter that, the series is indeed *uniformly convergent* and thus $g(x)$ is also continuous.

When the proper tools are in place, the proof that g is continuous is quite straightforward. The more difficult task is to show that g is not differentiable at any point in $\mathbb{R}$.

Let's first look at the point $x = 0$. The function g does not appear to be differentiable here, and a rigorous proof is not too difficult. Consider the sequence $x_m = 1/2^m$, where $m = 0, 1, 2, \dots$. Then

$$\frac{g(x_m) - g(0)}{x_m} = m + 1, \qquad \frac{g(-x_m) - g(0)}{-x_m} = -(m+1),$$

for each $m = 0, 1, 2, \dots$.

Exercise 11. (i) Modify the previous argument to show that $g'(1)$ does not exist. Show that $g'(1/2)$ does not exist.

Hint. Consider $\dfrac{g(1 + x_m) - g(1)}{x_m}$.

(ii) Show that $g'(x)$ does not exist for any rational number of the form $x = p/2^k$ where $p \in \mathbb{Z}$ and $k\mathbb{N} \cup \{0\}$.

A point of the form $x = p/2^k$ is a *dyadic* rational number. At such a point, the function $h_n(x)$ has a corner at x as long as $n \geq k$. Thus, it should not be too surprising that g fails to be differentiable at points of this form. The argument is more delicate at points between the dyadic points.

Assume x is not a dyadic number. For a fixed value of $m \in \mathbb{N} \cup \{0\}$, x falls between two adjacent dyadic points,

$$\frac{p_m}{2^m} < x < \frac{p_m + 1}{2^m}$$

Set $x_m = p_m/2^m$ and $y_m = (p_m + 1)/2^m$. Now

$$\lim_{m\to\infty} x_m = \lim_{m\to\infty} y_m = x, \qquad x_m < x < y_m.$$

Exercise 12. (i) First prove the following general lemma: Let f be defined on an open interval I and assume f is differentiable at $a \in I$. If $\{a^n\}$ and $\{b_n\}$ are sequences satisfying $a_n < x < b_n$ and $\lim a_n = \lim b_n = a$, show

$$\lim_{n\to\infty} \frac{f(x_n) - f(y_n)}{x_n - y_n} = f'(a).$$

(ii) Now use this lemma to show that $g'(x)$ does not exist.

Hint. Show that $g(x_m) = \sum_{n=0}^{m} \frac{1-(-1)^{p_m}}{2^n}$ and $g(y_m) = \sum_{n=0}^{m} \frac{1+(-1)^{p_m}}{2^n}$.

Weierstrass' original 1872 paper contained a demonstration that the infinite sum

$$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n x)$$

defined a continuous nowhere-differentiable function provided $0 < a < 1$ and b was an odd integer satisfying $ab > 1 + \frac{3\pi}{2}$. The condition on a is easy to understand. If $0 < a < 1$, then $\sum_{n=0}^{\infty} a^n$ is a convergent geometric series, and the forthcoming Weierstrass M-Test (next Chapter) can be used to conclude that f is continuous. The restriction on b is more mysterious. In 1916, G.H. Hardy extended Weierstrass' result to include any value of b for which $ab \geq 1$. Without looking at the details of either of these arguments, we nevertheless get a sense that the lack of a derivative is intricately tied to the relationship between the compression factor (the parameter a) and the rate at which the frequency of the oscillations increases (the parameter b).

Exercise 13. Review the argument for the nondifferentiability of $g(x)$ at nondyadic points. Does the argument still work if we replace $g(x)$ with the summation $\sum_{n=0}^{\infty} \frac{1}{2^n} h(3^n x)$? Does the argument work for the function $\sum_{n=0}^{\infty} \frac{1}{2^n} h(3^n x)$?