MAT2002 Ordinary Differential Equations
System of first order linear equations V

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Overview

1. Non-homogeneous linear systems
   - Method of undetermined coefficients
   - Variation of parameters

Outline

1. Non-homogeneous linear systems
   - Method of undetermined coefficients
   - Variation of parameters

Non-homogeneous linear systems

We now study for $\mathbf{A} \in \mathbb{R}^{n\times n}$ the non-homogeneous system

$$\mathbf{y}'(t) = \mathbf{A}\mathbf{y}(t) + \mathbf{g}(t), \tag{1}$$

and if $\mathbf{y}_1, \ldots, \mathbf{y}_n$ are n linearly independent solutions to the homogeneous system $\mathbf{y}'(t) = \mathbf{A}\mathbf{y}(t)$, and $\mathbf{Y}(t)$ is a particular solution to the non-homogeneous system, then the general solution is

$$\mathbf{y}(t) = c_1\mathbf{y}_1(t) + \cdots + c_n\mathbf{y}_n(t) + \mathbf{Y}(t),$$

where $\mathbf{y}_c(t) = c_1\mathbf{y}_1(t) + \cdots + c_n\mathbf{y}_n(t)$ is the complementary solution.

Method of undetermined coefficients

If we have a non-homogeneous term $\mathbf{g}(t)$ where each component has a sum or product of exponentials, cosine, sine and polynomials, then we can use the method of undetermined coefficients to obtain a particular solution to the non-homogeneous system

$$\mathbf{y}'(t) = \mathbf{A}\mathbf{y}(t) + \mathbf{g}(t).$$

One difference compared to second order equations and $n-$th order equations is that now the undetermined coefficients are <u>**vectors**</u>.

Method of undetermined coefficients

We now list the trial solutions for specific examples of $\mathbf{g}$:

$\mathbf{g}(t)$	Particular solution form $\mathbf{Y}(t)$	value of s
$\mathbf{P}_m(t)e^{\alpha t}$	$\mathbf{Q}_{m+s}(t)e^{\alpha t}$	alg. mult. of α
$\mathbf{P}_m(t)e^{\alpha t}\cos(\beta t)$	$\mathbf{Q}_{m+s}(t)e^{\alpha t}\cos(\beta t) + \mathbf{R}_{m+s}(t)e^{\alpha t}\sin(\beta t)$	alg. mult. of $\alpha + i\beta$
$\mathbf{P}_m(t)e^{\alpha t}\sin(\beta t)$	$\mathbf{Q}_{m+s}(t)e^{\alpha t}\cos(\beta t) + \mathbf{R}_{m+s}(t)e^{\alpha t}\sin(\beta t)$	alg. mult. of $\alpha + i\beta$

alg. mult. of α means the multiplicity of α as the eigenvalue of $\mathbf{A}$.

alg. mult. of $\alpha + i\beta$ means the multiplicity of $\alpha + i\beta$ as the eigenvalue of $\mathbf{A}$.

Method of undetermined coefficients

Here, we use the notation

$$\mathbf{P}_m(t) = \mathbf{a}_m t^m + \mathbf{a}_{m-1} t^{m-1} + \cdots + \mathbf{a}_1 t + \mathbf{a}_0,$$

where $\mathbf{a}_0, \ldots, \mathbf{a}_m$ are constant vectors, so that $\mathbf{P}_m(t)$ is a vector-valued polynomial of degree m.

Remark 1

In constrast to n-th order linear equations, where the form of the trial solution is $t^s \mathbf{Q}_m(t)$, so that the lowest order term is t^s, for linear systems we have to use a trial solution of the form $\mathbf{Q}_{m+s}(t)$, which is a polynomial of degree $m+s$ which includes all lower order terms $t^{s-1}, t^{s-2}, \ldots, t^1, t^0$.

In fact, if we try:

$$\mathbf{Y}(t) = e^{\alpha t} \left(\mathbf{q}_{m+s} t^{m+s} + \cdots + \mathbf{q}_1 t + \mathbf{q}_0 \right)$$

as the particular solution. One has

$$\begin{aligned} \mathbf{Y}'(t) =& \alpha e^{\alpha t} \left(\mathbf{q}_{m+s} t^{m+s} + \cdots + \mathbf{q}_1 t + \mathbf{q}_0 \right) + e^{\alpha t} \Big((m+s) \mathbf{q}_{m+s} t^{m+s-1} + \cdots + \mathbf{q}_1 \Big) \\ =& A e^{\alpha t} \left(\mathbf{q}_{m+s} t^{m+s} + \cdots + \mathbf{q}_1 t + \mathbf{q}_0 \right) + \Big(\mathbf{a}_m t^m + \mathbf{a}_{m-1} t^{m-1} + \cdots + \mathbf{a}_1 t + \mathbf{a}_0 \Big) e^{\alpha t} \end{aligned}$$

Method of undetermined coefficients

Then one has:

$$
\begin{aligned}
A\mathbf{q}_{m+s} &= \alpha\mathbf{q}_{m+s}, \quad \textit{coefficient of term } t^{m+s} \\
A\mathbf{q}_{m+s-1} &= \alpha\mathbf{q}_{m+s-1} + (m+s)\mathbf{q}_{m+s}, \quad \textit{coefficient of term } t^{m+s-1} \\
A\mathbf{q}_{m+s-2} &= \alpha\mathbf{q}_{m+s-2} + (m+s-1)\mathbf{q}_{m+s-1}, \quad \textit{coefficient of term } t^{m+s-2} \\
&\vdots \\
A\mathbf{q}_{m+1} &= \alpha\mathbf{q}_{m+1} + (m+2)\mathbf{q}_{m+2}, \quad \textit{coefficient of term } t^{m+1} \\
A\mathbf{q}_{m} + \mathbf{a}_m &= \alpha\mathbf{q}_{m} + (m+1)\mathbf{q}_{m+1}, \quad \textit{coefficient of term } t^{m} \\
&\vdots \\
A\mathbf{q}_{0} + \mathbf{a}_0 &= \alpha\mathbf{q}_{0} + \mathbf{q}_{1}, \quad \textit{coefficient of term } t^{0}
\end{aligned}
$$

The first s equations indeed gives that $\mathbf{q}_{m+s}, \cdots, \mathbf{q}_{m+1}$ all satisfies the linear system

$$(A - \alpha I)^s\mathbf{x} = \mathbf{0}$$

Method of undetermined coefficients

One can prove that $\mathbf{q}_{m+s}, \cdots, \mathbf{q}_{m+1}$ are linearly independent, and they are generalized eigenvectors. Since $dim(Null((A - \alpha I)^l)) = l$, where l is the alg. mult. of α as the eigenvalue of A. Therefore, one can take s to be l (the alg. mult. of α as the eigenvalue of A), so that $\mathbf{q}_{m+l}, \cdots, \mathbf{q}_{m+1}$ are linearly independent, and they are generalized eigenvectors. In the same time, one can see that the lower order terms $t^{s-1}, t^{s-2}, \ldots, t^1, t^0$ are needed.

Method of undetermined coefficients

Example 13.1

Example Consider

$$\mathbf{y}'(t) = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{y}(t) + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}.$$

We set

$$\mathbf{g}(t) = \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix},$$

and first find the solution to the homogeneous system. The characteristic equation for $\mathbf{A}$ is

$$\det(\mathbf{A} - r\mathbf{I}) = (r+3)(r+1) = 0 \Rightarrow r_1 = -3, \qquad r_2 = -1.$$

Computing

$$\mathbf{A} + 3\mathbf{I} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \mathbf{A} + \mathbf{I} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Method of undetermined coefficients

Example 13.1

So we can take as eigenvectors

$$\xi_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad \xi_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Therefore, the complementary solution to the homogeneous system $\mathbf{y}'(t) = \mathbf{A}\mathbf{y}(t)$ is

$$\mathbf{y}_c(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}.$$

Next, observe that

$$\mathbf{g}(t) = \underbrace{\begin{pmatrix} 2 \\ 0 \end{pmatrix} e^{-t}}_{\mathbf{g}_1(t)} + \underbrace{\begin{pmatrix} 0 \\ 3 \end{pmatrix} t}_{\mathbf{g}_2(t)}.$$

Method of undetermined coefficients

Example 13.1

Since we have a term $\mathbf{g}_1(t)$ involving e^{-t}, which forms part of the complementary solution, recalling the theory for second order equations - where if we encounter a non-homogeneous equation $ay'' + by' + cy = e^{\alpha t}$ and α is a root of the characteristic equation $ar^2 + br + c = 0$ we should try $Y(t) = Ate^{\alpha t}$, let's try a trial solution to the non-homogeneous system with $\mathbf{g}_1(t)$ of the form

$$\mathbf{x}(t) = \mathbf{a}te^{t}$$

for some undetermined vector $\mathbf{a}$. Substituting this into the equation gives

$$\mathbf{x}'(t) - \mathbf{A}\mathbf{x}(t) = -te^{-t}(\mathbf{A}\mathbf{a} + \mathbf{a}) + \mathbf{a}e^{-t} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} e^{-t}.$$

Comparing the coefficients, naturally we choose $\mathbf{a} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

Method of undetermined coefficients

Example 13.1

But, we also need to ensure that $\mathbf{Aa} + \mathbf{a} = \mathbf{0}$. A short computation shows that

$$\mathbf{Aa} + \mathbf{a} = \begin{pmatrix} -2 \\ 2 \end{pmatrix} \neq \mathbf{0}.$$

Therefore, the solution cannot be of the form $\mathbf{a}te^{-t}$.
To remedy this, let's try

$$\mathbf{x}(t) = \mathbf{a}te^{-t} + \mathbf{b}e^{-t},$$

and then

$$\mathbf{x}'(t) - \mathbf{Ax}(t) - te^{-t}(\mathbf{Aa} + \mathbf{a}) - e^{-t}(\mathbf{b} + \mathbf{Ab} - \mathbf{a}) = \mathbf{g}_1(t).$$

This means we should have

$$\mathbf{Aa} + \mathbf{a} = \mathbf{0}, \qquad \mathbf{b} + \mathbf{Ab} - \mathbf{a} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}.$$

Method of undetermined coefficients

Example 13.1

That is, $\mathbf{a}$ should be an eigenvector to the eigenvalue $r = -1$, and so we take $\mathbf{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Then

$$\mathbf{Ab} + \mathbf{b} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \Rightarrow -b_1 + b_2 = -1.$$

We can take $b_1 = 0, b_2 = -1$ and so a particular solution to $\mathbf{y}'(t) = \mathbf{Ay}(t) + \mathbf{g}_1(t)$ is

$$\mathbf{x}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{-t}.$$

For a particular solution to $\mathbf{y}'(t) = \mathbf{Ay}(t) + \mathbf{g}_2(t)$, we try a trial solution of the form

$$\mathbf{z}(t) = \mathbf{c}t + \mathbf{d}.$$

Method of undetermined coefficients

Example 13.1

Then,

$$\mathbf{z}'(t) - \mathbf{A}\mathbf{z}(t) = (\mathbf{c} - \mathbf{A}\mathbf{d}) - t\mathbf{A}\mathbf{c} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} t.$$

Hence, we require

$$\mathbf{A}\mathbf{c} = \begin{pmatrix} 0 \\ -3 \end{pmatrix}, \qquad \mathbf{A}\mathbf{d} = \mathbf{c}.$$

Solving these equations gives

$$\mathbf{c} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \mathbf{d} = \begin{pmatrix} -4/3 \\ -5/3 \end{pmatrix} \quad \Rightarrow \mathbf{z}(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \begin{pmatrix} -4/3 \\ -5/3 \end{pmatrix}.$$

Method of undetermined coefficients

Example 13.2

Find a particular solution to

$$\mathbf{y}'(t) = \begin{pmatrix} 1 & 4 \\ 1 & -2 \end{pmatrix} \mathbf{y}(t) + \begin{pmatrix} e^{-2t} \\ -2e^{t} \end{pmatrix}.$$

The eigenvalues of the matrix $\mathbf{A}$ are $r_1 = -3, r_2 = 2$ with corresponding eigenvectors

$$\xi_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad \xi_2 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}.$$

So the general solution to the homogeneous system is

$$\mathbf{y}_c(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{2t}, \qquad c_1, c_2 \in \mathbb{R}.$$

Method of undetermined coefficients

Example 13.2

Writing the term $\mathbf{g}(t)$ as

$$\mathbf{g}(t) = e^{-2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{t}\begin{pmatrix} 0 \\ -2 \end{pmatrix},$$

and since neither -2 nor 1 are eigenvalues of $\mathbf{A}$, we try a trial solution of the form

$$\mathbf{z}(t) = \mathbf{a}e^{-2t} + \mathbf{b}e^{t}.$$

Then, computing

$$\mathbf{z}'(t) - \mathbf{A}\mathbf{z}(t) = e^{-2t}(-2\mathbf{a} - \mathbf{A}\mathbf{a}) + e^{t}(\mathbf{b} - \mathbf{A}\mathbf{b}) = e^{-2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{t}\begin{pmatrix} 0 \\ -2 \end{pmatrix},$$

and upon comparing coefficients we need

$$(-2\mathbf{I} - \mathbf{A})\mathbf{a} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad (\mathbf{I} - \mathbf{A})\mathbf{b} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}.$$

Method of undetermined coefficients

Example 13.2

Solving these equations gives

$$\mathbf{a} = \begin{pmatrix} 0 \\ -0.25 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 2 \\ 0 \end{pmatrix},$$

and so a particular solution is

$$\mathbf{Y}(t) = \begin{pmatrix} 0 \\ -0.25 \end{pmatrix} e^{-2t} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} e^{t}.$$

Method of undetermined coefficients

Exercise: Find a particular solution to

$$\mathbf{y}'(t) = \begin{pmatrix} 1 & 5 \\ -1 & 1 \end{pmatrix} \mathbf{y}(t) + \begin{pmatrix} e^{2t} \\ \sin(2t) \end{pmatrix}.$$

Variation of parameters

We now consider more general non-homogeneous first order systems of the form

$$\mathbf{y}'(t) = \mathbf{P}(t)\mathbf{y}(t) + \mathbf{g}(t),$$

where the matrix $\mathbf{P}(t) = (p_{ij}(t))_{n\times n}$, and $p_{ij}(t), i, j = 1, \cdots, n$ are continuous on the interval I. First, we neglect the non-homogeneous term and study the homogeneous system $\mathbf{y}'(t) = \mathbf{P}(t)\mathbf{y}(t)$.

Definition 13.3

(Fundamental matrix). Let $\mathbf{y}_1(t), \ldots, \mathbf{y}_n(t)$ be a fundamental set of solutions to the homogeneous system $\mathbf{y}'(t) = \mathbf{P}(t)\mathbf{y}(t)$. The matrix $\mathbf{F}$ defined as

$$\mathbf{F}(t) = \begin{pmatrix} | & | & \cdots & | \\ \mathbf{y}_1(t) & \mathbf{y}_2(t) & \cdots & \mathbf{y}_n(t) \\ | & | & \cdots & | \end{pmatrix} = \begin{pmatrix} y_{11}(t) & y_{12}(t) & \ldots & y_{1n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1}(t) & y_{n2}(t) & \ldots & y_{nn}(t) \end{pmatrix}$$

is called a **fundamental matrix** for the system $\mathbf{y}'(t) = \mathbf{P}(t)\mathbf{y}(t)$.

Note that the fundamental matrix $\mathbf{F}(t)$ is invertible for all $t \in I$ since its columns forms a fundamental set of solutions(Wronskian $W(\mathbf{y}_1, \ldots, \mathbf{y}_n)[t] = \det(\mathbf{F}(t)) \neq 0$).

Variation of parameters

Propery:
Let $\mathbf{y}_1(t), \ldots, \mathbf{y}_n(t)$ be a fundamental set of solutions to the homogeneous system $\mathbf{y}'(t) = \mathbf{P}(t)\mathbf{y}(t)$. The matrix $\mathbf{F}$ defined as

$$\mathbf{F}(t) = \begin{pmatrix} | & | & \cdots & | \\ \mathbf{y}_1(t) & \mathbf{y}_2(t) & \cdots & \mathbf{y}_n(t) \\ | & | & \cdots & | \end{pmatrix}$$

satisfies the property

$$\frac{d\mathbf{F}(t)}{dt} = \mathbf{P}(t)\mathbf{F}(t)$$

Variation of parameters

Proof.

$$\frac{d}{dt}\mathbf{F}(t) = \frac{d}{dt}\begin{pmatrix} | & | & \cdots & | \\ \mathbf{y}_1(t) & \mathbf{y}_2(t) & \cdots & \mathbf{y}_n(t) \\ | & | & \cdots & | \end{pmatrix} = \begin{pmatrix} | & | & \cdots & | \\ \frac{d}{dt}\mathbf{y}_1(t) & \frac{d}{dt}\mathbf{y}_2(t) & \cdots & \frac{d}{dt}\mathbf{y}_n(t) \\ | & | & \cdots & | \end{pmatrix}$$

$$= \begin{pmatrix} | & | & \cdots & | \\ \mathbf{P}(t)\mathbf{y}_1(t) & \mathbf{P}(t)\mathbf{y}_2(t) & \cdots & \mathbf{P}(t)\mathbf{y}_n(t) \\ | & | & \cdots & | \end{pmatrix} = \mathbf{P}(t)\mathbf{F}(t).$$

Variation of parameters

Now Return to the non-homogeneous system

$$\mathbf{y}'(t) = \mathbf{P}(t)\mathbf{y}(t) + \mathbf{g}(t).$$

Assume we have a fundamental matrix $\mathbf{F}(t)$ to the homogeneous system with complementary solution

$$\mathbf{y}_c(t) = \mathbf{F}(t)\mathbf{c},$$

where $\mathbf{c}$ is a constant vector. The method of variation of parameters is to consider a trial solution

$$\mathbf{z}(t) = \mathbf{F}(t)\mathbf{u}(t),$$

where $\mathbf{u}(t)$ is a vector of functions. Then, if $\mathbf{z}$ is a solution to the non-homogeneous system, we find that

$$\mathbf{P}(t)\mathbf{F}(t)\mathbf{u}(t) + \mathbf{g}(t) = \mathbf{z}'(t) = \mathbf{F}(t)\mathbf{u}'(t) + \mathbf{F}'(t)\mathbf{u}(t).$$

Since $\mathbf{F}(t)$ is a fundamental matrix, i.e., $\mathbf{F}'(t) = \mathbf{P}(t)\mathbf{F}(t)$, we see that

$$\mathbf{F}(t)\mathbf{u}'(t) = \mathbf{g}(t) \Rightarrow \mathbf{u}'(t) = \mathbf{F}^{-1}(t)\mathbf{g}(t).$$

($\mathbf{F}(t)$ is invertible for any $t \in$I.)

Variation of parameters

Integrating this gives

$$\mathbf{u}(t) = \int \mathbf{F}^{-1}(t)\mathbf{g}(t)dt.$$

The particular solution is

$$\mathbf{z}(t) = \mathbf{F}(t) \int \mathbf{F}^{-1}(t)\mathbf{g}(t)dt.$$

Therefore the general solution to the non-homogeneous system is

$$\mathbf{y}(t) = \mathbf{y}_c(t) + \mathbf{z}(t) = \mathbf{F}(t)\mathbf{c} + \mathbf{F}(t)\left[\int \mathbf{F}^{-1}(t)\mathbf{g}(t)dt\right].$$

Variation of parameters

If we are also given initial conditions $\mathbf{y}(t_0) = \mathbf{v}$, then in the integral we write

$$\int_{t_0}^{t} \mathbf{F}^{-1}(s)\mathbf{g}(s)ds$$

so that

$$\mathbf{v} = \mathbf{y}(t_0) = \mathbf{F}(t_0)\mathbf{c} \Rightarrow \mathbf{c} = \mathbf{F}^{-1}(t_0)\mathbf{v}.$$

Hence, the unique solution to the IVP in the interval I is

$$\mathbf{y}(t) = \mathbf{F}(t)\mathbf{F}^{-1}(t_0)\mathbf{v} + \mathbf{F}(t)\left[\int_{t_0}^{t} \mathbf{F}^{-1}(s)\mathbf{g}(s)ds\right].$$

Variation of parameters

Example 13.4

Find a particular solution to

$$\mathbf{y}'(t) = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{y}(t) + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}.$$

Using the method of undetermined coefficients, we have that one particular solution is

$$\mathbf{Y}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-t} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} t - \frac{1}{3} \begin{pmatrix} 4 \\ 5 \end{pmatrix}.$$

Recalling that the complementary solution to the homogeneous system is

$$\mathbf{y}_c(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}.$$

Variation of parameters

Example 13.7

Computing the fundamental matrix

$$\mathbf{F}(t) = \begin{pmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{pmatrix},$$

its determinant $\det \mathbf{F}(t) = 2e^{-4t}$ and its inverse

$$\mathbf{F}^{-1}(t) = \frac{1}{2} \begin{pmatrix} e^{3t} & -e^{3t} \\ e^{t} & e^{t} \end{pmatrix}$$

we can then compute for the unknown coefficients by solving

$$\mathbf{u}'(t) = \mathbf{F}^{-1}(t)\mathbf{g}(t) \Rightarrow \begin{cases} u_1'(t) = e^{2t} - \frac{3}{2} t e^{3t}, \\ u_2'(t) = 1 + \frac{3}{2} t e^{t}. \end{cases}$$

Variation of parameters

Example 13.7

This gives

$$u_1(t) = \frac{1}{2}e^{2t} - \frac{1}{2}te^{3t} + \frac{1}{6}e^{3t}, \qquad u_2(t) = t + \frac{3}{2}te^{t} - \frac{3}{2}e^{t},$$

where we used

$$\int te^{\alpha t}dt = \frac{\alpha t - 1}{\alpha^2}e^{\alpha t}.$$

Hence, a particular solution is

$$\mathbf{Z}(t) = \mathbf{F}(t)\mathbf{u}(t) = te^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-t} + t\begin{pmatrix} 1 \\ 2 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} 4 \\ 5 \end{pmatrix}.$$

Variation of parameters

Example 13.7

Note that $\mathbf{Y}(t)$ obtained from the method of undetermined coefficients is different from the particular solution $\mathbf{Z}(t)$ obtained from the variation of parameters:

$$\mathbf{Y}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-t} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} t - \frac{1}{3} \begin{pmatrix} 4 \\ 5 \end{pmatrix},$$

$$\mathbf{Z}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-t} + \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} t - \frac{1}{3} \begin{pmatrix} 4 \\ 5 \end{pmatrix}.$$

One can check that both $\mathbf{Y}$ and $\mathbf{Z}$ are particular solutions, but the corresponding general solutions to the non-homogeneous system are equivalent:

$$\mathbf{y}(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + \mathbf{Y}(t) = d_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3t} + d_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + \mathbf{Z}(t)$$

if we choose

$$c_1 = d_1, \qquad c_2 = d_2 + \frac{1}{2}.$$

Variation of parameters

Example 13.5

Find a particular solution to

$$\mathbf{y}'(t) = \begin{pmatrix} 1 & 4 \\ 1 & -2 \end{pmatrix} \mathbf{y}(t) + \begin{pmatrix} e^{-2t} \\ -2e^{t} \end{pmatrix}.$$

From before, the eigenvalues of $\mathbf{A}$ are $r_1 = -3$ and $r_2 = 2$ with eigenvectors

$$\xi_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad \xi_2 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}.$$

From this we can write down the fundamental matrix

$$\mathbf{F}(t) = \begin{pmatrix} e^{-3t} & 4e^{2t} \\ -e^{-3t} & e^{2t} \end{pmatrix}.$$

The determinant is $\det \mathbf{F}(t) = 5e^{-t}$, with inverse

$$\mathbf{F}(t)^{-1} = \frac{1}{5} \begin{pmatrix} e^{3t} & -4e^{3t} \\ e^{-2t} & e^{-2t} \end{pmatrix}.$$

Variation of parameters

Example 13.8

Then, for the unknown coefficients, we solve

$$\mathbf{u}'(t) = \mathbf{F}^{-1}(t)\mathbf{g}(t) \Rightarrow \begin{cases} u_1'(t) = \frac{1}{5}(e^t + 8e^{4t}), \\ u_2'(t) = \frac{1}{5}(e^{-4t} - 2e^{-t}). \end{cases}$$

This gives

$$u_1(t) = \frac{1}{5}e^t + \frac{2}{5}e^{4t}, \qquad u_2(t) = -\frac{1}{20}e^{-4t} + \frac{2}{5}e^{-t},$$

and the particular solution is

$$\mathbf{Z}(t) = \mathbf{F}(t)\mathbf{u}(t) = \begin{pmatrix} 2e^t \\ -0.25e^{-2t} \end{pmatrix},$$

which coincides with particular solution obtained from the method of undetermined coefficients.