MAT2002 ODEs
Second-order linear equations–Homogeneous equations with constant coefficients

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Overview

1. Homogeneous equations with constant coefficients
    - Case 1: two distinct real roots
    - Complex roots
    - One repeated real root
  - Reduction of order
2. Euler equations

Outline

Motivation for homogeneous equations with constant coefficients

From previous lecture, we know that a fundamental set of solutions for the general second-order linear homogeneous ODE $y'' + p(t)y' + q(t)y = 0$ is important and completely determine the all the solutions of the ODE.

Although the fundamental set of solutions for the second-order linear homogeneous ODE $y'' + p(t)y' + q(t)y = 0$ always exist, but unfortunately, there is no method to find the fundamental set of solutions explicitly.

When p, q are constants, we can find the fundamental set of solutions for $y'' + py' + qy = 0$ explicitly.

Homogeneous equations with constant coefficients

Let's start with a previous example, we saw how to solve the equation

$$y'' + \frac{b}{a}y' = 0,$$

which has a general solution involving the exponential function.
Recall:

$$y'' + ey' = 0, \qquad e \in \mathbb{R}, \qquad e \neq 0.$$

Setting $v = y'$ the ODE satisfied by v is

$$v' + ev = 0 \Rightarrow v(t) = c\exp(-et), \quad c \in \mathbb{R}.$$

Hence

$$y'(t) = c\exp(-et) \Rightarrow y(t) = \frac{-c}{e}\exp(-et) + c_0, \quad c_0 \in \mathbb{R}.$$

Homogeneous equations with constant coefficients

The objective of this section is to study the solutions to the ODE

$$ay'' + by' + cy = 0 \tag{1}$$

for fixed real constants $a, b, c \in \mathbb{R}$ with $a \neq 0$.
Hence, let us consider substituting a **trial** function $y(t) = \exp(rt)$ for some constant r into the ODE (1) . This yields

$$(ar^2 + br + c)\exp(rt) = 0.$$

Since $\exp(rt)$ is positive, we obtain that

$$\boxed{ar^2 + br + c = 0}. \tag{2}$$

The equation (2) is known as the **characteristic equation** for the ODE (1). If we can find the roots of the characteristic equation, then we know that $\exp(rt)$, where r is a root, is a solution to (1).

Homogeneous equations with constant coefficients

Since (2) is a quadratic equation, by the well-known quadratic formula, we see that

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Three possibilities:

1. Two distinct real roots r_1, r_2 if $b^2 > 4ac$.
2. Two complex roots (complex conjugate pairs) $r_1, \overline{r_1}$ if $b^2 < 4ac$.
3. A repeated real root r if $b^2 = 4ac$.

Immediately we see that the explicit formula for the solution $y(t)$ to (1) will depend heavily on the **discriminant** $\Delta = b^2 - 4ac$.

Case 1: Two distinct real roots $\Delta > 0$

In the case $b^2 > 4ac$, we obtain two real roots

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

This gives us two solutions

$$y_1(t) = \exp(r_1 t), \quad y_2(t) = \exp(r_2 t).$$

Case 1: Two distinct real roots $\Delta > 0$

Recall:

Theorem 6.1

Let I be an open interval, p and q are continuous functions in I. Let y_1 and y_2 be two solutions to the ODE

$$y'' + p(t)y' + q(t)y = 0$$

for $I \in I$. Then, any solution y to the ODE can be expressed as

$$y(t) = c_1 y_1(t) + c_2 y_2(t) \tag{3}$$

for constants c_1 and c_2 if and only if there is a point $t_0 \in I$ such that the Wronskian $W(y_1, y_2)[t_0]$ is non-zero at t_0.

Case 1: Two distinct real roots $\Delta > 0$

Let's now check the Wronskian:

$$\begin{aligned} W(y_1, y_2)[t] =& y_1(t)y_2'(t) - y_2(t)y_1'(t) \\ =& r_2 \exp((r_1 + r_2)t) - r_1 \exp((r_1 + r_2)t) \\ =& (r_2 - r_1) \exp((r_1 + r_2)t). \end{aligned}$$

Since $r_1 \neq r_2$ and the exponential is never zero, we see that the Wronskian $W(y_1, y_2)[t]$ is non-zero for all $t \in \mathbb{R}$.

Thus, $y_1(t) = \exp(r_1 t), \quad y_2(t) = \exp(r_2 t)$ is the fundamental solution set of the ODE (2). By the above Theorem, any solution y to the ODE $ay'' + by' + cy = 0$ where $b^2 - 4ac > 0$ can be expressed as a linear combination of y_1 and y_2.

More precisely, any solution $y(t)$ to the ODE is of the form

$$\boxed{y(t) = c_1 \exp(r_1 t) + c_2 \exp(r_2 t)} \tag{4}$$

for some constants c_1 and c_2.

Case 1: Two distinct real roots $\Delta > 0$

Exercise: Check by differentiating to see that (4) is a solution to the ODE. To determine the values of c_1 and c_2, suppose we have the IVP

$$ay'' + by' + cy = 0, \quad y(t_0) = x_0, \quad y'(t_0) = x_1.$$

Then, a simple calculation shows that

$$\begin{aligned} c_1 \exp(r_1 t_0) + c_2 \exp(r_2 t_0) &= x_0, \\ r_1 c_1 \exp(r_1 t_0) + r_2 c_2 \exp(r_2 t_0) &= x_1. \end{aligned}$$

Upon rearranging leads to

$$c_1 = \frac{x_1 - x_0 r_2}{r_1 - r_2} \exp(-r_1 t_0), \quad c_2 = \frac{x_0 r_1 - x_1}{r_1 - r_2} \exp(-r_2 t_0).$$

As $r_1 \neq r_2$, the above expressions always make sense.

Case 1: Two distinct real roots $\Delta > 0$

Example 6.2

Find the general solution to the ODE

$$y'' + 9y' + 20y = 0.$$

As before we consider a trial function $y(t) = \exp(rt)$ and after substituting, we obtain the characteristic equation

$$r^2 + 9r + 20 = (r+4)(r+5) = 0.$$

This means that we have two real roots $r_1 = -4$ and $r_2 = -5$. Hence, the general solution is

$$y(t) = c_1 \exp(-4t) + c_2 \exp(-5t), \quad c_1, c_2 \in \mathbb{R}.$$

Note that, as $t \to \infty$, the solution $y(t)$ will tend to zero. This behaviour does not depend on the value of c_1 and c_2, since the exponents are both negative in this case.

Case 1: Two distinct real roots $\Delta > 0$

Example 6.3

Find the general solution to the ODE

$$y'' - y' - 42y = 0.$$

We obtain as the characteristic equation

$$r^2 - r - 42 = (r-7)(r+6) = 0.$$

This gives $r_1 = 7$ and $r_2 = -6$ and the general solution is

$$y(t) = c_1 \exp(7t) + c_2 \exp(-6t), \quad c_1, c_2 \in \mathbb{R}.$$

Note that as the function $c_2 \exp(-6t) \to 0$ as $t \to \infty$, and so we have

$$y(t) \to \begin{cases} \infty & \textit{if } \; c_1 > 0, \\ -\infty & \textit{if } \; c_1 < 0, \quad \textit{as} \quad t \to \infty \\ 0 & \textit{if } \; c_1 = 0. \end{cases}$$

The value of c_1 and c_2 are determined by the initial conditions. But it is important to point out that it is possible for the solution y to go to $\pm\infty$ with one of the exponent is positive.

Case 2: Complex roots $\Delta < 0$

We now consider the case $\Delta = b^2 - 4ac < 0$. Then, the roots to the characteristic equation $ar_2 + br + c = 0$ is a **complex-conjugate pair**:

$$r_1 = \lambda + i\mu,\ \lambda = \frac{-b}{2a},\ \mu = \sqrt{4ac - b^2},\ i := \sqrt{-1},\ r_2 = \overline{r_1} = \lambda - i\mu.$$

Since the characteristic equation is obtained by substituting the trial function $y(t) = \exp(rt)$, we obtain two functions

$$y_1(t) = \exp(r_1 t) = \exp((\lambda + i\mu)t), \quad y_2(t) = \exp(r_2 t) = \exp((\lambda - i\mu)t).$$

Euler's formula: Due to the imaginary number i appearing in the formulae for y_1 and y_2, we use the well-known **Euler's formula**: For any real number $x \in \mathbb{R}$,

$$\boxed{\exp(ix) = \cos(x) + i\sin(x)}.$$

As a consequence of the symmetries of cos and sin, we also have

$$\exp(-ix) = \cos(x) - i\sin(x).$$

Case 2: Complex roots $\Delta < 0$

Furthermore, applying the general rule $\exp(a+b) = \exp(a)\exp(b) \quad \textit{for } a, b \in \mathbb{C}$, we now arrived at

$$y_1(t) = \exp(\lambda t)(cos(\mu t) + i\sin(\mu t)), \;\; y_2(t) = \exp(\lambda t)(cos(\mu t) - i\sin(\mu t)).$$

Note that there is a <u>**common factor**</u> $\exp(\lambda t)$ appearing in both solutions. One can also check that

$$\overline{y_1(t)} = \exp(\lambda t)(\cos(\mu t) - i\sin(\mu t)) = y_2(t),$$

so that y_2 is the <u>**complex conjugate**</u> of y_1.

Let's check that y_1 and y_2 are linearly independent. Suppose there are constants $\alpha_1, \alpha_2 \in \mathbb{R}$ such that

$$\begin{aligned} &\alpha_1 y_1(t) + \alpha_2 y_2(t) = 0 \quad \forall t \in I \\ \Rightarrow & e^{\lambda t}((\alpha_1 + \alpha_2)\cos(\mu t) + i(\alpha_1 - \alpha_2)\sin(\mu t)) = 0. \end{aligned}$$

The exponential is non-zero for all $t \in \mathbb{R}$, and so to make the above expression zero, we need

$$\alpha_1 + \alpha_2 = 0, \quad \alpha_1 - \alpha_2 = 0 \Rightarrow \alpha_1 = \alpha_2 = 0$$

So y_1 and y_2 are linearly independent.

Case 2: Complex roots $\Delta < 0$

We can also check the Wronskian.

$$\begin{aligned} W(y_1, y_2)[t] &= y_1(t)y_2'(t) - y_2(t)y_1'(t) \\ &= e^{(\lambda+i\mu)t}(\lambda - i\mu)e^{(\lambda-i\mu)t} - (\lambda + i\mu)e^{(\lambda+i\mu)t}e^{(\lambda-i\mu)t} \\ &= e^{2\lambda t}(\lambda - i\mu - \lambda - i\mu) = -2i\mu e^{2\lambda t} \end{aligned}$$

Since the exponential is never zero for $t \in \mathbb{R}$, and μ is non-zero (otherwise we will not have $b^2 - 4ac < 0$), the Wronskian is non-zero for all $t \in \mathbb{R}$.

Case 2: Complex roots $\Delta < 0$

Then, by previous Theorem, any solution y to the ODE $ay'' + by' + cy = 0$ where $b^2 - 4ac < 0$ can be expressed as a linear combination of

$$y_1(t) = \exp(\lambda t)(cos(\mu t) + i\sin(\mu t)),\ \ y_2(t) = \exp(\lambda t)(cos(\mu t) - i\sin(\mu t)).$$

More precisely, any solution $y(t)$ to the ODE is of the form

$$\begin{aligned} y(t) &= e^{\lambda t}((c_1 + c_2)\cos(\mu t) + i(c_1 - c_2)\sin(\mu t)) \\ \text{or } y(t) &= e^{\lambda t}(d_1 \cos(\mu t) + d_2 i \sin(\mu t)) \end{aligned} \tag{5}$$

for some constants d_1 and d_2.

Case 2: Complex roots $\Delta < 0$

Although we have a solution expressed in (5) it is a complex-valued function. Since the coefficients of the ODE are real numbers, it would be better for us to obtain a real-valued function as a solution. It turns out that we can do such a thing with the following theorem.

Case 2: Complex roots $\Delta < 0$

Theorem 6.4

Given an ODE

$$y'' + p(t)y' + q(t)y = 0$$

with p and q are continuous **real-valued** *functions. If $y(t) = u(t) + iv(t)$ is a* **complex-valued** *solution to the ODE, where u and v are* **real-valued** *functions, then its real part $u(t)$ and its imaginary part $v(t)$ are both solutions to the ODE.*

Case 2: Complex roots $\Delta < 0$

Proof.

Substituting the complex-valued solution into the ODE gives

$$\begin{aligned} 0 &= u''(t) + iv''(t) + p(t)u'(t) + ip(t)v'(t) + q(t)u(t) + iq(t)v(t) \\ &= (u''(t) + p(t)u'(t) + q(t)u(t)) + i(v''(t) + p(t)v'(t) + q(t)v(t)). \end{aligned}$$

A complex number is zero if and only if its real part and imaginary part are both zero. On the LHS we have zero and on the RHS we have a complex number for every $t \in I$. Therefore we must have

$$u'' + p(t)u' + q(t)u = 0, \quad v'' + p(t)v' + q(t)v = 0.$$

Since

$$y_1(t) = \exp(\lambda t)(cos(\mu t) + i\sin(\mu t)),\ y_2(t) = \exp(\lambda t)(cos(\mu t) - i\sin(\mu t)),$$

we get the real-valued functions

$$u(t) = e^{\lambda t}\cos(\mu t), \quad v(t) = e^{\lambda t}\sin(\mu t)$$

Case 2: Complex roots $\Delta < 0$

Proof.

It is clear that u and v are linearly independent, and computing the Wronskian

$$\begin{aligned} W(u,v)[t] =& u(t)v'(t) - v(t)u'(t) \\ =& e^{\lambda t}\cos(\mu t)e^{\lambda t}(\lambda \sin(\mu t) + \mu \cos(\mu t)) \\ & - e^{\lambda t}\sin(\mu t)e^{\lambda t}(\lambda \cos(\mu t) - \mu \sin(\mu t)) \\ =& e^{2\lambda t}\mu(cos^2(\mu t) + sin^2(\mu t)) = \mu e^{2\lambda t} \end{aligned}$$

which is non-zero for all $t \in I$ as $\mu \neq 0$.
Thus by previous Theorem we see that any solution y to the ODE $ay'' + by' + cy = 0$ with $b^2 - 4ac < 0$ can be expressed as

$$\boxed{y(t) = c_1 e^{\lambda t}\cos(\mu t) + c_2 e^{\lambda t}\sin(\mu t).} \tag{6}$$

The advantage of this expression over (5) is that y is a real-valued function. □

Case 2: Complex roots $\Delta < 0$

Example 6.5

Solve the IVP

$$y'' + y' + \frac{37}{4}y = 0, \quad y(0) = 2, \quad y'(0) = 8.$$

The characteristic equation is

$$r^2 + r + \frac{37}{4} = 0,$$

with roots

$$r_1 = -\frac{1}{2} + 3i, \quad r_2 = -\frac{1}{2} - 3i, \quad \Rightarrow \quad \lambda = -\frac{1}{2}, \quad \mu = 3.$$

The general solution is

$$y(t) = e^{-\frac{1}{2}t}(c_1 \cos(3t) + c_2 \sin(3t))$$

for some constants $c_1, c_2 \in \mathbb{R}$. Using the initial conditions we have

$$y(0) = c_1 = 2, \quad y'(0) = -\frac{1}{2}c_1 + 3c_2 = 8 \quad \Rightarrow \quad c_1 = 2, \quad c_2 = 3.$$

Hence the particular solution is

$$y(t) = e^{-\frac{1}{2}t}(2\cos(3t) + 3\sin(3t))$$

Case 2: Complex roots $\Delta < 0$

Similar to the case of two real roots, we now investigate the possible behaviour of the following solution as $t \to \infty$.

$$\boxed{y(t) = c_1 e^{\lambda t} \cos(\mu t) + c_2 e^{\lambda t} \sin(\mu t).} \tag{7}$$

(1) If $\lambda = 0$, then (6) becomes

$$y(t) = c_1 \cos(\mu t) + c_2 \sin(\mu t)$$

In this case, the solution y is an **oscillation** with **constant amplitude**. The amplitude will depend on the values of c_1 and c_2, which depends on the initial conditions.

(2) If $\lambda > 0$, then due to the factor $e^{\lambda t}$, the amplitude of solution will oscillate and **grow** (exponentially) in time.

(3) If $\lambda < 0$, then due to the factor $e^{\lambda t}$, the amplitude of solution will oscillate and **decay** (exponentially) in time. In this case, $y(t) \to 0$ as $t \to \infty$.

Case 3: One repeated real root $\Delta = 0$

The last case is when $b^2 - 4ac = 0$ and we have a **repeated** root to the characteristic equation. The quadratic formula yields

$$r_1 = r_2 = -\frac{b}{2a}$$

as solutions to the characteristic equation $ar^2 + br + c = 0$. The problem is immediately apparent: both roots gives the same function

$$y_1(t) = y_2(t) = \exp\left(-\frac{b}{2a}t\right).$$

But for our developed theory we **require** at least two linearly independent solutions to the ODE. It is **not obvious** how to find a solution solution that is linearly independent to $y_1(t) = \exp\left(-\frac{b}{2a}t\right)$.

Case 3: One repeated real root $\Delta = 0$

Idea: Use the Wronskian (see Example 2). By Abel's theorem, if $y_1 = \exp\left(-\frac{b}{2a}t\right)$ and y_2 are two solutions to the ODE $ay'' + by' + cy = 0$ with $b^2 = 4ac$, then we know that the Wronskian is

$$W(y_1, y_2)[t] = d\exp\left(-\int \frac{b}{a}dt\right) = d\exp\left(-\frac{b}{a}t\right)$$

for some constant $d \in \mathbb{R}$. On the other hand we have

$$W(y_1, y_2)[t] = y_1(t)y_2'(t) - y_1'(t)y_2(t) = e^{-\frac{b}{2a}t}y_2'(t) + \frac{b}{2a}e^{-\frac{b}{2a}t}y_2(t).$$

Case 3: One repeated real root $\Delta = 0$

Choosing $d = 1$, and putting things together we have

$$e^{-\frac{b}{2a}t} y_2'(t) + \frac{b}{2a} e^{-\frac{b}{2a}t} y_2(t) = e^{-\frac{b}{a}t} \Rightarrow \boxed{y_2'(t) + \frac{b}{2a} y_2(t) = e^{-\frac{b}{2a}t}}.$$

This is a first order linear ODE for y_2, and using the method of integrating factors we have

$$\boxed{y_2(t) = te^{-\frac{b}{2a}t}}$$

where we have neglected any constants of integration.
Let us now check the linear independence for $y_1 = e^{-\frac{b}{2a}t}$ and $y_2 = te^{-\frac{b}{2a}t}$:
Suppose α_1 and α_2 are two constants such that

$$\begin{aligned} &\alpha_1 y_1(t) + \alpha_2 y_2(t) = 0 \quad \forall t \in I \\ \Rightarrow &e^{-\frac{b}{2a}t}(\alpha_1 + t\alpha_2) = 0. \end{aligned}$$

Since the exponential is never zero, for $\alpha_1 + t\alpha_2$ to be zero for all $t \in I$, we must have $\alpha_1 = \alpha_2 = 0$.

Case 3: One repeated real root $\Delta = 0$

For the Wronskian we compute and see that

$$W(y_1, y_2)[t] = e^{-\frac{b}{2a}t}(e^{-\frac{b}{2a}t} - \frac{b}{2a}te^{-\frac{b}{2a}t}) + \frac{b}{2a}te^{-\frac{b}{a}t} = e^{-\frac{b}{a}t} \neq 0.$$

Thus by Theorem 3 we see that any solution y to the ODE $ay'' + by' + cy = 0$ with $b^2 - 4ac = 0$ can be expressed as

$$\boxed{y(t) = c_1 e^{-\frac{b}{2a}t} + c_2 te^{-\frac{b}{2a}t}} \tag{8}$$

for constants $c_1, c_2 \in \mathbb{R}$.

Case 3: One repeated real root $\Delta = 0$

Example 6.6

Solve the IVP

$$y'' + 4y' + 4y = 0, \quad y(0) = 2, \quad y'(0) = 1.$$

The characteristic equation is

$$r^2 + 4r + 4 = (r + 2)^2 = 0.$$

This gives us a repeated root $r = 2$. The general solution is

$$y(t) = c_1 e^{-2t} + c_2 t e^{-2t}.$$

Using the initial conditions we find that

$$y(0) = c_1 = 2, \quad y'(0) = -2c_1 + c_2 = 1 \quad \Rightarrow \quad c_1 = 2, \quad c_2 = 5.$$

Hence the particular solution is

$$y(t) = 2e^{-2t} + 5te^{-2t}.$$

Long time behaviour

We now investigate the behaviour of the solution

$$\boxed{y(t) = c_1 e^{-\frac{b}{2a}t} + c_2 t e^{-\frac{b}{2a}t}} \tag{9}$$

as $t \to \infty$. Note that if $\frac{b}{2a} > 0$, then

$$e^{-\frac{b}{2a}t}, \quad t e^{-\frac{b}{2a}t} \to 0 \text{ as } t \to \infty.$$

Meanwhile, if $\frac{b}{2a} < 0$, then

$$e^{-\frac{b}{2a}t}, \quad t e^{-\frac{b}{2a}t} \to \infty \text{ as } t \to \infty.$$

and in this case, $t e^{-\frac{b}{2a}t} \gg e^{-\frac{b}{2a}t}$ as $t \to \infty$.
Roughly speaking we can summarise

$$y(t) = (c_1 + c_2 t) e^{-\frac{b}{2a}t} \to \begin{cases} 0 & \text{if } \frac{b}{2a} > 0, \\ \infty & \text{if } \frac{b}{2a} < 0, \qquad c_2 > 0, \\ -\infty & \text{if } \frac{b}{2a} < 0, \qquad c_2 < 0, . \end{cases}$$

Summary

For the second order linear ODE

$$ay'' + by' + cy = 0$$

with constants a, b, c. Let r_1 and r_2 be the roots to the characteristic equation

$$ar^2 + br + c = 0.$$

- If $b^2 > 4ac$, then r_1 and r_2 are real numbers, and the general solution is given as

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

- If $b^2 < 4ac$, then r_1 and r_2 are complex numbers such that $r_1 = \lambda + i\mu$ and $r_2 = \overline{r_1} = \lambda - i\mu$ for real numbers λ, μ. Then, the general solution is given as

$$y(t) = e^{\lambda t}(c_1 \cos(\mu t) + c_2 \sin(\mu t)).$$

- If $b^2 = 4ac$, then $r_1 = r_2 = r$.Then the general solution is given as

$$y(t) = c_1 e^{-\frac{b}{2a}t} + c_2 t e^{-\frac{b}{2a}t}.$$

Given one solution, we can find the other solution

In the above we used the Wronskian to deduce that $y_2 = te^{-\frac{b}{2a}t}$ is another solution to the ODE $ay'' + by' + cy = 0$ when $b^2 = 4ac$. There is also another method, called **reduction of order**, which actually can be applied to a second order homogeneous ODE with non-constant coefficient.

Remark: Although the general method for finding a fundamental set of solutions for $y'' + p(t)y' + q(t)y = 0$ is not available, but if we can find one nonzero-solution $y_1(t)$ of the ODE, then we can use the method of reduction order to find the other one $y_2(t)$ so that (y_1, y_2) forms a fundamental set of solutions.

Reduction of order

Consider the ODE

$$y'' + p(t)y' + q(t)y = 0.$$

Suppose we know $y_1(t)$ is a **non-zero** solution to the ODE. To find a second solution, consider the function

$$y(t) = v(t)y_1(t).$$

Then, product rule entails

$$y'(t) = v'(t)y_1(t) + v(t)y_1'(t), \quad y''(t) = v''(t)y_1(t) + 2v'(t)y_1'(t) + v(t)y_1''(t).$$

Reduction of order

If y is a solution to the ODE, we find that

$$\begin{aligned} 0 &= y'' + p(t)y' + q(t)y \\ &= v''y_1 + 2v'y_1' + vy_1'' + p(t)(v'y_1 + vy_1') + q(t)vy_1 \\ \Rightarrow 0 &= y_1 v'' + (2y_1' + p(t)y_1)v'. \end{aligned}$$

This gives us a second order ODE for v that only involves v'' and v'. Now we define a new function $z = v'$ leading to

$$y_1(t)z' + (2y_1'(t) + p(t)y_1(t))z = 0.$$

Here we treat y_1 and y_1' as given functions. Note that this is a first order linear ODE

$$\frac{dz}{dt} + \frac{2y_1' + py_1}{y_1} z = 0,$$

Since $y_1 \neq 0$.

Reduction of order

Solving this gives us

$$\begin{aligned} v'(t) = z(t) &= \exp\left(-\int \frac{2y_1' + py_1}{y_1} dt\right) \\ &= \exp\left(-\int p(t)dt - 2\ln(y_1(t))\right) = \frac{1}{y_1^2(t)} \exp\left(-\int p(t)dt\right). \end{aligned}$$

Integrating once more leads to

$$v(t) = \int (y_1(t))^{-2} e^{-\int p(t)dt} dt$$

and the second solution to the ODE is given as

$$y_2(t) = y_1(t) \int (y_1(t))^{-2} e^{-\int p(t)dt} dt.$$

Reduction of order

Example 6.7

Consider the ODE

$$ay'' + by' + cy = 0,$$

with $b^2 = 4ac$. We know that $y_1 = e^{-\frac{b}{2a}t}$ is a solution. Now written in standard form we see that

$$y'' + p(t)y' + q(t)y = 0 \qquad \text{with} \qquad p(t) = \frac{b}{a}, \quad q(t) = \frac{c}{a}.$$

So from the above formula for v, we have

$$v(t) = \int e^{-\frac{b}{a}t} e^{\frac{b}{a}t} dt = t \qquad \Rightarrow \qquad y_2(t) = ty_1(t) = te^{-\frac{b}{2a}t}.$$

Reduction of order

Remark 1

This method (Reduction of Order) can be used to find a second solution to the ODE **if you already have one solution**. *The difficulty actually lies in finding a first solution to the ODE.*

Outline

Euler equations

Euler equations are the differential equations of the form

$$x^2 \frac{d^2 y}{dx^2} + Ax\frac{dy}{dx} + By = 0, \qquad x > 0$$

where A and B are constants. Introducing a new independent variable

$$t = \ln x, \qquad \text{or} \qquad x = e^t,$$

and let

$$Y(t) = y(e^t) = y(x).$$

Taking the derivative we have

$$\frac{dy(x)}{dx} = \frac{dY(t)}{dx} = \frac{dY}{dt} \cdot \frac{dt}{dx} = Y'(t)\frac{1}{x}.$$

Then,

$$x\frac{dy(x)}{dx} = Y'(t).$$

Euler equations

Taking derivative again we get

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(Y'(t)\frac{1}{x}\right) \\
&= \frac{1}{x}\frac{d}{dx}Y'(t) + Y'(t)\left(-\frac{1}{x^2}\right) \\
&= \frac{1}{x}\frac{d}{dt}Y'(t)\frac{dt}{dx} - \frac{1}{x^2}Y'(t) \\
&= \frac{1}{x^2}\left(Y''(t) - Y'(t)\right).
\end{aligned}$$

Then,

$$x^2\frac{d^2y}{dx^2} = Y''(t) - Y'(t).$$

Euler equations

substituting $x\frac{dy}{dx}$ and $x^2\frac{d^2y}{dx^2}$ into the Euler equation we get

$$Y''(t) - Y'(t) + AY'(t) + BY(t) = 0,$$

or

$$Y''(t) + (A-1)Y'(t) + BY(t) = 0.$$

This is a constant coefficient linear equation, the general solution $Y(t)$ can be obtained. Then, the general solution of the Euler equation is

$$y(x) = Y(\ln x).$$

Euler equations

Example 6.8

Find the general solution of $x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 6y = 0$.
Using the change of variable $t = \ln x$ we get

$$Y'' + Y' - 6Y = 0.$$

The general solution is

$$Y(t) = c_1 e^{-3t} + c_2 e^{2t}.$$

Therefore, the general solution of the original equation is

$$\begin{aligned} y(x) &= c_1 e^{-3\ln x} + c_2 e^{2\ln x} \\ &= c_1 x^{-3} + c_2 x^2. \end{aligned}$$