MAT2006 Tutorial #9

1. Assume $f : [0,1] \to \mathbb{R}$ is a nonnegative continuous function and $f(0) = f(1) = 0$. Show that, for any y with $0 < y < 1$, there exists $x_0 \in [0,1]$ such that $f(x_0) = f(x_0 + y)$.

2. (a) Let $f(x) : I \to \mathbb{R}$ be a function, where I is an interval (bounded or not bounded). Show that $(i) \Longrightarrow (ii) \Longrightarrow (iii)$.

(i) $f(x)$ is differentiable and its derivative is bounded over I. That is, there exists $M > 0$ such that $|f'(x)| \leq M$ for all $x \in I$.

(ii) $f(x)$ is Lipschitz continuous over I. That is, there exists $L > 0$ such that $|f(x_1) - f(x_2)| \leq L|x_1 - x_2|$ for all $x_1, x_2 \in I$.

(iii) $f(x)$ is uniformly continuous over I.

(b) Show that $f(x) = \sin\sqrt{x}$ is uniformly continuous on $[0, \infty)$.

(c) Let I be a bounded open interval, $f(x)$ and $g(x)$ are both uniformly continuous function defined on I. Show that the product $f(x)g(x)$ is also uniformly continuously on I. Is the quotient $f(x)/g(x)$ (assume it is well-defined) uniformly continuous on I?

(d) Assume $f(x)$ and $g(x)$ are differentiable and their derivatives are bounded over an open interval I. Is their product $f(x)g(x)$ uniformly continuous on I? Is $f(x)g(x)$ uniformly continuous when I is bounded?

3. Show that the function

$$f(x) = \left(\frac{2}{\pi} - 1\right)\ln x - \ln 2 + \ln(1 + x)$$

has only one zero in $(0, 1)$.

— End —