Assignment 5

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Question 1

Use Laplace transform to solve the initial value problem

$$4y^{''} + y = g(t), y(0) = 3, y^{'}(0) = -7$$

Solution 1

Take the Laplace transform of all the terms and plug in the initial conditions to obtain

$$4(s^2Y(s) - 3s + 7) + Y(s) = G(s)$$

So

$$\begin{aligned} Y(s) &= \frac{12s - 28}{4\left(s^2 + \frac{1}{4}\right)} + \frac{G(s)}{4\left(s^2 + \frac{1}{4}\right)} \\ &= \frac{3s}{s^2 + (\frac{1}{2})^2} - 14\frac{\frac{1}{2}}{s^2 + \left(\frac{1}{2}\right)^2} + \frac{1}{2}G(s)\frac{\frac{1}{2}}{s^2 + \left(\frac{1}{2}\right)^2} \end{aligned}$$

Hence

$$y(t) = 3\cos\left(\frac{t}{2}\right) - 14\sin\left(\frac{t}{2}\right) + \frac{1}{2}\int_0^t \sin\left(\frac{\tau}{2}\right) g(t-\tau)d\tau$$

So, once we decide on a g(t) all we need to do is to evaluate the integral and we'll have the solution.

Question 2

Calcualte the Laplace transform of $f(t) = sin(\omega t + \theta)$

Solution 2

$$L[f(t)] = L[sint(\omega t + \theta)] = L[sin(\omega t)]cos(\theta) + L[cos(\omega t)]sin(\theta)$$

Then

$$L[sin(\omega t)] = \frac{\omega}{s^2 + \omega^2}$$

Similarly,

$$L[cos(\omega t)] = \frac{s}{s^2 + \omega^2}$$

Therefore,

$$L[f(t)] = \frac{\omega}{s^2 + \omega^2}cos(\theta) + \frac{s}{s^2 + \omega^2}sin(\theta)$$

Question 3

If $L[f(t)] = F(s)$, prove that $L[\frac{f(t)}{t}] = \int_s^{+\infty} F(u)du$

Solution 3

$$
\begin{aligned}
F(s) &= \int_0^{+\infty} e^{-st} f(t)dt \\
\int_s^{+\infty} F(u)du &= \int_s^{+\infty} \int_0^{+\infty} e^{-ut} f(t)dtdu \\
\int_s^{+\infty} F(u)du &= \int_0^{+\infty} (\int_s^{+\infty} e^{-ut}du) f(t)dt \\
\int_s^{+\infty} F(u)du &= \int_0^{+\infty} \frac{e^{-st}}{t} f(t)dt \\
\int_s^{+\infty} F(u)du &= L[\frac{f(t)}{t}]
\end{aligned}
$$

Question 4

If $L[f(t)] = F(s)$, calculate the laplace transform of $\int_0^t f(\tau)d\tau$

Solution 4

Let $g(t) = \int_0^t f(\tau)d\tau$, then $g(0) = 0$

We have known $L[g^{'}(t)] = sL[g(t)] - g(0) = sL[g(t)]$

So $L[g(t)] = \frac{1}{s}L[g^{'}(t)] = \frac{1}{s}L[f(t)] = \frac{1}{s}F(s)$

Question 5

If $f^{'}(t) + \int_0^t f(\tau)d\tau = 1$ and $f(0) = 1$, use Laplace transform to get $f(t)$

Solution 5

$$
\begin{aligned}
&L[f^{'}(t) + \int_0^t f(\tau)d\tau] = L[1] \\
&sF(s) - f(0) + \frac{1}{s}F(s) = \frac{1}{s} \\
&F(s) = \frac{1+s}{s^2+1} \\
&f(t) = L^{-1}[\frac{1}{s^2+1} + \frac{s}{s^2+1}] = sin(t) + cos(t)
\end{aligned}
$$

Question 6

Find the inverse Laplace transform of $F(s) = \frac{2s+1}{s^2+6s+13}$

Solution 6

By observation,

$$F(s) = \frac{2(s+3)-5}{(s+3)^2+4} = 2 \times \frac{s+3}{(s+3)^2+4} - \frac{5}{2} \times \frac{2}{(s+3)^2+4}.$$

Since

$$L[\sin(2t)] = \frac{2}{s^2+4}, L[\cos(2t)] = \frac{s}{s^2+4}$$

and

$$L[e^{-3t}f(t)] = F(s+3),$$

we have

$$L^{-1}[F(s)] = e^{-3t}\left(2\cos(2t) - \frac{5}{2}\sin(2t)\right).$$

Question 7

Find the inverse Laplace transform of $F(s) = \frac{1}{(s^2+2s+2)^2}$
Hint: $sin(\alpha)sin(\beta) = -\frac{1}{2}(cos(\alpha+\beta) - cos(\alpha-\beta))$

Solution 7

$$\begin{aligned}
L[f(t)] &= \frac{1}{(s^2+2s+2)^2} \\
&= \frac{1}{((s+1)^2+1)^2} \\
&= \frac{1}{(s+1)^2+1} \cdot \frac{1}{(s+1)^2+1} \\
\because L^{-1}[\frac{1}{(s+1)^2+1}] &= e^{-t}sin(t) \\
\therefore f(t) &= e^{-t}sin(t) * e^{-t}sin(t) \\
&= \int_0^t e^{-\tau}sin(\tau)e^{-(t-\tau)}sin(t-\tau)d\tau \\
&= e^{-t}\int_0^t sin(\tau)sin(t-\tau)d\tau \\
&= \frac{e^{-t}}{2}\int_0^t (cos(2\tau-t) - cos(t))d\tau \\
&= \frac{e^{-t}}{2}(sin(t) - tcos(t))
\end{aligned}$$

Question 8

Find the inverse Laplace transform of $F(s) = \frac{e^{-s}}{s^2} + \frac{e^{-3s}}{s-4}$ and write it as a right continuous piecewise-defined function.

Solution 8

The inverse Laplace transforms of $1/s^2$ and $1/(s-4)$ are t and e^{4t}, respectively. Thus, the inverse Laplace transform of $F(s)$ is

$$
\begin{aligned}
L^{-1}[F(s)] &= L^{-1}[e^{-s}\frac{1}{s^2}] + L^{-1}[e^{-3s}\frac{1}{s-4}] \\
&= u_1(t)(t-1) + u_3(t)e^{4t-12}
\end{aligned}
$$

On the interval $[0,1)$, both $t-1$ and e^{4t-12} are off. On the interval $[1,3)$, only $t-1$ is on. On the interval $[3,\infty)$, both $t-1$ and e^{4t-12} are on. Thus,

$$
L^{-1}[F(s)] = \begin{cases} 0 & \text{if } 0 \le t < 1 \\ t-1 & \text{if } 1 \le t < 3 \\ t-1+e^{4t-12} & \text{if } 3 \le t < \infty \end{cases}.
$$

Question 9

Use the Laplace transform to solve the following initial value problem:

$$
y' + 2y = f(t), \quad y(0) = 1,
$$

where

$$
f(t) = \begin{cases} 0 & \text{if } 0 \le t < 1 \\ t & \text{if } 1 \le t < \infty \end{cases}.
$$

Solution 9

Since $t \geq 0$, we first rewrite $f(t)$ as $f(t) = u_1(t)(t-1) + u_1(t)$. Thus, its Laplace transform is

$$L[f(t)] = e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right).$$

Let $Y(s) = L[y(t)]$ where $y(t)$ is the solution to the differential equation. Obviously, we can apply the Laplace transform to the differential equation and conclude

$$sY(s) - 1 + 2Y(s) = e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right).$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{1}{s+2} + e^{-s}\frac{s+1}{s^2(s+2)}.$$

A partial fraction decomposition gives

$$\frac{s+1}{s^2(s+2)} = \frac{1}{4} \times \frac{1}{s} + \frac{1}{2} \times \frac{1}{s^2} - \frac{1}{4} \times \frac{1}{s+2}.$$

Thus, we have

$$\begin{aligned} y(t) &= L^{-1}[Y(s)] \\ &= L^{-1}[\frac{1}{s+2}] + \frac{1}{4}L^{-1}[e^{-s}\frac{1}{s}] + \frac{1}{2}L^{-1}[e^{-s}\frac{1}{s^2}] - \frac{1}{4}L^{-1}[e^{-s}\frac{1}{s+2}] \\ &= e^{-2t} + \frac{1}{4}u_1(t) + \frac{1}{2}u_1(t)(t-1) - \frac{1}{4}u_1(t)e^{-2t+2}. \end{aligned}$$

Evaluating this piecewise gives

$$y(t) = \begin{cases} e^{-2t} & \text{if } 0 \leq t < 1 \\ e^{-2t} + \frac{1}{4}(2t-1) - \frac{1}{4}e^{-2t+2} & \text{if } 1 \leq t < \infty \end{cases}.$$

Question 10

Use Laplace transform to solve the following initial value problem

$$y' - y = \int_0^t (t-\lambda)e^{\lambda}d\lambda, \quad y(0) = -1$$

Solution 10

Note that $y' - y = t * e^t$. Taking Lalplace transform of both sides we find $sY(s) - (-1) - Y(s) = \frac{1}{s^2} \cdot \frac{1}{s-1}$. This implies that $Y(s) = -\frac{1}{s-1} + \frac{1}{s^2(s-1)^2}$. Using partial fractions decomposition we can write

$$\frac{1}{s^2(s-1)^2} = \frac{2}{s} + \frac{1}{s^2} - \frac{2}{s-1} + \frac{1}{(s-1)^2}.$$

Thus,

$$Y(s) = -\frac{1}{s-1} + \frac{2}{s} + \frac{1}{s^2} - \frac{2}{s-1} + \frac{1}{(s-1)^2} = \frac{2}{s} + \frac{1}{s^2} - \frac{3}{s-1} + \frac{1}{(s-1)^2}.$$

Finally,

$$y(t) = 2 + t - 3e^t + te^t, t \geq 0$$

Question 11

Use Laplace transform to solve the initial value problem

$$y'' + 2y' + y = \delta(t-2), \quad y(0) = 0, y'(0) = 1, 0 \leq t \leq 6 .$$

Graph the solution on the indicated interval.

Solution 11

Taking Laplace transform of both sides to obtain

$$s^2Y - 1 + 2sY + Y = e^{-2s}$$

Solving for $Y(s)$ we find

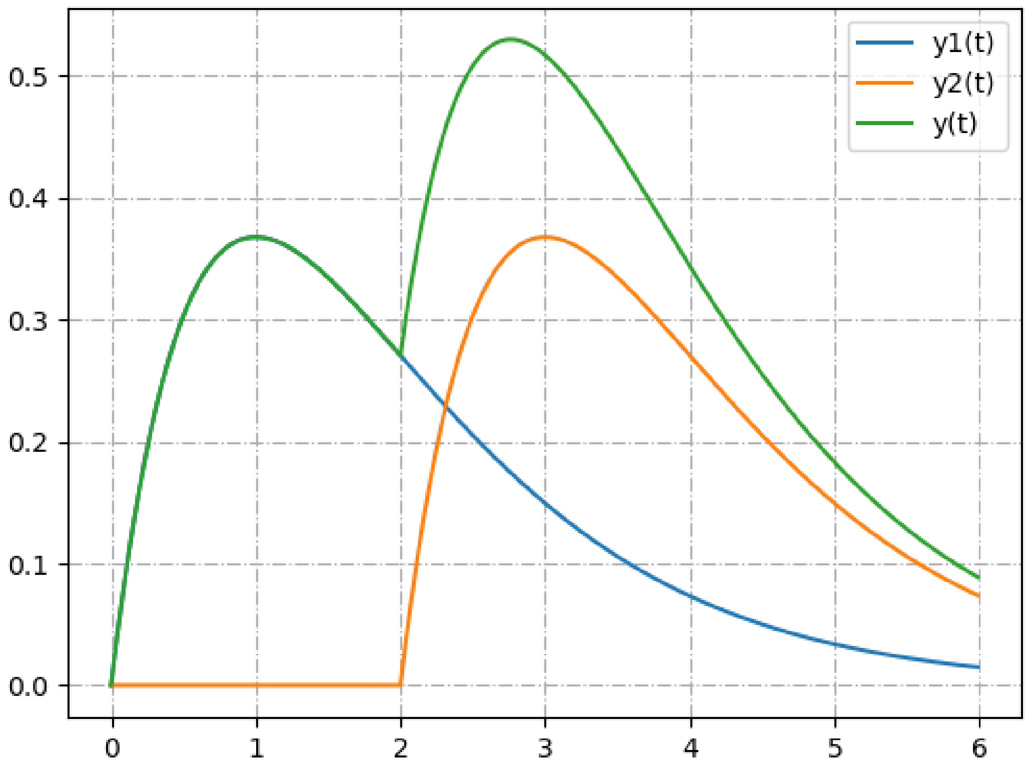
$$Y(s) = \frac{1}{(s+1)^2} + \frac{e^{-2s}}{(s+1)^2}$$

Therefore,

$$y_1(t) = L^{-1}[\frac{1}{(s+1)^2}] = te^{-t}$$

$$y_2(t) = L^{-1}[\frac{e^{-2s}}{(s+1)^2}] = (t-2)e^{-(t-2)}u_2(t)$$

$$y(t) = y_1(t) + y_2(t) = te^{-t} + (t-2)e^{-(t-2)}u_2(t)$$



y1(t)
y2(t)
y(t)
0.0
0.1
0.2
0.3
0.4
0.5
0
1
2
3
4
5
6