MAT 3253 Lecture 12

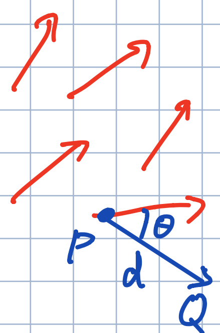
$f : \mathbb{R} \to \mathbb{C}$

$f(t) = u(t) + i\, v(t)$

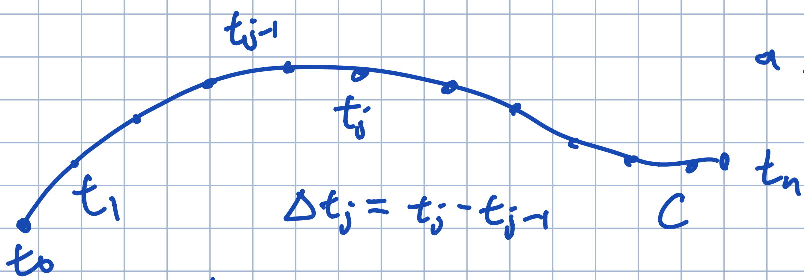
$\int_a^b f(t)\,dt \triangleq \int_a^b u(t)\,dt + i \int_a^b v(t)\,dt$

Work done / line integral

$\vec{F}(x,y) = \big(M(x,y),\ N(x,y)\big)$



$\vec{F} \cdot \vec{d}$ = work done



$a \leq t \leq b$

$\vec{r}(t)$

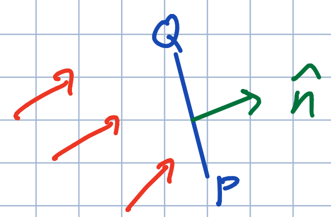
$$\sum_{j=1}^{n} \vec{F}(\vec{r}(t_j)) \cdot \vec{r}\,'(t_j)\,\Delta t_j$$

$$\to \int_C \vec{F} \cdot d\vec{r} = \int \vec{F} \cdot \vec{T}\, ds$$

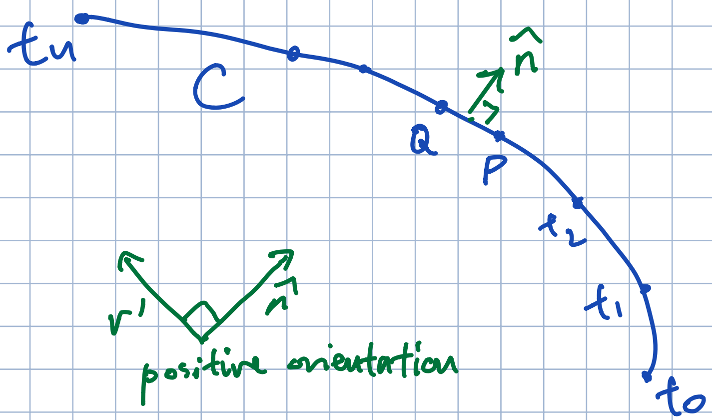
$$= \int M\,dx + N\,dy$$

Flux integral / flow integral

$\vec{F}(x,y) = (M(x,y), N(x,y))$



$\vec{F} \cdot \hat{n} \, |PQ|$



$a \le t \le b$

$\vec{r}(t)$ parametrizes C

$\vec{r}\,'(t)\Delta t = (\Delta x, \Delta y)$

$\hat{n} = (n_x, n_y)$

$$(n_x, n_y) = \left(\frac{\Delta y}{\sqrt{\Delta x^2 + \Delta y^2}}, \frac{-\Delta x}{\sqrt{\Delta x^2 + \Delta y^2}} \right), \quad \begin{vmatrix} n_x & n_y \\ \Delta x & \Delta y \end{vmatrix} > 0$$

$$\text{flux} = \int_C \vec{F} \cdot \hat{n} \, ds = \int_C M \, dy - N \, dx$$

$f : \mathbb{C} \to \mathbb{C}$

Assume f is continuous in a region $R \subseteq \mathbb{C}$

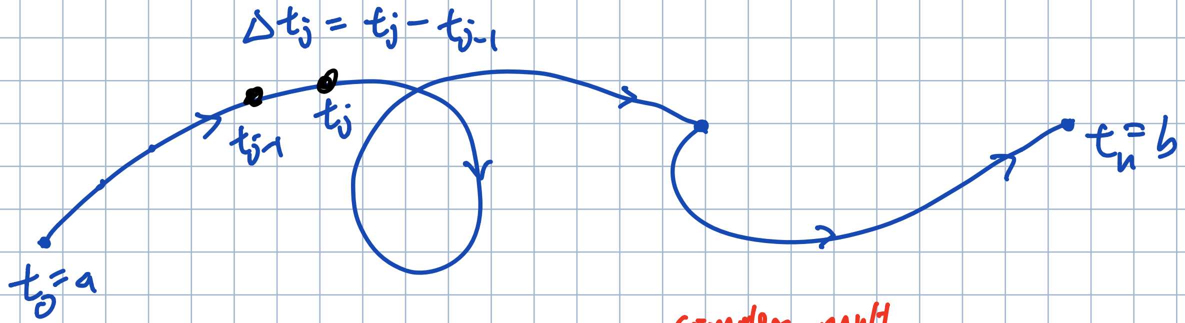
Curve / contour / path C

Parameterize C by $z(t)$ $\quad z : [a,b] \to \mathbb{C}$

$z(t) = x(t) + i\, y(t)$

Assume $x(t)$ and $y(t)$ are piece-wise differentiable, in C'

Assume C is smooth, $z'(t) \neq 0$ except for finitely many points.



complex mult.

$$\sum_{j=1}^{n} f(z(t_j)) \cdot z'(t_j)\, \Delta t_j$$

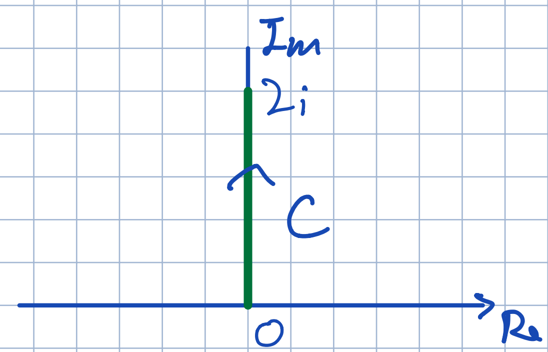
take limit as $n \to \infty$, $\max_j \Delta t_j \to 0$

Def $$\boxed{\int_C f(z)\, dz \triangleq \int_a^b f(z(t)) \cdot z'(t)\, dt}$$

Example $f(z) = \bar{z}$

$C: z(t) = 0 + it$

$0 \le t \le 2$

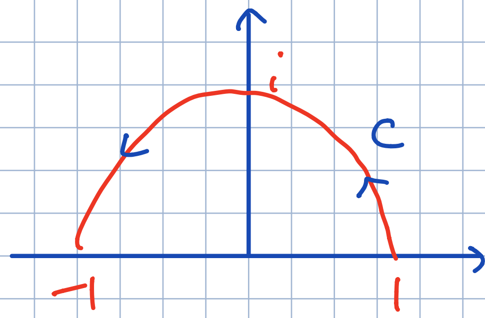


$$\int_C \bar{z}\, dz = \int_0^2 \underbrace{(0 + it)^*}_{f(z(t))} \cdot \underbrace{(i)}_{z'(t)}\, dt$$

$$= \int_0^2 (-it)\, i\, dt$$

$$= \left[\frac{t^2}{2}\right]_0^2 = 2$$

Example $\quad f(z)=z^2$



$$C:\ z(t)=\cos(t)+i\sin t,\qquad 0\le t\le \pi$$

$$z'(t)=-\sin(t)+i\cos t$$

$$\int_C z^2\,dz=\int_0^{\pi}\underbrace{(\cos t+i\sin t)^2}_{f(z)}\cdot(\overset{i^2}{-}\sin t+i\cos t)\,dt$$

$$=i\int_0^{\pi}(\cos 2t+i\sin 2t)(\cos t+i\sin t)$$

$$=i\int_0^{\pi}\cos 3t+i\sin 3t\,dt$$

$$=i\left[\frac{\sin 3t}{3}-i\frac{\cos 3t}{3}\right]_0^{\pi}$$

$$=-\frac{2}{3}$$

Physical interpretation

$$f(z)=u(x,y)+i\,v(x,y)$$

Define a vector field $\vec{F}(x,y)=(M(x,y),N(x,y))$

$$M(x,y)=u(x,y),\qquad N(x,y)=-v(x,y)$$

$$\int_C f(z)\,dz=\int_C(u+iv)\cdot(x'+iy')\,dt$$

$$=\int_C(u+iv)(dx+i\,dy)$$

$$= \int_C (u\,dx - v\,dy) + i(u\,dy + v\,dx)$$

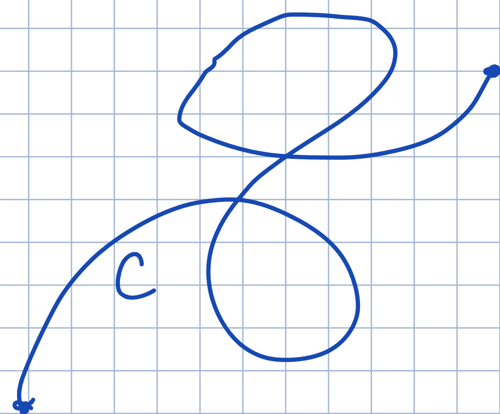
$$= \int_C \underbrace{(M\,dx + N\,dy)}_{\text{work integral}} + i \int_C \underbrace{(M\,dy - N\,dx)}_{\text{flux integral / flow integral}}$$

Independence of parameterization

$$f(z) = u(z) + i v(z)$$

C : $z(t)$ for $a \le t \le b$

$w(t)$ for $c \le t \le d$



We can find $\lambda(t) : [c, d] \to [a, b]$ monotonically

s.t. $w(t) = z(\lambda(t))$

$$\int_c^d f(w(t)) \cdot w'(t)\,dt$$

$$= \int_c^d f(z(\lambda(t))) \cdot \underbrace{z'(\lambda(t))\,\lambda'(t)}_{\text{chain rule}}\,dt$$

$$= \int_c^d \Big(u(z(\lambda(t))) + i v(z(\lambda(t))) \Big) \cdot \Big(x'(\lambda(t)) + i y'(\lambda(t)) \Big) \lambda'(t)\,dt$$

$$= \int_c^d \Big[u(z(\lambda(t)))\,x'(\lambda(t)) - v(z(\lambda(t)))\,y'(\lambda(t)) \Big] \lambda'(t)\,dt$$

$$+ i \int_c^d \Big[u(z(\lambda(t)))\,y'(\lambda(t)) + v(z(\lambda(t)))\,x'(\lambda(t)) \Big] \lambda'(t)\,dt$$

Substitute $\tau = \lambda(t)$

$d\tau = \lambda'(t)\, dt$

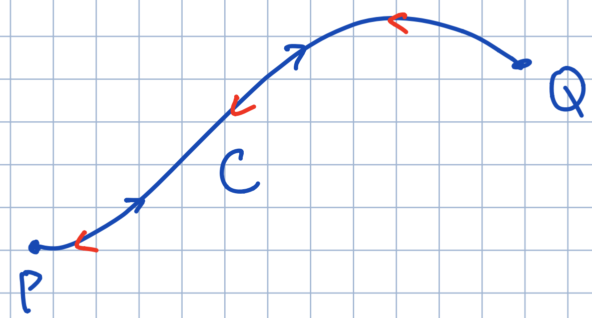
$$= \int_a^b u(z(\tau))\, x'(\tau) - v(z(\tau))\, y'(\tau)\, d\tau$$

$$+ i \int_a^b u(z(\tau))\, y'(\tau) + v(z(\tau))\, x'(\tau)\, d\tau$$

$$= \int_a^b f(z(\tau)) \cdot z'(\tau)\, d\tau \qquad \blacksquare$$

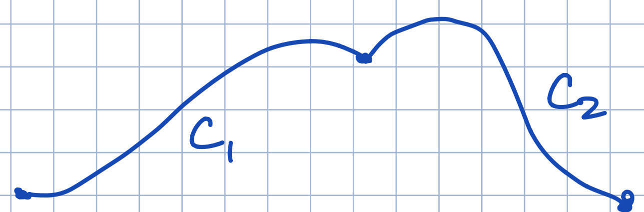
<u>Def</u> The negation of a curve C is the same locus with reverse direction

Negation of C is denoted by $-C$



<u>Theorem</u> $$\int_{-C} f(z)\, dz = -\int_C f(z)\, dz$$

$C_1 + C_2$



<u>Theorem</u> $$\int_{C_1 + C_2} f(z) = \int_{C_1} f\, dz + \int_{C_2} f\, dz$$

$$\int_C f+g\, dz = \int_C f\, dz + \int_C g\, dz$$

$$\int_C af\, dz = a \int_C f\, dz \qquad \forall\, a \in \mathbb{C}.$$