# MAT3253 Homework 8

Due date: 26 Mar.

**Question 1**. (Brown&Churchill Ex.42.6) Let $f(z)$ be the branch

$$z^{-1+i} = \exp((-1+i)\log z), \qquad (|z| > 0,\ 0 < \arg z < 2\pi)$$

of the indicated function. Evaluate the contour integral $\int_C f(z)\,dz$ when $C$ is the unit circle $z = e^{i\theta}$ $(0 \le \theta \le 2\pi)$.

**Question 2**. (Brown&Churchill Ex.42.9) Evaluate the integral

$$I = \int_C \bar{z}\,dz$$

using the parametric curve $C$

$$C:\ z(y) = \sqrt{4-y^2} + iy \qquad (-2 \le y \le 2).$$

**Question 3**. (Brown&Churchill Ex.43.4) Let $C_R$ denote the upper half of the circle $|z| = R$ $(R > 2)$, taken in the counterclockwise direction. Show that

$$\Big| \int_{C_R} \frac{2z^2-1}{z^4+5z^2+4} \Big| \le \frac{\pi R(2R^2+1)}{(R^2-1)(R^2-4)}$$

Then, by dividing the numerator and denominator on the right here by $R^4$, show that the value of the integral tends to zero as $R$ tends to infinity.

**Question 4**. (Brown&Churchill Ex.45.5) Show that

$$\int_{-1}^{1} z^i\,dz = \frac{1+e^{-\pi}}{2}(1-i),$$

where the integrand denotes the principal branch

$$z^i = \exp(i\mathrm{Log}(z)) \ \ (|z| > 0,\ -\pi < \mathrm{Arg}(z) < \pi)$$

(Here, the function $\mathrm{Log}(z)$ and $\mathrm{Arg}(z)$ denote the principal value of log and arg, respectively.)

Hint: Use an anti-derivative of the branch

$$z^i = \exp(i\log z) \qquad \big(|z| > 0, -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}\big)$$

of the same power function.

**Question 5**. (Bak&Newman Chapter 4 Ex.6) Show that, if $f$ is a continuous real-valued function and $|f| \le 1$, then

$$\Big|\int_{|z|=1} f(z)\,dz\Big| \le 4.$$

(The $ML$ inequality easily gives $|\int_{|z|=1} f(z)\,dz| \le 2\pi$. The purpose of this question is to strengthen this bound from $2\pi$ to 4.)

Hint: Show that

$$\Big|\int f\Big| \le \int_0^{2\pi} |\sin t|\,dt.$$

**Question 6**. (Bak&Newman Chapter 4 Ex.10) Evaluate

(a) $\int_0^i e^z\,dz$.

(b) $\int_{\pi/2}^{\pi/2+i} \cos 2z\,dz$.

**Question 7**. (Bak&Newman Chapter 4 Ex.11) Suppose $f$ is analytic in a convex region $D$ and $|f'| \le 1$ throughout $D$. Prove that $f$ is a "contraction"; i.e., show that

$$|f(b) - f(a)| \le |b - a|$$

for all $a, b \in D$.