MAT3253 Homework 15

Due date: 14 May.

Question 1. (Bak&Newman Chapter 7 Ex.1) Show that if f is analytic and non-constant on a compact domain, Ref and Imf assume their maxima and minima on the boundary.

Question 2. (Bak&Newman Chapter 7 Ex.6) Show that for any given rational function $f(z)$, with poles in the unit disc, it is possible to find another rational function $g(z)$, with no poles in the unit disc, and such that $|f(z)| = |g(z)|$ if $|z| = 1$.

Question 3. (Brown&Churchill Sec. 54 Ex.3) Let a function f be continuous on a closed bounded region R, and let it be analytic and not constant throughout the interior of R. Assuming that $f(z) \neq 0$ anywhere in R, prove that $|f(z)|$ has a *minimum value* m in R which occurs on the boundary of R and never in the interior. Do this by applying the corresponding result for maximum values to the function $g(z) = 1/f(z)$.

Question 4. (Brown&Churchill Sec. 54 Ex.5) Consider the function $f(z) = (z+1)^2$ and the closed triangular region R with vertices at the points $z = 0$, $z = 2$, and $z = i$. Find points in R where $|f(z)|$ has its maximum and minimum values,

Question 5. (Bak&Newman Chapter 7 Ex.8) Suppose that f is analytic in the annulus: $1 \leq |z| \leq 2$, that $|f| \leq 1$ for $|z| = 1$ and that $|f| \leq 4$ for $|z| = 2$. Prove $|f(z)| \leq |z|^2$ throughout the annulus.

Question 6. (Bak&Newman Chapter 13 Ex.1) Verify directly that $f(z) = z^k$ is locally one-to-one for $z \neq 0$, k any nonzero integer.

Question 7. (Bak&Newman Chapter 13 Ex.3) Find a conformal mapping f between the regions S and T, where

(i) $S = \{z = x + iy : -2 < x < 1\}$; $T = D(0; 1)$

(ii) $S = T =$ the upper half-plane; $f(-2) = -1, f(0) = 0$ and $f(2) = 2$

(iii) $S = \{re^{i\theta} : r > 0 \text{ and } 0 < \theta < \pi/4\}$; $T = \{x + iy : 0 < y < 1\}$

(iv) $S = D(0; 1) \setminus [0, 1]$; $T = D(0; 1)$.

Hint: For (iv) use a mapping of the upper semi-disc onto a quadrant.

Question 7. (Bak&Newman Chapter 13 Ex.10) Find the image of the circle $|z| = 1$ under the mappings

(a) $\omega = 1/z$,
(b) $\omega = 1/(z-1)$,
(c) $\omega = 1/(z-2)$.

Question 8. (Bak&Newman Chapter 13 Ex.14) What is the image of the upper half-plane under a mapping of the form

$$f(z) = \frac{az+b}{cz+d} \qquad a, b, c, d \text{ real}; \quad ad - bc < 0?$$

Question 9. (Bak&Newman Chapter 13 Ex.19) Find the fractional linear transformations which send

(a) $1,\ i,\ -1$ onto $-1,\ i,\ 1,$ respectively
(b) $-i,\ 0,\ i$ onto $0,\ i,\ 2i,$ respectively
(c) $-i,\ i,\ 2i$ onto $\infty,\ 0,\ 1/3,$ respectively.

Question 10. Derive the Fresnel integrals

$$\int_0^\infty \sin(x^2)\,dx = \int_0^\infty \cos(x^2)\,dx = \frac{\sqrt{2\pi}}{4}.$$

Hint: Integrate the function e^{-z^2} over a closed path consisting of three parts. The first part is a line segment from 0 to R on real axis, for some constant R. The second part is a portion of the circle from R to $Re^{i\pi/4}$, parameterized by $Re^{i\theta}$ for θ from 0 to $\pi/4$. The third part is a line segment from $Re^{i\pi/4}$ to the origin. You can use the fact that $\int_0^\infty e^{-x^2}\,dx = \sqrt{\pi}/2$.

Question 11. For any real constant $0 < a < 1$, show that

$$\int_{-\infty}^\infty \frac{e^{ax}}{e^{2x}+1}\,dx = \frac{\pi}{2\sin(\pi a/2)}.$$

Hint: Integrate the function $e^{az}/(e^{2z}+1)$ over a rectangle with vertices R, $R+2\pi i$, $-R+2\pi i$ and $-R$. Show that the integrals along the two vertical legs approach zero as $R \to \infty$.