Assignment 2

Deadline: 5pm, Mar 19th, 2021

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Question 1

Determine a formula for the n-th Picard approximation for the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ay, y(0) = 1, a \in \mathbb{R}$$

What is the limiting function $y(t) = \lim_{n\to\infty} y_n(t)$. Is it a solution? Are there other solutions that we may have missed?

Question 2

1. Find the exact solution of the initial value problem
$$\frac{\mathrm{d}y}{\mathrm{d}t} = y^2, y(0) = 1.$$
2. Calculate the first three Picard approximations $y_1(t)$, $y_2(t)$, and $y_3(t)$ and compare these results with the exact solution.

Question 3

1. Let $y_0(t)$ be a solution of the differential equation
$$y'' + p(t)y' + q(t)y = 0.$$
We assume that $p(t)$ and $q(t)$ are continuous functions on an interval I, so **the Existence and Uniqueness theorem** implies that a solution y_0 is defined on I. Show that if the curve of $y_0(t)$ is tangent to the t-axis at some point t_0 of I, then $y_0(t) = 0$ for all $t \in I$.

2. More generally, let $y_1(t)$ and $y_2(t)$ be two solutions of the differential equation
$$y'' + p(t)y' + q(t)y = f(t)$$
where we assume that $p(t)$, $q(t)$, and $f(x)$ are continuous functions on an interval I, so that **the Existence and Uniqueness theorem** implies that y_1 and y_2 are defined on I. Show that if the curves of $y_1(t)$ and $y_2(t)$ are tangent at some point t_0 of I, then $y_1(t) = y_2(t)$ for all $t \in I$.

Question 4

For each exercise below, verify that the functions f_1 and f_2 satisfy the given differential equation. Verify Abel's formula as given in **Abel's theorem** for the given initial point t_0. Determine the solution set.

1. $(t-1)y'' - ty' + y = 0, f_1(t) = e^t - t, f_2(t) = t, t_0 = 0$
2. $(1+t^2)y'' - 2ty' + 2y = 0, f_1(t) = 1 - t^2, f_2(t) = t, t_0 = 1$
3. $t^2y'' + ty' + 4y = 0, f_1(t) = \cos(2\ln t), f_2(t) = \sin(2\ln t), t_0 = 1$

Question 5

1. Verify that $y_1(t) = t^3$ and $y_2(t) = |t^3|$ are linearly independent on $(-\infty, +\infty)$.
2. Verify that $y_1(t)$ and $y_2(t)$ are solutions to the initial value problem
$$t^2y'' - 2ty' = 0, y(0) = 0, y'(0) = 0.$$
3. Explain why Parts (a) and (b) do not contradict **the Existence and Uniqueness theorem for the second-order linear equation**.
4. Show that the Wronskian, $W[y_1, y_2](t) = 0$, for all $t \in \mathbb{R}$.

Question 6

For each differential equation and the given solution, use reduction of order to find a second independent solution $y_2(t)$ and write down the general solution $y(t)$.

1. $t^2y'' - 3ty' + 4y = 0, y_1(t) = t^2$, where $t > 0$
2. $t^2y'' + 2ty' = 0, y_1(t) = \frac{1}{t}$, where $t \neq 0$
3. $t^2y'' - t(t+2)y' + (t+2)y = 0, y_1(t) = t$
4. $t^2y'' - 2ty' + (t^2+2)y = 0, y_1(t) = t\cos t$, where $t \neq k\pi + \pi/2, k \in \mathbb{Z}$

Question 7

Solve the following initial value problems.

1. $y'' - y = 0, y(0) = 0, y'(0) = 1$
2. $y'' - 10y' + 25y = 0, y(0) = 0, y'(0) = 1$
3. $y'' + 4y' + 13y = 0, y(0) = 1, y'(0) = -5$

Question 8

Find the general solution of each of the following Euler equations on $(0, \infty)$.

1. $x^2y'' + 7xy' + 9y = 0$
2. $x^2y'' + xy' - 4y = 0$
3. $x^2y'' + xy' + 4y = 0$

Question 9

Find the general solution of the following ODE.

$$x^2y'' + xy' - 4y = 3x$$

Question 10

Find $f(t)$ such that the following equation is valid.

$$f(x) = \int_0^x (x - t)f(t)dt + x$$