MAT2006: Elementary Real Analysis
Assignment #3

Deadline: Nov 14

1. Let A be nonempty and bounded above so that $s = \sup A$ exists.
(i) Show that $s \in \overline{A}$.
(ii) Can an open set contain its supremum?

2. (i) Prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
(ii) Does this result about closures extend to infinite unions of sets?

3. Let A be an uncountable set and let B be the set of real numbers that divides A into two uncountable sets; that is, $s \in B$ if both $\{x \,|\, x \in A \quad \text{and} \quad x < s\}$ and $\{x \,|\, x \in A \quad \text{and} \quad x > s\}$ are uncountable. Show B is nonempty and open.

4. Prove that the only sets that are both open and closed are $\mathbb{R}$ and the empty set $\emptyset$.

5. A dual notion to the closure of a set is the *interior* of a set. The interior of E is denoted E° and is defined as

$$E^\circ = \{x \in E \,|\, \text{there exists } V_\epsilon(x) \subset E\}.$$

Results about closures and interiors possess a useful symmetry.
(i) Show that E is closed if and only if $\overline{E} = E$. Show that E is open if and only if $E^\circ = E$.
(ii) Show that $(\overline{E})^c = (E^c)^\circ$ and $(E^\circ)^c = \overline{E^c}$.

6. Show that if a set $K \subset \mathbb{R}$ is closed and bounded, then it is sequentially compact.

7. Show that if K is sequentially compact and nonempty, then $\sup K$ and $\inf K$ both exist and are elements of K.

8 (NIP+AP implies HB)**.** Provide a proof of a bounded and closed set is compact using the Nested Interval Property.

Suppose $K \subset \mathbb{R}$ is closed and bounded, and let $\{O_\lambda \,|\, \lambda \in \Lambda\}$ be an open cover for K. For contradiction, let's assume that no finite subcover exists. Let I_0 be a closed interval containing K.

(a) Show that there exists a nested sequence of closed intervals $I_0 \supset I_1 \supset I_2 \supset \cdots$ with the property that, for each n, $I_n \cap K$ cannot be finitely covered and $\lim_{n\to\infty} |I_n| = 0$.
(b) Argue that there exists an $x \in K$ such that $x \in I_n$ for all n.
(c) Because $x \in K$, there must exist an open set O_{λ_0} from the original collection that contains x as an element. Explain how this leads to the desired contradiction.

9 (LUBP implies HB)**.** Consider the special case where K is a closed interval. Let $\{O_\lambda \mid \lambda \in \Lambda\}$ be an open cover for $[a, b]$ and define S to be the set of all $x \in [a, b]$ such that $[a, x]$ has a finite subcover from $\{O_\lambda \mid \lambda \in \Lambda\}$.

(a) Argue that S is nonempty and bounded, and thus $s = \sup S$ exists.

(b) Now show $s = b$, which implies $[a, b]$ has a finite subcover.

(c) Finally, prove the theorem for an arbitrary closed and bounded set K.

10 (HB implies BW)**.** Using the concept of open covers (and explicitly avoiding the Bolzano–Weierstrass Theorem), prove that every bounded infinite set has a limit point. Therefore, every bounded sequence has a convergent subsequence (BW).

11. Show that

(a) The countable union of F_σ sets is an F_σ set.

(b) The finite intersection of F_σ sets is an F_σ set.

(c) Give an example of the countable intersection of F_σ sets is not F_σ.

(d) The finite union of G_δ sets is a G_δ set.

(e) The countable intersection of G_δ sets is a G_δ set.

12. (i) For each of the following sets, determine whether it is an F_σ and/or G_δ set, explain why.

$(a)\quad (a, b);\qquad (b)\quad [a, b];\qquad (c)\quad (a, b];\qquad (d)\quad \mathbb{Q};\qquad (e)\quad \mathbb{I};$

(ii) [bonus question] We know that any open set is G_δ (why?).

(g) Show that any open set can be written as the union of at most countable intervals.

(h) Show that any open set is F_σ, and any closed set is G_δ.

13 (Infinite Limits)**.** *Definition*: $\lim_{x\to c} f(x) = \infty$ means that for all $M > 0$ we can find a $\delta > 0$ such that whenever $0 < |x - c| < \delta$, it follows that $f(x) > M$.

(i) Show $\lim\limits_{x\to 0} 1/x^2 = \infty$ in the sense described in the previous definition.

(ii) Now, construct a definition for the statement $\lim\limits_{x\to\infty} = L$. Show $\lim\limits_{x\to\infty} 1/x = 0$.

(iii) What would a rigorous definition for $\lim\limits_{x\to\infty} f(x) = \infty$ look like? Give an example of such a limit.

14 (Right and Left Limits)**.** Introductory calculus courses typically refer to the right-hand limit of a function as the limit obtained by "letting x approach c from the right-hand side."

(i) Give a proper ϵ–δ definition for the right-hand and left-hand limit statements:

$$\lim_{x\to c^+} f(x) = L, \qquad \lim_{x\to c^-} f(x) = M.$$

(ii) Prove that $\lim_{x\to c} f(x) = L$ if and only if both the right and left-hand limits equal L.

15 (Upper and Lower Limits)**.** As in the case of sequential limits, we have the upper and lower limits for a function,

$$\limsup_{x\to c} f(x) := \lim_{\delta\to 0^+} \sup_{0<|x-c|<\delta} f(x),$$

$$\liminf_{x\to c} f(x) := \lim_{\delta\to 0^+} \inf_{0<|x-c|<\delta} f(x).$$

Show that $\lim\limits_{x\to c} f(x)$ exists if and only if both $\limsup\limits_{x\to c}$ and $\liminf\limits_{x\to c}$ exist and they are equal to each other.

16 (Cauchy Criterion)**.** Let $f : A \to \mathbb{R}$ be a function and c a limit point of A. Show that $\lim\limits_{x\to c} f(x)$ exists if and only if for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - f(y)| < \epsilon \qquad \forall 0 < |x-c| < \delta, \quad \forall 0 < |y-c| < \delta.$$

17. Assume $h : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and let $K = \{x \,|\, h(x) = 0\}$. Show that K is a closed set.

18. Observe that if a and b are real numbers, then

$$\max\{a,b\} = \frac{(a+b) + |a-b|}{2}.$$

(i) Show that if $f_1, f_2, \ldots, f_n$ are continuous functions, then

$$g(x) = \max\{f_1(x), f_2(x), \ldots, f_n(x)\}$$

is a continuous function.

(ii) Let's explore whether the result in (i) extends to the infinite case. For each $n \in \mathbb{N}$, define f_n on $\mathbb{R}$ by

$$f_n(x) = \begin{cases} 1 & \text{if } |x| > 1/n \\ n|x| & \text{if } |x| \leq 1/n. \end{cases}$$

Now explicitly compute $h(x) = \sup\{f_1(x), f_2(x), f_3(x), \ldots\}$.

19. Let $F \subset \mathbb{R}$ be a nonempty closed set and define $g(x) = \inf\{|x-a| \,:\, a \in F\}$. Show that g is continuous on all of $\mathbb{R}$ and $g(x) \neq 0$ for all $x \notin F$.

20. Recall the theorem "A function that is continuous on a compact set K is uniformly continuous on K." Provide a proof by the definition "$K \subset \mathbb{R}$ is compact if every open cover of K has a finite subcover."

21. (i) Assume that g is defined on an open interval (a,c) and it is known to be uniformly continuous on $(a,b]$ and $[b,c)$, where $a < b < c$. Prove that g is uniformly continuous on (a,c).

(ii) Prove that $f(x) = \sqrt{x}$ is uniformly continuous on $[0,\infty)$.

(iii) Show that $f(x) = x^p$ with $p \in \mathbb{R}$ is uniformly continuous on $(0,\infty)$ if and only if $0 \leq p \leq 1$.

(iv) Assume $f(x)$ is a continuous function defined on $[0,\infty)$, and assume that $\lim_{x\to\infty} f(x) = L \in \mathbb{R}$. Show that $f(x)$ is uniformly continuous on $[0,\infty)$.

22. Give an example of each of the following, or provide a short argument for why the request is impossible.

(a) A continuous function defined on $[0,1]$ with range $(0,1)$.

(b) A continuous function defined on $(0,1)$ with range $[0,1]$.

(c) A continuous function defined on $(0,1]$ with range $(0,1)$.

23 (Continuous Extension Theorem)**.** (i) Show that a uniformly continuous function preserves Cauchy sequences; that is, if $f : A \to \mathbb{R}$ is uniformly continuous and $\{x_n\} \subset A$ is a Cauchy sequence, then show $f(x_n)$ is a Cauchy sequence.

(ii) Let g be a continuous function on the open interval (a, b). Prove that g is uniformly continuous on (a, b) if and only if it is possible to define values $g(a)$ and $g(b)$ at the endpoints so that the extended function g is continuous on $[a, b]$. (In the forward direction, first produce candidates for $g(a)$ and $g(b)$, and then show the extended g is continuous.)

24. Show that the following functions is not uniform continuous on (0,1).

$$(a) \quad f(x) = \sin\frac{1}{x}; \qquad (b) \quad g(x) = \ln x; \qquad (c) \quad h(x) = \frac{1}{1-x}.$$

25. Let $f : [0, 1] \to \mathbb{R}$ be continuous with $f(0) = f(1)$.

(i) Show that there must exist $x, y \in [0, 1]$ satisfying $|x - y| = 1/2$ and $f(x) = f(y)$.

(ii) Show that for each $n \in \mathbb{N}$ there exist $x_n, y_n \in [0, 1]$ with $|x_n - y_n| = 1/n$ and $f(x_n) = f(y_n)$.

(iii) If $h \in (0, 1/2)$ is not of the form $1/n$, there does not necessarily exist $|x - y| = h$ satisfying $f(x) = f(y)$. Provide an example that illustrates this using $h = 2/5$.

26. Let f be a continuous function on the closed interval $[0, 1]$ with range also contained in $[0, 1]$. Prove that f must have a fixed point; that is, show $f(x) = x$ for at least one value of $x \in [0, 1]$.

27 (Inverse functions)**.** If a function $f : A \to \mathbb{R}$ is one-to-one, then we can define the inverse function f^{-1} on the range of f in the natural way: $f^{-1}(y) = x$ where $y = f(x)$. Show that if f is continuous on an interval $[a, b]$ and one-to-one, then f^{-1} is also continuous.

28. (i) Given a countable set $A = \{a_1, a_2, a_3, \dots\}$, define $f(a_n) = 1/n$ and $f(x) = 0$ for all $x \notin A$. Find D_f.

(ii) Is it possible for a function f such that $D_f = \mathbb{I}$?

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