Chapter 4. Basic Topology of $\mathbb{R}$ *

1 Discussion: The Cantor Set

What follows is a fascinating mathematical construction, due to Georg Cantor, which is extremely useful for extending the horizons of our intuition about the nature of subsets of the real line.

Let C_0 be the closed interval $[0, 1]$, and define C_1 to be the set that results when the open middle third is removed; that is,

$$C_1 = C_0 \setminus \left(\frac{1}{3}, \frac{2}{3}\right) = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right].$$

Now, construct C_2 in a similar way by removing the open middle third of each of the two components of C_1:

$$C_2 = \left(\left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right]\right) \cup \left(\left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, \frac{1}{3}\right]\right)$$

If we continue this process inductively, then for each $n = 0, 1, 2, \cdots$ we get a set C_n consisting of $2n$ closed intervals each having length $1/3^n$. Finally, we define the *Cantor* set C (Fig. **??**) to be the intersection

$$C = \bigcap_{n=1}^{\infty} C_n .$$

It may be useful to understand C as the remainder of the interval $[0, 1]$ after the iterative process of removing open middle thirds is taken to infinity:

$$C = [0, 1] \setminus \left(\left[\frac{1}{3}, \frac{2}{3}\right] \cup \left[\frac{1}{9}, \frac{2}{9}\right] \cup \left[\frac{7}{9}, \frac{8}{9}\right] \cup \cdots\right)$$

*Lecture notes for CUHKSZ course MAT2006: Elementary Real Analysis.

The Nested Interval Property grantees that C is nonempty. In particular, $0 \in C$ since $0 \in C_n$ for every $n \in \mathbb{N}$. Also, if y is one endpoints for some C_n, it is also an endpoints of C_{n+1}. Thus C contains all the endpoints of C_n for every $n \in \mathbb{N}$, and so C is an infinite set.

Is there anything else? Is C countable? Does C contain any intervals? Any irrational numbers? These are difficult questions at the moment. All of the endpoints mentioned earlier are rational numbers (they have the form $m/3^n$), which means that if it is true that C consists of only these endpoints, then C would be a subset of $\mathbb{Q}$ and hence countable. We shall see about this. There is some strong evidence that not much is left in C if we consider the total length of the intervals removed. To form C_1, an open interval of length $1/3$ was taken out. In the second step, we removed two intervals of length $1/9$, and to construct C_n we removed 2^{n-1} middle thirds of length $1/3^n$. There is some logic, then, to defining the "length" of C to be 1 minus the total

$$\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \cdots \frac{2^{n-1}}{3^n} + \cdots = \frac{\frac{1}{3}}{1 - \frac{2}{3}} = 1.$$

The Cantor set has *zero length*[1].

To this point, the information we have collected suggests a mental picture of C as a relatively small, thin set. For these reasons, the set C is often referred to as Cantor "dust." But there are some strong counterarguments that imply a very different picture. First, C is actually *uncountable*, with cardinality equal to the cardinality of $\mathbb{R}$. One slightly intuitive but convincing way to see this is to create a 1-1 correspondence between C and the set of binary sequences, namely, sequences of the form $\{a_n\}_{n=1}^{\infty}$, where $a_n = 0$ or 1. For each $c \in C$, set $a_1 = 0$ if c falls in the left-hand component of C_1 and set $a_1 = 1$ if c falls in the right-hand component. Having established where in C_1 the point c is located, there are now two possible components of C_2 that might contain c. This time, we set $a_2 = 0$ or 1 depending on whether c falls in the left or right half of these two components of C_2. Continuing in this way, we come to see that every element $c \in C$ yields a sequence $\{a_1, a_2, a_3, \dots\}$ of zeros and ones that acts as a set of directions for how to locate c within C. Likewise, every such sequence corresponds to a point in the Cantor set. Because the set of sequences of zeros and ones is uncountable, we must conclude that C is uncountable as well.

What does this imply? In the first place, because the endpoints of the approximating sets C_n form a countable set, we are forced to accept the fact that not only are there other points in C but there are uncountably many of them. From the point of view of *cardinality*, C is quite large – as large as $\mathbb{R}$, in fact. This should be contrasted with the fact that from the point of view of *length*, C measures the same size as a single point. We conclude this

[1] More precisely, the Lebesgue measure of C is zero. Measure theory will be treated in the course Real Analysis.

discussion with a demonstration that from the point of view of *dimension*, C strangely falls somewhere in between.

There is a sensible agreement that a point has dimension zero, a line segment has dimension one, a square has dimension two, and a cube has dimension three. Without attempting a formal definition of dimension (of which there are several), we can nevertheless get a sense of how one might be defined by observing how the dimension affects the result of magnifying each particular set by a factor of 3. (The reason for the choice of 3 will become clear when we turn our attention back to the Cantor set). A single point undergoes no change at all, whereas a line segment triples in length. For the square, magnifying each length by a factor of 3 results in a larger square that contains 9 copies of the original square. Finally, the magnified cube yields a cube that contains 27 copies of the original cube within its volume. Notice that, in each case, to compute the "size" of the new set, the dimension appears as the exponent of the magnification factor.

Now, apply this transformation to the Cantor set. The set $C_0 = [0,1]$ becomes the interval $[0,3]$. Deleting the middle third leaves $[0,1] \cup [2,3]$, which is where we started in the original construction except that we now stand to produce an additional copy of C in the interval $[2,3]$. Magnifying the Cantor set by a factor of 3 yields two copies of the original set. Thus, if x is the dimension of C, then x should satisfy $2 = 3^x$, or $x = \log 2/\log 3 \approx 0.631$.

The notion of a noninteger or fractional dimension is the impetus behind the term "fractal," coined in 1975 by Benoit Mandlebrot to describe a class of sets whose intricate structures have much in common with the Cantor set. Cantor's construction, however, is over a hundred years old and for us represents an invaluable testing ground for the upcoming theorems and conjectures about the often elusive nature of subsets of the real line.

2 Open and Closed Sets

2.1 Open Sets

Given $a \in \mathbb{R}$ and $\epsilon > 0$, recall that the ϵ-neighborhood of a is the set

$$V_\epsilon(a) = \{x \in \mathbb{R} \mid a - \epsilon < x < a + \epsilon\}.$$

In other words, $V_\epsilon(a)$ is the open interval $(a-\epsilon, a+\epsilon)$, centered at a with radius ϵ.

Definition 1. A set $O \subset \mathbb{R}$ is open if for all points $a \in O$ there exists an ϵ-neighborhood $V_\epsilon(a) \subset O$.

Example 2.1. (i) $\mathbb{R}$ is an open set; the empty set $\emptyset$ is also open.

(ii) The open interval

$$(c,d) = \{x \in \mathbb{R} \mid c < x < d\}$$

is an open set. Let $x \in (c,d)$ be arbitrary. If we take $\epsilon = \min\{x-x, d-x\}$, then it follows that $V_\epsilon(x) \subset (c,d)$. It is important to see where this argument breaks down if the interval includes either one of its endpoints.

Theorem 1. (i) *The union of an arbitrary collection of open sets is open.*

(ii) *The intersection of a finite collection of open sets is open.*

Proof. To prove (i), we let $\{O_\lambda \mid \lambda \in \Lambda\}$ be a collection of open sets and let $O = \bigcup_{\lambda \in \Lambda} O_\lambda$. Let a be an arbitrary element of O. In order to show that O is open, we produce an ϵ-neighborhood of a completely contained in O. But $a \in O$ implies that a is an element of at least one particular $O_{\lambda'}$. Because we are assuming $O_{\lambda'}$ is open, there exists $V_\epsilon(a) \subset O_{\lambda'}$. The fact that $O_{\lambda'} \subset O$ implies that $V_\epsilon(a) \subset O$. This completes the proof of (i).

For (ii), let $\{O_1, O_2, \cdots, O_N\}$ be a finite collection of open sets. Now, if $a \in \bigcap_{n=1}^N O_n$, then a is an element of each of the open sets. By the definition of an open set, for each $1 \le k \le N$, there exists $V_{\epsilon_n}(a) \subset O_n$. We are in search of a single ϵ-neighborhood of a that is contained in every O_n, so the trick is to take the smallest one. Letting $\epsilon = \min\{\epsilon_1, \epsilon_2, \ldots, \epsilon_n\}$, it follows that $V_\epsilon(a) \subset V_{\epsilon_n}(a) \subset O_n$ for all n, and hence $V_\epsilon(a) \subset \bigcap_{n=1}^N O_n$, as desired. □

2.2 Closed Sets

Definition 2. A point x is a *limit point* of a set A if every ϵ-neighborhood $V_\epsilon(x)$ of x intersects the set A at some point other than x.

Limit points are also often referred to as "cluster points" or "accumulation points," but the phrase "x is a limit point of A" has the advantage of explicitly reminding us that x is quite literally the limit of a sequence in A.

Theorem 2. *A point x is a limit point of a set A if and only if $x = \lim_{n\to\infty} a_n$ for some sequence $\{a_n\}$ contained in A satisfying $a_n \neq x$ for all $n \in \mathbb{N}$.*

Proof. ($\Rightarrow$) Assume x is a limit point of A. In order to produce a sequence $\{a_n\}$ converging to x, we are going to consider the particular ϵ-neighborhoods obtained using $\epsilon = 1/n$. By Definition 2, every neighborhood of x intersects A in some point other than x. This means that, for each $n \in \mathbb{N}$, we are justified in picking a point

$$a_n \in V_{1/n}(x) \cap A$$

with $a_n \neq x$. It is straightforward to verify that $\{a_n\} \to x$. Given an arbitrary $\epsilon > 0$, choose N such that $1/N < \epsilon$. It follows that $|a_n - x| < \epsilon$ for all $n \ge N$.

($\Leftarrow$) For the reverse implication we assume $\lim_{n\to\infty} a_n = x$ where $a_n \in A$ but $a_n \neq x$, and let $V_\epsilon(x)$ be an arbitrary ϵ-neighborhood. The definition of convergence assures us that there exists a term a_N in the sequence satisfying $a_N \in V_\epsilon(x)$, and the proof is complete. □

The restriction that $a_n \neq x$ in Theorem 2 deserves a comment. Given a point $a \in A$, it is always the case that a is the limit of a sequence in A if we are allowed to consider the constant sequence $\{a, a, a, \cdots\}$. There will be occasions where we will want to avoid this somewhat uninteresting situation, so it is important to have a vocabulary that can distinguish limit points of a set from *isolated points*.

Definition 3. A point $a \in A$ is an *isolated point* of A if it is not a limit point of A.

As a word of caution, we need to be a little careful about how we understand the relationship between these concepts. Whereas an isolated point is always an element of the relevant set A, it is quite possible for a limit point of A not to belong to A. As an example, consider the endpoint of an open interval. This situation is the subject of the next important definition.

Definition 4. A set $F \subset \mathbb{R}$ is *closed* if it contains its limit points.

The adjective "closed" appears in several other mathematical contexts and is usually employed to mean that an operation on the elements of a given set does not take us out of the set. In linear algebra, for example, a vector space is a set that is "closed" under addition and scalar multiplication. In analysis, the operation we are concerned with is the limiting operation. Topologically speaking, a closed set is one where convergent sequences within the set have limits that are also in the set.

Theorem 3. *A set $F \subset \mathbb{R}$ is closed if and only if every Cauchy sequence contained in F has a limit that is also an element of F.*

Example 2.2. (i) Consider the set

$$A = \left\{ \frac{1}{n} \,|\, n \in \mathbb{N} \right\}$$

Each point in A is not a limit point, hence a isolated point. Although all of the points of A are isolated, the set does have one limit point, namely 0. This is because every neighborhood centered at zero, no matter how small, is going to contain points of A. Because $0 \notin A$, A is not closed. The set $F = A \cup \{0\}$ is an example of a closed set and is called the *closure* of A.

(ii) The closed interval

$$[c, d] = \{x \in \mathbb{R} \,|\, c \leq x \leq d\}$$

is a closed set. If x is a limit point of $[c, d]$, then by Theorem 2 there exists $c \leq x_n \leq d$ with $\{x_n\} \to x$. The Order Limit Theorem implies that $c \leq x \leq d$, hence $x \in [c, d]$.

(iii) Consider the set $\mathbb{Q} \subset \mathbb{R}$ of rational numbers. An extremely important property of $\mathbb{Q}$ is that its set of limit points is actually all of $\mathbb{R}$. To see why this is so, recall the theorem asserts that $\mathbb{Q}$ is dense in $\mathbb{R}$ from Chapter 1: for any two real numbers $x < y$, there exists $r \in \mathbb{Q}$ such that $x < r < y$. Hence, given any $y \in \mathbb{R}$ and any $\epsilon > 0$, there exists a rational number $r \neq y$ that falls in this neighborhood. Thus, y is a limit point of $\mathbb{Q}$.

The density property of $\mathbb{Q}$ can now be reformulated in the following way.

Theorem 4 (Density of $\mathbb{Q}$ in $\mathbb{R}$)**.** *For every $y \in \mathbb{R}$, there exists a sequence of rational numbers that converges to y.*

The same argument can also be used to show that every real number is the limit of a sequence of irrational numbers. Although interesting, part of the allure of the rational numbers is that, in addition to being dense in $\mathbb{R}$, they are countable. As we will see, this tangible aspect of $\mathbb{Q}$ makes it an extremely useful set, both for proving theorems and for producing interesting counterexamples.

2.3 Closure

Definition 5. Given a set $A \subset \mathbb{R}$, let L be the set of all limit points of A. The closure of A is defined to be $\overline{A} = A \cup L$.

In Example 2.2 (i), we saw that if $A = \{1/n \,|\, n \in \mathbb{N}\}$, then the closure of A is $\overline{A} = A \cup \{0\}$. Example 2.2 (iii) verifies that $\overline{\mathbb{Q}} = \mathbb{R}$. If A is an open interval (a, b), then $\overline{A} = [a, b]$. If A is a closed interval, then $\overline{A} = A$. It is not for lack of imagination that in each of these examples $\overline{A}$ is always a closed set.

Theorem 5. *For any $A \subset \mathbb{R}$, the closure A is a closed set and is the smallest closed set containing A.*

Proof. Assume x is a limit point of $A \cup L$, we shall show that x is limit point of A. Consider the ϵ-neighborhood $V_\epsilon(x)$ for an arbitrary $\epsilon > 0$. We know $V_\epsilon(x)$ must intersect $A \cup L$ and we would like to argue that it in fact intersects A. If $V_\epsilon(x)$ intersects A at a point different than x we are done, so let's assume that there exists an $\ell \in L$ with $\ell \in V_\epsilon(x)$ and $\ell \neq x$. We take $\epsilon' > 0$ small enough (e.g., $\epsilon' = \min\{\epsilon - |x - \ell|, |x - \ell|\}$) so that $V_{\epsilon'}(\ell) \subset V_\epsilon(x)$, and $x \notin V_{\epsilon'}(\ell)$. Because ℓ is a limit point of A we have that there exists an $a \in V_{\epsilon'}(\ell) \subset V_\epsilon(x)$ and thus $V_\epsilon(x)$ intersects A at some point other than x, as desired.

We have shown that any limit point x of $A \cup L$ is a limit point of A, and hence $x \in L \subset A \cup L$, which means $\overline{A} = A \cup L$ is closed.

Now, any closed set containing A must contain L as well. This shows that $\overline{A} = A \cup L$ is the smallest closed set containing A. $\square$

Exercise 1. Let L be the set of limit points of a given set A. Show that L is closed.

2.4 Complements

The mathematical notions of open and closed are not antonyms the way they are in standard English. If a set is not open, that does not imply it must be closed. Many sets such as the half-open interval $(c, d] = \{x \in \mathbb{R} \mid c < x \leq d\}$ are neither open nor closed. The sets $\mathbb{R}$ and $\emptyset$ are both simultaneously open and closed although, thankfully, these are the only ones with this disorienting property. There is, however, an important relationship between open and closed sets. Recall that the complement of a set $A \subset \mathbb{R}$ is defined to be the set

$$A^c = \mathbb{R} \setminus A = \{x \in \mathbb{R} \mid x \notin A\}.$$

Theorem 6. *A set O is open if and only if O^c is closed. Likewise, a set F is closed if and only if F^c is open.*

Proof. Given an open set $O \subset R$, let's first prove that O^c is a closed set, that is it contains all of its limit points. If x is a limit point of O^c , then every neighborhood of x contains some point of O^c. But that is enough to conclude that x cannot be in the open set O because $x \in O$ would imply that there exists a neighborhood $V_\epsilon(x) \subset O$. Thus, $x \in O^c$, as desired.

For the converse statement, we assume O^c is closed and argue that O is open. Thus, given an arbitrary point $x \in O$, we must produce an ϵ-neighborhood $V_\epsilon(x) \subset O$. Because O^c is closed, we can be sure that x is not a limit point of O^c. Looking at the definition of limit point, we see that this implies that there must be some neighborhood $V_\epsilon(x)$ of x that does not intersect the set O^c. But this means $V_\epsilon(x) \subset O$, which is precisely what we needed to show.

The second statement in this theorem follows quickly from the first using the observation that $(E^c)^c = E$ for any set $E \subset \mathbb{R}$.

□

Theorem 7. (i) *The union of a finite collection of closed sets is closed.*

(ii) *The intersection of an arbitrary collection of closed sets is closed.*

Proof. De Morgan's Laws state that for any collection of sets $\{E_\lambda\}_{\lambda \in \Lambda}$ it is true that

$$\left(\bigcup_{\lambda \in \Lambda} E_\lambda \right)^c = \bigcap_{\lambda \in \Lambda} E_\lambda^c, \qquad \left(\bigcap_{\lambda \in \Lambda} E_\lambda \right)^c = \bigcup_{\lambda \in \Lambda} E_\lambda^c.$$

The result follows directly from these statements and Theorem 1. □

Exercise 2. Prove the above version of De Morgan's law.

Exercise 3. Let A be nonempty and bounded above so that $s = \sup A$ exists.

(i) Show that $s \in \overline{A}$.

(ii) Can an open set contain its supremum?

Exercise 4. (i) Prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

(ii) Does this result about closures extend to infinite unions of sets?

Exercise 5. Let A be an uncountable set and let B be the set of real numbers that divides A into two uncountable sets; that is, $s \in B$ if both $\{x \,|\, x \in A \quad \text{and} \quad x < s\}$ and $\{x \,|\, x \in A \quad \text{and} \quad x > s\}$ are uncountable. Show B is nonempty and open.

Exercise 6. Prove that the only sets that are both open and closed are $\mathbb{R}$ and the empty set $\emptyset$.

Exercise 7. A dual notion to the closure of a set is the *interior* of a set. The interior of E is denoted E° and is defined as

$$E^\circ = \{x \in E \,|\, \text{there exists } V_\epsilon(x) \subset E\}.$$

Results about closures and interiors possess a useful symmetry.

(i) Show that E is closed if and only if $\overline{E} = E$. Show that E is open if and only if $E^\circ = E$.

(ii) Show that $(\overline{E})^c = (E^c)^\circ$ and $(E^\circ)^c = \overline{E^c}$.

3 Compact Sets

The central challenge in analysis is to exploit the power of the mathematical infinite – via limits, series, derivatives, integrals, etc. – without falling victim to erroneous logic or faulty intuition. A major tool for maintaining a rigorous footing in this endeavor is the concept of compact sets. In ways that will become clear, especially in our upcoming study of continuous functions, employing compact sets in a proof often has the effect of bringing a finite quality to the argument, thereby making it much more tractable.

3.1 Sequentially Compact Sets

Definition 6 (Sequentially Compact)**.** A set $K \subset \mathbb{R}$ is *sequentially compact* if every sequence in K has a subsequence that converges to a limit that is also in K.

Example 3.1. A bounded closed interval is sequentially compact.

To see this, notice that if $\{a_n\}$ is contained in an interval $[c, d]$, then the Bolzano–Weierstrass Theorem guarantees that we can find a convergent subsequence $\{a_{n_k}\}$. Because a closed interval is a closed set, we know that the limit of this subsequence is also in $[c, d]$.

What are the properties of closed intervals that we used in the preceding argument? The Bolzano–Weierstrass Theorem requires boundedness, and we used the fact that closed sets contain their limit points. As we are about to see, these two properties completely characterize (sequentially) compact sets in $\mathbb{R}$.

Definition 7 (Boundedness)**.** A set $A \subset \mathbb{R}$ is bounded if there exists $M > 0$ such that $|a| \leq M$ for all $a \in A$.

Theorem 8. *A set $K \subset \mathbb{R}$ is sequentially compact if and only if it is closed and bounded.*

Proof. Let K be sequentially compact. We will first prove that K must be bounded, so assume, for contradiction, that K is not a bounded set. The idea is to produce a sequence in K that marches off to infinity in such a way that it cannot have a convergent subsequence as the definition of sequentially compact requires. To do this, notice that because K is not bounded there must exist an element $x_1 \in K$ satisfying $|x_1 > 1|$. Likewise, there must exist $x_2 \in K$ with $|x_2| > 2$, and in general, given any $n \in \mathbb{N}$, we can produce $x_n \in K$ such that $|x_n| > n$.

Now, because K is assumed to be sequentially compact, $\{x_n\}$ should have a convergent subsequence $\{x_{n_k}\}$. But the elements of the subsequence must satisfy $|x_{n_k}| > n_k \geq k$, and consequently $\{x_{n_k}\}$ is unbounded. Because convergent sequences are bounded, we have a contradiction. Thus, K must at least be a bounded set.

Next, we will show that K is also closed. To see that K contains its limit points, we let $x = \lim_{n\to\infty} x_n$, where $\{x_n\}$ is contained in K and argue that x must be in K as well. By Definition 6, the sequence $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$, and we know $\{x_{n_k}\}$ converges to the same limit x. Finally, Definition 6 requires that $x \in K$. This proves that K is closed.

The proof of the converse statement is left as an exercise. □

Exercise 8. Show that if a set $K \subset \mathbb{R}$ is closed and bounded, then it is sequentially compact.

Exercise 9. Show that if K is sequentially compact and nonempty, then $\sup K$ and $\inf K$ both exist and are elements of K.

3.2 Open Covers and Compact Sets

Definition 8. Let $A \subset \mathbb{R}$. An *open cover* for A is a (possibly infinite) collection of open sets $\{O_\lambda \,|\, \lambda \in \Lambda\}$ whose union contains the set A; that is, $A \subset \bigcup_{\lambda\in\Lambda} O_\lambda$. Given an open cover for A, a *finite subcover* is a finite subcollection of open sets from the original open cover whose union still manages to completely contain A.

Example 3.2. Consider the open interval $(0, 1)$. For each point $x \in (0, 1)$, let O_x be the open interval $(x, 1)$. Taken together, the infinite collection $\{O_x \,|\, x \in (0, 1)\}$ forms an open cover for the open interval $(0, 1)$. Notice, however, that it is impossible to find a finite subcover. Given any proposed finite subcollection

$$\{O_{x_1}, O_{x_2}, \ldots, O_{x_n}\}$$

set $x' = \min\{x_1, x_2, \ldots, x_n\}$ and observe that any real number y satisfying $0 < y \leq x'$ is not contained in the union $\bigcup_{i=1}^{n} O_{x_i}$.

Now, consider a similar cover for the closed interval $[0, 1]$. For $x \in (0, 1)$, the sets $O_x = (x, 1)$ do a fine job covering $(0, 1)$, but in order to have an open cover of the closed interval $[0, 1]$, we must also cover the endpoints. To remedy this, we could fix $\epsilon > 0$, and let $O_0 = (-\epsilon, \epsilon)$ and $O_1 = (1 - \epsilon, 1 + \epsilon)$. Then, the collection

$$\{O_0, O_1, O_x, \,|\, x \in (0, 1)\}$$

is an open cover for $[0, 1]$. But this time, notice there is a finite subcover. Because of the addition of the set O_0, we can choose x' so that $x' < \epsilon$. It follows that $\{O_0, O_{x'}, O_1\}$ is a finite subcover for the closed interval $[0, 1]$.

Definition 9 (Compact Sets)**.** A set $K \subset \mathbb{R}$ is *compact* if every open cover of K has a finite subcover.

Theorem 9 (Heine–Borel Theorem)**.** *A set $K \subset \mathbb{R}$ is compact if and only if K is closed and bounded.*

Proof. ($\Rightarrow$) Assume $K \subset \mathbb{R}$ is compact, we shall show it is closed and bounded.

To show that K is bounded, we construct an open cover for K by defining O_x to be an open interval of radius 1 around each point $x \in K$. In the language of neighborhoods, $O_x = V_1(x)$. The open cover $\{O_x \,|\, x \in K\}$ then must have a finite subcover $\{O_{x_1}, O_{x_2}, \ldots, O_{x_n}\}$. Because K is contained in a finite union of bounded sets, K must itself be bounded.

The proof that K is closed is more delicate, and we argue it by contradiction. Let $\{y_n\}$ be a Cauchy sequence contained in K with $\lim_{n\to\infty} y_n = y$. To show that K is closed, we must demonstrate that $y \in K$, so assume for contradiction that this is not the case. If $y \notin K$, then every $x \in K$ is some positive distance away from y. We now construct an open cover by taking O_x to be an interval of radius $|x - y|/2$ around each point x in K. Because we are assuming K is compact, the resulting open cover $\{O_x \,|\, x \in K\}$ must have a finite subcover $\{O_{x_1}, O_{x_2}, \cdots, O_{x_n}\}$. The contradiction arises when we realize that, this finite subcover cannot contain all of the elements of the sequence y_n. To make this explicit, set

$$\epsilon_0 = \min\left\{\frac{|x_i - y|}{2}\,;\, 1 \leq i \leq n\right\}.$$

Because $\{y_n\} \to y$, we can certainly find a term y_N satisfying $|y_N - y| \leq \epsilon_0$. But such a y_N must necessarily be excluded from each O_{x_i}, meaning that

$$y_N \notin \bigcup_{i=1}^{n} O_{x_i}$$

Thus our supposed subcover does not actually cover all of K. This contradiction implies that $y \in K$, and hence K is closed and bounded.

The converse statement, that is a bounded and closed set is compact, is left as an exercise.

□

Exercise 10 (NIP+AP implies HB)**.** Provide a proof of a bounded and closed set is compact using the Nested Interval Property.

Suppose $K \subset \mathbb{R}$ is closed and bounded, and let $\{O_\lambda \,|\, \lambda \in \Lambda\}$ be an open cover for K. For contradiction, let's assume that no finite subcover exists. Let I_0 be a closed interval containing K.

(a) Show that there exists a nested sequence of closed intervals $I_0 \supset I_1 \supset I_2 \supset \cdots$ with the property that, for each n, $I_n \cap K$ cannot be finitely covered and $\lim_{n\to\infty} |I_n| = 0$.

(b) Argue that there exists an $x \in K$ such that $x \in I_n$ for all n.

(c) Because $x \in K$, there must exist an open set O_{λ_0} from the original collection that contains x as an element. Explain how this leads to the desired contradiction.

Exercise 11 (LUBP implies HB)**.** Consider the special case where K is a closed interval. Let $\{O_\lambda \,|\, \lambda \in \Lambda\}$ be an open cover for $[a, b]$ and define S to be the set of all $x \in [a, b]$ such that $[a, x]$ has a finite subcover from $\{O_\lambda \,|\, \lambda \in \Lambda\}$.

(a) Argue that S is nonempty and bounded, and thus $s = \sup S$ exists.

(b) Now show $s = b$, which implies $[a, b]$ has a finite subcover.

(c) Finally, prove the theorem for an arbitrary closed and bounded set K.

Exercise 12. (i) Provide an alternative proof of the statement "a compact set K is bounded" by considering the open cover $\{O_n = (-n, n) \,|\, n \in \mathbb{N}\}$.

(ii) Provide an alternative proof of "a compact set K is closed" by the following. Suppose, for contradiction, K is not closed, there is a limit point y of K such that $y \notin K$. Take $O_n = \left(\overline{V_{1/n}(y)}\right)^c := (-\infty, y - 1/n) \cup (y + 1/n, \infty)$. Show that $\{O_n \,|\, n \in \mathbb{N}\}$ is an open cover of K, and derive a contradiction from this.

Exercise 13 (HB implies BW)**.** Using the concept of open covers (and explicitly avoiding the Bolzano–Weierstrass Theorem), prove that every bounded infinite set has a limit point. Therefore, every bounded sequence has a convergent subsequence (BW).

Remark. Combining the Heine–Borel Theorem and Theorem 8 together, we notice that:

the following properties of $K \subset \mathbb{R}$ are equivalent:
(i) K is closed and bounded;
(ii) K is sequentially compact;
(iii) K is compact.

The above proposition is still true for sets in $\mathbb{R}^n$. However, there are examples:

(a) A compact set is not bounded in a infinite-dimensional space.

(b) In a metric space (a space with well-defined distance function, examples are $\mathbb{R}^n$, Banach spaces, etc.), compactness and sequentially compactness are equivalent. But, for general topological spaces, compactness neither implies nor is implied by sequentially compactness – there are examples of compact space but not sequentially compact, and examples of sequentially compact space but not compact.

For our purpose – doing analysis in $\mathbb{R}^n$ – we just need keep in mind that all three conditions are equivalent to each other.

Remark. We have met all seven often used propositions of completeness of real numbers, they are

AoC(LUBP), CP, NIP, MCT, BW, CC and HB.

They are all equivalent (while NIP or CC should be used along with AP), and each one of them can be used as description of completeness of real numbers.

4 Perfect Sets and Connected Sets

4.1 Perfect Sets

Definition 10. A set $P \in \mathbb{R}$ is perfect if it is closed and contains no isolated points.

Closed intervals (other than the singleton sets $[a, a]$) serve as the most obvious class of perfect sets, but there are more interesting examples.

Example 4.1 (Cantor Set)**.** The Cantor set is perfect.

Recall that the Cantor set is defined as

$$C = \bigcap_{n=1}^{\infty} C_n$$

where each C_n is a finite union of closed intervals. By Theorem 7, each C_n is closed, and by the same theorem, C is closed as well. It remains to show that no point in C is isolated.

Let $x \in C$ be arbitrary. To convince ourselves that x is not isolated, we must construct a sequence $\{x_n\}$ of points in C, different from x, that converges to x. Since $x \in C = \bigcap_{n=1}^{\infty} C_n$, so $x \in C_1$, and x is in one of the two subintervals that made C_1, choose $x_1 \neq x$ to be one endpoints of the subinterval where x belongs to. We see that $|x_1 - x| < 1/3$ and $x_1 \in C$ since it is one endpoints. In general, for each $n \in \mathbb{N}$, $x \in C_n$, and we can choose $x_n \neq x$ to be one of the endpoints of the subinterval of C_n where x belongs to. It then follows that $|x_n - x| < 1/3^n$ and $x_n \in C$. Thus the desired sequence $\{x_n\}$ is determined and x is a limit point of C.

Theorem 10. *A nonempty perfect set is uncountable*

4.2 Connected Sets

Definition 11. Two nonempty sets $A, B \subset \mathbb{R}$ are *separated* if $\overline{A} \cap B$ and $A \cap \overline{B}$ are both empty. A set $E \subset \mathbb{R}$ is *disconnected* if it can be written as $E = A \cup B$, where A and B are nonempty separated sets.

A set that is not disconnected is called a *connected* set.

Example 4.2. (i) If we let $A = (1, 2)$ and $B = (2, 5)$, then it is not difficult to verify that $E = (1, 2) \cup (2, 5)$ is disconnected. Notice that the sets $C = (1, 2]$ and $D = (2, 5)$ are not separated because $C \cap \overline{D} = \{2\}$ is not empty.

(ii) The set of rational numbers is disconnected. If we let

$$A = \mathbb{Q} \cap (-\infty, \sqrt{2}), \qquad B = \mathbb{Q} \cap (\sqrt{2}, \infty).$$

then we certainly have $\mathbb{Q} = A \cup B$. The fact that $A \subset (-\infty, \sqrt{2})$ implies (by the Order Limit Theorem) that any limit point of A will necessarily fall in $(-\infty, \sqrt{2}]$. Because this is disjoint from B, we get $\overline{A} \cap B = \emptyset$. We can similarly show that $A \cap \overline{B} = \emptyset$, which implies that A and B are separated.

Theorem 11. *A set $E \subset \mathbb{R}$ is connected if and only if, for all nonempty disjoint sets A and B satisfying $E = A \cup B$, there always exists a convergent sequence $\{x_n\} \to x$ with $\{x_n\}$ contained in one of A or B, and x an element of the other.*

Proof. Exercise. □

The concept of connectedness is more relevant when working with subsets of the plane and other higher-dimensional spaces. This is because, in $\mathbb{R}$, the connected sets coincide precisely with the collection of intervals.

Theorem 12. *A set $E \subset \mathbb{R}$ is connected if and only if whenever $a < c < b$ with $a, b \in E$, it follows that $c \in E$ as well.*

5 Baire's Theorem

Definition 12. A set $A \subset \mathbb{R}$ is called an F_σ set if it can be written as the countable union of closed sets. A set $B \subset \mathbb{R}$ is called a G_δ set if it can be written as the countable intersection of open sets.

Exercise 14. Show that a set A is a G_δ set if and only if A^c is an F_σ set.

Exercise 15. Replace each ____ with the word finite or countable, depending on which is more appropriate.
(a) The ____ union of F_σ sets is an F_σ set.
(b) The ____ intersection of F_σ sets is an F_σ set.
(c) The ____ union of G_δ sets is a G_δ set.
(d) The ____ intersection of G_δ sets is a G_δ set.

Exercise 16. (i) Show that a closed interval $[a, b]$ is a G_δ set.
(ii) Show that the half-open interval $(a, b]$ is both a G_δ and an F_σ set.
(iii) Show that $\mathbb{Q}$ is an F_σ set, and the set of irrationals $\mathbb{I}$ forms a G_σ set.

It is not readily obvious that the class F_σ does not include every subset of $\mathbb{R}$, but we are now ready to argue that $\mathbb{I}$ is not an F_σ set (and consequently $\mathbb{Q}$ is not a G_δ set). This will follow from a theorem due to René Louis Baire (1874–1932).

Recall that a set $G \subset \mathbb{R}$ is dense in $\mathbb{R}$ if, given any two real numbers $a < b$, it is possible to find a point $x \in G$ with $a < x < b$.

Theorem 13. *If $\{G_1, G_2, G_3, \ldots, \}$ is a countable collection of dense, open sets, then the intersection $\bigcap_{n=1}^{\infty} G_n$ is not empty.*

Exercise 17. Starting with $n = 1$, inductively construct a nested sequence of closed intervals $I_1 \supset I_2 \supset I_3 \supset \cdots$ satisfying $I_n \subset G_n$. Give special attention to the issue of the endpoints of each I_n. Show how this leads to a proof of the theorem.

Exercise 18. Show that it is impossible to write

$$\mathbb{R} = \bigcup_{n=1}^{\infty} F_n,$$

where for each $n \in \mathbb{N}$, F_n is a closed set containing no nonempty open intervals.
Hint. Show that each F_n^c is dense in $\mathbb{R}$.

Exercise 19. Show how the previous exercise implies that the set $\mathbb{I}$ of irrationals cannot be an F_σ set, and $\mathbb{Q}$ cannot be a G_δ set.

Exercise 20. Using Exercise 19 and versions of the statements in Exercise 15, construct a set that is neither in F_σ nor in G_δ.

Nowhere-Dense Sets

Recall that a set G is dense in $\mathbb{R}$ if and only if every point of $\mathbb{R}$ is a limit point of G. Because the closure of any set is obtained by taking the union of the set and its limit points, we have that

$$G \text{ is dense in } \mathbb{R} \text{ if and only if } \overline{G} = \mathbb{R}.$$

The set $\mathbb{Q}$ is dense in $\mathbb{R}$; the set $\mathbb{Z}$ is clearly not. In fact, in the jargon of analysis, $\mathbb{Z}$ is nowhere-dense in $\mathbb{R}$.

Definition 13. A set E is *nowhere-dense* if $\overline{E}$ contains no nonempty open intervals.

Exercise 21. Show that a set E is nowhere-dense in $\mathbb{R}$ if and only if the complement of $\overline{E}$ is dense in $\mathbb{R}$.

Exercise 22. Decide whether the following sets are dense in $\mathbb{R}$, nowhere-dense in $\mathbb{R}$, or somewhere in between.

(a) $A = \mathbb{Q} \cap [0, 5]$;
(b) $B = \{1/n \,|\, n \in \mathbb{N}\}$;
(c) $\mathbb{I}$;
(d) the Cantor set.

Theorem 14 (Baire's Theorem)**.** *The set of real numbers* $\mathbb{R}$ *cannot be written as the countable union of nowhere-dense sets.*

Exercise 23. Prove Baire's Theorem by contradiction.

Remark. Baire's Theorem is yet another statement about the size of $\mathbb{R}$. We have already encountered several ways to describe the sizes of infinite sets. In terms of cardinality, countable sets are relatively small whereas uncountable sets are large. We also briefly discussed the concept of "length," or "measure," in the discussion of the Cantor set. Baire's Theorem offers a third perspective. From this point of view, nowhere-dense sets are considered to be "thin" sets. Any set that is the countable union – i.e., a not very large union – of these small sets is called a "meager" set or a set of "first category." A set that is not of first category is of "second category." Intuitively, sets of the second category are the "fat" subsets. The Baire Category Theorem, as it is often called, states that $\mathbb{R}$ is of second category.

The Baire Category Theorem in its more general form states that any *complete metric space* must be too large to be the countable union of nowhere-dense subsets. One particularly interesting example of a complete metric space is the set of continuous functions defined on the interval $[0,1]$. (The distance between two functions f and g in this space is defined to be $\sup|f(x) - g(x)|$, where $x \in [0,1]$) Now, in this space we will see that the collection of continuous functions that are differentiable at even one point can be written as the countable union of nowhere-dense sets. Thus, a fascinating consequence of Baire's Theorem in this setting is that most continuous functions do not have derivatives at any point. Chapter 6 concludes with a construction of one such function. This odd situation mirrors the roles of $\mathbb{Q}$ and $\mathbb{I}$ as subsets of $\mathbb{R}$. Just as the familiar rational numbers constitute a minute proportion of the real line, the differentiable functions of calculus are exceedingly atypical of continuous functions in general.