1. **Question 3.29**

   Solution.

   For the simple linear regression model

$$y_i = \beta_0 + \beta_1 x_i + \varepsilon,$$

   we can see that

$$X = \begin{bmatrix} 1 & x_1 \\ \cdots & \cdots \\ 1 & x_n \end{bmatrix},\ X^T X = \begin{bmatrix} n & \sum_i x_i \\ \sum_i x_i & \sum_i x_i^2 \end{bmatrix},\ (X^T X)^{-1} = \frac{1}{S_{xx}} \begin{bmatrix} \sum_i x_i^2/n & -\bar{x} \\ -\bar{x} & 1 \end{bmatrix}.$$

   It follows that

$$\begin{aligned} h_{ij} =& \frac{1}{S_{xx}} \begin{bmatrix} 1 & x_i \end{bmatrix} \begin{bmatrix} \sum_i x_i^2/n & -\bar{x} \\ -\bar{x} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ x_j \end{bmatrix} \\ =& \frac{1}{S_{xx}} \left(\frac{\sum_i x_i^2}{n} - x_i \bar{x} - x_j \bar{x} + x_i x_j \right) \\ =& \frac{1}{S_{xx}} \left(\frac{\sum_i x_i^2}{n} - \bar{x}^2 + \bar{x}^2 - x_i \bar{x} - x_j \bar{x} + x_i x_j \right) \\ =& \frac{1}{S_{xx}} \left[\frac{\sum_i x_i^2 - n\bar{x}^2}{n} + (x_i - \bar{x})(x_j - \bar{x}) \right] \\ =& \frac{1}{n} + \frac{(x_i - \bar{x})(x_j - \bar{x})}{S_{xx}}. \end{aligned}$$

   Similarly,

$$\begin{aligned} h_{ii} =& \frac{1}{S_{xx}} \begin{bmatrix} 1 & x_i \end{bmatrix} \begin{bmatrix} \sum_i x_i^2/n & -\bar{x} \\ -\bar{x} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ x_i \end{bmatrix} \\ =& \frac{1}{S_{xx}} \left(\frac{\sum_i x_i^2}{n} - \bar{x}^2 + \bar{x}^2 - 2x_i \bar{x} + x_i^2 \right) \\ =& \frac{1}{n} + \frac{(x_i - \bar{x}^2)}{S_{xx}}. \end{aligned}$$

   It is clear that as x_i moves farther from $\bar{x}$, both h_{ij} and h_{ii} increase.

2. **Question 3.30**

   Solution.

   For the multiple linear regression model, LS estimator can be rewritten as

$$
\begin{aligned}
\hat{\beta} =& (X^T X)^{-1} X^T y \\
=& (X^T X)^{-1} X^T (X\beta + \varepsilon) \\
=& (X^T X)^{-1} X^T X\beta + (X^T X)^{-1} X^T \varepsilon \\
=& \beta + R\varepsilon.
\end{aligned}
$$

3. **Question 3.31**

   Solution.

   Equation (3.15b) gives that

$$e = (I - H)y.$$

   For a linear regression model $y = X\beta + \varepsilon$, we have

$$
\begin{aligned}
e =& (I - H)(X\beta + \varepsilon) \\
=& (I - H)X\beta + (I - H)\varepsilon \\
=& (X - X(X^T X)^{-1} X^T X)\beta + (I - H)\varepsilon \\
=& (I - H)\varepsilon.
\end{aligned}
$$

4. **Question 3.32**

Solution.

$$\begin{aligned}SS_{\mathrm{R}}(\beta) =& \hat{\beta}^T X^T y \\ =& y^T X(X^T X)^{-1} X^T y \\ =& y^T H y.\end{aligned}$$

<u>Note:</u>

Here $SS_{\mathrm{R}}(\beta) = \hat{\beta}^T X^T y$ denotes the regression sum of squares for the full model, which is mentioned in Page 89 of the textbook. It is different from our familiar representation form (3.24): $SS_{\mathrm{R}} = \hat{\beta}^T X^T y - (\sum_i y_i)^2/n$, which denotes the regression sum of squares due to regressors given β_0.

Consider the regression model with k regressors:

$$y = X\beta + \varepsilon,$$

where $y \in \mathbb{R}^n$, $X \in \mathbb{R}^{n\times p}$, $\beta \in \mathbb{R}^p$, $\varepsilon \in \mathbb{R}^n$ and $p = k+1$. Let the vector of regression coefficients be partitioned as

$$\beta = \left[\begin{array}{c} \beta_0 \\ \hline \beta_k \end{array}\right],$$

where $\beta_0 \in \mathbb{R}$ and $\beta_k \in \mathbb{R}^k$. Then the column vector of ones in X is associated with β_0, and the other columns are associated with β_k. Our model can be rewritten as

$$\begin{aligned}X =& \left[\begin{array}{c|c} 1 & X_k \end{array}\right] \\ y =& 1\beta_0 + X_k\beta_k + \varepsilon.\end{aligned}$$

Using the extra-sum-of-squares method, it can be seen that

$$\begin{aligned}SS_{\mathrm{R}}(\beta_0) =& \hat{\beta}_0^T 1^T y \\ =& y^T 1(1^T 1)^{-1} 1^T y \\ =& \frac{\sum_i y_i^2}{n}, \quad \text{since } (1^T 1)^{-1} = \frac{1}{n}, \\ SS_{\mathrm{R}}(\beta_0, \beta_k) =& \hat{\beta}^T X^T y \ (p \text{ degrees of freedom}), \\ SS_{\mathrm{R}}(\beta_k|\beta_0) =& SS_{\mathrm{R}}(\beta_0, \beta_k) - SS_{\mathrm{R}}(\beta_0) \\ =& \hat{\beta}^T X^T y - \frac{(\sum_i y_i)^2}{n} \ (k \text{ degrees of freedom}).\end{aligned}$$

5. **Question 3.33**

   Solution.

   The sample correlation coefficient between y and $\hat{y}$ is

$$
\begin{aligned}
(\mathrm{Corr}(y,\hat{y}))^2 =& \frac{[\frac{1}{n}\sum_i (y_i-\bar{y})(\hat{y}_i-\bar{y})]^2}{\frac{1}{n}\sum_i (y_i-\bar{y})^2 \frac{1}{n}\sum_i (\hat{y}_i-\bar{y})^2} \\
=& \frac{[(y-1\bar{y})^T(\hat{y}-1\bar{y})]^2}{SS_{\mathrm{T}}SS_{\mathrm{R}}}.
\end{aligned}
$$

   Applying

$$
H^T = H,\ H1 = 1,\ HH = H,
$$

   the numerator term can be rewritten as

$$
\begin{aligned}
(y-1\bar{y})^T(\hat{y}-1\bar{y}) =& (y-\hat{y}+\hat{y}-1\bar{y})^T(\hat{y}-1\bar{y}) \\
=& y^T(I_n-H)^T(H-1(1^T1)^{-1}1^T)y + (\hat{y}-1\bar{y})^T(\hat{y}-1\bar{y}) \\
=& (\hat{y}-1\bar{y})^T(\hat{y}-1\bar{y}) \\
=& SS_{\mathrm{R}}.
\end{aligned}
$$

   Then, we have

$$
(\mathrm{Corr}(y,\hat{y}))^2 = \frac{(SS_{\mathrm{R}})^2}{SS_{\mathrm{T}}SS_{\mathrm{R}}} = \frac{SS_{\mathrm{R}}}{SS_{\mathrm{T}}} = R^2.
$$