MAT 2002 Ordinary Differential Equations

Assignment 7 solution

Question 1 : Let $A = \begin{pmatrix} -2 & -1 \\ -\alpha & -2 \end{pmatrix}$

$\det(A - \lambda I) = (2+\lambda)(2+\lambda) - \alpha$

So the two eigenvalues are $\lambda_1 = -2-\sqrt{\alpha}$, $\lambda_2 = -2+\sqrt{\alpha}$ if $\alpha \geq 0$

$\lambda_1 = -2-\sqrt{-\alpha}\, i$, $\lambda_2 = -2+\sqrt{-\alpha}\, i$ if $\alpha < 0$

(a) When $\alpha = 3$, $\lambda_1 = -2-\sqrt{3} < 0$, $\lambda_2 = -2+\sqrt{3} < 0$

So the critical point 0 is a node and asymptotically stable

(b) when $\alpha = 5$, $\lambda_1 = -2-\sqrt{5} < 0$, $\lambda_2 = -2+\sqrt{5} > 0$

So the critical point 0 is a saddle point and unstable

(c) The transition happens when $-2+\sqrt{\alpha} = 0$

$\alpha = 4$

So the value of α where the transition happens is $\alpha = 4$

Notice there also exist another transition point at $\alpha = 0$.
(from asymptotically stable spiral to asymptotically stable node)
(No need to include this result in answer)

Question 2:

(a) Let $A = \begin{pmatrix} 1 & 1 \\ -5 & -3 \end{pmatrix}$

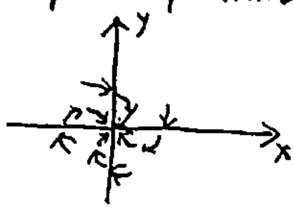
$\det(A - \lambda I) = 0 \Rightarrow (1-\lambda)(-3-\lambda)+5=0 \Rightarrow (\lambda+1)^2+1=0$

So the eigenvalues of A is $\lambda_1 = -1-i \quad \lambda_2 = -1+i$

0 is an asymptotically stable and spirial point.

At point $(0,1)$, $\frac{dy}{dt} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$, so the trajectory is clockwise

The phase portrait:



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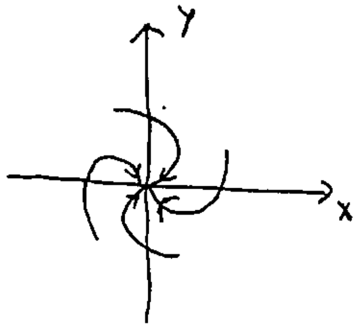
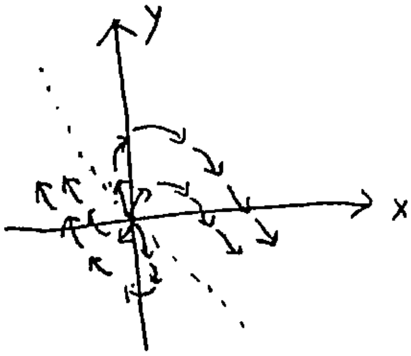


(b) Let $A = \begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix}$

$\det(A - \lambda I) = 0 \Rightarrow (3-\lambda)(-1-\lambda)+4=0 \Rightarrow (\lambda-1)^2=0 \Rightarrow \lambda_1=\lambda_2=1>0$

The matrix $(A-I) = \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}$ has only one eigen vector $\xi_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

From $(A-\lambda I)\eta = \xi_1$, we can find $\eta = \begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix}$



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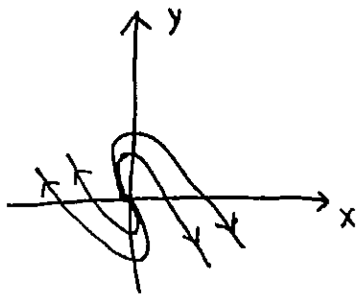


Note: the tendency (clockwise) can also be computed from some ~~particular points~~ particuler points.

Suggestion: use point $(0,1)$, whose derivative is $\frac{dy}{dt} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

So it is clockwise.

Question 3:

(a) First, we find the critical point where $\frac{dx}{dt}=0$

$$x^0 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

Let $x = x^0 + u$, then

$$\frac{du}{dt} = \frac{dx}{dt} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} u$$

$$\begin{vmatrix} 1-\lambda & 1 \\ 1 & -1-\lambda \end{vmatrix} = \lambda^2-2 \Rightarrow \text{the eigenvalues are } \lambda_1=\sqrt{2},\ \lambda_2=-\sqrt{2}$$

So the critical point is saddle and unstable

(b) First, we find the critical point where $\frac{dx}{dt}=0$

$$x^0 = \begin{pmatrix} -1 & -1 \\ 2 & -1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

Let $x = x^0 + u$, then

$$\frac{du}{dt} = \frac{dx}{dt} = \begin{pmatrix} -1 & -1 \\ 2 & -1 \end{pmatrix} u$$

$$\begin{vmatrix} -1-\lambda & -1 \\ 2 & -1-\lambda \end{vmatrix} = (\lambda+1)^2+2 \Rightarrow \text{the eigonvalues are } \lambda_1=-1+\sqrt{2}i,\ \lambda_2=-1-\sqrt{2}i$$

So the critical point is asymptotically stable and spirial point.

~~Q4~~ Q4:

(a) $\det(A-\lambda I) = \begin{pmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{pmatrix}$

$$= \lambda^2 - (a_{11}+a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21}$$

The solution of λ is

$$\lambda_{1,2} = \frac{-(a_{11}+a_{22}) \pm \sqrt{(a_{11}+a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21})}}{2}$$

If $\lambda_{1,2}$ are pure imaginary,

$$a_{11} + a_{22} = 0 \qquad a_{11}a_{22} - a_{12}a_{21} > 0$$

(b) Now $\frac{dy}{dx} = \frac{a_{21}x + a_{22}y}{a_{11}x + a_{12}y}$

$$(a_{21}x + a_{22}y) - (a_{11}x + a_{12}y)\frac{dy}{dx} = 0$$

Let $M = a_{21}x + a_{22}y$, $N = -(a_{11}x + a_{12}y)$ we have $\frac{\partial M}{\partial y} = +a_{22}$

$$\frac{\partial N}{\partial x} = -a_{11}$$

So this equation is exact

(c) ~~Since $\frac{\partial^2 M}{\partial x \partial y} = \frac{\partial^2 N}{\partial x}$~~ Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, There exists φ such that

$\frac{\partial \varphi}{\partial x} = M$ and $\frac{\partial \varphi}{\partial y} = N$

So $\varphi = \int M\,dx + f(y) = \frac{1}{2}a_{21}x^2 + a_{22}xy + f(y)$

$$\frac{\partial \varphi}{\partial y} = a_{22}x + f'(y) = -(a_{11}x + a_{12}y)$$

So $f(y) = -\frac{1}{2}a_{12}y^2$

Thus the solution is $a_{21}x^2 + 2a_{22}xy - a_{12}y^2 = k$

~~Notice the this equation can be transformed into~~

Notice Now we have $\Delta^2 = 4a_{22}^2 + 4a_{21}a_{12} = 4(-a_{11}a_{22} + a_{21}a_{12}) < 0$

So The trajectory must be ellipses.

Question 5:

(a) $\left.\begin{array}{l}\frac{dx}{dt}=0\\ \frac{dy}{dt}=0\end{array}\right\} \Rightarrow \left.\begin{array}{l}-y=0\\ -\gamma y - x(x-0.15)(x-2)=0\end{array}\right\} \Rightarrow$ 3 solutions $\begin{cases}x=0\\y=0\end{cases}$ $\begin{cases}x=0.15\\y=0\end{cases}$ $\begin{cases}x=2\\y=0\end{cases}$

Let $F=-y$, $G=-\gamma y - x(x-0.15)(x-2)$, the Jacobian matrix would be:

$$J=\begin{pmatrix}F_x & F_y\\ G_x & G_y\end{pmatrix}=\begin{pmatrix}0 & -1\\ -(3x^2-4.3x+0.3) & -\gamma\end{pmatrix}$$

So the linear system near $\binom{0}{0}$: $\begin{pmatrix}\frac{dx}{dt}\\ \frac{dy}{dt}\end{pmatrix}=\begin{pmatrix}0 & -1\\ -0.3 & -\gamma\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$

the linear system near $\binom{0.15}{0}$: $\begin{pmatrix}\frac{dx}{dt}\\ \frac{dy}{dt}\end{pmatrix}=\begin{pmatrix}0 & -1\\ 0.2775 & -\gamma\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$

the linear system near $\binom{2}{0}$: $\begin{pmatrix}\frac{dx}{dt}\\ \frac{dy}{dt}\end{pmatrix}=\begin{pmatrix}0 & -1\\ -3.7 & -\gamma\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$

the eigenvalue for the linear system near $\binom{0}{0}$: $\lambda_1=\frac{-\gamma-\sqrt{\gamma^2+1.2}}{2}<0$, $\lambda_2=\frac{-\gamma+\sqrt{\gamma^2+1.2}}{2}>0$

So the critical point $\binom{0}{0}$ is saddle point and unstable

the eigenvalue for the linear system near $\binom{0.15}{0}$: $\lambda=\frac{-\gamma\pm\sqrt{\gamma^2-1.11}}{2}$

So the critical point $\binom{0.15}{0}$ is

spiral point and asymptotically stable if $0<\gamma<\sqrt{1.11}$

node and asymptotically stable if $\gamma>\sqrt{1.11}$

node or spiral point and asymptotically stable if $\gamma=\sqrt{1.11}$

the eigenvalue for the linear system near $\binom{2}{0}$: $\lambda=\frac{-\gamma\pm\sqrt{\gamma^2+14.8}}{2}$

So the critical point $\binom{2}{0}$ is saddle point and unstable.

Note for Question 5:

We can also consider the case when $\gamma \le 0$

near $\binom{0}{0}$: $\lambda_1 < 0$ and $\lambda_2 > 0$ still holds, so it is saddle point and unstable

near $\binom{0.15}{0}$: When $\gamma = 0$, the eigen value is $\lambda = \pm \frac{\sqrt{1.11}}{2} i$

~~So it is center and stable~~

~~when $\gamma < \sqrt{1.11}$~~ Notice now the system is locally linear system

So the type is center or spiral point, while the stability is indeterminate

when $\gamma < -\sqrt{1.11}$, it is ~~asymptotically~~ node and unstable

when ~~$\sqrt{1.11} < \lambda < 0$~~ ~~$=\sqrt{1.11}$~~ $-\sqrt{1.11} < \gamma < 0$, it is spiral and unstable

when $\gamma = \sqrt{1.11}$, it is node or spiral point and unstable

near $\binom{2}{0}$: $\lambda_1 = \frac{-\gamma - \sqrt{\gamma^2 + 14.8}}{2} < 0$, $\lambda_2 = \frac{-\gamma + \sqrt{\gamma^2 + 14.8}}{2} > 0$

So it is saddle point and unstable

Question 6

(a)

Let $V = ax^2 + bxy + cy^2$

then $\dot{V} = (2ax+by)x' + (bx+2cy)y'$

$= (2ax+by)(-x^3+xy^2) + (bx+2cy)(-2x^2y - y^3)$

$= -2ax^4 - bx^3y + 2ax^2y^2 + bxy^3 - 2bx^3y - bxy^3 - 4cx^2y^2 - 2cy^4$

$= -2ax^4 - 3bx^3y + (2a-4c)x^2y^2 - 2cy^4$

Let $a=1,\ c=1,\ b=0$

then $V = x^2 + y^2$, positive definite

$\dot{V} = -2x^4 - 2x^2y^2 - 2y^4$, negative definite

So the critical point $\binom{0}{0}$ is asymptotically stable.

(b) Let $V = ax^2 + bxy + cy^2$

then $\dot{V} = (2ax+by)x' + (bx+2cy)y'$

$= (2ax+by)(x^3-y^3) + (bx+2cy)(2xy^2 + 4x^2y + 2y^3)$

$= 2ax^4 + bx^3y - 2axy^3 - by^4 + 2bx^2y^2 + 4bx^3y \cancel{+ 2bx^3y + 4cxy^3}$

$+ 2bxy^3 + 4cxy^3 + 8cx^2y^2 + 4cy^4$

$= 2ax^4 + 5bx^3y + (2b+8c)x^2y^2 + (2b-2a+4c)xy^3 + (4c-b)y^4$

Let $\cancel{a=2}\ b=0,\ c=1,\ a=2$

then $V = 2x^2 + y^2$, positive definite

$\dot{V} = 4x^4 + 8x^2y^2 + 4y^4$, positive definite

So the critical point $\binom{0}{0}$ is unstable

[redacted]

Question 7

(a)

1、 $\left.\begin{array}{l}\frac{dx}{dt}=0 \\ \frac{dy}{dt}=0\end{array}\right\} \Rightarrow \left.\begin{array}{l} x(1-x-y)=0 \\ y(1.5-x-y)=0\end{array}\right\} \Rightarrow$ 3 solutions $\begin{cases} x=0 \\ y=0 \end{cases}$ $\begin{cases} x=1 \\ y=0 \end{cases}$ $\begin{cases} x=0 \\ y=1.5 \end{cases}$

Let $F = x(1-x-y)$, $G = y(1.5-x-y)$, the Jacobian matrix would be:

$$J = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix} = \begin{pmatrix} 1-2x-y & -x \\ -y & 1.5-x-2y \end{pmatrix}$$

So the linear system near $\binom{0}{0}$: $\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1.5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

the linear system near $\binom{1}{0}$: $\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 0 & 0.5 \end{pmatrix}\left[\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]$

the linear system near $\binom{0}{1.5}$: $\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} -0.5 & 0 \\ -1.5 & -1.5 \end{pmatrix}\left[\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ 1.5 \end{pmatrix}\right]$

2、 the eigenvalues for the system near $\binom{0}{0}$: $\lambda_1 = 1 > 0$, $\lambda_2 = 1.5 > 0$

So the critical point $\binom{0}{0}$ is node and unstable

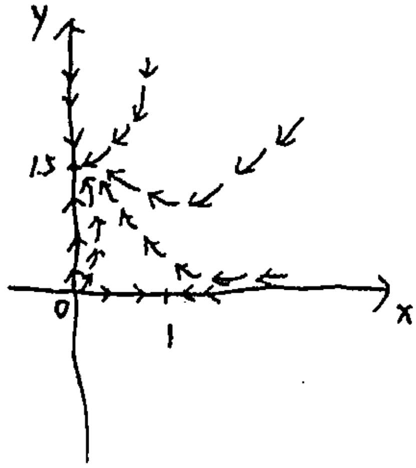
the eigenvalues for the system near $\binom{1}{0}$: $\lambda_1 = -1 < 0$, $\lambda_2 = 0.5 > 0$

So the critical point $\binom{1}{0}$ is saddle point and unstable

the eigenvalues for the system near $\binom{0}{1.5}$: $\lambda_1 = -0.5 < 0$, $\lambda_2 = -1.5 < 0$

So the critical point $\binom{0}{1.5}$ is node and asymptotically stable.

3、 the phase portrait:



4、 the solutions approaches the point $\binom{0}{1.5}$ as $t \to \infty$ if the initial $y \neq 0$ (point toward)

Otherwise ($y=0$), the solution will converge to the point $(1, 0)$. (move on x-axis)

5、 the term $1-x-y$ and $1.5-x-y$ is the limitation of food supply. The growth rate is proportional to the presence of species and the available food supply.

When $t \to \infty$, x species vanished, only y species exists. This is because y species has higher efficiency in utilizing food. ($1.5-x-y > 1-x-y$)

This is competition relationship

(b)

1. $\left.\begin{array}{l}\frac{dx}{dt}=0\\ \frac{dy}{dt}=0\end{array}\right\} \Rightarrow \left.\begin{array}{l}x(1-x+0.5y)=0\\ y(2.5-1.5y+0.25x)=0\end{array}\right\} \Rightarrow$ 4 solutions $\begin{cases}x=0\\y=0\end{cases}$ $\begin{cases}x=0\\y=\frac{5}{3}\end{cases}$ $\begin{cases}x=1\\y=0\end{cases}$ $\begin{cases}x=2\\y=2\end{cases}$

Let $F=x(1-x+0.5y)$, $G=y(2.5-1.5y+0.25x)$, the Jacobian matrix would be

$$J=\begin{pmatrix}F_x & F_y\\ G_x & G_y\end{pmatrix}=\begin{pmatrix}1-2x+0.5y & 0.5x\\ 0.25y & 2.5-3y+0.25x\end{pmatrix}$$

So the linear system near $\binom{0}{0}$: $\begin{pmatrix}dx/dt\\dy/dt\end{pmatrix}=\begin{pmatrix}1 & 0\\0 & 2.5\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$

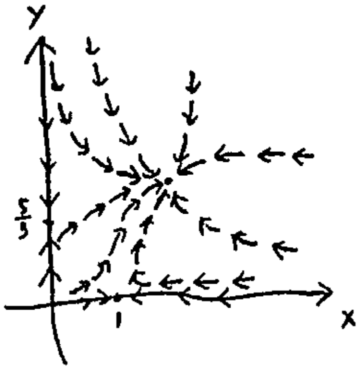
the linear system near $\binom{0}{\frac{5}{3}}$: $\begin{pmatrix}dx/dt\\dy/dt\end{pmatrix}=\begin{pmatrix}11/6 & 0\\5/12 & -2.5\end{pmatrix}\left[\begin{pmatrix}x\\y\end{pmatrix}-\begin{pmatrix}0\\\frac{5}{3}\end{pmatrix}\right]$

the linear system near $\binom{1}{0}$: $\begin{pmatrix}dx/dt\\dy/dt\end{pmatrix}=\begin{pmatrix}-1 & 0.5\\0 & 2.75\end{pmatrix}\left[\begin{pmatrix}x\\y\end{pmatrix}-\begin{pmatrix}1\\0\end{pmatrix}\right]$

the linear system near $\binom{2}{2}$: $\begin{pmatrix}dx/dt\\dy/dt\end{pmatrix}=\begin{pmatrix}-2 & 1\\0.5 & -3\end{pmatrix}\left[\begin{pmatrix}x\\y\end{pmatrix}-\begin{pmatrix}2\\2\end{pmatrix}\right]$

2. the eigenvalues for the system near $\binom{0}{0}$: $\lambda_1=1>0$ $\lambda_2=2.5>0$
So the critical point $\binom{0}{0}$ is node and unstable
the eigenvalues for the system near $\binom{0}{5/3}$: $\lambda_1=11/6>0$, $\lambda_2=-2.5<0$
So the critical point $\binom{0}{5/3}$ is saddle point and unstable
the eigenvalues for the system near $\binom{1}{0}$: $\lambda_1=-1<0$, $\lambda_2=2.75>0$
So the critical point $\binom{1}{0}$ is saddle point and unstable
the eigenvalues for the system near $\binom{2}{2}$: ~~$\lambda_1=-1<0$ $\lambda_2=-4<0$~~ $\lambda_1=\frac{-5-\sqrt{3}}{2}<0$, $\lambda_2=\frac{-5+\sqrt{3}}{2}<0$
So the critical point $\binom{2}{2}$ is node and asymptotically stable

3. Phase Portrait:



4. the solution approaches $(2,2)$ as $t\to\infty$
if $x, y\neq 0$ (point toward)
if $x=0, y>0$, then the solution will move on y-axis and approach $(0, 5/3)$
If $x>0, y=0$, then the solution will move on x-axis and approach $(1,0)$

5. the term $(1-x+0.5y)$ and $(2.5-1.5y+0.25x)$ represents the limitation of food supply. The growth rate is proportional to the presence of species and available food supply

When $t\to\infty$, x species and y species reachs an equilibrium since y species increase the food supply for x, so does x to y.
This is coexistence.

扫描全能王 创建