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CSC3001 Discrete Mathematics
Practice Exercise 2 **Sep 2020**

First order logic

1. Express the following sentence using first order logic.
   Define P(s,t) = student s goes to tutorial t
   Q(t,c) = tutorial t is in course c
   S = {Students}
   T = {Tutorials}
   C = {Courses} $\exists s \in S.\ \forall c \in C.\ \exists t \in T.\ P(s,t) \wedge Q(t,c)$
   Some student goes to at least one tutorial of each course.

2. Let $O(r)$ be " Router r is out of services", $B(r)$ be "Printer r is busy", $L(p)$ be "Packet p is lost" and $Q(p)$ be "Packet p is queued". Translate the following arguments and its negation into English.
   (a) $(\exists r, (O(r) \wedge B(r))) \rightarrow (\exists p, L(p))$.
   (b) $(\forall r, B(r)) \rightarrow (\exists p, Q(p))$.
   (c) $(\exists p, (Q(p) \wedge L(p))) \rightarrow (\exists r, O(r))$.
   (d) $((\exists r, B(r)) \wedge (\forall p, Q(p))) \rightarrow (\exists p, L(p))$.

3. Express the following mathematical statements into first order logic.
   (a) Any odd prime can be written as the sum of the square of two integers.
   Define: (a) $\forall p \in \mathbb{Z}^+,\ \neg oddprime(p) \vee sumofsquare(p)$
   $odd(p) := p$ is odd (e.g., $= \exists k \in \mathbb{Z}, p = 2k + 1$).
   $prime(p) := p$ is prime. (b) $\forall n \in \mathbb{Z},\ (n \le 0) \vee (\exists a,b,c,d \in \mathbb{Z},\ n = a^2+b^2+c^2+d^2)$
   $oddprime(p) := p$ is an odd prime.
   $sumofsquare(p) := p$ can be written as a sum of the square of two integers.
   $P(p) := p$ is an odd prime and can be written as a sum of the square of two integers.
   (b) Every positive integer can be written as the sum of the square of four integers.
   (c) Every three positive integers which satisfy $a + b = c$ must have a common factor greater than 1. (c) $\forall a,b,c \in \mathbb{Z}^+,\ \neg abc(a,b,c) \vee cf(a,b,c)$.
   Define:
   $abc(a, b, c) := (a + b = c)$
   $cf(a, b, c) := a$, b and c have a common factor greater than 1.
   $P(a, b, c) := a$, b and c satisfy $a + b = c$ and have a common factor greater than 1.
   (d) There exists a unique prime number that is even.
   Define: (d) $(\exists p \in \mathbb{Z}, P(p)) \wedge (\forall q \in \mathbb{Z}, (p=q) \vee \neg P(q))$
   $even(p) := p$ is even.
   $P(p) := p$ is prime and even.

4. **T/F** Let $P(x, y)$ be a predicate, where x and y has non empty domain. Which of the following is true?
   T (a) $\exists x, \exists y, P(x, y) \equiv \exists y, \exists x, P(x, y)$.
   T (b) $\forall x, \forall y, P(x, y) \equiv \forall y, \forall x, P(x, y)$.
   F (c) $\forall x, \exists y, P(x, y) \equiv \exists y, \forall x, P(x, y)$.
   F (d) $\forall x, \exists y, \neg P(x, y) \equiv \neg(\exists x, \forall y, \neg P(x, y))$.
   T (e) $\forall x, \exists y, \neg P(x, y) \equiv \forall x, \neg(\forall y, P(x, y))$.
   F (f) $\neg(\forall y, \forall x, P(x, y)) \equiv \forall x, \neg(\forall y, P(x, y))$.

T (g) $(\forall x, \exists y, P(x, y)) \rightarrow (\exists y, \exists x, P(x, y))$.

F (h) $\forall x \in Z^+, \exists y \in Z^+, x > y$.

F. (i) There are infinitely many prime numbers can be written as:
$\forall p \in N, \exists q \in N, (\text{prime}(p) \wedge \text{prime}(q) \wedge (q > p))$,
where prime(p) := p is a prime.

P	Q	P∨Q	(P∨Q)→(P∨Q)
T	T	T	T
F	F	F	T
T	F	T	T
F	T	T	T

5.
(a) Consider the following sentence in First Order Logic:

$\forall x$ (Married(Father(x), Mother(x)) $\rightarrow$
($\exists y$ (Certificate(y) $\wedge$ Names(y, Father(x), Mother(x))))).

In English, For every person who has a father and mother that are married, there exists a paper which is a wedding certificate and which contains the names of both father and the mother of this person.

Identify the functions, properties, binary relations and connectives in this sentence.

(b) Consider the sentence: $\exists x\, [p(x) \rightarrow q(x)]$. Assume we know that there is no value for x for which p(x) is true. Is it possible that the above sentence is true? Poissible .

(c) Provide the truth-table for $(P \vee Q) \rightarrow (P \vee Q)$ where P and Q are two propositions.

6. Consider the following sentence: F(x) is true when x is female, M(x) is true when x is male, D(x) is true when x lives in Disneyland and L(x,y) is true when x likes y. Translate the following sentence into first order logic:

(a) There is at least one male and female, both lives in Disneyland, that likes each other.
(b) All males and females live in Disneyland like each other.

(a) $(\exists x.\ M(x)) \wedge (\exists y.\ F(y)) \wedge D(x) \wedge D(y) \wedge L(x,y) \wedge L(y,x)$

(b) $\forall x.\ \forall y.\ \neg M(x) \vee \neg F(y) \vee \neg D(x) \vee \neg D(y) \vee (L(x,y) \wedge L(y,x))$.

Set Theory

1. Let $A = \{x \in Z | x = 6a + 4$ for some integer a$\}$, $B = \{y \in Z | y = 18b - 2$ for some integer b$\}$, and $C = \{z \in Z | z = 18c + 16$ for some integer c$\}$. Prove or disprove each of the following statements.
(a) $A \subseteq B$ (b) $B \subseteq A$ (c) $B = C$

$A = \{x \in \mathbb{Z} | x = 2(3a+1), a \in \mathbb{Z}\}$ $C = \{z \in \mathbb{Z} | z = 2(9c-1), c \in \mathbb{Z}\}$.
$B = \{y \in \mathbb{Z} | y = 2(9b-1), b \in \mathbb{Z}\}$

(a) ✗ (b) ✗ (c) ✓.

2. Indicate which of the following relationships are true and which are false:

T (a) $Z+ \subseteq Q$ F (b) $Q \subseteq Z$
F (c) $Z+ \cap Z \subseteq \emptyset$ T (d) $Q \cap R = Q$
T (e) $Q \cup Z = Q$ T (f) $Z+ \cap R = Z+$
F (g) $Z \cup Q = Z$

3. Find Pow(∅) and Pow(Pow(Pow(∅))).

$Pow(\emptyset) = \{\emptyset\}$. $Pow(Pow(\emptyset)) = Pow(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$
$Pow(Pow(Pow(\emptyset))) = Pow(\{\emptyset, \{\emptyset\}\})$
$= \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$.

✓ 4. Prove: for all sets A, B and C, if $A \subseteq B$ then $A \cup C \subseteq B \cup C$.

✓ 5. Prove: for all sets A, B and C, $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

✓ 6. The symmetric difference $A \oplus B$ of sets A and B is defined as $A \oplus B = (A - B) \cup (B - A)$.
(a) Show that $A \oplus B = B \oplus A$.
(b) Show that $A \oplus B = (A \cup B) - (A \cap B)$.

4. Proof. $A \subseteq B$. for $\forall x \in A \Rightarrow x \in B$.

$A \cup C = \{x \mid x \in A \text{ or } x \in C\}$, $B \cup C = \{x \mid x \in B \text{ or } x \in C\}$.

$\forall x \in A \cup C$. ① If $x \in A$, then $x \in B \Rightarrow x \in B \cup C$.

② If $x \in C$, then $x \in C \Rightarrow x \in B \cup C$.

5. Proof. $A \times (B \cap C) = \{(x,y) \mid x \in A, y \in B \cap C\}$.

$(A \times B) \cap (A \times C) = \{(x,y) \mid x \in A, y \in B\} \cap \{(x,y) \mid x \in A, y \in C\}$.

① Show "$\Rightarrow$".

$\forall (x,y) \in A \times (B \cap C)$. $\Rightarrow x \in A, y \in B \cap C \Rightarrow x \in A, y \in B \cap y \in C \Rightarrow (x,y) \in (A \times B) \cap (A \times C)$

$\Rightarrow A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$

② Show "$\Leftarrow$".

$\forall (x,y) \in (A \times B) \cap (A \times C) \Rightarrow x \in A, y \in B \cap y \in C \Rightarrow x \in A, y \in B \cap C \Rightarrow (x,y) \in A \times (B \cap C)$

$\Rightarrow (A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$

6. Proof. (a). $A \oplus B = (A-B) \cup (B-A)$.

$B \oplus A = (B-A) \cup (A-B)$.

$(A-B) \cup (B-A) = (B-A) \cup (A-B) \Rightarrow A \oplus B = B \oplus A$.

(b). W.T.S. $(A-B) \cup (B-A) = (A \cup B) - (A \cap B)$

① Show "$\Rightarrow$".

$\forall x \in (A-B) \cup (B-A) \Rightarrow x \in A-B$ or $x \in B-A$. $\Rightarrow$. $x \in A, x \notin B$ or $x \in B, x \notin A$

$\Rightarrow x \in A$ or $x \in B$, $x \notin A$ and $x \notin B$. $\Rightarrow x \in A \cup B$, $x \notin A \cap B$. $\Rightarrow x \in (A \cup B) - (A \cap B)$

$\Rightarrow (A-B) \cup (B-A) \subseteq (A \cup B) - (A \cap B)$.

② Show "$\Leftarrow$".

$\forall x \in (A \cup B) - (A \cap B)$. $\Rightarrow x \in A \cup B$, $x \notin A \cap B \Rightarrow x \in A$ or $x \in B$, $x \notin A$ and $x \notin B$

$\Rightarrow x \in A, x \notin B$ or $x \in B, x \notin A \Rightarrow x \in A-B$ or $x \in B-A \Rightarrow x \in (A-B) \cup (B-A)$.

$\Rightarrow (A \cup B) - (A \cap B) \subseteq (A-B) \cup (B-A)$

Thus, $A \oplus B = (A \cup B) - (A \cap B)$.