MAT 3253 Lecture 3

Example $(-1+i)^{20}$

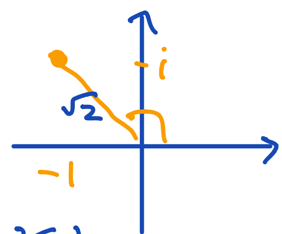
Method 1, expansion

$$(-1)^{20} + \binom{20}{1}(-1)^{19}\cdot i + \binom{20}{2}(-1)^{18}(i)^2 + \dots + (i)^{20}$$

binomial coef $\quad \boxed{\binom{n}{k}}, \quad C_n^k, \quad {}_nC_k, \quad C_k^n$

Method 2, Polar form

$$(-1+i) = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$



$$(-1+i)^{20} = 2^{\frac{20}{2}}\left(\cos 20\cdot\frac{3\pi}{4} + i\sin 20\cdot\frac{3\pi}{4}\right)$$
$$= 1024(\cos 15\pi + i\sin 15\pi)$$
$$= 1024(\cos\pi + i\sin\pi)$$
$$= -1024$$

Division $\quad \frac{a+bi}{c+di} = w \qquad a,b,c,d \in \mathbb{R}$

$w \in \mathbb{C}$

$$w\cdot(c+di) = a+bi$$

Method 1 $\quad x+iy = w$

$$(x+iy)(c+di) = cx - dy + i(xd+yc)$$
$$= a + bi$$

$$\begin{bmatrix} c & -d \\ d & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\underbrace{c^2+d^2}_{|c+id|^2}} \begin{bmatrix} c & d \\ -d & c \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

Assuming $c^2+d^2 \neq 0$

$$\boxed{\frac{1}{\sqrt{2}-1} = \frac{1}{\sqrt{2}-1} \cdot \frac{\sqrt{2}+1}{\sqrt{2}+1} = \frac{\sqrt{2}+1}{2-1} = \sqrt{2}+1}$$

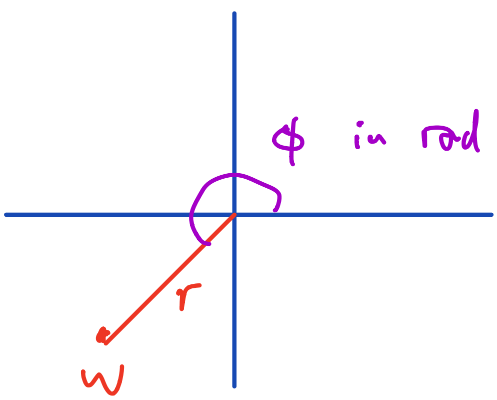
Method 2

$$\frac{a+bi}{c+di} \cdot \frac{c-di}{c-di}$$

$$= \frac{ac+bd+i(bc-ad)}{\underbrace{c^2+d^2}}$$ is a real number.

Method 3 Computer / python

Argument vs principal argument



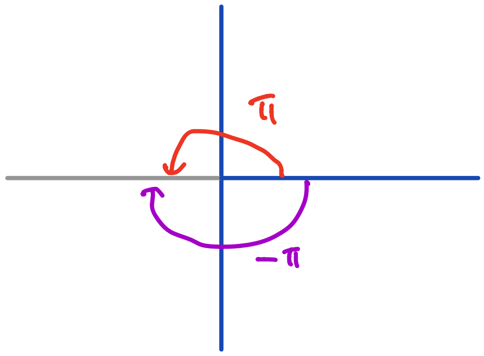
ϕ in rad

$\phi + 2\pi$

$\phi - 2\pi$

$\phi + 2\pi k \quad k \in \mathbb{Z}$

Principal arguement is in $(-\pi, \pi]$



Brown & Churchill : principal arguement is denoted by Arg

arg(z) is the argument

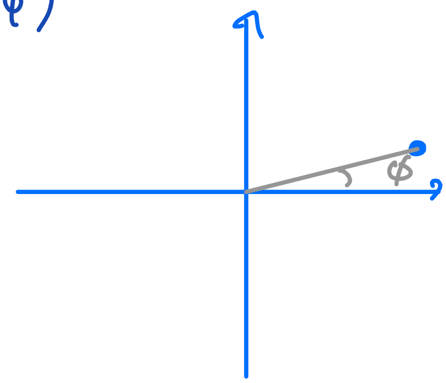
Bak & Newman : Arg(z) is the arguement

Remark : Complex number has no order

e.g. $i < 2i$ has no meaning

Square root

$$4+i = 17(\cos\phi + i\sin\phi)$$



$$\phi = \tan^{-1}\left(\tfrac{1}{4}\right)$$

$\sqrt{4+i}$ has modulus $(17)^{1/4}$

has argument $\frac{\phi}{2}$ or $\frac{\phi}{2}+\pi$

Finding θ s.t. $2\theta = \phi + 2\pi k \quad k \in \mathbb{Z}$

$$\theta = \frac{\phi}{2} + \pi k \quad k \in \mathbb{Z}$$

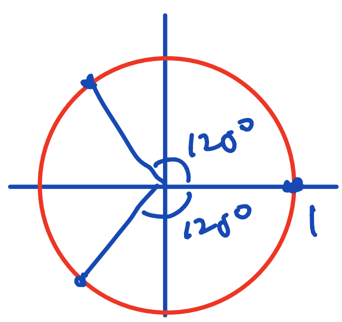
$$\sqrt{4+i} = (17)^{\frac{1}{4}}\Big(\cos\left(\tfrac{1}{2}\tan^{-1}(0.25)+\pi k\right)$$

$$+\, i\sin\left(\tfrac{1}{2}\tan^{-1}(0.25)+\pi k\right)\Big) \quad k = 0, 1.$$

$$= \pm (17)^{\frac{1}{4}}\left(\cos\tfrac{\phi}{2} + i\sin\tfrac{\phi}{2}\right)$$

Cube root

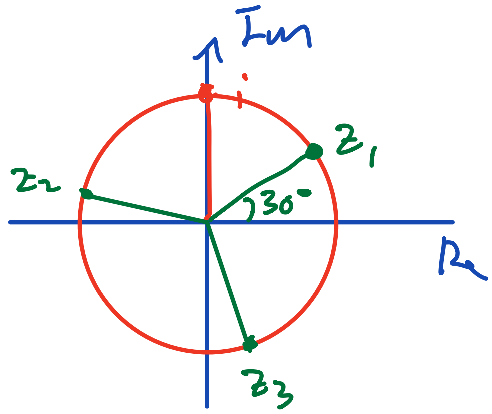
Example $\quad z^3 = 1$



Solutions are

$$1, \quad -\frac{1}{2} \pm \sqrt{\frac{3}{2}}\, i$$

Example Solve $z^3 = i$



$\phi = \frac{\pi}{2}$

$3\theta = \frac{\pi}{2} + 2\pi k \quad k \in \mathbb{Z}$

$\theta = \frac{\pi}{6} + \frac{2\pi k}{3}$

Solution $\cos\left(\frac{\pi}{6} + \frac{2\pi k}{3}\right) + i \sin\left(\frac{\pi}{6} + \frac{2\pi k}{3}\right) \quad k = 0, 1, 2.$