Solution to Sample 1

Question 1

(a) F: It can only be claimed that H_0 is 95% wrong, not 100% (surely).

(b) T: Rejecting H_0 at the 5% level provides sufficient evidence that H_0 is wrong, which implies correct H_1 since H_0 and H_1 are the only options considered.

(c) F: This only means insufficient evidence that H_0 is wrong in the sense that there is more than 10% chance that H_0 is correct.

Question 2

(a) T: A nonparametric procedure does not need any assumption or condition on the underlying probability distributions of the sample data, hence is valid whatever the sample distribution.

(b) F: A nonparametric procedure may still use binomial and normal distributions, which do not rely on any parametric assumptions. The former arises naturally from independent binary trials with a common probability of success, while the latter is derived from the central limit theorem.

(c) F: For example, a location parameter can be tested by a nonparametric procedure, such as the sign test.

Question 3

(a) T: $2^8 = 256$ and $T^+ = 7$ for (7), (1,6), (2,5), (3,4), (1,2,4)

(b) F: T^+ is symmetric about $8(9)/4 = 18$ (36 is the largest value of T^+).

(c) T: If $X_i < 0 < X_j$, then $R_i < R_j \iff -X_i = |X_i| < |X_j| = X_j \iff 0 < X_i + X_j$.

Question 4

(a) F: If $\Delta \neq 0$, then the Ansari-Bradley test is not justified.

(b) F: If equal dispersion is not justified, then neither are the Wilcoxon rank sum test and its result of different locations.

(c) T: The Wilcoxon rank sum test is not reliable because $\gamma^2 = 1$ is not justified; the Ansari-Bradley test is not reliable because $\Delta = 0$ is not justified.

Question 5

(a) The values of $Z_i = Y_i - X_i$, $i = 1,\ldots,12$, are calculated as

750, 1400, –300, 2400, –700, 800, 1300, –400, 1900, –1100, 1600, 300

The data have 8 positive values in the sample of size $n = 12$ and $B \sim Bin(12, 0.5)$ under $H_0 : \theta = 0$. Hence the exact p-value of testing $H_0 : \theta = 0$ against $H_1 : \theta > 0$ by the sign test is

$$\Pr(B \geq 8) = \Pr(B \leq 4) = \left[\binom{12}{0} + \binom{12}{1} + \binom{12}{2} + \binom{12}{3} + \binom{12}{4}\right](0.5)^{12} = 0.1938$$

(b) The approximate p-value with continuity correction is

$$\Pr(B \geq 8) \approx \Pr\left(Z > \frac{7.5 - 12(0.5)}{\sqrt{12(0.5)(1-0.5)}} \right) = \Pr(Z > 0.8660) = 0.1932$$

(c) The ordered values $Z_{(1)} \leq Z_{(2)} \leq \cdots \leq Z_{(12)}$ of $Z_1, \ldots, Z_{12}$ are:

–1100, –700, –400, –300, 300, 750, 800, 1300, 1400, 1600, 1900, 2400

Thus the estimate based on the sign statistic is

$$\tilde{\theta} = \frac{Z_{(6)} + Z_{(7)}}{2} = \frac{750 + 800}{2} = 775$$

For $B \sim Bin(12, 0.5)$, $\Pr(B \geq 10) = 0.0193 < 0.025$ and $\Pr(B \geq 9) = 0.0730 > 0.025$ (by MS Excel). Hence the minimum achievable confidence level above 95% is

$$1 - \alpha = 1 - 2(0.0193) = 1 - 0.0386 = 0.9614 \quad \text{with } \alpha = 0.0386$$

It follows that

$$b_{\alpha/2} = b_{0.0193} = 10 \quad \text{and} \quad C_\alpha = n + 1 - b_{\alpha/2} = 12 + 1 - 10 = 3$$

The 96.14% confidence interval of θ is given by $\left(Z_{(3)}, Z_{(10)}\right) = (-400, 1600)$.

For approximate 95% confidence interval, calculate

$$C_{0.05} \approx 0.5(12) - z_{0.025}\sqrt{12(0.5)(1-0.5)} = 6 - 1.96\sqrt{3} = 2.605$$

If we round it to 3, then an approximate 95% confidence interval is also given by $\left(Z_{(3)}, Z_{(10)}\right) = (-400, 1600)$.

A more conservative option is to take $C_{0.05} = 2$ and $\left(Z_{(2)}, Z_{(11)}\right) = (-700, 1900)$.

(d) Let θ denote the median of $Y-X$. We test $H_0:\theta=0$ (no difference between government and private sector salaries) against $H_1:\theta>0$ (private sector salaries are higher than government).

Calculate the values of $Z_i=Y_i-X_i$, $|Z_i|$, rank R_i of $|Z_i|$, $\psi_i=I_{\{Z_i>0\}}$ and $\psi_i R_i$, $i=1,\ldots,12$, in the following table:

i	Z_i	$\lvert Z_i\rvert$	R_i	ψ_i	$\psi_i R_i$
1	750	750	5	1	5
2	1400	1400	9	1	9
3	–300	300	1.5	0	0
4	2400	2400	12	1	12
5	–700	700	4	0	0
6	800	800	6	1	6
7	1300	1300	8	1	8
8	–400	400	3	0	0
9	1900	1900	11	1	11
10	–1100	1100	7	0	0
11	1600	1600	10	1	10
12	300	300	1.5	1	1.5

The Wilcoxon signed rank statistic is calculated as

$$T^+=\sum_{i=1}^{12}\psi_i R_i=5+9+12+6+8+11+10+1.5=62.5$$

By (2.7) and (2.10) of Topic 2 with $n=12$, under $H_0:\theta=0$,

$$\mathrm{E}_0[T^+]=\frac{12(13)}{4}=39 \quad\text{and}\quad \mathrm{Var}_0(T^+)=\frac{12(13)(25)}{12}-\frac{2(1)(3)}{48}=\frac{3897}{24}$$

(Note that there are two absolute values tied at 300, so that $g=1$ and $t_1=2$.)
It follows that

$$T^*=\frac{T^+-\mathrm{E}_0[T^+]}{\sqrt{\mathrm{Var}_0(T^+)}}=\frac{62.5-39}{\sqrt{3897/24}}=1.8442$$

and the approximate p-value of testing $H_0:\theta=0$ against $H_1:\theta>0$ is

$$\Pr\left(T^*>1.8442\right)\approx\Pr(N(0,1)>1.8442)=0.03258<0.05$$

Therefore, we reject $H_0:\theta=0$ in favour of $H_1:\theta>0$, which provides sufficient evidence that private sector salaries are higher than government at the 5% level of significance.

(e) Let $W_{(1)} \le W_{(2)} \le \cdots \le W_{(M)}$ be the ordered values of $\{(Z_i + Z_j)/2, 1 \le i \le j \le n\}$ (Walsh averages), where $M = n(n+1)/2 = 12(12+1)/2 = 78$. The values ordered Walsh averages are listed in the following table:

$W_{(1)} \le W_{(2)} \le \cdots \le W_{(78)}$											
k	$W_{(k)}$	k	$W_{(k)}$	k	$W_{(k)}$	k	$W_{(k)}$	k	$W_{(k)}$	k	$W_{(k)}$
1	–1100	14	–150	27	300	40	650	53	1050	66	1500
2	–900	15	–50	28	350	41	750	54	1075	67	1575
3	–750	16	0	29	400	42	750	55	1100	68	1600
4	–700	17	25	30	450	43	775	56	1100	69	1600
5	–700	18	50	31	450	44	800	57	1175	70	1600
6	–550	19	100	32	500	45	800	58	1200	71	1650
7	–500	20	150	33	500	46	800	59	1300	72	1750
8	–400	21	175	34	525	47	850	60	1325	73	1850
9	–400	22	200	35	550	48	850	61	1350	74	1900
10	–350	23	225	36	550	49	950	62	1350	75	1900
11	–300	24	250	37	600	50	1000	63	1350	76	2000
12	–200	25	250	38	600	51	1025	64	1400	77	2150
13	–175	26	300	39	650	52	1050	65	1450	78	2400

The estimate based on the signed ranks is

$$\hat{\theta} = \frac{W_{(M/2)} + W_{(M/2+1)}}{2} = \frac{W_{(39)} + W_{(40)}}{2} = \frac{650 + 650}{2} = 650$$

Since $\mathrm{E}_0[T^+] = 39$ and $\mathrm{Var}_0(T^+) = 3897/24$, by (2.14) of Topic 2

$$C_{0.05} \approx 39 - 1.96\sqrt{\frac{3897}{24}} = 14.024 \approx 14 \quad \text{and} \quad M + 1 - C_{0.05} \approx 79 - 14 = 65$$

Thus an approximate 95% confidence interval of θ based on the signed ranks is

$$\left(W_{(C_{0.05})}, W_{(M+1-C_{0.05})}\right) = \left(W_{(14)}, W_{(65)}\right) = (-150, 1450)$$

(f) The sign test has p-values over 19%, which shows insufficient evidence for $\theta > 0$ at the 10% level. The approximate p-value of the Wilcoxon signed rank test is 0.03258, which provides sufficient evidence at the 5% level for $\theta > 0$. This shows that the Wilcoxon signed rank test is more powerful and efficient than the sign test to detect the difference between paired samples based on the same set of data.

The 95% confidence interval (–150,1450) of the median θ based on the signed ranks is shorter, indicating more accurate estimation, than the 95% confidence interval (–400,1600) based on the sign statistic.

Question 6

Let $Z_{(1)} \le Z_{(2)} \le \cdots \le Z_{(m+n)}$ denote the ordered values of the combined X and Y values, and a_i the rank of $Z_{(i)}$, $i = 1,\ldots,m+n$. Then

$$(a_1, a_2, \ldots, a_{13}) = (1.5, 1.5, 3, 4, 5, 6, 7, 8, 9, 10, 11.5, 11.5, 13)$$

(a) Since $m = 8$, $n = 5$, $N = 8 + 5 = 13$, and there are two tied groups with $t_1 = t_2 = 2$, the mean and variance of the Wilcoxon rank sum statistic W under H_0 are

$$\mathrm{E}_0[W] = \frac{5(13+1)}{2} = 35 \quad \text{and} \quad \mathrm{Var}_0(W) = \frac{8(5)}{12}\left[14 - \frac{2(2-1)(2)(2+1)}{13(13-1)}\right] = 46.410$$

The observed value of W is $w = 4 + 3 + 1.5 + 6 + 5 = 19.5$. Hence

$$W^* = \frac{W - \mathrm{E}_0[W]}{\sqrt{\mathrm{Var}_0(W)}} = \frac{19.5 - 35}{\sqrt{46.410}} = -2.2752$$

Then the approximate p-value of testing $H_0 : \Delta = 0$ against $H_1 : \Delta < 0$ is given by $\Pr(N(0,1) \le -2.2752) = 0.0114$. This result shows there is strong evidence that the values of are significantly larger than those of Y, with an achieved significance level about 1.14%, which is substantially lower than 5% and close to 1%.

(b) From $(a_1, \ldots, a_{13}) = (1.5, 1.5, 3, 4, 5, 6, 7, 8, 9, 10, 11.5, 11.5, 13)$, the 5-tuples $(a_{i_1}, \ldots, a_{i_5})$ with $i_1 < \cdots < i_5$ such that $W = a_{i_1} + \cdots + a_{i_5} \le w = 19.5$ are shown below.

$(a_{i_1}, \ldots, a_{i_5})$, $1 \le i_1 < \cdots < i_5 \le 13$	W	No.
$(a_1, a_2, a_3, a_4, a_5) =$ (1.5,1.5,3,4,5)	15	1
$(a_1, a_2, a_3, a_4, a_6) =$ (1.5,1.5,3,4,6)	16	1
$(a_1, a_2, a_3, a_4, a_7) =$(1.5,1.5,3,4,7), $(a_1, a_2, a_3, a_5, a_6) =$(1.5,1.5,3,5,6)	17	2
(1.5,1.5,3,4,8), (1.5,1.5,3,5,7), (1.5,1.5,4,5,6)	18	3
(1.5,1.5,3,4,9), (1.5,1.5,3,5,8), (1.5,1.5,3,6,7), (1.5,1.5,4,5,7)	19	4
$(a_1, a_3, a_4, a_5, a_6) = (a_2, a_3, a_4, a_5, a_6) =$ (1.5,3,4,5,6)×2	19.5	2

The total number of $(a_{i_1}, \ldots, a_{i_5})$ with $i_1 < \cdots < i_5$ is

$$\binom{13}{5} = \frac{13(12)(11)(10)(9)}{5(4)(3)(2)} = 13(11)(9) = 1287$$

Thus the exact p-value of $H_0 : \Delta = 0$ versus $H_1 : \Delta < 0$ is

$$\Pr(W \le 19.5) = \frac{1+1+2+3+4+2}{1287} = \frac{13}{1287} = 0.0101$$

This again shows strong evidence for $\Delta < 0$.

(c) Calculate and order 40 values $\{Y_j - X_i, i = 1,\ldots,8, j = 1,\ldots,5\}$. The ordered values $U_{(1)} \le U_{(2)} \le \cdots \le U_{(40)}$ are provided in the table below.

$U_{(1)} \le U_{(2)} \le \cdots \le U_{(40)}$									
k	$U_{(k)}$	k	$U_{(k)}$	k	$U_{(k)}$	k	$U_{(k)}$	k	$U_{(k)}$
1	–0.87	9	–0.65	17	–0.52	25	–0.39	33	–0.21
2	–0.77	10	–0.62	18	–0.50	26	–0.39	34	–0.19
3	–0.77	11	–0.61	19	–0.49	27	–0.35	35	–0.14
4	–0.75	12	–0.60	20	–0.45	28	–0.35	36	0.00
5	–0.74	13	–0.60	21	–0.44	29	–0.32	37	0.12
6	–0.70	14	–0.57	22	–0.44	30	–0.32	38	0.17
7	–0.67	15	–0.56	23	–0.42	31	–0.26	39	0.35
8	–0.65	16	–0.55	24	–0.42	32	–0.25	40	0.42

An estimate of Δ based on Wilcoxon rank sum is given by

$$\hat{\Delta} = \frac{U_{(20)} + U_{(21)}}{2} = \frac{-0.45 - 0.44}{2} = -0.445$$

For the confidence interval, it follows from (3.18) that

$$C_{0.05} \approx \frac{mn}{2} - z_{0.025}\sqrt{\mathrm{Var}_0(W)} = \frac{8(5)}{2} - 1.96\sqrt{46.447} = 6.642$$

If we round this value to take $C_{0.05} = 7$, then an approximate 95% confidence interval of Δ is given by

$$\left(U_{(7)}, U_{(40+1-7)}\right) = \left(U_{(7)}, U_{(34)}\right) = (-0.67, -0.19)$$

A more conservative option is to take integer part $C_{0.05} = [6.642] = 6$ and so the confidence interval is $\left(U_{(6)}, U_{(35)}\right) = (-0.70, -0.14)$.

In either case, the 95% confidence interval cover negative values only, confirming the evidence for $\Delta < 0$ by the Wilcoxon rank sum test in parts (a) and (b).

Question 7

(a) As $m=8$, $n=4$, $N=m+n=12$, the scores of the ordered values $Z_{(1)} \le \cdots \le Z_{(12)}$ of combined samples for the Ansari-Bradley two-sample scale statistic C are

$$(a_1,\ldots,a_{12}) = (1,2,3,4,5,6,6,5,4,3,2,1)$$

Each possible value c of C is given by $c = a_i + a_j + a_k + a_l$ (sum of Y-scores), where a_i, a_j, a_k, a_l are drown from $(a_1,\ldots,a_{12})$ with $1 \le i < j < k < l \le 12$. Under $H_0: \mathrm{Var}(X) = \mathrm{Var}(Y)$, each (a_i, a_j, a_k, a_l) is equally likely from a total number

$$\binom{12}{4} = \frac{12(11)(10)(9)}{4(3)(2)} = 55(9) = 495 \text{ of possible outcomes}$$

By counting the number of (a_i, a_j, a_k, a_l) such that

$$c = a_i + a_j + a_k + a_l = 6,7,8,9 \quad \text{with } i<j<k<l,$$

the probabilities $\Pr(C=c)$ for $c=6,7,8,9$ are obtained and listed as follows:

(a_i,a_j,a_k,a_l), $1 \le i<j<k<l \le 12$	c	$\Pr(C=c)$
$(a_1,a_2,a_{11},a_{12}) = (1,2,2,1)$	6	1/495
$(a_1,a_2,a_{10},a_{12}) = (1,2,3,1)\times 2$, $(a_1,a_3,a_{11},a_{12}) = (1,3,2,1)\times 2$	7	4/495
$(1,2,3,2)\times 2$, $(1,2,4,1)\times 2$, $(1,3,3,1)$, $(1,4,3,1)\times 2$, $(2,3,2,1)\times 2$	8	9/495
$(1,2,3,3)$, $(1,2,4,2)\times 2$, $(1,2,5,1)\times 2$, $(1,3,3,2)$, $(1,3,4,1)\times 2$, $(1,4,3,1)\times 2$, $(1,5,2,1)\times 2$, $(2,3,3,1)$, $(2,4,2,1)\times 2$, $(3,3,2,1)$	9	16/495

Since $N=m+n=12$ is an even number, C has a symmetric distribution about

$$\mathrm{E}_0[C] = \frac{n(N+2)}{4} = \frac{4(12+2)}{4} = 14 \;\Rightarrow\; \Pr(C \ge c) = \Pr(C \le 28-c)$$

This together with $\Pr(C=c)$ in the above table lead to

$$\Pr(C \ge 20) = \Pr(C \ge 28-8) = \Pr(C \le 8) = \frac{1+4+9}{495} = \frac{14}{495} = 0.0283 < 0.05$$

and

$$\Pr(C \ge 19) = \Pr(C \ge 28-9) = \Pr(C \le 9) = \frac{14+16}{495} = \frac{30}{495} = 0.0606 > 0.05$$

Thus the largest achievable level under 5% is 0.0283 and $c_{0.0283} = 20$.

(b) To test $H_0: \gamma^2 = 1$ versus $H_1: \gamma^2 \neq 1$, the rejection rule is either $C \leq c_{1-\alpha/2} - 1$ or $C \geq c_{\alpha/2}$. The largest achievable level $\alpha \leq 10\%$ is $\alpha = 0.0283 \times 2 = 0.0566$ in this case, with $c_{\alpha/2} = c_{0.0283} = 20$ by the results in part (a) and $c_{1-\alpha/2} - 1 = 8$ due to the symmetry of $\Pr(C = c)$. Thus the rejection rule is either $C \leq 8$ or $C \geq 20$.

Order the combined sample as follows:

i	1	2	3	4	5	6
$Z_{(i)}$	–0.33	–0.22	–0.06	0.14	0.18	0.23
X or Y	Y	X	X	Y	X	X
i	7	8	9	10	11	12
$Z_{(i)}$	0.31	0.34	0.35	0.37	0.41	0.44
X or Y	X	Y	X	X	Y	X

Then the observed value of C from the data is

$$C = a_1 + a_4 + a_8 + a_{11} = 1 + 4 + 5 + 2 = 12$$

Hence neither $C \leq 8$ nor $C \geq 20$ holds. As a result, $H_0: \gamma^2 = 1$ is accepted at the 10% level of significance, which indicates insufficient evidence for $\gamma^2 \neq 1$, or $\text{Var}(X) \neq \text{Var}(Y)$.

(c) It has been calculated in part (a) that $\text{E}_0[C] = 14$. Furthermore,

$$\text{Var}_0(C) = \frac{mn(N+2)(N-2)}{48(N-1)} = \frac{8(4)(12+2)(12-2)}{48(12-1)} = \frac{2(14)(10)}{3(11)} = 8.485$$

Hence

$$C^* = \frac{C - \text{E}_0[C]}{\sqrt{\text{Var}_0(C)}} = \frac{12-14}{\sqrt{8.485}} = -0.6866$$

Thus the approximate p-value is

$$2\Pr(N(0,1) \leq -0.6866) = 2(0.24617) = 0.4923$$

This p-value is much larger than 10%, which shows there is little evidence for any difference in variability between the two samples.

Question 8

Wilcoxon signed rank test in Question 5(d):

```
> y<-c(125,223,145,323,208,192,158,175,233,421,168,145)
> x<-c(117.5,209,148,299,215,184,145,179,214,432,152,142)

>  wilcox.test(y,x,paired=TRUE, alternative = "greater")

        Wilcoxon signed rank test with continuity correction

data:  y and x
V = 62.5, p-value = 0.03554
alternative hypothesis: true location shift is greater than 0

Warning message:
In wilcox.test.default(y, x, paired = TRUE, alternative = "greater") :
  cannot compute exact p-value with ties
```

As p-value = 0.03553 < 0.05, reject $H_0 : \theta = 0$ in favor of $H_1 : \theta > 0$ at the 5% level.

This p-value is approximate due to ties.

With ties:

```
> pPairedWilcoxon(x,y)

Number of X values: 12 Number of Y values: 12
Wilcoxon T+ Statistic: 62.5

Monte Carlo (Using 10000 Iterations) upper-tail probability: 0.0359
```

Exact p-value conditional on ties is 0.0359

Wilcoxon rank sum test in Question 6(a):

```
> x<-c(0.89,0.76,0.63,0.69,0.58,0.79,0.02,0.79)
> y<-c(0.19,0.14,0.02,0.44,0.37)

> wilcox.test(y, x, alternative = "less")

        Wilcoxon rank sum test with continuity correction

data:  y and x
W = 4.5, p-value = 0.01384
alternative hypothesis: true location shift is less than 0

Warning message:
In wilcox.test.default(y, x, alternative = "less") :
  cannot compute exact p-value with ties
```

p-value = 0.01384, reject $H_0 : \Delta = 0$ in favor of $H_1 : \Delta < 0$ at the 5% level.

This p-value is approximate due to ties.

Ansari-Bradley in Question 7(b):

```
> x<-c(0.37,0.23,-0.06,0.18,0.44,0.31,-0.22,0.35)
> y<-c(0.34,0.14,-0.33,0.41)

> ansari.test(y,x)

        Ansari-Bradley test

data:  y and x
AB = 12, p-value = 0.6222
alternative hypothesis: true ratio of scales is not equal to 1
```

p-value = 0.6222, accept $H_0 : \gamma^2 = 1$. There is little evidence for $H_1 : \gamma^2 \neq 1$.