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Homework 11

Due by April 27, 2021

1. Consider a production system consisting of three single-server stations in series. Customer orders arrive at the system according to a Poisson process with rate 1 per hour. Each customer order immediate triggers a job that is released to the production system to be processed at station 1 first, and then at station 2. After being processed at station 2, a job has $p_1 = 10\%$ probability going back to station 1 for rework and $1 - p_1$ probability continuing onto station 3. After being processed at station 3, the job leaves the production system as a finished product. Assume that the processing times of jobs at each station are iid, having exponential distribution, regardless of the history of the jobs. The average processing times at stations 1, 2 and 3 are $m_1 = 0.8$, $m_2 = 0.70$ and $m_3 = 0.8$ hours, respectively.

   (a) Find the long-run fraction of time that there are 2 jobs at station 1, 1 job at station 2 and 4 jobs at station 3.

   (b) Find the long-run average (system) size at station 3.

   (c) Find the long-run average time in system for each job.

   (d) Reduce p_1 to 2.5%. Answer 1(c) again. What story can you tell?

2. Consider a special two-station open Jackson network. Each station has a single server and an infinite capacity waiting area. External arrivals to station 1 follow a Poisson process with rate α, and service times at station i are iid, following exponential distribution with rate μ_i, $i = 1, 2$. After being served at station 1, each customer goes to station 2, and after being served at station 2, the customer goes back to station 1 with probability 20% and leaves the system with probability 80%.

   Assume that

   $$(1.25)\alpha < \mu_1 \text{ and } (1.25)\alpha < \mu_2.$$

   Let $X_i(t)$ be the number of customers at station i at time t, $i = 1, 2$. Then $X = \{(X_1(t), X_2(t)), t \geq 0\}$ is a continuous time Markov chain.

   Prove that X is positive recurrent by proving that (a) X is irreducible, and (b) X has a stationary distribution. For (b), do the following three steps: (i) write the flow balance equation for *each state*, (ii) guess a form of the stationary distribution, and (iii) verify that the guessed stationary distribution does satisfy the flow balance equation for each state.

3. Consider a production line of two stations in tandem. Each station has a single server and a dedicated buffer (of infinite size) for its waiting jobs. The mean service times at station i is m_i minutes, $i = 1, 2$. Jobs arrive at station station 1 at rate λ jobs/minute. Each job is processed at station 1 first, at station 2 next, and then exits the system. When a server is busy, the arriving job to the station waits in the buffer. When a server completes the processing of a job, it picks the next job from the buffer to process. For each scenario below, find the utilization for each station and the system throughput, the rate at which jobs leave station 2.

   (a) $\lambda = 2$, $m_1 = .4$, $m_2 = .45$.
   (b) $\lambda = 2$, $m_1 = .6$, $m_2 = .45$.
   (c) $\lambda = 2$, $m_1 = .4$, $m_2 = .6$.
   (d) $\lambda = 2$, $m_1 = .6$, $m_2 = .8$.

4. Consider a production system consisting of three single-server stations in series. Customer orders arrive at the system according to a Poisson process with rate 1 per hour. Each customer order immediate triggers a job that is released to the production system to be processed at station 1 first, and then at station 2. After being processed at station 2, a job has $p_1 = 10\%$ probability going back to station 1 for rework and $1 - p_1$ probability continuing onto station 3. After being processed at station 3, the job leaves the production system as a finished product. Assume that the processing times of jobs at each station are iid, having exponential distribution, regardless of the history of the jobs. The average processing times at stations 1, 2 and 3 are $m_1 = 0.8$, $m_2 = 0.70$ and $m_3 = 0.8$ hours, respectively. The Manager decides to adopt the make-to-stock policy, using the CONWIP (Constant-Work-in-Process) job release policy defined as follows: only a job leaving station 3 triggers a new job to be released to station 1. Under this policy, the total number of jobs in the system is a constant N (also called the CONWIP level).

   (a) For $N = 2$, compute the throughput of the production system. What is the average time in system per job?

   (b) Is it possible to double the throughput? If so, what N is needed to achieve it? What is the corresponding average time in system per job?

5. Consider an $M/M/n$ system that models a customer call center with infinite waiting space. Assume the Poisson arrival process has rate of λ calls per minute. Assume that all n servers are identical. Their processing times are iid, exponentially distributed with mean 2 minutes. (Therefore, the each server's service rate is $\mu = 1/2$ calls per minute). Recall that $R = \lambda/\mu$ is the offered load to the system. Assume further that, as the arrival rate λ increases, the system manager chooses the number of servers n to be the smallest integer

$$n \geq R + .5\sqrt{R}.$$

Use **Python** or any other software to compute the average utilization per server and the long-run fraction of calls that do not need to wait before receiving service.

(a) $\lambda = 1$,

(b) $\lambda = 10$,

(c) $\lambda = 100$,

(d) $\lambda = 1000$.