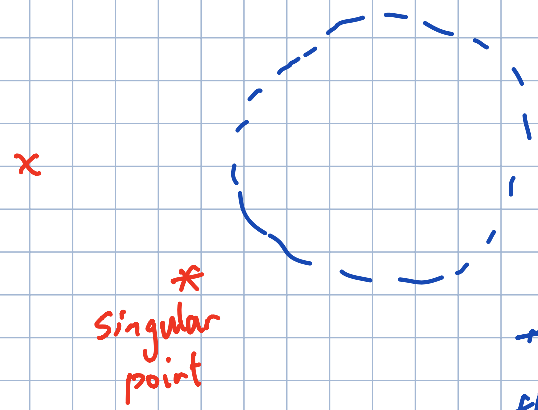
MAT 3253 Lecture 18

Theorem (existence of Local primitive (anti-derivative))



If $f(z)$ is analytic at every point inside this open disc,

then $\exists$ $F(z)$ defined on the open disc, s.t. $F'(z) = f(z)$.

Example $\int_C \sqrt{z}\, dz$

$\frac{2}{3} z^{3/2}$ is a primitive

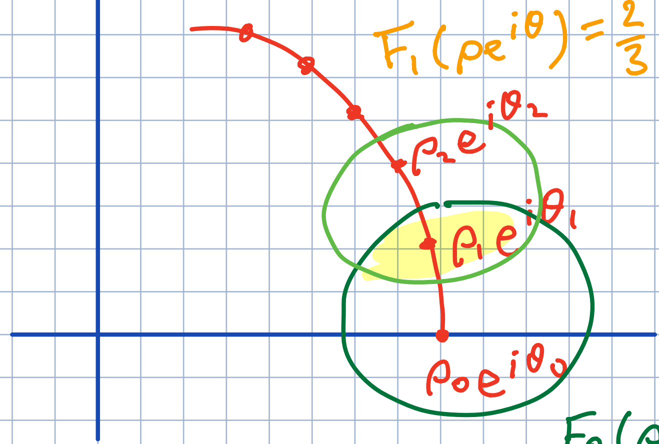
$\theta_2 = 0.2$

$\theta_1 = 0.1$

$\theta_0 = 0$

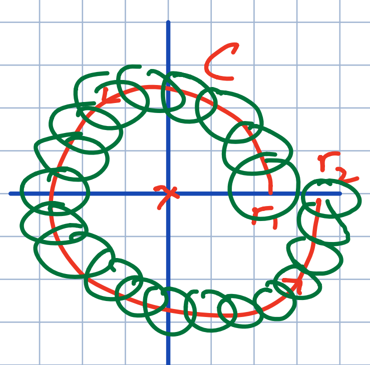
$F_1(\rho e^{i\theta}) = \frac{2}{3}\rho^{3/2} e^{i3\theta/2}$

$0.05 \le \theta \le 0.25$



$F_0(\rho e^{i\theta}) \triangleq \frac{2}{3}\rho^{3/2} e^{i\frac{3\theta}{2}}$

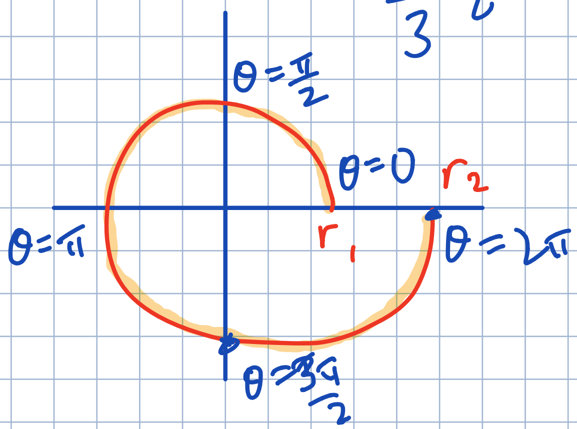
$-0.05 \le \theta \le 0.15$



$$\int_{\rho_0 e^{i\theta_0}}^{\rho_2 e^{i\theta_2}} \sqrt{z}\, dz = \cancel{F_0(\rho_1 e^{i\theta_1})} - F_0(\rho_0 e^{i\theta_0}) + F_1(\rho_2 e^{i\theta_2}) - \cancel{F_1(\rho_1 e^{i\theta_1})}$$

$$= F_1(\rho_2 e^{i\theta_2}) - F_0(\rho_0 e^{i\theta_0})$$

$$\frac{2}{3} z^{3/2}$$



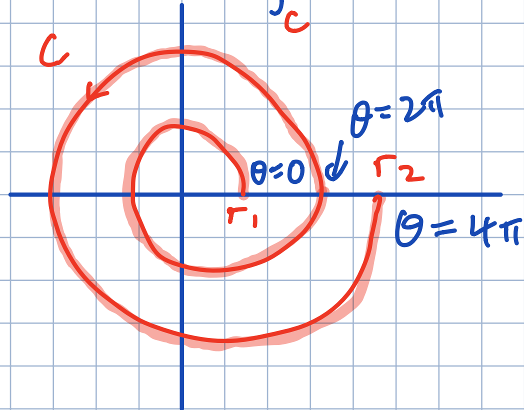
$$\int_C \sqrt{z}\, dz = \frac{2}{3} r_2^{3/2} \cdot \underbrace{e^{i\frac{3\cdot 2\pi}{2}}}_{= e^{i\pi}}$$

$$- \frac{2}{3} r_1^{3/2} e^{i\frac{3\cdot 0}{2}}$$

$$= -\frac{2}{3} r_2^{3/2} - \frac{2}{3} r_1^{3/2}$$

$$= -\frac{4}{3} r_1^{3/2} \quad \text{if } r_1 = r_2$$

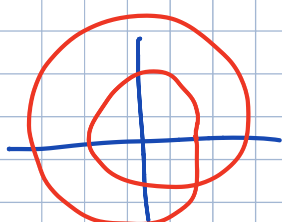
$$\int_C \sqrt{z}\, dz$$



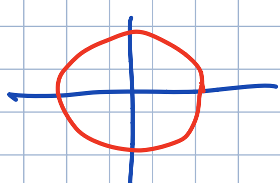
$$\text{Ans} = \frac{2}{3} r_2^{3/2} \overbrace{e^{i\frac{3}{2}(4\pi)}}^{1}$$

$$- \frac{2}{3} r_1^{3/2} \underbrace{e^{i\frac{3}{2}0}}_{1}$$

$$= +\frac{2}{3} r_2^{3/2} - \frac{2}{3} r_1^{3/2}$$



$$\int_C \sqrt{z}\, dz = 0$$

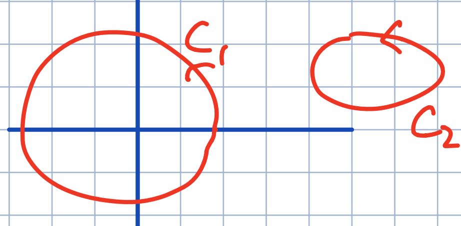


$$\int_C \sqrt{z}\, dz \neq 0$$

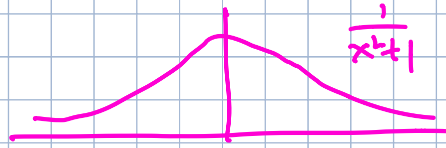
Example $\int_C \frac{1}{z^3}\, dz = 0$ primitive $\underbrace{-\frac{1}{2} z^{-2}}$ $= -\frac{1}{2} \cdot \frac{1}{z^2}$

well-defined for all

$z \in \mathbb{C} \setminus \{0\}$

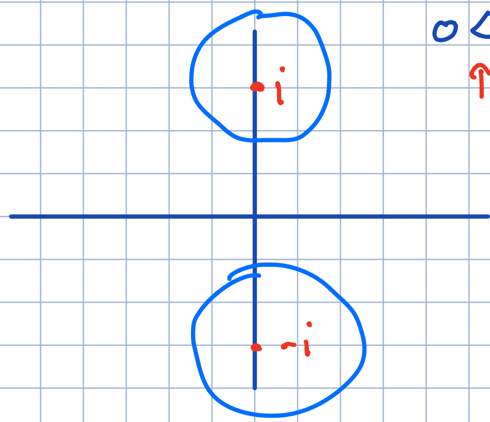


Singularity



$$f(z)=\frac{1}{z^2+1}=\frac{1}{(z-i)(z+i)}$$

two isolated singular points at $z=i$, $z=-i$.



$0<|z-i|<\varepsilon$

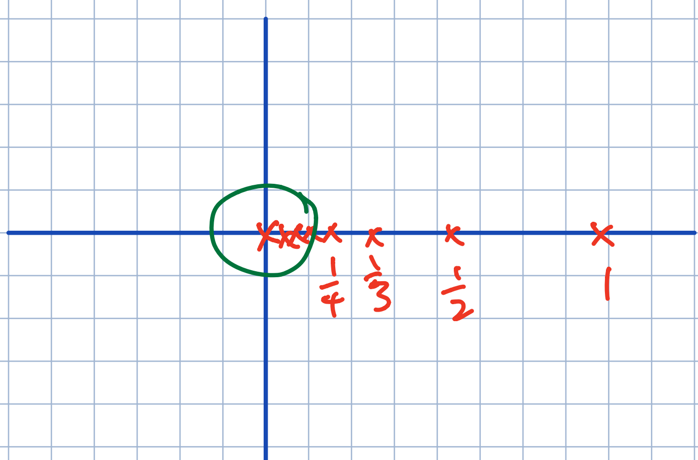
$0<|z+i|<\varepsilon$

Non-isolated singularity

① $\frac{1}{\sin\left(\frac{4}{z}\right)}$

$z=0$ non-isolated singularity

$z=\frac{1}{n}$



② Natural boundary

$$\sum_{n=0}^{\infty} z^{2^n} = z + z^2 + z^4 + z^8 + z^{16} + \dots$$

converges for $|z| < 1$

$$|z| + |z^2| + |z^4| + |z^8| \leq |z| + |z|^2 + |z|^3 + |z|^4 + \dots$$

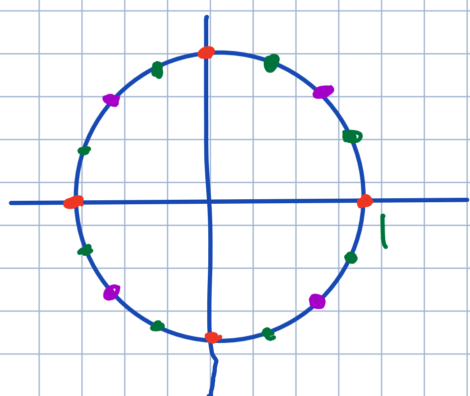
$z = 1$ $\quad 1 + 1 + 1 + \dots$

$z = -1$ $\quad -1 + 1 + 1 + \dots$

$z = i$ $\quad i + i^2 + 1 + 1 + 1 + \dots$

$z = -i$ $\quad -i + i^2 + 1 + 1 + 1 + \dots$

If z is a $(2^n)^{th}$ root of unit, then divergent



$z^4 = 1$

$z^8 = 1$

$z^{16} = 1$

$z^{32} = 1$

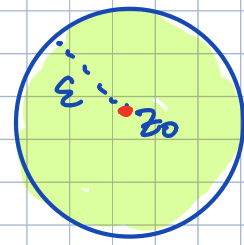
$z^{64} = 1$

$z^{128} = 1$

$\vdots$

The singular points are dense in the unit circle.

Three types of isolated singularity



(removable singularity)

① $f(z)$ is bounded near z_0

(pole)

② $|f(z)| \to \infty$ as $z \to z_0$

$f(z)$ is analytic in $0 < |z - z_0| < \varepsilon$

③ otherwise (essential singularity)

Example ① $f(z) = \dfrac{z^2 - 1}{z - 1} = \dfrac{(z+1)(z-1)}{z-1}$

$$= \begin{cases} z+1 & \text{if } z \neq 1 \\ \text{undefined} & \text{if } z = 1 \end{cases}$$

② $f(z) = \dfrac{1}{z(z+1)}$ has two poles

Theorem Suppose f is analytic in a punctured disc $\{ z \in \mathbb{C} : 0 < |z - z_0| < \varepsilon \}$.

If f is bounded in this open disc.

then $\exists$ an analytic $\tilde{f}(z)$ in $D(z_0 ; \varepsilon)$

so that $\tilde{f}(z) = f(z)$ for $z \in D(z_0 ; \varepsilon) \setminus \{z_0\}$