MAT 3253 Homework 8 118010350

1. Since $C=\{z=e^{i\theta}: 0\le\theta\le 2\pi\}$, and $f(z)=z^{-1+i}$, $|z|>0$, $0<\arg(z)<2\pi$. then $\int_C f(z)\,dz = \int_0^{2\pi} f(z(\theta))\cdot z'(\theta)\,d\theta$

$$= \int_0^{2\pi} e^{-\theta(1+i)}\cdot i\cdot e^{i\theta}\,d\theta$$

$$= \int_0^{2\pi} i\cdot e^{-\theta}\,d\theta$$

$$= i\cdot(-1)\cdot e^{-\theta}\Big|_0^{2\pi}$$

$$= i(1-e^{-2\pi})$$

2. Since $C=\{z=\sqrt{4-y^2}+iy: -2\le y\le 2\}$. and $f(z)=\bar{z}$.

then $I=\int_C f(z)\,dz = \int_{-2}^{2} f(z(y))\,z'(y)\,dy$

$$= \int_{-2}^{2}(\sqrt{4-y^2}-iy)\cdot\left(-\frac{y}{\sqrt{4-y^2}}+i\right)dy$$

$$= i\int_{-2}^{2} \sqrt{4-y^2}+\frac{y^2}{\sqrt{4-y^2}}\,dy$$

$$= 4i\int_{-2}^{2}\frac{1}{\sqrt{4-y^2}}\,dy$$

$$= 4\pi i$$

3. proof. ① Since $C_R=\{z=R\cdot e^{i\theta}: 0\le\theta\le\pi\}$, $R>2$, then

$$\left|\int_{C_R}\frac{2z^2-1}{z^4+5z^2+4}\,dz\right| \le \int_{C_R}\left|\frac{2z^2-1}{z^4+5z^2+4}\right|dz$$

$$= \int_0^{\pi}\left|\frac{2R^2\cdot e^{i2\theta}-1}{R^4e^{i4\theta}+5R^2\cdot e^{i2\theta}+4}\right|\cdot\left|i\cdot Re^{i\theta}\right|d\theta$$

$$= \int_0^{\pi}\frac{R\,|2R^2e^{i2\theta}-1|}{|R^4e^{i4\theta}+5R^2\cdot e^{i2\theta}+4|}\,d\theta$$

$$\le \int_0^{\pi}\frac{R\cdot(2R^2+1)}{R^4-5R^2+4}\,d\theta$$

$$= \int_0^{\pi}\frac{R\cdot(2R^2+1)}{(R^2-1)(R^2-4)}\,d\theta$$

$$= \frac{\pi R(2R^2+1)}{(R^2-1)(R^2-4)}$$

② Since $\lim_{R\to\infty} \frac{2\pi\cdot\frac{1}{R}+\pi\cdot\frac{1}{R^3}}{(1-\frac{1}{R^2})(1-\frac{4}{R^2})} = 0$, and $\left|\int_{C_R} \frac{2z^2-1}{z^4+5z^2+4}dz\right| \geq 0$.

then $0 \leq \lim_{R\to\infty} \left|\int_{C_R} \frac{2z^2-1}{z^4+5z^2+4}dz\right| \leq \lim_{R\to\infty} \frac{\pi R(2R^2+1)}{(R^2-1)(R^2-4)}$

thus, $\lim_{R\to\infty} \left|\int_{C_R} \frac{2z^2-1}{z^4+5z^2+4}dz\right| = 0$

4. prof. Since $C=\{z=x : -1\leq x\leq 1\}$, and $z^i = \exp(i\cdot \mathrm{Log}(z))$,

$\mathrm{Log}(z) = \log|z| + i\cdot \mathrm{Arg}(z)$, $|z|>0$, $-\pi < \mathrm{Arg}(z) < \pi$.

then
$$\begin{aligned}
\int_{-1}^{1} z^i dz &= \int_{-1}^{0} z^i dz + \int_0^1 z^i dz \\
&= \int_1^0 \exp(i\cdot\log(-x) - \pi)dx \\
&\quad + \int_0^1 \exp(i\cdot\log(x) - 0)dx \\
&= (1+e^{-\pi})\int_0^1 e^{i\log x}dx \\
&= (1+e^{-\pi})\int_0^1 (\cos(\log x) + i\sin(\log x))dx \\
&= (1+e^{-\pi})\int_{-\infty}^{0} e^y(\cos y + i\sin y)dy \qquad (y=\ln x) \\
&= (1+e^{-\pi})\cdot\int_{-\infty}^{0} e^{y(1+i)}dy \\
&= \frac{1+e^{-\pi}}{1+i} \\
&= \frac{1+e^{-\pi}}{2}(1-i)
\end{aligned}$$

5. prof. Since $C=\{z=\cos\theta + i\sin\theta : 0\leq\theta\leq 2\pi\}$, and

$f(z)$ is real-valued, then $\int_{|z|=1} f(z)dz$ is real-valued.

Since
$$\begin{aligned}
\int_{|z|=1} f(z)dz &= \int_0^{2\pi} f(z(\theta))z'(\theta)d\theta \\
&= \int_0^{2\pi} f(z(\theta))\cdot(-\sin\theta + i\cos\theta)d\theta \\
&= \int_0^{2\pi} f(z(\theta))\cdot(-\sin\theta)d\theta + i\int_0^{2\pi} f(z(\theta))\cdot\cos\theta d\theta \\
&= \int_0^{2\pi} f(z(\theta))\cdot(-\sin\theta)d\theta
\end{aligned}$$

then
$$\begin{aligned}
\left|\int_{|z|=1} f(z)dz\right| &= \left|\int_0^{2\pi} f(z(\theta))\cdot(-\sin\theta)d\theta\right| \\
&\leq \int_0^{2\pi} |f(z(\theta))|\cdot|\sin\theta|d\theta \\
&= \int_0^{2\pi} |\sin\theta|d\theta
\end{aligned}$$

$$= \int_0^{\pi} |\sin\theta|\, d\theta + \int_{\pi}^{2\pi} |\sin\theta|\, d\theta$$
$$= -\cos\theta \Big|_0^{\pi} + \cos\theta \Big|_{\pi}^{2\pi}$$
$$= 4$$

Thus, $\left| \int_{|z|=1} f(z)\,dz \right| \leq 4$.

6. (a) Since $(e^z)' = e^z$, then the integral is path independent.

$$\int_0^i e^z\, dz = e^i - e^0 = \cos 1 + i \sin 1 - 1$$

(b) Since $\left(\frac{1}{2}\sin 2z\right)' = \cos 2z$, then the integral is path independent.

$$\int_{\pi/2}^{\pi/2+i} \cos 2z\, dz = \frac{1}{2}\sin(\pi + 2i) - \frac{1}{2}\sin(\pi)$$
$$= \frac{e^2 - e^{-2}}{4i}$$

7. proof. Since f is analytic in a convex region D,

then for $\forall\, a, b \in D$, $at + b(1-t) \in D$, $0 \leq t \leq 1$.

Consider the direct line segment from a to b,

then $f(b) - f(a) = \int_a^b f'(z)\,dz$, where $C: a \to b$.

Since the path is a line segment between a and b,

$z = x + iy$, then there exists g, s.t. $y = g(x)$

Then $C = \{ z = x + ikx : x_1 \leq x \leq x_2 \}$, where $k \in \mathbb{R}$

and $z'(x) = 1 + ik$, $a = z(x_1)$, $b = z(x_2)$.

$$|f(b) - f(a)| = \left| \int_a^b f'(z)\, dz \right|$$
$$= \left| \int_{x_1}^{x_2} f'(z(x)) \cdot z'(x)\, dx \right|$$
$$\leq \int_{x_1}^{x_2} |f'(z(x))| \cdot |z'(x)|\, dx$$
$$\leq \int_{x_1}^{x_2} |z'(x)|\, dx$$
$$= \int_{x_1}^{x_2} |1 + ik|\, dx$$
$$= \sqrt{k^2+1}\,(x_2 - x_1)$$

$= |b-a|$.

Thus, $|f(b)-f(a)| \le |b-a|$ for $\forall\, a, b \in D$.