MAT2006: Elementary Real Analysis
Assignment #2

Deadline: Oct. 24

1 (Squeeze Theorem)**.** Show that if $x_n \le y_n \le z_n$ for all $n \in \mathbb{N}$, and if $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = \ell$, then $\lim_{n\to\infty} y_n = \ell$ as well.

2. Show that

$$\text{(i)} \qquad \lim_{n\to\infty} \sqrt[n]{1+\frac{a}{n}} = 1, \quad \text{where} \quad a > 0.$$

$$\text{(ii)} \qquad \lim_{n\to\infty} \frac{n^k}{n!} = 0, \quad \text{where} \quad k \in \mathbb{N}.$$

$$\text{(iii)} \qquad \lim_{n\to\infty} \frac{n^k}{a^n} = 0, \quad \text{where} \quad a > 1, \quad k \in \mathbb{N}.$$

$$\text{(iv)} \qquad \lim_{n\to\infty} \frac{a^n}{n!} = 0, \quad \text{where} \quad a \in \mathbb{R}.$$

$$\text{(v)} \qquad \lim_{n\to\infty} \sqrt[n]{\frac{a^n}{n}+\frac{b^n}{n^2}} = b, \quad \text{where} \quad b \ge a > 0.$$

$$\text{(vi)} \qquad \lim_{n\to\infty} \frac{\sqrt[3]{n^2}\sin n!}{n+1} = 0.$$

$$\text{(vii)} \qquad \lim_{n\to\infty} \frac{n^2+\cos n}{[n+(-1)^n]^2} = 1.$$

3 (Cesaro Means)**.** (i) Show that if $\{x_n\}$ is a convergent sequence, then the sequence given by the averages

$$y_n = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

also converges to the same limit.

(ii) Give an example to show that it is possible for the sequence $\{y_n\}$ of averages to converge even if $\{x_n\}$ does not.

4. Show that the sequence

$$\sqrt{2}, \quad \sqrt{2+\sqrt{2}}, \quad \sqrt{2+\sqrt{2+\sqrt{2}}}, \quad \cdots,$$

is convergent and find its limit.

5. Set $x_1 = 2$ and
$$x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}, \qquad \forall n \in \mathbb{N}.$$
Show that $\{x_n\}$ is convergent and find its limit.

6. For a bounded sequence $\{x_n\}$, the Bolzano–Weierstrass Theorem says that there exists a convergent subsequence. Let E be the set of real numbers s such that $x_{n_k} \to s$ for some subsequence $\{x_{n_k}\}$. Show that
$$\limsup_{n\to\infty} x_n = \sup E \qquad \text{and} \qquad \liminf_{n\to\infty} x_n = \inf E.$$

7. For the following sequences, find their upper and lower limits.
$$\text{(i)} \quad \{(-1)^n\}_{n=1}^{\infty}, \qquad \text{(ii)} \quad \{(-1)^n n\}_{n=1}^{\infty}, \qquad \text{(iii)} \quad \left\{(-1)^n \frac{1}{n}\right\}_{n=1}^{\infty}.$$

8. Find the sup, inf, max and min for the following sets
$$(a) \quad A = \left\{\frac{1}{n} \,|\, n \in \mathbb{N}\right\}; \qquad (b) \quad B = \left\{1 - \frac{1}{n} \,|\, n \in \mathbb{N}\right\}.$$

9. Show that a sequence $\{x_n\}$ is convergent if and only if $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n$. In this case, all three share the same value.

10 (Order Properties for Upper and Lower Limits)**.** Assume there exists $M \in \mathbb{N}$ such that $x_n \le y_n$ for each $n \ge M$. Show that
$$\liminf_{n\to\infty} x_n \le \liminf_{n\to\infty} y_n, \qquad \limsup_{n\to\infty} x_n \le \limsup_{n\to\infty} y_n.$$

11. Assume $0 \le x_{n+m} \le x_n + x_m$ for all $n, m \in \mathbb{N}$. Show that the sequence $\left\{\frac{x_n}{n}\right\}$ converges.
Hint. Apply the result about upper and lower limits in the above two problems.

12. Assume $\lim_{n\to\infty} x_n = A$. Show that
$$\lim_{n\to\infty} \frac{\frac{1}{2}x_1 + \frac{2}{3}x_2 + \cdots + \frac{n}{n+1}x_n}{n} = A.$$

13. Assume $x_n > 0$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \ell < \infty$. Show that $\lim_{n\to\infty} \sqrt[n]{x_n} = \ell$.

14. Assume $x_n > 0$ for every $n \in \mathbb{N}$. Show that
$$\limsup_{n\to\infty} \sqrt[n]{x_n} \le \limsup_{n\to\infty} \frac{x_{n+1}}{x_n}.$$

15. (i) Use the Monotone Convergence Theorem to prove the Archimedean Property without making any use of Least Upper Bound Property.

(ii) Use the Monotone Convergence Theorem to prove the Nested Interval Property without making any use of Least Upper Bound Property.

16. Assume the Nested Interval Property is true. Use the technique in proving the Bolzano–Weierstrass Theorem to provide a proof of the Lest Upper Bound Property. To prevent the argument from being circular, assume also that $1/2^n \to 0$ (which is a consequence of the Archimedean Property).

17. Assume the Bolzano–Weierstrass Theorem is true and use it to construct a proof of the Monotone Convergence Theorem without making any appeal to the Archimedean Property.

18. Use the Cauchy Criterion to prove the Bolzano–Weierstrass Theorem, and find the point in the argument where the Archimedean Property is implicitly required.

19. Assume $\sum_{n=1}^{\infty} a_n^2$ and $\sum_{n=1}^{\infty} b_n^2$ converge. Show that

$$\sum_{n=1}^{\infty} |a_n b_n|, \qquad \sum_{n=1}^{\infty} (a_n + b_n)^2, \qquad \sum_{n=1}^{\infty} \frac{|a_n|}{n}$$

also converge.

20. Show that if $\lim\limits_{n\to\infty} n a_n = a \neq 0$ then $\sum\limits_{n=1}^{\infty} a_n$ diverges.

21. Proving the Alternating Series Test amounts to showing that the sequence of partial sums

$$s_n = a_1 - a_2 + a_3 + \cdots + (-1)^{n+1} a_n$$

converges. Different characterizations of completeness lead to different proofs.

(a) Prove the Alternating Series Test by showing that $\{s_n\}$ is a Cauchy sequence.

(b) Supply another proof for this result using the Nested Interval Property.

(c) Consider the subsequences $\{s_{2n}\}$ and $\{s_{2n+1}\}$, and show how the Monotone Convergence Theorem leads to a third proof for the Alternating Series Test.

22. Discuss the convergence (absolute, conditional convergence or divergence) of the following series

$$\text{(i)} \quad \sum_{n=1}^{\infty} \frac{n \cos \frac{n\pi}{3}}{2^n}; \qquad \text{(ii)} \quad \sum_{n=1}^{\infty} (-1)^n \frac{\sin^2 n}{n}.$$

23 (Abel's test)**.** Abel's Test for convergence states that if the series $\sum_{k=1}^{\infty} x_k$ converges, and if $\{y_k\}$ is a sequence satisfying $y_1 \geq y_2 \geq y_3 \geq \cdots \geq 0$, then the series $\sum_{k=1}^{\infty} x_k y_k$ converges.

(i) Prove the *summation by parts* formula. Let $s_0 = 0$ and $s_n = x_1 + x_2 + \cdots x_n$ for $n \in \mathbb{N}$. Then

$$\sum_{k=m}^{n} x_k y_k = s_n y_{n+1} - s_{m-1} y_m + \sum_{k=m}^{n} s_k (y_k - y_{k+1})$$

Hint. Note that $x_k = s_k - s_{k-1}$.

(ii) Use the Comparison Test to argue that $\sum_{k=m}^{\infty} s_k (y_k - y_{k+1})$ converges absolutely, and show how this leads directly to a proof of Abel's Test.

24 (Dirichlet's Test)**.** Dirichlet's Test for convergence states that if the partial sums of $\sum_{k=1}^{\infty} x_k$ are bounded (but not necessarily convergent), and if $\{y_k\}$ is a sequence satisfying $y_1 \geq y_2 \geq y_3 \geq \cdots \geq 0$, with $\lim_{k\to\infty} y_k = 0$, then the series $\sum_{k=1}^{\infty} x_k y_k$ converges.

(i) Point out how the hypothesis of Dirichlet's Test differs from that of Abel's Test, but show that essentially the same strategy can be used to provide a proof.

(ii) Show how the Alternating Series Test can be derived as a special case of Dirichlet's Test.

— End —