MAT2006 Tutorial #11

1. Assume $\{f_n\}$ and $\{g_n\}$ are uniformly convergent sequences of functions defined on A.

(a) Show that $\{f_n + g_n\}$ is a uniformly convergent sequence of functions.

(b) Give an example to show that the product $\{f_n g_n\}$ may not converge uniformly.

(c) Prove that if there exists an $M > 0$ such that $|f_n(x)| \leq M$ and $|g_n| \leq M$ for all $n \in \mathbb{N}$ and $x \in A$, then $\{f_n g_n\}$ does converge uniformly.

2. Consider the sequence of functions defined by

$$g_n(x) = \frac{x^n}{n}.$$

(a) Show $\{g_n\}$ converges uniformly on $[0, 1]$ and find $g = \lim_{n\to\infty} g_n$. Show that g is differentiable and compute $g'(x)$ for all $x \in [0, 1]$.

(b) Now, show that g_n' converges on $[0, 1]$. Is the convergence uniform? Set $h = \lim_{n\to\infty} g_n'$ and compare h and g'. Are they the same?

3. (a) Show that

$$g(x) = \sum_{n=1}^{\infty} \frac{\cos(2^n x)}{2^n}$$

is continuous on all of $\mathbb{R}$.

(b) The function g was cited previously as an example of a continuous nowhere differentiable function. What happens if we try to use the Differentiable Limit Theorem to explore whether g is differentiable?

4. Recall that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots, \qquad \forall |x| < 1.$$

Using the above formula to find values for $\sum_{n=1}^{\infty} n/2^n$ and $\sum_{n=1}^{\infty} n^2/2^n$.

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