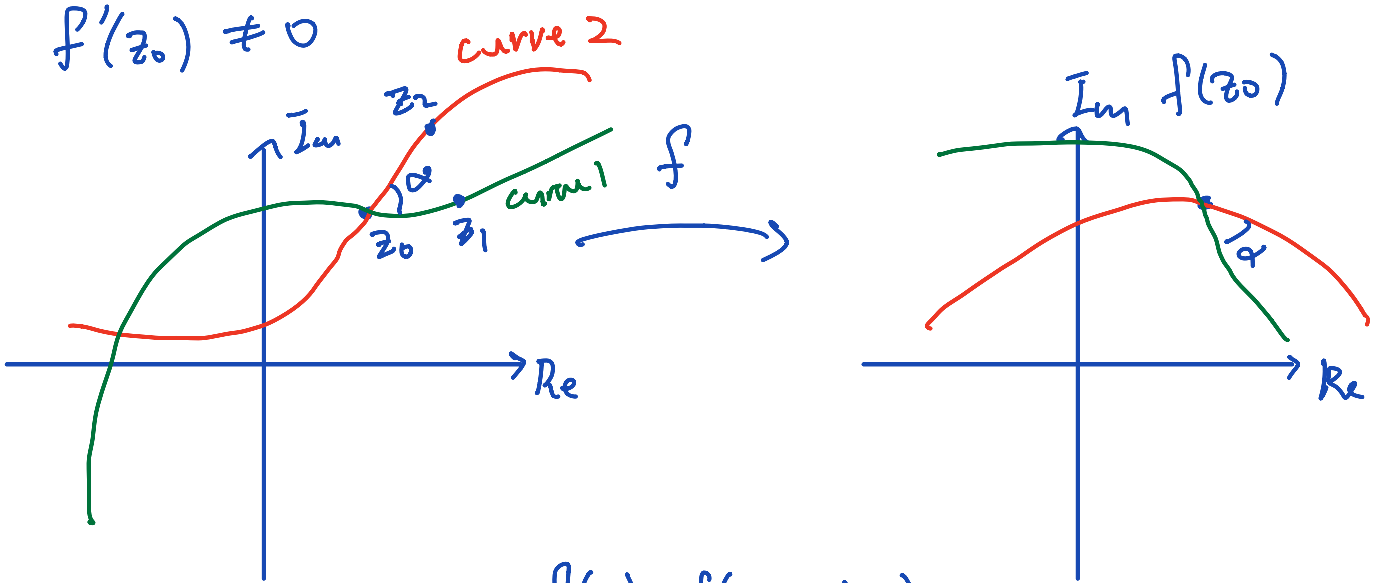
MAT 3253 Lecture 7

Angle-preserving property

$f'(z_0) \neq 0$



$$\Delta z_1 = z_1 - z_0$$
$$\Delta z_2 = z_2 - z_0$$

$$\alpha = \arg\left(\frac{z_2 - z_0}{z_1 - z_0}\right)$$

$$f(z_1) = f(z_0 + \Delta z_1) \doteq f(z_0) + f'(z_0)\cdot\Delta z_1$$

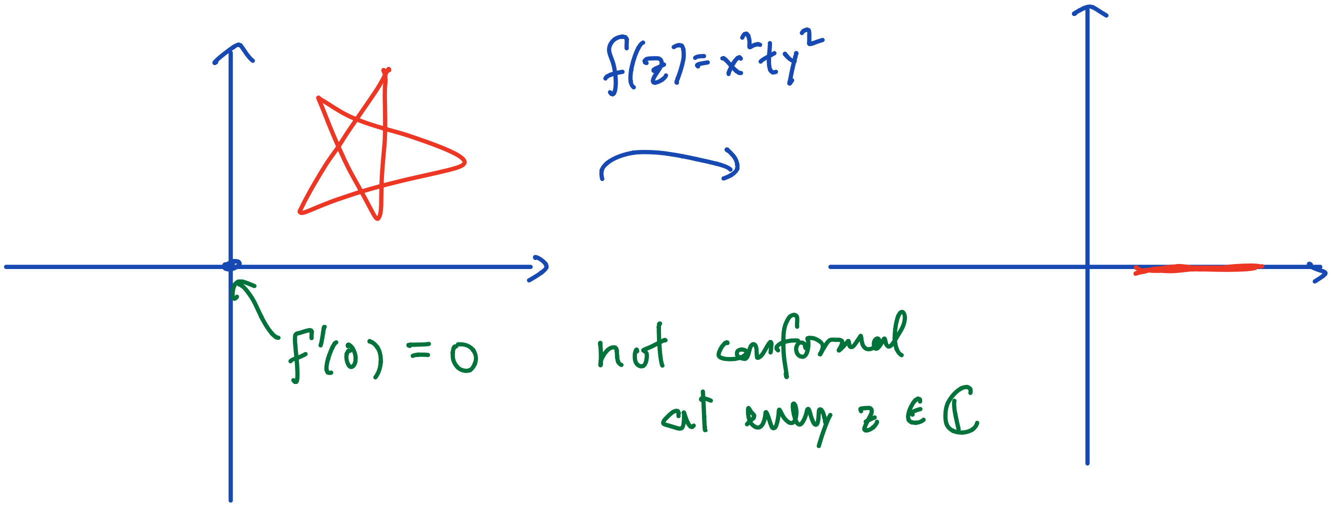
$$f(z_2) = f(z_0 + \Delta z_2) \doteq f(z_0) + f'(z_0)\,\Delta z_2$$

$$\arg\left(\frac{f(z_2) - f(z_0)}{f(z_1) - f(z_0)}\right)$$

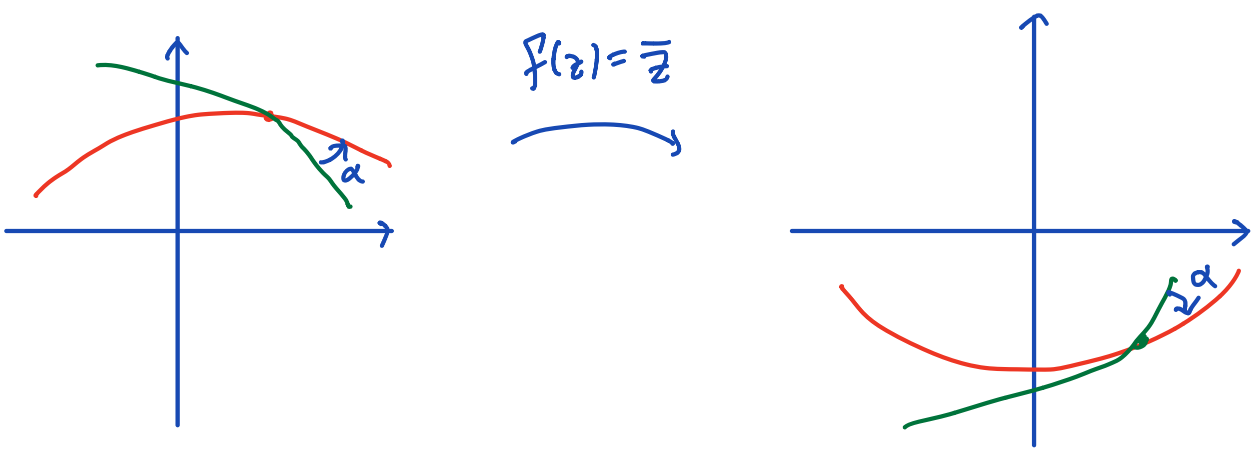
$$\doteq \arg\left(\frac{\cancel{f'(z_0)}\cdot\Delta z_2}{\cancel{f'(z_0)}\cdot\Delta z_1}\right)$$

$$\longrightarrow \arg\left(\frac{\Delta z_2}{\Delta z_1}\right)$$

Example $f(z) = f(x+iy) = x^2+y^2$



Example $f(z) = \bar{z}$



orientation is not preserved $\Rightarrow$ not analytic

Remark: holomorphic / analytic / regular

holomorphic / analytic + $f'(z_0) \neq 0 \Rightarrow$ conformal

but conformal $\not\Rightarrow$ holomorphic / analytic in general

Laplace equation $u(x,y)$

$$u_{xx} + u_{yy} = 0$$

Def A function that satisfies Laplace equation is called a <u>harmonic function</u>.

$f(z)$ is analytic $\Rightarrow \begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$

$$u_{xx} = v_{yx} \qquad u_{yy} = -v_{xy}$$

$$u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0$$

Similarly, $v(x,y)$ is harmonic function.

Task Given a harmonic $u(x,y)$, find a <u>harmonic conjugate function</u> $v(x,y)$ s.t.

$u + iv$ is analytic.

Find v s.t. $u_x = v_y$ and $u_y = -v_x$.

$$dv = v_x\,dx + v_y\,dy$$
$$= \underbrace{-u_y}_{M}\,dx + \underbrace{u_x}_{N}\,dy$$

Assume D is simply connected.

We apply Green's theorem to show that $\int dv$ is independent of path.

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = \frac{\partial}{\partial x} u_x - \left(\frac{\partial}{\partial y}(-u_y)\right) = 0$$

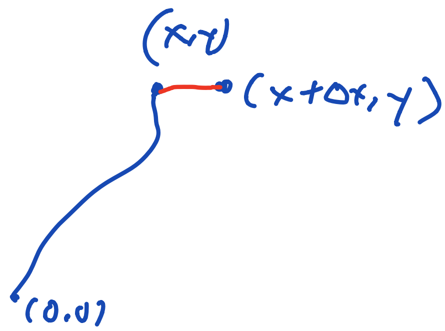
because u is harmonic.

Define

$$v(x,y) \triangleq \int_{(0,0)}^{(x,y)} -u_y\,dx + u_x\,dy \qquad \text{(line integral)}$$

is well-defined because the line integral is indep. of path.

Check CR equation



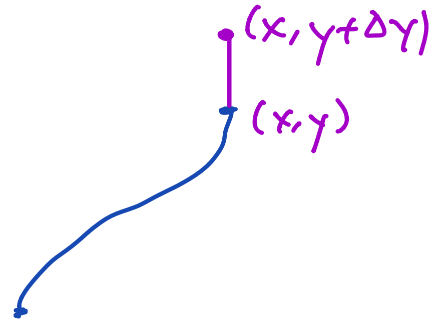
$$\frac{\partial v}{\partial x} = \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_{(x,y)}^{(x+\Delta x, y)} \underbrace{-u_y\,dx + u_x\,dy}_{dv}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_{(x,y)}^{(x+\Delta x, y)} -u_y\,dx$$

$$= -u_y$$

$$\frac{\partial v}{\partial y} = \lim_{\Delta y \to 0} \frac{1}{\Delta y} \int_{(x,y)}^{(x, y+\Delta y)} -u_y\,dx + u_x\,dy$$



$$v_y = u_x$$

$$\therefore\ v(x,y) = \int_{(x_0,y_0)}^{(x,y)} -u_y\,dx + u_x\,dy + C$$

when D is simply connected.

Example $u(x,y) = -2x^2 + x^3 + 2y^2 - 3xy^2$

Find the harmonic conjugate of $u(x,y)$.

$u(x,y)$ satisfies Laplace equation, because

$$u_x = -4x + 3x^2 - 3y^2$$
$$u_{xx} = -4 + 6x$$

$$u_y = 4y - 6xy$$
$$u_{yy} = 4 - 6x$$

$$u_{xx} + u_{yy} = 0$$

METHOD 1

$$\begin{tikzcd}
 & v(x,y) \arrow[dl, shift right, "\partial/\partial x"'] \arrow[dr, "\partial/\partial y"] & \\
-4y+6xy \arrow[ur, shift right, "\int dx"'] & & -4x+3x^2-3y^2
\end{tikzcd}$$

$$\boxed{\begin{aligned} V_x &= -u_y \\ V_y &= u_x \end{aligned}}$$

$$v(x,y) = -4xy + 3x^2y + C(y)$$

$$v_y = -4x + 3x^2 + C'(y) = -4x + 3x^2 - 3y^2$$

$$\Rightarrow C'(y) = -3y^2 \Rightarrow C(y) = -y^3 + C$$

$\therefore$ $v(x,y) = -4xy + 3x^2y - y^3 + C$ for some constant C

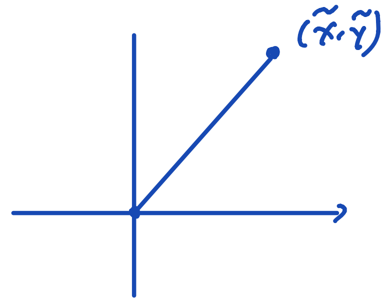
$u(x,y) + i v(x,y) = -2x^2 + x^3 + 2y^2 - 3xy^2 + i(-4xy + 3x^2y - y^3) + C$

is analytic

In fact it is the same as $z^3 - 2z^2$.

METHOD 2

$$v(\tilde{x},\tilde{y}) = \int_{(0,0)}^{(\tilde{x},\tilde{y})} (6xy-4y)\,d \; + (-4x+3x^2-3y^2)\,d$$



We can draw a direct path from

$(0,0)$ to $(\tilde{x},\tilde{y})$

Parameterize the line segment by

$$\begin{cases} x = t\tilde{x} \\ y = t\tilde{y} \end{cases} \quad \text{for} \quad 0 \le t \le 1$$

$$v(\tilde{x},\tilde{y}) = \int_0^1 (6t^2\tilde{x}\tilde{y} - 4t\tilde{y})\tilde{x} + (-4t\tilde{x} + 3t^2\tilde{x}^2 - 3t^2\tilde{y}^2)\tilde{y} \; dt$$

$$= 2\tilde{x}^2\tilde{y} - 2\tilde{x}\tilde{y} - 2\tilde{x}\tilde{y} + \tilde{x}^2\tilde{y} - \tilde{y}^3 + C$$

$$= 3\tilde{x}\tilde{y} - 4\tilde{x}\tilde{y} - \tilde{y}^3 + C$$

The answer is the same by method 1.