MAT 2002 Assignment 4

Deadline: Friday 5:00 pm., 23 April

1. Let $\boldsymbol{x}(t)$ be the complex solution of $\frac{d\boldsymbol{x}(t)}{dt} = \mathbf{A}(t)\boldsymbol{x}(t)$, where $\mathbf{A}(t)$ is a real matrix. Proof that the real part, imaginary part and complex conjugation of $\boldsymbol{x}(t)$ are the solutions of the linear system.

2. Solve the following systems or the initial value problem:

(1) $\boldsymbol{x}' = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \boldsymbol{x}$,

(2) $\boldsymbol{x}' = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix} \boldsymbol{x}$;

(3) $\boldsymbol{x}' = \begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix} \boldsymbol{x}$, $\boldsymbol{x}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

3. Let

$$\boldsymbol{J} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix},$$

where λ is an arbitrary real number.

(1) Find $\boldsymbol{J}^2$, $\boldsymbol{J}^3$, $\boldsymbol{J}^4$; Use the mathematical induction to prove that

$$\boldsymbol{J}^n = \begin{bmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{bmatrix}.$$

(2) Determine $\exp(\boldsymbol{J}t)$; Use $\exp(\boldsymbol{J}t)$ to solve the initial value problem $\boldsymbol{x}' = \boldsymbol{J}\boldsymbol{x}, \boldsymbol{x}(0) = \boldsymbol{x}_0 = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, c_1, c_2 are constant real numbers.

4. Calculate $e^{\mathbf{A}t}$ if

(1) $\mathbf{A} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$,

(2) $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$,

(3) $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$.

Verify directly that $\boldsymbol{x} = e^{\mathbf{A}t}\boldsymbol{x}_0$ is the solution to $\boldsymbol{x}' = \mathbf{A}\boldsymbol{x}, \boldsymbol{x}(0) = \boldsymbol{x}_0 = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$. Note that a, b, c_1, c_2 are constant real numbers.

5. Recall that for the case when 3×3 coefficient matrix $\mathbf{A}$ has only one eigenvalue, we can use $S - N$ decomposition for computing $e^{\mathbf{A}t}$ to obtain the general solution for $\boldsymbol{x}'(t) = \mathbf{A}\boldsymbol{x}(t)$. Define

$$\mathbf{A} = \begin{bmatrix} 5 & -1 & -2 \\ 2 & 1 & 0 \\ 1 & -1 & 3 \end{bmatrix}.$$

Solve the system $\boldsymbol{x}'(t) = \mathbf{A}\boldsymbol{x}(t)$ by using $S - N$ decomposition.

6. Find the general solution and write down a fundamental matrix of the given system of equations. Please express the general solution in terms of real-valued function. $\boldsymbol{x}'(t) = \mathbf{A}\boldsymbol{x}(t)$, where

(1) $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix}$,

(2) $\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$,

(3) $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 2 & 0 & 1 \end{bmatrix}$.

7. Find the general solution of the given system of equations $\boldsymbol{x}' = \mathbf{A}\boldsymbol{x} + \boldsymbol{f}(t)$.

(1) $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$, $\boldsymbol{f} = \begin{bmatrix} e^t \\ t \end{bmatrix}$,

(2) $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ -5 & -2 \end{bmatrix}$, $\boldsymbol{f} = \begin{bmatrix} -\cos t \\ \sin t \end{bmatrix}$,

(3) $\mathbf{A} = \begin{bmatrix} 4 & 8 \\ -2 & -4 \end{bmatrix}$, $\boldsymbol{f} = \begin{bmatrix} t^{-3} \\ -t^{-2} \end{bmatrix}$, $t > 0$,

(4) $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$, $\mathbf{f}(t) = \begin{bmatrix} 0 \\ 0 \\ e^{-t} \end{bmatrix}$.

8. Assume that m is not an eigenvalue of matrix A. Proof nonhomogeneous linear differential equations

$$\boldsymbol{x}' = \mathbf{A}\boldsymbol{x} + \boldsymbol{c}e^{mt}$$

has a solution

$$\boldsymbol{x}(t) = \boldsymbol{p}e^{mt}$$

where $\boldsymbol{c}, \boldsymbol{p}$ are constant vectors.