MAT2006 Tutorial #7

1. Decide whether the following sets are dense in $\mathbb{R}$, nowhere-dense in $\mathbb{R}$, or somewhere in between.

(a) $A = \mathbb{Q} \cap [0, 5]$;

(b) $B = \{1/n \,|\, n \in \mathbb{N}\}$;

(c) $\mathbb{I}$;

(d) the Cantor set.

2. In the lecture, we have mentioned that $t(x)$ is discontinuous at any rational point. Show that $t(x)$ is continuous at any irrational point. Here,

$$t(x) = \begin{cases} 1 & \text{if} \quad x = 0, \\ 1/n & \text{if} \quad x = m/n \in \mathbb{Q} \setminus \{0\} \text{ is in lowest terms with } n > 0, \\ 0 & \text{if} \quad x \notin \mathbb{Q}. \end{cases}$$

3. Let $f(x) : \mathbb{R} \to \mathbb{R}$ be a function. Show that $f(x)$ is continuous if and only if $f^{-1}(O)$ is open for any open set $O \subset \mathbb{R}$.

4. Assume f and g are defined on all of $\mathbb{R}$ and that $\lim_{x \to p} f(x) = q$ and $\lim_{x \to q} g(x) = r$

(a) Give an example to show that it may not be true that

$$\lim_{x \to p} g(f(x)) = r.$$

(b) Show that the result in (a) does follow if we assume f and g are continuous.

(c) Does the result in (a) hold if we only assume f is continuous? How about if we only assume that g is continuous?

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