STA4001 | Stochastic Processes | Spring 2021

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Homework 4

Due: March 2, 2021

1. For Problem 1(a)(b) in Homework 3, suppose demand D_n follows Poisson distribution with mean 1.

   (a) Just write down the transition matrix of the DTMC $\{X_n\}$. (No need to write the problem-solving process, and make sure that the Problem 1(a) in Homework 3 is correct if you use the transition function.)

   (b) Let $g(x)$ be the expected profit (revenue minus cost) in period $n+1$ given that the ending inventory at the end of period n is x. Calculate $g(x)$ for different values of x.

   (c) Assume $\beta = .9$. Compute $\mathbb{E}[\sum_{n=0}^{\infty} \beta^n g(X_n)|X_0 = 1]$. (You may use software like Python. You need to write the problem-solving process and main results, but do not need to submit the code.)

   (d) Now assume the product has a maximum lifetime of three periods, instead of two assumed previously. Demand is satisfied by inventory from the oldest to the newest. Let $X_n = (X_{n1}, X_{n2})$ be ending inventory profile of period n, where X_{n1} is the amount of inventory with remaining one period lifetime, X_{n2} be the amount of inventory with remaining two period lifetime. Suppose the manager uses the same $S = 4$ order up to policy, i.e., orders enough to bring the total inventory level up to S at the beginning of next period. Show that $\{X_n\}$ is a DTMC and use Monte Carlo method to estimate $\mathbb{E}[\sum_{n=0}^{\infty} \beta^n g(X_n)|X_0 = (0,0)]$, where $g(X_n)$ is the expected profit in period $n+1$. Please provide CI with the confidence level defined by you. (To avoid infinite sampling, you may choose a large T so that $\mathbb{E}[\sum_{n=0}^{T} \beta^n g(X_n)|X_0 = (0,0)] \approx \mathbb{E}[\sum_{n=0}^{\infty} \beta^n g(X_n)|X_0 = (0,0)]$.)

   (e) For the Monto-Carlo method in (d), how many episodes are needed in order for the half width of the CI is within \$.1? Give the estimate of your CI.

   For part (d), please also submit your code for implementing Monte-Carlo method.

2. For Problem 5 in Homework 3,

   (a) Assume $X_0 = 2$. Find the probability that the DTMC first reaches 0 before reaching 4.

   (b) Find $T_{2,3}$, the expected number of timeslots needed for the DTMC to first reach state 3 when initially starting from state 2.

3. Let $X = \{X_n : n = 0, 1, \ldots, \}$ be a DTMC on state space S. Define $Y_n = (X_n, X_{n+1})$. Prove that $Y = \{Y_n : n = 0, 1, 2, ...\}$ is a DTMC. Specify its state space and the transition matrix.

4. Consider a jewelry store that only sells diamond rings and operates as follows. Each week, the store is open from Monday-Friday. The weekly (5-day) demand is random, and has distribution

$$D = \begin{cases} 0, & \text{w.p. } 1/6 \\ 1, & \text{w.p. } 1/6 \\ 2, & \text{w.p. } 1/6 \\ 3, & \text{w.p. } 1/6 \\ 4, & \text{w.p. } 1/6 \\ 5, & \text{w.p. } 1/6. \end{cases}$$

Assume that each ring sells for \$100, and any rings unsold by the end of Friday require cleaning immediately, which costs \$10 per ring. After a week's worth of sales, the store owner reviews inventory on Saturday morning, and decides how many rings to order. The ring supplier offers two shipping options: standard or express shipping. Standard shipping costs \$15 per ring, and the order arrives on the following Friday evening (a week after it is placed) after the store closes. Express shipping costs \$35 per ring, but the order arrives on the evening of the next day (Sunday).

Consider the following ordering policy: each Saturday morning, the store owner looks at the inventory and sees x rings. She then orders $(3-x)^+$ rings via standard shipping, and then places an express order to ensure she starts out on Monday with 5 rings (if she has more than 5 rings on Saturday morning, no order is placed). Let X_n be the number of rings in inventory on the morning of the nth Saturday, $n = 0, 1, 2, 3 \ldots$

   a. Prove that $\{X_n\}$ is a DTMC.
   b. Write down the state space and transition matrix of this DTMC.
   c. Calculate the expected profit in 10 weeks of the store starting with 0 ring. (You may use software like Python. You need to write the problem-solving process and main results, but do not need to submit the code.)

5. Consider the same setup as in the previous problem, except that the express shipping option is now replaced by a "rustic" shipping option. They now transport the rings in person for a fraction of the cost (\$5 per ring), but also take 3 weeks to deliver the ring, i.e. an order placed on the nth Saturday morning will arrive on the Friday before the $(n+3)$rd Saturday morning. Let $R_n^{(1)}$ be the number of rings to be delivered via rustic shipping one week from the nth Saturday morning. Similarly, let $R_n^{(2)}$ be the number of rings to be delivered two weeks from the nth Saturday morning.

Consider the following ordering policy: on the morning of the nth Saturday, the store owner orders $(5 - X_n)^+$ rings via rustic shipping. She also orders $(5 - X_n - R_n^{(1)})^+$ rings via standard shipping.

a. Model the system as a DTMC (it is no longer enough to keep track of just X_n).