STA4001 Stochastic Processes Spring 2021
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Midterm
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1. (20 points) Let us consider a soccer team. Each game, they will either win or lose. If they win both the past two games, they will win in the next game with probability 0.8. If they just win one of the past two games, they will win in the next game with probability 0.6. If they lose both the past two games, they will win in the next game with probability 0.4.

   Suppose X_n represents the state of the team after the n-th game. ($X_n = 1$ if the team win and $X_n = 0$ otherwise)

   (a) (10 points) Check whether $\{X_n, n = 1, 2, \dots\}$ is a Markov chain. If so, please prove it. If not, please model this process as a DTMC and prove it.

   (b) (10 points) Find the transition matrix and the stationary distribution.

   Solution.

   (a) Assume X_n is a Markov chain.

   $$0.8 = \mathbb{P}(X_{n+2} = 1 \mid X_{n+1} = 1, X_n = 1) = \mathbb{P}(X_{n+2} \mid X_{n+1} = 1) = \mathbb{P}(X_{n+2} \mid X_{n+1} = 1, X_n = 0) = 0.6$$

   Contradiction! Thus, X_n is not a Markov chain.
   We define $Y_n = (X_n, X_{n+1})$ and i.i.d. sequence $\{U_n, n = 1, 2, \dots\}$ uniformly distributed on $[0, 1]$. Then we define a function f:

   $$f((1,1),U) = \begin{cases} (1,1) & 0 \le U < 0.8 \\ (1,0) & 0.8 \le U \le 1 \end{cases} \qquad f((1,0),U) = \begin{cases} (0,1) & 0 \le U < 0.6 \\ (0,0) & 0.6 \le U \le 1 \end{cases}$$

   $$f((0,1),U) = \begin{cases} (1,1) & 0 \le U < 0.6 \\ (1,0) & 0.6 \le U \le 1 \end{cases} \qquad f((0,0),U) = \begin{cases} (0,1) & 0 \le U < 0.4 \\ (0,0) & 0.4 \le U \le 1 \end{cases}$$

   Thus, we can have $Y_{n+1} = f(Y_n, U_n)$. $\{Y_n, n = 1, 2, \dots\}$ is a Markov chain.
   (5 points for DTMC judgement; 3 points for DTMC model; 2 points for proof)

   (b) Let us list the state as $\{(0,0), (0,1), (1,0), (1,1)\}$. Then we have the transition matrix as follows:

   $$P = \begin{bmatrix} 0.6 & 0.4 & 0 & 0 \\ 0 & 0 & 0.4 & 0.6 \\ 0.4 & 0.6 & 0 & 0 \\ 0 & 0 & 0.2 & 0.8 \end{bmatrix}$$

   Using $\pi P = \pi$, we can get the stationary distribution $\pi = (\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{2})$.
   (5 points for transition matrix; 5 points for stationary distribution)

2. (25 points) As we know, the Quanjiafu convenience store sells "fruitfishing", which contains multiple types of fruits (which is definitely perishable). Facing random demand, the store's manager wants to find the optimal order of different fruits to optimize the revenue. For example, the manager just asks for an optimal solution of one fruit, pineapple. Suppose that the weekly demand D (kg) for the product is i.i.d with uniform distribution with p.d.f $f(x) = \frac{1}{20}, x \in [0, 20]$.

   Sales happen from Monday-Friday. Unused items from one week *cannot* be used in the following week. Suppose that each item sells at 20 CNY. Each item costs 10 CNY to order, and each leftover item by Friday evening has a salvage value of 5 CNY. The store makes a weekly order from a supplier each Friday evening for the product to arrive early Monday morning (before sales commence).

   Note that, although consumers will buy cut fruit, the store can only order the raw fruit, which means that the order quantity should be **integer**.

(a) (8 points) Find the optimal order quantity and compute the expected profit per week, given that the optimal order quantity is ordered.

(b) (8 points) Compute the 95% confidence interval of the cumulative profit in 100 weeks, given that the optimal order quantity from part (a) is ordered. (you need to state the calculation procedures clear and leaving your answer as an expression is fine.)

(c) (9 points) Suppose that on the evening of a fixed Friday, the store receives as donation of 6 kg fresh items of pineapple, free of charge. Furthermore, there is now a 40 CNY fixed cost to make an order from the supplier. Should the store place an order that Friday evening? If so, how many items should it order, in order to maximize the expected profit for the following week? Show your work to support your claim.

Solution.

(a)

$$\frac{c_p - c_v}{c_p - c_s} = \frac{20 - 10}{20 - 5} = \frac{10}{15} = \frac{2}{3}.$$

The c.d.f of D is $F(x) = \frac{x}{20}$. Let $F(x) = \frac{c_p - c_v}{c_p - c_s}$, we can have the optimal decimal order is $\frac{40}{3} = 13.33$.

We have the optimal decimal oder quantity is 13.33. However, our order quantity should be integer. Thus, we need to compare the expected profit when the order is 13 and 14. When the order is 13, the expected profit per week is

$$20\mathbb{E}[\min(D, 13)] + 5\mathbb{E}[(13 - D)^+] - 10 \times 13 = 66.625$$

When the order is 14, the expected profit per week is

$$20\mathbb{E}[\min(D, 14)] + 5\mathbb{E}[(14 - D)^+] - 10 \times 14 = 66.5 < 66.625$$

So the optimal order quantity is 13 and the corresponding expected profit is 66.625.

(4 points for the decimal quantity; 2 points for the expected profit; 2 points for the integer order quantity)

(b) Let $P(13)$ be the single period profit when the order quantity is 13. The desired C.I. is $[6662.5 - 1.96\sqrt{100}\sigma, 6662.5 + 1.96\sqrt{100}\sigma]$, where $\sigma^2 = \mathbb{E}P^2(13) - (\mathbb{E}P(13))^2$.

(4 points for the expression of variance; 4 points for the expression of CI)

(c) **If we do not order,** the expected profit is

$$20\mathbb{E}[\min(6, D)] + 5\mathbb{E}(6 - D)^+ = 106.5$$

If order, then order 7 items. The expected profit is

$$20\mathbb{E}[\min(13, D)] + 5\mathbb{E}(13 - D)^+ - 7 \times 10 - 40 = 86.625$$

The solution is do not order.

(4 points for expected profit of each case; 1 point for the result)

3. (25 points) Joe, together with N friends, will play the game *pass the parcel* on Saturday. In the game, they all sit face to face in a circle, and are marked clockwise as $0, 1, \ldots, N$ (where Joe is marked as 0). And for each step, each kid who has the parcel will give it to the kid on the right with the same probability p, and give it to the kid on the left with the same probability $1 - p$. Let X_n be the state in which the parcel is at the step n.

(a) (3 points) Draw a transition diagram.

(b) (5 points) What is the stationary distribution? Explain your answer.

(c) (5 points) If $N \geq 3$, does $\lim_{n\to\infty} P^n_{1,3}$ exist? Explain your answer.

On Sunday, Joe will play the game *Super Mario Bros.* Suppose the game has infinite levels from $1, 2, \ldots$. If Joe wins in the level k, he will go to next level $k+1$, otherwise he will go to the level 1 again. Now we know that Joe wins level 1 with probability 1, and level k with probability $1 - \frac{1}{k}$ for each $k \geq 2$. Let $\{X_n, n = 0, 1, \ldots\}$ be the level Joe is in at the n-th try.

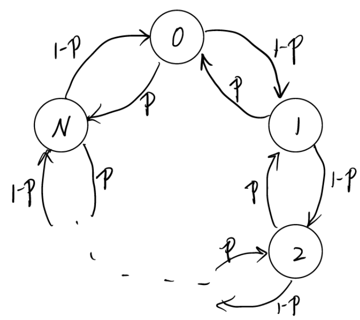
(d) (3 points) Write down the transition matrix.

(e) (4 points) Is the DTMC irreducible? Prove your answer.

(f) (5 points) What is the type for each state, that is, transient, positive recurrent or null recurrent? Prove your answer.

Solution.

(a) The transition diagram is a cycle:



(2 points for transition diagram; 1 point for transition probability)

(b) From the symmetric property, all the states have the same stationary distribution, i.e.

$$\pi_0 = \pi_1 = \cdots = \pi_N.$$

From $\sum_{i=0}^{N} = 1$, we have

$$\pi_0 = \pi_1 = \cdots = \pi_N = \frac{1}{N+1}.$$

(4 points for relation: Case 1: 3 points if only listing the detail of $\pi = \pi P$, Case 2: 2 points if wrongly using cutting method ; 1 point for the result)

(c) If $N+1$ is even or N is odd, the DTMC is periodic, and the limit does not exit.

If $N+1$ is odd or N is even, the DTMC is aperiodic, and the limit exits.

(1 point for the 2 cases; 2 points for periodic and 2 points for aperiodic)

(d)

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 & \ldots \\ 1/3 & 0 & 0 & 2/3 & 0 & 0 & \ldots \\ 1/4 & 0 & 0 & 0 & 3/4 & 0 & \ldots \\ 1/5 & 0 & 0 & 0 & 0 & 4/5 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$

(e) Yes.

For state k ($k \neq 1$), $P^{k-1}_{1,k} = \prod_{i=2}^{k-1} \frac{i-1}{i} = \frac{1}{k-1} > 0$, so $1 \Rightarrow k$. Besides, $P_{k,1} = \frac{1}{k} > 0$, so $k \Rightarrow 1$. That is $1 \Leftrightarrow k$ for any $k \neq 1$. Therefore, all states are communicative so irreducible.

(2 points for "Yes"; 2 point for proof)

(f) Firstly, note that $\{X_n\}$ is **irreducible**, we only need to show the state 1 is null recurrent. Define $f_{ij}^n = \Pr\{T_j = n | X_0 = i\}$. Notice there is only one path that firstly arrive at the state 1 given the initial state is state 1, so

$$f_{1,1}^{(n)} = p_{1,n}^{n-1} \cdot p_{n,1} = \frac{1}{n-1} \cdot \frac{1}{n} \qquad n \geq 2.$$

From $f_{1,1}^{(1)} = 0$,

$$f_1 = \sum_{n=2}^{\infty} f_{1,1}^{(n)} = \sum_{n=2}^{\infty} \frac{1}{n-1} \cdot \frac{1}{n} = \sum_{n=2}^{\infty} (\frac{1}{n-1} - \frac{1}{n}) = 1.$$

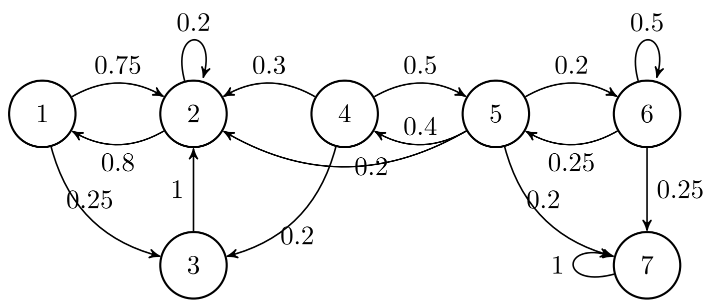
So state 1 is recurrent.

$$\mathbb{E}[T_1 \mid X_0 = 1] = \sum_{n=2}^{\infty} n \cdot f_{1,1}^{(n)} = \sum_{n=2}^{\infty} \frac{1}{n-1} = \infty$$

So state 1 is null recurrent.

(2 points for the conclusion; 1 point for using irreducible property, 1 point for proof of recurrent state and 1 point for the infinite expected time)

4. (25 points) Consider the Markov Chain specified in the following.



(a) (4 points) Find $\mathbb{P}\{X_2 = 2 \mid X_0 = 4\}$.

(b) (4 points) List all positive recurrent states, null recurrent states and the transient states.

(c) (4 points) Does the distribution of X_n has a limit given that the process starts in state 1? If so, what is it?

(d) (5 points) If it will get i CNY when arriving at the state i for each time, what is the long run average reward given that the process starts in state 5?

(e) (4 points) What is the expectation of the first time arriving at any recurrent state given that the process starts in state 5?

(f) (4 points) What is the expected number of times that the Markov chain is in state 6, given that it starts in state 5?

Solution.

(a)

$$\begin{aligned}\mathbb{P}\{X_2 = 2 \mid X_0 = 4\} &= \sum_{i=1}^{7} \mathbb{P}\{X_2 = 2 \mid X_1 = i\}\mathbb{P}\{X_1 = i \mid X_0 = 4\} \\ &= p_{4,2}p_{2,2} + p_{4,3}p_{3,2} + p_{4,5}p_{5,2} = 0.06 + 0.2 + 0.1 = 0.36\end{aligned}$$

(3 points for the expression and 1 points for result)

(b) Positive recurrent states: $\{1,2,3\},\{7\}$; Null recurrent states: $\emptyset$; Transient states: $4,5,6$.
(2 points for the p.r. and 2 points for transient states)

(c) We just need to look at the irreducible closed set $\{1,2,3\}$.

$$\begin{aligned}
\pi_1 &= 0.8\pi_2 \\
\pi_3 &= 0.25\pi_1 = 0.2\pi_2 \\
1 &= \pi_1 + \pi_2 + \pi_3 = 0.8\pi_2 + \pi_2 + 0.2\pi_2 = 2\pi_2
\end{aligned}$$

Hence,

$$\pi_1 = 0.4,\ \pi_2 = 0.5,\ \pi_3 = 0.1$$

Since restricted to $\{1,2,3\}$, it is aperiodic and p.r., the distribution of X_n converges to $(\pi_1,\pi_2,\pi_3) = (0.4,0.5,0.1)$.
(3 points for the expression and 1 points for result)

(d) We need to see the probability finally arriving at a closed irreducible set. Let μ_i be the probability absorbed into state 7 given that the initial state is i. Using the first step method,

$$\begin{aligned}
\mu_4 &= 0.5\mu_5 \\
\mu_5 &= 0.4\mu_4 + 0.2\mu_6 + 0.2 \\
\mu_6 &= 0.5\mu_6 + 0.25\mu_5 + 0.25
\end{aligned}$$

Then,

$$\mu_4 = 3/14,\ \mu_5 = 3/7,\ \mu_6 = 5/7$$

So the long run fraction of time for each state is (given initial state 5)

$$[4/7\times 0.4, 4/7\times 0.5, 4/7\times 0.1, 0, 0, 0, 3/7] = [8/35, 2/7, 2/35, 0, 0, 0, 3/7],$$

and the long run average reward is

$$1\times 8/35 + 2\times 2/7 + 3\times 2/35 + 0 + 0 + 0 + 7\times 3/7 = 139/35.$$

(2 points for the expression and 1 points for result of equations, 1 point for the long run fraction of time for each state and 1 point for the long run average reward)

(e) Let t_i be the expected time arriving at any recurrent state given that the initial state is i. Using the first step method,

$$\begin{aligned}
t_4 &= 1 + 0.5t_5 \\
t_5 &= 1 + 0.4t_4 + 0.2t_6 \\
t_6 &= 1 + 0.5t_6 + 0.25t_5
\end{aligned}$$

Then,

$$t_4 = 16/7,\ t_5 = 18/7,\ t_6 = 23/7$$

(3 points for the expression and 1 points for result)

(f) Let N_i be the expected number of times that the Markov chain is in state 6, given that it starts in state i. Using the first step method,

$$\begin{aligned}
N_4 &= 0.5N_5 \\
N_5 &= 0.4N_4 + 0.2(1 + N_6) \\
N_6 &= 0.5(1 + N_6) + 0.25N_5
\end{aligned}$$

Then,

$$N_4 = 2/7,\ N_5 = 4/7,\ N_6 = 9/7$$

(3 points for the expression and 1 points for result)

5. (5 points) Prove that for a finite irreducible DTMC, it has at most one stationary distribution.

Solution. Suppose we have two stationary distributions π and π'. Let $\alpha = \max_j \frac{\pi(j)}{\pi'(j)}$ and $j^* = \arg\max \frac{\pi(j)}{\pi'(j)}$. Then

$$
\begin{aligned}
\pi(j^*) &= \sum_i \pi(i) P^n_{ij^*} = \sum_i \pi'(i) \frac{\pi(i)}{\pi'(i)} P^n_{ij^*} \\
&\leq \sum_i \pi'(i) \alpha P^n_{ij^*} \\
&= \alpha \pi'(j^*) = \pi(j^*)
\end{aligned}
$$

The inequality turning into equality implies that $P^n_{ij^*}(\alpha - \frac{\pi(i)}{\pi'(i)}) = 0$ for all n, i. Since it is irreducible DTMC, for any i we an find n such that $P^n_{ij^*} > 0$, leading to $\frac{\pi(i)}{\pi'(i)} = \alpha$. We must have $\alpha = 1$ and hence $\pi = \pi'$.

(3 points for using contradiction and 2 points for remaining logics)