CSC3001: Discrete Mathematics

Assignment 1

Instructions:

1. Print out this question paper (**two-sided**) and write down your full working on the blank area.
2. You can have discussions with your classmates. However, make sure all the solutions you submit are your own work. Any plagirism will be given **ZERO** mark.
3. Submission of this assignment should **NOT** be later than **5pm on 11th of October**.
4. Before your submission, please **make a softcopy** of your work for further discussion in a tutorial.
5. After making your softcopy, submit your assignment to the dropbox located on the 4th floor in Chengdao Building.

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1. (*20 points*) Given statements p, q, r, s, which of the following arguments are valid? (**Note:** you need to give your arguments in order to obtain full mark.)

(i)
$$\begin{array}{l} (p \vee q) \to \neg r \\ p \to \neg q \\ \neg q \to p \\ \hline \therefore\ \neg r \end{array}$$

(ii)
$$\begin{array}{l} p \to q \\ q \to \neg p \\ \hline \therefore\ p \leftrightarrow q \end{array}$$

(iii)
$$\begin{array}{l} (q \wedge r) \to p \\ (p \vee q) \to r \\ \hline \therefore\ s \leftrightarrow s \end{array}$$

(i). Suppose the conclusion $\neg r$ is false.

$(p \vee q) \to \neg r$ (true), $\neg r$ (false) $\Rightarrow$ $p \vee q$ (false).

$p \vee q$ (false) $\Rightarrow$ p (false), q (false)

Then $p \to \neg q$ is true, $\neg q \to p$ is false.

Not all the assumptions are true, thus argument (i) is valid.

(ii). Suppose the conclusion $p \leftrightarrow q$ is false.

$\Rightarrow$ p (true), q (false) or p (false), q (true).

① If p is true, q is false.

$\Rightarrow$ $p \to q$ (false), $q \to \neg p$ (true). Situation ① is valid

② If p is false, q is true

$\Rightarrow$ $p \to q$ (true), $q \to \neg p$ (true). Situation ② is not valid.

Thus argument (ii) is not valid.

(iii). ① If s is true, then $s \to s$ is true, thus the conclusion $s \leftrightarrow s$ is true.

② If s is false, then $s \to s$ is true, thus the conclusion $s \leftrightarrow s$ is true

Since the conclusion is always true, argument (iii) is valid.

2. (*20 points*) Let $a \in \mathbb{Z}, b \in \mathbb{Z}^+$. Use Well Ordering Principle to prove that there exist $q, r \in \mathbb{Z}$ such that

$$a = qb + r \qquad \text{and} \qquad 0 \leq r < b$$

Proof. Let $a \in \mathbb{Z}$, $b \in \mathbb{Z}^+$. Define $S = \{a - bk \mid k \in \mathbb{Z} \text{ and } a - bk \geq 0\}$.

① If $a \geq 0$, take $k = 0$, then $a - bk = a - b \cdot 0 = a \geq 0$. $\Rightarrow S \neq \phi$.

② If $a < 0$, take $k = 2a$, then $a - bk = a - b \cdot 2a$
$= a(1 - 2b) > 0 \Rightarrow S \neq \phi$.

Since $S \subseteq \mathbb{N}$ and $S \neq \phi$, by Well Ordering Principle,

there exists a smallest $r = a - bq \in S$. $(r \geq 0)$.

(Show that $r < b$). Suppose for contradiction that $r \geq b$,

then $a - b(q+1) = a - bq - b = r - b \geq 0$. $\Rightarrow a - b(q+1) \in S$.

Since $b > 0$, then $r - b < r$, contradict with r

is the smallest member of S. $\Rightarrow r < b$.

Thus, there exist $q, r \in \mathbb{Z}$, s.t $a = qb + r$, and $0 \leq r < b$.

3. *(20 points)*

(a) Translate the following statement into logical formula without predicates.

> For each $a, b \in \mathbb{Z}^+$ with $a \leq b$, we have
> $$\frac{a}{b} = \frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_m}$$
> for some mutually distinct $d_1, \ldots, d_m \in \mathbb{Z}^+$.

(b) Use mathematical induction to prove the statement in (a).
(Full mark will be given **ONLY** if you use mathematical induction.)

(a) $\forall a, b, m, d_i, d_j, i, j \in \mathbb{Z}^+.\ (a \leq b) \wedge (\frac{a}{b} = \frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_m}) \wedge (d_i \neq d_j) \wedge (i \neq j)$
$\wedge\ (i \leq m) \wedge (j \leq m)$

(b). Proof by induction.

① $P(n)$, when $a = n$, for each $b \geq a$, we have $\frac{n}{b} = \frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_m}$
for some mutally distinct $d_1, \ldots, d_m \in \mathbb{Z}^+$.

② Case $P(1)$, $a = 1$, $b \geq a = 1$, then $\frac{a}{b} = \frac{1}{b}$, $P(1)$ is true.

③ Case $P(2)$, $a = 2$, $b \geq a = 2$, then $\frac{a}{b} = \frac{2}{b}$.

if b is even, $b = 2p$, then $\frac{a}{b} = \frac{2}{2p} = \frac{1}{p}$

if b is odd, $b = 2p+1$, then $\frac{a}{b} = \frac{2}{2p+1}$

$$\frac{2}{2p+1} = \frac{2 \cdot r}{(2p+1) \cdot r} = \frac{2r-1}{(2p+1) \cdot r} + \frac{1}{(2p+1) \cdot r}, \quad r \in \mathbb{Z}^+ \text{ and } r \geq 2.$$

We can always find r, such that $2r-1 = 2p+1$.

$$\Rightarrow \frac{2}{2p+1} = \frac{1}{r} + \frac{1}{(2p+1) \cdot r}$$

That is, $\frac{2}{b} = \frac{1}{r} + \frac{1}{b \cdot r}$, $P(2)$ is true.

④ Suppose $P(t)$ is true for $t \in \{1, 2, \cdots k\}$.

that is $\forall t \in \{1, 2, \cdots k\}$. $a = t$, $b \geq a$, $\frac{a}{b} = \frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_m}$.

for some mutually distinct $d_1 \ldots d_m \in \mathbb{Z}^+$.

Then, show that $P(t+1)$ is true for $t \in \{1, 2, \cdots k\}$.

i.e. show that $P(k+1)$ is true.

$a = k+1$, $b \geq a$, $\frac{a}{b} = \frac{k+1}{b}$.

If $k+1$ and b are both even, assume $k+1 = 2p$, $b = 2s$.

then $\frac{a}{b} = \frac{2p}{2s} = \frac{p}{s}$, and $p = \frac{k+1}{2} < k$,

By $P(t)$, $\frac{a}{b} = \frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_m}$, $d_1 \ldots d_m \in \mathbb{Z}^+$

mutually distinct.

If $k+1$ and b are not both even.

By $P(t)$, $\frac{k}{b} = \frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_m}$, $d_1 \ldots d_m \in \mathbb{Z}^+$.

mutually distinct

when $d_1 \ldots d_m \neq b$. then $\frac{k+1}{b} = \frac{k}{b} + \frac{1}{b} = \frac{1}{d_1} + \cdots + \frac{1}{d_m} + \frac{1}{b}$.

when $d_k = b$, for $1 \leq k \leq m$, W.L.O.G. assume $d_m = b$.

$\frac{k+1}{b} = \frac{k}{b} + \frac{1}{b} = \frac{1}{d_1} + \cdots + \frac{1}{d_{m-1}} + \frac{2}{b}$.

By $P(2)$, $\frac{2}{b} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, $p_1 \ldots p_m \in \mathbb{Z}^+$

mutually distinct.

$\Rightarrow$ If $d_1 \ldots d_{m-1}, p_1 \ldots p_m$ are mutally distinct,

then $\frac{k+1}{b} = \frac{1}{d_1} + \cdots + \frac{1}{d_{m-1}} + \frac{1}{p_1} + \cdots + \frac{1}{p_m}$

$\Rightarrow$ If not, then we get some $\frac{2}{d_k}$, $1 \leq k \leq m-1$.

repeat above 6 step, and $P(1) \cdots P(k)$ are true,

finally we can find $\frac{k+1}{b} = \frac{1}{d_1} + \cdots + \frac{1}{d_m}$, $d_1 \ldots d_m \in \mathbb{Z}^+$

mutually distinct.

Thus, $P(t+1)$ is true for $t \in \{1, 2, \cdots k\}$.

4. (*20 points*) Prove that

$$A = \{5a \mid a \in \mathbb{Z}\}, \qquad B = \left\{5\left\lfloor \frac{4b}{3} \right\rfloor - 2 \;\middle|\; b \in \mathbb{Z}\right\}, \qquad C = \{20c - 7 \mid c \in \mathbb{Z}\}$$

form a partition for the set $X = \left\{ \left\lfloor \frac{5x+1}{2} \right\rfloor \;\middle|\; x \in \mathbb{Z}\right\}$.

Proof. ① If x is even, let $x = 2k$, $k \in \mathbb{Z}$

then $\lfloor \frac{5x+1}{2} \rfloor = \lfloor \frac{5 \cdot 2k+1}{2} \rfloor = \lfloor 5k + \frac{1}{2} \rfloor = 5k$.

Then $A = \{5a \mid a \in \mathbb{Z}\} \subseteq X$.

② If x is odd. let $x = 2k+1$, $k \in \mathbb{Z}$.

then $\lfloor \frac{5x+1}{2} \rfloor = \lfloor \frac{5 \cdot (2k+1)+1}{2} \rfloor = 5k+3$.

when $k = 4p+2$. $5k+3 = 5(4p+2)+3 = 20p+13$, $p \in \mathbb{Z}$.
$= 20c - 7$, $c \in \mathbb{Z}$.

Then $C = \{20c-7 \mid c \in \mathbb{Z}\} \subseteq X$.

when $k \neq 4p+2$, i.e. $k = 4p$, $4p+1$, $4p+3$.

$B = \{5\lfloor \frac{4b}{3} \rfloor - 2 \mid b \in \mathbb{Z}\}$, $b = 3t$, $3t+1$, $3t+2$, $t \in \mathbb{Z}$

if $b = 3t$, then $5\lfloor \frac{4b}{3} \rfloor - 2 = 5 \cdot \lfloor \frac{4 \cdot 3t}{3} \rfloor - 2 = 5 \cdot 4t - 2$
$= 5 \cdot 4p + 18 = 5(4p+3)+3$. $p \in \mathbb{Z}$.

if $b = 3t+1$, then $5 \cdot \lfloor \frac{4b}{3} \rfloor - 2 = 5 \cdot \lfloor \frac{4 \cdot (3t+1)}{3} \rfloor - 2 = 5 \cdot 4t + 3$
$= 5 \cdot 4p + 3$, $p \in \mathbb{Z}$.

if $b = 3t+2$, then $5 \cdot \lfloor \frac{4b}{3} \rfloor - 2 = 5 \cdot \lfloor \frac{4 \cdot (3t+2)}{3} \rfloor - 2 = 5 \cdot 4t + 8$.
$= 5 \cdot (4p+1)+3$, $p \in \mathbb{Z}$.

Since $k = 4p,\ 4p+1,\ 4p+3,\ p \in \mathbb{Z}$ in X, then.

$$\lfloor \frac{5x+1}{2} \rfloor = 5k+3 = 5 \cdot 4p + 3$$

$$\lfloor \frac{5x+1}{2} \rfloor = 5k+3 = 5 \cdot (4p+1)+3$$

$$\lfloor \frac{5x+1}{2} \rfloor = 5k+3 = 5(4p+3)+3.$$

which corresponds with elements in B.

Then $B = \{ 5 \lfloor \frac{4b}{3} \rfloor - 2 \mid b \in \mathbb{Z} \}$.

Since we have considered all the situations in X,

then A, B, C form a partition for set X.

5. (*20 points*) Let $\alpha, \beta \in \mathbb{R}$ be such that none of them is a root of a nonzero polynomial with integer coefficients (that is, $c_n x^n + c_{n-1}x^{n-1} + \ldots + c_1 x + c_0$, where $c_i \in \mathbb{Z}$). Show that there are at least two irrational numbers contained in the following set

$$S = \{\alpha + \beta, \alpha - \beta, \alpha\beta\}$$

Proof. Define $X = \{x \mid c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0 = 0,\ c_i \in \mathbb{Z},\ n \in \mathbb{N}\}$.

① Show that if $x \in \mathbb{Q}$, then $x \in X$.

Suppose $x \in \mathbb{Q}$, then $\exists\, p, q \in \mathbb{Z}$, $q \neq 0$.

s.t. $x = \frac{p}{q}$, then let $c_1 = q$, $c_0 = p$,

we have $c_1 x + c_0 = 0 \Rightarrow px + q = 0$. $x = \frac{p}{q}$.

Thus, $x \in X$.

② Show that if $x = a + bi \in \mathbb{R} \setminus \mathbb{Q}$, $\bar{x} = a - bi \in \mathbb{R} \setminus \mathbb{Q}$.

then $x \in X$, $\bar{x} \in X$. $(a \in \mathbb{Q},\ b^2 \in \mathbb{Q})$.

consider polynomial $c_2 x^2 + c_1 x + c_0 = 0$.

which is equivalent to $c_2 (x + \frac{c_1}{2c_2})^2 + c_0 - \frac{c_1^2}{4c_2} = 0$.

$$\Rightarrow (x + \frac{c_1}{2c_2})^2 + \frac{c_0}{c_2} - \frac{c_1^2}{4c_2^2} = 0.$$

Let $\frac{c_1}{2c_2} = -a$, $\frac{c_0}{c_2} - \frac{c_1^2}{4c_2^2} = b^2$, then we have.

$$c_2 x^2 + c_1 x + c_0 = 0 \Rightarrow (x-a)^2 + b^2 = 0 \Rightarrow x = a \pm bi.$$

Thus, $x \in X$, $\bar{x} \in X$.

③ Show that S contains at least two irrational numbers.

Since $\alpha, \beta \in \mathbb{R}$ and $\alpha, \beta \notin X$, then α, β must be irrational numbers and not in the form of $a \pm bi$.

W.L.O.G, Assume the irrational component in α, β is $\sqrt{\pi}$.

if $\alpha = c+\sqrt{\pi}$, $\beta = d-\sqrt{\pi}$ then $\alpha+\beta = c+d$

$(c, d \in \mathbb{Q})$ $\alpha-\beta = c-d+2\sqrt{\pi}$

$\alpha\cdot\beta = c\cdot d - (c-d)\sqrt{\pi} - \pi$.

$\Rightarrow$ $\alpha-\beta$, $\alpha\cdot\beta$ are irrational

if $\alpha = c+\sqrt{\pi}$, $\beta = d+\sqrt{\pi}$, then $\alpha+\beta = c+d+2\sqrt{\pi}$

$(c, d \in \mathbb{Q})$ $\alpha-\beta = c-d$

$\alpha\cdot\beta = cd+(c+d)\sqrt{\pi}+\pi$.

$\Rightarrow$ $\alpha+\beta$, $\alpha\cdot\beta$ are irrational.

if $\alpha = c\cdot\sqrt{\pi}$, $\beta = \frac{d}{\sqrt{\pi}}$, then $\alpha+\beta = c\sqrt{\pi}+\frac{d}{\sqrt{\pi}}$

$(c, d \in \mathbb{Q})$. $\alpha-\beta = c\cdot\sqrt{\pi} - \frac{d}{\sqrt{\pi}}$.

$\alpha\cdot\beta = cd$.

$\Rightarrow$ $\alpha+\beta$, $\alpha-\beta$ are irrational.

Or if $\alpha = \sqrt{\pi}+\frac{1}{\sqrt{\pi}}$, $\beta = \sqrt{\pi}-\frac{1}{\sqrt{\pi}}$, then $\alpha+\beta = 2\sqrt{\pi}$

$\alpha-\beta = 2\frac{1}{\sqrt{\pi}}$

$\alpha\beta = \pi - \frac{1}{\pi}$.

$\Rightarrow$ $\alpha+\beta$, $\alpha-\beta$, $\alpha\cdot\beta$ are irrational.

Thus, at least two numbers in S are irrational.

6. (*10 points*) **[bonus question]** A kid is playing a game on a 4×4 table whose entries are filled with mutually distinct numbers. He needs to make a reshuffle on these numbers so that the numbers on the same line (only consider horizontal, vertical, and two diagonal directions) also appear on the same line after the reshuffle. After trying a few times he conjectures that the ordering of the numbers are always preserved, that is, if b is a number between a, c on a line, then b is also a number between a, c on the new line after the reshuffle. (1) Is this conjecture true? (2) And is this conjecture true for any $n \times n$ table?

Initial configuration

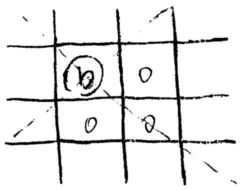
	(,1)	(,2)	(,3)	(,4)
(1,)	1	2	3	4
(2,)	5	6	7	8
(3,)	9	10	11	12
(4,)	13	14	15	16

A fesible reshuffle

4	3	2	1
8	7	6	5
12	11	10	9
16	15	14	13

(1). Suppose in the initial configuration. b is a number between a and C on a row. $b(i,j)$. $a(i,j-1)$, $C(i,j+1)$. $i=1,2,3,4$, $j=2,3$.

①



Since the structure of table is symmetric, the situations of positions are similar in $(2,2), (2,3) \cdot (3,2)\ (3,3)$

W.L.O.G. we consider b in position $(2,2)$.

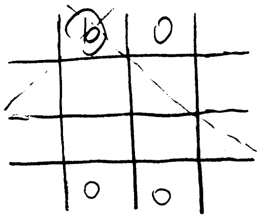
Consider two diagonal directions of b, after reshuffle.

b could be in position $(2,2)$, $(2,3)$ $(3,2)$ $(3,3)$.

If b in $(2,2)$, then $\frac{a(2,1)}{C(2,3)}$ Not OK, the same as original a, c. $\frac{a(1,2)}{c(3,2)}$ Ok. $\frac{a(2,4)}{\times}$ $\frac{a(4,2)}{\times}$ Not OK, no place for C.

If b in $(2,3)$, then $\frac{a(1,3)}{C(3,3)}$ OK. $\frac{a(2,4)}{C(2,2)}$ ok. $\frac{a(2,1)}{\times}$ $\frac{a(4,3)}{\times}$ Not ok, no place for C.

Similar for b in $(3,2)$ and $(3,3)$, we can find that b is always in between a and C.

② 

Since the structure of table is symmetric, the situation of positions are similar in (1,2) (1,3) (4,2) (4,3).

W.L.O.G. we consider b in position (1,2)

consider two diagonal directions of b, after reshuffle,

b could be in position (1,2) (1,3) (4,2) (4,3) (2,1) (3,1). (2,4) (3,4).

If b in (1,2), then $\begin{array}{c} a(1,1) \\ \downarrow \\ c(1,3) \end{array}$ $\begin{array}{c} a(1,4) \\ \downarrow \\ \times \end{array}$

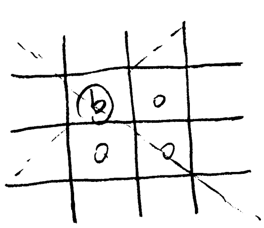
Not ok, the same as original a, c. Not ok, no place for c.

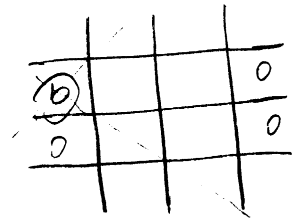
If b in (1,3), then $\begin{array}{c} a(1,4) \\ \downarrow \\ c(1,2) \end{array}$

ok.

Similar for b in (4,2), (4,3), ... we can find that b is always in between a and c.

Suppose in the initial configuration, b is a number between a and c on a column, b(i,j), a(i-1,j), c(i+1,j), i=2,3, j=1,2,3,4.

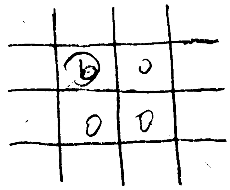
① 

② 

Similar in row case, we can find b is always in between a and c.

Suppose in the initial configuration, b is a number between a and c on diagonal, b(i,j), a(i-1,j-1), c(i+1,j+1), i=2,3, j=2,3.

or a(i+1,j-1), c(i-1,j+1).



Since the structure of table is symmetric, the situation of positions are similar in (2,2) (2,3) (3,2) (3,3).

W.L.O.G. we consider b in position (2,2).

Consider two diagonal directions of b, after reshuffle, b could be in position (2,2) (2,3) (3,2) (3,3).

If b in (2,2), then a(1,1) → c(3,3); a(3,1) → c(1,3)

↓

Not ok, the same as original a, c.

If b in (2,3), then

a(1,4) → c(3,2) → ok.

a(1,2) → c(3,4) → ok.

a(3,4) → c(1,2) → ok

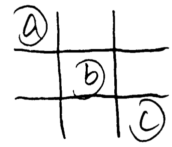
a(4,1) → × Not ok, no place for c.

Similar for b in (3,2), (3,3). we can find that b is always in between a and c.

By considering all three cases, we can find that the conjecture is true when n=4.

(2) ① when $n=3$, in this case, b can only be in center.

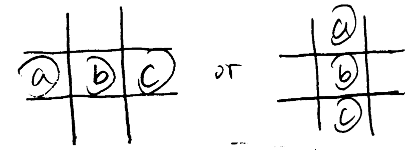
If a, b, c in diagonal direction,



after reshuffle, a(1,3) c(3,1) or a(3,1) c(1,3) or a(3,3) c(1,1)

If a, b, c in vertical, or horizontal direction.

we can also find b is in between a and c after reshuffle.



a(1,2) c(3,2) or a(3,2) c(1,2) or ...

The conjecture is true when $n=3$.

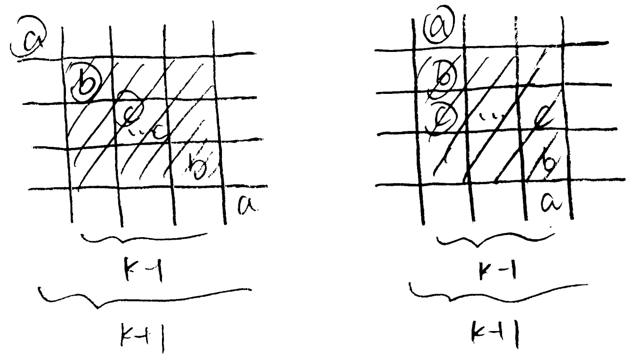
② By (1), the conjecture is true when $n=4$.

③ Suppose the conjecture is true when $n=t$, $t\in\{3,4,\dots k\}$.

Want to show it is true when $n=t+1$. (i.e. $n=k+1$).

When $n=k+1$, the inner part of $(k+1)\times(k+1)$ table is the same as $(k-1)\times(k-1)$ table.

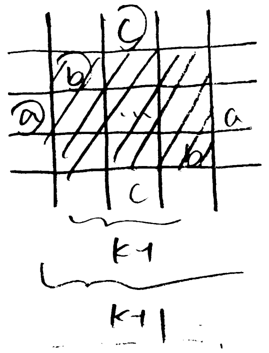
If b, c are in $(k-1)\times(k-1)$ table, after reshuffle, since $n=k$ is true, then b and c are always adjacent. then a can only be in certain place which enables b in between a and c.



a(k+1, k+1) c(k-1, k-1) or... a(k+1, k) c(k-1, k) or ...

Then b is in between a and c.

If only b is in $(k-1)\times(k-1)$ table, after reshuffle, Since b



can only be in corner of inner table,

then a and c are in adjacent of b.

in diagonal direction.

Then b is in between a and c.

$a(k, k+1)$

$c(k+1, k)$

or ...

The conjecture is true when $n=t+1$ for $t\in\{3,4,\dots k\}$.

Thus, by induction, the conjecture is true for any $n\times n$ table.