STA4001: Lecture 8

Stationary distribution, Irreducibility

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Gambler's ruin problem

- Consider a DTMC with state space $S = \{0, 1, 2, 3, 4\}$ and transition probabilities

$$P_{00} = P_{44} = 1, P_{i,i+1} = .2, \quad P_{i,i-1} = .8.$$

- States 0 and 4 are absorbing states.
- Compute the probability that starting from state 3, the DTMC is eventually absorbed into state 0.

The first-step method

- Let P_i be the probability that starting from state i, the DTMC eventually is absorbed into state 0.
- First step method:

$$P_3 = .8P_2 + .2(0) \tag{1}$$
$$P_2 = .8P_1 + .2P_3 \tag{2}$$
$$P_1 = .8 + .2P_2 \tag{3}$$

- In vector form,

$$\begin{pmatrix} 1 & -.2 & 0 \\ -.8 & 1 & -.2 \\ 0 & -.8 & 1 \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix} = \begin{pmatrix} 0.8 \\ 0 \\ 0 \end{pmatrix}, \Rightarrow \begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix} = \begin{pmatrix} 0.98824 \\ 0.94118 \\ 0.75294 \end{pmatrix}$$

Expected time to end the game

- N_i is the expected number of steps to end the game starting from state i, $i = 1, 2, 3$.

$$
\begin{aligned}
N_3 &= 1 + .2(0) + (.8)N_2 \\
N_1 &= 1 + (.2)N_1 + .8(0) \\
N_2 &=
\end{aligned}
$$

- In vector form,

$$
\begin{pmatrix} 1 & -.2 & 0 \\ -.8 & 1 & -.2 \\ 0 & -.8 & 1 \end{pmatrix} \begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \Rightarrow \begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix} = \begin{pmatrix} 1.5882 \\ 2.9412 \\ 3.3529 \end{pmatrix}.
$$

Stationary distribution

- $\pi = (\pi_i, i \in S)$ is a stationary distribution of a DTMC with state space S and transition matrix P if

$$\begin{aligned} &\pi = \pi P, \\ &\pi \geq 0, \\ &\sum_{i \in S} \pi_i = 1. \end{aligned}$$

- Examples

$$P = \begin{pmatrix} .2 & .8 \\ .5 & .5 \end{pmatrix} \Rightarrow \pi = \left(\frac{5}{13}, \frac{8}{13} \right).$$

Examples

- Simple, symmetric random walk on a circle (4 nodes)
- Simple, symmetric random walk on a circle (3 nodes)
- X_n is the ending inventory at the end of week n using $(s, S) = (2, 3)$ inventory policy with demand distribution

d	0	1	2	3
$\mathbb{P}\{D = d\}$	.1	.4	.3	.2

$$P = \begin{pmatrix} .2 & .3 & .4 & .1 \\ .2 & .3 & .4 & .1 \\ .5 & .4 & .1 & 0 \\ .2 & .3 & .4 & .1 \end{pmatrix}$$

$$\pi_0 = \frac{38}{130}, \quad \pi_1 = \frac{43}{130}, \quad \pi_2 = \frac{40}{130}, \quad \pi_3 = \frac{9}{130} \approx .0692.$$

Cost structure in the inventory model

- Holding cost for each item left by the end of a Friday is \$100.
- Variable cost (C_v) is \$1000.
- Fixed cost (C_f) is \$1500.
- Each item sells \$2000, C_p.
- Let $g(i)$ be the expected profit of the following week, given that this week's inventory ends with i items.

$$\begin{aligned} g(0) =& -\text{Cost} + \text{Revenue} \\ =& [-3(\$1000) - \$1500] + [3(\$2000)(.2) + 2(\$2000)(.3) + \\ & + 1(\$2000)(.4) + 0(.1)] = -\$1300 \\ g(2) =& [-2(\$100)] + [(\$0)(.1) + (\$2000)(.4) + (\$4000)(.3 + .2)] \\ =& \$2600 \\ g(1) =& -\$400, \quad g(3) = \$2900. \end{aligned}$$

Long-run average profit per week

- Long-run average profit per week is

$$\$488.39$$

- Connection to stationary distribution?

Irreducible DTMC

Definition

(a) State i reaches state j there exists a path

$$i_0 = i, i_1, i_2, \ldots, i_n = j$$

such that $P_{i_k, i_{k+1}} > 0$ for $k = 0, \ldots, n-1$.

(b) A DTMC is said to be irreducible if every pair of states i and j reach each other.

(c) i and j communicate if i reaches j and j reaches i.

Communications

Lemma

(a) State $i \to j$ *if there exists an integer* $n \geq 1$ *such that* $P_{ij}^n > 0$.
(b) States $i \to j$ *and* $j \to k$ *imply that* $i \to k$.

Stationary distribution (uniqueness)

Theorem (Theorem 1)

If a DTMC is irreducible, it has at most one stationary distribution.

- Proof:

Recurrence and Transience

$$T_i = \min\{n \geq 1 | X_n = i\} = \text{first time to reach } i$$

Definition

A state i of a DTMC is recurrent if

$$\mathbb{P}\{T_i < \infty | X_0 = i\} = 1;$$

otherwise, state i is transient.

$$N_i = \text{number of visits to state } i = \sum_{n=1}^{\infty} 1_{\{X_n = i\}}.$$

Theorem

- *If state i is recurrent, then $\mathbb{E}[N_i | X_0 = i] = \infty$.*
- *If state i is transient, then $\mathbb{E}[N_i | X_0 = i] < \infty$.*

Stopping times

- Let $X = \{X_n : n = 0, 1, \dots\}$ be a DTMC on space space S.
- A $\{0, 1, \dots\} \cup \{\infty\}$-valued random variable is called a *stopping time of* the DTMC if the event $\{T = n\}$ depends only on X_0,X_1, ..., X_n for $n = 0$, ...
- For a set $A \subset S$, the first passage time to A,

$$T_A = \inf\{n \geq 1 : X_n \in A\}.$$

- $T_i = T_{\{i\}}$.
- Last passage time

$$L_A = \text{ the last time to visit } A.$$