MAT3253 Complex Variables
Lecture Notes

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Contents

This is a set of notes for MAT3253 Complex variables. The followings are the reference books.

References

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[BrownChurchill] J. W. Brown and R. V. Churchill, *Complex Variables and Applications*, 8th edition, McGraw-Hill, New York, 2009.

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[Rudin2] W. Rudin, *Real and Complex Analysis*, 3rd edition, McGraw-Hill, New York, 1987.

[Ahlfors] Ahlfors, *Complex Analysis – An Introduction to the Theory of Analytic Functions of One Complex Variable*, 3rd edition, McGraw-Hill, New York, 1979.

1 Lecture 1 (Complex numbers)

Summary:

- Construction of complex field using pairs of real numbers.
- Construction of complex field using 2×2 matrices.

Complex numbers were first invented to solve algebraic equations. As a vector space, the set of complex numbers is an extension of the real numbers of dimension 2. It is also equipped with a multiplication operator that extends the multiplication of real numbers.

We start with the axioms of a complex field.

Definition 1.1. A number system $(F, +, \cdot)$ is called a *field* if

1. (closed) $a + b \in F$ for all $a, b \in F$.

2. (associative) $(a+b)+c = a+(b+c)$, for all $a, b, c \in F$.

3. (commutative) $a+b = b+a$, for all $a, b \in F$.

4. (existence of zero) $\exists 0 \in F$ such that $0+a = a+0 = a$, for all $a \in F$.

5. (additive inverse) for all $a \in F$, $\exists a' \in F$ such that $a + a' = 0$.

6. (closed) $a \cdot b \in F$ for all $a, b \in F$.

7. (associative) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$, for all $a, b, c \in F$.

8. (commutative) $a \cdot b = b \cdot a$, for all $a, b \in F$.

9. (existence of one) $\exists 1 \in F$ such that $1 \cdot a = a \cdot 1 = a$, for all $a \in F$.

10. (multiplicative inverse) for all $a \in F \setminus \{0\}$, $\exists a'' \in F$ such that $a \cdot a'' = 1$.

11. (distributive) $a \cdot (b+c) = a \cdot b + a \cdot c$, for all $a, b, c \in F$.

A subset K of a field F is called a *subfield* of F if the elements in K satisfy the all axioms of a field, and F is called an *extension* of K. Examples of fields include the rational numbers $\mathbb{Q}$ and the real numbers $\mathbb{R}$. A field F containing $\mathbb{R}$ as a subfield and a special element I that satisfies $I^2 + 1 = 0$ is called a *complex field*. Complex field is denoted by $\mathbb{C}$.

Using this terminology, we say that the complex field is obtained by extending $\mathbb{R}$ so that the equation $x^2 + 1 = 0$ has a solution.

Remark. The definition of complex field above is not 100% accurate. To be precise, we should say that $\mathbb{C}$ is *generated* by the real numbers in $\mathbb{R}$ and the special number I. The term “generated” roughly means that all numbers in $\mathbb{C}$ can be obtained from “mixing” real numbers and the number I by addition, subtraction, multiplication and division.

Remark. In general, a field needs not be infinite. (None of the axioms require that there are infinitely many elements in a field.) We can construct number systems consisting of finitely many elements satisfying the axioms of field. To construct an example of a field of size 3, we can label the elements by 0, 1, 2, and define the addition and multiplication by the following tables

$+$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$\cdot$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

The addition and multiplication are addition and multiplication modulo 3.

The special number I in a complex number is usually called the *imaginary unit*. However, the calculations with complex numbers is very concrete and not imaginary. We provide two constructions of complex field below. In the first construction a complex number is a pair of real numbers. In the second one a complex number is a 2×2 matrix over the real numbers.

Construction of complex field (I)

Let

$$F_1 \triangleq \{(a,b) : \ a, b \in \mathbb{R}\}. \tag{1.1}$$

A "complex number" is thus regarded as a point on a plane, called the *complex plane* or *Argand plane*. The addition and multiplication operators are defined by

$$\begin{aligned}(a,b) + (c,d) &\triangleq (a+b,\ c+d), \\ (a,b) \cdot (c,d) &\triangleq (ac-bd,\ ad+bc).\end{aligned}$$

The additive and multiplicative identities are $(0,0)$ and $(1,0)$, respectively. The real numbers are embedded in F_1 by $x \mapsto (x,0)$. Real-number calculation can be carried out in F_1. By identifying x_1 with $(x_1,0)$ and x_2 with $(x_2,0)$, the sum and product of x_1 and x_2 are respectively

$$\begin{aligned}(x_1,0) + (x_2,0) &= (x_1+x_2,0), \ \text{and} \\ (x_1,0) \cdot (x_2,0) &= (x_1x_2,0).\end{aligned}$$

The special number I in this representation is $I = (0,1)$. We can check that

$$I^2 = (0,1) \cdot (0,1) = ((0)(0) - (1)(1),\ (0)(1) + (1)(0)) = (1,0).$$

F_1 is therefore a complex field.

Construction of complex field (II)

Let

$$F_2 \triangleq \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} : a, b \in \mathbb{R} \right\}. \tag{1.2}$$

Addition and multiplication are performed using the usual matrix addition and multiplication. The additive and multiplicative identities are the zero matrix and identity matrix, respectively. The "imaginary unit" I is $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. The field F_2 contains $\mathbb{R}$ as a subfield because the subset

$$\left\{ \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} : x \in \mathbb{R} \right\} \tag{1.3}$$

can be identified with the set of real numbers. Real numbers are represented as diagonal matrices with equal diagonal entries. The matrix addition and multiplication reduces to real-number addition and multiplication when restricted to matrices in (1.3),

$$\begin{bmatrix} x_1 & 0 \\ 0 & x_1 \end{bmatrix} + \begin{bmatrix} x_2 & 0 \\ 0 & x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 & 0 \\ 0 & x_1 + x_2 \end{bmatrix}$$
$$\begin{bmatrix} x_1 & 0 \\ 0 & x_1 \end{bmatrix} \cdot \begin{bmatrix} x_2 & 0 \\ 0 & x_2 \end{bmatrix} = \begin{bmatrix} x_1 x_2 & 0 \\ 0 & x_1 x_2 \end{bmatrix}.$$

We can use F_2 as a numerical model for calculating complex numbers.

The two constructions are essentially the same (meaning that F_1 and F_2 are isomorphic). The first construction emphasizes that a complex number is a pair of real numbers. The second construction emphasizes that complex multiplication is the same as multiplying by matrix in a special form. One can check that

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} ac - bd & -(ad + bc) \\ ad + bc & ac - bd \end{bmatrix}.$$

We will write $a + bi$ as a notation for (a, b) or $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. Using the $a + bi$ notation, the multiplication of complex numbers can be written as

$$(a + bi)(c + di) = ac - bd + i(ad + bc).$$

The complex numbers as a vector space has dimension 2 over $\mathbb{R}$. We can pick 1 and i as a basis. Using the first construction method, a complex number (x, y) can be written as

$$(x, y) = x(1, 0) + y(0, 1),$$

with $(1, 0)$ and $(0, 1)$ serving as the standard basis vectors. If we using the second construction method, we can use $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ as a basis,

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

2 Lecture 2 (Basic notions and operations in $\mathbb{C}$)

Summary

- Complex conjugate, modulus
- Complex division
- Polar form of complex numbers, DeMoivre formula

Definition 2.1. For a complex number $z = a + bi$ in $\mathbb{C}$, define the *real* and *imaginary part* of z as

$$\mathrm{Re}(z) \triangleq a \quad \text{and} \quad \mathrm{Im}(z) \triangleq b.$$

Define the *complex conjugate* of z by

$$\bar{z} \triangleq z^* \triangleq a - bi.$$

The *modulus* of z is defined as

$$|z| = \sqrt{a^2 + b^2}.$$

The modulus of z is also called the *absolute value* or the *radius*.

Geometrically, the complex conjugate of z is the reflection of z along the real axis. The modulus is the distance between the origin and the point z in the complex plane. The next proposition says that the complex conjugate, as a mapping from $\mathbb{C}$ to $\mathbb{C}$, is compatible with complex addition and multiplication.

Proposition 2.2.

(i) $(z^*)^* = z$ *for any* $z \in \mathbb{C}$.

(ii) Given any two complex numbers z_1 and z_2 in $\mathbb{C}$,

$$(z_1 + z_2)^* = z_1^* + z_2^* \quad \textit{and} \quad (z_1 z_2)^* = z_1^* z_2^*.$$

Part (ii) in Prop. 2.2 says that the reflection of the sum (resp. product) of two complex numbers is the same as the sum (resp. product) of the two points obtained by reflection. The proof is simple and is omitted. Using Prop. 2.2, we can show that the modulus is a multiplicative function.

Proposition 2.3. *For any two complex numbers $z_1, z_2 \in \mathbb{C}$, $|z_1 z_2| = |z_1|\,|z_2|$.*

Proof. Use the fact that $|z|^2 = z\bar{z}$ for any $z \in \mathbb{C}$, and complex multiplication is commutative

$$|z_1 z_2|^2 = (z_1 z_2)(z_1 z_2)^* = z_1 z_2 z_1^* z_2^* = z_1 z_1^* z_2 z_2^* = |z_1|^2\,|z_2|^2.$$

□

The relationship between the real part, imaginary part and complex conjugate are

$$\mathrm{Re}(z) = \frac{z + z^*}{2} \quad \text{and} \quad \mathrm{Im}(z) = \frac{z - z^*}{2i}. \tag{2.1}$$

We can use complex conjugate to perform division in complex numbers. Suppose we want to divide $z_1 = a + bi$ by $z_2 = c + di$, where c and d are not zero. We multiply and divide by the conjugate of z_2,

$$\frac{z_1}{z_2} = \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di} = \frac{ac + bd}{c^2 + d^2} + i\frac{bc - ad}{c^2 + d^2}. \tag{2.2}$$

We can also do complex division using the 2×2 representation of complex numbers. Division is the same as taking matrix inverse,

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & -d \\ d & c \end{bmatrix}^{-1} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & d \\ -d & c \end{bmatrix} \frac{1}{c^2 + d^2} = \frac{1}{c^2 + d^2} \begin{bmatrix} ac + bd & ad - bc \\ bc - ad & ac + bd \end{bmatrix}.$$

The answer is the same as in (2.2). We note that $c^2 + d^2$ is the determinant of the matrix $\begin{bmatrix} c & -d \\ d & c \end{bmatrix}$ and is the same as the square of the absolute value of $c + di$.

Definition 2.4. The *argument* of a nonzero complex number z is defined as the angle from the positive real axis to the straight line from 0 to z. We write $\arg(z)$ to denote the argument function. The argument of $z = 0$ is not defined. The argument of a nonzero complex number is defined only up to integral multiples of 2π.

Definition 2.5. The points in the complex plane with modulus equal to 1 is called the *unit circle*.

A complex number $z = x + iy$ can be written in polar form

$$z = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta),$$

where r is the modulus of z and θ is an argument of z. Note that $\cos\theta + i\sin\theta$ lies on the unit circle for any θ. There are more than one way to write a complex number in polar form, because we can always add $2\pi k$ to θ, for any integer k, and get the same point on the complex plane.

Using the polar form, complex multiplication can calculated in terms of the modulus and the argument.

Proposition 2.6. *Given $z_1 = r_1\cos\theta_1 + ir_1\sin\theta_1$ and $z_2 = r_2\cos\theta_2 + ir_2\sin\theta_2$ in polar form, their product can be computed by*

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)].$$

Proof. The proof follows from the definition of complex multiplication and basic trigonometric identities,

$$\begin{aligned} z_1 z_2 &= r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2\big[\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)\big] \\ &= r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]. \end{aligned}$$

□

The polar form suggests that the operation of complex multiplication can be decomposed into two parts. Using the second construction of complex numbers (1.2), the 2×2 matrix corresponding to a complex number $a + bi$ can be factorized as

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = r \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

where $r = \sqrt{a^2 + b^2}$ and $\theta = \arg(a + bi)$. The matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is a rotation matrix. The matrix-vector product

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}$$

is the point obtained by rotating (x, y) counter-clockwise by angle θ. The geometric meaning of multiplication by $a + bi = r(\cos\theta + i\sin\theta)$ is thus, (i) first rotate by θ counter-clockwise, then (ii) scale up (or down) by a factor of r.

Using the polar form, complex conjugate and complex division are computed by

$$(r\cos\theta + ir\sin\theta)^* = (r\cos(-\theta) + ir\sin(-\theta))$$
$$(r_1\cos\theta_1 + ir_1\sin\theta_1)/(r_2\cos\theta_2 + ir_2\sin\theta_2) = (r_1/r_2)(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)),$$

provided that $r_2 \neq 0$.

Theorem 2.7 (DeMoivre formula)**.** *For any $n \in \mathbb{Z}$ and $\theta \in \mathbb{R}$, we have*

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta). \tag{2.3}$$

Proof. The formula is obviously true when $n = 1$. When $n = 2$, it follows directly from Prop. 2.6,

$$(\cos\theta + i\sin\theta)^2 = \cos(\theta + \theta) + i\sin(\theta + \theta) = \cos(2\theta) + i\sin(2\theta).$$

We apply mathematical induction to establish (2.3) for all positive integers n and for all real numbers θ.

For negative n, we first note that

$$\begin{aligned}(\cos\theta + i\sin\theta)^{-1} &= \frac{1}{\cos\theta + i\sin\theta} \\ &= \frac{1}{\cos\theta + i\sin\theta} \cdot \frac{\cos\theta - i\sin\theta}{\cos\theta - i\sin\theta} \\ &= \cos\theta - i\sin\theta \\ &= \cos(-\theta) + i\sin(-\theta).\end{aligned}$$

Hence for positive integer m, we have

$$\begin{aligned}(\cos\theta + i\sin\theta)^{-m} &= ((\cos\theta + i\sin\theta)^{-1})^m \\ &= (\cos(-\theta) + i\sin(-\theta))^m \\ &= \cos(-m\theta) + i\sin(-m\theta).\end{aligned}$$

$\square$

Using the matrix representation of complex numbers, the DeMoivre's formula can be stated as

$$\begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}^n = \begin{bmatrix}\cos(n\theta) & -\sin(n\theta) \\ \sin(n\theta) & \cos(n\theta)\end{bmatrix}.$$

Geometrically speaking, this says that rotating n times by an angle θ is the same as rotating once by angle $n\theta$.

Example 2.1. Compute $(-1 + i\sqrt{3})^8$.

The complex number $-1 + i\sqrt{3}$ in polar form is

$$2(-1/2 + i\sqrt{3}/2) = 2(\cos(2\pi/3) + i\sin(2\pi/3)).$$

Hence, with the use of DeMoivre's formula, we get

$$(-1 + i\sqrt{3})^8 = 2^8(\cos(8\cdot 2\pi/3) + i\sin(8\cdot 2\pi/3)) = 256(\cos(4\pi/3) + i\sin(4\pi/3)).$$

In Cartesian form, the answer is $128(-1 - i\sqrt{3})$.

Example 2.2. Compute $(-1 + i)^{20}$.

Express $-1 + i$ in polar form $\sqrt{2}(\cos(3\pi/4) + i\sin(3\pi/4))$. By DeMoivre's formula,

$$\begin{aligned}(-1 + i)^{20} &= 20^{20/2}(\cos(20\cdot 3\pi/4) + i\sin(20\cdot 3\pi/4)) \\ &= 1024(\cos\pi + i\sin\pi) \\ &= -1024.\end{aligned}$$

Example 2.3. Express $\sin(5\theta)$ as a polynomial in $\sin(\theta)$.

By DeMoivre's formula,

$$\begin{aligned}&(\cos(5\theta) + i\sin(5\theta)) = (\cos\theta + i\sin\theta)^5 \\ &= \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta + 5\cos^4\theta\sin\theta + i\sin^5\theta.\end{aligned}$$

Equating the imaginary parts, we obtain

$$\begin{aligned}\sin(5\theta) &= 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta \\ &= 5(1-\sin^2\theta)^2\sin\theta - 10(1-\sin^2\theta)\theta\sin^3\theta + \sin^5\theta \\ &= 5\sin\theta - 10\sin^3\theta + 5\sin^5\theta - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta \\ &= 5\sin\theta - 20\sin^3\theta + 16\sin^5\theta.\end{aligned}$$

3 Lecture 3 (n-th roots of complex number)

Summary

- Complex division
- Principal argument
- Extracting the n-th roots of a complex number

Dividing a complex number $a+ib$ by $c+di$, where a, b, c, and d are arbitrary real numbers, means finding a complex number $w = x+iy$ such that $(x+iy)(c+di) = a+bi$. The problem can be reduced to a system of linear equations. By equating real and imaginary partsin

$$(x+iy)(c+di) = (cx-dy) + i(xd+yc) = a+bi,$$

we obtain

$$\begin{bmatrix} c & -d \\ d & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}.$$

The solution can be obtained by multiplying both sides by the inverse of $\begin{bmatrix} c & -d \\ d & c \end{bmatrix}$,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{c^2+d^2} \begin{bmatrix} c & d \\ -d & c \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

A faster method is to apply the following trick using complex conjugate

$$\frac{a+bi}{c+di} = \frac{a+bi}{c+di} \cdot \frac{c-di}{c-di} = \frac{(ac+bd)+i(bc-ad)}{c^2+d^2}.$$

Definition 3.1. Given a nonzero complex number z, the *principal argument* of z is the *unique* angle θ_0 (in radian) in $(-\pi, \pi]$ such that $z = |z|(\cos(\theta_0) + i \sin(\theta_0))$.

We follow the notation in [BrownChurchill] and denote the principal argument of z by $\text{Arg}(z)$. In general, the argument function is multi-valued; it can be equal to $\text{Arg}(z) + 2k\pi$ for any integer k.

Example 3.1. Compute the square roots of $4 + i$.

Write $4+i$ as $\sqrt{17}(\cos\phi + i \sin\phi)$, where $\phi = \tan^{-1}(1/4)$. A complex number w is called a square root of $4 + i$ if $w^2 = 4 + i$. By DeMoivre's formula, the modulus of w must be $\sqrt{\sqrt{17}}$. If we denote the argument of w by θ, then $2\theta = \phi + 2k\pi$ for some integer k. This gives

$$\theta = \frac{\phi}{2} + k\pi,$$

and we can take $k = 0, 1$ (because adding an integral multiple of 2π to the argument gives the same complex number.) We can write the answer as

$$\sqrt{4+i} = (17)^{1/4}(\cos(\phi/2 + k\pi) + i \sin(\phi/2 + k\pi)), \qquad \text{for } k = 0, 1,$$

or

$$\sqrt{4+i} = \pm(17)^{1/4}(\cos(\phi/2) + i \sin(\phi/2)).$$

Example 3.2. Compute the cube roots of unit.

Method 1. It amounts to solving $z^3 - 1 = 0$. After factorizing the polynomial into

$$(z-1)(z^2 + z + 1) = 0,$$

the solutions are 1, and the two roots of $z^2 + z + 1$, namely $-\frac{1}{2} \pm i\sqrt{\frac{3}{2}}$.

Method 2. We find all complex numbers with unit modulus and argument θ such that $3\theta = 0$. There are three possible values for θ, and they are 0, $2\pi/3$ and $-2\pi/3$. The cube roots of unity are

$$1, \; \cos(2\pi/3) + i \sin(2\pi/3), \; \cos(2\pi/3) - i \sin(2\pi/3).$$

Example 3.3. Compute the cube roots of i.

The principal argument of i is $\pi/2$. We want to find the values of θ such that

$$3\theta = \frac{\pi}{2} + 2\pi k, \qquad \text{for } k \in \mathbb{Z}.$$

There are three choices for θ, namely, $\pi/6 + 2\pi k/3$, for $k = 0, 1, 2$. The cube roots of i are

$$\cos(\frac{\pi}{6} + \frac{2\pi k}{3}) + i\sin(\frac{\pi}{6} + \frac{2\pi k}{3}).$$

for $k = 0, 1, 2$.

In general, there are n solutions when taking the n-th root of a nonzero number. The method is the same as in the above examples. If we plot the n solutions in the complex plane, they form a regular n-gon with the origin as the center.

4 Lecture 4 (Complex plane as metric space and topological space)

Summary

- Point at infinity
- Concepts from metric space

In a 3-dimensional space, identify the points in the horizontal plane as the complex numbers. A point in the 3-D space has coordinates (ξ, η, ζ). A complex number $x + iy$ is thus located at $(x, y, 0)$. Put a sphere of radius 1 on the x-y plane, touching the x-y plane at the origin. The equation of the sphere is

$$\xi^2 + \eta^2 + (\zeta - \frac{1}{2})^2 = (\frac{1}{2})^2.$$

If we draw a straight line connecting $(0, 0, 1)$ and a point $(x, y, 0)$ on the horizontal plane, there is a unique intersection point on the sphere. This gives a one-to-one correspondence between the points on the complex plane and the points on the sphere, except the north pole. This mapping is called the *stereographical projection*. The totality of all complex numbers can be represented as the points on a punctured sphere. The picture can be completed by adjoining an extra point to the complex numbers.

Definition 4.1. The *extended complex numbers* as a set is defined as $\mathbb{C} \cup \{\infty\}$, where ∞ is a symbol called the *point at infinity*. The symbol ∞ corresponds to the north pole in the stereographic projection. The sphere in the stereographical projection is called the *Riemann sphere*.

Remark. The importance of the Riemann sphere is that it is a compact set, and compact set has nice topological properties.

In this lecture we study complex numbers as points on a metric space, with the metric induced by the complex absolute value; the distance between two complex numbers z_1 and z_2 is $|z_1 - z_2|$. We readily check that the triangular inequality is satisfied.

Proposition 4.2 (Triangle inequality)**.**

$$|z_1 + z_2| \le |z_1| + |z_2|$$

for any two complex numbers z_1 and z_2.

Proof. Take the square of the left-hand side,

$$\begin{aligned} |z_1 + z_2|^2 &= (z_1 + z_2)(z_1^* + z_2^*) \\ &= |z_1|^2 + 2\mathrm{Re}(z_1 z_2^*) + |z_2|^2 \\ &\le |z_1|^2 + 2|\mathrm{Re}(z_1 z_2^*)| + |z_2|^2. \end{aligned}$$

It is sufficient to prove

$$|\mathrm{Re}(z_1 z_2^*)| \le |z_1|\,|z_2|, \tag{4.1}$$

because it will immediately give $|z_1 + z_2|^2 \le (|z_1|^2 + |z_2|)^2$.

To prove (4.1), suppose $z_1 = a + bi$ and $z_2 = c + di$, and write $\mathrm{Re}(z_1 z_2^*) = ac + bd$. We want to prove $(ac + bd)^2 \le (a^2 + b^2)(c^2 + d^2)$. This inequality holds for any real numbers a, b, c and d because

$$(a^2 + b^2)(c^2 + d^2) - (ac + bd)^2 = (ad - bc)^2 \ge 0.$$

□

Notation: a *sequence* of complex numbers $z_1, z_2, z_3, \dots$ is denoted by $(z_k)_{k=1}^{\infty}$ or $\{z_k\}$.

Definition 4.3. Given a complex sequence $(z_n)_{n=1}^{\infty}$, we say that z_n *converges* to w if

$$\forall \epsilon > 0\ \exists N,\ s.t.\ |z_n - w| < \epsilon,\ \forall n \ge N.$$

It is equivalent to requiring that $(|z_n - w|)_{n=1}^{\infty}$ as a real sequence is converging to 0 as $n \to \infty$. We write $z_n \to w$ if z_n converges to w, and

$$\lim_{n\to\infty} z_n = w.$$

Example 4.1.

$$\frac{1}{n} + \frac{i}{n^2} \to 0.$$

$$(0.5)^n(\cos n + i \sin n) \to 0.$$

Example 4.2. Compute $\lim_{n\to\infty} \frac{n}{n+i}$.

We can first make a guess that the limit should be 1, because when n is large, adding i to n has negligible effect. To make the argument rigorous, we write

$$\frac{n}{n+i} = \frac{n+i-i}{n+i} = 1 - \frac{i}{n+i}.$$

Then

$$\left|\frac{n}{n+i} - 1\right| = \left|\frac{i}{n+i}\right| = \frac{1}{\sqrt{n^2+1}} \to 0, \qquad \text{as } n \to \infty.$$

Therefore $n/(n+i) \to 1$ as $n \to \infty$.

Example 4.3. The sequence $((2i)^n)_{n=1}^{\infty}$ does not converge to any complex number in $\mathbb{C}$. However, if we look at the projection of $(2i)^n$ on the Riemann sphere, it is converging to the point at infinity ∞. Hence we can say that $(2i)^n \to \infty$.

More generally, we say that a sequence of complex numbers $(z_n)_{n=1}^{\infty}$ *converges to the point at infinity* if $z_n^{-1} \to 0$.

Definition 4.4. A sequence $(z_k)_{k=1}^{\infty}$ is called a *Cauchy sequence* if for all $\epsilon > 0$, there exists an integer N such that

$$|z_m - z_n| \le \epsilon \quad \text{whenever } m, n \ge N.$$

The basic property of Cauchy sequence for real numbers extends to the complex case.

Theorem 4.5. *A complex sequence $(z_k)_{k=1}^{\infty}$ converges if and only if $(z_k)_{k=1}^{\infty}$ is Cauchy.*

We have the following relationship between convergence of complex sequence and real sequences.

Theorem 4.6. *A complex sequence $(z_k)_{k=1}^{\infty}$ converges if and only if both $(\mathrm{Re}(z_k))_{k=1}^{\infty}$ and $(\mathrm{Im}(z_k))_{k=1}^{\infty}$ converge.*

The notion of infinite series for complex numbers is the same as in calculus.

Definition 4.7. An infinite series of complex numbers $\sum_{k=1}^{\infty} z_k$ *converges* if the sequence of partial sums

$$\Big(\sum_{k=1}^{n} z_k\Big)_{n=1}^{\infty}$$

is convergent.

Proposition 4.8. *Given a sequence of complex numbers $(z_k)_{k=1}^{\infty}$, if the real infinite series $\sum_{k=1}^{\infty} |z_k|$ converges, then $\sum_{k=1}^{\infty} z_k$ also converges.*

The proof can be done by consider the real and imaginary parts of z_k, and reduced to the real case.

Definition 4.9. We say that a series $(z_k)_{k=1}^{\infty}$ is *absolutely convergent* if $\sum_{k=1}^{\infty} |z_k|$ is convergent.

Example 4.4. (Complex geometric series) Evaluate $\sum_{k=1}^{\infty}(0.5i)^k$.

We can check that this is absolutely convergent. Because $|0.5i| = 0.5$,

$$\sum_{k=1}^{\infty} |0.5i|^k = \sum_{k=1}^{\infty} (0.5)^k$$

is a geometric series with common ratio strictly less than 1, and hence is convergent.

For any finite n, we have

$$\sum_{k=1}^{n} (0.5i)^k = \frac{(0.5i)^{n+1} - 0.5i}{0.5i - 1}.$$

We take limit as $n \to \infty$,

$$\sum_{k=1}^{\infty} (0.5i)^k = \lim_{n\to\infty} \frac{(0.5i)^{n+1} - 0.5i}{0.5i - 1} = \frac{-0.5i}{0.5i - 1} = \frac{-1 + 2i}{5}.$$

Definition 4.10. An *open disc* centered at z_0 with radius r is defined as

$$D(z_o; r) \triangleq \{z \in \mathbb{C} : |z - z_0| < r\}.$$

A *circle* centered at z_0 with radius r is

$$C(z_o; r) \triangleq \{z \in \mathbb{C} : |z - z_0| \leq r\}.$$

A set S in the complex plane is said to be *open* if for any $z \in S$, we can find $\delta > 0$ such that $D(z; \delta_0) \subseteq S$.

It can be shown that an open disc is indeed open.

Definition 4.11. The *boundary* of a set S, denoted by ∂S, is defined as

$$\{z \in \mathbb{C} : \forall \delta > 0,\ D(z;\delta) \cap S \neq \emptyset \text{ and } D(z;\delta) \cap S^c \neq \emptyset\}.$$

A set is said to be a *closed set* if the complement is open.

A set is *bounded* if it is contained in $D(0;M)$ for some larger M.

A set is *compact* if it is closed and bounded.

5 Lecture 5 (Complex function)

Summary

- Domain of a function
- Continuous function
- Complex differential function

In complex analysis, a *domain/region* is an open and connected set in $\mathbb{C}$.

In MAT3253, we can understand "connected" as "path-connected", i.e., any two points in the set are connected by a path. The following is a useful fact for two-dimensional region.

Proposition 5.1. *Suppose R is an open set, and A and B are points in R that are connected by a path, then there exists a polygonal path from A to B with finitely many linear parts.*

The above proposition means that when we consider connectedness, it is sufficient to consider piece-wise linear paths.

Definition 5.2. A function $f : \mathbb{C} \to \mathbb{C}$ is said to be *continuous at z_0* if for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$|z - z_0| < \delta \ \Rightarrow\ |f(z) - f(z_0)| < \epsilon.$$

A function f is said to be *continuous* in a domain D if f is continuous at every point in D.

By consider the real and imaginary part separately, we can prove the following

Theorem 5.3. *A complex function f is continuous if and only if the real and imaginary parts are continuous.*

Example 5.1. Show that $f(z) = 1/z$ is continuous in the domain $\mathbb{C} \setminus \{0\})$.

Suppose $z = x + iy$ and $z \neq 0$.

$$\frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$$

The real part of $f(z)$ is $x/(x^2+y^2)$ and the imaginary part is $-y/(x^2+y^2)$. Both of them are continuous functions in the domain $\mathbb{C} \setminus \{0\})$. Hence $f(z)$ is continuous by the previous theorem.

A complex function $f(z)$ can be interpreted as a two-dimensional vector field,

$$f(x+iy) = u(x,y) + iv(x,y).$$

When we say that $f(x+iy)$ is *real differentiable*, we mean that the vector-valued function $(u(x,y), v(x,y))$ is differentiable as in multivariable calculus. By the definition of differentiability, if $(u(x,y), v(x,y)$ is differentiable at a point (x_0, y_0), we can approximate the effect of a small change in x and y by linear function,

$$\begin{bmatrix} u(x_0+\Delta x, y_0+\Delta y) \\ v(x_0+\Delta x, y_0+\Delta y) \end{bmatrix} \approx \begin{bmatrix} u(x_0,y_0) \\ v(x_0,y_0) \end{bmatrix} + \begin{bmatrix} u_x(x_0,y_0) & u_y(x_0,y_0) \\ v_x(x_0,y_0) & v_y(x_0,y_0) \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}. \tag{5.1}$$

The entries in the 2×2 matrix are the partial derivatives of u and v evaluated at (x_0, y_0). The symbol "$\approx$" means that the higher-order terms are converging to zero faster than the linear term. More precisely, it means that the limit

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\|\text{Difference between L.H.S and R.H.S. of (5.1)}\|}{\sqrt{\Delta x^2 + \Delta y^2}} = 0.$$

Example 5.2. The function $f(z)$ defined by $x^2 + i(x+y)$ is real differentiable. Partial derivatives of the real part $u(x,y) = x^2$ and imaginary part $v(x,y) = x + y$ exist, and we have

$$\begin{bmatrix} (x+\Delta x)^2 \\ x + \Delta x + y + \Delta y \end{bmatrix} \approx \begin{bmatrix} x^2 \\ x+y \end{bmatrix} + \begin{bmatrix} 2x & 0 \\ y & x \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} \tag{5.2}$$

for any x and y.

Definition 5.4. A complex function f is said to be *complex differentiable at z_0* if the limit

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta) - f(z_0)}{\Delta z} \tag{5.3}$$

exists. This is equivalent to requiring that

$$f(z_0 + \Delta z) \approx f(z_0) + w_0 \Delta z,$$

were w_0 is a complex constant and is the limit in (5.3). The limit in (5.3) is denoted by $f'(z_0)$.

We can understand this by interpreting complex multiplication as matrix multiplication. If a function f is complex differentiable at a point z_0, then the 2×2 matrix in (5.1) must be in the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, so that we can realize the matrix multiplication in (5.1) by complex multiplication.

Example 5.3. Consider the function

$$f(z) = z^3 = (x + iy)^3 = (x^3 - 3xy^2) + i(3x^2 y - y^3).$$

The real and imaginary parts are $u(x, y) = x^3 - 3xy^2$ and $v(x, y) = 3x^2 y - y^3$, respectively. Suppose we fix a base point $(x_0, y_0) = (1, 1)$. The linear approximation in (5.1) at $(1, 1)$ can be written as

$$\begin{bmatrix} u(1 + \Delta x, 1 + \Delta y) \\ v(1 + \Delta x, 1 + \Delta y) \end{bmatrix} \approx \begin{bmatrix} -2 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 & -6 \\ 6 & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}.$$

For general $z = x + iy$, the 2×2 derivative matrix is

$$\begin{bmatrix} 3x^2 - 3y^2 & -6xy \\ 6xy & 3x^2 - 3y^2 \end{bmatrix}.$$

We can use complex arithemtic to realize the linear approximation

$$f(z + \Delta x) = f(z) + (3x^2 - 3y^2 + i(6xy)) \cdot \Delta z,$$

and the complex derivative turns out to be equal to $3x^2 - 3y^2 + i(6xy) = 3z^2$.

The function in Example 5.2 is real differentiable everywhere but not complex differentiable in general. In fact it is complex differentiable only at $(x, y) = (0, 0)$. However, the function in Example 5.3 is complex differentiable at all points in $\mathbb{C}$, and the complex derivative is $3z^2$.

6 Lecture 6 (Analytic functions)

Summary

- Cauchy-Riemann equation
- Definition of analytic function

We recall two basic results from multivariable calculus.

Theorem 6.1. *Suppose $\vec{f}(x,y) = (u(x,y), v(x,y))$ be a two-dimensional vector field.*

1. *A necessary condition for $\vec{f}$ to be real differentiable at a point (x_0, y_0) is that all partial derivatives u_x, u_y, v_x and v_y exists in a neighborhood of (x_0, y_0).*
2. *A sufficient condition for $\vec{f}$ to be real differentiable at (x_0, y_0) is (i) partial derivatives u_x, u_y, v_x and v_y exists in a neighborhood of (x_0, y_0), and (ii) the partial derivatives u_x, u_y, v_x are continuous at (x_0, y_0).*

We first derive an important necessary condition for complex differentiability.

Theorem 6.2 (Cauchy-Riemann equations)**.** *Suppose $f(x+iy) = u(x,y) + iv(x,y)$ is complex differentiable at z_0 (see Definition 5.4), where z_0 is in the domain of $f(z)$. Then*

$$u_x = v_y, \quad \textit{and} \quad v_y = -v_x.$$

Proof. The limit in computing complex derivative does not depend on how we approach the point z_0. We can approach z_0 horizontally or vertically, and the results must be the same if the function is complex differentiable.

Let $\Delta z = \Delta x$ and take $\Delta x \to 0$.

$$\begin{aligned}
&\lim_{\Delta x \to 0} \frac{u(x_0+\Delta x, y_0) + iv(x_0+\Delta x, y_0) - u(x_0,y_0) - iv(x_0,y_0)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{u(x_0+\Delta x, y_0) - u(x_0,y_0)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x_0+\Delta x, y_0) - v(x_0,y_0)}{\Delta x} \\
&= \frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}(x_0,y_0).
\end{aligned}$$

Next suppose $\Delta z = i\Delta y$ and take $\Delta y \to 0$.

$$\begin{aligned}
&\lim_{\Delta y \to 0} \frac{u(x_0, y_0 + \Delta y) + iv(x_0, y_0 + \Delta y) - u(x_0, y_0) - iv(x_0, y_0)}{i\Delta y} \\
&= \lim_{\Delta y \to 0} \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{i\Delta y} + i \lim_{\Delta y \to 0} \frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{i\Delta y} \\
&= -i\frac{\partial u}{\partial x}(x_0, y_0) + \frac{\partial v}{\partial x}(x_0, y_0).
\end{aligned}$$

By equating real and imaginary parts of the two limits, we get $u_x(x_0, y_0) = v_y(x_0, y_0)$ and $u_y(x_0, y_0) = -v_x(x_0, y_0)$. □

Similar to Theorem 6.1, we have the following sufficient condition for complex differentiability.

Theorem 6.3. *A complex function f is complex differentiable at z_0 if*

1. *The partial derivatives u_x, u_y, v_x, v_y exists in a neighborhood of z_0.*
2. *Cauchy-Riemann equations are satisfied at z_0.*
3. *u_x, u_y, v_x, v_y are continuous at z_0.*

If the above conditions hold, the complex derivative of f is given by $u_x + iv_x = v_y - iu_y$.

Example 6.1. The function $f(z) = az + b$, for any $a, b \in \mathbb{C}$ is complex differentiable at any $z \in \mathbb{C}$. The complex derivative is $f'(z) = a$.

Example 6.2. The conjugate function $f(z) = z^*$ is not complex differentiable anywhere. It is because $u_x = 1$ and $v_y = -1$, and $u_x \neq v_y$ at any point in $\mathbb{C}$.

Example 6.3. Consider the square function $f(z) = z^2$. The real and imaginary parts are $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$, respectively. We check that the partial derivatives

$$u_x = 2x,\ u_y = -2y,\ v_x = 2y,\ v_y = 2x$$

exist and are continuous at every point in $\mathbb{C}$. This check conditions 1 and 3 in Theorem 6.3. Furthermore, the Cauchy-Riemann equalities are satisfied everywhere, because

$$u_x = v_y = 2x, \quad \text{and} \quad u_y = -v_x = -2y.$$

By Thoeorem 6.3, $f(z) = z^2$ is differentiable, and the complex derivative is $u_x + iv_x = 2z$.

Example 6.4. The function $f(z) = |z|^2 = x^2 + y^2$ has zero imaginary part. As a real-valued function it is real differentiable. However it is complex differentiable only at $z = 0$. We see this by computing the partial derivatives

$$\begin{aligned} u_x &= 2x, \qquad v_x = 0, \\ u_y &= 2y, \qquad v_y = 0. \end{aligned}$$

The Cauchy-Riemann equations are satisfied only at $z = 0$. Therefore it is not complex differentible if $z \neq 0$. By Theorem 6.3, it is indeed complex differentiable at $z = 0$.

Example 6.5. The function $f(z) = 1/z$ is defined in the domain $\mathbb{C} \setminus \{0\}$. It is complex differentiable everywhere in the domain because, for $z \neq 0$,

$$\begin{aligned} \frac{\frac{1}{z+h} - \frac{1}{z}}{h} &= \frac{1}{h}\Big(\frac{z-(z+h)}{(z+h)z}\Big) \\ &= -\frac{1}{z(z+h)}. \end{aligned}$$

When $h \to 0$, the limit of $-\frac{1}{z(z+h)}$ is $-1/z^2$. Therefore $f(z) = 1/z$ is complex differentiable for $z \in \mathbb{C} \setminus \{0\}$, and the complex derivative is $-1/z^2$.

The function in Example 6.4 is complex differentiable only at one time, and is considered as pathological. The main theorems in complex analysis usually require that the function is complex differentiable in a domain (the interior is nonempty).

Definition 6.4. A function f is said to be *analytic*/*holomorphic*/*regular* at a point z_0 if there is a neighborhood of z_0 such that f is complex differentiable at every point in the neighborhood. A function is said to be *entire* if it is complex differentiable at every point in $\mathbb{C}$.

For example, the function in Example 6.1 and 6.3 is entire. The function $1/z$ in Example 6.5 is analytic in the domain of definition. Example 6.2 and 6.4 are not analytic anywhere.

7 Lecture 7 (Conformal property, Harmonic conjugate)

Summary

- Angle-preserving property of analytic functions
- Harmonic functions
- Harmonic conjugate

Complex differentiable functions are very special. This lecture investigates two such special properties.

Conformal property

Suppose $f(z)$ is complex differentiable at a given point z_0 in $\mathbb{C}$, and suppose $f'(z_0)$ is nonzero. The function f is *conformal*, or *angle-reserving*. It is based on the fact that multiplication by a nonzero complex constant can be interpreted geometrically as a rotation.

Draw two parametric curves $\gamma_1(t)$ and $\gamma_2(t)$ through z_0. By "parametric curve" we means a smooth map from an interval in $\mathbb{R}$ to to $\mathbb{C}$. Let the range of the parameter t in $\gamma_1(t)$ be (a, b), where $a < 0 < b$, and $\gamma_1(0) = z_0$. Likewise, let the range of t in $\gamma_2(t)$ be (a', b'), where $a' < 0 < b'$ and $\gamma_2(0) = z_0$. In the domain of f, the angle between the lines from $\gamma_1(0)$ to $\gamma_1(\Delta t)$ and the line from $\gamma_2(0)$ to $\gamma_2(\Delta 2)$ is

$$\arg\Big(\frac{\gamma_2(\Delta t) - \gamma_2(0)}{\gamma_1(\Delta t) - \gamma_1(0)}\Big),$$

which converges to

$$\arg(\gamma_2'(0)/\gamma_1'(0))$$

as $\Delta t \to 0$. In the range of f, the angle between the lines from $f(\gamma_1(0))$ to $f(\gamma_1(\Delta t))$ and from $f(\gamma_2(0))$ to $f(\gamma_2(\Delta t))$ is

$$\arg\Big(\frac{f(\gamma_2(\Delta t)) - f(\gamma_2(0))}{f(\gamma_1(\Delta t)) - f(\gamma_1(0))}\Big).$$

Since f is assumed to be complex differentiable, when we take limit as Δt approaches 0, and write the limit as

$$\lim_{\Delta t \to 0} \arg\Big(\frac{f'(z_0)\cdot(\gamma_2(\Delta t) - \gamma_2(0))}{f'(z_0)\cdot(\gamma_1(\Delta t) - \gamma_1(0))}\Big).$$

Because $f'(z_0)$ is non-zero, the limit is the same as $\arg(\gamma_2'(0)/\gamma_1'(0))$. Therefore the angle between $\gamma_1(t)$ and $\gamma_2(t)$ at the point z_0 is the same as the angle between the images $f(\gamma_2(t))$ and $f(\gamma_1(t))$ at the point $f(z_0)$. If we draw some perpendicular grid lines in the domain of $f(z)$, then the images of these lines will intersect at 90 degrees.

The conformal property explains why the conjugate function $f(z) = \bar{z}$ is not complex differentiable anywhere. The conjugate function is a reflection geometrically, and reflection reverses the orientation of angles.

Harmonic functions

Definition 7.1. A function $u(x, y)$ in two variables is called a *harmonic function* if it satisfies the Laplace equation

$$u_{xx} + u_{yy} = 0.$$

Proposition 7.2. *Write $z = x + iy$ and suppose $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain. If $u(x, y)$ and $v(x, y)$ are twice differentiable, the second-order partial derivatives are continuous, then the $u(x, y)$ and $v(x, y)$ are harmonic functions.*

Proof. If $f(z)$ is analytic, then it satisfies the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$. Take partial derivatives again, we obtain $u_{xx} = v_{yx}$ and $u_{yy} = -v_{xy}$. Since we assume the order of the partial derivatives can be exchanged,

$$u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0.$$

Similarly, we can show that $v(x, y)$ is harmonic. $\square$

Remark. The assumption that $u(x, y)$ and $v(x, y)$ are twice differentiable can be relaxed. We will show that this conditions are automatically satisfied if $f(z)$ is analytic.

In the rest of this lecture we assume that all functions are in C^2, i.e., the second partial derivatives exist and are continuous.

Definition 7.3. Given a harmonic function $u(x, y)$, a function $v(x, y)$ is called a *harmonic conjugate* of $u(x, y)$ if $u(x, y)$ and $v(x, y)$ satisfy the Cauchy-Riemann equations.

Suppose $u(x, y)$ is harmonic function in a simply connected domain D, then we can find a harmonic conjugate of $u(x, y)$ using path integral. The differential form

$$v_x dx + v_y dy = -u_y dx + u_x dy$$

is exact, because

$$\frac{\partial}{\partial x} u_x - \frac{\partial}{\partial y}(-u_y) = u_{xx} + u_{yy} = 0.$$

By Green's theorem, the line integral

$$\int -u_y dx + u_x dy$$

only depends on the start point and the end point of the path. We can define a function $v(x,y)$ by first fixing a point (x_0, y_0) in the domain D, and let

$$v(\tilde{x}, \tilde{y}) = \int_{(x_0,y_0)}^{(\tilde{x},\tilde{y})} -u_y dx + u_x dy \tag{7.1}$$

for any point $(\tilde{x}, \tilde{y})$ in D. This is well-defined because the line integral is independent of path. (Here $\tilde{x}$ and $\tilde{y}$ are fixed real constants, and x and y are the dummy variables used in the integral.)

We can check that the function $v(x,y)$ so defined is a harmonic conjugate of $u(x,y)$. Let $(\tilde{x}, \tilde{y})$ be any point in D. The partial derivative of v with respect to x at this point is

$$\begin{aligned}
v_x(\tilde{x}, \tilde{y}) &= \lim_{\Delta x \to 0} \frac{v_x(\tilde{x} + \Delta x, \tilde{y}) - v_x(\tilde{x}, \tilde{y})}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_{(\tilde{x},\tilde{y})}^{(\tilde{x}+\Delta x,\tilde{y})} -u_y dx + u_x dy \\
&= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_{(\tilde{x},\tilde{y})}^{(\tilde{x}+\Delta x,\tilde{y})} -u_y dx \\
&= -u_y(\tilde{x}, \tilde{y}).
\end{aligned}$$

In the last step we have used the assumption that u_y is continuous at $(\tilde{x}, \tilde{y})$.

Likewise, by consider the partial derivative of v with respect to y, we get

$$\begin{aligned}
v_y(\tilde{x}, \tilde{y}) &= \lim_{\Delta y \to 0} \frac{v_x(\tilde{x}, \tilde{y} + \Delta y) - v_x(\tilde{x}, \tilde{y})}{\Delta y} \\
&= \lim_{\Delta y \to 0} \frac{1}{\Delta y} \int_{(\tilde{x},\tilde{y})}^{(\tilde{x},\tilde{y}+\Delta y)} -u_y dx + u_x dy \\
&= \lim_{\Delta y \to 0} \frac{1}{\Delta y} \int_{(\tilde{x},\tilde{y})}^{(\tilde{x},\tilde{y}+\Delta y)} u_x dy \\
&= u_x(\tilde{x}, \tilde{y}).
\end{aligned}$$

Again, we have used the continuity of u_x in the last step.

This proves that the Cauchy-Riemann equations are satisfied. In summary, we can conclude that a harmonic conjugate of $u(x, y)$ can be written as in (7.1) when D is simply connected. We note that $v(x, y)$ is defined up to a constant, because we can always add an integration constant to (7.1).

Example 7.1. Find a harmonic conjugate of the function

$$u(x, y) = -2x^2 + x^3 + 2y^2 - 3xy^2.$$

The function $u(x, y)$ is defined on the whole complex plane. We check that it is harmonic:

$$\begin{aligned} u_x &= -4x + 3x^2 - 3y^2 \\ u_{xx} &= -4 + 6x \\ u_y &= 4y - 6xy \\ u_{yy} &= 4 - 6x. \end{aligned}$$

Hence $u_{xx} + u_{yy}$ is identically equal to zero.

<u>METHOD 1</u> Using path integral, we can obtain the function v by

$$v(\tilde{x}, \tilde{y}) := \int_{(0,0)}^{(\tilde{x},\tilde{y})} (6xy - 4y)dx + (-4x + 3x^2 - 3y^2)dy.$$

for any $(\tilde{x}, \tilde{y}) in \mathbb{R}^2$. To calculate the path integral, we can simply take the direct path from $(0, 0)$ to $(\tilde{x}, \tilde{y})$. Parametrized this path by

$$\begin{cases} x = t\tilde{x} \\ y = t\tilde{y} \end{cases}$$

for $0 \leq t \leq 1$. We then calculate the path integral

$$\begin{aligned} v(\tilde{x}, \tilde{y}) &= \int_0^1 (6t^2\tilde{x}\tilde{y} - 4t\tilde{y})\tilde{x} + (-4t\tilde{x} + 3t^2\tilde{x}^2 - 3t^2\tilde{y}^2)\tilde{y}\, dt \\ &= \int_0^1 -8\tilde{x}\tilde{y}t + (9\tilde{x}^2\tilde{y} - 3\tilde{y}^3)t^2 \\ &= -4\tilde{x}\tilde{y} + 3\tilde{x}^2\tilde{y} - \tilde{y}^3 + C \end{aligned}$$

where C is a constant.

METHOD 2. The second method is ad hoc and is the same as in Calculus II. We first integrate

$$v_x = -u_y = -4y + 6xy$$

with respect to x and get

$$v(x,y) = \int -4y + 6xy\,dx = -4xy + 3x^2y + C(y),$$

where $C(y)$ is a constant that may involve y. Differentiate the above with respect to y,

$$v_y = -4x + 3x^2 + C'(y).$$

After comparing with u_x, we see that $C'(y) = -3y^2$, and hence $C(y) = -y^3 + C$ for some constant C. The answer is

$$v(x,y) = -4xy + 3x^2y - y^3 + C.$$

8 Lecture 8 (Complex exponential function)

Summary

- Complex exponential function (first definition)
- Complex log function
- Complex powers

We want an analytic function that extends the real exponential function. By "extension" we means that if we restrict the input a real number to this complex function, the output is the same as the result obtained from the real exponential function.

We start with a function $u(x,y) = e^x\cos(y)$. It is easily check that this function is harmonic, and $u(x,0) = e^x$. Using the method of path integral, we consider the path integral

$$\int_{(0,0)}^{(\tilde{x},\tilde{y})} -u_y dx + u_x dy = \int_{(0,0)}^{(\tilde{x},\tilde{y})} (e^x \sin y)dx + (e^x\cos y)dy.$$

This path integral is conservative, because we can find a "potential function"

$$v(x,y) = e^x \sin y$$

such that

$$v_x = e^x \sin y, \quad \text{and} \quad v_y = e^x \cos y.$$

Therefore

$$u(x,y) + iv(x,y) = e^x \cos y + ie^x \sin y$$

is an analytic function, and it is defined and analytic on the whole complex plane.

Definition 8.1. We define the *complex exponential function* by

$$\exp(z) \triangleq e^x(\cos y + i \sin y).$$

We often write e^z as a short-hand notation.

Definition 8.1 is adopted in [BakNewman] and [BrownChurchill] as the definition of complex exponential function.

We note that the complex exponential function has a complex period $2\pi i$; that is, for any $z \in \mathbb{C}$,

$$e^{z+2\pi k i} = e^z, \quad \text{for } k \in \mathbb{Z}.$$

As a result, the inverse function of e^z is multi-valued. Given a complex number $r(\cos\theta + i\sin\theta)$ in polar form, the complex log function of $r(\cos\theta + i\sin\theta)$ can take values

$$\log r + i(\theta + 2\pi k)$$

for $k \in \mathbb{Z}$. More formally we have the following

Definition 8.2. For nonzero $w \in \mathbb{C}$, the *complex log function* is defined as

$$\log(w) = \log|w| + i(\arg(w) + 2\pi k)$$

where $k = 0,\ \pm 1,\ \pm 2, \ldots$ Here the function $\arg(w)$ is the multi-valued argument function. If we take the principal argument, we have a uniquely defined function

$$\log(w) = \log|w| + i\mathrm{Arg}(w),$$

and it is called the *principal complex log function*.

Example 8.1. Compute 2^i by calculating $e^{i\log 2}$.

$$\begin{aligned}\exp(i\log 2) &= \exp(i(\log 2 + i2\pi k)) \\ &= \exp(-2\pi k + i\log 2) \\ &= e^{-2\pi k}(\cos(\log 2) + i\sin(\log 2)),\end{aligned}$$

where k can take any integer as its value.

Example 8.2. Compute i^i by calculating $e^{i\log i}$.

$$\begin{aligned}\exp(i\log i) &= \exp(i(\log 1 + i(\frac{\pi}{2} + 2\pi k))) \\ &= \exp(-(\frac{\pi}{2} + 2\pi k)),\end{aligned}$$

where $k \in \mathbb{Z}$. If we take $k = 0$, then we can say that the principal value of i^i is $e^{-\pi/2}$, which is a real number.

If we want to define the log function as an analytic function, we have to make a branch cut. Usually we take the negative real axis

$$\{(x, 0) : x \leq 0\}$$

as the branch cut. We can see why we cannot define a complex log function on the whole complex plane by trying to get a harmonic conjugate of

$$u(x, y) = \log(\sqrt{x^2 + y^2}).$$

We can calculate

$$u_x = \frac{x}{x^2 + y^2} \quad \text{and } u_y = \frac{y}{x^2 + y^2}.$$

The path integral

$$\int -\frac{y}{x^2 + y^2}dx + \frac{x}{x^2 + y^2}dy$$

is not conservative for paths in the punctured plane $\mathbb{C} \setminus \{(0, 0)\}$.

If we take

$$D = \mathbb{C} \setminus \{(x, 0) : x \leq 0\}$$

as the domain, then we can define the complex log function as

$$Log(z) = \log(|z|) + i\text{Arg}(z).$$

where $\text{Arg}(z)$ denotes the principal argument function. The complex derivative of $Log(z)$ is

$$Log'(z) = u_x + iv_x = u_x - iu_y = \frac{x - iy}{x^2 + y^2} = \frac{1}{z}.$$

9 Lecture 9 (Euler's formula)

Summary

- Complex exponential function (second definition)
- Complex sine and cosine function
- Euler's formula

For real number x, the meaning of e^x can be defined in several ways.

- First define the constant e by $e = \lim_{n\to\infty}(1 + 1/n)^n$. For integer a, define e^a by $\underbrace{e \cdot e \cdots e}_{a \text{ factors}}$. For integer b, let $e^{1/b}$ be the number y such that $y^b = e$. For rational number a/b, define $e^{a/b}$ by $(e^a)^{(1/b)}$. For irrational number x, we approximate x by taking a sequence of rational numbers $(a_k/b_k)_{k=1}^{\infty}$ that converges to x, and defined e^x by
$$e^x \triangleq \lim_{k\to\infty} e^{a_k/b_k}.$$
- First define the log function by
$$\log(y) \triangleq \int_1^y \frac{1}{y}\, dy$$
for $y > 0$, and define e^x be the inverse function of $\log(y)$. That is, given x_0, let e^{x_0} be the number y_0 such that
$$x_0 = \int_1^{y_0} \frac{1}{y}\, dy.$$
The number y_0 is uniquely determined because the function $log(y)$ is monotonic.

- Given a real number x, define e^x by power series

$$e^x \triangleq \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

All three definitions are equivalent. We will use the third one to extend e^x to complex numbers. The definition of complex exponential function using power series is adopted in more advanced texts such as [Rudin2] and [Ahlfors].

Definition 9.1. Given a complex number z, define e^z by power series

$$e^z \triangleq \exp(z) \triangleq \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots .$$

It is obvious that when z is a real number the above definition reduces to the real exponential function. It is also easy to see $\exp(0) = 1$. However it is not obvious from this definition that e^z is never equal to zero.

Remark. In this lecture we first neglect all the convergence issue. Assume that the series are convergent for all z, and furthermore, assume that the convergence is absolute. This is analogous to the real power series. Moreover, we will assume in this lecture that the convergence is uniform, so that we can re-arrange the order of terms and compute derivative term-wise. We shall return to these questions in the next lecture.

We establish below a fundamental property of the function $\exp(z)$.

Theorem 9.2. *For any complex numbers z_1 and z_2,*

$$\exp(z_1 + z_2) = \exp(z_1)\exp(z_2).$$

Proof. We use a fact about product of infinite series: Given two absolutely convergent series $\sum_k a_k$ and $\sum_k b_k$, the product $(\sum_k a_k)(\sum_k b_k)$ is equal to $\sum_k c_k$, where c_k is defined by the convolution

$$c_k \triangleq a_0 b_k + a_1 b_{k-1} + a_2 b_{k-2} + \cdots + a_{k-1} b_1 + a_k b_0.$$

Apply this fact to $a_k = z_1^k/k!$ and $b_k = z_2^k/k!$. For $k \geq 0$, the coefficient c_k is

$$\begin{aligned} c_k &= \sum_{\ell=0}^{k} \frac{z_1^\ell}{\ell!} \frac{z_2^{k-\ell}}{(k-\ell)!} \\ &= \frac{1}{k!} \sum_{\ell=0}^{k} \binom{k}{\ell} z_1^k z_2^{k-\ell} \\ &= \frac{1}{k!} (z_1 + z_2)^k. \end{aligned}$$

Hence

$$\exp(z_1)\exp(z_2) = \sum_{n=1}^{\infty} \frac{(z_1+z_2)^n}{n!} \triangleq \exp(z_1+z_2).$$

□

Using this theorem we can derive some immediate corollaries.

Theorem 9.3. *For any $z \in \mathbb{C}$,*

- $e^{-z} = (e^z)^{-1}$*;*
- $e^z \neq 0$*;*
- $e^{nz} = (e^z)^n$ *for integer n.*

Proof. In the previous theorem, let $z_1 = z$ and $z_2 = -z$,

$$e^z e^{-z} = e^{z-z} = e^0 = 1.$$

Therefore $(e^z)^{-1} = e^{-z}$. We see that the value of e^z cannot equal to 0 because it is the reciprocal of some complex number.

For positive integer n, we can prove $e^{nz} = (e^z)^n$ by repeatedly applying the previous theorem. For negative integer n, we apply the first part in this theorem. □

Definition 9.4. Let the *complex sine* and *complex cosine function* be defined by

$$\begin{aligned} \sin(z) &\triangleq \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \\ \cos(z) &\triangleq \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}. \end{aligned}$$

Since the coefficients of the sine and cosine power series are the same as in the real case, $\sin(z)$ and $\cos(z)$ reduce to the real sine and cosine function when z is a real number. The main relationship between the exponential and the sinusoidal functions is recorded in the next theorem.

Theorem 9.5 (Euler's formula)**.**

$$e^{iz} = \cos(z) + i\sin(z). \tag{9.1}$$

Proof. The power series expansion of e^{iz} is

$$e^{iz} = 1 + iz - \frac{z^2}{2!} - \frac{iz^3}{3!} + \frac{z^4}{4!} + \frac{iz^5}{5!} - \frac{z^6}{6!} + \cdots$$

Assuming that we can exchange the order of adding the terms in the powers, we can separate the terms without i and the terms with i. By direct comparison, we can re-arrange the sum to $\cos(z) + i\cos(z)$. □

In particular when z is a real number ϕ, we have

$$e^{i\phi} = \cos\phi + i\sin\phi. \tag{9.2}$$

Using Theorem 9.5, we can express sin and cos in terms of exp.

Theorem 9.6. *For any* $z \in \mathbb{C}$*,*

$$\begin{aligned}\cos(z) &= \frac{e^{iz} + e^{iz}}{2}\\ \sin(z) &= \frac{e^{iz} - e^{iz}}{2i}.\end{aligned}$$

Proof. We use the properties that $\cos(z)$ is an even function and $\sin(z)$ is an odd function. This can be easily seen because the complex cosine function is even because all the terms in the power series that defines $\cos(z)$ in Def. 9.4 have even power. The terms in the power series that defines $\sin(z)$ all have odd powers. Hence

$$e^{-iz} = \cos(z) - i\sin(z) \tag{9.3}$$

By adding (9.3) to (9.1), we get

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

after some re-arrangement of terms. Similarly, by subtracting (9.3) from (9.1), we get

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}.$$

□

The result in Theorem 9.6 holds for any complex numbers. In particular, when we restrict z to a real number ϕ, we can get

$$\cos(\phi) = \frac{e^{i\phi} + e^{i\phi}}{2}$$
$$\sin(\phi) = \frac{e^{i\phi} - e^{i\phi}}{2i}.$$

We end this lecture by showing that the definition of exp by power series is the same as that given in previous lecture.

Theorem 9.7. *For any complex number $z = x + iy$,*

$$e^{x+iy} = e^x(\cos y + i \sin y).$$

Proof. Apply Theorem 9.2 to $z_1 = x$ and $z_2 = iy$,

$$e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y).$$

The last equality follows from (9.2). □

10 Lecture 10 (Convergence of complex power series)

<u>Summary</u>

- Ratio test
- Region of convergence

- Root test / Hadamard formula for radius of convergence

In Definition 9.1 and 9.4 we define complex exponential function and sinusoidal functions by power series,but the issue of convergence is neglected in Lecture 9. We discuss some convergence criteria in this lecture.

A power series centered at $z_0 \in \mathbb{C}$ has the form

$$\sum_{k=0}^{k} a_k (z - z_0)^k,$$

where a_k's are the coefficients. The coefficients may take complex values in general. For the ease of notation, we assume $z_0 = 0$ in this lecture.

Theorem 10.1 (Limit ratio test). *Consider a sequence of complex number $(b_k)_{k=1}^{\infty}$. Suppose*

$$\lim_{k\to\infty} \left| \frac{b_{k+1}}{b_k} \right| = L.$$

- *If $L > 1$, then $\sum_k b_k$ is divergent.*
- *If $L < 1$, then $\sum_k b_k$ is convergent.*
- *If $L = 1$, there is no conclusion.*

Proof. First suppose $L > 1$. There exists a sufficiently large N such that

$$\frac{|b_{k+1}|}{|b_k|} > 1$$

for all $k \geq N$. In particular, we have

$$|b_N| < |b_{N+1}| < |b_{N+2}| < \cdots$$

By the n-th term test, the series $\sum_k b_k$ must be divergent.

Suppose $L < 1$. Let M be any number such that $L < M < 1$. There exists a sufficiently large N such that

$$\frac{|b_{k+1}|}{|b_k|} < M$$

for all $k \geq N$. This gives

$$\begin{aligned}
|b_{N+1}| &< |b_N| M \\
|b_{N+2}| &< |b_{N+1}| M < |b_N| M^2 \\
\vdots \quad & \quad \vdots
\end{aligned}$$

In general, we have $|b_{N+k}| < |b_N| M^k$, for $k \geq 1$. By comparing with $\sum_{k=N}^{\infty} |b_N| M^{k-N}$, which is a convergent geometric series, the power series $\sum_{k=N}^{\infty} |b_k|$ is convergent. Hence $\sum_{k=0}^{\infty} b_k$ converges absolutely. $\square$

Example 10.1. $\sum_{n=0}^{\infty} n! z^n$ diverges for all $z \neq 0$, because for any $z \neq 0$, the ratio

$$\Big| \frac{(n+1)! z^{n+1}}{n! z^n} \Big| = (n+1)|z|$$

diverges to ∞ as $n \to \infty$.

Example 10.2. $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges for all $z \in \mathbb{C}$. We can apply the ratio test by calculating

$$\frac{\frac{z^n}{(n+1)!}}{\frac{z^n}{n!}} = \frac{|z|}{n+1}.$$

For any fixed $z \in \mathbb{C}$, this ratio approach 0 as $n \to \infty$. Hence it converges for any z.

Example 10.3. Consider the series

$$\sum_{n=0}^{\infty} \frac{1}{2n+1} z^n.$$

The ratio of the moduli of two consecutive terms is

$$\frac{2n-1}{2n+1}|z|.$$

It converge to a complex number with modulus strictly less 1 if and only if $|z| < 1$. Hence the series converges inside the unit circle.

Theorem 10.2. *Suppose $\sum_{k=0}^{\infty} a_k z^k$ converges at $z_1 \neq 0$, then it converges for all z with $|z| < |z_1|$.*

Proof. Fix $\epsilon > 0$. Since $|a_k z_1^k| \to 0$ as $k \to \infty$, there is a sufficiently large N such that

$$|a_k z_1^k| < \epsilon, \qquad \forall k \geq N.$$

Hence, for all $k \geq 0$, we have

$$|a_k z_1^k| \leq \max(\epsilon, |a_0|, |a_1 z_1|, |a_2 z_2^2|, \ldots, |a_{N-1} z_1^{N-1}|) \triangleq C.$$

For each k, re-write $|a_k z^k|$ as

$$|a_k z^k| = |a_k z_1^k| \frac{|z|^k}{|z_1|^k} \leq C\rho^k,$$

where ρ is defined as $\rho \triangleq |z/z_1| < 1$. Because $\sum_k C\rho^k$ is a convergent geometric series, we can apply comparison test and conclude that $\sum_k a_k z^k$ is convergent absolutely for $|z| < |z_1|$. □

Corollary 10.3. *If $\sum_k a_k z^k$ diverges at z_1, then $\sum_k a_k z^k$ diverges whenever $|z| > |z_1|$.*

Proof. We prove by contradiction. Suppose $\sum_k a_k z^k$ converges for some point $z = z_2$ with modulus $|z_2|$ strictly larger than $|z_1|$. By the previous theorem, the power series must converge (absolutely) at $z = z_1$. This contradicts the assumption that $\sum_k a_k z^k$ diverges at z_1. □

Radius of convergence

Given a power series $\sum_k a_k z^k$, there are only two logical possibilities: it diverges for all $z \neq 0$, or it converges for some $z \neq 0$. In the former case, we have divergence on the whole complex plane, and we do not need to consider it as far as convergence is concerned.

Suppose $\sum_k a_k z^k$ converges at some point $z_1 \in \mathbb{C}$ other than the origin. Let

$$R \triangleq \sup \Big\{ |z| : \sum_k a_k z^k \text{ converges } \Big\} \tag{10.1}$$

We have $R > 0$. By Theorem 10.2 and its corollary,

$$\sum_k a_k z^k \text{ converges whenever } |z| < R,$$
$$\sum_k a_k z^k \text{ diverges whenever } |z| > R.$$

Notation: If $\sum_k a_k z^k$ converges for all z, then we write $R = \infty$, and say that the radius of convergence is infinite. If $\sum_k a_k z^k$ converges only at $z = 0$, then $R = 0$.

Definition 10.4. Given a power series $\sum_{k=0}^{\infty} a_k (z - z_0)^k$ with center z_0, the value of R in (10.1) is called the *radius of convergence* of $\sum_{k=0}^{\infty} a_k (z - z_0)^k$. The region $|z - z_0| < R$ is called the *region of convergence*.

Example 10.4. The complex functions $\exp(z)$, $\sin(z)$ and $\cos(z)$ all have radius of convergence $R = \infty$. The radius of convergence in Example 10.3 is 1.

Theorem 10.5 (Hadamard formula for radius of convergence)**.** *Given a complex power series $\sum_{k=0}^{\infty} a_k z^k$, the radius of convergence can be computed by*

$$R = \frac{1}{\limsup_k |a_k|^{1/k}}.$$

Proof. Suppose z is a complex number with $|z| < R$. Pick a real number ρ between $|z|$ and R, i.e., $|z| < \rho < R$. By the definition of R in the theorem,

$$\frac{1}{\rho} > \limsup |a_k|^{1/k}.$$

By the property of limsup, there exists a sufficiently large N such that for all $k \geq N$,

$$|a_k|^{1/k} < 1/\rho \qquad \Rightarrow \qquad |a_k| < 1/\rho^k.$$

Apply this upper bound to get

$$|a_k z_k| < \frac{|z|^k}{\rho^k}.$$

Since $|z|^k/\rho^k < 1$, the geometric series $\sum_k |z|^k/\rho^k$ is convergent. By comparison test, $\sum_k a_k z^k$ is convergent.

Now suppose that $|z| > R$. Pick a real number ρ such that $|z| > \rho > R$. Taking reciprocal,

$$\frac{1}{\rho} < \limsup |a_k|^{1/k}.$$

By the defining property of limsup,

$$\frac{1}{\rho} < |a_k|^{1/k} \quad \text{for infinitely many } k.$$

Hence $|a_k z^k| > \left|\frac{z^k}{\rho^k}\right| > 1$ for infinitely many k. By the nth-term test, the power series diverges when $|z| > R$. □

As in real analysis, absolute convergence of an infinite series implies that we can rearranging the order of summation [Rudin1, Theorem 3.55]. The same is true for absolutely convergent complex series. We state this property formally below

Theorem 10.6. *Let $\sum_{k=0}^{\infty} w_k$ be an absolutely convergent complex power series. For any bijective mapping b from $\{0, 1, 2, \ldots\}$ to itself, the series $\sum_{k=0}^{\infty} c_{b(k)}(z-z_0)^{b(k)}$ obtained by re-arranging the order of summation is convergent and*

$$\sum_{k=0}^{\infty} c_{b(k)}(z-z_0)^{b(k)} = \sum_{k=0}^{\infty} c_k (z-z_0)^k.$$

Proof. The proof is basically the same for the real case and is omitted. □

We now give a complete proof of the Euler's formula in Theorem 9.5.

Proof of Theorem 9.5. In view of Theorem 10.6, we just need to verify that $\sum_n (iz)^n/n!$ is absolutely convergent. Given any $z \in \mathbb{C}$, let z_1 be any complex number with modulus strictly larger than $|z|$. Since the radius of convergence of the complex exponential function is infinity, $\sum_n z_1^n/n!$ is convergent. By Theorem 10.2, the series $\sum_n (iz)^n/n!$ converges absolutely, and hence we can re-arrange the order of summation in $\sum_n (iz)^n/n!$ to get

$$\begin{aligned}
e^{iz} &\triangleq \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} \\
&= \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!} + i \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!} \\
&= \cos(z) + i \sin(z).
\end{aligned}$$

□

11 Lecture 11 (Properties of power series)

Summary

- Uniform convergence
- The uniqueness of coefficients of power series
- Term-wise differentiation of power series
- Analyticity of power series

Definition 11.1. An infinite series of functions $\sum_{k=0}^{\infty} f_k(z)$ is said to converge *uniformly* to $g(z)$ in a region $R \subseteq \mathbb{C}$ if for all $\epsilon > 0$, there exists $N(\epsilon)$ such that

$$\Big|\sum_{k=0}^{n} f_k(z) - g(z)\Big| < \epsilon$$

for all $n \geq N(\epsilon)$ and for all $z \in R$. The number $N(\epsilon)$ could be a function of ϵ but should not depend on z.

The following theorem gives a sufficient condition for uniform convergence.

Theorem 11.2. *Suppose a complex power series $\sum_{k=0}^{\infty} c_k(z-z_0)^k$ converges absolutely at $z = z_1$, then $\sum_{k=0}^{\infty} c_k(z-z_0)^k$ converges uniformly in the closed disc $S: |z-z_0| \leq |z_1 - z_0|$.*

It is tacitly assumed that z_1 is in the interior of the region of convergence. Very often when we talk about uniform convergence, the region in which the function is converging uniformly is a compact set. We note that the region S in the theorem is compact.

Proof. Let $g(z) \triangleq \sum_k c_k(z-z_0)^k$ be the limit function. From Theorem 10.2 in Lecture 10, $g(z)$ converges (absolutely) in $|z-z_0| < |z_1 - z_0|$. We repeat the same proof idea to show that it is also convergent on the boundary of S.

$$\begin{aligned}\sum_{k=0}^{\infty} |c_k||z-z_0|^k &= \sum_{k=0}^{\infty} |c_k||z_1-z_0|^k \frac{|z-z_0|^k}{|z_1-z_0|^k} \\ &\leq \sum_{k=0}^{\infty} |c_k||z_1-z_0|^k.\end{aligned}$$

In the last step we use the assumption that $z \in S$. Since it is assumed that $\sum_k c_k(z-z_0)^k$ is absolutely convergent at $z = z_1$, the series $\sum_{k=0}^{\infty} |c_k||z_1 - z_0|^k$ is finite and hence by comparison test, $\sum_{k=0}^{\infty} c_k(z_1 - z_0)^k$ is absolutely convergent for all $z \in S$.

For $k \geq 0$, let $M_k \triangleq |c_k||z_1 - z_0|^k$. By assumption, $\sum_{k=0}^{\infty} M_k$ converges. Given any $\epsilon > 0$, there exists an integer $N(\epsilon)$ such that

$$\sum_{k=n+1}^{\infty} M_k \leq \epsilon, \qquad \text{for all } n \geq N(\epsilon).$$

So, for this choice of $N(\epsilon)$, we have

$$\begin{aligned}\Big|\sum_{k=0}^{\infty} c_k(z-z_0)^k - g(z)\Big| &= \Big|\sum_{k=n+1}^{\infty} c_k(z-z_0)^k\Big| \\ &\leq \sum_{k=n+1}^{\infty} |c_k||z-z_0|^k \\ &\leq \sum_{k=n+1}^{\infty} M_k \leq \epsilon\end{aligned}$$

for all $z \in S$. This proves that $\sum_k c_k(z-z_0)^k$ is uniformly convergent to $g(z)$ for $z \in S$. □

Similar to power series with real variable, complex power series has many nice properties, including

(i) term-wise differentiability from uniform convergence [Rudin1, Theorem 7.17],

(ii) exchanging the order of limit and summation [Rudin1, Theorem 7.11].

The proof of in the complex case is similar to the real case, and will not be repeated in the lecture note. In property (ii), Theorem 7.11 in [Rudin1] requires uniform convergence of power series. For term-wise differentiation, we need the following property.

Theorem 11.3. *Given a power series $\sum_n c_n(z-z_0)^n$, the series $\sum_n nc_n(z-z_0)^{n-1}$ obtained by term-wise differentiation has the same radius of convergence as $\sum_n c_n(z-z_0)^n$.*

Proof. By Theorem 10.5, the radius of convergence of $\sum_n c_n(z-z_0)^n$ is given by

$$R = \frac{1}{\limsup_n |c_n|^{1/n}}.$$

We want to show that the radius of convergence of $\sum_n nc_n(z-z_0)^{n-1}$ is R.

To simplify calculations, we note that the radius of convergence of $\sum_n nc_n(z-z_0)^{n-1}$ is the same as the radius of convergence of $\sum_n nc_n(z-z_0)^n$, and the latter can be computed by

$$\frac{1}{\limsup_n |nc_n|^{1/n}} = \frac{1}{\limsup_n \sqrt[n]{n}|c_n|^{1/n}} = \frac{1}{\limsup_n |c_n|^{1/n}} = R.$$

In the second equality, we have used the fact that $\lim_{n\to\infty} \sqrt[n]{n} = 1$. □

From Theorem 11.3 and Theorem 10.2, we obtain the following

Theorem 11.4 (Analyticity of power series)**.** *Let R be the radius of convergence of a complex power series $\sum_n c_n(z-z_0)^n$. The function $f(z)$ defined by the power series $\sum_n c_n(z-z_0)^n$ is complex differentiable at every point in the open disc $D(z_0;R)=\{z\in\mathbb{C}: |z-z_0|<R\}$, i.e., it is analytic in the region of convergence.*

Proof. Let ρ to be a positive real number strictly less than R (in case $R=\infty$, we just can pick any positive real number). Pick two complex numbers z_1 and z_2 such that

$$\rho = |z_1-z_0| < |z_2-z_0| < R.$$

By Theorem 11.3, the power series $\sum_n nc_n(z_2-z_0)^{n-1}$ converges (because z_2 is inside the region of convergence). By Theorem 10.2, $\sum_n nc_n(z_1-z_0)^{n-1}$ converges absolutely at $z=z_1$, and by Theorem 11.2, $\sum_n nc_n(z-z_0)^{n-1}$ converges uniformly in the $D(z_0,\rho)$. We can now adapt the proof of Theorem 7.17 of [Rudin1] to conclude that

$$f'(z)=\sum_{n=0}^{\infty} nc_n(z-z_0)^{n-1}. \tag{11.1}$$

Since this holds for any z in the open disc $D(z_0;\rho)$ and ρ can be arbitrarily close to R, (11.1) holds for all z in the open disc $D(z_0;R)$. □

By repeatedly apply Theorem 11.4, we can differentiate arbitrarily many times.

Corollary 11.5. *For any positive integer j, the function $f(z)$ defined by a power series $\sum_{n=0}^{\infty} c_n(z-z_0)^n$ can be differentiated j-th time. The j-th derivative is*

$$f^{(j)}(z)=\sum_{n=j}^{\infty} n(n-1)(n-2)\cdots(n-j+1)c_n(z-z_0)^{n-j},$$

and the region of convergence is the same as that of $\sum_{n=0}^{\infty} c_n(z-z_0)^n$.

Theorem 11.6 (Uniqueness of coefficients)**.** *Suppose $(z_j)_{j=1}^{\infty}$ is a sequence of complex numbers converging to z_0 (with $z_j\neq z_0$ for all j). If two power series $\sum_{k=0}^{\infty} a_k(z-z_0)$ and $\sum_{k=0}^{\infty} b_k(z-z_0)$ takes the same values at $z=z_j$, for $j=0,1,2,\ldots$, then $a_k=b_k$ for all k.*

Proof. By assumption we have $\sum_{k=0}^{\infty} a_k(z_j-z_0)$ and $\sum_{k=0}^{\infty} b_k(z_j-z_0)$ for $j=1,2,3,\ldots$.

Take limit as $j \to \infty$ and exchange the order of limit and summation,

$$\begin{aligned}\lim_{j\to\infty} \sum_{k=0}^{\infty} a_k (z_j - z_0)^k &= \lim_{j\to\infty} \sum_{k=0}^{\infty} b_k (z_j - z_0)^k \\ \sum_{k=0}^{\infty} a_k \lim_{j\to\infty} (z_j - z_0)^k &= \sum_{k=0}^{\infty} b_k \lim_{j\to\infty} (z_j - z_0)^k \\ \Rightarrow \qquad a_0 &= b_0.\end{aligned}$$

Subtract a_0 and b_0 from both sides, we get

$$(z_j - z_0) \sum_{k=1}^{\infty} a_k (z_j - z_0)^{k-1} = (z_j - z_0) \sum_{k=1}^{\infty} b_k (z_j - z_0)^{k-1}$$

for all $j = 1, 2, \ldots$. Since $z_j \neq z_0$ for all j, we can repeat the same argument as in the previous paragraph to

$$\sum_{k=0}^{\infty} a_{k+1} (z_j - z_0)^k = \sum_{k=0}^{\infty} b_{k+1} (z_j - z_0)^k$$

and obtain $a_1 = b_1$. The theorem can then be proved by induction. □

Corollary 11.7. *If $\sum_{k=0}^{\infty} a_k (z - z_0)^k = \sum_{k=0}^{\infty} b_k (z - z_0)^k$ for all z in an open disc $D(z_0; r)$ with center z_0 and radius r (where r is smaller than the radius of convergence), then $a_k = b_k$ for all k.*

Proof. For $j = 1, 2, 3, \ldots$, take z_j to be any complex number one the circle $C(z_0; r/j)$ with center z_0 and radius r/j, and apply the previous theorem. □

From this result we also see that a power series with nonzero coefficients and positive radius of convergence defines a nonzero analytic function.

Corollary 11.8. *If $\sum_{k=0}^{\infty} a_k (z - z_0)^k = 0$ for all z in an open disc $D(z_0; r)$ with center z_0 and radius r, then $a_k = 0$ for all k.*

We illustrate an application of the uniqueness of coefficients by solving a famous recurrence relation.

Example 11.1. Solve the recurrence relation

$$x_{n+2} = x_{n+1} + x_n$$

for $n \geq 0$ with a given initial condition $x_0 = 1$ and $x_1 = 1$. The solutions are the Fibonacci numbers $F_0 = 1$, $F_1 = 1$, $F_2 = 2$, $F_3 = 3$, $F_4 = 5$, $F_5 = 8, \ldots$. The purpose of this example is to find a closed form expression for the Fibonacci numbers, using the method of generating function.

Let $g(z)$ be a function defined by a power series whose coefficients are the Fibonacci numbers,

$$g(z) \triangleq \sum_{n=0}^{\infty} F_n z^n.$$

The radius of convergence is positive because F_n's grow slower than 2^n. Hence $g(1/2)$ is a convergent series. The radius of convergence is at least equal to $1/2$. From

$$\begin{aligned} g(z) &= F_0 + F_1 z + F_2 z^2 + F_3 z^3 + F_4 z^4 + \cdots \\ zg(z) &= \qquad\quad F_0 z + F_1 z^2 + F_2 z^3 + F_3 z^4 + \cdots \\ z^2 g(z) &= \qquad\qquad\qquad\quad F_0 z^2 + F_1 z^3 + F_2 z^4 + \cdots \end{aligned}$$

we can get

$$g(z) - zg(z) - z^2 g(z) = F_0 + (F_1 - F_0)z = 1.$$

Therefore, as a function, $g(z)$ can be computed by

$$g(z) = \frac{1}{1 - z - z^2}$$

for z in a sufficient small disk $D(0; r)$ (we can take $r = 1/2$ in this example).

We next apply the method of partial fraction to expand $g(z)$ as

$$g(z) = \frac{A}{1 - \phi z} + \frac{B}{1 - \mu z}$$

where A and B are some numbers, and ϕ and μ are the reciprocal roots of $1 - z - z^2$. It is more convenient to consider the "reciprocal polynomial" $w^2 - w - 1$. The roots are

$(1 \pm \sqrt{5})/2$. Let $\phi = (1+\sqrt{5})/2$ and $\mu = (1-\sqrt{5})/2$. Using linear algebra, we can find $A = \phi/\sqrt{5}$ and $B = -\mu/\sqrt{5}$ by solving a 2×2 linear system. We thus get

$$g(z) = \frac{\phi}{\sqrt{5}} \frac{1}{1-\phi z} - \frac{\mu}{\sqrt{5}} \frac{1}{1-\mu z}.$$

Expand both term by geometric series,

$$g(z) = \frac{\phi}{\sqrt{5}} \sum_{n \geq 0} \phi^n z^n - \frac{\mu}{\sqrt{5}} \sum_{n \geq 0} \mu^n z^n.$$

By the uniqueness of coefficients, we can equate the n-th coefficients with F_n,

$$F_n = \frac{1}{\sqrt{5}} \Big((\frac{1+\sqrt{5}}{2})^{n+1} - (\frac{1-\sqrt{5}}{2})^{n+1} \Big).$$

Since $\frac{1-\sqrt{5}}{2}$ has absolute value strictly less than 1, the second term converges to zero exponentially fast. We can see that the ratio of two consecutive Fibonacci numbers approaches the limit ϕ, which is the Golden ratio.