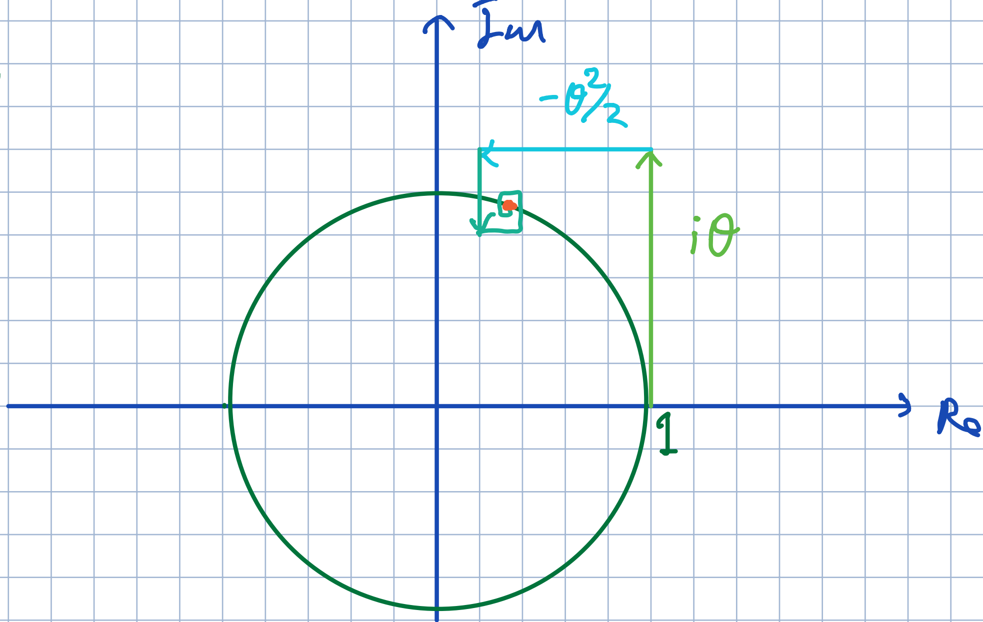
MAT 3253 Lecture 10

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad \theta \in \mathbb{R}$$

$$e^{i\theta} \triangleq 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots$$

$\theta > 0$



Power series $\displaystyle\sum_{k=0}^{\infty} a_k (z - z_0)^k$

$a_k \in \mathbb{C}$

z_0 is center.

take $z_0 = 0$ for simplicity.

Limit ratio test

Consider a sequence of complex no. $a_1, a_2, \ldots$

Assume $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = L$

if $L > 1$, then $\sum a_k$ is divergent

if $L < 1$, then $\sum a_k$ is convergent.

if $L = 1$, no conclusion.

Pf Suppose $L > 1$

$\exists N$, $\frac{|a_{k+1}|}{|a_k|} > 1 \quad \forall k \geq N$

$|a_N| < |a_{N+1}| < |a_{N+2}| < \cdots$

The nth-term test $\Rightarrow$ $\sum a_k$ is divergent.

Suppose $L < 1$.

Let M be a no. $L < M < 1$.

$\exists N$ s.t. $\frac{|a_{k+1}|}{|a_k|} < M < 1 \quad \forall k \geq N$

$|a_{N+1}| < |a_N| \cdot M$

$|a_{N+2}| < |a_{N+1}| \cdot M = |a_N| \cdot M^2$

$\vdots \qquad \vdots \qquad \vdots$

$\sum_{k=N}^{\infty} |a_k|$ converges by comparing with $\sum_{k=N}^{\infty} |a_k| \cdot M^{k-1}$

$\therefore \sum_{k=0}^{\infty} a_k$ converges absolutely $\blacksquare$

Example $\sum_{n=0}^{\infty} n!\, z^n$ diverges $\forall\ z \neq 0$

Fix $z \neq 0$ $\left| \dfrac{(n+1)!\ z^{n+1}}{n!\ \ z^n} \right| = (n+1)|z| \to \infty$

as $n \to \infty$

Example $\sum_{n=0}^{\infty} \dfrac{z^n}{n!}$ converges $\forall\ z \in \mathbb{C}$

Fix z $\dfrac{\frac{1}{(n+1)!} \cdot |z|^{n+1}}{\frac{1}{n!}\, |z|^n} = \dfrac{|z|}{n+1} \to 0$ as $n \to \infty$

$\therefore \sum_{n=0}^{\infty} \dfrac{z^n}{n!}$ converges absolutely.

Example $\sum_{n=0}^{\infty} \dfrac{1}{2n+1} \cdot z^n$

Fix z. $\dfrac{\frac{1}{2n+1}|z|^n}{\frac{1}{2n-1}|z|^{n-1}} = \dfrac{2n-1}{2n+1}|z| \to |z|$

It converges when $|z| < 1$.

Theorem Suppose $\sum_{k=0}^{\infty} a_k z^k$ converges at $z_1 \neq 0$

then it converges for all z with $|z| < |z_1|$

Proof $|a_k z_1^k| \to 0$ as $k \to \infty$

Fix $\varepsilon > 0$, $\exists N$ s.t. $|a_k z_1|^k < \varepsilon$

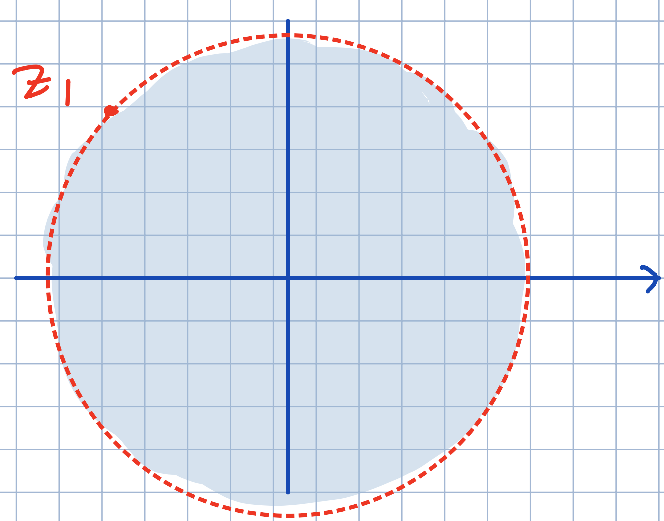
$\forall k \geq N$

$\forall k \quad |a_k z_1^k| \le \max\left(\varepsilon, |a_0 z_1^0|, |a_1 z_1|, |a_2 z_1^2|, \dots, |a_{N-1} z_1^{N-1}|\right) \triangleq C$

$$|a_k z^k| = |a_k z_1^k| \frac{|z|^k}{|z_1|^k} \le C \rho^k \qquad \rho \triangleq \left|\frac{z}{z_1}\right| < 1$$

$\sum C\rho^k$ is convergent.

By comparison test $\sum a_k z^k$ is convergent absolutely for $|z| < |z_1|$ $\blacksquare$



Corollary: If $\sum a_k z^k$ diverges at z_1 then $\sum_{k=0}^{\infty} a_k z^k$ diverges whenever $|z| > |z_1|$.

Radius of convergence

Suppose $\sum a_k z^k$ converges for some $z_1 \neq 0$

Let $R = \sup \underbrace{\left\{ |z| : \sum_{k=0}^{\infty} a_k z^k \text{ converges} \right\}}_{\text{It set is not empty.}}$

Then $\sum a_k z^k$ converges whenver $|z| < R$

$\sum a_k z^k$ diverges whenever $|z| > R$

Notation: $\sum a_k z^k$ converges $\forall z \Rightarrow$ we say $R = \infty$

$\sum a_k z^k$ converges only at $z = 0 \Rightarrow R = 0$.

Example $\exp(z)$ has $R = \infty$

$\cos(z)$, $\sin(z)$ have $R = \infty$

Hadamard formula for radius of convergence.

$$R = \frac{1}{\limsup |a_k|^{1/k}}$$

Proof Suppose $|z| < R$

$|z| < \rho < R$

$$\frac{1}{\rho} > \limsup |a_k|^{1/k}$$

$\exists N$ s.t. $\forall k \geq N$ $\quad |a_k|^{1/k} < \frac{1}{\rho}$

$$|a_k| < \frac{1}{\rho^k}$$

$$|a_k z^k| < \frac{|z|^k}{\rho^k}$$

$\left|\frac{z}{\rho}\right| < 1 \Rightarrow \sum \left|\frac{z}{\rho}\right|^k$ converges.

by comparison test $\sum_{k=0}^{\infty} a_k z^k$ is convergent.

Suppose $|z| > R$

$$|z| > \rho > R$$

$$\frac{1}{\rho} < \limsup |a_k|^{1/k}$$

$\frac{1}{\rho} < |a_k|^{1/k}$ for infinitely many k.

$$\frac{1}{\rho^k} < |a_k|$$

$|a_k z^k| > \left|\frac{z^k}{\rho^k}\right| > 1$ for infinitely many k.

n^{th}-term test fails $\Rightarrow$ divergent. $\blacksquare$