Supplementary Notes on Trees

Some Terms

An **acyclic** graph is a graph with no cycles
A **tree** is a connected acyclic graph
An edge of G is a **bridge** if $G - e$ is disconnected

Lemma: A connected graph with n vertices must have at least $n-1$ edges.

Proof: (Induction on n)
Obviously this is true for $n = 1$. Assume this is true for connected graphs with n vertices, and add a further vertex to such a graph. The new vertex must be connected to the existing graph, so that it requires at least one edge to be connected to it. Thus, by induction hypothesis, a connected graph with n+1 vertices must have at least $(n-1) + 1 = n$ edges, completing the induction. ❑

Theorem: Let G be a graph with n vertices. Then the following are equivalent:

(i) G is a tree
(ii) There is a unique path between every pair of vertices in G
(iii) G is connected and every edge in G is a bridge
(iv) G is connected and has $n - 1$ edges
(v) G is acyclic and has $n - 1$ edges

Proof:

(i) ↔ (ii)

Let G be a tree. Then G is connected, and there exists a path between every pair of vertices. Let there be two distinct paths between two vertices u and v of G. The union of these two paths contains a cycle, which contradicts the fact that G is a tree.

Conversely, let G be a graph and let there be exactly one path between every pair of vertices in G. Therefore, G is connected. If G is not a tree, then there is a cycle, say between vertices u and v. Thus, there are two distinct paths between u and v, which contradicts the hypothesis. Thus, G is connected and is acyclic, and therefore it is a tree.

(i) ↔ (iii)

Let G be a tree, then it is connected. Consider an arbitrary edge e along a path P connecting two vertices u and v. From (ii), P is unique, and deleting e will result in no path between u and v, and disconnects the graph. Hence, e is a bridge.

Conversely, suppose G is connected and every edge in G is a bridge. If G has a cycle C, then deleting an edge in C will not disconnect the graph, which contradicts the hypothesis. Thus G is acyclic and hence a tree.

(i) ↔ (iv)

Let G be a tree with n vertices, then it is connected. We prove by induction on n. If n=1, it is obviously true. Assume this is true for all $m < n$. Since from (iii) every edge is a bridge, the subgraph G' obtained from G after deleting an edge will have two components G_1 and G_2 with n_1 and n_2 vertices respectively, where $n_1 + n_2 = n$, and as there were no cycles to begin with, each component is a tree. By induction hypothesis, the number of edges in both the components together is $(n_1-1) + (n_2-1) = n-2$. Thus the number of edges in G will be $(n-2) + 1 = n-1$.

Conversely, suppose the connected graph G with n vertices and $n-1$ edges is not a tree. Then it has a cycle containing an edge e. If e is deleted, then the resulting graph G' is still a connected graph with $n-2$ edges. Thus, we have a connected graph with n vertices and less than $n-1$ edges, contradicting the Lemma.

(i) ↔ (v)

Let G be a tree with n vertices, then it is acyclic by definition, and it has $n-1$ edges from (iv).

Conversely, consider an acyclic graph G with n vertices and has $n - 1$ edges. Suppose G is not connected. Let the components of G be G_i ($i = 1, 2, \ldots, k$) and $k > 1$, such that and G_i has n_i vertices, where $n_1 + n_2 + \ldots + n_k = n$. Now, each component G_i is acyclic and connected and is therefore a tree, and from (iv), has n_i-1 edges. Thus, the total number of edges in G is $n-k$, where $k > 1$, which contradicts that G has $n - 1$ edges. Thus, G is connected and is thus a tree. ❑

Corollary: Any tree with at least two vertices has at least two vertices of degree one.

Let the number of vertices in a given tree G be n ($n > 1$). Therefore, the number of edges in G is $n-1$. Therefore, by the Handshaking Lemma, the degree sum of the tree is $2(n-1)$. This degree sum is to be divided among the n vertices. This implies there exists some vertex v_1 with degree < 2. Since the degree of v_1 must be at least one for the graph to be connected, thus the degree of v_1 is one. Assume G has exactly one vertex v_1 of degree one, while all the other $n-1$ vertices have degree ≥ 2. Then sum of degrees is $d(v_1) + d(v_2) + \ldots + d(v_n) \geq 1 + 2 + 2 + \ldots + 2 = 1 + 2(n-1)$, which contradicts the degree sum of the tree is $2(n-1)$. Hence G has at least two vertices of degree one. ❑