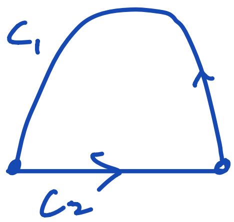
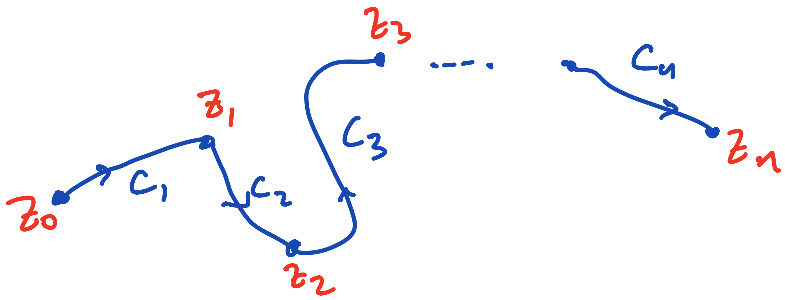
MAT 3253 Lecture 14

Piece-wise smooth curve



$$\int_{C_1+C_2} f\,dz = \int_{C_1} f\,dz + \int_{C_2} f\,dz$$



$C = C_1 + C_2 + C_3 + \dots C_n$. Suppose $F'(z) = f(z)$

$$\int_C f\,dz = \sum_{j=1}^{n} \int_{C_j} f(z)\,dz$$

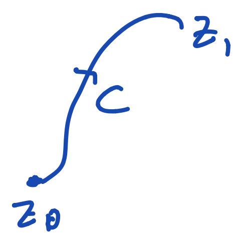
$$= \sum_{j=1}^{n} F(z_j) - F(z_{j-1})$$

$$= -F(z_0) + F(z_1) - F(z_1) + F(z_2) - \dots$$

$$\dots - F(z_{n-1}) + F(z_n)$$

$$= F(z_n) - F(z_0)$$

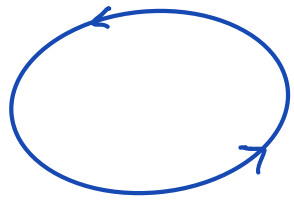
Example $\displaystyle\int_C z^3 - z\,dz$



$$= \int_{z_0}^{z_1} z^3 - z\,dz$$

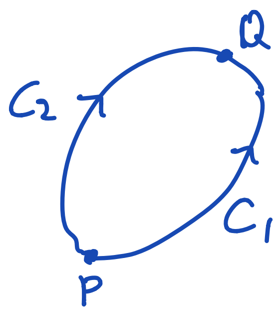
$$= \left[\frac{z^4}{4} - \frac{z^2}{2}\right]_{z_0}^{z_1} = \frac{z_1^4}{4} - \frac{z_1^2}{2} - \frac{z_0^4}{4} + \frac{z_0^2}{2} .$$

Closed curve



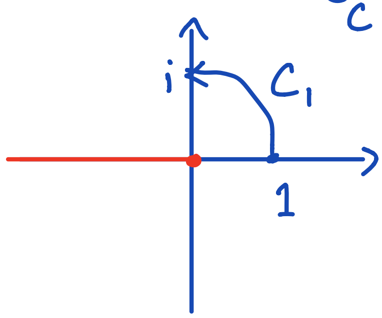
Theorem

If $\int_C f\,dz$ is path independent, then for closed curve C, $\oint_C f\,dz = 0$, and vice versa.

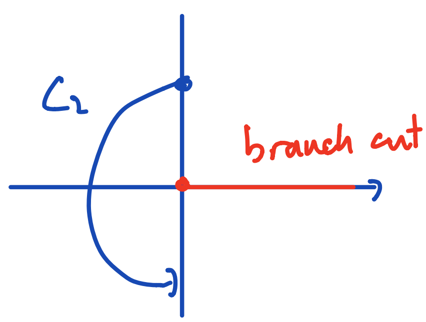


$$\int_C = \int_{C_1} + \int_{-C_2} = \int_{C_1} - \int_{C_2} = 0$$

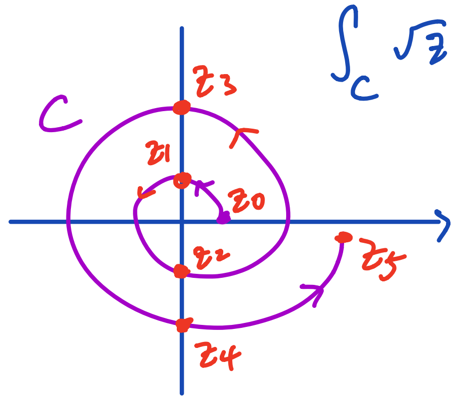
Example $\int_C \sqrt{z}\,dz$ $\qquad \sqrt{z} = \frac{2}{3} z^{3/2}$.



$$\int_{C_1} \sqrt{z}\,dz$$



take another branch cut

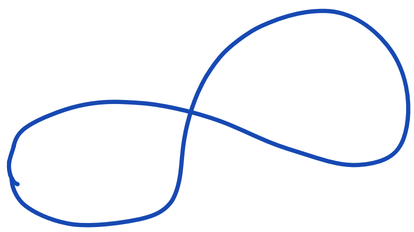


$$\int_C \sqrt{z}\, dz$$

Divide the contour into several parts.

Def^n A curve is <u>simple</u> if there is no self-intersection.

Non-simple



(1789–1857) (1858–1936)

Cauchy - Goursat theorem for rectangular contour.

Remark:

The original version Cauchy theorem requires that derivative is continuous.

Recall: Cantor intersection theorem

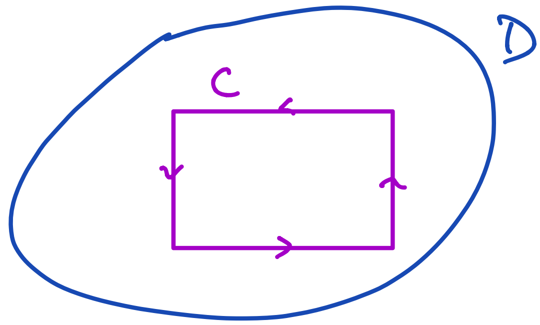
Suppose $K_1, K_2, K_3, \dots$ are compact sets

$K_1 \supseteq K_2 \supseteq K_3 \supseteq K_4 \dots$ each $K_j \neq \phi$.

then $\bigcap_{j=1}^{\infty} K_j$ is not empty.

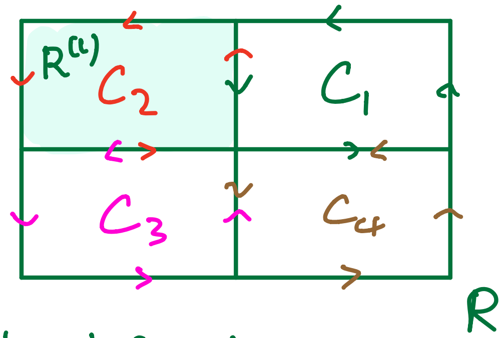
Theorem Suppose $f(z)$ is analytic in a domain D and C is the boundary of a rectangle R with sides parallel to the real and imaginary axis.

Then
$$\oint_C f(z)\,dz = 0$$



Proof Suppose $\int_C f(z)\,dz = I$

Want to show $I = 0$



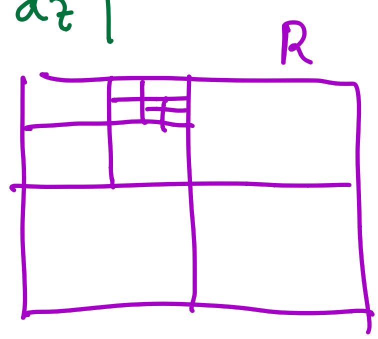
Let $\int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} = I$

$$|I| \le \left|\int_{C_1} f\right| + \left|\int_{C_2} f\right| + \left|\int_{C_3} f\right| + \left|\int_{C_4} f\right|$$

Suppose $R^{(1)}$ is the rectangle s.t.

$$\left|\int_{\partial R^{(1)}} f\,dz\right| = \max_j \left|\int_{C_j} f\,dz\right|$$

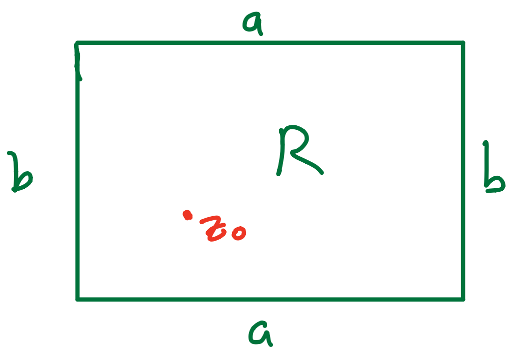
$$\Rightarrow \frac{|I|}{4} \le \left|\int_{\partial R^{(1)}} f\,dz\right|$$



Recursively, pick $R \supseteq R^{(1)} \supseteq R^{(2)} \supseteq R^{(3)} \supseteq \ldots$

$$* \quad \frac{|I|}{4^k} \le \left|\int_{\partial R^{(k)}} f\,dz\right| \qquad \forall k = 1, 2, 3, \ldots$$

"∂" denotes the boundary operator
$\partial R^{(k)}$ is the boundary of $R^{(k)}$



$*$ $R^{(k)}$ has perimeter $\frac{L}{2^k}$

$R^{(1)}$ has perimeter $L/2$

$R^{(2)}$ has perimeter $L/4$

$L = 2a + 2b$

Let $z_0 \in \bigcap_{k=1}^{\infty} R^{(k)}$ (exists by intersection thm)

$z_0 \in R^{(k)} \quad \forall k$

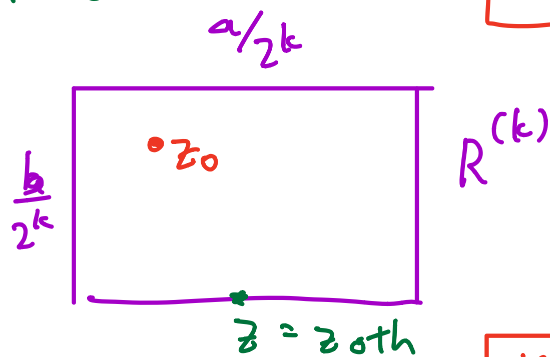
f is differentiable at $z = z_0$

$$f(z_0 + h) = f(z_0) + f'(z_0)\cdot h + \varepsilon \cdot h$$

$|\varepsilon| \to 0$ as $|h| \to 0$

Fix δ, choose k $\quad |\varepsilon| < \delta$

$\boxed{a > b}$



$|z - z_0|$

$\leq$ diagonal of the $R^{(k)}$

$$\leq \sqrt{\left(\frac{a}{2^k}\right)^2 + \left(\frac{a}{2^k}\right)^2} = \frac{a\sqrt{2}}{2^k}$$

replace
$z_0 + h = z$
$h = z - z_0$

$$\int_{\partial R^{(k)}} f(z)\, dz = \int_{\partial R^{(k)}} \underbrace{f(z_0) + f'(z_0)\cdot(z - z_0)}_{\text{have anti-derivatives}} + \varepsilon(z - z_0)\, dz$$

$$= \int_{\partial R^{(k)}} \varepsilon(z - z_0)\, dz$$

$$\left(f(z_0)\cdot z\right)' = f(z_0)$$

$$\left(f'(z_0)\frac{(z - z_0)^2}{2}\right)' = f'(z_0)(z - z_0)$$

$$\frac{|I|}{4^k} \le \left| \int_{\partial R^{(k)}} \varepsilon (z - z_0)\, dz \right| \le \delta \, \frac{a\sqrt{2}}{2^k} \cdot \underbrace{\frac{L}{2^k}}_{\text{perimeter of } R^{(k)}}$$

ML inequality

$$|I| \le \delta \underbrace{a\sqrt{2}\, L}_{\text{constant}}$$

Since δ is arbitrary, $|I| = 0$

$$\Rightarrow I = 0 \qquad \boxed{}$$

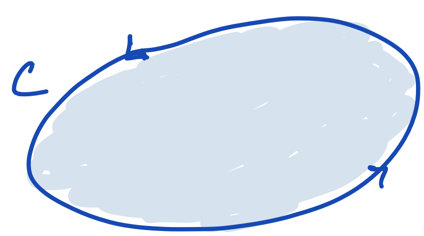
Area $R^{(k)} = \frac{1}{4}$ Area $R^{(k-1)}$

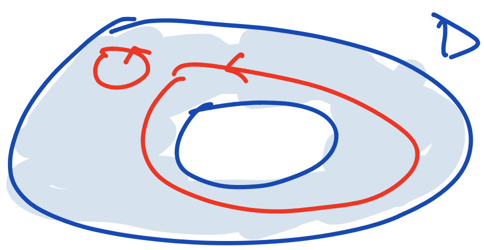
Area of $R^{(k)} > 0 \Rightarrow R^{(k)} \neq \phi$

<u>More general version of Cauchy theorem</u>

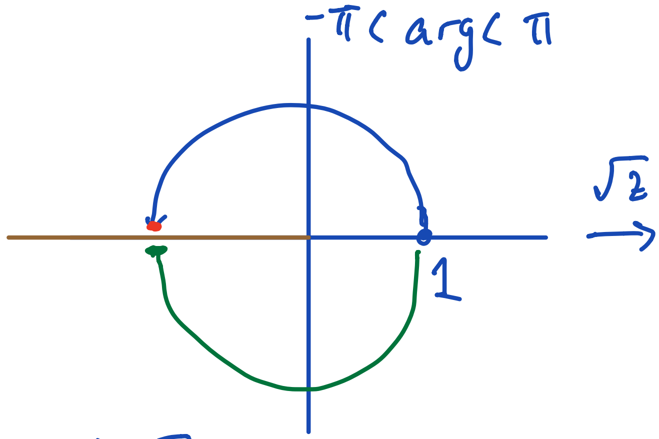
If $f(z)$ is analytic in a domain D, and C is a simple closed curve in D s.t. $f'(z)$ exists in the interior of C

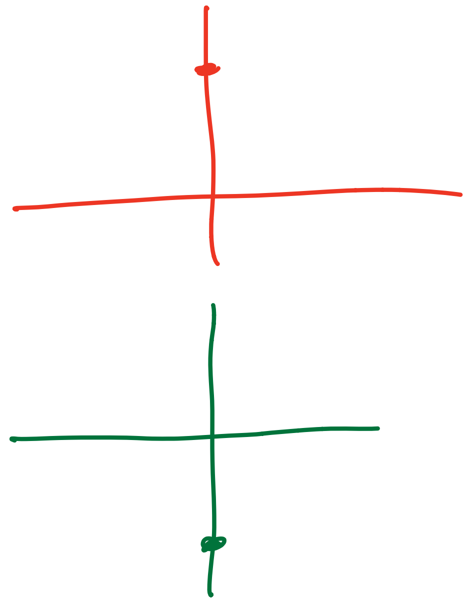
$$\int_C f(z)\, dz = 0$$





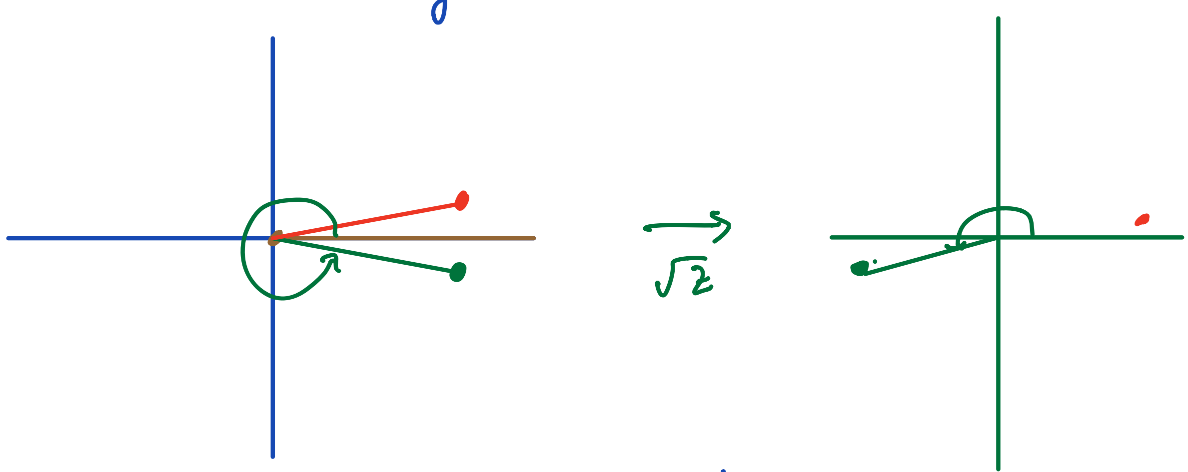
$\sqrt{z}$





This version of $\sqrt{z}$ is not continuous on the negative real axis.

$0 < \arg < 2\pi$



This version of $\sqrt{z}$ is not continuous on the positive real axis.