Solution to Assignment 2

Question 1

(a) T: The total number of rank assignments is $N!/(n_1!\cdots n_k!)$. When $n_1 = \cdots = n_k = n$, H is invariant with the order of $(R_1,\ldots,R_k)$. Thus the number of required rank assignments can be reduced to

$$\frac{1}{k!}\cdot\frac{N!}{n_1!\cdots n_k!} = \frac{(nk)!}{k!(n!)^k}$$

(b) F: The value of J varies with the order of treatments even if $n_1 = \cdots = n_k$, hence the number of required rank assignments is $6!/(2!2!2!) = 90$.

It is easy to see that the value of J differs between the 6 rank assignments in Example 5.1 (on page 5 of Lecture Notes Section 5).

(c) F: It is possible that at the same level of significance, H_0 is rejected by J, but not by the Kruskal-Wallis test. In Example 5.4, the approximate p-value of J is 2.1%, which is well below 5%. But

$$H = \frac{12}{18(18+1)}\cdot\frac{40.5^2 + 52.5^2 + 78^2}{6} - 3(18+1) = 4.289$$

and

$$A = \frac{2(2^3-2)+3^3-3+4^3-4}{18^3-18} = 0.0165 \;\Rightarrow$$

$$H' = \frac{H}{1-A} = \frac{4.289}{1-0.0165} = 4.361 \;\Rightarrow$$

$$p\text{-value of } H' \approx \Pr\left(\chi_2^2 \ge 4.361\right) = 0.1130 > 0.05$$

Thus at the 5% level of significance, H_0 is rejected by J, but not by H'.

Remark: The reason for this phenomenon is, when the data match an ordered alternative H_1 (such as $R_1 < R_2 < R_3$ in Example 5.4), they may give stronger evidence to this H_1 than general alternatives that include all possible scenarios against H_0, not just the ordered H_1.

Question 2

(a) F: The alternative in a hypothesis test is prespecified based on the interest of the problem before data are analysed or even collected. The one-sided multiple comparison procedure, on the other hand, makes decisions based on the results of data analyses without a prespecified alternative.

(b) T: Under H_0, if W_{ij} is asymptotically normal, then $\mathrm{E}_0[W_{ij}^*]=0$ and $\mathrm{Var}_0(W_{ij}^*)=2$ imply $W_{ij}^* \sim N(0,2)$ asymptotically. On the other hand, $\mathrm{E}[Z_i - Z_j]=0$ and

$$\mathrm{Var}(Z_i - Z_j) = \frac{1}{n_i} + \frac{1}{n_j} = 2\frac{n_i + n_j}{2n_i n_j} \Rightarrow \frac{Z_i - Z_j}{\sqrt{(n_i + n_j)/(2n_i n_j)}} \sim N(0,2)$$

This proves the statement.

(c) F: By the definition of $w_{0.05}^*$, it is possible that under H_0,

$$\Pr(\text{Decision}) = \Pr(|W_{12}^*| < w_{0.05}^*, |W_{13}^*| < w_{0.05}^*, |W_{23}^*| \ge w_{0.05}^*) < \Pr(|W_{23}^*| \ge w_{0.05}^*)$$

$$< \Pr(|W_{uv}^*| \ge w_{0.05}^* \text{ for some } 1 \le u < v \le 3) = 0.05$$

Question 3

(a) F: Since the total number of observations in block i is k,

$$\sum_{j=1}^{g_i} t_{i,j} = k \Rightarrow \sum_{j=1}^{g_i} t_{i,j}^3 - k = \sum_{j=1}^{g_i} (t_{i,j}^3 - t_{i,j}) = \sum_{j=1}^{g_i} t_{i,j}(t_{i,j} - 1)(t_{i,j} + 1)$$

(b) T: The minimum value of L is

$$L_{\min} = \sum_{j=1}^{k} jn(k+1-j) = n(k+1)\sum_{j=1}^{k} j - n\sum_{j=1}^{k} j^2 = \frac{nk(k+1)^2}{2} - \frac{nk(k+1)(2k+1)}{6}$$

$$= \frac{nk(k+1)(3k+3-2k-1)}{6} = \frac{nk(k+1)(k+2)}{6} > \frac{nk(k+1)^2}{6} = \frac{2}{3}\mathrm{E}_0[L]$$

(c) T: Let s be the common number of observations in all blocks. Then by (6.21),

$$A_j = \sum_{i=1}^{n} \sqrt{\frac{12}{s+1}}\left(r_{ij} - \frac{s+1}{2}\right) = \sqrt{\frac{12}{s+1}}\sum_{i=1}^{n}\left(r_{ij} - \frac{s+1}{2}\right) = \sqrt{\frac{12}{s+1}}\left[R_j - \frac{n(s+1)}{2}\right]$$

$= aR_j + b$, $j = 1, \ldots, k$, where

$$a = \sqrt{\frac{12}{s+1}} \quad \text{and} \quad b = \sqrt{\frac{12}{s+1}}\left[-\frac{n(s+1)}{2}\right] = -n\sqrt{3(s+1)}$$

Question 4 [20 marks]

(a) Calculate D_i^2, $i=0,1,\ldots,11$, and E_j^2, $j=0,1,\ldots,10$, according to the formulae for the Miller's Jackknife test. Then the values of

$$S_i = \ln D_i^2, \quad A_i = 11S_0 - 10S_i, \quad i=1,\ldots,11,$$

and

$$T_j = \ln E_j^2, \quad B_j = 10T_0 - 9T_j, \quad j=1,\ldots,10,$$

are calculated and shown in the table below:

i	S_i	A_i	j	T_j	B_j
0	3.10175		0	3.12300	
1	3.06025	3.51670	1	2.70918	6.84735
2	3.09884	3.13085	2	2.75559	6.42968
3	3.11330	2.98622	3	3.23721	2.09510
4	3.14626	2.65662	4	3.23819	2.08627
5	3.18803	2.23893	5	3.23022	2.15796
6	3.20558	2.06336	6	3.22321	2.22106
7	3.20435	2.07567	7	3.20364	2.39722
8	3.20059	2.11326	8	3.18388	2.57503
9	3.19192	2.20000	9	3.16389	2.75494
10	2.86581	5.46113	10	3.11917	3.15747
11	2.73174	6.80182			

It follows that

$$\bar{A} = \frac{1}{11}\sum_{i=1}^{11} A_i = 3.20405, \quad \bar{B} = \frac{1}{10}\sum_{j=1}^{10} B_j = 3.27221$$

$$V_1 = \sum_{i=1}^{11} \frac{(A_i - \bar{A})^2}{11(10)} = 0.21993, \quad V_2 = \sum_{i=1}^{10} \frac{(B_i - \bar{B})^2}{10(9)} = 0.32691$$

Then

$$Q = \frac{\bar{A} - \bar{B}}{\sqrt{V_1 + V_2}} = \frac{3.20405 - 3.27221}{\sqrt{0.21993 + 0.32691}} = -0.09217$$

and so

$$p\text{-value} = \Pr(|Q| \ge 0.09217) \approx 2\Pr(Z \ge 0.09217) = 2(0.4633) = 0.9266$$

Thus H_0 is accepted with a very large p-value, which shows there is little evidence for $\mathrm{Var}(X) \ne \mathrm{Var}(Y)$.

(b) The ranks for W and scores for C are listed below with $m = 11$, $n = 10$, $N = 21$:

Sample	Y	Y	X	X	X	X	X	X	X	X	X
Value	2.2	2.5	2.8	3.5	3.8	4.6	6.1	7.5	8.8	9.2	9.8
Rank	1	2	3	4	5	6	7	8	9	10	11
Score	1	2	3	4	5	6	7	8	9	10	11
Sample	Y	Y	Y	Y	Y	Y	Y	Y	X	X	
Value	10.1	11.6	12.3	12.7	13.5	14.1	14.6	15.5	15.7	16.8	
Rank	12	13	14	15	16	17	18	19	20	21	
Score	10	9	8	7	6	5	4	3	2	1	

Calculate

$$W = 1 + 2 + 12 + 13 + \cdots + 19 = 127,\ C = 1 + 2 + 10 + 9 + \cdots + 3 = 55,$$

$$\mathrm{E}_0[W] = \frac{n(N+1)}{2} = \frac{10(22)}{2} = 110, \quad \mathrm{Var}_0(W) = \frac{11(10)(22)}{12} = 201.67,$$

$$\mathrm{E}_0[C] = \frac{10(22)^2}{4(21)} = 57.62 \ \text{ and } \ \mathrm{Var}_0(C) = \frac{11(10)(22)(21^2+3)}{48(21^2)} = 50.76$$

Then the Lepage test statistic is

$$D = \left(W^*\right)^2 + \left(C^*\right)^2 = \frac{(127-110)^2}{201.67} + \frac{(55-57.62)^2}{50.76} = 1.568 < 1.833 = \chi^2_{2,0.4}$$

Thus H_0 is accepted at the 40% level of significance, which shows little evidence for the two samples to have different location and/or dispersion parameters.

(c) The values of $F_{11}(t)$ and $G_{10}(t)$ at $Z_{(1)} < \cdots < Z_{(21)}$ are listed below.

X/Y	Y	Y	X	X	X	X	X	X	X	X	X
$Z_{(i)}$	2.2	2.5	2.8	3.5	3.8	4.6	6.1	7.5	8.8	9.2	9.8
$F_{11}(t)$	0		1/11	2/11	3/11	4/11	5/11	6/11	7/11	8/11	9/11
$G_{10}(t)$	1/10	2/10 = 1/5 = 0.2									

X/Y	Y	Y	Y	Y	Y	Y	Y	Y	X	X	
$Z_{(i)}$	10.1	11.6	12.3	12.7	13.5	14.1	14.6	15.5	15.7	16.8	
$F_{11}(t)$	9/11								10/11	1	
$G_{10}(t)$	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1			

It shows that the largest difference between $F_{11}(t)$ and $G_{10}(t)$ at $Z_{(1)} < \cdots < Z_{(21)}$ is

$$D = \max_{1 \le i \le 21} \left| F_{11}\left(Z_{(i)}\right) - G_{10}\left(Z_{(i)}\right) \right| = \left| F_{11}(9.8) - G_{10}(9.8) \right| = \frac{9}{11} - \frac{1}{5} = \frac{34}{55}$$

Thus the two-sample Kolmogorov-Smirnov test statistic value from the data is

$$J = \frac{mn}{d} D = \frac{11(10)}{1} \cdot \frac{34}{55} = \frac{340}{5} = 68$$

The R command and output below show that the p-value of the test is given by

$$p\text{-value} = \Pr(J \geq 68) = \Pr(D \geq 0.61818) = 0.0242 < 0.025$$

This is less than half of the 5% level and thus provides strong evidence that the distributions of X and Y are different.

```
> cKolSmirn(0.025,11,10)
```

```
Number of X values: 11 Number of Y values: 10
For the given alpha=0.025, the upper cutoff value is Kolmogorov-Smirnov J=68,
with true alpha level=0.0242
```

(d) Comments on the results of parts (a) – (c):

1) Parts (a) and (b) indicate that the two samples may have equal medians and variances, but part (c) shows their distributions are significantly different. Thus there is sufficient evidence for general differences between the distributions of the two samples, but not in their location/dispersion parameters.

2) If the location-scale parameter model were correct, then the results of (a) – (c) would imply the same distribution of X and Y. But that would contradict the result of part (c), which is valid without additional conditions. Therefore the location-scale parameter model is not appropriate.

Question 5 [24 marks]

(a) $(n_1,\ldots,n_5)=(6,5,7,4,6) \Rightarrow N=28$. The Jonckheere-Terpstra test statistic is

$$J=U_{12}+U_{13}+U_{14}+U_{15}+U_{23}+U_{24}+U_{25}+U_{34}+U_{35}+U_{45}$$
$$=26+36+20+32+28+11+18+16+16+10=213$$

By R, the p-value of the test is $\Pr(J\geq 213)=0.0099 \Rightarrow$ Reject H_0 at $\alpha=0.01$.

```
> cJCK(0.01,c(6,5,7,4,6))
Group sizes: 6 5 7 4 6
For the given alpha=0.01, the upper cutoff value is Jonckheere-Terpstra J=213, with
true alpha level=0.0099
```

Thus there is very strong evidence for $\tau_1\leq\tau_2\leq\tau_3\leq\tau_4\leq\tau_5$ at the 1% level.

The normal approximation gives the same test result shown below:

$$\mathrm{E}_0[J]=\frac{N^2-n_1^2-\cdots-n_5^2}{4}=\frac{28^2-6^2-5^2-7^2-4^2-6^2}{4}=\frac{622}{4}=155.5 \quad \text{and}$$

$$\mathrm{Var}_0(J)=\frac{28^2(56+3)-6^2(12+3)-\cdots-6^2(12+3)}{72}=\frac{7307}{12}=608.92 \Rightarrow$$

$$J^*=\frac{213-155.5}{\sqrt{608.92}}=2.330>2.326=z_{0.01} \Rightarrow \text{Reject } H_0 \text{ at } \alpha=0.01$$

(b) First calculate $U_{vu}=n_un_v-U_{uv}$ for $1\leq u<v\leq 5$ to obtain U_{uv} for all $u\neq v$:

		v				
	u	1	2	3	4	5
	1	–	26	36	20	32
	2	4	–	28	11	18
U_{uv}	3	6	7	–	16	16
	4	4	9	12	–	10
	5	4	12	26	14	–

Then for known peak $p=3$, the Mack-Wolfe statistic is calculated by

$$A_3=U_{12}+U_{13}+U_{23}+U_{43}+U_{53}+U_{54}=26+36+28+12+26+14=142$$

By the result from R shown below, p-value $=\Pr(J\geq 142)=0.0203$.

```
cUmbrPK(0.023, c(6,5,7,4,6),3)
Group sizes: 6 5 7 4 6
For the given alpha=0.023, the upper cutoff value is Mack-Wolfe Peak Known A 3=142,
with true alpha level=0.0203
```

Thus there is strong evidence for $\tau_1\leq\tau_2\leq\tau_3\geq\tau_4\geq\tau_5$ at about 2% level.

Next, $(n_1,\ldots,n_5)=(6,5,7,4,6) \Rightarrow N_1=6+5+7=18,\ N_2=7+4+6=17 \Rightarrow$

$$\mathrm{E}_0[A_3]=\frac{N_1^2+N_2^2-n_1^2-\cdots-n_5^2-n_3^2}{4}=\frac{18^2+17^2-6^2-\cdots-6^2-7^2}{4}=100.5$$

$$\mathrm{Var}_0(A_3)=\frac{2(18^3+17^3)+3(18^2+17^2)-6^2(12+3)-\cdots-6^2(12+3)-7^2(14+3)}{72}$$

$$+\frac{7(18)(17)-7^2(28)}{6}=\frac{20082}{72}+\frac{770}{6}=\frac{1629}{4}=407.25$$

$$\Rightarrow A_3^*=\frac{142-100.5}{\sqrt{407.25}}=2.056>2.054=z_{0.02} \Rightarrow \text{ Reject } H_0 \text{ at } \alpha=0.02$$

Thus the normal approximation gives the same test result.

(c) From the values of U_{uv} for all $u\neq v$ in part (b), calculate

$U_{\bullet 1}=U_{21}+\cdots+U_{51}=4+6+4+4=18$, $U_{\bullet 2}=26+7+9+12=54$,

$U_{\bullet 3}=36+28+12+26=102$, $U_{\bullet 4}=20+11+16+14=61$, $U_{\bullet 5}=32+18+16+10=76$

Then by the formulae for $\mathrm{E}_0[U_{\bullet q}]$ and $\mathrm{Var}_0(U_{\bullet q})$,

$$\mathrm{E}_0[U_{\bullet 1}]=\frac{6(28-6)}{2}=66,\quad \mathrm{Var}_0(U_{\bullet 1})=\frac{6(22)(29)}{12}=319 \Rightarrow U_{\bullet 1}^*=\frac{18-66}{\sqrt{319}}=-2.69$$

Similarly, $U_{\bullet 2}^*=-0.21$, $U_{\bullet 3}^*=1.51$, $U_{\bullet 4}^*=0.85$, $U_{\bullet 5}^*=0.76$. Thus $U_{\bullet 3}^*=1.51>U_{\bullet q}^*$ for $q=1,2,4,5 \Rightarrow \hat{p}=3$.

The results of R with target $\alpha=0.077$ are shown below:

```
> cUmbrPU(0.077, c(6,5,7,4,6))
Group sizes: 6 5 7 4 6
For the given alpha=0.077, the upper cutoff value is Mack-Wolfe Peak Unknown
A*(p-hat)=2.0623637675, with true alpha level=0.075

> cUmbrPU(0.077, c(6,5,7,4,6))
Monte Carlo Approximation (with 10000 Iterations) used:
Group sizes: 6 5 7 4 6
For the given alpha=0.077, the upper cutoff value is Mack-Wolfe Peak Unknown
A*(p-hat)=2.0465032914, with true alpha level=0.0764
```

Thus $A_{\hat{p}}^*=A_3^*=2.056$ is between $a_{\hat{p},0.0764}^*=2.0465$ and $a_{\hat{p},0.075}^*=2.0623$.

This shows moderate evidence for umbrella alternatives with unknown peak at around 7.6% level of significance.

(d) Part (a) shows that the achieved level of significance for ordered alternatives is just below 1%. Part (b) shows the achieved level for umbrella alternatives with $p=3$ is around 2%, while part (c) shows the level close to 7.6%. Hence according to the level of significance, the data provide stronger evidence for ordered alternatives than umbrella alternatives.

The test results in parts (a) – (c) are consistent, not contradictive, since ordered alternatives are special cases of umbrella alternatives (with $p=k$). In this problem, it is possible that $\tau_1<\tau_2<\tau_3=\tau_4=\tau_5$, which are consistent with both ordered and umbrella alternatives. The tests suggest that strict inequalities are likely to hold in treatments 1 – 3, but not in treatments 3 – 5.

Question 6 [20 marks]

(a) By the formulae of $\mathrm{E}[r_{ij}]$ and $\mathrm{Var}(r_{ij})$, and the independence of $\{r_{ij}\}$ over i,

$$\mathrm{E}[R_j]=\sum_{i=1}^{n}\mathrm{E}[r_{ij}]=\sum_{i=1}^{n}\frac{s+1}{2}I_{\{c_{ij}=1\}}=\frac{s+1}{2}\sum_{i=1}^{n}I_{\{c_{ij}=1\}}=\frac{p(s+1)}{2}$$

and

$$\mathrm{Var}(R_j)=\sum_{i=1}^{n}\mathrm{Var}(r_{ij})=\frac{(s+1)(s-1)}{12}\sum_{i=1}^{n}I_{\{c_{ij}=1\}}=\frac{p(s+1)(s-1)}{12}$$

The Durbin-Skillings-Mack test statistic in BIBD is defined by

$$D=\frac{12}{\lambda k(s+1)}\sum_{j=1}^{k}\left(R_j-\frac{p(s+1)}{2}\right)^2=\frac{12}{\lambda k(s+1)}\sum_{j=1}^{k}R_j^2-\frac{3(s+1)p^2}{\lambda}$$

Thus

$$\mathrm{E}[D]=\frac{12}{\lambda k(s+1)}\sum_{j=1}^{k}\mathrm{E}\left[\left(R_j-\frac{p(s+1)}{2}\right)^2\right]=\frac{12}{\lambda k(s+1)}\sum_{j=1}^{k}\mathrm{E}\left[\left(R_j-\mathrm{E}[R_j]\right)^2\right]$$

$$=\frac{12}{\lambda k(s+1)}\sum_{j=1}^{k}\mathrm{Var}(R_j)=\frac{12}{\lambda k(s+1)}\sum_{j=1}^{k}\frac{p(s+1)(s-1)}{12}=\frac{1}{\lambda k}kp(s-1)$$

These together with $p(s-1)=\lambda(k-1)$ for BIBD imply

$$\mathrm{E}[D]=\frac{p(s-1)}{\lambda}=\frac{\lambda(k-1)}{\lambda}=k-1$$

(b) This block design is BIBD with $k = n = 5$, $s = p = 4$ and $\lambda = 3$. The in-block ranks $\{r_{ij}\}$ and $R_1,\ldots,R_5$ are shown in the table below:

Block	Treatment j				
i	1	2	3	4	5
1	3	1	2	0	4
2	2	0	1	4	3
3	3	1	2	4	0
4	0	2	1	3	4
5	2	1	0	4	3
R_j	10	5	6	15	14

Then the Durbin-Skillings-Mack test statistic D is calculated by

$$D = \frac{12}{\lambda k(s+1)}\sum_{j=1}^{k} R_j^2 - \frac{3(s+1)p^2}{\lambda} = 12\frac{10^2+5^2+6^2+15^2+14^2}{3(5)(4+1)} - \frac{3(4+1)4^2}{3}$$

$$= 12\frac{582}{75} - 5(16) = \frac{2328}{25} - 80 = \frac{2328-2000}{25} = \frac{328}{25} = 13.12$$

Thus the approximate p-value is given by

$$p\text{-value} \approx \Pr\left(\chi^2_{k-1} \geq 13.12\right) = \Pr\left(\chi^2_4 \geq 13.12\right) = 0.0107$$

This shows strong evidence (at the level close to 1%) that the treatment effects $\tau_1,\ldots,\tau_5$ are different.

(c) The Skillings-Mack multiple two-sided all-treatment comparison procedure decides $R_u \neq R_v$ for $u < v$ at $\alpha = 0.1$ if

$$|R_u - R_v| \geq q_{0.1}\sqrt{\frac{(s+1)(ps-p+\lambda)}{12}} = 3.479\sqrt{\frac{(4+1)(4\cdot 4-4+3)}{12}} = 8.70$$

Therefore,

$|R_2 - R_4| = 15-5 = 10 > 8.70 \;\Rightarrow\; \tau_2 \neq \tau_4$,

$|R_2 - R_5| = 14-5 = 9 > 8.70 \;\Rightarrow\; \tau_2 \neq \tau_5$,

$|R_3 - R_4| = 15-6 = 9 > 8.70 \;\Rightarrow\; \tau_3 \neq \tau_4$, and

$|R_u - R_v| \leq 14-6 = 8 < 8.70 \;\Rightarrow\; \tau_u = \tau_v$ for all other pairs $(u,v) = (1,2)$, (1,3), (1,4), (1,5), (2,3), (3,5), (4,5) with $u < v$.