MAT2002 Ordinary Differential Equations
System of first order linear equations III

Dongdong He

The Chinese University of Hong Kong (Shenzhen)

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Overview

Outline

Fundamental matrices and matrix exponential

Recall:

$$\mathbf{x}'(t) = \mathbf{P}(t)\mathbf{x}(t) \tag{1}$$

Definition 11.1

Suppose that $\mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(n)}(t)$ form a fundamental set of solutions for the homogeneous linear system (1). Then the matrix

$$\Psi(t) = \begin{pmatrix} | & | & \cdots & | \\ \mathbf{x}^{(1)}(t) & \mathbf{x}^{(2)}(t) & \cdots & \mathbf{x}^{(n)}(t) \\ | & | & \cdots & | \end{pmatrix} \tag{2}$$

whose columns are the vectors $\mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(n)}(t)$ is called a **fundamental matrix** of the system (1).

Fundamental matrices and matrix exponential

Suppose that $\mathbf{x}^{(1)}(t), \dots, \mathbf{x}^{(n)}(t)$ form a fundamental set of solutions for the homogeneous constant coefficient linear system (1), the general solution of the system (1)

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t)$$

can be written as

$$\mathbf{x} = \boldsymbol{\Psi}(t)\mathbf{c}, \tag{3}$$

where $\mathbf{c}$ is the vector with components $c_1, \cdots, c_n$. If

$$\mathbf{x}(t_0) = \mathbf{x}_0,$$

then the solution of the corresponding initial value problem is

$$\mathbf{x} = \boldsymbol{\Psi}(t)\boldsymbol{\Psi}^{-1}(t_0)\mathbf{x}_0. \tag{4}$$

Since each column of $\boldsymbol{\Psi}$ is a solution of the system (6), it is easy to check that $\boldsymbol{\Psi}$ satisfies the matrix differential equation

$$\boldsymbol{\Psi}' = \mathbf{P}(t)\boldsymbol{\Psi}, \tag{5}$$

where $\boldsymbol{\Psi}' = \frac{d\boldsymbol{\Psi}}{dt} = (\frac{d\psi_{ij}(t)}{dt})_{n\times n}$, $\psi_{ij}(t)$ is the (i,j)-entry of $\boldsymbol{\Psi}$.

Matrix exponential.

In this slide, we will use the fundamental matrix and matrix exponential to solve the following IVP

$$\mathbf{x}' = \mathbf{A}\mathbf{x} \tag{6}$$

$$\mathbf{x}(t_0) = \mathbf{x}_0 \tag{7}$$

Matrix exponential.

Matrix exponential. For a scalar a we have the power series expansion

$$e^{at} = 1 + \sum_{n=1}^{\infty} \frac{a^n t^n}{n!}.$$

For any $n \times n$ constant matrix $\mathbf{A}$, we can show that the following

$$\mathbf{I} + \sum_{n=1}^{\infty} \frac{\mathbf{A}^n}{n!}$$

converges to a matrix (the proof will not be shown in this course). Thus, we can defined as the matrix exponential:

$$e^{\mathbf{A}} \triangleq \mathbf{I} + \sum_{n=1}^{\infty} \frac{\mathbf{A}^n}{n!}. \tag{8}$$

And for any t, $\mathbf{I} + \sum_{n=1}^{\infty} \frac{\mathbf{A^n t^n}}{n!}$ also converges, we can define

$$e^{\mathbf{At}} \triangleq \mathbf{I} + \sum_{n=1}^{\infty} \frac{\mathbf{A^n t^n}}{n!}. \tag{9}$$

Matrix exponential.

Property

1. $\exp(\mathbf{O}_{n\times n}) = \mathbf{I}$.
2. If $\Lambda = diag(\lambda_1, \cdots, \lambda_n)$, then $e^{\Lambda} = diag(e^{\lambda_1}, \cdots, e^{\lambda_n})$.

Matrix exponential.

Theorem 11.2

1. *If* $\mathbf{AB} = \mathbf{BA}$, *then* $e^{\mathbf{A}+\mathbf{B}} = e^{\mathbf{A}}e^{\mathbf{B}}$.
2. $(e^{\mathbf{A}})^{-1} = e^{-\mathbf{A}}$
3. *For non-singular matrix* $\mathbf{P}$, *we have*

$$\exp(\mathbf{PAP}^{-1}) = \mathbf{P}e^{\mathbf{A}}\mathbf{P}^{-1}.$$

Matrix exponential.

Proof.

For (1), we have (using Binomial Theorem)

$$\exp(\mathbf{A}+\mathbf{B}) = \sum_{n=0}^{\infty} \frac{(\mathbf{A}+\mathbf{B})^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{j=0}^{n} \binom{n}{j} \mathbf{A}^j \mathbf{B}^{n-j} \qquad \text{By condition } \mathbf{AB} = \mathbf{BA}.$$

It follows that

$$\begin{aligned}
\exp(\mathbf{A}+\mathbf{B}) &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} \mathbf{A}^j \mathbf{B}^{n-j} \\
&= \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{1}{j!(n-j)!} \mathbf{A}^j \mathbf{B}^{n-j} = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{\mathbf{A}^j}{j!} \frac{\mathbf{B}^{n-j}}{(n-j)!} \\
&= \sum_{j=0}^{\infty} \sum_{n=j}^{\infty} \frac{\mathbf{A}^j}{j!} \frac{\mathbf{B}^{n-j}}{(n-j)!} = \sum_{j=0}^{\infty} \frac{\mathbf{A}^j}{j!} \left[\sum_{n=j}^{\infty} \frac{\mathbf{B}^{n-j}}{(n-j)!} \right] \\
&= \sum_{j=0}^{\infty} \frac{\mathbf{A}^j}{j!} e^{\mathbf{B}} = e^{\mathbf{A}} e^{\mathbf{B}}.
\end{aligned}$$

□

Matrix exponential.

Proof.

For (2), we only need to show $e^{\mathbf{A}}e^{-\mathbf{A}} = \mathbf{I}$. Since

$$e^{\mathbf{A}}(e^{\mathbf{A}})^{-1} = \mathbf{I} = \exp(\mathbf{O}_{n\times n}) = e^{\mathbf{A}-\mathbf{A}} = e^{\mathbf{A}}e^{-\mathbf{A}}. \text{ since } \mathbf{A}(-\mathbf{A}) = (-\mathbf{A})\mathbf{A}.$$

For (3), we observe

$$\begin{aligned} \exp(\mathbf{PAP}^{-1}) &= \sum_{n=0}^{\infty} \frac{(\mathbf{PAP}^{-1})^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{\mathbf{PA}^n\mathbf{P}^{-1}}{n!} \\ &= \mathbf{P}\left[\sum_{n=0}^{\infty} \frac{\mathbf{A}^n}{n!}\right]\mathbf{P}^{-1} \\ &= \mathbf{P}e^{\mathbf{A}}\mathbf{P}^{-1} \end{aligned}$$

□

Matrix exponential.

Corollary

If $\mathbf{A}$ is diagonalizable, then there exists an invertible matrix $\mathbf{P}$ such that $\mathbf{A} = \mathbf{P}diag(\lambda_1, \cdots, \lambda_n)\mathbf{P}^{-1}$, then $e^{\mathbf{A}} = \mathbf{P}diag(e^{\lambda_1}, \cdots, e^{\lambda_n})\mathbf{P}^{-1}$.

Matrix exponential.

$$e^{\mathbf{A}t} = \sum_{p=0}^{\infty} \frac{\mathbf{A}^p t^p}{p!}.$$

Since $e^{\mathbf{A}t}$ is defined as a convergent matrix power series, it is differentiable and can be differentiated term by term (The proof will not be shown in this course). Since

$$\begin{aligned}\frac{d}{dt}(e^{\mathbf{A}t}) =& \frac{d}{dt}\sum_{p=0}^{\infty} \frac{\mathbf{A}^p t^p}{p!} = \sum_{p=0}^{\infty} \frac{d}{dt}\frac{\mathbf{A}^p t^p}{p!} = \sum_{p=1}^{\infty} p\frac{\mathbf{A}^p t^{p-1}}{p!} \\ =& \mathbf{A}\sum_{p=1}^{\infty} \frac{\mathbf{A}^{p-1} t^{p-1}}{(p-1)!} = \mathbf{A}e^{\mathbf{A}t} = \sum_{p=1}^{\infty} \frac{\mathbf{A}^{p-1} t^{p-1}}{(p-1)!}\mathbf{A} = e^{\mathbf{A}t}\mathbf{A}\end{aligned}$$

Indeed, we have the following theorem.

Theorem 11.3

$$\frac{d(e^{\mathbf{A}t})}{dt} = (e^{\mathbf{A}t})' = \mathbf{A}e^{\mathbf{A}t} = e^{\mathbf{A}t}\mathbf{A}. \tag{10}$$

Matrix exponential.

Thus, each column of $e^{\mathbf{A}t}$ is a solution for $\mathbf{x}' = \mathbf{A}\mathbf{x}$, moreover, $e^{\mathbf{A}t}|_{t=0} = I$. $W[e^{\mathbf{A}t}](t=0) = 1 \neq 0$. Thus, $e^{\mathbf{A}t}$ is the fundamental matrix for the system

$$\mathbf{x}' = \mathbf{A}\mathbf{x}.$$

The general solution for above ODE is

$$\mathbf{x} = e^{\mathbf{A}t}\mathbf{c}, - - - - - - - - - (*)$$

where $\mathbf{c} = [c_1, \cdots, c_n]^T$ is an arbitary constant vector.
Now we look for the solution for IVP

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \quad \mathbf{x}(t_0) = \mathbf{x}_0.$$

Substituting the initial condition for the above general solution (*), one has

$$\mathbf{x}_0 = e^{\mathbf{A}t_0}\mathbf{c}.$$

Thus, $\mathbf{c} = (e^{\mathbf{A}t_0})^{-1}\mathbf{x}_0 = e^{-\mathbf{A}t_0}\mathbf{x}_0$. Therefore,

$$\mathbf{x} = e^{\mathbf{A}t}\mathbf{c} = e^{\mathbf{A}t}e^{-\mathbf{A}t_0}\mathbf{x}_0 = e^{\mathbf{A}(t-t_0)}\mathbf{x}_0.$$

Matrix exponential for 2×2 matrix

Example 11.4

For

$$\mathbf{y}' = \mathbf{A}\mathbf{y}, \quad \mathbf{A} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$$

with eigenvalues and corresponding eigenvectors

$$r_1 = 3, \qquad \xi_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \qquad r_2 = -1, \qquad \xi_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix},$$

A is diagonalizable,

$$\mathbf{A} = \mathbf{P}\,diag(3, -1)\mathbf{P}^{-1}, \quad \mathbf{P} = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}, \quad \mathbf{P}^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{4} \end{pmatrix}$$

Matrix exponential for 2×2 matrix

Example continue

$$
\begin{aligned}
e^{\mathbf{A}t} =& \mathbf{P} e^{diag(3,-1)t} \mathbf{P}^{-1} \\
=& \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{4} \end{pmatrix} = \begin{pmatrix} \frac{e^{3t}+e^{-t}}{2} & \frac{e^{3t}-e^{-t}}{4} \\ e^{3t} - e^{-t} & \frac{e^{3t}+e^{-t}}{2} \end{pmatrix}
\end{aligned}
$$

The general solution is

$$
\begin{aligned}
\mathbf{y}(t) =& e^{\mathbf{A}t}\mathbf{c} \\
=& \mathbf{P} e^{diag(3,-1)t} \mathbf{P}^{-1}\mathbf{c} \\
=& \mathbf{P} e^{diag(3,-1)t} \mathbf{d} \\
=& \begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = d_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + d_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{t}
\end{aligned}
$$

Fundamental matrix

Remark 1

If $\mathbf{A}$ is diagonalizable, then $\mathbf{A} = \mathbf{P}\mathbf{\Lambda}\mathbf{P}^{-1}$, then $e^{\mathbf{A}t} = \mathbf{P}e^{\mathbf{\Lambda t}}\mathbf{P}^{-1}$ is the fundamental matrix for the system

$$\mathbf{x}' = \mathbf{A}\mathbf{x}$$

satisfying $e^{\mathbf{A}t}|_{t=0} = I$. Moreover, $\Psi(t) = \mathbf{P}e^{\mathbf{\Lambda t}}$ is also a fundamental matrix.

Matrix exponential for 3×3 matrix

Example 11.5

Find the fundamental solution matrix $\Phi(t)$ satisfying $\Phi(0) = \mathbf{I}$ and the general solution for system

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \mathbf{x}$$

You can verify that the three eigen-pairs of $\mathbf{A}$ is $(r_1, \xi^{(1)}), (r_2, \xi^{(2)}), (r_3, \xi^{(3)})$, where

$$r_1 = 2, \xi^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}; \; r_2 = -1, \xi^{(2)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}; \; r_3 = -1, \xi^{(3)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix};$$

Hence we set

$$\mathbf{\Lambda} = diag(r_1, r_2, r_3), \qquad \mathbf{P} = \begin{bmatrix} \xi^{(1)} & \xi^{(2)} & \xi^{(3)} \end{bmatrix}$$

Matrix exponential for 3×3 matrix

Example 11.6

For fundamental solution matrix satisfying $\Phi(0) = \mathbf{I}$, we have:

$$\Phi(t) = e^{\mathbf{A}t} = \mathbf{P}e^{\Lambda t}\mathbf{P}^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} e^{-2t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-t} \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} e^{2t} & e^{-t} & 0 \\ e^{2t} & 0 & e^{-t} \\ e^{2t} & -e^{-t} & -e^{-t} \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t} & \frac{1}{3}e^{2t} - \frac{1}{3}e^{-t} & \frac{1}{3}e^{2t} - \frac{1}{3}e^{-t} \\ \frac{1}{3}e^{2t} - \frac{1}{3}e^{-t} & \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t} & \frac{1}{3}e^{2t} - \frac{1}{3}e^{-t} \\ \frac{1}{3}e^{2t} - \frac{1}{3}e^{-t} & \frac{1}{3}e^{2t} - \frac{1}{3}e^{-t} & \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t} \end{pmatrix}.$$

However, if A is not diagonalizable, compute e^{A} is quite involving.

Semisimple-Nilpotent decomposition (S-N decomposition)

Definition 11.7

A square matrix is called Semisimple if it is diagonalizable.
And a square matrix A is called Nilpotent if there is some positive integer k s.t. $A^k = O$.

S-N decomposition (Semisimple-Nilpotent decomposition)

Theorem 11.8

(Semisimple-Nilpotent decomposition)
Let $\mathbf{A}$ *be an* $n \times n$ *matrix. Then, there exist two* $n \times n$ *matrices* $\mathbf{S}$ *and* $\mathbf{N}$ *such that*
(a) $\mathbf{S}$ *is diagonalizable (semisimple),*
(b) $\mathbf{N}$ *is Nilpotent,*
(c) $\mathbf{A} = \mathbf{S} + \mathbf{N}$,
(d) $\mathbf{SN} = \mathbf{NS}$.
The two matrices $\mathbf{S}$ *and* $\mathbf{N}$ *are uniquely determined by these four conditions.*

We skip the proof for this theorem, you could check the book "*Basic Theory of Ordinary Differential Equations, Po-Fang Hsieh, Yasutaka Sibuya, Springer, 1999*" .

S-N decomposition

Although $S - N$ decomposition exists for any $n \times n$ matrix, but the construction of S and N for a general $n \times n$ matrix is related to Jordan form of the matrix (I think most students don't known). Thus, for simplicity, in this course, we only focus on the $S - N$ decomposition for 2×2 and 3×3 matrices with one single eigenvalue:

Theorem 11.9 (S-N decomposition)

Let $\mathbf{A}$ *be a* 2×2 *or* 3×3 *matrix which has only one distinct eigenvalue* r*. Then* $\mathbf{A}$ *could be decomposed as* $\mathbf{A} = \mathbf{S} + \mathbf{N}$ *such that*

1. $\mathbf{S} = r\mathbf{I}$.
2. $\mathbf{N} = \mathbf{A} - \mathbf{S}$.
3. $\mathbf{N}^2 = \mathbf{O}$ *or* $\mathbf{N}^3 = \mathbf{O}$.
4. $\mathbf{SN} = \mathbf{NS}$

We also skip the proof for this theorem, you could check the book "*Basic Theory of Ordinary Differential Equations, Po-Fang Hsieh, Yasutaka Sibuya, Springer, 1999*" .

S-N decomposition

Fact

For $\mathbf{A}$ be a 2×2 or 3×3 matrix which has only one distinct eigenvalue r, we decompose it as

$$\mathbf{A} = \mathbf{S} + \mathbf{N},$$

where $\mathbf{S} = r\mathbf{I}$, $\mathbf{N} = \mathbf{A} - \mathbf{S}$, $\mathbf{N}^2 = \mathbf{O}$ or $\mathbf{N}^3 = \mathbf{O}$. It follows that

$$e^{\mathbf{A}t} = e^{(\mathbf{S}+\mathbf{N})t} = e^{\mathbf{S}t+\mathbf{N}t}.$$

Since $\mathbf{SN} = \mathbf{NS}$, we derive:

$$e^{\mathbf{A}t} = e^{\mathbf{S}t}e^{\mathbf{N}t}$$

where

$$e^{\mathbf{S}t} = e^{\mathbf{I}(rt)} = e^{rt}\mathbf{I}, \quad e^{\mathbf{N}t} = \sum_{k=0}^{\infty} \frac{(\mathbf{N}t)^k}{k!} = \sum_{k=0}^{2} \frac{(\mathbf{N}t)^k}{k!} = \mathbf{I} + \mathbf{N}t + \frac{1}{2}\mathbf{N}^2t^2.$$

Hence

$$e^{\mathbf{A}t} = e^{rt}\left(\mathbf{I} + \mathbf{N}t + \frac{1}{2}\mathbf{N}^2t^2\right).$$

S-N decomposition for 2×2 matrix

Example 11.10

Find a fundamental solution matrix of

$$\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \mathbf{x}.$$

The eigenvalues of $\mathbf{A}$ must satisfy:

$$\det(\mathbf{A} - r\mathbf{I}) = \begin{vmatrix} 1-r & -1 \\ 1 & 3-r \end{vmatrix} = (r-2)^2 \implies r = 2.$$

We perform the S-N decompositon for $\mathbf{A}$:

$$\mathbf{A} = \mathbf{S} + \mathbf{N} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}$$

S-N decomposition for 2×2 matrix

Example continue

And we observe
$$e^{\mathbf{S}t} = e^{2t} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\mathbf{N}^2 = \mathbf{O} \implies e^{\mathbf{N}t} = \mathbf{I} + \mathbf{N}t = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -t & -t \\ t & t \end{pmatrix} = \begin{pmatrix} 1-t & -t \\ t & 1+t \end{pmatrix}$$

The fundamental matrix is given by:

$$e^{\mathbf{A}t} = e^{\mathbf{S}t} e^{\mathbf{N}t} = e^{2t} \begin{pmatrix} 1-t & -t \\ t & 1+t \end{pmatrix}.$$

The general solution is given by:

$$\mathbf{x} = c_1 \begin{pmatrix} 1-t \\ t \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -t \\ 1+t \end{pmatrix} e^{2t}.$$

S-N decomposition for 3×3 matrix

Example 11.11

Find a fundamental solution matrix of

$$\mathbf{x}' = \begin{pmatrix} 5 & -3 & -2 \\ 8 & -5 & -4 \\ -4 & 3 & 3 \end{pmatrix} \mathbf{x}$$

The eigenvalues of $\mathbf{A}$ must satisfy:

$$\det(\mathbf{A} - r\mathbf{I}) = \begin{vmatrix} 5-r & -3 & -2 \\ 8 & -5-r & -4 \\ -4 & 3 & 3-r \end{vmatrix} = -(r-1)^3 \implies r = 1.$$

We perform the S-N decompositon for $\mathbf{A}$:

$$\mathbf{A} = \mathbf{S} + \mathbf{N} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{pmatrix}$$

S-N decomposition for 3×3 matrix

Example continue

And we observe

$$e^{\mathbf{S}t} = e^t \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\mathbf{N}^2 = \mathbf{O} \implies e^{\mathbf{N}t} = \mathbf{I} + \mathbf{N}t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 4t & -3t & -2t \\ 8t & -6t & -4t \\ -4t & 3t & 2t \end{pmatrix}$$

$$= \begin{pmatrix} 4t+1 & -3t & -2t \\ 8t & -6t+1 & -4t \\ -4t & 3t & 2t+1 \end{pmatrix}$$

S-N decomposition for 3×3 matrix

Example continue

The fundamental matrix is given by:

$$e^{\mathbf{A}t} = e^{\mathbf{S}t} e^{\mathbf{N}t} = e^t \begin{pmatrix} 4t+1 & -3t & -2t \\ 8t & -6t+1 & -4t \\ -4t & 3t & 2t+1 \end{pmatrix}$$

Thus the fundamental solution matrix is given by:

$$\mathbf{x} = c_1 \begin{pmatrix} 4t+1 \\ 8t \\ -4t \end{pmatrix} e^t + c_2 \begin{pmatrix} -3t \\ -6t+1 \\ 3t \end{pmatrix} e^t + c_3 \begin{pmatrix} -2t \\ -4t \\ 2t+1 \end{pmatrix} e^t.$$

S-N decomposition for 3×3 matrix

Example 11.12

The general solution is given by

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{pmatrix} \mathbf{x}$$

The eigenvalues of $\mathbf{A}$ must satisfy:

$$\det(\mathbf{A} - r\mathbf{I}) = \begin{vmatrix} 1-r & 1 & 1 \\ 2 & 1-r & -1 \\ -3 & 2 & 4-r \end{vmatrix} = -(r-2)^3 \implies r = 2.$$

We perform the S-N decompositon for $\mathbf{A}$:

$$\mathbf{A} = \mathbf{S} + \mathbf{N} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} + \begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -3 & 2 & 2 \end{pmatrix}$$

S-N decomposition for 3×3 matrix

Example continue

And we observe

$$e^{\mathbf{S}t} = e^{2t} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\mathbf{N}^2 = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 1 \\ 1 & -1 & -1 \end{pmatrix}, \mathbf{N}^3 = \mathbf{O}$$

It follows that

$$e^{\mathbf{N}t} = \mathbf{I} + \mathbf{N}t + \frac{1}{2}\mathbf{N}^2 t^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} -t & t & t \\ 2t & -t & -t \\ -3t & 2t & 2t \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ -\frac{t^2}{2} & \frac{t^2}{2} & \frac{t^2}{2} \\ \frac{t^2}{2} & -\frac{t^2}{2} & -\frac{t^2}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1-t & t & t \\ 2t - \frac{t^2}{2} & 1 - t + \frac{t^2}{2} & -t + \frac{t^2}{2} \\ -3t + \frac{t^2}{2} & 2t - \frac{t^2}{2} & 1 + 2t - \frac{t^2}{2} \end{pmatrix}.$$

S-N decomposition for 3×3 matrix

Example continue

The fundamental matrix is given by:

$$e^{\mathbf{A}t} = e^{\mathbf{S}t} e^{\mathbf{N}t} = e^{2t} \begin{pmatrix} 1-t & t & t \\ 2t - \frac{t^2}{2} & 1 - t + \frac{t^2}{2} & -t + \frac{t^2}{2} \\ -3t + \frac{t^2}{2} & 2t - \frac{t^2}{2} & 1 + 2t - \frac{t^2}{2} \end{pmatrix}$$

Thus the general solution is given by:

$$\mathbf{x} = c_1 \begin{pmatrix} 1-t \\ 2t - \frac{t^2}{2} \\ -3t + \frac{t^2}{2} \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} t \\ 1 - t + \frac{t^2}{2} \\ 2t - \frac{t^2}{2} \end{pmatrix} e^{2t} + c_3 \begin{pmatrix} t \\ -t + \frac{t^2}{2} \\ 1 + 2t - \frac{t^2}{2} \end{pmatrix} e^{2t}.$$

S-N decomposition for matrix

Up to now, we can solve $e^{\mathbf{A}t}$ when $\mathbf{A}$ is a 2×2 or 3×3 matrix with a single eigenvalue.

For a general $n \times n$ matrix $\mathbf{A}$, to compute the matrix exponential $e^{\mathbf{A}t}$, we need to use the Jordan form of the matrix $\mathbf{A}$ or find the $S - N$ composition of $\mathbf{A}$. However, I think most students are not familiar with Jordan form of the matrix and also the construction of $S - N$ decomposition for the general $n \times n$ matrix has also not been taught in the linear algebra course.

Thus, in this course, instead of computing the matrix exponential $e^{\mathbf{A}t}$ directly for a general $n \times n$ matrix $\mathbf{A}$, we will provide another method to solve the general $n \times n$ linear system of ODEs.

Outline

Appendix:Matrix exponential.

Definition 11.13 (Matrix Norm)

The *norm* of a matrix $\mathbf{A}$ is the number

$$||\mathbf{A}|| = \sup_{\mathbf{x} \neq \mathbf{0}} \frac{||\mathbf{Ax}||}{||\mathbf{x}||}$$

Or equivalently,

$$||\mathbf{A}|| = \sup_{||\mathbf{x}||=1} ||\mathbf{Ax}||$$

the *sphere* $||\mathbf{Ax}||$ for $||\mathbf{x}|| = 1$ is compact, and $||\mathbf{Ax}||$ is continuous, thus $0 < ||\mathbf{A}|| < \infty$.

Appendix:Matrix exponential.

Definition 11.14 (Matrix Norm)

A matrix norm on the set of all $n \times n$ matrices is a real-valued function $\|\cdot\|$, defined on this set, satisfying for all $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ and all real numbers α:

(i) $\|\mathbf{A}\| \geq 0$,
(ii) $\|\mathbf{A}\| = 0$ if and only if $\mathbf{A} = \mathbf{O}$,
(iii) $\|\alpha \mathbf{A}\| = |\alpha| \|\mathbf{A}\|$,
(iv) $\|\mathbf{A} + \mathbf{B}\| \leq \|\mathbf{A}\| + \|\mathbf{B}\|$.
(v) $\|\mathbf{AB}\| \leq \|\mathbf{A}\| \|\mathbf{B}\|$.

Appendix:Matrix exponential.

Definition 11.15 (Vector Norm)

Let $\mathbf{x} = [x_1, \cdots, x_n]^T \in \mathbb{R}^n$, then

$$\|\mathbf{x}\|_1 = \sum_{1 \le i \le n} |x_i|.$$

$$\|\mathbf{x}\|_\infty = \max_{1 \le i \le n} |x_i|.$$

$$\|\mathbf{x}\|_2 = \sqrt{\mathbf{x}^T \mathbf{x}} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

Theorem 11.16 (Induced Matrix Norm)

Let $\mathbf{A} \in \mathbb{R}^{n \times n}$, if $\| \cdot \|_v$ is a vector norm defined in $\mathbb{R}^n$, then

$$\|\mathbf{A}\|_v = \max_{\mathbf{x} \in \mathbb{R}^n} \frac{\|\mathbf{A}\mathbf{x}\|_v}{\|\mathbf{x}\|_v} = \max_{\|\mathbf{y}\|_v = 1, \mathbf{y} \in \mathbb{R}^n} \|\mathbf{A}\mathbf{y}\|_v$$

defines a matrix norm. Matrix norms defined from vector norms are called the induced matrix norm.

Appendix:Matrix exponential.

Theorem 11.17 (Matrix 1-Norm, 2-norm, ∞-norm)

Let $\mathbf{A} = [a_{ij}]_{n\times n} \in \mathbb{R}^{n\times n}$*, then the induced matrix 1-norm, 2-norm,* ∞*-norm are given as follows:*

$$\|\mathbf{A}\|_\infty = \max_{1\le i\le n} \sum_{1\le j\le n} |a_{ij}|.\text{(row norm)}$$

$$\|\mathbf{A}\|_1 = \max_{1\le j\le n} \sum_{1\le i\le n} |a_{ij}|.\text{(column norm)}.$$

$$\|\mathbf{A}\|_2 = \sqrt{\lambda_{max}(A^T A)},$$

where $\lambda_{max}(A^T A)$ *is the maximum eigenvalue of* $A^T A$*.*

The proof for this theorem is skipped. You can find the above results in the book: Richard L. Burden, J. Douglas Faires, Annete M. Burden, Numerical Analysis, 10th ed, Cengage Learning, 2015.

Appendix:Matrix exponential.

Example 11.18 (Matrix 1-Norm, 2-norm, ∞-norm)

$$A = \begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix}.$$

Then

$$\begin{aligned} \|A\|_1 &= \max\{1 + |-3|, |-2| + 4\} = 6 \\ \|A\|_\infty &= \max\{1 + |-2|, |-3| + 4\} = 7 \\ \|A\|_2 &= \sqrt{15 + \sqrt{221}} \approx 5.46. \end{aligned}$$

Appendix:Matrix exponential.

Definition 11.19

A sequence of $r \times r$ matrices, $\{\mathbf{A}_n\}$, is called convergent if for any given $\epsilon > 0$ there exists $N > 0$, such that

$$\|\mathbf{A}_n - \mathbf{A}_m\| < \epsilon, \qquad \forall m, n > N,$$

where the matrix norm $\|\cdot\|$ could be the 1-norm, 2-norm, or ∞-norm for matrices.

Appendix:Matrix exponential.

Theorem 11.20

Every convergent sequence of matrices $\{\mathbf{A}_n\}$ has a limit.

Proof.

Let a_{ij}^n, $1 \le i, j \le r$, $n = 1, 2, \cdots$, be the components of $\mathbf{A}_n$. For any given $\epsilon > 0$ there exists $N > 0$, such that

$$|a_{ij}^n - a_{ij}^m| \le \|\mathbf{A}_n - \mathbf{A}_m\| < \epsilon, \qquad \forall 1 \le i, j \le r, \quad \forall m, n > N,$$

where the matrix norm $\|\cdot\|$ could be the 1-norm, 2-norm, or ∞-norm.
Then $\{a_{ij}^n\}$ is a Cauchy sequence, and converges, *i.e.*, $a_{ij}^n \to a_{ij}$, as $n \to \infty$. Let $\mathbf{A} = [a_{ij}]$, then $\mathbf{A}$ is the limit of $\{\mathbf{A}_n\}$. □

Appendix:Matrix exponential.

Theorem 11.21

The series

$$\mathbf{I} + \sum_{n=1}^{\infty} \frac{\mathbf{A}^n t^n}{n!}. \tag{11}$$

is convergent for any finite number t, the limit matrix is defined as $e^{\mathbf{A}t}$.

Proof

Let

$$\mathbf{S}_n = \sum_{k=0}^{n} \frac{\mathbf{A}^k t^k}{k!},$$

then for $n > m$ we have

$$\begin{aligned}\|\mathbf{S}_n - \mathbf{S}_m\| &= \left\| \sum_{k=0}^{n} \frac{\mathbf{A}^k t^k}{k!} - \sum_{k=0}^{m} \frac{\mathbf{A}^k t^k}{k!} \right\| = \left\| \sum_{k=m+1}^{n} \frac{\mathbf{A}^k t^k}{k!} \right\| \\ &\le \sum_{k=m+1}^{n} \frac{\|\mathbf{A}^k\| |t|^k}{k!} \le \sum_{k=m+1}^{n} \frac{\|\mathbf{A}\|^k |t|^k}{k!}.\end{aligned}$$

where the matrix norm $\|\cdot\|$ could be the 1-norm, 2-norm, or ∞-norm.

Appendix:Matrix exponential.

Proof continue

Since the series

$$\sum_{k=0}^{\infty} \frac{\|\mathbf{A}\|^k |t|^k}{k!} = e^{\|\mathbf{A}\||t|}$$

converges for any $\|\mathbf{A}\||t|$ (t is finite), then for any given $\epsilon > 0$ there exists N, such that

$$\sum_{k=m+1}^{n} \frac{\|\mathbf{A}\|^k |t|^k}{k!} < \epsilon, \qquad \forall n > m > N,$$

i.e., $\{\mathbf{S}_n\}$ converges. The limit matrix is defined as $e^{\mathbf{A}t}$.