MAT3253 Homework 14

Due date: 7 May.

Question 1. (Bak&Newman Chapter 10 Ex.8)

(a) Show that Rouche's Theorem remains valid if the condition $|f| > |g|$ on γ is replaced by $|f| \geq |g|$ and $f + g \neq 0$ on γ.

(b) Find the number of zeroes of $z^5 + 2z^4 + 1$ in the unit disc.

Question 2. (Bak&Newman Chapter 10 Ex.9) Find the number of zeros of

a. $f_1(z) = 3e^z - z \quad \text{in } |z| \leq 1$

b. $f_2(z) = \dfrac{1}{3}e^z - z \quad \text{in } |z| \leq 1$

c. $f_3(z) = z^4 - 5z + 1 \quad \text{in } 1 \leq |z| \leq 2$

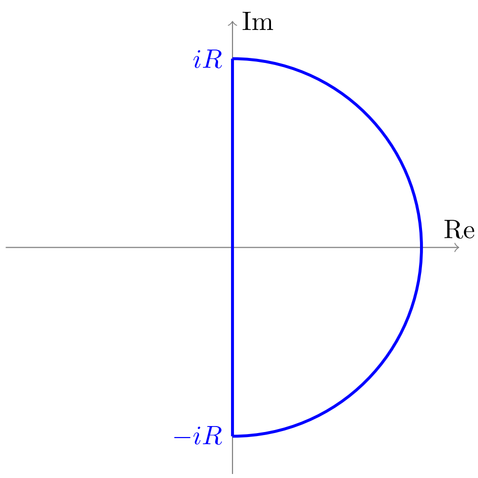
d. $f_4(z) = z^6 - 5z^4 + 3z^2 - 1 \quad \text{in } |z| \leq 1$

Question 3. (Bak&Newman Chapter 10 Ex.10) Suppose $\lambda > 1$. Show that

$$\lambda - z - e^{-z} = 0$$

has exactly one root (which is a real number) in the right half-plane.

Hint: Consider the following contour with clockwise orientation and show that the function $\lambda - z - e^{-z}$ has exactly one zero inside the contour for all sufficiently larger R.



Question 4. (Bak&Newman Chapter 11 Ex.1) Evaluate the following definite integrals

a. $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)^2}\,dx,$

b. $\displaystyle\int_{0}^{\infty} \frac{x^2}{(x^2+4)^2(x^2+9)}\,dx,$

c. $\displaystyle\int_{0}^{\infty} \frac{1}{x^4+x^2+1}\,dx,$

d. $\displaystyle\int_{0}^{\infty} \frac{\sin x}{x(1+x^2)}\,dx,$

e. $\displaystyle\int_{0}^{\infty} \frac{\cos x}{1+x^2}\,dx,$

f. $\displaystyle\int_{0}^{\infty} \frac{1}{x^3+8}\,dx,$

g. $\displaystyle\int_{0}^{\infty} \frac{x^{\alpha-1}}{1+x}\,dx, \quad 0<\alpha<1,$

h. $\displaystyle\int_{0}^{2\pi} \frac{1}{(2+\cos x)^2}\,dx,$

i. $\displaystyle\int_{0}^{2\pi} \frac{\sin^2 x}{5+3\cos x}\,dx,$

j. $\displaystyle\int_{0}^{2\pi} \frac{1}{a+\cos x}\,dx \qquad (a \text{ real}), |a|>1.$

Question 5. Suppose f is a function in the form $p(z)/q(z)$ with a *simple pole* at z_0 and $p(z)$ is analytic in a neighborhood of z_0. Show that the residue of $f(z)$ at z_0 is

$$Res(\frac{p(z)}{q(z)}; z_0) = \frac{p(z_0)}{q'(z_0)}.$$

Question 6. Determine whether the following functions are analytic at the point at infinity. If it is not analytic at the point at infinity, classify the type of singularity.

(a). $\dfrac{1}{z^4+z}$

(b). z^3+1

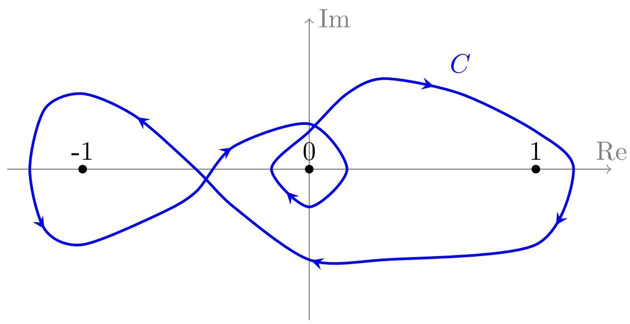
(c). $\sin(z)$

(d). $\dfrac{z^3+z^2+i}{z-2}$

Question 7. Evaluate the contour integral

$$\int_C \frac{1}{z(z^2-1)}\,dz$$

where C is the contour below.



Question 8. Use residue at infinity to compute the integral in Question 7 with the contour replaced by $|z|=2$ in the counter-clockwise direction.