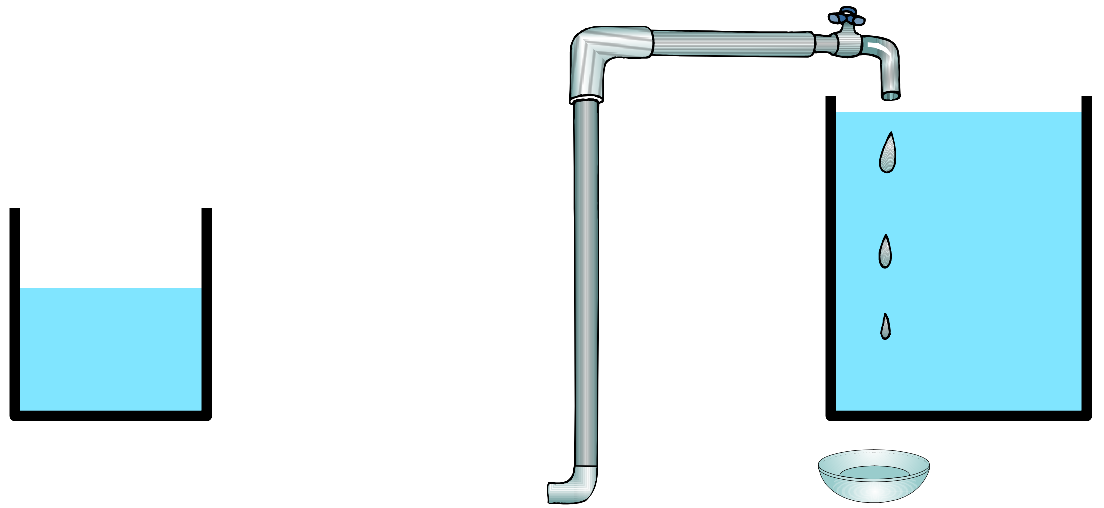
Greatest Common Divisors



3 Gallon Jug 5 Gallon Jug

Common Divisors

c is a *common divisor* of a and b means $c|a$ and $c|b$.

$\gcd(a,b)$ ::= the **greatest common divisor** of a and b.

Say $a=8$, $b=10$, then 1,2 are common divisors, and $\gcd(8,10)=2$.

Say $a=10$, $b=30$, then 1,2,5,10 are common divisors, and $\gcd(10,30)=10$.

Say $a=3$, $b=11$, then the only common divisor is 1, and $\gcd(3,11)=1$.

Claim. If p is prime, and p does not divide a, then $\gcd(p,a) = 1$.

The Quotient-Remainder Theorem

For b > 0 and any a, there are *unique* integers

q ::= quotient(a,b),   r ::= remainder(a,b), such that

a = qb + r   and   0 ≤ r < b.

We also say   **q = a div b**   and   **r = a mod b.**

When b=2, there is a unique q such that

a=2q or a=2q+1.

$$q = \lfloor \frac{a}{2} \rfloor$$

When b=3, there is a unique q such that

a=3q or a=3q+1 or a=3q+2.

$$q = \lfloor \frac{a}{3} \rfloor$$

Floor function $\lfloor x \rfloor$ **:** the greatest integer that is ≤ x

The Quotient-Remainder Theorem

For $b > 0$ and any a, there are *unique* integers

q ::= quotient(a,b), r ::= remainder(a,b), such that

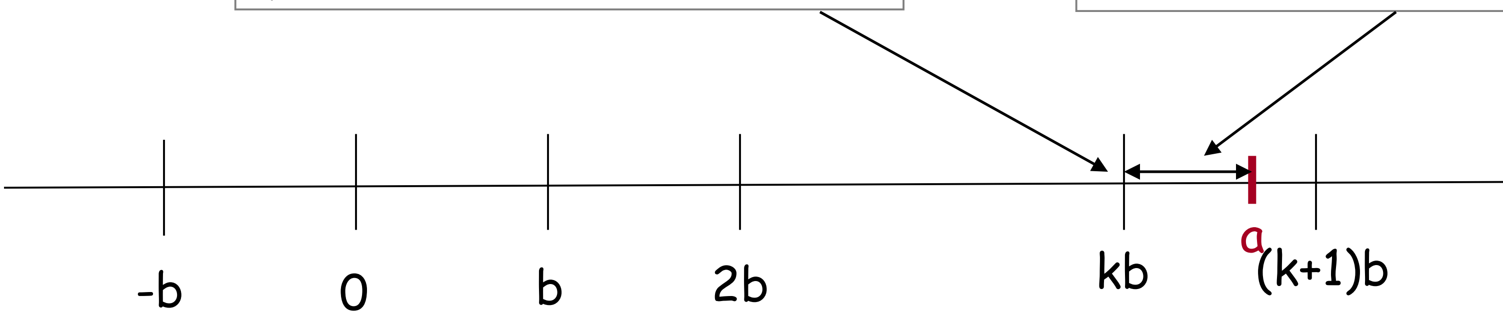
$$a = qb + r \quad \text{and} \quad 0 \le r < b.$$

Given any b, we can partition the integers into blocks of b numbers.

For any a, there is a unique "position" for this number.

q = the block which a lies in

r = the offset in this block



Clearly, given a and b, the numbers q and r are uniquely determined.

Greatest Common Divisors

Given a and b, how to compute gcd(a,b)?

Maybe try every number? Not easy for large numbers…

Do we have a better way to do it?

Let's say $a \geq b > 0$.

1. If a=kb, then gcd(a,b)=b, and we are done.
2. Otherwise, by the Division Theorem, a = qb + r where r>0.

Greatest Common Divisors

Let's say $a \geq b$.

1. If $a=kb$, then $\gcd(a,b)=b$, and we are done.
2. Otherwise, by the Division Theorem, $a = qb + r$ where $r>0$.

$a=12, b=8 \Rightarrow 12 = 8 + 4$	$\gcd(12,8) = 4$	$\gcd(8,4) = 4$
$a=21, b=9 \Rightarrow 21 = 2\times 9 + 3$	$\gcd(21,9) = 3$	$\gcd(9,3) = 3$
$a=99, b=27 \Rightarrow 99 = 3\times 27 + 18$	$\gcd(99,27) = 9$	$\gcd(27,18) = 9$

Euclid: $\gcd(a,b) = \gcd(b,r)$!

Euclid's GCD Algorithm

a = qb + r

Euclid: gcd(a,b) = gcd(b,r)!

Assumption: $a > b \geq 0$.

gcd(a,b)

if b = 0, then answer = a.

else

  write a = qb + r ← $q = \lfloor \frac{a}{b} \rfloor$ $r = a - qb$

  answer = gcd(b,r)

Example 1

```
gcd(a,b)
if b = 0, then answer = a.
else
  write a = qb + r
  answer = gcd(b,r)
```

GCD(102, 70)	102 = 70 + 32
= GCD(70, 32)	70 = 2x32 + 6
= GCD(32, 6)	32 = 5x6 + 2
= GCD(6, 2)	6 = 3x2 + 0
= GCD(2, 0)	

Return value: 2.

Example 2

```
gcd(a,b)
if b = 0, then answer = a.
else
  write a = qb + r
  answer = gcd(b,r)
```

$GCD(252, 189)$	$252 = 1 \times 189 + 63$
$= GCD(189, 63)$	$189 = 3 \times 63 + 0$
$= GCD(63, 0)$	

Return value: 63.

Example 3

```
gcd(a,b)
if b = 0, then answer = a.
else
  write a = qb + r
  answer = gcd(b,r)
```

GCD(662, 414)	662 = 1x414 + 248
= GCD(414, 248)	414 = 1x248 + 166
= GCD(248, 166)	248 = 1x166 + 82
= GCD(166, 82)	166 = 2x82 + 2
= GCD(82, 2)	82 = 41x2 + 0
= GCD(2, 0)	

Return value: 2.

Correctness of Euclid's GCD Algorithm

$\boxed{a = qb + r}$

$\boxed{\text{Euclid: } gcd(a,b) = gcd(b,r)}$

When r = 0:

Then $a = qb$, so $gcd(a, b) = b$;

$r = 0$, so $gcd(b, r) = gcd(b, 0) = b$.

Therefore, $gcd(a,b) = gcd(b,r)$.

Correctness of Euclid's GCD Algorithm

$a = qb + r$

Euclid: $gcd(a,b) = gcd(b,r)$

When $r > 0$:

Let d be a common divisor of b, r

$\Rightarrow b = k_1 d$ and $r = k_2 d$ for some k_1, k_2.

$\Rightarrow a = qb + r = qk_1 d + k_2 d = (qk_1 + k_2)d$ => d is a common divisor of a, b

Let d be a common divisor of a, b

$\Rightarrow a = k_3 d$ and $b = k_1 d$ for some k_1, k_3.

$\Rightarrow r = a - qb = k_3 d - qk_1 d = (k_3 - qk_1)d$ => d is a common divisor of b, r

So, {common factors of a, b} = {common factors of b, r}

$\Rightarrow gcd(a, b) = gcd(b, r)$.

Is Euclid's GCD Algorithm fast?

Naive algorithm: try every number.

Assumption: $a > b \geq 0$.

gcd(a,b)

Let d=1

1. If d|a and d|b, then store d.
2. Let d=d+1
3. If $d \leq b$, return to 1.

   else the answer = max of all stored "d"s

So the running time is about b iterations.

Is Euclid's GCD Algorithm fast?

Euclid's algorithm:

In two iterations, a, b are decreased by half. (why?)

$a = bq + r \geq b + r > 2r$

=> $gcd(a,b) = gcd(b,r)$ where $r < a/2$

Similarly, if $b = rq' + r'$, then

$gcd(b,r) = gcd(r,r')$ where $r' < b/2$

Supposing $b \approx 2^d$, then in the worst case, b keeps reducing until it gets down to roughly 1; so $b/2^d \approx 1$, or $d \approx \log_2 b$. Since the above shows since both the divisor and dividend has to be reduced, each reduction by $\frac{1}{2}$ is counted as 2 iterations; thus the number of iterations is $2d \approx 2\log_2 b$. So the running time is about $2\log_2 b$ iterations.

Exponentially faster!!

Linear Combination vs Common Divisor

Greatest common divisor

d is a common divisor of a and b if d|a and d|b

gcd(a,b) = greatest common divisor of a and b

Smallest positive integer linear combination

d is an **integer linear combination** of a and b if d=sa+tb for integers s,t.

spc(a,b) = **smallest** **positive** integer linear **combination** of a and b

Theorem. gcd(a,b) = spc(a,b)

Linear Combination vs Common Divisor

Theorem. $\gcd(a,b) = \text{spc}(a,b)$

The above is sometimes called Bezout's Identity.

For example, the *greatest* common divisor of 52 and 44 is 4.
And 4 is an integer linear combination of 52 and 44:

$$6 \cdot 52 + (-7) \cdot 44 = 4$$

Furthermore, no integer linear combination of 52 and 44 is equal to a smaller positive integer.

To prove the theorem, we will prove:

$\gcd(a,b) \leq \text{spc}(a,b)$

$\gcd(a,b) \mid \text{spc}(a,b)$

$\gcd(a,b) \geq \text{spc}(a,b)$

$\text{spc}(a,b)$ divides a and b

GCD ≤ SPC

Claim. If $d \mid a$ and $d \mid b$, then $d \mid sa + tb$ for any s,t.

Proof.

$d \mid a \Rightarrow a = dk_1$

$d \mid b \Rightarrow b = dk_2$

$sa + tb = sdk_1 + tdk_2 = d(sk_1 + tk_2)$

$\Rightarrow d \mid (sa+tb)$

GCD | SPC

Let $d = \gcd(a,b)$. By definition, $d \mid a$ and $d \mid b$.

Let $f = \mathrm{spc}(a,b) = sa+tb$

According to the claim, $d \mid f$. So $\gcd(a,b) \le \mathrm{spc}(a,b)$.

GCD $\geq$ SPC

We will prove that spc(a,b) is actually a common divisor of a and b.

First, show that spc(a,b) | a.

1. By the Division Theorem (since $a \geq spc(a,b)$),

$$a = q \times spc(a,b) + r \qquad \text{and} \qquad spc(a,b) > r \geq 0$$

2. Let $spc(a,b) = sa + tb$.
3. Then $r = a - q \times spc(a,b) = a - q \times (sa + tb) = (1-qs)a + qtb$.
4. So r is an integer linear combination of a and b with $spc(a,b) > r$.
5. This is only possible when $r = 0$.

Similarly, spc(a,b) | b.

Thus, spc(a,b) divides both a and b, which follows $spc(a,b) \leq gcd(a,b)$.

Application of Bezout's Identity

Theorem. $\gcd(a,b) = \mathrm{spc}(a,b)$

Lemma. If $\gcd(a,b)=1$ and $\gcd(a,c)=1$, then $\gcd(a,bc)=1$.

By Bezout's identity, there exist s,t,u,v such that

$$sa + tb = 1$$
$$ua + vc = 1$$

So $(sa + tb)(ua + vc) = 1$

Expanding LHS gives

$$saua + savc + tbua + tbvc = 1$$

$\Rightarrow (sau + svc + tbu)a + (tv)bc = 1$

This implies $\mathrm{spc}(a,bc)=1$. By Bezout's identity, we have $\gcd(a,bc)=1$.

Prime Divisibility

Theorem. $\gcd(a,b) = \mathrm{spc}(a,b)$

Lemma. p prime and $p|ab$ implies $p|a$ or $p|b$.

proof. W.l.o.g, assume p does not divide a. Then $\gcd(p,a)=1$.

So by Bezout's identity, there exist s and t such that

$$sa + tp = 1$$

$$\underbrace{(sa)b}_{p|ab} + \underbrace{(tp)b}_{p|p} = b$$

$\boxed{\text{Hence } p|b}$

Corollary. If p is prime, and $p|\, a_1 \cdot a_2 \cdots a_m$ then $p|a_i$ for some i.

Fundamental Theorem of Arithmetic

Every integer $n>1$ has a *unique* factorization into primes:

$$p_1 \le p_2 \le \cdots \le p_k$$

$$n = p_1 p_2 \cdots p_k$$

Example:

$61394323221 = 3 \cdot 3 \cdot 3 \cdot 7 \cdot 11 \cdot 11 \cdot 37 \cdot 37 \cdot 37 \cdot 53$

Unique Factorization

Theorem. There is a unique factorization.

Proof. Suppose there is a number with two different factorizations.

By Well Ordering Principle, we choose the smallest such $n > 1$:

$$n = p_1 \cdot p_2 \cdots p_k = q_1 \cdot q_2 \cdots q_m$$

Since n is smallest, we must have that $p_i \neq q_j$ all i,j

(Otherwise, if any $p_i = q_j$ then, by cancellation, $n/p_i = n/q_j$ would be another positive integer, smaller than n, which also has two distinct factorizations, contradicting that n is the smallest – the reduced factorizations, resulting from deleting identical factors on both sides, is distinct since if it is not, then the original factorization cannot be distinct)

contradiction!

Since $p_1 | n = q_1 \cdot q_2 \cdots q_m$, so by Corollary $p_1 | q_i$ for some i.

Since both p_1 , q_i are prime numbers, we must have $p_1 = q_i$.

Extended GCD Algorithm

How can we write gcd(a,b) as an integer linear combination?

This can be done by extending the Euclidean algorithm.

Example: $a = 259$, $b=70$

$259 = 3 \cdot 70 + 49$ $\quad$ $49 = a - 3b$

$70 = 1 \cdot 49 + 21$ $\quad$ $21 = 70 - 49$

$21 = b - (a-3b) = -a+4b$

$49 = 2 \cdot 21 + 7$ $\quad$ $7 = 49 - 2 \cdot 21$

$7 = (a-3b) - 2(-a+4b) = \underline{3a - 11b}$

$21 = 7 \cdot 3 + 0$ $\quad$ done, gcd = 7

Extended GCD Algorithm

Example: $a = 899$, $b=493$

$899 = 1\cdot 493 + 406$ so $406 = a - b$

$493 = 1\cdot 406 + 87$ so $87 = 493 - 406$

$= b - (a-b) = -a + 2b$

$406 = 4\cdot 87 + 58$ so $58 = 406 - 4\cdot 87$

$= (a-b) - 4(-a+2b) = 5a - 9b$

$87 = 1\cdot 58 + 29$ so $29 = 87 - 1\cdot 58$

$= (-a+2b) - (5a-9b) = \underline{-6a + 11b}$

$58 = 2\cdot 29 + 0$ done, gcd = 29

Die Hard

Simon says: On the fountain, there are 2 jugs, one is 5-gallon and the other is 3-gallon. Fill one with exactly 4 gallons of water and place it on the scale then the timer will stop. You must be precise; one ounce more or less will result in detonation. If you're still alive in 5 minutes, we'll speak.

Die Hard

Bruce: Wait, wait a second. I don't get it. Do you get it?

Samuel: No.

Bruce: Get the jugs. Obviously, we can't fill the 3-gallon jug with 4 gallons of water.

Samuel: Obviously.

Bruce: All right. I know, here we go. We fill the 3-gallon jug exactly to the top, right?

Samuel: Uh-huh.

Bruce: Okay, now we pour this 3 gallons into the 5-gallon jug, giving us exactly 3 gallons in the 5-gallon jug, right?

Samuel: Right, then what?

Bruce: All right. We take the 3-gallon jug and fill it a third of the way...

Samuel: No! He said, "Be precise." Exactly 4 gallons.

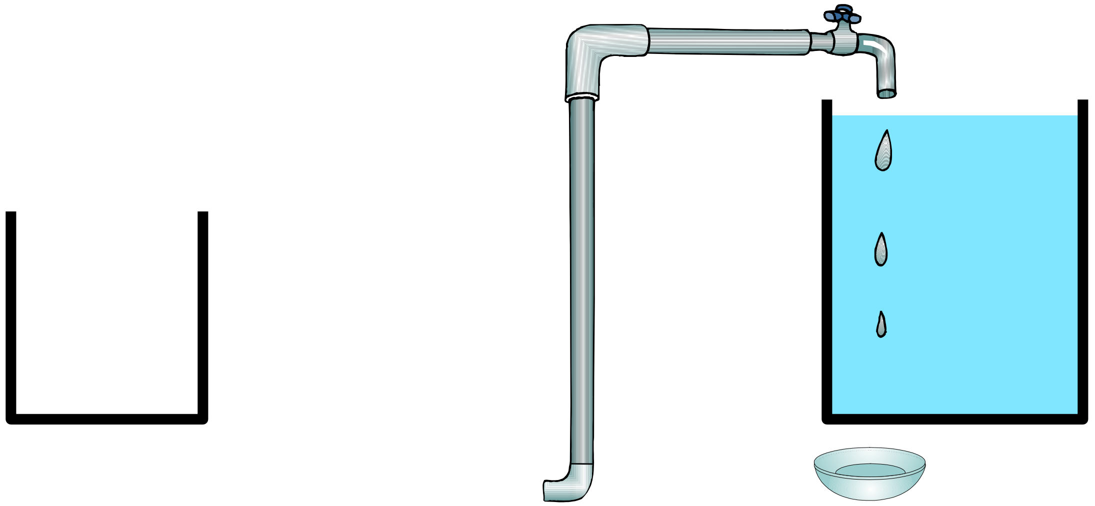
Bruce: Sh - -. Every cop within 50 miles is running his a** off and I'm out here playing kids games in the park.

Samuel: Hey, you want to focus on the problem at hand?

Die Hard

Start with empty jugs: (0,0)

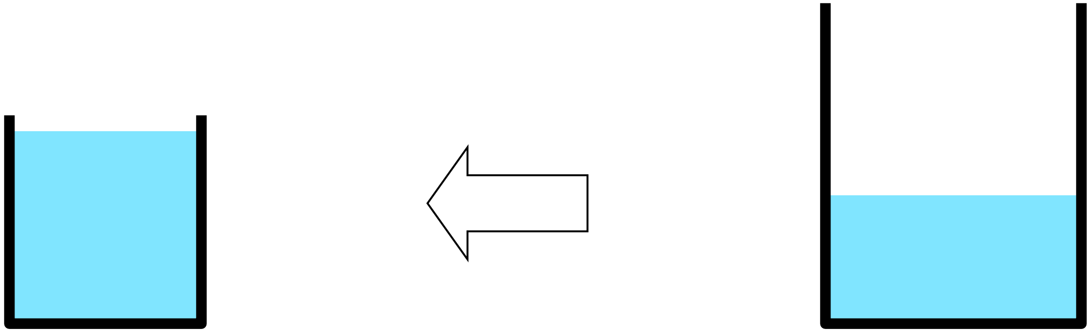
Fill the big jug: (0,5)



3-Gallon Jug 5-Gallon Jug

Die Hard

Pour from big to little: (3,2)

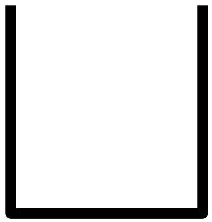


3-Gallon Jug

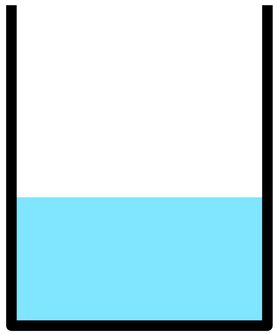
5-Gallon Jug

Die Hard

Empty the little: (0,2)



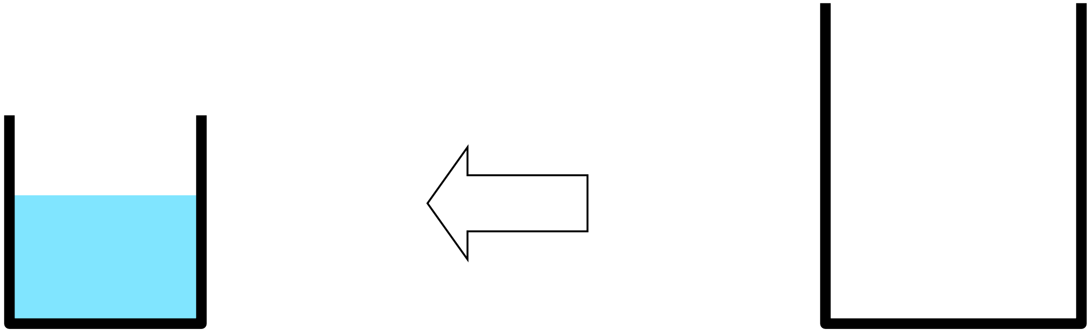
3-Gallon Jug



5-Gallon Jug

Die Hard

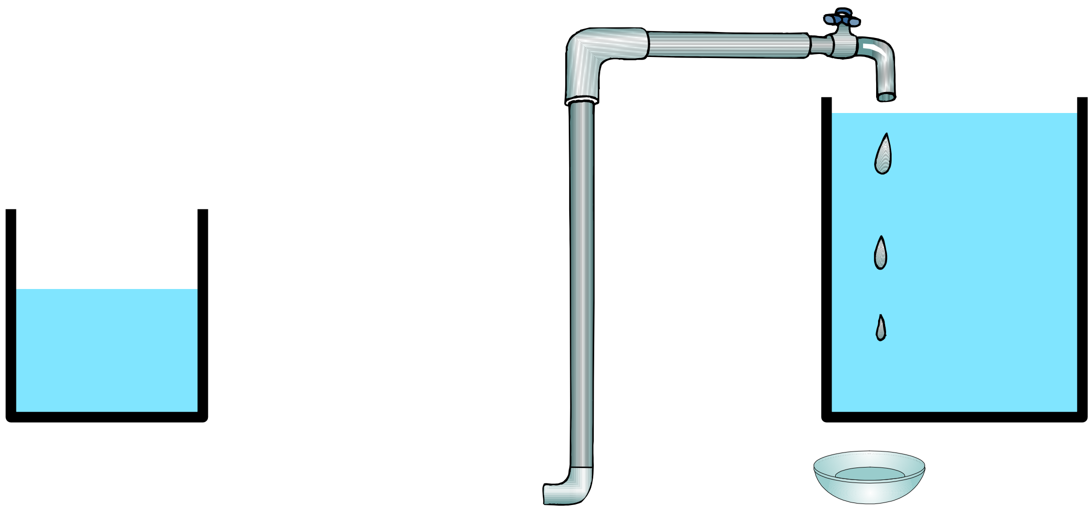
Pour from big to little: (2,0)



3-Gallon Jug

5-Gallon Jug

Fill the big jug: (2,5)

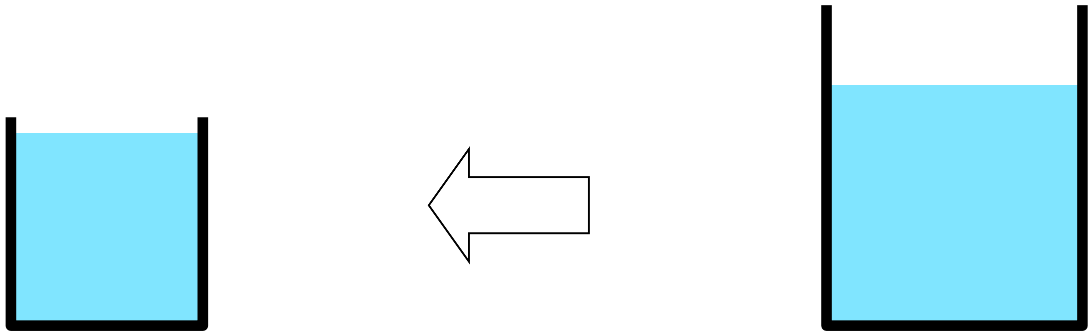


3-Gallon Jug

5-Gallon Jug

Die Hard

Pour from big to little: (3,4)

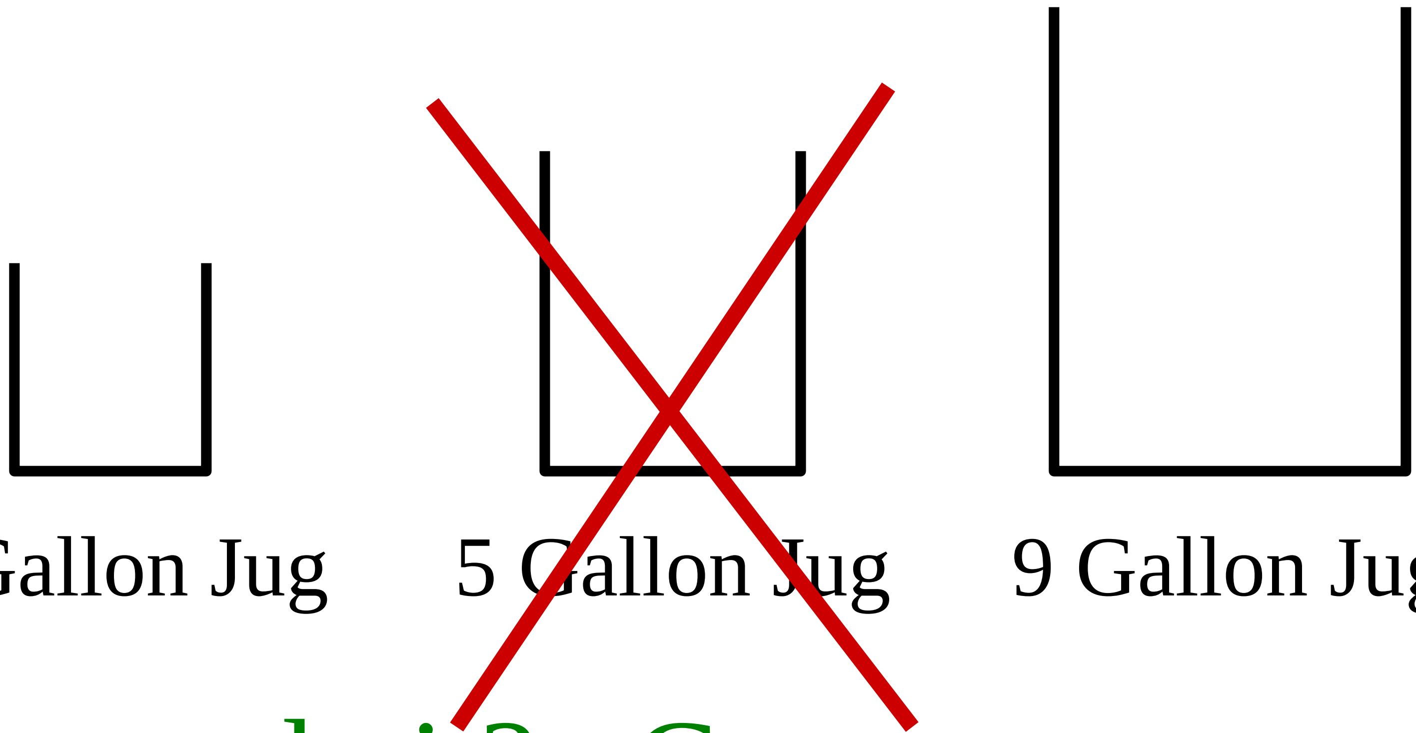


3-Gallon Jug　　　　5-Gallon Jug

Done!!

Die Hard

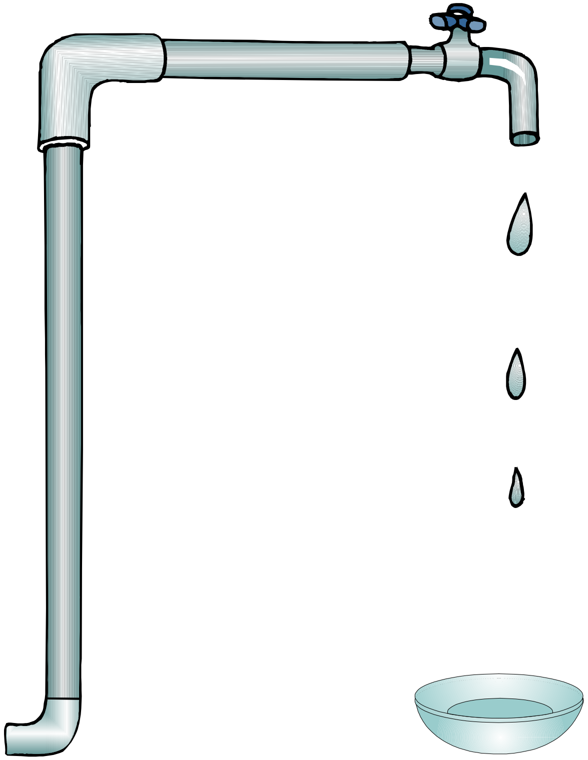
What if you have a 9 gallon jug instead?



Can you do it? Can you prove it?

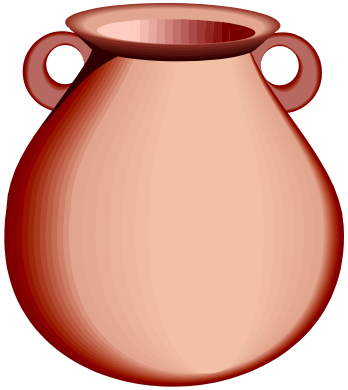
Die Hard

Supplies:



Water

3-Gallon Jug



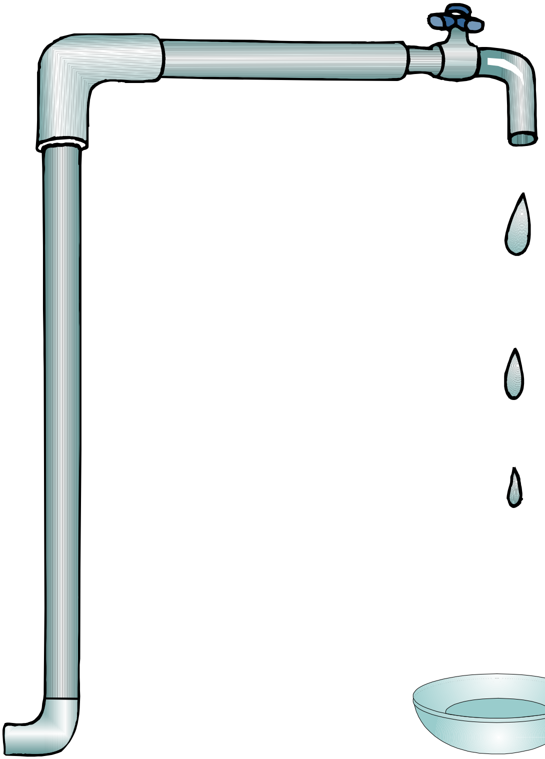
9-Gallon Jug

Invariant Method

Invariant: the number of gallons in each jug is a multiple of 3.

i.e., $3|L$ and $3|B$ (3 divides both L and B)

Corollary. It is impossible to have exactly 4 gallons in one jug.



Bruce Dies!

Generalized Die Hard

Can Bruce form 3 gallons using 21 and 26-gallon jugs?

This question is not so easy to answer without number theory.

The Amount of Water in Each Jug

The amount of water in each jug is always an integer linear combination of their capacities

Suppose we have two jugs with capacities a and b, respectively, with $a < b$. We shall carry out a few operations and see what happens. The state of the system at each step is represented by (x, y), where x is the amount of water in the first jug, and y, the amount in the second jug.

$(0, 0)$	$\rightarrow (a, 0)$	fill first jug
	$\rightarrow (0, a)$	pour first into second
	$\rightarrow (a, a)$	fill first jug
	$\rightarrow (2a - b, b)$	pour first into second
	$\rightarrow (2a - b, 0)$	empty second jug
	$\rightarrow (0, 2a - b)$	pour first into second
	$\rightarrow (a, 2a - b)$	fill first
	$\rightarrow (3a - 2b, b)$	pour first into second

Thus, we see that the amount of water in each jug is always an linear combination of their capacities

General Solution for Die Hard

Invariant in Die Hard Transition:

Suppose that we have water jugs with capacities B and L.

Then the amount of water in each jug is always an integer linear combination of B and L.

Lemma. gcd(a, b) divides any integer linear combination of a and b.

Let d = gcd(a,b). Then

$$d|a \quad \text{and} \quad d|b$$

So $d|ax+by$.

Corollary. The amount of water in each jug is a multiple of gcd(a,b).

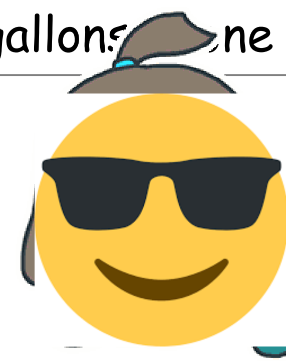
General Solution for Die Hard

Corollary. The amount of water in each jug is a multiple of gcd(a,b).

Given jug of 3 and jug of 9, is it possible to have exactly 4 gallons in one jug?

NO, because gcd(3,9)=3, and 4 is not a multiple of 3.

Given jug of 21 and jug of 26, is it possible to have exactly 3 gallons in one jug?



gcd(21,26)=1, and 3 is a multiple of 1,

so this means possible??

Theorem. Given water jugs of capacity a and b with a ≤ b,
it is possible to have exactly k (≤ b) gallons in one jug
if and only if k is a multiple of gcd(a,b).

General Solution for Die Hard

Theorem. Given water jugs of capacity a and b with $a \leq b$, it is possible to have exactly k ($\leq b$) gallons in one jug if and only if k is a multiple of gcd(a,b).

Given jug of 21 and jug of 26, is it possible to have exactly 3 gallons in one jug?

$$\gcd(21,26) = 1$$
$$\Rightarrow 5 \times 21 - 4 \times 26 = 1$$
$$\Rightarrow 15 \times 21 - 12 \times 26 = 3$$

Repeat 15 times:

1. Fill the 21-gallon jug.
2. Pour all the water in the 21-gallon jug into the 26-gallon jug.
   Whenever the 26-gallon jug becomes full, empty it out.

General Solution for Die Hard

$15 \times 21 - 12 \times 26 = 3$

Repeat 15 times:

1. Fill the 21-gallon jug.
2. Pour all the water in the 21-gallon jug into the 26-gallon jug.

   Whenever the 26-gallon jug becomes full, empty it out.

Claim. There must be exactly 3 gallons left after this process.

1. Totally we have filled 15x21 gallons.
2. We pour out t multiple of 26 gallons.
3. The 26 gallon jug can only hold the volume between 0 and 26.
4. So t must be 12.
5. And there is exactly 3 gallons left.

General Solution for Die Hard

Given two jugs with capacity A and B with $A \le B$, the target is C.

If gcd(A,B) does not divide C, then it is impossible.

Otherwise, compute $C = sA + tB$. (We can always make $s > 0$.)

Repeat s times:

1. Fill the A-gallon jug.
2. Pour all the water in the A-gallon jug into the B-gallon jug.

   Whenever the B-gallon jug becomes full, empty it out.

The B-gallon jug will be emptied exactly t times.

After that, there will be exactly C gallons in the B-gallon jug.