Lecture 3: Newsvendor Model (Continued)

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Optimal order quantity

- $y*$ is the smallest y such that

$$F(y) \geq \frac{c_p - c_v}{c_p - c_s}.$$

- Example,

$$\frac{c_p - c_v}{c_p - c_s} = \frac{1 - .25}{1 - 0} = .75$$

d	10	15	20	25	30
$\mathbb{P}(D = d)$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{4}$
$F(d)$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{3}{4}$	1

- $y* = 25$

Proof when D is discrete

- If $h(y+1) - h(y) > 0$, y cannot be optimal, and the optimal order quantity $y* > y$.
-

$$\begin{aligned} h(y+1) - h(y) &= \mathbb{E}[\text{Profit}(y+1, D) - \text{Profit}(y, D)] \\ &= \mathbb{E}[\text{revenue}(y+1, D) - \text{revenue}(y, D)] - c_v \end{aligned}$$

- If $D \geq y+1$, there will be no leftover items in both "systems".

$$\text{revenue}(y+1, D) - \text{revenue}(y, D) = c_p$$

- If $D \leq y$,

$$\text{revenue}(y+1, D) - \text{revenue}(y, D) = c_s$$

Optimal quantity: Discrete case

- Note that

$$
\begin{aligned}
h(y+1) - h(y) &= c_p \mathbb{P}\{D \geq y+1\} + c_s \mathbb{P}\{D \leq y\} - c_v \\
&= (c_p - c_v)\mathbb{P}\{D \geq y+1\} + (c_s - c_v)\mathbb{P}\{D \leq y\} \\
&= c_p - c_v - (c_p - c_s)\mathbb{P}\{D \leq y\}.
\end{aligned}
$$

- Thus

$$
\begin{aligned}
h(y+1) > h(y) &\Longleftrightarrow \mathbb{P}\{D \leq y\} < \frac{c_p - c_v}{c_p - c_s}, \\
h(y+1) < h(y) &\Longleftrightarrow \mathbb{P}\{D \leq y\} > \frac{c_p - c_v}{c_p - c_s}.
\end{aligned}
$$

- If $\mathbb{P}\{D \leq y\} < \frac{c_p - c_v}{c_p - c_s}$, y cannot be optimal.
- If $\mathbb{P}\{D \leq y\} > \frac{c_p - c_v}{c_p - c_s}$, $y+1$ cannot be optimal.

Optimality y^*

- Thus, optimal y^* must satisfy

$$F(y^*) \geq \frac{c_p - c_v}{c_p - c_s}.$$

- Suppose that y^* is the smallest y such that

$$F(y) \geq \frac{c_p - c_v}{c_p - c_s}.$$

  Then y^* is optimal.
- If $F(y^*) = \frac{c_p - c_v}{c_p - c_s}$, then both y^* and $y^* + 1$ are optimal.

Holding cost and fixed cost

- Suppose that each left over item costs h. It is equivalent to $c_s = -h = -.1$. The expected profit is

$$\mathbb{E}\text{Profit}(q, D) = \mathbb{E}[\min(q, D)]c_p - c_v q - h\mathbb{E}(q - D)^+.$$

- In addition, there is a fix cost c_f to make an order. The expected profit is

$$\mathbb{E}\text{Profit}(q, D) = \mathbb{E}[\min(q, D)]c_p - c_v q - c_f \mathbb{1}_{\{q>0\}} - h\mathbb{E}(q - D)^+.$$

Optimal order quantity maximizing the expected profit

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$\mathbb{P}(D = d)$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{4}$
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- $y* = 25$, $\mathbb{E}[\text{profit}(25, D)] = 13.125$

Projected Profit

- Limitation of using expectation as objective function

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- Order $q = 25$ copies:

Demand	Profit(25)	Probability
10	3.75	1/4
15	8.75	1/8
20	13.75	1/8
25	18.75	1/4
30	18.75	1/4

- Expected profit

$$\begin{aligned} h(25) = & 3.75(1/4) + 18.75(1/8) + 13.75(1/8) + 18.75(1/4) \\ & + 18.75(1/4) = 13.125. \end{aligned}$$

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$$\begin{aligned} h(25) =& 3.75(1/4) + 18.75(1/8) + 13.75(1/8) + 18.75(1/4) \\ &+ 18.75(1/4) = 13.125. \end{aligned}$$

- You, as the CEO of this company, need to report the projected profit in the next three months (quarter). What number should you report?

Confidence interval

- With 95% confidence, if 25 copies are ordered everyday, the 90 day profit lies in the interval

$$[1181.2 - 1.96\sigma\sqrt{90}, [1181.2 + 1.96\sigma\sqrt{90}],$$

- $P_1(25)$, ..., $P_{90}(25)$ are profits for day 1 through day 90. They are i.i.d. with mean 13.125 and standard deviation σ.
- Let

$$R(25) = P_1(25) + \ldots + P_{90}(25)$$

be the 3-month profit. Then $R(25)$ has mean 1181.2, standard deviation $\sqrt{90}\sigma$, and is approximately normal, where σ is the stanadrd deviation of the daily profit. Thus,

$$\frac{R(25) - 1181.2}{\sqrt{90}\sigma} \approx N(0, 1).$$

Maximize what?

- Thus,

$$\mathbb{P}\left\{\left|\frac{R(25)-1181.2}{\sqrt{90}\sigma}\right|<1.96\right\}\approx .95.$$

- Namely, with 95% level of confidence,

$$1181.2-1.96\sqrt{90}\sigma<R(25)<1181.2+1.96\sqrt{90}\sigma,$$
$$90\mathbb{E}(P_1(25))-1.96\sqrt{90}\,\mathrm{std}(P_1(25))<R(25)<\dots$$

- $\mathbb{E}(P_1(25)^2)=212.50$, $\mathrm{Var}(P_1(25))=40.234$, $\mathrm{std}(P_1(25))=6.3431$.
- Confidence interval is

$$(1063.3, \qquad 1299.1)$$

- Choose y to maximize

Perishable products with two periods lifetime

- Consider a product with two periods lifetime. $c_p = 1.0$, $c_v = .25$
- Suppose that each left over (not expired) item cost $h = .1$. But each left over item (if not expired) can be used for the following period.
- There is a disposal cost $c_d = 0.5$ for each unit of expired item.
- New order arrives at the beginning of each day.
- Demand is consumed from the least fresh to the freshest.
- You need to decide the ordering quantity at the end of period n.
- X_n is the ending inventory level for period n.

Order-up-to inventory policy for perishable products

- Order enough to bring the inventory level to S at the beginning of the next period.
- For example $S = 30$.
- $\mathbb{P}\{X_{10} = 5 | X_9 = 10\}$ and $\mathbb{P}\{X_{10} = 5 | X_9 = 25\}$

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What if there are multiple classes of customers?