MAT 2006 Assignment 3 118010350

1. proof. (i). Since $S = \sup A$, then for $\forall \varepsilon > 0$, $\exists a \in A$. s.t $S - \varepsilon < a \le S$.

Take $\varepsilon_n = \frac{1}{n}$, then $\exists \{a_n\} \subset A$, s.t $S - \frac{1}{n} < a_n \le S$.

$\Rightarrow \{a_n\} \to S$ as $n \to \infty$. $\Rightarrow S$ is a limit p.t. of A.

For $\bar{A}$ contains all limit pts of A, thus $S \in \bar{A}$.

(ii). No. Suppose A is an open set, and $S = \sup A \in A$.

Then $\exists V_\varepsilon(S) \subset A$. $\Rightarrow S < S + \varepsilon/2 \in A$. But $S \ge a$, $\forall a \in A$.

contradiction! Thus open set contains no supremum.

2. proof. (i). ① "$\Rightarrow$" Let $x \in \overline{A \cup B}$, if $x \in A \cup B$, then $x \in \bar{A} \cup \bar{B}$.

if x is a limit p.t of $A \cup B$, then $\exists \{x_n\} \subset A \cup B$.

s.t. $\{x_n\} \to x$ as $n \to \infty$. Then there exists infinite terms in A or B (or both), s.t.

$\{x_{n_k}\} \to x$, as $k \to \infty$, $\{x_{n_k}\} \subset A$ or $\{x_{n_k}\} \subset B$.

Thus, $x \in \bar{A}$ or $x \in \bar{B} \Rightarrow x \in \bar{A} \cup \bar{B}$.

$\Rightarrow \overline{A \cup B} \subset \bar{A} \cup \bar{B}$.

② "$\Leftarrow$" Let $x \in \bar{A} \cup \bar{B}$. If $x \in A$ or $x \in B$, then $x \in \overline{A \cup B}$.

if x is a limit p.t of A, then $\exists \{x_n\} \subset A \subset A \cup B$.

s.t $\{x_n\} \to x$ as $n \to \infty$. $\Rightarrow x$ is a limit pt of $A \cup B$.

$\Rightarrow x \in \overline{A \cup B}$. Similarly, if x is a limit pt of B,

then x is a limit p.t of $A \cup B \Rightarrow x \in \overline{A \cup B}$.

$\Rightarrow \overline{A \cup B} \supset \bar{A} \cup \bar{B}$.

Thus by ① & ②, $\overline{A \cup B} = \bar{A} \cup \bar{B}$.

(ii). No. eg. $\bar{\mathbb{Q}} = \mathbb{R}$, but since $\mathbb{Q}$ is countable, then

$\mathbb{Q} = \bigcup_{i=1}^{\infty} \{a_i\}$, the closure of $\{a_i\}$ is still $\{a_i\}$.

thus, $\bigcup_{i=1}^{\infty} \overline{\{a_i\}} = \bigcup_{i=1}^{\infty} \{a_i\} = \mathbb{Q} \neq \mathbb{R}$

3. proof ① Prove B is nonempty.

Suppose $B = \phi$, then $\forall s \in \mathbb{R}$, one of $\{x \mid x \in A, x < s\}$,

$\{x \mid x \in A, x > s\}$ is uncountable.

Let $S_1 \in \mathbb{R}$ s.t $\{x \mid x \in A, x < S_1\}$ is unctb.

$S_2 \in \mathbb{R}$ s.t $\{x \mid x \in A, x > S_2\}$ is unctb.

Then $\exists S' \in \mathbb{R}$. $S_1 < S' < S_2$, s.t. $\{x \mid x \in A, x < S_1 < S'\}$ is unctb and $\{x \mid x\in A,\ x>S_2>S'\}$ is also unctb. Contradiction!

Thus, B is nonempty.

② Prove B is open.

Since $\exists S\in\mathbb{R}$ s.t both $\{x\mid x\in A, x<S\}$ and $\{x\mid x\in A, x>S\}$ are uncountable, let $A_1=\{x\mid x\in A, x<S\}$, $A_2=\{x\mid x\in A, x>S\}$.

By ①, we can also similarly know that

$\exists S_1\in\mathbb{R}$ s.t $\{x\mid x\in A_1, x<S_1\}$ and $\{x\mid x\in A_1, x>S_1\}$ are unctb

$\exists S_2\in\mathbb{R}$ s.t $\{x\mid x\in A_2\ x<S_2\}$ and $\{x\mid x\in A_2, x>S_2\}$ are unctb.

Let $\varepsilon_1=|S-S_1|$, $\varepsilon_2=|S-S_2|$, take $\varepsilon=\min\{\varepsilon_1,\varepsilon_2\}$.

$\Rightarrow \forall S_0\in V_\varepsilon(S)$, $S_0\in B$. $\Rightarrow V_\varepsilon(S)\subset B$.

Thus, B is open.

4. proof. Assume a set A is both open and closed.

① if $A=\phi$, then we are done.

② if $A\neq\phi$, W.T.S. $A=\mathbb{R}$ that is, for $a\in A$, $a+\varepsilon\in A$ for $\forall\varepsilon>0$ and $a-\varepsilon\in A$, for $\forall\varepsilon>0$.

Suppose not, then $\exists\varepsilon>0$, s.t $a+\varepsilon\notin A$ for $a\in A$.

then $a+\varepsilon$ is an U.B for $(a-\varepsilon, a+\varepsilon)\cap A$

let $S=(a-\varepsilon, a+\varepsilon)\cap A$, $S\subset A$. since $S\neq\phi$, S is bdd

By L.U.B.P. there exists $\alpha=\sup S$.

Since A is open, then $\exists\varepsilon_1>0$, s.t $V_{\varepsilon_1}(a)\subset A$

$\Rightarrow \exists\{a_n\}\subset S$, s.t $\alpha-\frac{1}{n}<a_n\leq\alpha$.

$\Rightarrow \{a_n\}\to\alpha$, as $n\to\infty$. $\Rightarrow \alpha$ is a limit pt of $S\subset A$.

Since A is closed, then $\alpha\in A$. $\exists\varepsilon_2>0$, s.t $V_{\varepsilon_2}(\alpha)\subset S\subset A$

which contradicts with α is the L.U.B of S.

Thus, $\forall\varepsilon>0$. $a+\varepsilon\in A$. Similarly, $\forall\varepsilon>0$. $a-\varepsilon\in A$. $\Rightarrow A=\mathbb{R}$

5. Proof. (1). ① "$\Rightarrow$". Since E is closed, then $L_E\subset E$,

thus $\bar{E}=E\cup L_E=E$.

"$\Leftarrow$". Since $E=\bar{E}=E\cup L_E$, then E contains all its limit pts by definition. E is closed.

② "$\Rightarrow$". Since E is open, then $\forall x\in E$, $\exists V_\varepsilon(x)\subset E$.

then $\forall x \in E$, $x \in E^\circ \Rightarrow E \subset E^\circ$ and $E^\circ \subset E$. $\Rightarrow E^\circ = E$

"$\Leftarrow$" Since $E^\circ = E$, then $\forall x \in E$, $\exists V_\varepsilon(x) \subset E$.

by definition, E is open.

(iii). ① $\forall x \in (\bar{E})^c$, $\Rightarrow x \notin \bar{E} \Rightarrow x \notin E$ and $x \notin L_E$

$\Rightarrow x \in E^c$ and $\exists V_\varepsilon(x) \cap E = \phi$.

$\Rightarrow x \in E^c$ and $\exists V_\varepsilon(x) \subset E^c$. $\Rightarrow x \in (E^c)^\circ \Rightarrow (\bar{E})^c \subset (E^c)^\circ$.

$\forall x \in (E^c)^\circ \Rightarrow x \in E^c$ and $\exists V_\varepsilon(x) \subset E^c$.

$\Rightarrow x \notin E$ and $\exists V_\varepsilon(x) \cap E = \phi$.

$\Rightarrow x \notin E$ and $x \notin L_E \Rightarrow x \notin \bar{E}$. $\Rightarrow (E^c)^\circ \subset (\bar{E})^c$.

Thus, $(\bar{E})^c = (E^c)^\circ$.

② Since $(\bar{E})^c = (E^c)^\circ$, then $\bar{E} = \left((E^c)^\circ\right)^c$.

Substitute E by E^c, then $\overline{E^c} = (E^\circ)^c$.

6. proof. Let $K \subset \mathbb{R}$, and K is closed and bounded.

Since K is bounded, then $\forall \{X_n\} \subset K$ is bounded.

By B.W. Theorem, $\exists \{X_{nk}\} \subset \{X_n\}$, s.t $\{X_{nk}\} \to X$, as $k \to \infty$.

Since K is closed, then $\{X_{nk}\} \to X \in K$.

By definition. K is seq. compact.

7. proof. Since K is nonempty and seq. compact.

then K is closed and bounded.

By L.U.B.P., $\sup K$ exists. let $S_1 = \sup K$.

then for $\forall \varepsilon_n = \frac{1}{n} > 0$, $\exists a_n \in K$. s.t $S_1 - \frac{1}{n} < a_n \le S_1$.

thus, $\{a_n\} \to S_1$ as $n \to \infty$. $\Rightarrow S_1$ is the limit p.t of K.

Since K is closed, then $S_1 \in K$

Similarly. Let $K' = -K$, $\inf K = -\sup K'$ exists.

let $S_2 = \inf K$, then for $\forall \varepsilon_n = \frac{1}{n} > 0$.

$\exists a_n \in K$ s.t $S_2 \le a_n < S_2 + \frac{1}{n}$.

thus, $\{a_n\} \to S_2$ as $n \to \infty$, $\Rightarrow S_2$ is the limit pt of K.

Since K is closed, then $S_2 \in K$.

8. proof. (NIP + AP $\Rightarrow$ HB).

Assume $K \subset \mathbb{R}$ is closed and bounded. let $\{O_\lambda | \lambda \in \Lambda\}$ be an open cover for K. Suppose for contradiction, there is no finite subcover exists.

(a). let $K \subset I_0 = [a,b]$, then at least one of $[a, \frac{a+b}{2}] \cap K$, $[\frac{a+b}{2}, b] \cap K$ has no finite subcover, choose that interval as I_1, thus inductively, we can get I_{n+1} from I_n. Then $I_1 \supset I_2 \supset \cdots \supset I_k \supset I_{k+1} \supset \cdots$ with property that $I_n \cap K$ cannot be finitely covered. and $\lim_{n\to\infty} |I_n| = 0$, for $I_n = \frac{1}{2^{n-1}}(b-a)$.

(b). By NIP., $\bigcap_{n=1}^{\infty} I_n \neq \phi$. Since $I_n \cap K$ has no finite cover for $\forall n \in \mathbb{N}$, then $I_n \cap K \neq \phi$. for $\forall n \in \mathbb{N}$. Then exists $x \in K$. s.t. $x \in I_n$ for $\forall n \in \mathbb{N}$.

(c). Since $x \in K$, then there exists $O_{\lambda_0} \in \{O_\lambda | \lambda \in \Lambda\}$. s.t. $x \in O_{\lambda_0}$. Since O_{λ_0} is open, then $\exists V_\delta(x) \subset O_{\lambda_0}$. For $x \in I_n$, for $\forall n \in \mathbb{N}$. then there exists $|I_N| < \delta$, $\forall n \geq N$. and thus $I_N \subset V_\delta(x)$, $\forall n \geq N$. which contradicts with $I_n \cap K$ has no finite subcover for $\forall n \in \mathbb{N}$.

9. proof. (LUBP $\Rightarrow$ HB).

(a). let $x = a$, then $[a,x] = \{a\}$. Take any $O_{\lambda_0} \in \{O_\lambda | \lambda \in \Lambda\}$. s.t. $a \in O_{\lambda_0}$. thus, O_{λ_0} is a finite cover of $[a,x]$. Thus. $a \in S$, and S is nonempty.
Since $\forall x \in S$, $a \leq x \leq b$. thus S is bounded.
Thus $s = \sup S$ exists.

(b) Suppose $s > b$, since $x \leq b$. $\forall x \in S$. and $s = \sup S$.
let $\varepsilon = s - b$. then $\exists x \in S$. s.t. $x > s - \varepsilon/2 = \frac{1}{2}b + \frac{1}{2}s = b + \varepsilon/2$.
Contradiction!!
Suppose $s < b$. $[a,s]$ has a finite subcover $\{O_{\lambda_n}\}_{n=1}^{N}$
let $O_{\lambda_0} \in \{O_{\lambda_n}\}_{n=1}^{N}$, s.t. $s \in O_{\lambda_0}$
Since O_{λ_0} is open, then $\exists V_\varepsilon(s) \subset O_{\lambda_0}$.
Take $s' = \min\{s+\varepsilon, b\}$. thus, $[a,s'] \subset \bigcup_{n=1}^{N} O_{\lambda_n}$
Thus. $[a,s']$ has a finite subcover.

Contradiction!

Thus $s=b$. which implies $[a,b]$ has a finite subcover.

(c). Assume $K\subset\mathbb{R}$ is closed and bounded. K is nonempty.
Let $s=\sup K$, $l=\inf K$. By previous result,
we get $s\in K$, $l\in K$. thus $s=\max K$, $l=\min K$
Then $K\subset[l,s]$. Since $[l,s]$ is an closed interval,
then any open cover of $[l,s]$ has a finite subcover.
for $K\subset[l,s]$, then any open cover of K has a
finite subcover. By definition, K is compact.

10. proof. (HB $\Rightarrow$ BW).
Assume $K\subset\mathbb{R}$ is bounded and infinite.
Suppose for contradiction that K has no limit pt.
Then K is closed and bounded, K is compact.
Since $\forall x\in K$, x is not a limit p.t of K,
Then $\exists\,\delta_x>0$. s.t $\mathring{V}_{\delta_x}(x)\cap K=\phi$. Thus $V_{\delta_x}(x)\cap K=\{x\}$.
we get $\{V_{\delta x}(x)\}_{x\in K}$ is an open cover of K.
Since K is compact, then there exists a finite
subcover $\{V_{\delta x_n}(x_n)\}_{n=1}^{N}$ s.t. $K\subset\bigcup_{n=1}^{N}V_{\delta x_n}$
thus $K\subset\bigcup_{n=1}^{N}V_{\delta x_n}(x_n)\cap E=\bigcup_{n=1}^{N}\{x_n\}$
since K is infinite, but $\bigcup_{n=1}^{N}\{x_n\}$ is finite,
we find a contradiction!
Thus K has a limit point, that is, every bounded
sequence has a convergent subseq.

11. proof. (a). $\bigcup_{i=1}^{\infty}F_{\sigma i}=\bigcup_{i=1}^{\infty}\left(\bigcup_{n=1}^{\infty}F_n^i\right)$, where F_n^i is closed, $\forall i,n\in\mathbb{N}$.
Since $F_{\sigma i}=\bigcup_{n=1}^{\infty}F_n^i$, then $F_{\sigma i}$ is an F_σ set, $\forall i\in\mathbb{N}$.
and the countable union of countable set is still
countable, thus $\bigcup_{i=1}^{\infty}F_{\sigma i}$ is a countable union of
closed sets, so $\bigcup_{i=1}^{\infty}F_{\sigma i}$ is an F_σ set.

(b) $\bigcap_{i=1}^{N}F_{\sigma i}=\bigcap_{i=1}^{N}\left(\bigcup_{n=1}^{\infty}F_n^i\right)$, where F_n^i is closed, $\forall i,n\in\mathbb{N}$.

By distribution law, $\bigcap_{i=1}^{N} F_{\sigma i}$ is a union of $\bigcap_{i=1}^{N} F_n^i$,
where n can be arbitrarily chosen from $\mathbb{N}$.
Since we can list all the elements in $\{\bigcap_{i=1}^{N} F_n^i\}$ by order
then the union is countable, and $\bigcap_{i=1}^{N} F_n^i$ is closed.
thus $\bigcap_{i=1}^{N} F_{\sigma i}$ is an F_σ set

(c). $\mathbb{I} = \bigcap_{n=1}^{\infty} \{q_n\}^c$, $\mathbb{I}$ is not an F_σ set.

(d). By (b), we have $\bigcap_{i=1}^{N} F_{\sigma i}$ is an F_σ set,
Then $(\bigcap_{i=1}^{N} F_{\sigma i})^c = \bigcup_{i=1}^{N} F_{\sigma i}^c$ is a G_δ set.
Thus, finite union of G_δ sets is a G_δ set.

(e). By (a). we have $\bigcup_{i=1}^{\infty} F_{\sigma i}$ is an F_σ set.
Then $(\bigcup_{i=1}^{\infty} F_{\sigma i})^c = \bigcap_{i=1}^{\infty} F_{\sigma i}^c$ is a G_δ set.
Thus, countable intersection of G_δ sets is a G_δ set.

12. (i). (a) (a,b) is F_σ and G_δ.

$(a,b) = \bigcup_{n=1}^{\infty} [a+\frac{1}{n}, b-\frac{1}{n}] \Rightarrow (a,b)$ is F_σ.

$(a,b) = \bigcap_{n=1}^{\infty} (a,b) \Rightarrow (a,b)$ is G_δ.

(b). $[a,b]$ is F_σ and G_δ.

$[a,b] = \bigcap_{n=1}^{\infty} (a-\frac{1}{n}, b+\frac{1}{n}) \Rightarrow [a,b]$ is G_δ.

$[a,b] = \bigcup_{n=1}^{\infty} [a,b] \Rightarrow [a,b]$ is F_σ.

(c). $(a,b]$ is F_σ and G_δ.

$(a,b] = \bigcup_{n=1}^{\infty} [a+\frac{1}{n}, b] \Rightarrow (a,b]$ is F_σ.

$(a,b] = \bigcap_{n=1}^{\infty} (a, b+\frac{1}{n}) \Rightarrow (a,b]$ is G_δ.

(d). $\mathbb{Q}$ is F_σ. $\mathbb{Q} = \bigcup_{n=1}^{\infty} \{q_n\}$.

(e). $\mathbb{I}$ is G_δ. $\mathbb{I} = \bigcap_{n=1}^{\infty} \{q_n\}^c$

(f). Let $A \subset \mathbb{R}$ is open. $A = \bigcap_{n=1}^{\infty} A \Rightarrow A$ is G_δ.

(g) proof. Let $A \subset \mathbb{R}$ is open then $\forall a \in A$. $\exists V_{\varepsilon_a}(a) \subset A$.
we can get a set of intervals $\{I_n\}$, such that all intervals are mutally disjoint and each of them is the largest interval contains some $a \in A$.
Since each interval contains rational numbers, thus, they are at most countable.

(h). proof. By (g), since any open set can be written as the union of at most countable intervals, and any interval is F_σ, thus any open set is F_σ.
By taking the complement of any open set, we get any closed set is G_δ.

13. proof. (i). For $\forall M>0$, $\exists \delta = \sqrt{\frac{1}{M}}$, s.t. $f(x) = \frac{1}{x^2} > M$, $\forall\, 0<|x-0|<\delta$.
Thus, $\lim_{x \to 0} \frac{1}{x^2} = \infty$.

(ii). Definition: for $\forall \varepsilon>0$. $\exists M>0$. s.t $|f(x)-L|<\varepsilon$. $\forall x>M$.
For $\forall \varepsilon>0$. $\exists M = \frac{1}{\varepsilon}$. s.t $|\frac{1}{x}-0|<\varepsilon$. $\forall x>M$.
Thus $\lim_{x\to\infty} \frac{1}{x} = 0$.

(iii). Definition: for $\forall M_1>0$. $\exists M_2>0$. s.t $f(x)>M_1$. $\forall x>M_2$.
Let $f(x) = x^2$, $\lim_{x\to\infty} f(x) = \infty$.

14. (i) Definition (right-hand):
$\forall \varepsilon>0$. $\exists \delta>0$. s.t. $|f(x)-L|<\varepsilon$. $\forall\, 0<x-c<\delta$.
Definition (left-hand):
$\forall \varepsilon>0$ $\exists \delta>0$ s.t $|f(x)-M|<\varepsilon$. $\forall -\delta<x-c<0$.

(ii). proof. "$\Rightarrow$". Since $\lim_{x\to c} f(x) = L$. then for $\forall \varepsilon>0$.
$\exists \delta>0$. s.t $|f(x)-L|<\varepsilon$. $\forall\, 0<|x-c|<\delta$.
then $|f(x)-L|<\varepsilon$. $\forall -\delta<x-c<0$ and $\forall\, 0<x-c<\delta$.
Thus, $\lim_{x\to c^-} f(x) = L$ and $\lim_{x\to c^+} f(x) = L$

"$\Leftarrow$" Since $\lim_{x\to c^-} f(x)=L$, and $\lim_{x\to c^+} f(x)=L$, then.

$\forall \varepsilon>0$. $\exists \delta_1>0$. s.t $|f(x)-L|<\varepsilon$. $\forall -\delta_1<x-c<0$.

$\exists \delta_2>0$, s.t $|f(x)-L|<\varepsilon$. $\forall\ 0<x-c<\delta_2$.

Take $\delta=\min\{\delta_1,\delta_2\}$ then for $\forall\varepsilon>0$. $\exists\delta>0$.

s.t $|f(x)-L|<\varepsilon$, $\forall\ 0<|x-c|<\delta \Rightarrow \lim_{x\to c} f(x)=L$.

15. proof. "$\Rightarrow$". Suppose $\lim_{x\to c} f(x)=L$, then for $\forall\varepsilon>0$.

$\exists\delta_0>0$. s.t $|f(x)-L|<\varepsilon$. $\forall\ 0<|x-c|<\delta_0$.

then $|\sup_{0<|x-c|<\delta} f(x)-L|<\varepsilon$. $\forall\delta\le\delta_0$.

and $|\inf_{0<|x-c|<\delta} f(x)-L|<\varepsilon$. $\forall\delta\le\delta_0$

Then for $\forall\varepsilon>0$. $\exists\delta_0>0$. s.t $|\sup_{0<|x-c|<\delta} f(x)-L|<\varepsilon$. $\forall\ 0<\delta-0<\delta_0$

and $|\inf_{0<|x-c|<\delta} f(x)-L|<\varepsilon$. $\forall\ 0<\delta-0<\delta_0$.

Thus, $\lim_{\delta\to0^+}\sup_{0<|x-c|<\delta} f(x)=L=\lim_{\delta\to0^+}\inf_{0<|x-c|<\delta} f(x)$.

Thus, $\lim_{x\to c}\sup f(x)=\lim_{x\to c}\inf f(x)$.

"$\Leftarrow$". Suppose $\lim_{x\to c}\sup f(x)=\lim_{x\to c}\inf f(x)=L$,

then for $\forall\varepsilon>0$. $\exists\delta_1>0$. s.t $|\sup_{0<|x-c|<\delta} f(x)-L|<\varepsilon$, $\forall\ 0<\delta-0<\delta_1$.

$\exists\delta_2>0$. s.t $|\inf_{0<|x-c|<\delta} f(x)-L|<\varepsilon$, $\forall\ 0<\delta-0<\delta_2$.

Take $\delta_0=\min\{\delta_1,\delta_2\}$. then $|\underset{0<|x-c|<\delta}{f(x)}-L|<\varepsilon$. $\forall\ 0<\delta-0<\delta_0$.

That is $|f(x)-L|<\varepsilon$, $\forall\ 0<|x-c|<\delta_0$.

Thus, for $\forall\varepsilon>0$. $\exists\delta_0$. s.t $|f(x)-L|<\varepsilon$. $\forall\ 0<|x-c|<\delta_0$.

So we get $\lim_{x\to c} f(x)=L$.

16. proof. "$\Rightarrow$". Suppose $\lim_{x\to c} f(x)=L$. then for $\forall\varepsilon/2>0$.

$\exists\delta>0$. s.t $|f(x)-L|<\varepsilon/2$, $\forall\ 0<|x-c|<\delta$.

$|f(y)-L|<\varepsilon/2$, $\forall\ 0<|y-c|<\delta$.

$\Rightarrow |f(x)-f(y)|=|f(x)-L+L-f(y)|$

$\le|f(x)-L|+|f(y)-L|=\varepsilon$.

Thus $\exists\delta>0$. s.t $|f(x)-f(y)|<\varepsilon$. $\forall\ 0<|x-c|<\delta$. $\forall\ 0<|y-c|<\delta$.

"$\Leftarrow$". Since for $\forall\varepsilon>0$. $\exists\delta>0$. s.t. $|f(x)-f(y)|<\varepsilon$.

$\forall 0<|x-c|<\delta,\ \forall 0<|y-c|<\delta.$

Then let $\{a_n\} \subset A$ be an arbitrary set s.t. $\{a_n\} \to c$ as $n \to \infty$.

Then $\exists N \in \mathbb{N}$. s.t $0<|a_n-c|<\delta$, $0<|a_m-c|<\delta$. $\forall n, m \geq N$.

so we have $|f(a_n)-f(a_m)|<\varepsilon$, for $\forall n, m \geq N$.

Thus, $\{f(a_n)\}$ is a Cauchy Sequence. it is convergent.

Suppose $\{f(a_n)\} \to L$, as $n \to \infty$, then for $\forall \{a_n\} \subset A$.

$\lim_{n\to\infty} a_n = c$, we have $\lim_{n\to\infty} f(a_n) = L$. $\Rightarrow \lim_{x\to c} f(x) = L$.

17. proof. If K contains no limit p.t, then K is closed.

If K contains some limit p.t, let $l \in K$ be a limit p.t.

W.T.S. $l \in K$, that is $h(l)=0$.

Suppose for contradiction that $h(l) \neq 0$.

Since $\exists \{a_n\} \subset K$. s.t $\{a_n\} \to l$. as $n \to \infty$

and h is cts on $\mathbb{R}$, then $\{f(a_n)\} \to f(l)$ as $n \to \infty$.

Since $f(a_n)=0$ $\forall n \in \mathbb{N}$. then for $\forall \varepsilon>0$. $\exists N \in \mathbb{N}$.

s.t $|f(a_n)-f(l)| = |0-f(l)| < \varepsilon \Rightarrow |f(l)|<\varepsilon$, $\forall \varepsilon>0$.

Contradiction !! Thus $f(l)=0$. and $l \in K$.

Thus K is a closed set.

18. (i) proof. Prove by induction.

when $n=1$, $f(x)$ is cts $\Rightarrow g(x) = \max\{f_1(x)\} = f_1(x)$ is cts.

when $n=2$, $f_1(x), f_2(x)$ are cts

$\Rightarrow g(x) = \max\{f_1(x), f_2(x)\} = \dfrac{(f_1(x)+f_2(x)) + |f_1(x)-f_2(x)|}{2}$

Since $f_1(x)+f_2(x)$ is cts, and $|f_1(x)-f_2(x)|$ is cts.

then $g(x)$ is also cts.

Suppose, $g(x)$ is cts when $n=k$.

Then when $n=k+1$, $f_1 \cdots f_k, f_{k+1}$ are cts function.

Let $h(x) = \max\{f_1(x), \cdots, f_k(x)\}$. is cts.

Then $g(x) = \max\{h(x), f_{k+1}(x)\}$ by $n=2$, is cts

Thus when $n=k+1$, $g(x)$ is still cts

By induction, we have $g(x) = \{f_1(x), \cdots f_n(x)\}$ is

a cts function for $\forall n \in \mathbb{N}$.

(ii). Since $f_n(x) = \begin{cases} 1, & |x| > \frac{1}{n} \\ n|x|, & |x| \le \frac{1}{n} \end{cases}$ then $f_n(x) \le 1$, $\forall n \in \mathbb{N}$, $\forall x \in \mathbb{R}$.

and $h(x) = \sup\{f_1(x), \cdots\}$, thus, $h(x) \le 1$, $\forall x \in \mathbb{R}$

Then W.T.S. $h(x) \ge 1$, $\forall x \in \mathbb{R}$.

Suppose for contradiction that $\exists m \in \mathbb{R}$ s.t $h(m) = a < 1$.

By A.P. $\exists N \in \mathbb{N}$ s.t. $\frac{1}{N} < |m|$.

Then $f_N(x) = \begin{cases} 1, & |x| > \frac{1}{N} \\ n|x|, & |x| \le \frac{1}{N} \end{cases}$, Since $|m| > \frac{1}{N}$, then $f_N(m) = 1 > h(m)$

Since $h(x) = \sup\{f_1(x), \cdots\}$, then $h(x) \ge f_N(x)$, $\forall x \in \mathbb{R}$

which contradicts with $h(m) = a < f_N(m) = 1$

Thus, $h(x) \ge 1$, $\forall x \in \mathbb{R}$.

So we get $h(x) = \sup\{f_1(x), \cdots\} = 1$.

19. proof. ① Let $x, y \in \mathbb{R}$. Since $g(x) = \inf\{|x-a| : a \in F\}$,

Then for $\forall \varepsilon > 0$, $\exists m \in F$, s.t $|x-m| < g(x) + \varepsilon$.

$\Rightarrow g(y) \le |y-m| \le |y-x| + |x-m| < |y-x| + g(x) + \varepsilon$.

$\Rightarrow g(y) - g(x) < |y-x| + \varepsilon$

And for $\forall \varepsilon > 0$, $\exists n \in F$, s.t $|y-n| < g(y) + \varepsilon$.

$\Rightarrow g(x) \le |x-m| \le |x-y| + |y-m| < |x-y| + g(y) + \varepsilon$.

$\Rightarrow g(x) - g(y) < |x-y| + \varepsilon$

Thus, $|g(x) - g(y)| < |x-y| + \varepsilon$, for $\forall \varepsilon > 0$

$\Rightarrow |g(x) - g(y)| \le |x-y|$.

Then for $\forall \varepsilon > 0$, $|g(x) - g(y)| < \varepsilon$, $\forall |x-y| < \varepsilon$, $x, y \in \mathbb{R}$

Thus, $g(x)$ is uniformly cts on $\mathbb{R}$., (cts on $\mathbb{R}$.)

② Suppose for contradiction that $\exists x_0 \notin F$. s.t $g(x_0) = 0$.

Then $g(x_0) = \inf\{|x_0 - a|, a \in F\} = 0$.

Then for $\forall \varepsilon = \frac{1}{n} > 0$, $\exists a_n \in F$, s.t. $0 \le |x_0 - a_n| < 0 + \frac{1}{n}$

Thus, $\{|x_0 - a_n|\} \to 0$ as $n \to \infty$.

That is $\lim_{n\to\infty} |x_0 - a_n| = 0$. $\Rightarrow \lim_{n\to\infty} a_n = x_0$.

$\Rightarrow$ x_0 is a limit pt of $\{a_n\} \subset F$.

Since F is closed, then $x_0 \in F$. Contradiction!

$\Rightarrow g(x) \ne 0$ for all $x \notin F$.

20. proof. Since $f(x)$ is cts on K, then $\forall y \in K$, $\forall \varepsilon > 0$.

$\exists \delta_y = \delta(y,\varepsilon)$, s.t. $|f(x)-f(y)| < \varepsilon/2$, $\forall |x-y| < \delta_y$

Since K is compact, for $\bigcup_{y\in K} V_{\delta_y}(y) \supset K$, there exists

a finite subcover s.t. $\bigcup_{n=1}^{N} V_{y_n}(y_n) \supset K$

Choose $\delta = \frac{1}{2}\min\{\delta_{y_1}, \delta_{y_2}, \cdots, \delta_{y_N}\}$

For $\forall |x-y| < \delta$, $|f(x)-f(y)| = |f(x)-f(y_n)+f(y_n)-f(y)|$

$\leq |f(x)-f(y_n)| + |f(y)-f(y_n)| < \varepsilon$.

Thus, $f(x)$ is uniformly cts on K

21 proof. (i). Since g is uniformly cts on $(a,b]$, then for $\forall \varepsilon > 0$

$\exists \delta_1 > 0$. s.t $|g(x_1)-g(x_2)| < \varepsilon$, $\forall |x_1-x_2| < \delta_1$, $x_1, x_2 \in (a,b]$

$\exists \delta_2 > 0$. s.t. $|g(x_1)-g(b)| < \varepsilon/2$, $\forall |x_1-b| < \delta_2$, $x_1 \in (a,b]$

Since g is uniformly cts on $[b,c)$, then for $\forall \varepsilon > 0$.

$\exists \delta_3 > 0$. st. $|g(y_1)-g(y_2)| < \varepsilon$, $\forall |y_1-y_2| < \delta_3$, $y_1, y_2 \in [b,c)$

$\exists \delta_4 > 0$. s.t $|g(y_1)-g(b)| < \varepsilon/2$, $\forall |y_1-b| < \delta_4$, $y_1 \in [b,c)$.

Take $\delta' = \delta_2 + \delta_4$ then $\exists \delta' > 0$. s.t.

$|g(x_1)-g(y_1)| = |g(x_1)-g(b)+g(b)-g(y_1)|$

$\leq |g(x_1)-g(b)| + |g(y_1)-g(b)| < \varepsilon$.

$\forall |x_1-y_1| < \delta'$ $x_1 \in (a,b]$, $y_1 \in [b,c)$.

Take $\delta = \min\{\delta', \delta_1, \delta_3\}$, then $\exists \delta > 0$. s.t

$|g(x)-g(y)| < \varepsilon$. $\forall |x-y| < \delta$. $x, y \in (a,b)$.

$\Rightarrow$ g is uniformly cts on (a,c).

(ii). Since $[0,1]$ is closed and bounded,

then $[0,1]$ is compact.

$f(x) = \sqrt{x}$ is cts on $[0,1]$ $\Rightarrow f(x) = \sqrt{x}$ is uniformly cts on $[0,1]$.

Then consider $x, y \in [1,\infty)$.

Since $|\sqrt{x}-\sqrt{y}| = \frac{|x-y|}{\sqrt{x}+\sqrt{y}} < |x-y|$, then for $\forall \varepsilon > 0$.

$\exists \delta = \varepsilon$, s.t $|f(x)-f(y)| = |\sqrt{x}-\sqrt{y}| < \varepsilon$. $\forall |x-y| < \delta$.

$\Rightarrow f(x) = \sqrt{x}$ is uniformly cts on $[1,\infty)$.

By (i), we get $f(x) = \sqrt{x}$ is uniformly cts on $[0,\infty)$.

(iii) "$\Rightarrow$" Let $f(x) = x^p$ be uniformly cts on $(0,\infty)$.

If $p > 1$. then take $x_n = n^{\frac{1}{p-1}} + \frac{1}{n}$, $y_n = n^{\frac{1}{p-1}}$, $n \in \mathbb{N}$

Then $|x_n - y_n| = \frac{1}{n} \to 0$ as $n \to \infty$.

$|f(x_n) - f(y_n)| = (n^{\frac{1}{p-1}} + \frac{1}{n})^p - (n^{\frac{1}{p-1}})^p$

$= n^{\frac{p}{p-1}} + \binom{p}{1}(n^{\frac{1}{p-1}})^{p-1}\frac{1}{n} + \cdots - n^{\frac{p}{p-1}} \ge 1$.

$\Rightarrow$ $f(x)$ is not uniformly cts when $p>1$.

If $p<0$. take $x_n = \frac{1}{n-\frac{1}{p}}$. $y_n = \frac{1}{n}$, $n \in \mathbb{N}$.

Then $|x_n - y_n| \to 0$ as $n \to \infty$.

$|f(x_n) - f(y_n)| = (n - \frac{1}{p})^{-p} - (n)^{-p}$

$= n^{-p} + \binom{-p}{1} \cdot n^{-p-1}(\frac{1}{-p}) + \cdots - n^{-p} \ge 1$.

$\Rightarrow$ $f(x)$ is not uniformly cts when $p<0$.

Thus, $0 \le p \le 1$.

"$\Leftarrow$". Suppose $0 \le p < 1$, W.T.S $f(x) = x^p$ is uniformly cts on $(0,\infty)$

Let $g(x) = f(x) - x = x^p - x$, $g'(x) = px^{p-1} - 1$.

when $x \in (p^{\frac{1}{1-p}}, \infty)$, $g'(x) < 0$, then for $x > y \ge p^{\frac{1}{1-p}}$

$f(x) < f(y) \Rightarrow x^p - x < y^p - y \Rightarrow |x^p - y^p| < |x-y|$.

$\Rightarrow$ $f(x)$ is uniformly cts on $[p^{\frac{1}{1-p}}, \infty)$.

$[0, p^{\frac{1}{1-p}}]$ is compact $\Rightarrow$ $f(x)$ is uniformly cts on $[0, p^{\frac{1}{1-p}}]$

$\Rightarrow$ $f(x)$ is uniformly cts on $[0,\infty)$.

Thus, $f(x)$ is uniformly cts on $(0,\infty)$.

(iv). Since $\lim_{x\to\infty} f(x) = L \in \mathbb{R}$, then for $\forall \varepsilon > 0$.

$\exists N \in \mathbb{N}$. s.t $|f(x) - L| < \frac{\varepsilon}{3}$, $\forall x \ge N$.

$|f(y) - L| < \frac{\varepsilon}{3}$, $\forall y \ge N$.

$\Rightarrow |f(x) - f(y)| \le |f(x) - L| + |f(y) - L| < \frac{2}{3}\varepsilon$. for $\forall x.y \ge N$.

Since f is cts, and $[0,N]$ is compact,

then f is uniformly cts on $[0,N]$

then $\exists \delta > 0$. s.t $|f(x) - f(y)| < \frac{\varepsilon}{3}$, $\forall |x-y| < \delta$, $x.y \in [0,N]$

Thus, if $x,y \in [0,N]$ or $x,y \in [N,\infty)$.

$|f(x) - f(y)| < \varepsilon$, $\forall |x-y| < \delta$.

if $x < N < y$, then for $|x-y| < \delta$. $\Rightarrow |x-N| < \delta$. $|y-N| < \delta$.

$\Rightarrow |f(x) - f(N)| < \frac{\varepsilon}{3}$, $|f(y) - L| < \frac{\varepsilon}{3}$.

$\Rightarrow |f(x) - f(y)| = |f(x) - f(N) + f(N) - L + L - f(y)|$

$\le |f(x) - f(N)| + |f(N) - L| + |f(y) - L| < (\varepsilon/3) \cdot 3 = \varepsilon$.

$\Rightarrow$ $f(x)$ is uniformly cts on $[0,\infty)$

22. (a). $K=[0,1]$ is compact, f is cts on K, then $f(K)$ is also compact. but $(0,1)$ is open (not compact).

(b). e.g. $f(x)=\begin{cases} 0, & x\in(0,\frac{1}{4}) \\ 2x-\frac{1}{2}, & x\in[\frac{1}{4},\frac{3}{4}] \\ 1, & x\in(\frac{3}{4},1). \end{cases}$

(c). e.g. $f(x)=\frac{1}{2}(1-x)\sin\frac{1}{x}+\frac{1}{2}$.

23. proof (i) Since f is uniformly cts on A, then for $\forall\varepsilon>0$
$\exists\delta>0$. s.t. $|f(x)-f(y)|<\varepsilon$, $\forall\ |x-y|<\delta$
Since $\{x_n\}\subset A$ is a Cauchy sequence, then for $\forall\delta>0$.
$\exists N\in\mathbb{N}$. s.t. $|x_n-x_m|<\delta$. $\forall\ n,m\geq N$.
let $x=x_n$ $y=x_m$ then we get for $\forall\varepsilon>0$.
$\exists\delta>0$. s.t $|f(x_n)-f(x_m)|<\varepsilon$, $\forall\ n,m\geq N$.
$\Rightarrow$ $f(x_n)$ is a Cauchy sequence.

(ii) "$\Rightarrow$" Since g is uniformly cts on (a,b),
for $\forall\{a_n\}\subset(a,b)$, and $\{a_n\}\to a$, then $\forall\{g(a_n)\}$ cvgs to the same limit. Suppose not, $\{g(a_n)\}$ and $\{g(a_m)\}$ cvgs to diffrent limit, then $\exists\varepsilon_0$, $\exists n_k, m_k$. s.t $|g(a_{n_k})-g(a_{m_k})|\geq\varepsilon_0$, $\forall k\in\mathbb{N}$. which is a contradiction for g is uniformly cts on (a,b).
Thus, we can define $g(a)=\lim_{n\to\infty}g(a_n)$, by definition. g is cts on point a. Similary, we can define $g(b)=\lim_{n\to\infty}g(b_n)$, and g is cts on b by definition.
$\Rightarrow g$ is cts on $[a,b]$

"$\Leftarrow$" Since $[a,b]$ is compact, and g is cts on $[a,b]$, then g is uniformly cts on $[a,b]$. thus g is uniformly cts on (a,b).

24. prof. (a) Take $\varepsilon_0=1$. let $x_n=\frac{1}{2n\pi}$, $y_n=\frac{1}{2n\pi+\frac{\pi}{2}}\in(0,1)$. $n\in\mathbb{N}$.
Then $|x_n-y_n|\to0$ as $n\to\infty$, but
$|f(x_n)-f(y_n)|=|0-1|=1\geq\varepsilon_0$.
$\Rightarrow f(x)=\sin\frac{1}{x}$ is not uniformly cts on $(0,1)$.

(b). Take $\varepsilon_0=1$, let $x_n=e^{-n}$, $y_n=e^{-(n+1)}\in(0,1)$, $n\in\mathbb{N}$.

Then $|x_n-y_n|\to 0$ as $n\to\infty$, but

$|f(x_n)-f(y_n)|=|-n+n+1|=1\geq\varepsilon_0$.

$\Rightarrow f(x)=\ln x$ is not uniformly cts on $(0,1)$.

(c). Take $\varepsilon_0=1$, let $x_n=\frac{n}{n+1}$, $y_n=\frac{n-1}{n}\in(0,1)$, $n\in\mathbb{N}$.

Then $|x_n-y_n|\to 0$ as $n\to\infty$. but

$|f(x_n)-f(y_n)|=|n+1-n|=1\geq\varepsilon_0$

$\Rightarrow f(x)=\frac{1}{1-x}$ is not uniformly cts on $(0,1)$

25. proof (i). let $g(x)=f(x+\frac{1}{2})-f(x)$. then $g(0)\cdot g(\frac{1}{2})=-[f(1)-f(\frac{1}{2})]^2\leq 0$.

If $f(1)=f(\frac{1}{2})$, then we are done.

If $f(1)\neq f(\frac{1}{2})$. then $g(0)$ and $g(\frac{1}{2})$ have different sign.

Since $g(x)$ is cts, then by IVT, $\exists x_0\in(0,\frac{1}{2})$. s.t $g(x_0)=0$.

$\Rightarrow f(x_0+\frac{1}{2})-f(x_0)=0$.

Thus, there exists $x,y\in[0,1]$, s.t $|x-y|=\frac{1}{2}$, $f(x)=f(y)$

(ii). let $g(x)=f(x)-f(x-\frac{1}{n})$. then $g(1)=f(1)-f(\frac{n-1}{n})$,

$g(\frac{n-1}{n})=f(\frac{n-1}{n})-f(\frac{n-2}{n}),\ \dots\ g(\frac{1}{n})=f(\frac{1}{n})-f(0)$.

If $g(\frac{k}{n})=0$, then we are done

if $g(\frac{k}{n})\neq 0$, for all $k\leq n$. Then suppose for contradiction that $g(\frac{k}{n})>0$, for all k, then $f(1)>f(\frac{n-1}{n})>\dots>f(0)$.

which contradicts with $f(0)=f(1)$. Similarly, it is also impossible that $g(\frac{k}{n})<0$ for all k. Thus, $\exists k_1, k_2\leq n$. s.t $g(\frac{k_1}{n})<0$, $g(\frac{k_2}{n})>0$. By IVT. $\exists x_0$. s.t $g(x_0)=0$.

$\Rightarrow f(x_0)=f(x_0-\frac{1}{n})$.

Thus, there exists $x_n, y_n\in[0,1]$. s.t $|x_n-y_n|=\frac{1}{n}$. $f(x_n)=f(y_n)$

(iii). e.g. let $f(x)=\cos(5\pi x)+2x$, $f(0)=f(1)=1$,

but $f(x+\frac{2}{5})-f(x)=\frac{4}{5}$.

26. proof. let $g(x)=f(x)-x$. Then $g(0)=f(0)$, $g(1)=f(1)-1$.

$\Rightarrow g(0)\cdot g(1)=f(0)\cdot(f(1)-1)$. Since $f(0)\in[0,1]$

and $f(1)-1\in[-1,0]$, then $g(0)\cdot g(1)\leq 0$.

If $g(0)\cdot g(1)=0$, then $f(0)=0$ or $f(1)=1$. we are done.

If $g(0)\cdot g(1)<0$. then by IVT. $\exists x_0 \in (0,1)$. s.t $g(x_0)=0$.

$\Rightarrow f(x_0)-x_0=0. \Rightarrow f(x_0)=x_0$

Thus. f must have a fixed point.

27. proof W.T.S f is monotone on $[a,b]$.

W.L.O.G. Suppose for contradiction that f is increasing on $[a_1,b_1]$ and decreasing on $[b_1,a_2]$ where, $[a_1,b_1]$, $[b_1,a_2] \subset [a,b]$

Since f is cts on $x=b_1$, then for $\forall \varepsilon>0$.

$\exists \delta_1>0$, s.t. $f(b_1)-f(x)=\varepsilon$, $b_1-x=\delta_1$, $x\in[a_1,b_1]$.

$\exists \delta_2>0$, s.t $f(y)-f(b_1)=\varepsilon$, $y-b_1=\delta_2$, $x\in[b_1,a_2]$

Thus, $\exists x\in[a_1,b_1]$, $y\in[b_1,a_2]$ s.t $f(x)=f(y)$.

Contradicts with f is one-to-one.

$\Rightarrow f$ is strictly monotone on $[a,b]$

W.L.O.G. Assume f is increasing on $[a,b]$.

Then. for $\forall x_1,x_2\in[a,b]$, $\frac{f(x_1)-f(x_2)}{x_1-x_2}>0$

$\Rightarrow \frac{x_1-x_2}{f(x_1)-f(x_2)}=\frac{f^{-1}(f(x_1))-f^{-1}(f(x_2))}{f(x_1)-f(x_2)}>0$.

$\Rightarrow f^{-1}(x)$ is strictly increasing.

Since the range of $f(x)$ is an interval,

Thus, $f^{-1}(x)$ is also cts.

28. (i). Since f is not on A, then $D_f=A$.

(ii). Impossible. Since $\mathbb{I}$ is a G_δ set but not F_σ set, then. it is impossible to find f. s.t. $D_f=\mathbb{I}$.