MAT2002: Ordinary Differential Equations

Assignment 6. Deadline: 10:00pm, May 14th, 2021

For questions containing phase portrait, you are recommended to use numerical tools like python or Matlab to obtain a better comprehension (not necessary to include it in your submission). Your submission should contain the sketch of phase portrait, which means, you should draw it without the help of these numerical tools. The grading of phase portrait focuses on the tendency around the critical points, you do not need to care about the interaction between critical points.

Question 1.
Consider the system:

$$\frac{dy}{dt} = \begin{pmatrix} -2 & -1 \\ -\alpha & -2 \end{pmatrix} y$$

(a) Solve the above system for $\alpha = 3$ and classify the critical 0 of the system as to type and stability.
(b) Repeat part (a) for $\alpha = 5$, notice now the behavior is different.
(c) Find the value of α where the transition from one behavior to the other occurs.

Question 2.
Sketch the phase portrait of the following linear systems of first order differential equations with critical points $(0, 0)$, then state whether this critical point is stable or not.
(a)

$$\frac{dy}{dt} = \begin{pmatrix} 1 & 1 \\ -5 & -3 \end{pmatrix} y$$

(b)

$$\frac{dy}{dt} = \begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix} y$$

Question 3.
Determine the critical point of the following linear systems of first order differential equations, and then classify its type and examine its stability. Hint: you can use the transformation of $x = x^0 + u$, where x^0 is the critical point.
(a)

$$\frac{dx}{dt} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} x - \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

(b)

$$\frac{dx}{dt} = \begin{pmatrix} -1 & -1 \\ 2 & -1 \end{pmatrix} x - \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

Question 4.
In this problem, we indicate how to show that the trajectories are ellipses when the eigenvalues are pure imaginary. Consider the system:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

(a) Show that the eigenvalues of the coefficient matrix are pure imaginary if and only if

$$a_{11} + a_{22} = 0 \quad a_{11}a_{22} - a_{12}a_{21} > 0$$

(b) The trajectories of this system can be found by the single equation:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

Show that this equation is exact.
(c) Solve the equation in (b). Then conclude the trajectories are ellipses when the eigenvalues are pure imaginary.

Question 5.
The system

$$\begin{aligned} x' &= -y \\ y' &= -\gamma y - x(x - 0.15)(x - 2) \end{aligned}$$

results from an approximation to Hodgkin-Huxley equations, which model the transmission of neural impulses along an axon. Here the parameter γ is a real number and $\gamma > 0$. Find all critical points, and classify them by investigating the approximate linear system near each one.

Question 6.
In each of the following nonlinear system of first order differential equations, construct a suitable Liapunov function, then state the stability of the critical point at the origin. (Asumptotically stable, stable or unstable)
(a)

$$\begin{aligned} x' &= -x^3 + xy^2 \\ y' &= -2x^2y - y^3 \end{aligned}$$

(b)

$$x' = x^3 - y^3$$
$$y' = 2xy^2 + 4x^2y + 2y^3$$

Question 7.
The following nonlinear system of first order differential equations can be interpreted as describing the interaction of two spcies with population x and y (only $x, y \geq 0$ case are under consideration). For these nonlinear systems of first order differential equations:
1. Find all critical points and the corresponding linear system near each critical points.
2. Determine the type and the stability of each critical points.
3. Draw the phase portrait.
4. Describe the behavior of the solutions and the limiting behavior of x and y as $t \to \infty$.
5. Please briefly explain why this nonlinear system of first order differential equations can be interpreted as describing the interaction of two spcies with population x and y. Interpret the results when $t \to \infty$ in terms of the populations of the two species.
(a)

$$dx/dt = x(1 - x - y)$$
$$dy/dt = y(1.5 - x - y)$$

(b)

$$x' = x(1 - x + 0.5y)$$
$$y' = y(2.5 - 1.5y + 0.25x)$$