MAT2002 ODEs
Nonlinear Differential Equations and Stability I

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Overview

Outline

Non-homogeneous linear systems

In all of our previous study, we have mainly focused on linear equations to compute explicit solutions. For nonlinear equations, we have been restricted to techniques for separable equations, exact equations and Bernoulli equations (in the first-order nonlinear ODE case).
For a general nonlinear equation

$$y^{(n)} = F(t, y, y', \ldots, y^{(n-1)}),$$

or nonlinear systems of differential equations

$$\frac{d\mathbf{y}(t)}{dt} = \mathbf{P}(t, \mathbf{y}(t)),$$

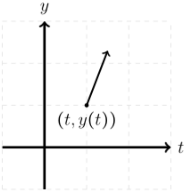
we can only say something about existence and uniqueness of solutions in a small time interval if F or $\mathbf{P}$ are continuous with continuous derivatives. Unfortunately, explicit formulas for solutions are usually not available, but we can use geometric methods to deduce more information about the solutions. This will be the focus of this chapter (section).

First-order equation

Given a first order nonlinear ODE

$$y'(t) = f(t, y),$$

for continuous functions f and $\frac{\partial f}{\partial y}$, we know that there is exactly one solution to the IVP when initial conditions are given. Furthermore we can plot the graph of t vs y using the equation.



At each point (t, y) we can draw a line segment with the slope $f(t, y)$. This gives a **direction field** for the ODE. Putting an arrow at the end of each line segment, this gives a **vector field** in the $t - y$ plane.

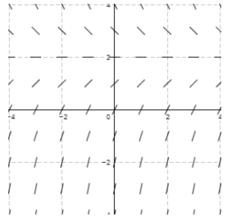
First-order equation

Example 15.1

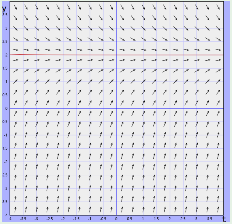
Consider the first order ODE

$$y' = f(y) = 2 - y.$$

As f does not depend on t, the slopes of the line segments at a fixed y-coordinate are all the same. We plot the direction field (vector field) in the following Figure. Notice that the line segments have zero slope whenever the points lie on the line $\{y = 2\}$.



(a) Direction field



(b) Vector field

Vector field of $y' = 2 - y$ in (t, y) plane with arrows indicating the change of y

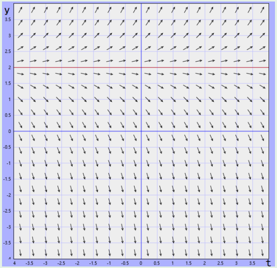
First-order equation

Example 15.2

Similarly, for the first order ODE

$$y' = y - 2,$$

we have the vector field as the following figure. Also notice that the line segments have zero slope whenever the points lie on the line $\{y = 2\}$.



Vector field of $y' = y - 2$ in (t, y) plane with arrows indicating the change of y

Trajectory for ODE

If we put a particle inside this vector field at initial position (t_0, y_0), the particle will move along with the vector field, this traces out a **trajectory** $\{(t, y(t)) : t \in I\}$ for the solution to the ODE.

In the first example $y' = 2 - y$, all trajectories will go to the line $\{y = 2\}$ as $t \to \infty$, while for the second example $y' = y - 2$, if trajectories start with initial position $y_0 = 2$, then the trajectories will stay on the line $\{y = 2\}$ as $t \to \infty$. However, if $y_0 > 2$, then the trajectories will move up and away from $\{y = 2\}$, and correspondingly in $y_0 < 2$, the trajectories will move down and away from $\{y = 2\}$.

In the above examples, the line $\{y = 2\}$ is what we will call **equilibrium/stationary** solutions to the ODE, as the values of y do not change as time progresses.

Critical point and stationary solutions

Definition 15.3

(Critical point and stationary solutions). Given a continuous function $f(t, y)$, suppose $y_* \in \mathbb{R}$ is a point such that

$$\boxed{f(t, y_*) = 0 \quad \forall t \in I},$$

then y_* is a **critical point** of f. We call the constant function

$$\phi(t) = y_* \quad \forall t \in I$$

a **stationary solution** to the ODE $y' = f(t, y)$.

Autonomous ODE

Besides the direction fields, another useful graphic for the nonlinear **autonomous** ODE

$$y' = f(y)$$

is the graph y vs $f(y)$.

Example 15.4

Recall the Logistic equation: for positive constants r and K,

$$y' = f(y) = ry\left(1 - \frac{y}{K}\right)$$

for population dynamics. It is easy to see that $y = 0$ and $y = K$ are stationary solutions, and if y is not equal to 0 or K, then

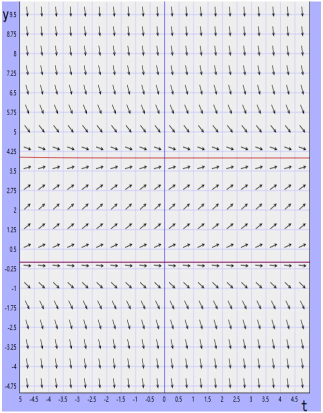
$$y(t) = \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}} \quad \textit{for } y(0) = y_0.$$

Hence, we can deduce that for nonnegative initial values y_0,

$$y(t) \to K \textit{ as } t \to \infty \textit{ if } y_0 > 0, \quad y(t) = 0 \textit{ for all } t > 0 \textit{ if } y_0 = 0.$$

Logistic equation

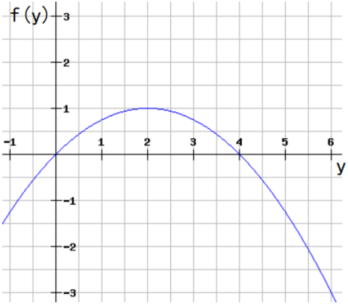
We can plot the vector field for the case $r = 1$ and $K = 4$ in the following figure, and observe that the line segments with y-coordinate equal to 0 or 4 ($y = 0$ or $y = 4$) have zero slope.



Vector field for the Logistic equation in (t, y) plane with $r = 1$ and $K = 4$ with arrows indicating the changing of y.

Logistic equation

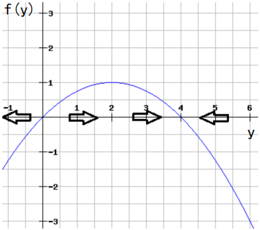
We now plot the graph y vs $f(y)$ in the following figure, which is a parabola that intersects the horizontal axis at two points $y = 0$ and $y = K$. The points that intersect the horizontal axis are the stationary solutions.



The plot y vs $f(y)$ for the Logistic equation.

Logistic equation

From this plot we can also deduce some behaviour of the solution to the ODE. Suppose we start with an initial condition x_0 in between 0 and K, then $f(x_0)$ is positive and so the solution y will increase in value, until it reaches $y = K$ where the derivative y' is zero. Similarly, if we start with an initial condition $x_1 > K$, then $f(x_1)$ is negative. Hence, the solution y will decrease in value, until it reaches $y = K$. Similarly, if we start with an initial value $x_2 < 0$, then $f(x_2)$ is negative and the solution y will decrease, moving away from the stationary solution $y = 0$. This can be summarized in the following figure, where we include arrows to demonstrate the behaviour of the solution.



The plot y vs $f(y)$ for the Logistic equation. Arrows indicate the behaviour of the solution.

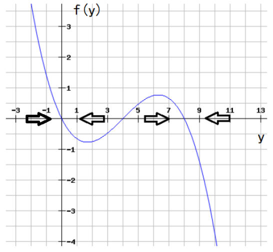
Modified Logistic equation

Example 15.5

We study a modification of the Logistic equation, called Logistic equation with threshold. Let $r > 0, 0 < T < K$ be positive constants, and consider the equation

$$y' = f(y) = -r\left(1 - \frac{y}{T}\right)\left(1 - \frac{y}{K}\right)y.$$

First we identify the critical points, which are $y_1 = 0, y_2 = T$ and $y_3 = K$. Next, plotting the graph y vs $f(y)$ (see the following Figure for $r = 1$, $T = 4$ and $K = 8$) leads to a cubic graph



The plot y vs $f(y)$ for the modified Logistic equation with threshold. Arrows indicate the behaviour of the solution.

Modified Logistic equation

Example 15.5 continue

We have the following observation for the ODE

$$y' = f(y) = -r\left(1 - \frac{y}{T}\right)\left(1 - \frac{y}{K}\right)y.$$

- if initial condition $y(0) = y_0 \in (0, T)$, then $f(y_0)$ is negative and the solution y should decrease;
- if initial condition $y(0) = y_0 \in (T, K)$, then $f(y_0)$ is positive and the solution y should increase;
- if initial condition $y(0) = y_0 > K$, then $f(y_0)$ is negative and the solution y should decrease.

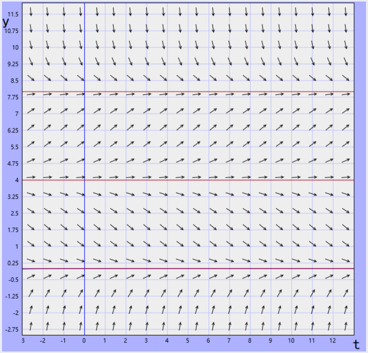
From this we deduce that

$$y(t) \to 0 \ if \ 0 < y_0 < T, \quad y(t) \to T \ if \ y_0 = T, \quad y(t) \to K \ if \ y_0 > T.$$

Modified Logistic equation

Example 15.5 continue

Furthermore, the vector field plot in the following figure also supports our observations on the solution behaviour.



Vector field for the Logistic equation in (t, y) plane with threshold.

Using vector field to study ODE

The idea of using direction field (vector field) and the plot y vs $f(y)$ to study the behaviour of the solution without actually solving the ODE is the essential idea of this chapter (section).

In the above examples we saw that there are instances where if we start "close" to a stationary solution, we either converge to the stationary solution, or we move away to another stationary solution, or even possibly the solution $y(t)$ goes to $\pm\infty$ as $t \to \infty$.

Definition of stability

For the stationary point y_*, if we start close to y_*, and the trajectory moves towards y_* as time progresses, then we call this the stationary point y_* is **asymptotically stable**.

Definition 15.6

(Stability). Given an autonomous first order ODE $y' = f(y)$, and a stationary solution y_*. We say that y_* is **asymptotically stable** if there is an $\delta_0 > 0$ (depending only on y_*) such that for any solution $\phi(t)$ to the IVP

$$y' = f(y) \text{ for } t \in I, \quad y(t_0) = y_0 \text{ with } t_0 \in I,$$

the following property is satisfied:

$$|y_0 - y_*| < \delta_0 \quad \Longrightarrow \quad \phi(t) \to y_* \text{ as } t \to \infty,$$

which means if we start close to y_*, we will move towards y_* as time progresses.

Definition of stability

For the stationary point y_*, if we start close to y_*, the trajectory moves far away from y_* as time progresses, then we call this the stationary point is **unstable**.

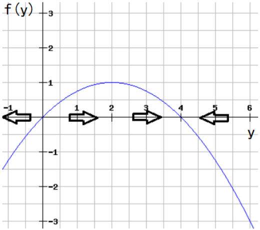
Denote $\phi(t)$ be the solution to the IVP

$$y' = f(y) \textit{ for } t \in I, \quad y(t_0) = y_0 \textit{ with } t_0 \in I,$$

Then for an unstable stationary solution y_*, except for $y_0 = y_*$(which implies that $\phi(t) = y_*$ for all $t \geq t_0$), any other initial condition would lead to $|\phi(t) - y_*| \nrightarrow 0$ $(t \to +\infty)$, and so the solution $\phi(t)$ will never reach y_* for an unstable stationary solution $y = y_*$.

Example for stability

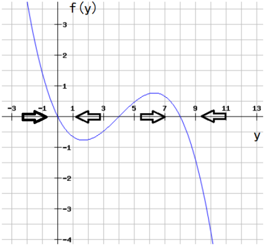
For the Logistic equation, the stationary solution $y_* = 0$ is unstable, and $y_* = K$ is asymptotically stable for any initial condition $y_0 > 0$.



The plot y vs $f(y)$ for the Logistic equation. Arrows indicate the behaviour of the solution.

Example for stability

For the Logistic equation with threshold (Modified Logistic equation), $y_* = 0$ and $y_* = K$ are asymptotically stable and $y_* = T$ is unstable.



The plot y vs $f(y)$ for the modified Logistic equation with threshold. Arrows indicate the behaviour of the solution.