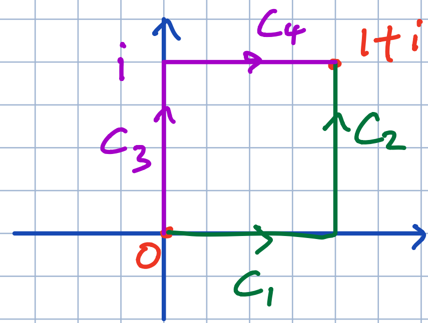
MAT 3253 Lecture 13

Example $f(z) = x^2 + iyx \qquad (z = x+iy)$



C_1 : $z(x) = x + 0i$

$0 \le x \le 1$

$z'(x) = 1$

$$\int_{C_1} f\,dz \triangleq \int_0^1 x^2 \cdot \overset{z'(x)}{1}\,dx = 1/3$$

C_2 : $z(y) = 1 + iy, \quad 0 \le y \le 1$

$z'(y) = i$

$$\int_{C_2} f\,dz \triangleq \int_0^1 (1 + iy\cdot 1)\cdot \overset{z'(y)}{i}\,dy = i - \frac{1}{2}$$

$$\boxed{\int_{C_1} + \int_{C_2} = i - \frac{1}{6}}$$

C_3 : $z(y) = iy, \quad 0 \le y \le 1$

$z'(y) = i$

$$\int_{C_3} f\,dz \triangleq \int_0^1 (0 + i\cdot 0)\,i\,dy = 0$$

C_4 : $z(x) = x + i, \quad 0 \le x \le 1$

$z'(x) = 1$

$$\int_{C_4} f\,dz \triangleq \int_0^1 x^2 + ix\,dx = \frac{1}{3} + \frac{i}{2}$$

$$\boxed{\int_{C_3} + \int_{C_4} = \frac{1}{3} + \frac{i}{2}}$$

Theorem Suppose $g(t)$ is continuous complex function from $[a,b]$ to $\mathbb{C}$. Then

$$\left| \int_a^b g(t)\,dt \right| \le \int_a^b |g(t)|\,dt$$

modulus

Proof $\alpha = \int_a^b g(t)\,dt$

$\alpha = re^{i\theta}$

$$e^{-i\theta}\int_a^b g(t)\,dt = r \in \mathbb{R}$$

$$= \int_a^b \underbrace{e^{-i\theta} g(t)}_{u(t)+iv(t)}\,dt$$

$$\int_a^b u(t)\,dt = r, \quad \int_a^b v(t)\,dt = 0$$

$$u(t) \le \sqrt{u^2(t)} \le \sqrt{u^2(t)+v^2(t)}$$
$$= |e^{-i\theta} g(t)|$$
$$= |g(t)|$$

By monotonic property for real functions

$$\left|\int_a^b g(t)\,dt\right| = r = \int_a^b u(t)\,dt \le \int_a^b |g(t)|\,dt \qquad \blacksquare$$

ML inequality / ML formula

Theorem If $|f(z)| \le M$ for z on a smooth curve C, and the length of $C = L$. then $\left|\int_C f(z)\,dz\right| \le ML$.

Def The length of of a curve C, represented $z(t)$, $a \le t \le b$, is defined as
$$\int_a^b |z'(t)|\,dt$$

Proof $$\left|\int_C f(z)\,dz\right| = \left|\int_a^b f(z(t))\cdot z'(t)\,dt\right|$$

$$\boxed{|z_1 z_2| = |z_1|\cdot|z_2|}$$

$$\leq \int_a^b |f(z(t))|\cdot|z'(t)|\,dt$$

$$\leq M\int_a^b |z'(t)|\,dt$$

$$= ML$$

$\blacksquare$

Example $$\left|\int_a^b e^{it}\,dt\right| \leq \int_a^b |e^{it}|\,dt$$

$$= \int_a^b dt = b-a$$

Theorem Suppose $f(z)$ is a continuous complex function in a region R.

The followings are equivalent

(a) f is the derivative of a function $F(z)$ in R
(F is an anti-derivative or primitive of f)

(b) for any contour C in R from z_1 to z_2

$$\int_C f(z)\,dz = F(z_2) - F(z_1)$$

Proof $(a \Rightarrow b)$

Suppose C is a smooth curve in R

Use chain rule for complex function

Let $g(t) = F(z(t))$ $\qquad z: [a,b] \to \mathbb{C}$

$g'(t) = F'(z(t))\cdot z'(t)$ $\qquad F: \mathbb{C} \to \mathbb{C}$

$$z(a) = z_1 \qquad z(b) = z_2$$

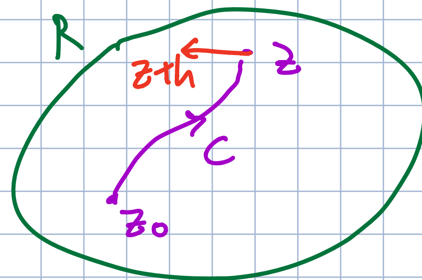
$$\int_C f(z)\,dz \triangleq \int_a^b \underbrace{f(z(t))}_{F'(z(t))} \cdot z'(t)\,dt$$

$$= \int_a^b g'(t)\,dt$$

$$= g(b) - g(a)$$

$$= F(z_2) - F(z_1)$$

$(b \Rightarrow a)$



Define $F(z) \triangleq \int_C f(z)$

$$= \int_{z_0}^{z} f(w)\,dw$$

Want to show $F'(z) = f(z)$.

$$\frac{F(z+h) - F(z)}{h} - f(z) = \frac{1}{h}\int_z^{z+h} f(w)\,dw - f(z)$$

$$= \frac{1}{h}\int_z^{z+h} f(w) - f(z)\,dw$$

Because f is continuous at z

Given any $\varepsilon > 0$, $\exists\, \delta > 0$ s.t.

$|f(w) - f(z)| < \varepsilon$ whenever $|w - z| < \delta$.

$$\left| \frac{F(z+h) - F(z)}{h} - f(z) \right| \overset{\text{ML inequality}}{\leq} \frac{1}{|h|} \underset{\uparrow_M}{\varepsilon}\, \underset{\uparrow_L}{|h|} \leq \varepsilon$$

for all $|h| < \delta$

$$\therefore \lim_{h \to 0} \frac{F(z+h) - F(z)}{h} = f(z)$$

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