MAT2006: Elementary Real Analysis
Assignment #1

Deadline: Sep. 30

1. Given an $n \in \mathbb{N}$ being not a square number, (i.e., $n \neq 1, 4, 9, 25, \dots$). Show that $\sqrt{n}$ is irrational.
Hint. The Fundamental Theorem of Arithmetic says that any natural number has a unique factorization

$$n = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k},$$

where $p_1, p_2, \dots, p_k$ are prime numbers and $n_1, n_2, \dots, n_k$ are natural numbers.

2. Given any rational number r and irrational number i, why $r + i$ and ri are irrational? (For the latter, we also assume $r \neq 0$.)

3. The *direct (Cartesian) product* of two sets X and Y is the set

$$X \times Y = \{(x, y) \mid x \in X \text{ and } y \in Y\}.$$

A *relation* between X and Y is a subset R of $X \times Y$, and x and y are said to be R-related if $(x, y) \in R$.

(i) Show that a function $f : X \to Y$ can be regarded as a relation. (**Note.** the corresponding R is called the graph of f).

(ii) Show that the order $\leq$ of real numbers is a relation, and illustrate R in the $\mathbb{R}^2$ plane.

(iii) Show that the "is a subset of" (or, "is contained in") $\subset$ among all subsets of a set M is a relation.

4. (i) Show that the relation in part (iii) of the last problem '$\subset$' is not a total order but a partial order. (Assume M has at least two elements.)

(ii) Given two points (x_1, y_1) and (x_2, y_2) on the $\mathbb{R}^2$ plane, define a relation $\preceq$ by the following

$$(x_1, y_1) \preceq (x_2, y_2) \qquad \text{if and only if} \quad x_1 < x_2 \quad \text{or} \quad (x_1 = x_2,\ y_1 \leq y_2).$$

Show that $\preceq$ is a total order.

(iii) Given two points (x_1, y_1) and (x_2, y_2) on the $\mathbb{R}^2$ plane, define a relation $\prec$ by the following

$$(x_1, y_1) \prec (x_2, y_2) \qquad \text{if and only if} \quad x_1 \leq x_2 \quad \text{and} \quad y_1 \leq y_2.$$

Show that $\prec$ is a partial order.

5. Let A and B be nonempty bounded above subsets of $\mathbb{R}$, and let $A+B$ be the set of all sums $a+b$ where $a \in A$ and $b \in B$. Show that $\sup(A+B) = \sup A + \sup B$.

6. Find the sup, inf, max and min for the following sets

$$(a) \quad A = \left\{ \frac{1}{n} \,|\, n \in \mathbb{N} \right\}; \qquad (b) \quad B = \left\{ 1 - \frac{1}{n} \,|\, n \in \mathbb{N} \right\}.$$

7. If $A \subset B$ and B is countable, then A is either finite or countable.
Hint. Assume A is infinite, find a 1–1 and onto function f from $\mathbb{N}$ to A. Assume g is a 1–1 and onto from $\mathbb{N}$ to B. Let $n_1 = \min\{n \in \mathbb{N} \,|\, g(n) \in A\}$ and set $f(1) = g(n_1)$. Continue this construction of f.

8. (i) If $A_1, A_2, \ldots, A_m$ are each countable sets, then the union $A_1 \cup A_2 \cup \cdots \cup A_m$ is countable. [The union of finite many of countable sets is countable.]
Hint. Use the induction argument. For the case $A_1 \cup A_2$, let $B = A_2 \setminus A_1$, then A_1 and B are disjoint and $A_1 \cup B = A_1 \cup A_2$.

(ii) If A_n is countable for each $n \in \mathbb{N}$, then $\bigcup_{n=1}^{\infty} A_n$ is countable. [The union of countable many of countable sets is countable.]
Hint. The induction argument doesnot apply here, (why?)

(iii) Show that the set of lattice points $\mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$ is countable.

9. In the lecture notes, a sketch proof of the Schröder–Bernstein Theorem is given. Complete the proof with details.

10. Show the following two sets have the same cardinality using the Schröder–Bernstein Theorem.

$$(a) \quad [0,1] \sim (0,1) \qquad (b) \quad [0,1] \sim [0,1] \times [0,1].$$

Hint. For (b), to find a 1–1 function from $[0,1] \times [0,1]$ to $[0,1]$, consider the decimal representation of real numbers.

11. Let S be the set consisting of all sequences of 0's and 1's. Observe that S is not a particular sequence, but rather a large set whose elements are sequences; namely,

$$S = \{(a_1, a_2, a_3, \ldots) \,|\, a_n = 0 \text{ or } 1\}$$

As an example, the sequence $(1,0,1,0,1,0,1,0,\ldots)$ is an element of S, as is the sequence $(1,1,1,1,1,1,\ldots)$. Show that S is uncountable.
Hint. Consider Cantor's digitalization method.

12. Answer each of the following by establishing a 1–1 correspondence with a set of known cardinality.

(i) Is the set of all functions from {0,1} to $\mathbb{N}$ countable or uncountable?

(ii) Is the set of all functions from $\mathbb{N}$ to {0, 1} countable or uncountable?

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