MAT2006: Elementary Real Analysis
Assignment #2

Reference Solution

1 (Squeeze Theorem)**.** Show that if $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$, and if $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = \ell$, then $\lim_{n\to\infty} y_n = \ell$ as well.

Proof. It follows from $n \in \mathbb{N}$, and if $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = \ell$ that, for any $\epsilon > 0$, there exists $N_x, N_z \in \mathbb{N}$ such that

$$|x_n - \ell| < \epsilon \qquad \forall n \geq N_x,$$

and

$$|z_n - \ell| < \epsilon \qquad \forall n \geq N_z,$$

which, together with the hypothesis $x_n \leq y_n \leq z_n$, yield

$$-\epsilon < x_n - \ell \leq y_n - \ell \leq z_n - \ell < \epsilon, \qquad \forall n \geq N := \max\{N_x, N_y\}.$$

That is

$$|y_n - \ell| \leq \epsilon \qquad \forall n \geq N,$$

with $N = \max\{N_x, N_y\}$, which completes the proof. □

2. Show that

$$\text{(i)} \qquad \lim_{n\to\infty} \sqrt[n]{1 + \frac{a}{n}} = 1, \quad \text{where} \quad a > 0.$$

$$\text{(ii)} \qquad \lim_{n\to\infty} \frac{n^k}{n!} = 0, \quad \text{where} \quad k \in \mathbb{N}.$$

$$\text{(iii)} \qquad \lim_{n\to\infty} \frac{n^k}{a^n} = 0, \quad \text{where} \quad a > 1, \quad k \in \mathbb{N}.$$

$$\text{(iv)} \qquad \lim_{n\to\infty} \frac{a^n}{n!} = 0, \quad \text{where} \quad a \in \mathbb{R}.$$

$$\text{(v)} \qquad \lim_{n\to\infty} \sqrt[n]{\frac{a^n}{n} + \frac{b^n}{n^2}} = b, \quad \text{where} \quad b \geq a > 0.$$

$$\text{(vi)} \qquad \lim_{n\to\infty} \frac{\sqrt[3]{n^2}\sin n!}{n+1} = 0.$$

$$\text{(vii)} \qquad \lim_{n\to\infty} \frac{n^2 + \cos n}{[n + (-1)^n]^2} = 1.$$

Proof. (i) Note that

$$1 \le \sqrt[n]{1+\frac{a}{n}} \le \sqrt[n]{2}$$

whenever $n \ge a$. Recall that $\sqrt[n]{p} \to 1$ for any $p > 0$. It then follows from the Squeeze Theorem that

$$\sqrt[n]{1+\frac{a}{n}} = 1, \quad \text{where} \quad a > 0.$$

(ii) Note that when $n \ge k$,

$$0 \le \frac{n^k}{n!} \le \frac{1}{n-k} \cdot \frac{n}{n-k+1} \cdot \frac{n}{n-k+2} \cdots \frac{n}{n-1} \cdot \frac{n}{n},$$

and that

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{n-k} \cdot \frac{n}{n-k+1} \cdot \frac{n}{n-k+2} \cdots \frac{n}{n-1} \cdot \frac{n}{n} \\ =&\lim_{n\to\infty} \frac{1}{n-k} \cdot \lim_{n\to\infty} \frac{n}{n-k+1} \cdots \lim_{n\to\infty} \frac{n}{n} \\ =&0 \cdot 1 \cdot 1 \cdots 1 = 0. \end{aligned}$$

The desired identity of limit holds by the Squeeze Theorem.

(iii) When $k = 1$, let $a = 1 + b$ with $b > 0$. Then, when $n \ge 2$,

$$a^n = (1+b)^n = 1 + nb + \frac{n(n-1)}{2} b^2 + \cdots > \frac{n(n-1)}{2} b^2$$

Therefore

$$0 < \frac{n^k}{a^n} = \frac{n}{a^n} < \frac{2}{(n-1)b^2} \to 0$$

as $n \to \infty$. By the Squeeze Theorem, we have

$$\lim_{n\to\infty} \frac{n}{a^n} = 0.$$

When $k \ge 2$, we then have

$$\lim_{n\to\infty} \frac{n^k}{a^n} = \lim_{n\to\infty} \left[\frac{n}{(a^{1/k})^n} \right]^k = 1.$$

Here, we have made use of the fact that $a^{1/k} > 1$ whenever $a > 1$ and $k > 0$.

(iv) Let $k > 2|a|$ be a natural number. When $n > k$, we have

$$0 \le \left| \frac{a^n}{n!} \right| = \left(\frac{|a|}{1} \cdot \frac{|a|}{2} \cdots \frac{|a|}{k} \right) \left(\frac{|a|}{k+1} \cdot \frac{|a|}{k+2} \cdots \frac{|a|}{n} \right) < |a|^k \left(\frac{1}{2} \right)^{n-k} = \frac{(2|a|)^k}{2^n} \to 0$$

as $n \to \infty$. Thus the desired limit identity holds by the Squeeze Theorem.

(v) Note that

$$\sqrt[n]{\frac{b^n}{n^2}} \le \sqrt[n]{\frac{a^n}{n} + \frac{b^n}{n^2}} \le \sqrt[n]{\frac{2b^n}{n}} \le b, \qquad \forall n \ge 2,$$

and that

$$\lim_{n\to\infty} \sqrt[n]{\frac{b^n}{n^2}} = \frac{b}{(\lim_{n\to\infty} \sqrt[n]{n})^2} = b.$$

Thus, the desired limit identity follows from the Squeeze Theorem.

(vi) Note that

$$0 \le \left| \frac{\sqrt[3]{n^2}\sin n!}{n+1} \right| \le \frac{\sqrt[3]{n^2}}{n} = \frac{1}{\sqrt[3]{n}} \to 0$$

as $n \to \infty$, the desired limit identity thus holds by the Squeeze Theorem.

(vii) Note that

$$\frac{n^2-1}{(n+1)^2} \le \frac{n^2+\cos n}{[n+(-1)^n]^2} \le \frac{n^2+1}{(n-1)^2}$$

and that

$$\lim_{n\to\infty} \frac{n^2-1}{(n+1)^2} = \lim_{n\to\infty} \frac{n^2+1}{(n-1)^2} = 1.$$

Therefore, the desired limit identity thus holds by the Squeeze Theorem. □

3 (Cesaro Means)**.** (i) Show that if $\{x_n\}$ is a convergent sequence, then the sequence given by the averages

$$y_n = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

also converges to the same limit.

(ii) Give an example to show that it is possible for the sequence $\{y_n\}$ of averages to converge even if $\{x_n\}$ does not.

Proof. (i) Assume $\{x_n\} \to L$. Given any $\epsilon > 0$, there exists $N_1 > 0$ such that $|x_n - L| \le \epsilon/2$ for every $n \ge N_1$. The convergence of $\{x_n\}$ implies it is bounded, and so is $\{x_n - L\}$. There exists $M > 0$ such that $|x_n - L| \le M$ for all $n \in \mathbb{N}$. Now, when $n \ge N_1$, we have

$$\begin{aligned}
|y_n - L| &= \left| \frac{x_1 + x_2 + \cdots + x_n - nL}{n} \right| \\
&= \left| \frac{(x_1 - L) + (x_2 - L) + \cdots + (x_n - L)}{n} \right| \\
&\le \frac{|x_1 - L| + |x_2 - L| + \cdots + |x_{N_1 - 1} - L|}{n} + \frac{|x_{N_1} - L| + \cdots |x_n - L|}{n} \\
&\le \frac{M(N_1 - 1)}{n} + \frac{(n - N_1)\epsilon}{2n}
\end{aligned}$$

Because N_1 and M are fixed constants at this point, we may choose N_2 so that $M(N_1-1)/n < \epsilon/2$ for all $n \ge N_2$. Now, let $N = \max\{N_1, N_2\}$. Then

$$|y_n - L| \le \frac{M(N_1 - 1)}{n} + \frac{(n - N_1)\epsilon}{2n} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Whenever $n \ge N$, which completes the proof.

(ii) The sequence $x_n = (-1)^n$ does not converge, but the averages satisfy $y_n \to 0$ as $n \to \infty$. □

4. Show that the sequence

$$\sqrt{2}, \quad \sqrt{2+\sqrt{2}}, \quad \sqrt{2+\sqrt{2+\sqrt{2}}}, \quad \cdots,$$

is convergent and find its limit.

Proof. We shall show that this sequence is increasing and bounded. First rewrite the sequence in a recursive way:

$$x_1 = \sqrt{2}, \qquad x_{n+1} = \sqrt{2x_n}$$

Let's prove that the sequence is increasing and bounded above by 2 by induction. Note that

$$x_1 = \sqrt{2} < \sqrt{2\sqrt{2}} = x_2 < 2.$$

so we just need to prove that $x_n < x_{n+1} < 2$ implies $x_{n+1} < x_{n+2} < 2$. If $x_n < x_{n+1} < 2$, then $\sqrt{2x_n} < \sqrt{2x_{n+1}} < \sqrt{4}$, which is $x_{n+1} < x_{n+2} < 2$ and the sequence is increasing and bounded above by 2.

Therefore this sequence converges by Monotone Convergence Theorem, and we have both $\{x_n\}$ and $\{x_{n+1}\}$ converge to some real number L. Taking limits across the recursive equation $x_{n+1} = \sqrt{2x_n}$, or equivalently $x_{n+1} = 2x_n$, yields $L^2 = 2L$, which implies $L = 2$ or $L = 0$. By the Order Limit Theorem, $L = 2$. (Argue that $x_n > 1$. Or, argue that 0 can not be the sup of x_n.) $\square$

5. Set $x_1 = 2$ and

$$x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}, \qquad \forall n \in \mathbb{N}.$$

Show that $\{x_n\}$ is convergent and find its limit.

Proof. We first observe that a simple induction argument shows that x_n is positive for all n. We also have

$$x_{n+1}^2 = \left(\frac{x_n}{2} + \frac{1}{x_n}\right)^2 = \left(\frac{x_n}{2} - \frac{1}{x_n}\right)^2 + 2 \geq 2,$$

which means $x_n^2 \geq 2$ for all $n \in \mathbb{N}$. Thus $\{x_n\}$ is bounded below (for example, $x_n \geq 1$ for all n).

Now we show that $\{x_n\}$ is decreasing. Now

$$x_{n+1} - x_n = \frac{x_n}{2} + \frac{1}{x_n} - x_n = \frac{2 - x_n^2}{2x_n} < 0,$$

for all $n \in \mathbb{N}$. Because $\{x_n\}$ is decreasing and bounded below, it must be convergent. We may set $x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1}$. Taking limits across the recursive equation we find

$$x = \frac{x}{2} + \frac{1}{x},$$

which is $x^2 = 2$. Thus $x = \sqrt{2}$. The case $x = -\sqrt{2}$ is ruled out since $x_n > 0$ and the Order Limit Theorem. $\square$

6. For a bounded sequence $\{x_n\}$, the Bolzano–Weierstrass Theorem says that there exists a convergent subsequence. Let E be the set of real numbers s such that $x_{n_k} \to s$ for some subsequence $\{x_{n_k}\}$. Show that

$$\limsup_{n\to\infty} x_n = \sup E \qquad \text{and} \qquad \liminf_{n\to\infty} x_n = \inf E.$$

Proof. Let $y_m = \sup\{x_n\}_{n=1}^{\infty}$. Then it is clear that $\{y_m\}$ is a decreasing sequence. Recall the definition

$$\limsup_{n\to\infty} x_n = \lim_{m\to\infty} \sup\{x_n\}_{n=m}^{\infty} = \lim_{m\to\infty} y_m := s.$$

We shall show that s is the least upper bound of E.

We first show that s is an upper bound of E. For every $a \in E$, there exists a subsequence $\{x_{n_k}\} \to a$ as $k \to \infty$. Note that $n_k \geq k$ and so

$$x_{n_k} \leq \sup\{x_n\}_{n=k}^{\infty} = y_k,$$

Taking $k \to \infty$ and by the Order Limit Theorem, we have

$$a = \lim_{k\to\infty} x_{n_k} \leq \lim_{k\to\infty} y_k = s.$$

Therefore, s is an upper bound of E.

Next we show that s is the least upper bound of E.

Method I. Assume b is also an upper bound of E, we shall show that $s \leq b$. Given any $\epsilon > 0$, we assert that there exists $N \in \mathbb{N}$ such that $x_n < b+\epsilon$ for $n \geq N$. Suppose this is not true. Then, these exists a subsequence $x_{n_p} \geq b + \epsilon$ for each $p \in \mathbb{N}$. Since $\{x_n\}$ is bounded and so is its subsequence $\{x_{n_p}\}$, and the Bolzano-Weierstrass Theorem implies there exists a subsequence of $\{x_{n_p}\}$ converges to a point $a \in E$. Now the Order Limit Theorem implies that $a \geq b + \epsilon$, which is a contradiction with the assumption that b is an upper bound of E. Thus, we have shown there exists $N \in \mathbb{N}$ such that $b + \epsilon$ is an upper bound of $\{x_n\}_{n=N}^{\infty}$, and hence, whenever $m \geq N$, we have $y_m \leq y_N = \sup x_n{}_{n=N}^{\infty} \leq b + \epsilon$. Then Order Limit Theorem grantees $s = \lim_{m\to\infty} y_m \leq b + \epsilon$. Since $\epsilon > 0$ is arbitrary, we must have $s \leq b$, which completes the proof.

Method II. Given any $\epsilon > 0$, we shall show that there exists $a \in E$ such that $a > s - \epsilon$. Recall that $y_m = \sup\{x_n\}_{n=m}^{\infty}$, thus there exists x_{n_m} such that $n_m \geq m$ and $x_{n_m} > y_m - \epsilon/2$. Now the boundedness of $\{x_{n_m}\}$ as a subsequence of the bounded sequence x_n and the Bolzano-Weierstrass theorem grantee the existence of a subsequence $\{x_{n_{m_k}}\}$ of $\{x_{n_m}\}$ that converges to a point $a \in E$. Now $x_{n_{m_k}} > y_{m_k} - \epsilon/2$, then by sending $k \to \infty$, it follows from the Order Limit Theorem that $a \geq \lim_{k\to\infty} y_{m_k} - \epsilon/2 = s - \epsilon/2 > s - \epsilon$. Here, we also have made use the fact that the subsequence of a convergent sequence converges to the same limit.

The proof of the result for the lower limit is similar and omitted here. □

7. For the following sequences, find their upper and lower limits.

$$\text{(i)} \quad \{(-1)^n\}_{n=1}^{\infty}, \qquad \text{(ii)} \quad \{(-1)^n n\}_{n=1}^{\infty}, \qquad \text{(iii)} \quad \left\{(-1)^n \frac{1}{n}\right\}_{n=1}^{\infty}.$$

Solution.

$$\text{(i)} \quad \limsup_{n\to\infty}(-1)^n = 1, \qquad \liminf_{n\to\infty}(-1)^n = -1,$$

$$\text{(ii)} \quad \limsup_{n\to\infty}(-1)^n n = \infty, \qquad \liminf_{n\to\infty}(-1)^n n = -\infty,$$

$$\text{(iii)} \quad \limsup_{n\to\infty}(-1)^n \frac{1}{n} = 0, \qquad \liminf_{n\to\infty}(-1)^n \frac{1}{n} = 0.$$

$\square$

8. Find the sup, inf, max and min for the following sets

$$(a) \quad A = \left\{\frac{1}{n} \,|\, n \in \mathbb{N}\right\}; \qquad (b) \quad B = \left\{1 - \frac{1}{n} \,|\, n \in \mathbb{N}\right\}.$$

Solution. See Assignment 1. $\square$

9. Show that a sequence $\{x_n\}$ is convergent if and only if $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n$. In this case, all three share the same value.

Proof. Denote $y_m = \sup\{x_n\}_{n=m}^\infty$ and $z_m = \inf\{x_n\}_{n=m}^\infty$. Then $\limsup_{n\to\infty} x_n = \lim_{m\to\infty} y_m$ and $\liminf_{n\to\infty} x_n = \lim_{m\to\infty} z_m$.

($\Rightarrow$) Assume $\lim_{n\to\infty} x_n = L$, we shall show that $\lim_{n\to\infty} y_n = L$. The proof of $\lim_{n\to\infty} z_n = L$ is similar. Given any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|x_n - L| < \epsilon$ for all $n \geq N$. Thus, $L - \epsilon < x_n < L + \epsilon$ for all $n \geq N$, and therefore $L + \epsilon < \sup\{x_n\}_{n=m}^\infty \leq L + \epsilon$ for all $m \geq N$. That is $|y_m - L| \leq \epsilon$ for all $m \geq N$. Thus $\lim_{m\to\infty} u_m = L$.

($\Leftarrow$) Assume $\lim_{m\to\infty} y_m = \lim_{m\to\infty} z_m = L$, we shall show that $\lim_{n\to\infty} x_n = L$. This follows immediately from the fact that $z_m \leq x_m \leq y_m$ and the Squeeze Theorem. $\square$

10 (Order Properties for Upper and Lower Limits)**.** Assume there exists $M \in \mathbb{N}$ such that $x_n \leq y_n$ for each $n \geq M$. Show that

$$\liminf_{n\to\infty} x_n \leq \liminf_{n\to\infty} y_n, \qquad \limsup_{n\to\infty} x_n \leq \limsup_{n\to\infty} y_n.$$

Proof. Assume $x_n \leq y_n$ for all n. Then we have, whenever $m \in \mathbb{N}$,

$$x_n \leq y_n \leq \sup\{y_n\}_{n=m}^\infty, \qquad \forall n \geq m$$

which means $\sup\{y_n\}_{n=m}^\infty$ is an upper bound of $\{x_n\}_{n=m}^\infty$ and thus,

$$\sup\{x_n\}_{n=m}^\infty \leq \sup\{y_n\}_{n=m}^\infty, \qquad \forall m \in \mathbb{N}.$$

Sending $m \to \infty$ in the above inequality and by the Order Limit Theorem, we have

$$\limsup_{n\to\infty} x_n \leq \limsup_{n\to\infty} y_n.$$

The proof to the inequality of the lower limits is similar. $\square$

11. Assume $0 \le x_{n+m} \le x_n + x_m$ for all $n, m \in \mathbb{N}$. Show that the sequence $\left\{\frac{x_n}{n}\right\}$ converges.
Hint. Apply the result about upper and lower limits in the above two problems.

Proof. Fixed $n \in \mathbb{N}$. Then for any $p \in \mathbb{N}$ and $p \ge n$, we have $p = kn + m$, where $0 \le m < n$. It is clear that

$$x_p = x_{kn+m} \le x_{kn} + x_m \le x_{(k-1)n} + x_n + x_m \le x_{(k-2)n} + 2x_n + x_m \le \cdots \le kx_n + x_m.$$

Thus,

$$0 \le \frac{x_p}{p} \le \frac{kx_n}{kn+m} + \frac{x_m}{p} \le \frac{x_n}{n} + \frac{M_n}{p}$$

where $M_n = \max\{x_1, x_2, \ldots, x_n\}$. Note that $\{\frac{x_p}{p}\}$ is a bounded sequence. Sending $p \to \infty$, we have

$$0 \le \limsup_{p\to\infty} \frac{x_p}{p} \le \frac{x_n}{n} + \limsup_{p\to\infty} \frac{M_n}{p} = \frac{x_n}{n}.$$

Notice that the above inequality holds for any fixed $n \in \mathbb{N}$. Taking the lower limits as $n \to \infty$ now yields

$$0 \le \limsup_{p\to\infty} \frac{x_p}{p} \le \liminf_{n\to\infty} \frac{x_n}{n},$$

which, combined with

$$\liminf_{n\to\infty} \frac{x_n}{n} \le \limsup_{n\to\infty} \frac{x_n}{n},$$

leads to

$$\liminf_{n\to\infty} \frac{x_n}{n} = \limsup_{n\to\infty} \frac{x_n}{n}.$$

Thus $\lim_{n\to\infty} \frac{x_n}{n}$ exists. $\square$

12. Assume $\lim\limits_{n\to\infty} x_n = A$. Show that

$$\lim_{n\to\infty} \frac{\frac{1}{2}x_1 + \frac{2}{3}x_2 + \cdots + \frac{n}{n+1}x_n}{n} = A.$$

Proof. Assume $\lim\limits_{n\to\infty} x_n = A$. Given any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$A - \epsilon < x_n < A + \epsilon, \qquad \forall n \ge N.$$

Hence,

$$\begin{aligned} y_n =& \frac{\frac{1}{2}x_1 + \frac{2}{3}x_2 + \cdots + \frac{n}{n+1}x_n}{n} \\ =& \frac{1}{n}\left(\frac{1}{2} + \cdots + \frac{N-1}{N}x_{N-1}\right) + \frac{1}{n}\left(\frac{N}{N+1}x_N + \cdots \frac{n}{n+1}x_n\right) \\ \le& \frac{1}{n}\left(\frac{1}{2} + \cdots + \frac{N-1}{N}x_{N-1}\right) + \frac{n-N+1}{n}(A+\epsilon). \end{aligned}$$

Taking upper limits to both sides, we have

$$\limsup_{n\to\infty} y_n \leq A + \epsilon.$$

In a similar manner, we also have

$$\liminf_{n\to\infty} y_n \geq A - \epsilon.$$

Therefore,

$$A - \epsilon \leq \liminf_{n\to\infty} y_n \leq \limsup_{n\to\infty} y_n \leq A + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we have

$$A = \limsup_{n\to\infty} y_n = \liminf_{n\to\infty} y_n = \lim_{n\to\infty} y_n.$$

Method II. Note that

$$\lim_{n\to\infty} \frac{n}{n+1} x_n = \lim_{n\to\infty} \frac{n}{n+1} \lim_{n\to\infty} x_n = A.$$

Then applies Problem 3 to the sequence $\{\frac{n}{n+1} x_n\}$. □

13. Assume $x_n > 0$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \ell < \infty$. Show that $\lim_{n\to\infty} \sqrt[n]{x_n} = \ell$.

Proof. Assume $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \ell$ and $0 < \ell < \infty$. Given any $\epsilon \in (0, \ell)$, there exists $N \in \mathbb{N}$ such that

$$\ell - \epsilon < \frac{x_{k+1}}{x_k} < \ell + \epsilon, \qquad \forall k \geq N.$$

For any $n \geq N$, multiplying the above inequality for each $n = N, N+1, \cdots n-1$, and taking the n-th root of the resulting inequality gives

$$\sqrt[n]{x_N}(\ell - \epsilon)^{(n-N)/n} < \sqrt[n]{x_n} \leq \sqrt[n]{x_N}(\ell + \epsilon)^{(n-N)/n}$$

In the above inequality, sending $n \to \infty$ and taking the lower and upper limits, we get

$$\ell - \epsilon \leq \liminf_{n\to\infty} \sqrt[n]{x_n} \leq \limsup_{n\to\infty} \sqrt[n]{x_n} \leq \ell + \epsilon.$$

Since $\epsilon \in (0, \ell)$ is arbitrary, we ahve

$$\liminf_{n\to\infty} \sqrt[n]{x_n} = \limsup_{n\to\infty} \sqrt[n]{x_n} = \ell,$$

and therefore,

$$\lim_{n\to\infty} \sqrt[n]{x_n} = \ell.$$

When $\ell = 0$, choosing $\epsilon > 0$ arbitrary, and replacing all $\ell - \epsilon$ by 0 in the above argument. □

14. Assume $x_n > 0$ for every $n \in \mathbb{N}$. Show that

$$\limsup_{n\to\infty} \sqrt[n]{x_n} \le \limsup_{n\to\infty} \frac{x_{n+1}}{x_n}.$$

Proof. Assume $\limsup\limits_{n\to\infty} \dfrac{x_{n+1}}{x_n} = A$. If $A = +\infty$, the conclusion is obviously true. We now assume that $0 \le A < \infty$ and shall show that $\limsup\limits_{n\to\infty} \sqrt[n]{x_n} \le A$.

For any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\frac{x_{m+1}}{x_m} \le \sup\left\{ \frac{x_{n+1}}{x_n} \,\middle|\, n \ge m \right\} < A + \epsilon, \qquad \forall m \ge N.$$

Given any $n \ge N$. Taking $m = N, N+1, \ldots, n-1$ and multiplying the resulting inequalities, we obtain

$$\frac{x_n}{x_N} < (A+\epsilon)^{n-N},$$

which leads to

$$\sqrt[n]{x_n} \le \sqrt[n]{x_N (A+\epsilon)^{-N}}(A+\epsilon).$$

Taking the upper limit as $n \to \infty$, we get

$$\limsup_{n\to\infty} \sqrt[n]{x_n} \le A + \epsilon.$$

It then follows form the fact that $\epsilon > 0$ is arbitrary that

$$\limsup_{n\to\infty} \sqrt[n]{x_n} \le A = \limsup_{n\to\infty} \frac{x_{n+1}}{x_n}.$$

This completes the proof. □

15. (i) Use the Monotone Convergence Theorem to prove the Archimedean Property without making any use of Least Upper Bound Property.

(ii) Use the Monotone Convergence Theorem to prove the Nested Interval Property without making any use of Least Upper Bound Property.

Proof. (i) Assume, for contradiction, that the AP is not true. That is $\mathbb{N}$ is bounded above. Now consider the two sequences $x_n = n$ and $y_n = x_n - 1 = n - 1$ for $n \in \mathbb{N}$. Both $\{x_n\}$ and $\{y_n\}$ are increasing and bounded above sequences. By MCT, they are convergent. Since $y_{n+1} = x_n$, hence $\{x_n\}$ and $\{y_n\}$ converges to the same limits. We have

$$1 = \lim_{n\to\infty}(x_n - y_n) = \lim_{n\to\infty} x_n - \lim_{n\to\infty} y_n = 0,$$

which is a contradiction. Thus AP must be true. (Note that, here we just made use of the Algebraic Limit Theorem, which is a consequence of the $\epsilon - N$ definition and the fact that $\mathbb{R}$ is an ordered field, nothing more. In particular, the argument doesnot depend on any completeness axioms.)

(ii) Let $I_n = [a_n, b_n]$ form a sequence of nested closed intervals,

$$I_1 \supset I_2 \supset I_3 \supset \cdots .$$

Clearly, $\{a_n\}$ is an increasing sequence and bounded above, since each b_n is an upper bound. By MCT, $\{a_n\} \to a$ for some $a \in \mathbb{R}$. [Caution: here we cannot say $a = \sup\{a_n\}$ because we don't assume LUBP, the existence of sup is still in question!]

Note that $a_n \leq b_m$ for all $n, m \in \mathbb{N}$. The Order Limit Theorem yields $a \leq b_m$ for each $m \in \mathbb{N}$. To show that $a_n \leq a$ for all $n \in \mathbb{N}$, we prove it by contradiction. Assume there exists n_0 such that $a_{n_0} = c > a$. Then $a_n \geq a_{n_0} > \frac{c+a}{2} > a$ for all $n \geq n_0$. Then the Order Limit Theorem leads to

$$a \geq \frac{a+c}{2} > a,$$

which is a contradiction. Thus $a_n \leq a \leq b$, that is $a \in I_n$ for each $n \in \mathbb{N}$. Thus

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset,$$

which proves the NIP. $\square$

16. Assume the Nested Interval Property is true. Use the technique in proving the Bolzano–Weierstrass Theorem to provide a proof of the Lest Upper Bound Property. To prevent the argument from being circular, assume also that $1/2^n \to 0$ (which is a consequence of the Archimedean Property).

Proof. Assume $E \subset \mathbb{R}$ is a nonempty bounded above set. We shall show that $\sup E$ exists. Let $a_1 \in E$ and b_1 be an upper bound of E, and set $I_1 = [a_1, b_1]$.

Set $c_1 = \frac{a_1+b_1}{2}$. If c_1 is an upper bound of E, and we take $a_2 = a_1$ and $b_2 = c_1$. Otherwise, we then take $a_2 = c_1$ and $b_2 = b_1$. We then set $I_2 = [a_2, b_2]$.

In general, if we have choose $I_n = [a_n, b_n]$ for $n \in \mathbb{N}$, b_n is an upper bound of E, and a_n is not an upper bound of E. Let $c_n = \frac{a_n+b_n}{2}$. If c_n is an upper bound of E, set $a_{n+1} = a_n$ and $b_{n+1} = c_n$. Otherwise, set $a_{n+1} = c_n$ and $b_{n+1} = b_n$. Then take $I_{n+1} = [a_{n+1}, b_{n+1}]$.

Then I_n's form a nested sequence of closed intervals

$$I_1 \supset I_2 \supset I_3 \supset I_4 \supset \cdots ,$$

thus by the Nested Interval Property, there exists $s \in \mathbb{R}$ such that

$$s \in \bigcap_{n=1}^{\infty} I_n .$$

Set $M = b_1 - a_1$. Then the length of I_n, $|I_n| = \frac{M}{2^{n-1}} \to 0$ as $n \to \infty$. (Here, we implicitly used the AP). Therefore

$$0 \leq |a_n - s| \leq |b_n - a_n| = \frac{1}{2^{n-1}} \to 0, \qquad 0 \leq |b_n - s| \leq |b_n - a_n| = \frac{1}{2^{n-1}} \to 0,$$

and hence Squeeze theorem yields

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = s.$$

(Note that Squeeze theorem depends on the definition of limit and the fact that $\mathbb{R}$ is an ordered field, and in particular, doesnot depend on any completeness axiom. Recall the proof of Problem 1.)

We assert that $s = \sup A$. (1) Since each b_n is an upper bound of A, we have $a \leq b_n$. Then the Order Limit Theorem leads to $a \leq s$ and hence s is an upper bound of A. (2) If b is an upper bound of A, then $a_n \leq b$ for each $n \in \mathbb{N}$ and the Order Limit Theorem now implies that $s \leq b$, which means s is the least upper bound of A. Therefore, we have shown that for each nonempty and bounded above set a, there exists a least upper bound $s = \sup A$. (We emphasize again here, the Order Limit Theorem doesnot depend on any completeness axiom.) □

17. Assume the Bolzano–Weierstrass Theorem is true and use it to construct a proof of the Monotone Convergence Theorem without making any appeal to the Archimedean Property.

Proof. Assume $\{x_n\}$ is an increasing sequence and bounded above. The case when $\{x_n\}$ is decreasing sequence and bounded below can be shown in a similar manner.

Clearly $\{x_n\}$ is a bounded sequence, and by the Bolzano–Weierstrass theorem, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges, and we assume it converges to $x \in \mathbb{R}$. Then, for any $\epsilon > 0$, there exists $K \in \mathbb{N}$ such that

$$|x_{n_k} - x| < \epsilon \qquad \forall k \geq K.$$

Set $N = n_K$, then the fact that $n_m \geq m$ and $\{x_n\}$ is increasing leads to

$$x - \epsilon < x_{n_K} \leq x_m \leq x_{n_m} \leq x + \epsilon, \qquad \forall m \geq N,$$

that is

$$|x_m - x| < \epsilon, \qquad \forall m \geq N.$$

Thus the sequence $\{x_n\}$ converges. □

18. Use the Cauchy Criterion to prove the Bolzano–Weierstrass Theorem, and find the point in the argument where the Archimedean Property is implicitly required.

Proof. Assume $\{x_n\}$ is a bounded sequence – there exists $M > 0$ such that $|x_n| \leq M$ for each $n \in \mathbb{N}$. In the lecture notes, we have used the NIP to show the Bolzano–Weierstrass theorem, where by successively bisecting the interval $[-M, M]$, we construction I_n. Take the same procedure, and we shall prove the Bolzano–Weierstrass theorem using Cauchy Criterion instead of NIP.

Given $\epsilon > 0$. By construction, the length of I_k is $M/2^{k-1}$ which converges to zero. Choose N so that $k \geq N$ implies that the length of I_k is less than ϵ. So for any $p, q \geq N$, because x_{n_p} and x_{n_q} are in I_k, it follows that $|x_{n_p} - x_{n_q}| < \epsilon$, which means that $\{x_{n_k}\}$ is a Cauchy sequence, and by the Cauchy Criterion, it converges. We thus proved the BW by CC. Here the Archimedean Property is used at the point when we claim that $M/2^{k-1}$ converges to zero. □

19. Assume $\sum_{n=1}^{\infty} a_n^2$ and $\sum_{n=1}^{\infty} b_n^2$ converge. Show that

$$\sum_{n=1}^{\infty} |a_n b_n|, \qquad \sum_{n=1}^{\infty} (a_n + b_n)^2, \qquad \sum_{n=1}^{\infty} \frac{|a_n|}{n}$$

also converge.

Proof. (a) Note that

$$|a_n b_n| \le \frac{a_n^2 + b_n^2}{2},$$

and that

$$\sum_{n=1}^{\infty} \frac{a_n^2 + b_n^2}{2}$$

converges by the Algebraic Limit Theorem for series. It then follows that $\sum_{n=1}^{\infty} |a_n b_n|$ also converges by the Comparison Test.

(b) Similar as part (a) by noting that

$$(a_n + b_n)^2 \le 2(a_n^2 + b_n^2).$$

(c) Take $b_n = \frac{1}{n}$ and recall that $\sum_{n=1}^{\infty} b_n^2 = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. Then apply the result in part (a). □

20. Show that if $\lim\limits_{n\to\infty} n a_n = a \neq 0$ then $\sum\limits_{n=1}^{\infty} a_n$ diverges.

Proof. Without loss of generality, we may assume $a > 0$. Then, $\lim\limits_{n\to\infty} n a_n = a$ implies that there exists $N \in \mathbb{N}$ such that

$$n a_n > a - \frac{a}{2} = \frac{a}{2} > 0, \qquad \forall n \ge N.$$

That is

$$a_n > \frac{a/2}{n}, \qquad \forall n \ge N.$$

Recall that $\sum \frac{1}{n}$ diverges, so does $\sum \frac{a/2}{n}$. Therefore, $\sum_{n=1}^{\infty} a_n$ also diverges by the Comparison Test. □

21. Proving the Alternating Series Test amounts to showing that the sequence of partial sums

$$s_n = a_1 - a_2 + a_3 - \cdots + (-1)^{n+1} a_n$$

converges. Different characterizations of completeness lead to different proofs.

(a) Prove the Alternating Series Test by showing that $\{s_n\}$ is a Cauchy sequence.

(b) Supply another proof for this result using the Nested Interval Property.

(c) Consider the subsequences $\{s_{2n}\}$ and $\{s_{2n+1}\}$, and show how the Monotone Convergence Theorem leads to a third proof for the Alternating Series Test.

Proof. (a) Assume that $\{a_n\}$ is positive and deceasing and that $\{a_n\} \to 0$. Note that when $n - m > 0$ is odd, one has

$$a_{m+1} - a_{m+2} + \cdots - a_{n+1} + a_n = a_{m+1} - (a_{m+2} - a_{m+3}) - (a_{m+4} + a_{m+5}) - \cdots - (a_{n-1} - a_n) \leq a_{m+1}$$

and

$$a_{m+1} - a_{m+2} + \cdots - a_{n+1} + a_n = (a_{m+1} - a_{m+2}) + (a_{m+3} - (a_{m+4}) + \cdots + (a_{n-2} - a_{n-1}) + a_n \geq 0$$

Similarly, when $n - m > 0$ is even,

$$a_{m+1} - a_{m+2} + \cdots + a_{n-1} - a_n = a_{m+1} - (a_{m+2} - a_{m+3}) - (a_{m+4} + a_{m+5}) - \cdots - (a_{n-2} - a_{n-1}) - a_n \leq a_{m+1}$$

and

$$a_{m+1} - a_{m+2} + \cdots - a_{n-1} - a_n = (a_{m+1} - a_{m+2}) + (a_{m+3} - (a_{m+4}) + \cdots + (a_{n-1} - a_n) \geq 0$$

Thus,

$$|s_n - s_m| = |a_{m+1} - a_{m+1} + \cdots + (-1)^{n-m} a_n| \leq a_{m+1} \qquad \forall n > m \geq 1.$$

Since $\{a_n\} \to 0$, given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $0 \leq a_n \leq \epsilon$ for all $n \geq N$. Therefore,

$$|s_n - s_m| \leq a_{m+1} < \epsilon, \qquad \forall n > m \geq N,$$

which means $\{s_n\}$ is a Cauchy sequence and therefore converges.

(b) Denote $I_1 = [0, s_1]$ and $I_2 = [s_2, s_1]$, Then $I_1 \supset I_2$ since $\{a_n\}$ is decreasing. In general, $I_{2m} = [s_{2m}, s_{2m-1}]$ and $I_{2m+1} = [s_{2m}, s_{2m+1}]$, for each $m \in \mathbb{N}$. and we have a sequence of nested closed intervals

$$I_1 \supset I_2 \supset I_3 \supset \cdots.$$

By the Nested Interval Property there exists at least one point S satisfying $S \in I_n$ for every $n \in \mathbb{N}$. Note that

$$0 \leq |s_n - S| \leq |s_n - s_{n-1}| = a_n \to 0$$

as $n \to \infty$. By the Squeeze Theorem, we have

$$\lim_{n \to \infty} s_n = S.$$

(c) Note that the subsequence $\{s_{2n}\}$ is increasing and bounded above since $s_{2n} \leq a_1$. The Monotone Convergence Theorem implies that $\lim_{n \to \infty} s_{2n} = S$ for some $S \in \mathbb{R}$. Now, the Algebraic Limit Theorem yields

$$\lim_{n \to \infty} s_{2n+1} = \lim_{n \to \infty} [s_{2n} + a_{2n+1}] = S + \lim_{n \to \infty} a_{2n+1} = S + 0 = S.$$

The fact that both $\{s_{2n}\}$ and $\{s_{2n+1}\}$ converge to S implies that $\{s_n\} \to S$ as well. $\square$

22. Discuss the convergence (absolute, conditional convergence or divergence) of the following series

$$\text{(i)} \quad \sum_{n=1}^{\infty} \frac{n \cos \frac{n\pi}{3}}{2^n}; \qquad \text{(ii)} \quad \sum_{n=1}^{\infty} (-1)^n \frac{\sin^2 n}{n}.$$

Solution. (i) Note that

$$\left|\frac{n\cos\frac{n\pi}{3}}{2^n}\right| = \frac{n}{2^n} := a_n.$$

Moreover,

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{2n} \to \frac{1}{2} < 1.$$

By the Ratio Test, the series is absolutely convergent.

(ii) The series is convergent but not absolutely convergent.

Note that

$$\sin^2 n = \frac{1-\cos(2n)}{2}$$

then

$$\sum_{n=1}^{\infty}(-1)^n\frac{\sin^2 n}{n} = \frac{1}{2}\sum_{n=1}^{\infty}(-1)^n\frac{1-\cos(2n)}{n}$$

Alternating series test tells

$$\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$$

converges, we just need to check the convergence of the other series $\sum_{n=1}^{\infty}(-1)^n\frac{\cos(2n)}{n}$.

(a) Recall the formula

$$\cos\alpha\cos\beta = \frac{\cos(\alpha-\beta)+\cos(\alpha+\beta)}{2}.$$

Then

$$\begin{aligned}
&2\cos(1)\sum_{n=1}^{N}(-1)^n\cos(2n)\\
=&\sum_{n=1}^{N}(-1)^n 2\cos(2n)\cos 1\\
=&\sum_{n=1}^{N}(-1)^n\{\cos(2n-1)+\cos(2n+1)\}\\
=&-[\cos 1+\cos 3]+[\cos 3+\cos 5]-[\cos 5+\cos 7]+\cdots+(-1)^N[\cos(2N-1)+\cos(2N+1)]\\
=&-\cos 1+(-1)^N\cos(2N+1).
\end{aligned}$$

Thus, the sequence of partial sums

$$s_n = \sum_{k=1}^{\infty}(-1)^k\cos(2k) = \frac{-\cos 1+(-1)^n\cos(2n+1)}{2\cos 1}$$

is bounded. Then Dirichlet's test yields

$$\sum_{n=1}^{\infty}(-1)^n\frac{\cos(2n)}{n}$$

is convergent. (Try by yourself.) Thus the series

$$\sin^2 n = \frac{1-\cos(2n)}{2}$$

is convergent because it is the sum of two convergent series.

(b) To check the absolute convergence, that is the convergence of

$$\sum_{n=1}^{\infty} \frac{\sin^2 n}{n} = \frac{1}{2} \sum \frac{1-\cos(2n)}{n}.$$

The harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

is divergent. We need to check

$$\sum_{n=1}^{\infty} \frac{\cos(2n)}{n}$$

is convergent. In a similar manner as above and using the formula

$$2\cos\alpha \sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta)$$

we have

$$\begin{aligned}
&2\sin(1) \sum_{n=1}^{N} \cos(2n) \\
=& \sum_{n=1}^{N} 2\cos(2n)\sin 1 \\
=& \sum_{n=1}^{N} \{-\sin(2n-1) + \sin(2n+1)\} \\
=&[-\sin 1 + \sin 3] + [-\sin 3 + \sin 5] + [-\sin 5 + \sin 7] + \cdots + [-\sin(2N-1) + \sin(2N+1)] \\
=& -\sin 1 + \sin(2N+1).
\end{aligned}$$

Thus, the sequence of partial sums

$$t_n = \sum_{k=1}^{\infty} \cos(2k) = \frac{-\sin 1 + \sin(2n+1)}{2\sin 1}$$

is bounded. Again, Dirichlet's test tells us the series

$$\sum_{n=1}^{\infty} \frac{\cos(2n)}{n}$$

is convergent. (Check this by yourself). Then argue that

$$\sum_{n=1}^{\infty} \frac{\sin^2 n}{n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1-\cos(2n)}{n}$$

is divergent. □

23 (Abel's test)**.** Abel's Test for convergence states that if the series $\sum_{k=1}^{\infty} x_k$ converges, and if $\{y_k\}$ is a sequence satisfying $y_1 \geq y_2 \geq y_3 \geq \cdots \geq 0$, then the series $\sum_{k=1}^{\infty} x_k y_k$ converges.

(i) Prove the *summation by parts* formula. Let $s_0 = 0$ and $s_n = x_1 + x_2 + \cdots x_n$ for $n \in \mathbb{N}$. Then

$$\sum_{k=m}^{n} x_k y_k = s_n y_{n+1} - s_{m-1} y_m + \sum_{k=m}^{n} s_k (y_k - y_{k+1})$$

Hint. Note that $x_k = s_k - s_{k-1}$.

(ii) Use the Comparison Test to argue that $\sum_{k=m}^{\infty} s_k (y_k - y_{k+1})$ converges absolutely, and show how this leads directly to a proof of Abel's Test.

Proof. (i) Note that

$$\sum_{k=m}^{n} x_k y_k = \sum_{k=m}^{n} (s_k - s_{k-1}) y_k = \sum_{k=m}^{n} s_k y_k - \sum_{k=m}^{n} s_{k-1} y_k,$$

and the second term can be rewritten as

$$\sum_{k=m}^{n} s_{k-1} y_k = \sum_{k=m-1}^{n-1} s_k y_{k+1}.$$

Therefore,

$$\sum_{k=m}^{n} x_k y_k = s_n y_{n+1} - s_{m-1} y_m + \sum_{k=m}^{n} s_k (y_k - y_{k+1}).$$

(ii) Since $\sum_{n=1}^{\infty} x_n$ converges, so does its partial sum $\{s_n\}$, and hence $\{s_n\}$ is bounded. Assume $|s_n| \leq M$ for each $n \in \mathbb{N}$. Now,

$$|s_k (y_k - y_{k+1})| \leq M (y_k - y_{k+1}).$$

Since $\{y_k\}$ is decreasing and bounded below, it follows that $\{y_k\} \to y \geq 0$ by the Monotone Convergence Theorem. Then

$$\sum_{k=m}^{\infty} M (y_k - y_{k+1}) = M (y_m - \lim_{n \to \infty} y_n) = M (y_m - y)$$

is convergent for each $m \in \mathbb{N}$. By the Comparison Test,

$$\sum_{k=m}^{\infty} s_k (y_k - y_{k+1})$$

is absolutely convergent for each $m \in \mathbb{N}$.

Using the result in part (i), the partial sum of $\sum x_k y_k$ satisfies

$$t_n = \sum_{k=1}^{n} x_k y_k = s_n y_{n+1} + \sum_{k=1}^{n} s_k (y_k - y_{k+1}) := s_n y_{n+1} + r_n$$

where

$$r_n = \sum_{k=1}^{n} s_k(y_k - y_{k+1})$$

is the partial sum of $\sum_{k=1}^{n} s_k(y_k - y_{k+1})$ which converges to some $R \in \mathbb{R}$. Sending $n \to \infty$, we then have

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} (s_n y_{n+1} + r_n) = Sy + R,$$

by the Algebraic Limit Theorem. Therefore, $\sum_{k=1}^{\infty} x_k y_k$ converges. □

24 (Dirichlet's Test)**.** Dirichlet's Test for convergence states that if the partial sums of $\sum_{k=1}^{\infty} x_k$ are bounded (but not necessarily convergent), and if $\{y_k\}$ is a sequence satisfying $y_1 \geq y_2 \geq y_3 \geq \cdots \geq 0$, with $\lim_{k\to\infty} y_k = 0$, then the series $\sum_{k=1}^{\infty} x_k y_k$ converges.

(i) Point out how the hypothesis of Dirichlet's Test differs from that of Abel's Test, but show that essentially the same strategy can be used to provide a proof.

(ii) Show how the Alternating Series Test can be derived as a special case of Dirichlet's Test.

Proof. (i) In the Abel Test, one requires $\sum_{k=1}^{\infty} x_k$ converges while in the Dirichlet Test one only needs the partial sum of $\sum_{k=1}^{\infty} x_k$ is bounded (not necessarily convergent); and for the sequence $\{y_n\}$, it should be decreasing in Abel's test while not only decreasing and but also tend to 0 in Dirichlet's Test.

All the proof will follow exactly line by line as in the last Problem part (ii) until to the end, that $\lim_{n\to\infty} s_n$ doesnot exist here. But,

$$0 \leq |s_n y_{n+1}| \leq M y_{n+1} \to 0$$

as $n \to \infty$ by hypothesis, and the Squeeze Theorem yields

$$\lim_{n\to\infty} s_n y_{n+1} = 0.$$

Therefore, we still have the limit

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} (s_n y_{n+1} + r_n) = 0 + R = R$$

exists, and hence $\sum_{k=1}^{\infty} x_k y_k$ converges.

(ii) The Alternating Series Test is a special case when $x_k = (-1)^{k+1}$. Note that the partial sum of $\sum_{k=1}^{\infty}(-1)^{k+1}$ has an upper bound 1. □

— End —