MAT 2006 Tutorial 1. 9.14.

$\sqrt{2}, \sqrt{3}, \sqrt{6}$

1. Proof. Suppose, for contradiction that $\sqrt{3}$ is a rational number

$\sqrt{3} = \frac{p}{q}$, $p, q \in \mathbb{Z}$ $\Rightarrow 3 = \frac{p^2}{q^2} \Rightarrow p^2 = 3q^2$.

(p and q have no common factors except 1).

$\Rightarrow$ p^2 has 3 as a factor.

$\Rightarrow$ p has 3 as a factor

If $p = 3k+1$, then $p^2 = (3k+1)^2 = 9k^2 + 6k + 1$ is not a multiple of 3.

If $p = 3k+2$, then $p^2 = (3k+2)^2 = 9k^2 + 6k + 4$ is not a multiple of 3.

$\Rightarrow (3k)^2 = 3q^2 \Rightarrow 9k^2 = 3q^2 \Rightarrow q^2 = 3k^2$.

(similar) $\Rightarrow$ q^2 has 3 as a factor.

$\Rightarrow$ q has 3 as a factor.

contradiction!

$\Rightarrow \sqrt{3}$ is not a rational number.

Exercise ① $\sqrt{6}$? ② $\sqrt{4}$? ③ $\sqrt{n}$? (n is not a square number).

2. (1) Proof ① Show $(A \cup B)^c \subset A^c \cap B^c$.

$x \in (A\cup B)^c \Rightarrow x \notin A \cup B \Rightarrow x \notin A$ and $x \notin B$.

$\Rightarrow x \in A^c$ and $x \in B^c \Rightarrow x \in A^c \cap B^c$.

② Show $A^c \cap B^c \subset (A \cup B)^c$

$x \in A^c \cap B^c \Rightarrow x \in A^c$ and $x \in B^c \Rightarrow x \notin A$ and $x \notin B$

$\Rightarrow x \notin A \cup B \Rightarrow x \in (A \cup B)^c$.

(2). Proof ① $n=1$. $A_1^c = A_1^c$ ✓

$n=2$ $(A_1 \cup A_2)^c = A_1^c \cap A_2^c$ ✓

✓ finite $n \in \mathbb{N}$.

② Suppose that it holds for $n=k$, i.e.

$(A_1 \cup A_2 \cup \cdots \cup A_k)^c = A_1^c \cap A_2^c \cap \cdots \cap A_k^c$.

when $n=k+1$,

$$(A_1 \cup A_2 \cup \cdots \cup A_k \cup A_{k+1})^c = ((A_1 \cup \cdots \cup A_k) \cup A_{k+1})$$

(By ①) $= (A_1 \cup \cdots \cup A_k)^c \cap A_{k+1}^c$

$= A_1^c \cap A_2^c \cap \cdots \cap A_{k+1}^c$

Does $\left(\bigcup_{n=1}^{\infty} A_n\right)^c = \bigcap_{n=1}^{\infty} A_n^c$ holds? Yes! (But does not hold for induction).

3. (1) Proof. ① $\max\{\sup A, \sup B\}$ is an u.B. of $A \cup B$.

$\forall x \in A \cup B$, $x \in A$ or $x \in B$.

If $x \in A$, then $x \le \sup A \le \max\{\sup A, \sup B\}$.

If $x \in B$, then $x \le \sup B \le \max\{\sup A, \sup B\}$

② $\max\{\sup A, \sup B\}$ is L.U.B of $A \cup B$.

If S is an u.B of $A \cup B$. then is u.B of A. B.

$S \ge \sup A$. $S \ge \sup B$.

$\Rightarrow S \ge \max\{\sup A, \sup B\}$

(2) Proof. Prove by induction.

① $n=1$ ✓. $n=2$ ✓.

② Suppose it holds for $n=k$. i.e

$\sup(A_1 \cup \cdots \cup A_k) = \max\{\sup A_1, \cdots \sup A_k\}$.

when $n=k+1$,

$\sup(A_1 \cup \cdots \cup A_{k+1})$

$= \sup((A_1 \cup \cdots \cup A_k) \cup A_{k+1})$

$= \max\{\sup(A_1 \cup \cdots \cup A_k), \sup A_{k+1}\}$.

$= \max\{\max\{\sup A_1 \cdots \sup A_k\}, \sup A_{k+1}\}$.

$= \max\{\sup A_1 \cdots \sup A_{k+1}\}$.

$\sup(\bigcup_{n=1}^{\infty} A_n)$. $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n\in\mathbb{N}} A_n$.

(3) Proof. $\sup(\bigcup_{n=1}^{\infty} A_n) = \max_{n\in\mathbb{N}}\{\sup A_n\}$ $\bigcup_{n=1}^{\infty} A_n = [0,1)$.

Check (a): $A_n = [0, 1-\frac{1}{n}]$. $\Rightarrow \sup(\bigcup_{n=1}^{\infty} A_n) = 1$

$\sup A_n = 1-\frac{1}{n}$: $\max_{n\in\mathbb{N}}(1-\frac{1}{n})$ does not exist !! ✗

(*) $\left(\sup(\bigcup_{n=1}^{\infty} A_n) = \sup_{n\in\mathbb{N}}(\sup A_n)\right)$ It's ok for (a).

Check (b): $A_n = [0,n]$ $\bigcup_{n=1}^{\infty} A_n = [0,\infty)$.

$\Rightarrow \sup(\bigcup_{n=1}^{\infty} A_n)$ does not exist !! ✗

Exercise: If $\sup(\bigcup_{n=1}^{\infty} A_n)$ exists, then (*) is true !! ✓

(4) Proof. Show that $\sup B$ is an u.B of A

$A\subset B$. $A \neq \emptyset \Rightarrow B \neq \emptyset$.

B is bdd above & $B\neq\emptyset \Rightarrow \sup B$ exists !!

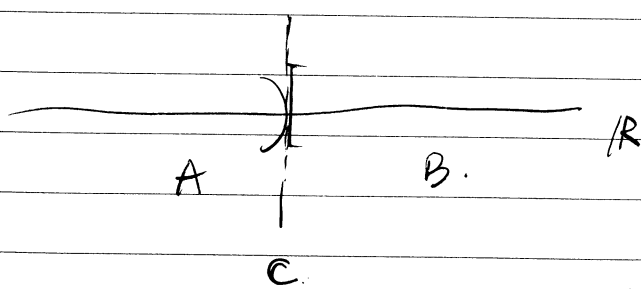
$\forall x\in A$, then $x\in B$. $x \le \sup B$.

$\Rightarrow \sup B$ is an u.B of A.

Since A is not $\emptyset$, by L.U.B.P. $\sup A$ exists !!

$\Rightarrow \sup A \le \sup B$.

4. Proof



$A\cap B = \emptyset$.
$A\cup B = \mathbb{R}$

L.U.B.P $\overset{\text{i.e.}}{\Leftrightarrow}$ Dedekind's Cut property

This does not hold for $\mathbb{Q}$.

$A = \{q\in\mathbb{Q} \mid q<\sqrt{2}\}$

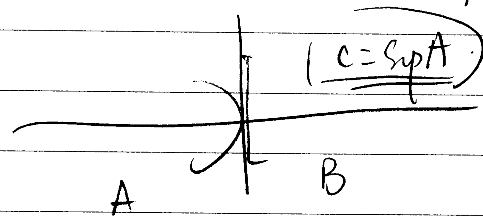
$\left(\begin{array}{l} A\cap B = \emptyset \\ A\cup B = \mathbb{Q} \end{array} \right) \Leftarrow B = \{q\in\mathbb{Q} \mid q>\sqrt{2}\}$.

$\mathbb{R}$ $\mathbb{Q}$ $\mathbb{R}$ $\quad \frac{1}{n} > 0 \ \forall n \in \mathbb{N}$

No C can be chose to seperete A & B !

✓ ① (LUBP $\Rightarrow$ Cut property) Suppose (A, B) is a cut.

W.T.S $\exists c \in \mathbb{R}$ s.t $a \le c \ \forall a \in A$ and $b \ge c \ \forall b \in B$



intuition !

$B \neq \phi$, $\forall b \in B$, $b > a$, $\forall a \in A$. $\Rightarrow \forall b \in B$ is an U.B of A.

$A \neq \phi \Rightarrow$ By L.U.B.P, Sup A exists !

Take $c = \text{Sup} A$. ✓

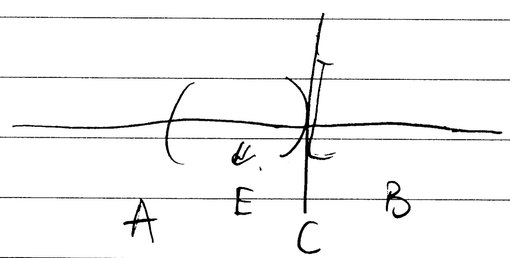
$\Rightarrow \forall a \in A$, $a \le \text{Sup} A = c$.

$\Rightarrow \forall b \in B$, b is an U.B of A.

So, $b \ge \text{Sup} A = c$. ✓

✓ ② (Cut property $\Rightarrow$ LUBP) Suppose $E \subset \mathbb{R}$ is nonempty and bdd above.

W.T.S Sup E exists.



Take $B = \{x \in \mathbb{R} \mid x \text{ is an U.B of } E\}$

$A = \mathbb{R} \setminus B = \{x \in \mathbb{R} \mid x < e \text{ for some } e \in E\}$.

E has an U.B $\Rightarrow B \neq \phi$

If max E exists, Sup E = max E, nothing to prove !

If max E does not exist, $\forall e \in E$, e is not U.B of E. ✓

$\Rightarrow E \subset A$. $(A \neq \phi)$.

clearly, $A \cap B = \phi$. $A \cup B = \mathbb{R}$.

$\forall a \in A$, $\forall b \in B$, $\exists e \in E$. s.t. $a < e \le b$. $\Rightarrow (A, B)$ is a cut.

By cut property, $\exists c \in \mathbb{R}$. s.t. $a \le c$, $\forall a \in A$, & $b \ge c$, $\forall b \in B$.

$\Rightarrow$ W.T.S. $c = \sup E$.

Suppose for contradiction, c is not an U.B of E.

$\exists e_0 \in E$, s.t. $e_0 > c$. $\Rightarrow$ $c < \underbrace{\frac{e_0 + c}{2}}_{\in A} < e_0$.

$c \ge a$ $\forall a \in A$

contradiction !!

So c is an U.B of E.

B contains all U.B of E, $c \le b$, $\forall b \in B$.

$\Rightarrow c = \sup E$.