Stochastic Processes

Lecture 15: Poisson processes (III)

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Thinning

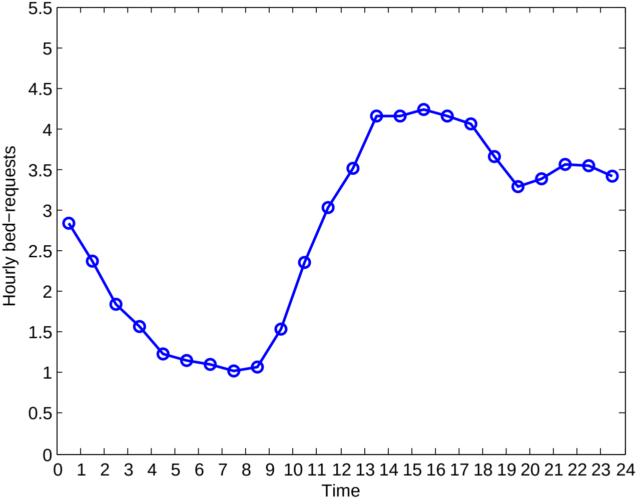
- Let $N = \{N(t), t \geq 0\}$ be a Poisson process with rate λ.
- Each arrival flips a coin with probability of p getting a head.
- $N_1(t)$ is the number of heads in $(0, t]$,
- $N_2(t)$ is the number of tails in $(0, t]$.
- $N_i = \{N_i(t), t \geq 0\}$ is a Poisson process with rate λ_i, $i = 1, 2$, where $\lambda_1 = \lambda p$ and $\lambda_2 = \lambda(1 - p)$.
- Furthermore N_1 and N_2 are independent.

Time-nonhomogenous Poisson processes

Definition

A stochastic process $N = \{N(t), t \geq 0\}$ is said to be a (time-nonhomogeneous) Poisson process with rate function $\{\lambda(t), t > 0\}$ if (a) it has independent increments, (b) $N(s,t] \sim \text{Poisson}(\int_s^t \lambda(u)\, du)$ for any $0 \leq s < t$, (c) $N(0) = 0$.

Bed-request patterns



Similar patterns observed in Armony et al. (2015), Griffin et al. (2011), Powell et al. (2012)

Simulating a non-homogeneous Poisson process

- time-change: G is a rate-1 Poisson process

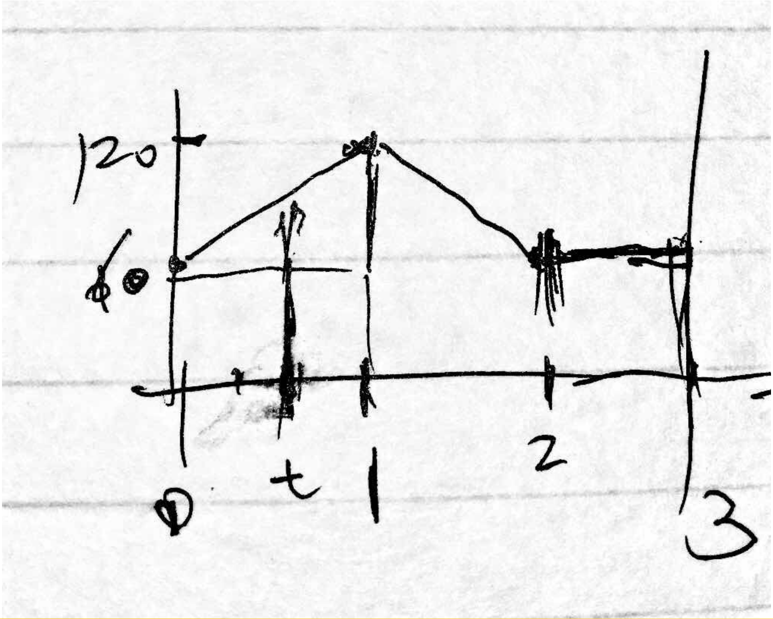
$$N(t) = G\left(\int_0^t \lambda(u)\,du\right), \quad t \geq 0.$$

- N is a Poisson process with rate function $\{\lambda(t), t \geq 0\}$.
- accept-reject: Suppose that $\lambda(t) \leq \Lambda$ for all $t \geq 0$. Let G be a Poisson process with rate Λ. At each arrival time t of G, flip a coin with probability of $\lambda(t)/\Lambda$ getting a head.

$$N(t) = \#\text{ of heads in } (0, t].$$

- Is N what we expect to be?

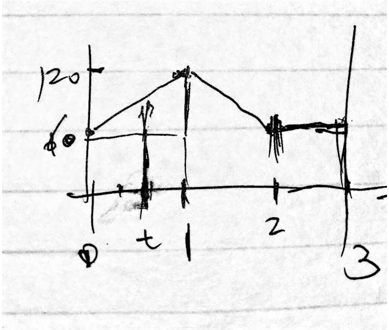
An example



Rate function $\lambda(t)$

- $\lambda(t) = 60 + 60t$ for $0 \leq t \leq 1$
- $\lambda(t) = 120 - 60(t-1)$ for $1 \leq t \leq 2$
- $\lambda(t) = 60$ for $t \leq 2$.

$\Lambda(s) = \int_0^s \lambda(t)dt$



- $\Lambda(s) = \frac{60+(60+60s)}{2}s$ for $0 \le s \le 1$.
- $\Lambda(1) = 90$.
- $\Lambda(s) = 90 + \frac{60+(120+60(2-s))}{2}(s-1)$ for $1 \le s \le 2$.
- $\Lambda(2) = 180$.
- $\Lambda(s) = 180 + 60(s-2)$ for $2 \le s \le 3$.

$\Lambda^{-1}(t)$

- For $0 \le t \le 90$,

$$\Lambda^{-1}(t) = \sqrt{1 + t/30} - 1.$$

- For $90 \le t \le 180$,

$$30s^2 - 90s + t - 120 = 0,$$
$$s = 3 - \sqrt{9 - (t + 60)/30}$$

- For $180 \le t$,

$$s = 2 + \frac{t - 180}{60}.$$

Example, $A = 120$

- For $0 \le t \le 1$

$$\frac{\lambda(t)}{A} = \frac{60 + 60t}{120} = .5 + .5t.$$

Markov Property

Theorem

Let $\{N(t)\}_{t\geq 0}$ be a Poisson process of rate λ. Then, for any $s \geq 0$, $\{N(s+t) - N(s)\}_{t\geq 0}$ is also a Poisson process of rate λ, independent of $\{N(r) : r \leq s\}$.

Comapre with Markov Property for DTMC.

Renewal Process

Inspection Paradox