Appendix

Distribution of Wilcoxon signed ranks

By Assumption 2.2, $X_1,\ldots,X_n$ are symmetric about 0 under $H_0:\theta=0$, so that

$$\Pr(X_i<0)=\Pr(X_i>0)=0.5 \quad \text{and} \quad X_i\sim -X_i,\ i=1,\ldots,n.$$

Let R_i be the rank of X_i in absolute order, $\psi_i=I_{\{X_i>0\}}$,

$$T^+=R_1\psi_1+R_2\psi_2+\cdots+R_n\psi_n \quad \text{and} \quad S=\psi_1+2\psi_2+\cdots+n\psi_n$$

Before proving $T^+\sim S$ generally, let us first look at a special case to demonstrate the idea of the proof. Consider $n=3$. Then

$$\Pr(S=1)=\Pr(\psi_1=1,\psi_2=\psi_3=0)=\Pr(X_1>0,X_2<0,X_3<0)=0.5^3$$

and

$$\Pr(T^+=1)=\Pr(R_1=1,\psi_1=1,\psi_2=\psi_3=0)+\Pr(R_2=1,\psi_2=1,\psi_1=\psi_3=0)$$
$$+\Pr(R_3=1,\psi_1=\psi_2=0,\psi_3=1)$$

Since $X_i \sim -X_i$ and R_1 is not affected by replacing X_i with $-X_i$,

$$\Pr(R_1 = 1, \psi_1 = 1, \psi_2 = \psi_3 = 0) = \Pr(R_1 = 1, X_1 > 0, X_2 < 0, X_3 < 0)$$

$$= \Pr(R_1 = 1, X_1 > 0, -X_2 < 0, -X_3 < 0) = \Pr(R_1 = 1, X_1 > 0, X_2 > 0, X_3 > 0)$$

Similarly,

$$\Pr(R_2 = 1, \psi_2 = 1, \psi_1 = \psi_3 = 0) = \Pr(R_2 = 1, X_1 > 0, X_2 > 0, X_3 > 0) \text{ and}$$

$$\Pr(R_3 = 1, \psi_1 = \psi_2 = 0, \psi_3 = 1) = \Pr(R_3 = 1, X_1 > 0, X_2 > 0, X_3 > 0)$$

It follows that

$$\Pr(T^+ = 1) = \sum_{i=1}^{3} \Pr(R_i = 1, X_1 > 0, X_2 > 0, X_3 > 0) = \Pr(X_1 > 0, X_2 > 0, X_3 > 0)$$

$$= 0.5^3 = \Pr(X_1 > 0, X_2 < 0, X_3 < 0) = \Pr(S = 1)$$

By similar arguments we can show $\Pr(T^+ = t) = \Pr(S = t)$ for all $t \in \{0, 1, \ldots, 6\}$. Thus $T^+ \sim S$ for $n = 3$. The proof for general sample size n is provided next.

Let $i(1),\ldots,i(B) \in \{1,\ldots,n\}$ with $i(1) < \cdots < i(B)$. Define

$$\begin{cases} E = \{i(1),\ldots,i(B)\}, \quad A(E) = i(1) + \cdots + i(B) & \text{if } 1 \le B \le n \\ E = \phi \text{ (empty set)}, \quad A(E) = 0 & \text{if } B = 0 \end{cases} \tag{A.1}$$

For $t \in \{0,1,\ldots,M = n(n+1)/2\}$, define

$$\mathcal{E}_t = \{E \text{ defined in (A.1) such that } A(E) = t\} \tag{A.2}$$

For example, if $n \ge 3$, then $\mathcal{E}_0 = \{\phi\}$, $\mathcal{E}_2 = \{\{2\}\}$, $\mathcal{E}_3 = \{\{3\},\{1,2\}\}$, and so on. Then it follows from (A.1) – (A.2) that

$$\begin{aligned} \Pr(S = t) &= \sum_{E \in \mathcal{E}_t} \Pr\left(\psi_1 + \cdots + n\psi_n = t = i(1) + \cdots + i(B), E = \{i(1),\ldots,i(B)\}\right) \\ &= \sum_{E \in \mathcal{E}_t} \Pr\left(\psi_{i(1)} = \cdots = \psi_{i(B)} = 1, \psi_i = 0 : i \notin \{i(1),\ldots,i(B)\}\right) \\ &= \sum_{E \in \mathcal{E}_t} \Pr\left(X_i > 0 : i \in E, X_u < 0 : u \notin E\right) \end{aligned}$$

This together with $\Pr(X_i < 0) = \Pr(X_i > 0) = 0.5$ imply

$$\Pr(S = t) = \sum_{E \in \mathcal{E}_t} \Pr(X_i > 0 : i \in E, -X_u < 0 : u \notin E) = \sum_{E \in \mathcal{E}_t} \Pr(X_1 > 0, \ldots, X_n > 0)$$

$$= \sum_{E \in \mathcal{E}_t} \prod_{i=1}^{n} \Pr(X_i > 0) = \sum_{E \in \mathcal{E}_t} 0.5^n = 0.5^n \left|\mathcal{E}_t\right| \quad \text{for } 1 \le B \le n, \tag{A.3}$$

where $\left|\mathcal{E}_t\right|$ represents the number of elements (sets E) in $\mathcal{E}_t$.

Next, define

$$j(b) = i \;\text{ if and only if }\; R_i = i(b), \text{ so that } R_{j(b)} = i(b),\;\; b = 1, \ldots, B.$$

Then $\omega = (j(1), \ldots, j(B))$ is a permutation of B elements in $\{1, 2, \ldots, n\}$.

Denote by $\Omega = \Omega(B)$ the set of all such permutations, and define

$$J = \begin{cases} \{j(1), \ldots, j(B)\} & \text{if } 1 \le B \le n \\ \phi & \text{if } B = 0 \end{cases} \tag{A.4}$$

Given $E = \{i(1),\ldots,i(B)\}$ defined in (A.1) with

$$i(1)+\cdots+i(B)=t\in\{0,1,\ldots,M\},$$

if $\omega=\big(j(1),\ldots,j(B)\big)\in\Omega$ satisfies

$$\big(R_{j(1)},\ldots,R_{j(B)}\big)=(i(1),\ldots,i(B)),\ X_u>0 \text{ for } u\in J \ \text{ and } X_u<0 \text{ for } u\notin J,$$

where J is defined in (A.4), then

$$T^{+}=R_1\psi_1+\cdots+R_n\psi_n=R_{j(1)}+\cdots+R_{j(B)}=i(1)+\cdots+i(B)=t$$

It follows that

$$\Pr\big(T^{+}=t\big)=\sum_{E\in\mathcal{E}_t}\sum_{\omega\in\Omega}\Pr\Big(\big(R_{j(1)},\ldots,R_{j(B)}\big)=(i(1),\ldots,i(B));i(1)+\cdots+i(B)=t\Big)$$

$$=\sum_{E\in\mathcal{E}_t}\sum_{\omega\in\Omega}\Pr\big(R_{j(b)}=i(b),b=1,\ldots,B;X_u>0:u\in J,X_u<0:u\notin J\big)$$

Since $X_u \sim -X_u$ for $u = 1,\ldots,n$ due to the symmetry of X_u about 0, and the ranks $R_1,\ldots,R_n$ are not affected by replacing X_u with $-X_u$, the above equation leads to

$$\begin{aligned}
\Pr(T^+ = t) &= \sum_{E\in\mathcal{E}_t}\sum_{\omega\in\Omega}\Pr\left(R_{j(b)} = i(b), b = 1,\ldots,B; X_u > 0 : u \in J, -X_u < 0 : u \notin J\right) \\
&= \sum_{E\in\mathcal{E}_t}\sum_{\omega\in\Omega}\Pr\left(R_{j(b)} = i(b), b = 1,\ldots,B; X_1 > 0,\ldots,X_n > 0\right) \\
&= \sum_{E\in\mathcal{E}_t}\Pr(X_1 > 0,\ldots,X_n > 0) = 0.5^n\left|\mathcal{E}_t\right| \quad \text{for } 1 \le B \le n. \qquad \text{(A.5)}
\end{aligned}$$

Compare (A.5) with (A.3), we get

$$\Pr(T^+ = t) = \Pr(S = t),\ t = 1,\ldots,M\ \ (1 \le B \le n)$$

It is obvious that

$$\Pr(T^+ = 0) = \Pr(X_1 < 0,\ldots,X_n < 0) = \Pr(S = 0)$$

This completes the proof of $T^+ \sim S$.

To help understand the notations used in the above proof, consider the case of $n=3$ and $t=1,3$ for illustration.

For $t=1$, $\mathcal{E}_t=\mathcal{E}_1=\{E=\{i(1)\}=\{1\}\}$ with $B=1$ and $|\mathcal{E}_1|=1$. Hence

$$R_i=i(1)=1 \iff J=\{j(1)\}=\{i\},\ i=j(b)=j(1)\in\{1,2,3\} \text{ for } b=1,$$

so that $R_{j(1)}=R_{j(b)}=i(b)=i(1)=1$. Then (A.5) becomes

$$\Pr(T^+=1)=\sum_{\omega\in\Omega}\Pr\left(R_{j(1)}=i(1)=1; X_u>0: u\in J=\{i\}, X_u<0: u\notin J\right)$$

$$=\sum_{i=1}^{3}\Pr\left(X_i>0, R_i=1, -X_u<0, u\neq i\right)=\sum_{i=1}^{3}\Pr\left(X_u>0, u=1,2,3; R_i=1\right)$$

$$=\Pr(X_1>0, X_2>0, X_3>0)=0.5^3=\Pr(X_1>0, X_2<0, X_3<0)$$

$$=\Pr(\psi_1=1, \psi_2=\psi_3=0)=\Pr(S=1)$$

This was shown at the start without using the notations defined in (A.1) – (A.3).

For $t = 3$, $\mathcal{E}_t = \mathcal{E}_3 = \{E_1 = \{3\}, E_2 = \{1,2\}\}$ with $|\mathcal{E}_3| = 2$. For $E_1 = \{3\}$ $(B = 1)$,

$$\Pr(T^+ = 3, E_1) = \sum_{\omega \in \Omega} \Pr\left(R_{j(1)} = i(1) = 3; X_u > 0 : u \in J, X_u < 0 : u \notin J\right) = 0.5^3$$

by similar arguments as above. For $E_2 = \{1,2\}$ $(B = 2)$, $J = \{j(1), j(2)\} = \{i, j\}$,

$$\Pr(T^+ = 3, E_2) = \sum_{\omega \in \Omega} \Pr\left(R_{j(1)} = 1, R_{j(2)} = 2; X_u > 0 : u \in J, X_u < 0 : u \notin J\right)$$

$$= \sum_{i \neq j} \Pr\left(R_i = 1, R_j = 2; X_u > 0 : u \in \{i, j\}, -X_u < 0 : u \notin \{i, j\}\right)$$

$$= \Pr(X_1 > 0, X_2 > 0, X_3 > 0) = 0.5^3$$

It follows that

$$\Pr(T^+ = 3) = \Pr(T^+ = 3, E_1) + \Pr(T^+ = 3, E_2) = 0.5^3 + 0.5^3$$

$$= \Pr(X_1 > 0, X_2 > 0, X_3 < 0) + \Pr(X_1 < 0, X_2 < 0, X_3 > 0)$$

$$= \Pr(\psi_1 = \psi_2 = 1, \psi_3 = 0) + \Pr(\psi_1 = \psi_2 = 0, \psi_3 = 1) = \Pr(S = 3)$$

Connection between T^+ and $W_{(k)}$

To see why $W_{(k)} < 0 < W_{(k+1)} \;\Leftrightarrow\; T^+ = M - k$, we first look at a simple example for $n = 5$. Suppose that the ordered data are $\left(X_{(1)},\ldots,X_{(5)}\right) = (-5,-2,1,3,8)$ with absolute ranks $\left(R_{(1)},\ldots,R_{(5)}\right) = (4,2,1,3,5)$. Then $M = 5(6)/2 = 15$ and

$\left(W_{(1)},\ldots,W_{(15)}\right) = \left(-5,\ -3.5,\ -2\ -2,\ -1,\ -0.5,\ 0.5,\ 1,\ 1.5,\ 2,\ 3,\ 3,\ 4.5,\ 5.5,\ 8\right)$

Thus $W_{(k)} = -0.5 < 0 < 0.5 = W_{(k+1)}$ for $k = 6$, $T^- = R_{(1)} + R_{(2)} = 4 + 2 = 6 = k$ and

$$T^+ = R_{(3)} + R_{(4)} + R_{(5)} = 1 + 3 + 5 = 9 = 15 - 6 = M - k$$

Note also that $X_{(m)} < 0 < X_{(m+1)}$ for $m = 2$, and there are $p = 3$ pairs (i, j) with $i \le m = 2 < j$ and $R_{(i)} > R_{(j)}$: $(i, j) = (1,3)$ with $R_{(1)} = 4 > 1 = R_{(3)}$, $(i, j) = (1,4)$ with $R_{(1)} = 4 > 3 = R_{(4)}$, and $(i, j) = (2,3)$ with $R_{(2)} = 2 > 1 = R_{(3)}$.

It is easy to check in this example that

$$k = 6 = \frac{2(3)}{2} + 3 = \frac{m(m+1)}{2} + p = T^- \quad \text{and so} \quad T^+ = M - k$$

Proof of $W_{(k)} < 0 < W_{(k+1)} \;\Leftrightarrow\; T^{+} = M - k$

Let $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$ be the ordered values of $X_1, \ldots, X_n$, $M = n(n+1)/2$, and $W_{(1)} < W_{(2)} < \cdots < W_{(M)}$ the ordered values of Walsh averages $\left(X_i + X_j\right)/2$ for $1 \le i \le j \le n$.

Let k and m satisfy

$$W_{(k)} < 0 < W_{(k+1)} \text{ and } X_{(m)} < 0 < X_{(m+1)}$$

Denote by $R_{(i)}$ the rank of $X_{(i)}$ in increasingly ordered values of $|X_1|, \ldots, |X_n|$. Then

$$X_{(1)} < \cdots < X_{(m)} < 0 \;\Rightarrow\; X_{(i)} + X_{(j)} < 0 \text{ for all } 1 \le i \le j \le m.$$

If $i \le m < j$, then

$$X_{(i)} + X_{(j)} < 0 \;\Leftrightarrow\; 0 < X_{(j)} < -X_{(i)} \;\Leftrightarrow\; \left|X_{(j)}\right| < \left|X_{(i)}\right| \;\Leftrightarrow\; R_{(i)} > R_{(j)}$$

Let $p =$ Number of pairs (i, j) with $i \le m < j, R_{(i)} > R_{(j)}$. Then

$$k = \text{No.}\left\{(i,j): \frac{X_i + X_j}{2} < 0\right\} = \text{No.}\left\{(i,j): X_{(i)} + X_{(j)} < 0\right\}$$

$$= \text{No.}\left\{(i,j): 1 \le i \le j \le m\right\} + p = \frac{m(m+1)}{2} + p$$

Next, since $X_{(1)} < \cdots < X_{(m)} < 0 < X_{(m+1)} < \cdots < X_{(n)}$, we have

$$\left|X_{(m)}\right| < \left|X_{(m-1)}\right| < \cdots < \left|X_{(1)}\right| \quad \text{and} \quad \left|X_{(m+1)}\right| < \left|X_{(m+2)}\right| < \cdots < \left|X_{(n)}\right|$$

If $\left|X_{(1)}\right| < \left|X_{(m+1)}\right|$, then $\left|X_{(i)}\right| < \left|X_{(j)}\right|$, so that $R_{(i)} < R_{(j)}$, for all $i \le m < j$, which imply $p = 0$ and $\left(R_{(m)}, R_{(m-1)}, \ldots, R_{(1)}\right) = (1, 2, \ldots, m)$. Thus

$$T^- = \sum_{i=1}^{n} R_i I_{\{X_i < 0\}} = R_{(1)} + \cdots + R_{(m)} = 1 + \cdots + m = \frac{m(m+1)}{2} + p = k$$

If $\left|X_{(1)}\right| > \left|X_{(m+1)}\right|$, then $\left|X_{(i)}\right| > \left|X_{(j)}\right|$ and $R_{(i)} > R_{(j)}$ for some $i \le m < j$.

For each $j > m$, if $|X_{(j)}| < |X_{(1)}|$, then there exists $t \le m$ such that

$$\begin{cases} |X_{(m)}| < \cdots < |X_{(t+1)}| < |X_{(j)}| < |X_{(t)}| < \cdots < |X_{(1)}| & \text{if } t < m, \\ |X_{(j)}| < |X_{(t)}| = |X_{(m)}| < \cdots < |X_{(1)}| & \text{if } t = m. \end{cases}$$

This adds 1 to each $R_{(1)},\ldots,R_{(t)}$, and hence t to $T^- = R_{(1)} + \cdots + R_{(t)} + \cdots$, compared with $|X_{(j)}| > |X_{(1)}|$. On the other hand, $t \le m$ and $|X_{(j)}| < |X_{(t)}|$ imply $l \le t \le m < j$ and $R_{(l)} \ge R_{(t)} > R_{(j)}$ for $l = 1,\ldots,t$, which add t to p. For example, if

$$\left(X_{(1)},\ldots,X_{(5)}\right) = (-5,-2,1,3,8) \quad \text{with} \quad \left(R_{(1)},\ldots,R_{(5)}\right) = (4,2,1,3,5),$$

then $|X_{(1)}| = 5 > 1 = |X_{(3)}| = |X_{(m+1)}|$ and $p = 3$.

For $j = 3 > 2 = m$, $|X_{(j)}| = |X_{(3)}| = 1 < 4 = |X_{(1)}|$ and $t = 2 = m$ satisfies

$$|X_{(j)}| = |X_{(3)}| = 1 < 2 = |X_{(t)}| = |X_{(m)}| = |X_{(2)}| < |X_{(1)}| = 5$$

This adds $t = 2$ to $T^- = R_{(1)} + R_{(2)}$ compared with $|X_{(j)}| = |X_{(3)}| > |X_{(1)}|$.

Also for $j=3$, there are two pairs $(i,j)=(1,3)$ and $(2,3)$ with $i \le m = 2 < j$ and $R_{(i)} > R_{(j)}$, adding $t=2$ to p.

Similarly for $j=4>m$, $\left|X_{(j)}\right| = \left|X_{(4)}\right| = 3 < 4 = \left|X_{(1)}\right|$ and $t=1<m$ satisfies

$$\left|X_{(m)}\right| = \left|X_{(t+1)}\right| = \left|X_{(2)}\right| = 2 < \left|X_{(j)}\right| = \left|X_{(4)}\right| = 3 < 4 = \left|X_{(t)}\right| = \left|X_{(1)}\right|$$

This adds $t=1$ to T^- compared with $\left|X_{(j)}\right| = \left|X_{(4)}\right| > \left|X_{(1)}\right|$. $j=4$ also adds 1 to p by 1 pair $(i,j)=(1,4)$ with $i \le m = 2 < j$ and $\left|X_{(i)}\right| = \left|X_{(1)}\right| = 4 > 3 = \left|X_{(4)}\right| = \left|X_{(j)}\right|$.

Thus totally the case $\left|X_{(1)}\right| > \left|X_{(m+1)}\right| = \left|X_{(3)}\right|$ adds 3 to both T^- and p.

In general, the case $\left|X_{(1)}\right| > \left|X_{(m+1)}\right|$ adds the same value to T^- and p compared to $\left|X_{(1)}\right| < \left|X_{(m+1)}\right|$. Consequently,

$$W_{(k)} < 0 < W_{(k+1)} \iff T^- = \frac{m(m+1)}{2} + p = k \iff T^+ = M - k$$

The mean of the Kruskal-Wallis statistic H

In a one-way layout with k treatments and ranks $\{r_{ij} : i = 1,\ldots,n_j; j = 1,\ldots,k\}$, if for a given $j \in \{1,\ldots,k\}$, we treat $\{r_{ij} : i = 1,\ldots,n_j\}$ as the Y-ranks and the rest of $\{r_{ij}\}$ as the X-ranks, then the R_j defined in (5.1) is the Wilcoxon rank sum statistic.

Hence by (3.8) and (3.9), assuming no ties in $\{r_{ij}\}$,

$$\mathrm{E}_0[R_j] = \frac{n_j(N+1)}{2} \quad \text{and} \quad \mathrm{Var}_0(R_j) = \frac{n_j(N-n_j)(N+1)}{12}, \quad j = 1,\ldots,k.$$

Then by (5.2), the Kruskal-Wallis statistic H has the mean under H_0:

$$\mathrm{E}_0[H] = \frac{12}{N(N+1)}\sum_{j=1}^{k}\frac{1}{n_j}\mathrm{E}_0\left[\left(R_j - \frac{n_j(N+1)}{2}\right)^2\right] = \frac{12}{N(N+1)}\sum_{j=1}^{k}\frac{1}{n_j}\mathrm{Var}_0(R_j)$$

$$= \frac{12}{N(N+1)}\sum_{j=1}^{k}\frac{(N-n_j)(N+1)}{12} = \frac{1}{N}\sum_{j=1}^{k}(N-n_j) = \frac{kN-N}{N} = k-1$$

If there are ties among $\{r_{ij}\}$, then the variance of R_j is adjusted by (3.11) to

$$\mathrm{Var}_0(R_j) = \frac{n_j(N-n_j)}{12}\left[N+1-\frac{1}{N(N-1)}\sum_{u=1}^{g} t_u(t_u-1)(t_u+1)\right]$$

$$= \frac{n_j(N-n_j)}{12}\left[N+1-\frac{N^3-N}{N(N-1)}A\right] = \frac{n_j(N-n_j)(N+1)}{12}(1-A)$$

where g is the number of groups with tied ranks in $\{r_{ij}\}$, t_u is the number of tied points in group u, $u=1,\ldots,g$, and A is defined in (5.4). It follows that

$$\mathrm{E}_0[H] = \frac{12}{N(N+1)}\sum_{j=1}^{k}\frac{1}{n_j}\mathrm{Var}_0(R_j) = \frac{1-A}{N}\sum_{j=1}^{k}(N-n_j) = (1-A)(k-1)$$

and the mean of $H' = H/(1-A)$ under H_0 is given by

$$\mathrm{E}_0[H'] = \mathrm{E}_0\left[\frac{H}{1-A}\right] = \frac{\mathrm{E}_0[H]}{1-A} = \frac{(1-A)(k-1)}{1-A} = k-1$$

This justified the adjustment in (5.4).

Mean and variance of Jonckheere-Terpstra statistic

The Jonckheere-Terpstra test statistic is defined by

$$J = \sum_{u<v} U_{uv} = \sum_{v=2}^{k} \sum_{u=1}^{v-1} U_{uv} \quad \text{with} \quad U_{uv} = \sum_{i=1}^{n_u} \sum_{j=1}^{n_v} I_{\{X_{iu} < X_{jv}\}}, \ 1 \le u < v \le k.$$

First, note that if Z_1, Z_2, Z_3 are i.i.d. continuous random variables, then

$$\Pr(Z_1 < Z_2) = \Pr(Z_2 < Z_1) = \frac{1}{2} \tag{A.6}$$

and for $(i, j, k) = (1,2,3)$, (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1),

$$\Pr(Z_i < Z_j < Z_k) = \Pr(Z_1 < Z_2 < Z_3) = \frac{1}{6} \ \Rightarrow \tag{A.7}$$

$$\Pr(Z_i, Z_j < Z_k) = \Pr(Z_i < Z_j < Z_k) + \Pr(Z_j < Z_i < Z_k) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3} \tag{A.8}$$

and

$$\Pr(Z_i < Z_j, Z_k) = \Pr(Z_i < Z_j < Z_k) + \Pr(Z_i < Z_k < Z_j) = \frac{1}{3} \tag{A.9}$$

Let $1 \le u < v \le k$ throughout the arguments below. Write $I_{ij} = I_{ij}(u,v) = I_{\{X_{iu} < X_{jv}\}}$ for simplicity. Then by (A.6), under H_0,

$$\mathrm{E}[I_{ij}] = \Pr(X_{iu} < X_{jv}) = \frac{1}{2} \text{ for all } 1 \le i \le n_u \text{ and } 1 \le j \le n_v \tag{A.10}$$

Hence

$$\mathrm{E}[U_{uv}] = \mathrm{E}\left[\sum_{i=1}^{n_u}\sum_{j=1}^{n_v} I_{ij}\right] = \sum_{i=1}^{n_u}\sum_{j=1}^{n_v} \mathrm{E}[I_{ij}] = \frac{1}{2} n_u n_v \tag{A.11}$$

Since

$$\sum_{u<v} n_u n_v = \frac{1}{2}\sum_{u \ne v} n_u n_v = \frac{1}{2}\left(\sum_{u=1}^{k} n_u \sum_{v=1}^{k} n_v - \sum_{u=1}^{k} n_u^2\right) = \frac{1}{2}\left(N^2 - \sum_{u=1}^{k} n_u^2\right), \tag{A.12}$$

(A.11) implies

$$\mathrm{E}_0[J] = \sum_{u<v} \mathrm{E}[U_{uv}] = \frac{1}{2}\sum_{u<v} n_u n_v = \frac{1}{4}\left(N^2 - \sum_{u=1}^{k} n_u^2\right)$$

This proves (5.6).

Next, by (A.8) – (A.10), $\mathrm{E}\left[I_{ij}^2\right] = \mathrm{E}[I_{ij}] = 1/2$,

$$\mathrm{E}[I_{ij} I_{pj}] = \Pr(X_{iu}, X_{pu} < X_{jv}) = \frac{1}{3}, \quad \mathrm{E}[I_{ij} I_{iq}] = \Pr(X_{iu} < X_{jv}, X_{qv}) = \frac{1}{3},$$

and $\mathrm{E}[I_{ij} I_{pq}] = \mathrm{E}[I_{ij}]\mathrm{E}[I_{pq}] = 1/4$ for $i \neq p$, $j \neq q$. Hence

$$\begin{aligned}
\mathrm{E}\left[U_{uv}^2\right] &= \mathrm{E}\left[\left(\sum_{i=1}^{n_u} \sum_{j=1}^{n_v} I_{ij}\right)^2\right] = \sum_{i=1}^{n_u} \sum_{j=1}^{n_v} \sum_{p=1}^{n_u} \sum_{q=1}^{n_v} \mathrm{E}[I_{ij} I_{pq}] \\
&= \sum_{i,j} \mathrm{E}\left[I_{ij}^2\right] + \sum_{i \neq p; j} \mathrm{E}[I_{ij} I_{pj}] + \sum_{i; j \neq q} \mathrm{E}[I_{ij} I_{iq}] + \sum_{i \neq p, j \neq q} \mathrm{E}[I_{ij} I_{pq}] \\
&= \frac{n_u n_v}{2} + \frac{n_u (n_u - 1) n_v}{3} + \frac{n_u n_v (n_v - 1)}{3} + \frac{n_u (n_u - 1) n_v (n_v - 1)}{4} \\
&= \frac{n_u n_v}{12}\left(6 + 4n_u - 4 + 4n_v - 4 + 3n_u n_v - 3n_u - 3n_v + 3\right) \\
&= \frac{n_u n_v}{12}\left(3n_u n_v + n_u + n_v + 1\right) \qquad \text{(A.13)}
\end{aligned}$$

It follows from (A.11) and (A.13) that

$$\mathrm{Var}(U_{uv}) = \mathrm{E}\left[U_{uv}^2\right] - \left(\mathrm{E}[U_{uv}]\right)^2 = \frac{n_u n_v}{12}\left(3n_u n_v + n_u + n_v + 1\right) - \frac{1}{4}(n_u n_v)^2$$

$$= n_u n_v\left(\frac{3n_u n_v + n_u + n_v + 1}{12} - \frac{n_u n_v}{4}\right) = \frac{n_u n_v (n_u + n_v + 1)}{12} \qquad \text{(A.14)}$$

Let $I'_{pq} = I'_{pq}(u,t) = I_{\{X_{pu} < X_{qt}\}}$ for $v \neq t > u$. Then under H_0, by (A.9),

$$\mathrm{Cov}(I_{ij}, I'_{iq}) = \mathrm{E}[I_{ij} I'_{iq}] - \mathrm{E}[I_{ij}]\mathrm{E}[I'_{iq}] = \Pr(X_{iu} < X_{jv}, X_{qt}) - \frac{1}{4} = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

and for $i \neq p$,

$$\mathrm{Cov}(I_{ij}, I'_{pq}) = \mathrm{E}[I_{ij} I'_{pq}] - \frac{1}{4} = \mathrm{E}[I_{ij}]\mathrm{E}[I'_{pq}] - \frac{1}{4} = \frac{1}{4} - \frac{1}{4} = 0$$

Consequently,

$$\mathrm{Cov}(U_{uv}, U_{ut}) = \sum_{i,j,p,q} \mathrm{Cov}(I_{ij}, I'_{pq}) = \sum_{i,j,q} \frac{1}{12} = \frac{n_u n_v n_t}{12}, \quad v \neq t > u. \qquad \text{(A.15)}$$

Let $I''_{pq} = I''_{pq}(s,u) = I_{\{X_{ps} < X_{qu}\}}$ for $s < u$. Then by (A.6) and (A.7), under H_0,

$$\mathrm{Cov}(I_{ij}, I''_{pi}) = \Pr(X_{ps} < X_{iu} < X_{jv}) - \frac{1}{4} = \frac{1}{6} - \frac{1}{4} = -\frac{1}{12},$$

$$\mathrm{Cov}(I_{ij}, I''_{pq}) = \mathrm{E}[I_{ij} I''_{pq}] - \frac{1}{4} = \mathrm{E}[I_{ij}]\mathrm{E}[I''_{pq}] - \frac{1}{4} = \frac{1}{4} - \frac{1}{4} = 0 \quad \text{if } i \neq q.$$

Thus

$$\mathrm{Cov}(U_{uv}, U_{su}) = \sum_{i,j,p,q} \mathrm{Cov}(I_{ij}, I''_{pq}) = \sum_{i,j,p} -\frac{1}{12} = -\frac{n_u n_v n_s}{12}, \quad s < u \qquad \text{(A.16)}$$

Similarly,

$$\mathrm{Cov}(U_{uv}, U_{vt}) = -\frac{n_u n_v n_t}{12}, \ v < t; \ \ \mathrm{Cov}(U_{uv}, U_{sv}) = \frac{n_u n_v n_s}{12}, \quad u \neq s < v. \qquad \text{(A.17)}$$

If u, v, s, t are distinct, then U_{uv} (determined by treatments u, v) is independent of U_{st} (determined by treatments s, t), hence

$$\mathrm{Cov}(U_{uv}, U_{st}) = 0 \quad \text{for all distinct } u, v, s, t. \qquad \text{(A.18)}$$

It follows from (A.14) – (A.18) that

$$\mathrm{Var}_0(J) = \sum_{u<v}\left[\mathrm{Var}(U_{uv}) + \sum_{s<t,(s,t)\neq(u,v)} \mathrm{Cov}(U_{uv},U_{st})\right]$$

$$= \sum_{u<v}\left[\mathrm{Var}(U_{uv}) + \sum_{u<t\neq v} \mathrm{Cov}(U_{uv},U_{ut}) + \sum_{t>v}\mathrm{Cov}(U_{uv},U_{vt})\right]$$

$$+\sum_{u<v}\left[\sum_{u\neq s<v} \mathrm{Cov}(U_{uv},U_{sv}) + \sum_{s<u}\mathrm{Cov}(U_{uv},U_{su})\right]$$

$$= \frac{1}{12}\sum_{u<v} n_u n_v\left[n_u + n_v + 1 + \sum_{u<t\neq v} n_t - \sum_{t>v} n_t + \sum_{u\neq s<v} n_s - \sum_{s<u} n_s\right]$$

$$= \frac{1}{12}\sum_{u<v} n_u n_v\left[n_u + n_v + 1 + \sum_{u<t<v} n_t + \sum_{u<s<v} n_s\right]$$

$$= \frac{1}{12}\sum_{u<v} n_u n_v\left[n_u + n_v + 1 + 2\sum_{t=u+1}^{v-1} n_t\right] \qquad \text{(A.19)}$$

Similar to (A.12), by the symmetry of multiplication,

$$\sum_{u<v} n_u n_v (n_u + n_v) = \frac{1}{2}\sum_{u\neq v} n_u n_v (n_u + n_v) = \frac{1}{2}\sum_{u=1}^{k}\sum_{v=1}^{k} n_u n_v (n_u + n_v) - \sum_{u=1}^{k} n_u^3$$

$$= 2\times\frac{1}{2}\sum_{u=1}^{k} n_u^2 \sum_{v=1}^{k} n_v - \sum_{u=1}^{k} n_u^3 = N\sum_{u=1}^{k} n_u^2 - \sum_{u=1}^{k} n_u^3, \tag{A.20}$$

and as there are $3! = 6$ ways to order 3 distinct values u, v, t,

$$6\sum_{u<t<v} n_u n_v n_t = \sum_{u\neq t\neq v\neq u} n_u n_v n_t = \sum_{u=1}^{k} n_u \sum_{v=1}^{k} n_v \sum_{t=1}^{k} n_t - \sum_{u=1}^{k} n_u^3 - 3\sum_{u=1}^{k} n_u^2 \sum_{v\neq u} n_v$$

$$= N^3 - \sum_{u=1}^{k} n_u^3 - 3\sum_{u=1}^{k} n_u^2 (N - n_u) = N^3 + 2\sum_{u=1}^{k} n_u^3 - 3N\sum_{u=1}^{k} n_u^2,$$

which implies

$$\sum_{u<v} n_u n_v \sum_{t=u+1}^{v-1} n_t = \sum_{u<t<v} n_u n_v n_t = \frac{1}{6}\left(N^3 + 2\sum_{u=1}^{k} n_u^3 - 3N\sum_{u=1}^{k} n_u^2 \right) \tag{A.21}$$

Substitute (A.20), (A.12) and (A.21) into (A.19), we obtain

$$\mathrm{Var}_0(J) = \frac{1}{12}\left[\sum_{u<v} n_u n_v (n_u + n_v) + \sum_{u<v} n_u n_v + 2\sum_{u<v} n_u n_v \sum_{t=u+1}^{v-1} n_t\right]$$

$$= \frac{1}{12}\left[N\sum_{u=1}^{k} n_u^2 - \sum_{u=1}^{k} n_u^3 + \frac{1}{2}\left(N^2 - \sum_{u=1}^{k} n_u^2\right) + \frac{2}{6}\left(N^3 + 2\sum_{u=1}^{k} n_u^3 - 3N\sum_{u=1}^{k} n_u^2\right)\right]$$

$$= \frac{1}{72}\left[2N^3 + 3N^2 + (-6+4)\sum_{u=1}^{k} n_u^3 + (6N - 3 - 6N)\sum_{u=1}^{k} n_u^2\right]$$

$$= \frac{1}{72}\left[N^2(2N+3) - 2\sum_{u=1}^{k} n_u^3 - 3\sum_{u=1}^{k} n_u^2\right]$$

$$= \frac{1}{72}\left[N^2(2N+3) - \sum_{u=1}^{k} n_u^2(2n_u + 3)\right]$$

This proves (5.7).

Mean and variance of Mack-Wolfe statistic

Express the Mack-Wolfe test statistic as $A_p = A_{1p} + A_{2p}$, where

$$A_{1p} = \sum_{u<v\le p} U_{uv} \quad \text{and} \quad A_{2p} = \sum_{p\le u<v} U_{vu} \tag{A.22}$$

Since $U_{vu} = n_u n_v - U_{uv}$ we can write

$$A_{2p} = M_p - \overline{A}_{2p} \ \text{ with } M_p = \sum_{p\le u<v} n_u n_v \ \text{ and } \ \overline{A}_{2p} = \sum_{p\le u<v} U_{uv} \tag{A.23}$$

Then A_{1p} and $\overline{A}_{2p}$ are the Jonckheere-Terpstra statistics for treatments $1,2,\ldots,p$ and $p,p+1,\ldots,k$, with sizes $N_1 = 1+\cdots+n_p$ and $N_2 = n_p+\cdots+n_k$, respectively. Hence by (5.6), (5.7) and (A.23),

$$\mathrm{E}_0[A_{1p}] = \frac{1}{4}\left(N_1^2 - \sum_{i=1}^{p} n_i^2\right),\ \mathrm{E}_0[\overline{A}_{2p}] = \frac{1}{4}\left(N_2^2 - \sum_{i=p}^{k} n_i^2\right), \tag{A.24}$$

$$M_p = \frac{1}{2}\sum_{p\le u\ne v\ge p} n_u n_v = \frac{1}{2}\left(N_2^2 - \sum_{i=p}^{k} n_i^2\right) = 2\mathrm{E}_0[\overline{A}_{2p}], \tag{A.25}$$

$$\operatorname{Var}_0(A_{1p}) = \frac{1}{72}\left[N_1^2(2N_1+3) - \sum_{i=1}^{p} n_i^2(2n_i+3) \right] \tag{A.26}$$

and

$$\operatorname{Var}_0(A_{2p}) = \operatorname{Var}_0\left(\overline{A}_{2p}\right) = \frac{1}{72}\left[N_2^2(2N_2+3) - \sum_{i=p}^{k} n_i^2(2n_i+3) \right] \tag{A.27}$$

It follows from (A.23) – (A.25) that

$$\begin{aligned}
\mathrm{E}_0[A_p] &= \mathrm{E}_0[A_{1p} + M_p - \overline{A}_{2p}] = \mathrm{E}_0[A_{1p}] + 2\mathrm{E}_0\left[\overline{A}_{2p}\right] - \mathrm{E}_0\left[\overline{A}_{2p}\right] \\
&= \mathrm{E}_0[A_{1p}] + \mathrm{E}_0\left[\overline{A}_{2p}\right] = \frac{1}{4}\left(N_1^2 - \sum_{i=1}^{p} n_i^2 \right) + \frac{1}{4}\left(N_2^2 - \sum_{i=p}^{k} n_i^2 \right) \\
&= \frac{1}{4}\left(N_1^2 + N_2^2 - \sum_{i=1}^{k} n_i^2 - n_p^2 \right)
\end{aligned}$$

This proves (5.9).

Next, by (A.17), (A.18) and (A.23),

$$\mathrm{Cov}\left(A_{1p},A_{2p}\right)=\mathrm{Cov}\left(A_{1p},M_p-\overline{A}_{2p}\right)=-\mathrm{Cov}\left(A_{1p},\overline{A}_{2p}\right)$$

$$=\sum_{u<v\le p\le s<t}-\mathrm{Cov}(U_{uv},U_{st})=\sum_{u<p<t}-\mathrm{Cov}(U_{up},U_{pt})$$

$$=\sum_{u<p<t}\frac{n_u n_t n_p}{12}=\frac{n_p}{12}\left(n_1+\cdots+n_{p-1}\right)\left(n_{p+1}+\cdots+n_k\right)$$

$$=\frac{n_p}{12}\left(N_1-n_p\right)\left(N_2-n_p\right)=\frac{n_p}{12}\left[N_1N_2-(N_1+N_2)n_p+n_p^2\right]$$

$$=\frac{n_p}{12}\left[N_1N_2-\left(N+n_p\right)n_p+n_p^2\right]=\frac{1}{12}\left(n_pN_1N_2-n_p^2N\right)\quad\text{(A.28)}$$

Substituting (A.26) – (A.28) into

$$\mathrm{Var}_0\left(A_p\right)=\mathrm{Var}_0\left(A_{1p}+A_{2p}\right)=\mathrm{Var}_0\left(A_{1p}\right)+\mathrm{Var}_0\left(A_{2p}\right)+2\mathrm{Cov}\left(A_{1p},A_{2p}\right)$$

Then (5.10) follows.

Mean and variance of $U_{.q}$ in (5.11)

Note that since $\Pr(X_{iu} > X_{jv}) = 1/2$, (A.11) holds for $u > v$ as well. Hence

$$E_0[U_{.q}] = \sum_{i \neq q} E[U_{iq}] = \sum_{i \neq q} \frac{n_i n_q}{2} = \frac{n_q}{2} \sum_{i \neq q} n_i = \frac{n_q (N - n_q)}{2}$$

This proves the mean in (5.11). For the variance in (5.11), by (A.14) – (A.17) and the relation $U_{iq} = n_i n_q - U_{qi}$ for $i > q$,

$$\mathrm{Var}(U_{iq}) = \mathrm{Var}(-U_{iq}) = \frac{n_i n_q (n_i + n_q + 1)}{12} \quad \text{for } 1 \leq i \neq q \leq k$$

and

$$\mathrm{Cov}(U_{iq}, U_{jq}) = \left\{ \begin{array}{ll} \mathrm{Cov}(U_{iq}, U_{jq}), & i \neq j < q \\ \mathrm{Cov}(-U_{qi}, -U_{qj}) = \mathrm{Cov}(U_{qi}, U_{qj}) & i \neq j > q \\ \mathrm{Cov}(U_{iq}, -U_{qj}) = -\mathrm{Cov}(U_{iq}, U_{qj}) & i < q < j \end{array} \right\} = \frac{n_i n_j n_p}{12}$$

for $i \neq j \neq q$ (all distinct $i, j, q \in \{1, \ldots, k\}$).

It follows that

$$\mathrm{Var}_0(U_{\cdot q}) = \mathrm{Var}_0\left(\sum_{i\neq q} U_{iq}\right) = \sum_{i\neq q} \mathrm{Var}(U_{iq}) + \sum_{i\neq j\neq q} \mathrm{Cov}(U_{iq}, U_{jq})$$

$$= \sum_{i\neq q} \frac{n_i n_q (n_i + n_q + 1)}{12} + \sum_{i\neq j\neq q} \frac{n_i n_j n_q}{12} = \frac{n_q}{12}\left[\sum_{i\neq q} n_i \left(n_q + 1\right) + \sum_{i,j\neq q} n_i n_j\right]$$

$$= \frac{n_q}{12}\left[\left(N - n_q\right)\left(n_q + 1\right) + \sum_{i\neq q} n_i \sum_{j\neq q} n_j\right]$$

$$= \frac{n_q}{12}\left[\left(N - n_q\right)\left(n_q + 1\right) + \left(N - n_q\right)\left(N - n_q\right)\right]$$

$$= \frac{n_q}{12}\left(N - n_q\right)\left(n_q + 1 + N - n_q\right) = \frac{n_q (N - n_q)(N + 1)}{12}$$

This proves the variance in (5.11).

Distribution of $A_{\hat{p}}^{*}$ with unknown p

Let $k=3$, $(n_1,n_2,n_3)=(1,2,1)$, $N=4$, $N!/(n_1!\ldots n_4!)=4!/(1\cdot 2\cdot 1)=24/2=12$.

$$\mathrm{E}_0[U_{\bullet 1}]=\frac{n_1(N-n_1)}{2}=\frac{4-1}{2}=\frac{3}{2}=1.5=\frac{n_3(N-n_3)}{2}=\mathrm{E}_0[U_{\bullet 3}]$$

$$\mathrm{Var}_0(U_{\bullet 1})=\frac{n_1(N-n_1)(N+1)}{12}=\frac{3(5)}{12}=\frac{5}{4}=1.25=\mathrm{Var}_0(U_{\bullet 3})$$

$$\mathrm{E}_0[U_{\bullet 2}]=\frac{n_2(N-n_2)}{2}=\frac{4-2}{2}=1,\quad \mathrm{Var}_0(U_{\bullet 2})=\frac{2(4-2)(4+1)}{12}=\frac{5}{3}$$

If $p=1$, then $n_p=n_1=1$, $N_1=n_1=1$, $N_2=n_1+n_2+n_3=4$. Hence

$$\mathrm{E}_0[A_1]=\frac{N_1^2+N_2^2-n_1^2-n_2^2-n_3^2-n_1^2}{4}=\frac{1+4^2-3-2^2}{4}=\frac{10}{4}=2.5$$

$$\mathrm{Var}_0(A_1)=\frac{2(1+4^3)+3(1+4^2)-3\times 5-2^2\times 7}{72}+\frac{4-4}{6}=\frac{138}{72}=\frac{23}{12}$$

Similarly, $p=3 \Rightarrow n_p = n_3 = 1$, $N_1 = n_1 + n_2 + n_3 = 4$, $N_2 = n_3 = 1 \Rightarrow$

$$\mathrm{E}_0[A_3] = 2.5 \quad \text{and} \quad \mathrm{Var}_0(A_3) = \frac{23}{12}$$

If $p=2$, then $n_p = n_2 = 2$, $N_1 = n_1 + n_2 = 1+2=3$, $N_2 = n_2 + n_3 = 2+1=3$.

Hence

$$\mathrm{E}_0[A_2] = \frac{N_1^2 + N_2^2 - n_1^2 - n_2^2 - n_3^2 - n_p^2}{4} = \frac{2\times 3^2 - 2 - 2\times 2^2}{4} = \frac{8}{4} = 2$$

$$\mathrm{Var}_0(A_2) = \frac{2\times 2\times 3^3 + 3\times 2\times 3^2 - 2\times 5 - 2\times 2^2 \times 7}{72} + \frac{2\times 3^2 - 2^2 \times 4}{6}$$

$$= \frac{108+54-10-56}{72} + \frac{18-16}{6} = \frac{96+24}{72} = \frac{120}{72} = \frac{5}{3}$$

Consider the following cases of ranks for treatments I, II, III.

Case 1.

I	II	III
1	2	4
	3	

$$U_{\bullet 1} = U_{21} + U_{31} = 0 + 0 = 0 \;\Rightarrow\; U_{\bullet 1}^{*} = (0 - 1.5)\big/\sqrt{1.25} = -1.342$$

$$U_{\bullet 2} = U_{12} + U_{32} = 2 + 0 = 2 \;\Rightarrow\; U_{\bullet 2}^{*} = (2 - 1)\big/\sqrt{5/3} = 0.775$$

$$U_{\bullet 3} = U_{13} + U_{23} = 1 + 2 = 3 \;\Rightarrow\; U_{\bullet 3}^{*} = (3 - 1.5)\big/\sqrt{1.25} = 1.342$$

Thus $U_{\bullet 3}^{*} > U_{\bullet 2}^{*} > U_{\bullet 1}^{*} \;\Rightarrow\; \hat{p} = 3 \;\Rightarrow\; A_3 = U_{12} + U_{13} + U_{23} = 2 + 1 + 2 = 5 \;\Rightarrow$

$$A_{\hat{p}}^{*} = A_3^{*} = \frac{A_3 - \mathrm{E}_0[A_3]}{\sqrt{\mathrm{Var}_0(A_3)}} = \frac{5 - 2.5}{\sqrt{23/12}} = 1.806$$

Similarly, we can calculate $A_{\hat{p}}^{*}$ for other cases in Comment 36 on page 246:

Case 2.

I	II	III
4	2	1
	3	

$\hat{p} = 1,\; A_1 = U_{21} + U_{31} + U_{32} = 2 + 1 + 2 = 5$

$$A_{\hat{p}}^{*} = A_1^{*} = \frac{A_1 - \mathrm{E}_0[A_1]}{\sqrt{\mathrm{Var}_0(A_1)}} = \frac{5 - 2.5}{\sqrt{23/12}} = 1.806$$

Case 3.

I	II	III
1	2	3
	4	

$$\hat{p}=2,\ A_2=U_{12}+U_{21}=2+1=3$$

$$A_{\hat{p}}^{*}=A_2^{*}=\frac{A_2-\mathrm{E}_0[A_2]}{\sqrt{\mathrm{Var}_0(A_2)}}=\frac{3-2}{\sqrt{5/3}}=0.775$$

Case 5.

I	II	III
3	1	4
	2	

$$\hat{p}=3,\ A_3=U_{12}+U_{13}+U_{23}=0+1+2=3$$

$$A_{\hat{p}}^{*}=A_3^{*}=\frac{A_3-\mathrm{E}_0[A_3]}{\sqrt{\mathrm{Var}_0(A_3)}}=\frac{3-2.5}{\sqrt{23/12}}=0.361$$

Note that $A_{\hat{p}}^{*}$ depends on the value of $\hat{p}$ for unknown peak p, whereas A_p^{*} must use the same p in all cases if the peak is assumed known. Hence $A_{\hat{p}}^{*}$ and A_p^{*} have different distributions even if $\hat{p}=p$. For example, if $\hat{p}=2$ is estimated from the data, then $A_{\hat{p}}^{*}=A_2^{*}$ numerically, but the distributions of $A_{\hat{p}}^{*}$ and A_2^{*} are different.

Arbitrary incomplete block and BIBD

In BIBD, $\lambda_{qt} = \lambda$ is the same for all $t \neq q \in \{1,\ldots,k\}$. Hence by (6.21),

$$\sigma_{qt} = -\lambda \ \text{ for } 1 \le t \neq q \le k-1 \ \text{ and } \ \sigma_{qq} = (k-1)\lambda,\ q = 1,\ldots,k-1. \tag{A.29}$$

and so by (6.22),

$$\Sigma_0 = (\sigma_{qt})_{(k-1)\times(k-1)} = \lambda\begin{bmatrix} k-1 & -1 & \cdots & -1 \\ -1 & k-1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & k-1 \end{bmatrix} \tag{A.30}$$

Let $\mathbf{1} = [1 \;\; 1 \; \cdots \; 1]^{\mathsf{T}}$ be a $(k-1)\times 1$ column vector of 1's. Then

$$\mathbf{1}\cdot\mathbf{1}^{\mathsf{T}} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}\begin{bmatrix} 1 & \cdots & 1 \end{bmatrix} = (1)_{(k-1)\times(k-1)}, \quad \mathbf{1}^{\mathsf{T}}\mathbf{1} = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = k-1 \tag{A.31}$$

It follows from (6.22) and (A.30) – (A.31) that

$$\Sigma_0 = \lambda\left(kI_{k-1} - \mathbf{1}\cdot\mathbf{1}^\mathsf{T}\right) \quad \text{and} \quad \mathbf{1}\cdot\mathbf{1}^\mathsf{T}\mathbf{1}\cdot\mathbf{1}^\mathsf{T} = (k-1)\mathbf{1}\cdot\mathbf{1}^\mathsf{T},$$

where I_{k-1} is the $(k-1)\times(k-1)$ identity matrix. Consequently,

$$\left(I_{k-1} + \mathbf{1}\cdot\mathbf{1}^\mathsf{T}\right)\left(kI_{k-1} - \mathbf{1}\cdot\mathbf{1}^\mathsf{T}\right) = kI_{k-1} + (k-1)\mathbf{1}\cdot\mathbf{1}^\mathsf{T} - \mathbf{1}\cdot\mathbf{1}^\mathsf{T}\mathbf{1}\cdot\mathbf{1}^\mathsf{T} = kI_{k-1} \;\Rightarrow$$

$$I_{k-1} + \mathbf{1}\cdot\mathbf{1}^\mathsf{T} = k\left(kI_{k-1} - \mathbf{1}\cdot\mathbf{1}^\mathsf{T}\right)^{-1} \;\Rightarrow$$

$$\lambda k\Sigma_0^{-1} = k\left(kI_{k-1} - \mathbf{1}\cdot\mathbf{1}^\mathsf{T}\right)^{-1} = I_{k-1} + \mathbf{1}\cdot\mathbf{1}^\mathsf{T} \tag{A.32}$$

Since $\mathbf{A}^\mathsf{T}\mathbf{A} = A_1^2 + \cdots + A_{k-1}^2$ and $\mathbf{A}^\mathsf{T}\mathbf{1} = \mathbf{1}^\mathsf{T}\mathbf{A} = A_1 + \cdots + A_{k-1} = -A_k$, it follows from (6.23) and (A.32) that

$$\lambda k(SM) = \lambda k\mathbf{A}^\mathsf{T}\Sigma_0^{-1}\mathbf{A} = \mathbf{A}^\mathsf{T}\left(\lambda k\Sigma_0^{-1}\right)\mathbf{A} = \mathbf{A}^\mathsf{T}\left(I_{k-1} + \mathbf{1}\cdot\mathbf{1}^\mathsf{T}\right)\mathbf{A}$$

$$= \mathbf{A}^\mathsf{T}\mathbf{A} + (\mathbf{A}^\mathsf{T}\mathbf{1})^2 = A_1^2 + \cdots + A_{k-1}^2 + (-A_k)^2 = \sum_{j=1}^{k} A_j^2 \tag{A.33}$$

In BIBD, $s_i = s$ for all $i = 1,\ldots,n$ and with $r_{ij} = (s+1)/2$ for $c_{ij} = 0$,

$$R_j + (n-p)\frac{s+1}{2} = \sum_{i:c_{ij}=1}^{n} r_{ij} + (n-p)\frac{s+1}{2} = \sum_{i=1}^{n} r_{ij}, \quad j = 1,\ldots,k. \tag{A.34}$$

By (6.19) and (A.34), if the data satisfy BIBD, then for $j = 1,\ldots,k$,

$$A_j = \sum_{i=1}^{n} \sqrt{\frac{12}{s+1}}\left(r_{ij} - \frac{s+1}{2}\right) = \sqrt{\frac{12}{s+1}}\left(\sum_{i=1}^{n} r_{ij} - \frac{n(s+1)}{2}\right)$$

$$= \sqrt{\frac{12}{s+1}}\left(R_j + (n-p)\frac{s+1}{2} - \frac{n(s+1)}{2}\right) = \sqrt{\frac{12}{s+1}}\left(R_j - \frac{p(s+1)}{2}\right) \tag{A.35}$$

Thus (A.33) and (A.35) imply

$$SM = \frac{1}{\lambda k}\sum_{j=1}^{k} A_j^2 = \frac{12}{\lambda k(s+1)}\sum_{j=1}^{k}\left(R_j - \frac{p(s+1)}{2}\right)^2 = D \text{ in (6.16)}$$

Least square estimate of equal slope

For linear regression lines with an equal slope β:

$$Y_{ij} = \alpha_i + \beta x_{ij} + e_{ij}, \quad j = 1,\ldots,n_i, \quad i = 1,\ldots,k. \tag{A.36}$$

Denote by S the sum of squared errors:

$$S = \sum_{i=1}^{k}\sum_{j=1}^{n_i} e_{ij}^2 = \sum_{i=1}^{k}\sum_{j=1}^{n_i}(Y_{ij} - \alpha_i - \beta x_{ij})^2$$

Then the least square estimates of α_i and β are the solutions to minimize S, or to satisfy the equations:

$$\frac{\partial S}{\partial \alpha_i} = 2\sum_{j=1}^{n_i}(Y_{ij} - \alpha_i - \beta x_{ij})(-1) = 0, \quad i = 1,\ldots,k, \tag{A.37}$$

and

$$\frac{\partial S}{\partial \beta} = 2\sum_{i=1}^{k}\sum_{j=1}^{n_i}(Y_{ij} - \alpha_i - \beta x_{ij})(-x_{ij}) = 0 \tag{A.38}$$

From (A.37) we obtain

$$\sum_{j=1}^{n_i}(Y_{ij}-\alpha_i-\beta x_{ij})=\sum_{j=1}^{n_i}Y_{ij}-n_i\alpha_i-\beta\sum_{j=1}^{n_i}x_{ij}=0$$

As a result,

$$\alpha_i=\frac{1}{n_i}\sum_{j=1}^{n_i}Y_{ij}-\frac{1}{n_i}\beta\sum_{j=1}^{n_i}x_{ij}=\overline{Y}_i-\beta\overline{x}_i,\ \ i=1,\ldots,k. \tag{A.39}$$

Substitution of $\alpha_i=\overline{Y}_i-\beta\overline{x}_i$ from (A.39) into (A.38) then leads to

$$\sum_{i=1}^{k}\sum_{j=1}^{n_i}(Y_{ij}-\overline{Y}_i-\beta x_{ij}+\beta\overline{x}_i)x_{ij}=\sum_{i=1}^{k}\sum_{j=1}^{n_i}(Y_{ij}-\overline{Y}_i)x_{ij}-\beta\sum_{i=1}^{k}\sum_{j=1}^{n_i}(x_{ij}-\overline{x}_i)x_{ij}=0$$

It follows that

$$\beta=\frac{\sum_{i=1}^{k}\sum_{j=1}^{n_i}(Y_{ij}-\overline{Y}_i)x_{ij}}{\sum_{i=1}^{k}\sum_{j=1}^{n_i}(x_{ij}-\overline{x}_i)x_{ij}} \tag{A.40}$$

Since

$$\sum_{j=1}^{n_i}(x_{ij}-\overline{x}_i)=0 \quad \text{and} \quad \sum_{j=1}^{n_i}(Y_{ij}-\overline{Y}_i)=0, \quad i=1,\ldots,k,$$

we have

$$\sum_{i=1}^{k}\sum_{j=1}^{n_i}(x_{ij}-\overline{x}_i)x_{ij}=\sum_{i=1}^{k}\sum_{j=1}^{n_i}(x_{ij}-\overline{x}_i)x_{ij}-\sum_{i=1}^{k}\overline{x}_i\sum_{j=1}^{n_i}(x_{ij}-\overline{x}_i)=\sum_{i=1}^{k}\sum_{j=1}^{n_i}(x_{ij}-\overline{x}_i)^2$$

and similarly,

$$\sum_{i=1}^{k}\sum_{j=1}^{n_i}(Y_{ij}-\overline{Y}_i)x_{ij}=\sum_{i=1}^{k}\sum_{j=1}^{n_i}(Y_{ij}-\overline{Y}_i)(x_{ij}-\overline{x}_i)=\sum_{i=1}^{k}\sum_{j=1}^{n_i}(x_{ij}-\overline{x}_i)Y_{ij}$$

Then (A.40) shows that the least square estimate of β in (A.36) is

$$\overline{\beta}=\frac{\sum_{i=1}^{k}\sum_{j=1}^{n_i}(x_{ij}-\overline{x}_i)Y_{ij}}{\sum_{i=1}^{k}\sum_{j=1}^{n_i}(x_{ij}-\overline{x}_i)^2}$$