# MAT2006: Elementary Real Analysis
# Assignment #3

Reference Solution

**1.** Let $A$ be nonempty and bounded above so that $s = \sup A$ exists.
(i) Show that $s \in \overline{A}$.
(ii) Can an open set contain its supremum?

*Proof.* (i) Assume $s = \sup A$. Then, for any $\epsilon > 0$, there exists $a \in A$ such that $a > s - \epsilon$, thus $V_\epsilon(s) \cap A \neq \emptyset$. If $s \in A$, then $s \in \overline{A}$. If $s \notin A$, the above property says that $s$ is a limit point of $A$, and hence again $s \in \overline{A}$.

(ii) No. Assume $s = \sup A$ and $A$ is open. Suppose $s \in A$, then there exists a neighbourhood $V_\epsilon(s)$ of $s$ contained entirely in $A$, this means that $s + \epsilon/2 \in A$ which is a contraction with $s$ being an upper bound of $A$. □

**2.** (i) Prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
(ii) Does this result about closures extend to infinite unions of sets?

*Proof.* (i) Let $L_A$, $L_B$ and $L_{A\cup B}$ denote the sets of limits points of $A$, $B$ and $A\cup B$ respectively. We claim that $L_A \cup L_B = L_{A\cup B}$.

Firstly, if $x \in L_A \cup L_B$, we may assume $x \in L_A$ and the case $x \in L_B$ is similar. Then, for any $\epsilon > 0$, $V_\epsilon^0(x) \cap A \neq \emptyset$, and hence $V_\epsilon^0(x) \cap (A \cup B) = (V_\epsilon^0(x) \cap A) \cup (V_\epsilon^0(x) \cap B) \neq \emptyset$. Thus $x \in L_{A\cup B}$.

Secondly, if $x \in L_{A\cup B}$, we must have $x \in L_A \cup L_B$. Suppose this is not true. There exists $\epsilon_1 > 0$ and $\epsilon_2 > 0$ such that $V_{\epsilon_1}^0(x) \cap A = \emptyset$ and $V_{\epsilon_2}^0(x) \cap B = \emptyset$. Choose $\epsilon = \min\{\epsilon_1, \epsilon_2\} > 0$. Then $V_\epsilon^0 \cap (A \cup B) = (V_\epsilon \cap A) \cup (V_\epsilon \cap B) \subset (V_{\epsilon_1} \cap A) \cup (V_{\epsilon_2} \cap B) = \emptyset$, which is a contraception with $x \in L_{A\cup B}$. Thus $x \in L_A \cup L_B$.

Now, we have $L_{A\cup B} = L_A \cup L_B$ and thus

$$\overline{A \cup B} = (A \cup B) \cup L_{A\cup B} = (A \cup B) \cup (L_A \cup L_B) = (A \cup L_A) \cup (B \cup L_B) = \overline{A} \cup \overline{B}.$$

(ii) No. Let $A_n = [\frac{1}{n}, 1 - \frac{1}{n}]$. Then

$$\overline{\bigcup_{n=1}^{\infty} A_n} = \overline{(0,1)} = [0,1] \qquad \text{but} \qquad \bigcup_{n=1}^{\infty} \overline{A_n} = \bigcup_{n=1}^{\infty} \left[\frac{1}{n}, 1 - \frac{1}{n}\right] = (0,1).$$ □

**3.** Let $A$ be an uncountable set and let $B$ be the set of real numbers that divides $A$ into two uncountable sets; that is, $s \in B$ if both $\{x \,|\, x \in A \quad \text{and} \quad x < s\}$ and $\{x \,|\, x \in A \quad \text{and} \quad x > s\}$ are uncountable. Show $B$ is nonempty and open.

*Proof.* For each $x \in \mathbb{R}$, define two sets

$$C_s = (-\infty, s) \cap A, \quad D_s = (s, \infty) \cap A.$$

Note that

$$C_s \cup D_s = A \setminus \{s\}.$$

For each $s$, one of $C_s$ and $D_s$ must be uncountable. For otherwise, if both $C_s$ and $D_s$ are at most countable, so is $A$.

Suppose, for a contradiction, that $B = \emptyset$. Define

$$E = \{s \in \mathbb{R} \,|\, C_s \text{ is at most countable}\}.$$

Then $E$ is nonempty. For otherwise, $-n \notin E$ for all $n \in \mathbb{N}$, that is all $C_{-n}$ are uncountable, but noting that $-n \notin B = \emptyset$, $D_{-n}$ must be at most countable. Thus,

$$A = \bigcup_{n=1}^{\infty} D_{-n}$$

is at most countable, which is a contradiction. Thus $E$ is nonempty.

Similarly, $E$ is bounded above. Suppose not, $n \in E$ for all $n \in \mathbb{N}$, which implies that each $C_n$ is at most countable. Then

$$A = \bigcup_{n=1}^{\infty} C_n$$

is at most countable, a contradiction. Therefore, $E$ is a nonempty, bounded above set. By the Least Upper Bound Property, there exists $s \in \mathbb{R}$ such that $x = \sup E$. Note that if $a \in E$ and $b < a$, then $b \in E$. Thus, for any $y > x$, it follows that $y \notin E$, that is $C_y$ is uncountable, by noting $y \notin B = \emptyset$ that, $D_y$ must be at most countable. Thus

$$A \setminus \{x\} = C_x \cup D_x = \left( \bigcup_{n=1}^{\infty} C_{x-(1/n)} \right) \bigcup \left( \bigcup_{n=1}^{\infty} D_{x+(1/n)} \right)$$

is at most countable, since each $C_{x-(1/n)}$ and $D_{x+(1/n)}$ is countable. This is a contradiction with that $A$ is uncountable. Therefore, $B$ is nonempty.

Now, we shall show that $B$ is open. For any $s \in B$, then both $C_s$ and $D_s$ are uncountable. Noting that

$$C_s = \bigcup_{n=1}^{\infty} C_{s-(1/n)},$$

there exists $n_1 \in \mathbb{N}$ such that $C_{s-(1/n_1)}$ is uncountable. For otherwise, $C_s$ is at most countable, contradicts with the fact that $s \in B$. Now $t \in B$ for any $t \in [s - 1/(n_1), s]$. First $C_t$ is uncountable for such a $t$, since

$$C_t \supset C_{s-(1/n_1)} \qquad \forall s - \frac{1}{n_1} \le t \le s,$$

and the latter is uncountable. Second, $D_t$ is also uncountable for such a $t$, since

$$D_t \supset D_s \qquad \forall s - \frac{1}{n_1} \le t \le s$$

and the latter is uncountable. That is $[s - (1/n_1), s] \subset B$.

Similarly, there exists $n_2 \in \mathbb{N}$ such that $[s, s + (1/n_2)] \subset B$. Take

$$\epsilon = \min\{1/n_1, 1/n_2\}.$$

Then $V_\epsilon(s) \in B$. Therefore, $B$ is open. □

*Method II, NIP, sketch.* Assume, for contradiction, that $B = \emptyset$. Define $A_n = A \cap [-n, n]$, then

$$A = \bigcup_{n=1}^{\infty} A_n,$$

and use this to show that there exists $M \in \mathbb{N}$ such that $A_M$ is uncountable. (Explain this point.)

Then set $I_1 = [-M, M]$. Bisect this interval. Among of the two halves $[-M, 0]$ and $[0, M]$, one of them intersects $A$ with an uncountable set and the other intersects $A$ with an at-most-countable set. (Explain why.) Let choose the one intersecting $A$ with an uncountable set and denote it as $I_2$.

In general, if we have choose $I_n = [a_n, b_n]$, which intersects $A$ with an uncountable set, then we can choose one half of $I_n$, denoting by $I_{n+1} = [a_{n+1}, b_{n+1}]$, intersects $A$ with an uncountable set while the other half intersects $A$ with an at-most-countable set.

Then there exists $x \in \mathbb{R}$ such that

$$\{x\} = \bigcap_{n=1}^{\infty} I_n$$

(Explain why.) Show that

$$\bigcup_{n=1}^{\infty} (A_M \cap I_n^c) = A_M \cap \left( \bigcup_{n=1}^{\infty} I_n^c \right) = A_M \cap \left( \bigcap_{n=1}^{\infty} I_n \right)^c = A_M \setminus \{x\},$$

and the left-hand side is at most countable, and thus we arrive at a contradiction. (Explain this.) Therefore, $B \neq \emptyset$. □

**4.** Prove that the only sets that are both open and closed are $\mathbb{R}$ and the empty set $\emptyset$.

*Proof.* It is known that $\mathbb{R}$ and $\emptyset$ are both open and closed. Suppose $A$ is both open and closed and $A \neq \emptyset$, $A \neq \mathbb{R}$. Then $A^c$ is both open and closed and it is not $\emptyset$ neither $\mathbb{R}$. Choose $x \in A$ and $y \in A^c$. We may assume $x < y$ and the proof for the case $x > y$ is similar. Let $B := A \cap (-\infty, y)$. Then $x \in B$ thus $B$ is nonempty, and $y$ is an upper bounded of $B$. Hence, by the Least Upper Bound Property, there exists $s = \sup B$ and it is clear that $s \le y$. Note that $s \in \overline{A} = A$ according to Problem 1 and the fact $A$ is closed, thus $s < y$. Now $(s, y] \subset A^c$, and thus $s \in \overline{A^c} = A^c$ since $A^c$ is also closed. But $s \in A$ and $s \in A^c$ is a contradiction, and thus the only sets that are both open and closed are $\mathbb{R}$ and the empty set $\emptyset$. □

**5.** A dual notion to the closure of a set is the *interior* of a set. The interior of $E$ is denoted $E^\circ$ and is defined as

$$E^\circ = \{x \in E \,|\, \text{there exists } V_\epsilon(x) \subset E\}.$$

Results about closures and interiors possess a useful symmetry.

(i) Show that $E$ is closed if and only if $\overline{E} = E$. Show that $E$ is open if and only if $E^\circ = E$.

(ii) Show that $(\overline{E})^c = (E^c)^\circ$ and $(E^\circ)^c = \overline{E^c}$.

*Proof.* (i) ($\Rightarrow$) If $E$ is closed, then $L_E \subset E$ and thus $\overline{E} = E \cup L_E \subset E$. Note that $E \subset (E \cup L_E) = \overline{E}$, we must have $\overline{E} = E$.

($\Leftarrow$) If $\overline{E} = (E \cup L_E) = E$, we have $L_E \subset E$ and thus $E$ is closed.

($\Rightarrow$) If $E$ is open. Then for any $x \in E$, there exists a neighborhood of $x$ contained entirely in $E$, thus $x \in E^\circ$ and hence $E \subset E^\circ$. By its definition, we have $E^\circ \subset E$. Hence $E^\circ = E$.

($\Leftarrow$) If $E^\circ = E$. Then any $x \in E = E^\circ$, there exists a neighborhood of $x$ contained entirely in $E$, thus $E$ is open.

(ii) Let $x \in (\overline{E})^c$. Then $x \notin \overline{E} = E \cup L_E$, which implies that $x \in E^c$ and there exists a neighborhood of $x$ such that $V_\epsilon(x) \cap E = \emptyset$. Thus $V_\epsilon(x) \subset E^c$, and we have $x \in (E^c)^\circ$. We also see that the above argument can be reversed, that is $x \in (E^c)^\circ$ also implies $x \in (\overline{E})^c$. Thus $(\overline{E})^c = (E^c)^\circ$.

For the second identity, set $A = E^c$ and applying the first identity to $A$, we have

$$(E^\circ)^c = ((A^c)^\circ)^c = ((\overline{A})^c)^c = \overline{A} = \overline{E^c}.$$

□

**6.** Show that if a set $K \subset \mathbb{R}$ is closed and bounded, then it is sequentially compact.

*Proof.* Assume $K$ is closed and bounded, and let $\{x_n\}$ be a sequence in $K$ and thus a bounded sequence. By the Bolzano–Weierstrass Theorem, there exists a subsequence $\{x_{n_k}\}$ converges to a real number $x \in \mathbb{R}$. Thus $x$ is a limit point of $K$ and then $x \in K$ since $K$ is closed. Therefore, $K$ is sequentially compact. □

**7.** Show that if $K$ is sequentially compact and nonempty, then $\sup K$ and $\inf K$ both exist and are elements of $K$.

*Proof.* Since $K$ is sequentially compact, it is closed and bounded. Then the AoC implies $\sup K$ exists since $K$ is not empty and bound above. Let $s = \sup K$. Then $s \in \overline{K} = K$ according to Problem 1 (i). In a similar manner, $\inf K$ exists and belongs to $K$. □

**8** (NIP+AP implies HB)**.** Provide a proof of a bounded and closed set is compact using the Nested Interval Property.

Suppose $K \subset \mathbb{R}$ is closed and bounded, and let $\{O_\lambda \,|\, \lambda \in \Lambda\}$ be an open cover for $K$. For contradiction, let's assume that no finite subcover exists. Let $I_0$ be a closed interval containing $K$.

(a) Show that there exists a nested sequence of closed intervals $I_0 \supset I_1 \supset I_2 \supset \cdots$ with the property that, for each $n$, $I_n \cap K$ cannot be finitely covered and $\lim_{n\to\infty} |I_n| = 0$.

(b) Argue that there exists an $x \in K$ such that $x \in I_n$ for all $n$.

(c) Because $x \in K$, there must exist an open set $O_{\lambda_0}$ from the original collection that contains $x$ as an element. Explain how this leads to the desired contradiction.

*Proof.* (a) Since $K$ is bounded, there exists $M > 0$ such that $|x| \leq M$ for all $x \in K$. Take $I_0 = [-M, M]$. Now bisect $I_0$ into two half closed intervals $H_l = [-M, 0]$ and $H_r = [0, M]$. Since we assumed there is no finite subcover of $\{O_\lambda \,|\, \lambda \in \Lambda\}$ that covers $K$, then there must be one of the two halves, say $H_r$, such that $K \cap H_r$ cannot be finitely covered, for otherwise, $K = (K \cap H_l) \cup (K \cap H_r)$ can be finitely covered. Then set $I_1 = H_r$. We may define a sequence of nested closed intervals

$$I_0 \supset I_1 \supset I_2 \supset \cdots ,$$

with each of $I_n \cap K$ cannot be finitely covered. It is constructed inductively by the following. Suppose $I_n$ is a closed interval such that $I_n \cap K$ cannot be finitely covered, by splitting it into two halves with each being a closed interval, we can find one half such that its intersection with $K$ cannot be finitely covered, denote that half by $I_{n+1}$. By the NIP, there exists $x \in \mathbb{R}$ such that

$$x \in \bigcap_{n=0}^{\infty} I_n .$$

Clearly,

$$|I_n| = \frac{2M}{2^n} \to 0 \qquad \text{as } n \to \infty.$$

(Here we made use of the AP).

(b) We claim that $x \in K$. Suppose, for a contradiction, $x \notin K$. Since $K$ is closed and thus $K^c$ is open, $x$ is in the open set $K^c$. There exists $\epsilon_0 > 0$ such that $V_{\epsilon_0}(x) \subset K^c$. A similar manner as in part (i) shows the existence of $I_N$ such that $I_N \subset V_{\epsilon_0}(x)$ and thus $I_N \cap K = \emptyset$, which is a contradiction with $I_N \cap K$ cannot be finitely covered.

(c) Since $x \in K$ and $\{O_\lambda \,|\, \lambda \in \Lambda\}$ is an open cover of $K$, there exists $\lambda_0$ such that $x \in O_{\lambda_0}$. By the openness, there exists an $\epsilon_0 > 0$ such that $V_{\epsilon_0}(x) \subset O_{\lambda_0}$. Since $|I_n| \to 0$, there exists $N \in \mathbb{N}$ such that $|I_N| < \epsilon_0$. Now, $I_N \cap K$ can be finitely covered, indeed $O_{\lambda_0}$, as a single open set covers $I_N \cap K$, we arrive at a contradiction.

Thus, any open cover of $K$ must have a finite subcover, that is $K$ is compact. □

**9** (LUBP implies HB)**.** Consider the special case where $K$ is a closed interval. Let $\{O_\lambda \,|\, \lambda \in \Lambda\}$ be an open cover for $[a, b]$ and define $S$ to be the set of all $x \in [a, b]$ such that $[a, x]$ has a finite subcover from $\{O_\lambda \,|\, \lambda \in \Lambda\}$.

(a) Argue that $S$ is nonempty and bounded, and thus $s = \sup S$ exists.

(b) Now show $s = b$, which implies $[a, b]$ has a finite subcover.

(c) Finally, prove the theorem for an arbitrary closed and bounded set $K$.

*Proof.* (a) Note that $a \in [a, b] \subset \bigcup_{\lambda \in \Lambda} O_\lambda$. There exists $\lambda_a \in \Lambda$ such that $a \in O_{\lambda_a}$, and thus a single open set $O_{\lambda_a}$ suffices to cover $[a, a] = \{a\}$, which means $a \in S$, and thus $S$ is not empty. $S$ is bounded above since $x \leq b$ for all $x \in S$. By the LUBP, $s = \sup S$ exists.

(b) Note that $b$ is an upper bound of $S$, thus $s \leq b$. Suppose, for a contradiction, $s \neq b$, that is $s < b$. Since $s \in [a, b]$ which is covered by $\{O_\lambda \,|\, \lambda \in \Lambda\}$, there exists $\lambda_s \in \Lambda$, such that $s \in O_{\lambda_s}$, an open set. Thus, there exists, $\varepsilon > 0$ such that $s \in V_\varepsilon(s) \subset O_{\lambda_s}$. Since $s - \varepsilon/2 \in S$, thus $[a, s - \varepsilon/2]$ can be finitely covered, with one additional open set $O_{\lambda_s}$, we see that $[a, s + \varepsilon/2]$ can also be finitely covered. That is, $s + \varepsilon/2 \in S$, which is a contradiction with $s$ being the least upper bound of $S$. Therefore, $s = b$.

With a similar argument as above, there exists a neighborhood $V_\epsilon(b) \subset O_{\lambda_b}$ for some $\lambda_b \in \Lambda$. now, since $b - \epsilon/2 \in S$ indicates that $[a, b - \epsilon/2]$ can be finitely covered. With an additional open set $O_{\lambda_b}$, we see that $[a, b]$ can be finitely covered. By the definition, $[a, b]$ is compact.

(c) Since $K$ is bounded, there exists $M > 0$ such that $K \subset [-M, M] := I$. Since $K$ is closed, thus its complement $K^c$ is open. Assume $\{O_\lambda\}_{\lambda\in\Lambda}$ is an open cover of $K$, then adding an open set $K^c$ to this collection will yield an open cover for $\mathbb{R}$, thus an open cover for $[-M, M]$. By parts (a)-(b), there exists a subcover of $[-M, M]$, and thus a subcover of $K$. By deleting the possible open set $K^c$ in this subcover, it is readily seen that it is a subcollection of $\{O_\lambda\}_{\lambda\in\Lambda}$ that covers $K$, hence $K$ is compact by definition. $\square$

**10** (HB implies BW)**.** Using the concept of open covers (and explicitly avoiding the Bolzano–Weierstrass Theorem), prove that every bounded infinite set has a limit point. Therefore, every bounded sequence has a convergent subsequence (BW).

*Proof.* For contradiction, we assume $K$ is a bounded infinite set which has no limit point. Then $K$ is closed. Thus HB implies that $K$ is compact and thus any open cover of $K$ has a finite subcover. For any $x \in K$, since $x$ is not a limit point of $K$, there exists a neighborhood of $x$ such that $V_{\epsilon_x}(x) \cap K = \{x\}$. Note that $\{V_{\epsilon_x}(x)\}_{x\in K}$ is an open cover of $K$, thus there is a finite subcover of $K$, that is there exists $N \in \mathbb{N}$ such that

$$K \subset \bigcup_{n=1}^{N} V_{\epsilon_n}(x_n), \qquad \epsilon_n := \epsilon_{x_n}.$$

But this implies that $K = \{x_1, x_2, \ldots, x_N\}$ which is a contradiction with the fact that $K$ is infinite. Thus every bounded infinite set has a limit point.

Let $\{x_n\}$ be a bounded sequence. If as a set, $\{x_n\}$ is finite, then there exist a value $x$ and a subsequence $\{x_{n_k}\}$ such that $x_{n_k} = x$ for all $k \in \mathbb{N}$. If as a set, $\{x_n\}$ is infinite, there is a limit point $x$ of this set. We may choose a subsequence $\{x_{n_k}\}$ by starting with $n_1 = 1$, and in general, after choosing $n_k$, let

$$n_{k+1} = \min\{n > n_k \,:\, |x_n - x| < 1/(k+1)\}.$$

Then $\{x_{n_k}\}$ is a subsequence that converges to $x$. $\square$

**11.** Show that

(a) The countable union of $F_\sigma$ sets is an $F_\sigma$ set.
(b) The finite intersection of $F_\sigma$ sets is an $F_\sigma$ set.
(c) Give an example of the countable intersection of $F_\sigma$ sets is not $F_\sigma$.
(d) The finite union of $G_\delta$ sets is a $G_\delta$ set.
(e) The countable intersection of $G_\delta$ sets is a $G_\delta$ set.

*Proof.* (a) Assume $\{F_n\}_{n=1}^{\infty}$ is a countable collection of $F_\sigma$ sets, thus $F_n = \cup_{m=1}^{\infty} F_{nm}$ for each $n \in \mathbb{N}$, where $F_{nm}$ is a closed set for each $n, m \in \mathbb{N}$. Since the set $\mathbb{N} \times \mathbb{N}$ is countable, there is a one-to-one correspondence $g$ from $k \in \mathbb{N}$ to $(n, m) \in \mathbb{N} \times \mathbb{N}$ thus if we denote $E_k = F_{g(k)} := F_{nm}$, we have

$$\bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} \left( \bigcup_{m=1}^{\infty} F_{nm} \right) = \bigcup_{n,m=1}^{\infty} F_{nm} = \bigcup_{k=1}^{\infty} E_{g(k)}.$$

Thus The countable union of $F_\sigma$ sets is an $F_\sigma$ set.

(b) It suffices to show that the intersection of two $F_\sigma$ sets is still $F_\sigma$ and then apply an induction argument.

Assume $F_1$ and $F_2$ are two $F_\sigma$ sets, with

$$F_1 = \bigcup_{n=1}^{\infty} F_{1n}, \qquad F_2 = \bigcup_{n=1}^{\infty} F_{2n},$$

where all $F_{in}$ ($i = 1, 2$, $n \in \mathbb{N}$) are closed. Now, we claim

$$F_1 \cap F_2 = \left( \bigcup_{n=1}^{\infty} F_{1n} \right) \bigcap \left( \bigcup_{m=1}^{\infty} F_{2m} \right) = \bigcup_{n=1}^{\infty} \left( \bigcup_{m=1}^{\infty} (F_{1n} \cap F_{2m}) \right).$$

To see this, we first show that

$$A \cap \left( \bigcup_{n=1}^{\infty} B_n \right) = \bigcup_{n=1}^{\infty} (A \cap B_n).$$

For

$$\begin{aligned} x \in A \cap \left( \bigcup_{n=1}^{\infty} B_n \right) \quad &\iff x \in A \text{ and } x \in \bigcup_{n=1}^{\infty} B_n \\ &\iff x \in A \text{ and } x \in B_{n_0} \quad \text{for some } n_0 \in \mathbb{N} \\ &\iff x \in A \cap B_{n_0} \quad \text{for some } n_0 \in \mathbb{N} \\ &\iff x \in \bigcup_{n=1}^{\infty} (A \cap B_n). \end{aligned}$$

Therefore, regarding first $(\bigcup_{m=1}^{\infty} F_{2m})$ as $A$ and $F_{1n}$ as $B_n$ in the previous step, we have

$$F_1 \cap F_2 = \left( \bigcup_{n=1}^{\infty} F_{1n} \right) \bigcap \left( \bigcup_{m=1}^{\infty} F_{2m} \right) = \bigcup_{n=1}^{\infty} \left( F_{1n} \bigcap \left( \bigcup_{m=1}^{\infty} F_{2m} \right) \right)$$

which, by applying the previous step again, yields

$$F_1 \cap F_2 = \bigcup_{n=1}^{\infty} \left( \bigcup_{m=1}^{\infty} (F_{1n} \cap F_{2m}) \right).$$

This is a $F_\sigma$ set by part (a), since it is a countable union of $F_\sigma$ sets – $\bigcup_{m=1}^{\infty}(F_{1n} \cap F_{2m})$. An induction argument immediately shows that any finite intersection of $F_\sigma$ sets is still $F_\sigma$.

(c) Note that the set of irrational numbers $\mathbb{I}$ is not $F_\sigma$. We may write is as a countable intersection of $F_\sigma$ sets as follows. Write $\mathbb{Q} = \{r_1, r_2, r_3, \dots\}$ and define

$$F_n = \mathbb{R} \setminus \{r_n\} = (-\infty, r_n) \cup (r_n, \infty), \qquad \forall n \in \mathbb{N}.$$

Each $F_n$ is $F_\sigma$ since it is a union of two $F_\sigma$ sets – two open intervals. Then, we have

$$\bigcap_{n=1}^{\infty} F_n = \mathbb{I}$$

is not $F_\sigma$.

(d) Assume $\{G_k\}_{k=1}^{n}$ is a finite collection of $G_\delta$ sets. Recall that a set is $F_\sigma$ if and only if its complement is $G_\delta$, and vice versa. Then the complement $G_k^c$ is $F_\sigma$ for each $k$. By the de Morgan law, we have

$$\left(\bigcup_{k=1}^{n} G_n\right)^c = \bigcup_{k=1}^{n} G_n^c,$$

which is $F_\sigma$ according to part (b). Thus the complement, $\bigcup_{k=1}^{n} G_n$ is $G_\delta$.

(e) A similar manner as what we did in part (d), where we apply part (a) instead of part (b). $\square$

**12.** (i) For each of the following sets, determine whether it is an $F_\sigma$ and/or $G_\delta$ set, explain why.

$(a)\quad (a,b); \qquad (b)\quad [a,b]; \qquad (c)\quad (a,b]; \qquad (d)\quad \mathbb{Q}; \qquad (e)\quad \mathbb{I};$

(ii) [bonus question] We know that any open set is $G_\delta$ (why?).

(g) Show that any open set can be written as the union of at most countable intervals.

(h) Show that any open set is $F_\sigma$, and any closed set is $G_\delta$.

*Solution.* (i).

(a) The interval $(a,b)$ is both $F_\sigma$ and $G_\delta$, since

$$(a,b) = \bigcup_{n=1}^{\infty} \left[a - \frac{1}{n}, b + \frac{1}{n}\right]; \qquad (a,b) = \bigcap_{n=1}^{\infty} I_n, \quad I_n = (a,b).$$

Also note that the empty set is a closed set.

(b) The interval $[a,b]$ is both $F_\sigma$ and $G_\delta$, since

$$[a,b] = \bigcup_{n=1}^{\infty} I_n, \quad I_n = [a,b]; \qquad [a,b] = \bigcap_{n=1}^{\infty} \left(a - \frac{1}{n}, b + \frac{1}{n}\right).$$

(c) The interval $(a,b]$ is both $F_\sigma$ and $G_\delta$, since

$$(a,b] = \bigcup_{n=1}^{\infty} \left[a - \frac{1}{n}, b\right]; \qquad (a,b] = \bigcap_{n=1}^{\infty} \left(a, b + \frac{1}{n}\right).$$

(d) $\mathbb{Q}$ is $F_\sigma$, since $\mathbb{Q} = \{r_1, r_2, r_3, \dots\} = \bigcup_{n=1}^\infty \{r_n\}$ and each finite set is closed.

$\mathbb{Q}$ is not $G_\delta$, since $\mathbb{I}$ is not $F_\sigma$. For otherwise, if $\mathbb{I}$ is $F_\sigma$, then $\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$ could be written as a countable union of closed sets, each of which doesnot contain any open interval, thus each of those closed sets is nowhere dense. This is a contradiction with the Baire category theorem.

(e) $\mathbb{I}$ is $G_\delta$ but not $F_\sigma$, which follows from $\mathbb{I} = \mathbb{Q}^c$ and part (d).

The set $\bigcup_{n=1}^\infty \{\mathbb{Q} + \sqrt{n}\}$ is not $G_\delta$, since its complement is not $F_\sigma$, which can be shown by a similar argument for $\mathbb{I}$ is not $F_\sigma$.

(ii) Let $A$ be an open set, then $A = \bigcap_{n=1}^\infty G_n$, with $G_n = A$ for each $n \in \mathbb{N}$. Thus any open set is $G_\delta$.

(g) Let $A$ be a nonempty open set. We may define a binary relation over $A$ by, for any $x, y \in A$

$$x \sim y \quad \text{if} \quad (x, y) \subset A.$$

It is readily seen that, for any $x, y, z \in A$,

$$(1) \quad x \sim x; \qquad (2) \quad \text{if } x \sim y \text{ then } y \sim x; \qquad (3) \quad \text{if } x \sim y \text{ and } y \sim z \text{ then } x \sim z.$$

Thus the defined relation is an equivalent relation, and we define the equivalent class of $x \in A$ as

$$[x] = \{y \in A \,|\, x \sim y\}.$$

Note that $[x]$ is connected thus it is an interval. Moreover, it is open – for any $y \in [x] \subset A$, there exists $V_\epsilon(y) \subset A$, and thus $V_\epsilon(y) \subset [x]$. Moreover, since each nonempty open interval must contain a rational number, we may assume $x$ is rational.

Thus the set $A$ can be written as a union of disjoint open intervals of the form $[x]$, and such equivalent classes are at most countable, since $\mathbb{Q}$ is countable.

(h) Recall that each open interval is $F_\sigma$, and by (g) and Problem 11(a), any open set is also $F_\sigma$. Thus any closed set, as the complement of an open set, is $G_\delta$. □

**13** (Infinite Limits). *Definition*: $\lim_{x \to c} f(x) = \infty$ means that for all $M > 0$ we can find a $\delta > 0$ such that whenever $0 < |x - c| < \delta$, it follows that $f(x) > M$.

(i) Show $\lim_{x \to 0} \frac{1}{x^2} = \infty$ in the sense described in the previous definition.

(ii) Now, construct a definition for the statement $\lim_{x \to \infty} f(x) = L$. Show $\lim_{x \to \infty} 1/x = 0$.

(iii) What would a rigorous definition for $\lim_{x \to \infty} f(x) = \infty$ look like? Give an example of such a limit.

*Proof.* (i) Given any $M > 0$, choose $\delta = \frac{1}{\sqrt{M}}$. Then whenever $|x| < \delta$,

$$f(x) = \frac{1}{x^2} > M,$$

as desired.

(ii) If for any $\epsilon > 0$, there exists an $M > 0$ such that

$$|f(x) - L| < \epsilon \qquad \forall x > M,$$

then we say

$$\lim_{x\to\infty} f(x) = L.$$

Now consider the function $f(x) = 1/x$. Given any $\epsilon > 0$, choose $M = \frac{1}{\epsilon}$. Then, we have

$$|f(x) - 0| = \left|\frac{1}{x}\right| < \epsilon \qquad \forall x > M.$$

Thus, by definition, $\lim_{x\to\infty} 1/x = 0$.

(iii) If for any $M > 0$ there exists $N > 0$ such that

$$f(x) > M \qquad \forall x > N,$$

then we say

$$\lim_{x\to\infty} f(x) = \infty. \qquad \square$$

**14** (Right and Left Limits)**.** Introductory calculus courses typically refer to the right-hand limit of a function as the limit obtained by "letting $x$ approach $c$ from the right-hand side."

(i) Give a proper $\epsilon$–$\delta$ definition for the right-hand and left-hand limit statements:

$$\lim_{x\to c^+} f(x) = L, \qquad \lim_{x\to c^-} f(x) = M.$$

(ii) Prove that $\lim_{x\to c} f(x) = L$ if and only if both the right and left-hand limits equal $L$.

*Proof.* (i) If, given any $\epsilon > 0$, there exists a $\delta_1 > 0$ such that

$$|f(x) - L| < \epsilon \qquad \forall c < x < c + \delta_1,$$

then we say

$$\lim_{x\to c^+} f(x) = L.$$

If, given any $\epsilon > 0$, there exists a $\delta_2 > 0$ such that

$$|f(x) - M| < \epsilon \qquad \forall c - \delta_2 < x < c,$$

then we say

$$\lim_{x\to c^-} f(x) = M.$$

(ii) ($\Rightarrow$) Assume $\lim_{x\to c} f(x) = L$. Then for any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - L| < \epsilon, \qquad \forall x \in (c - \delta, c) \cup (c, c + \delta).$$

Thus, by the $\epsilon$-$\delta$ definition of the right-hand and left-hand limits, we have

$$\lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x) = L.$$

($\Leftarrow$) Assume

$$\lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x) = L.$$

Now, by the definitions, for any given $\epsilon > 0$, there exists a $\delta_1 > 0$ and a $\delta_2 > 0$ such that

$$|f(x) - L| < \epsilon \qquad \forall c < x < c + \delta_1,$$
$$|f(x) - L| < \epsilon \qquad \forall c - \delta_2 < x < c.$$

Now let $\delta = \min\{\delta_1, \delta_2\}$. We have

$$|f(x) - L| < \epsilon, \qquad \forall 0 < |x - c| < \delta,$$

that is

$$\lim_{x\to c} f(x) = L. \qquad \square$$

**15** (Upper and Lower Limits)**.** As in the case of sequential limits, we have the upper and lower limits for a function,

$$\limsup_{x\to c} f(x) := \lim_{\delta\to 0^+} \sup_{0<|x-c|<\delta} f(x),$$

$$\liminf_{x\to c} f(x) := \lim_{\delta\to 0^+} \inf_{0<|x-c|<\delta} f(x).$$

Show that $\lim_{x\to c} f(x)$ exists if and only if both $\limsup_{x\to c} f(x)$ and $\liminf_{x\to c} f(x)$ exist and they are equal to each other.

*Proof.* ($\Rightarrow$) Assume

$$\lim_{x\to c} f(x) = L.$$

By definition, for any $\epsilon > 0$, there exists a $\delta_\epsilon := \delta(\epsilon) > 0$ such that

$$L - \epsilon < f(x) < L + \epsilon \qquad \forall 0 < |x - c| < \delta_\epsilon,$$

Thus,

$$L - \epsilon \le \inf_{0<|x-c|<\delta_\epsilon} f(x) \le \sup_{0<|x-c|<\delta_\epsilon} f(x) \le L + \epsilon$$

If we denote

$$g(\delta) := \inf_{0<|x-c|<\delta} f(x) \quad \text{and} \quad h(\delta) := \sup_{0<|x-c|<\delta} f(x), \qquad \text{for } \delta > 0.$$

It is readily seen that $g(\delta)$ is a decreasing function and $h(\delta)$ is an increasing function, and that for any $0 < \delta_1 < \delta_2$

$$g(\delta_2) \le g(\delta_1) \le h(\delta_1) \le h(\delta_2).$$

Therefore, whenever $0 < \delta < \delta_\epsilon$

$$L - \epsilon \le g(\delta_\epsilon) \le g(\delta) \le h(\delta) \le h(\delta_\epsilon) \le L + \epsilon,$$

By the definition of upper and lower limits,

$$\liminf_{x\to c} f(x) := \lim_{\delta\to 0^+} g(\delta) = L$$

and

$$\limsup_{x\to c} f(x) := \lim_{\delta\to 0^+} h(\delta) = L.$$

($\Leftarrow$) Now assume

$$\liminf_{x\to c} f(x) := \lim_{\delta\to 0^+} g(\delta) = L$$

and

$$\limsup_{x\to c} f(x) := \lim_{\delta\to 0^+} h(\delta) = L,$$

where $g(\delta)$ and $h(\delta)$ are defined as previously. Now, given any $\epsilon > 0$, there exists a $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$L - \epsilon < g(\delta) < L + \epsilon \qquad \forall 0 < \delta < \delta_1$$

and

$$L - \epsilon < h(\delta) < L + \epsilon \qquad \forall 0 < \delta < \delta_2.$$

Note that,

$$g(\delta) := \inf_{0<|x-c|<\delta} f(x) \le f(x) \le \sup_{0<|x-c|<\delta} f(x) := h(\delta) \qquad \forall 0 < |x-c| < \delta.$$

Choose $\delta_0 = \min\{\delta_1, \delta_2\}$. Then, it follows by the monotonicity of $g$ and $h$ that

$$L - \epsilon < g(\delta_0) \le f(x) \le h(\delta_0) < L + \epsilon \qquad \forall 0 < |x-c| < \delta_0,$$

that is

$$\lim_{x\to c} f(x) = L. \qquad \square$$

**16** (Cauchy Criterion)**.** Let $f : A \to \mathbb{R}$ be a function and $c$ a limit point of $A$. Show that $\lim_{x\to c} f(x)$ exists if and only if for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - f(y)| < \epsilon \qquad \forall 0 < |x-c| < \delta, \quad \forall 0 < |y-c| < \delta.$$

*Proof.* ($\Rightarrow$) Assume $\lim_{x\to c} f(x)$ exists and equals to $L$. For any $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - L| < \frac{\epsilon}{2}, \qquad \forall 0 < |x-c| < \delta.$$

Thus, whenever $0 < |x-c| < \delta$ and $0 < |y-c| < \delta$. we have

$$|f(x) - f(y)| = |(f(x) - L) - (f(y) - L)| \le |f(x) - L| + |f(y) - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

($\Leftarrow$) Assume that, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - f(y)| < \frac{\epsilon}{2} \qquad \forall 0 < |x-c| < \delta, \quad \forall 0 < |y-c| < \delta.$$

Assume $\{x_n\} \subset A$, $x_n \neq c$ and $\{x_n\} \to c$. Then there exists $N \in \mathbb{N}$ such that

$$|x_n - c| < \delta, \qquad \forall n \ge N.$$

Thus
$$|f(x_n) - f(x_m)| < \frac{\epsilon}{2}, \qquad \forall n > m \geq N.$$
Thus $\{f(x_n)\}$ is a Cauchy sequence and hence a convergent sequence, say $\{f(x_n)\} \to L$.

Assume $\{y_n\} \subset A$, $y_n \neq c$ and $\{y_n\} \to c$. We must also have $f(y_n)$ converges to some value, say $\{f(x_n)\} \to M$. Note that $\{z_n\} = \{x_1, y_1, x_2, y_2, x_3, y_3, \dots\}$ is also a sequence in $A$, not equal to $c$ and converging to $c$. Thus $f(z_n)$ converges for the same reason, and as two subsequences $\{f(x_n)\}$ and $\{f(x_n)\}$ must converge to the same value, that is $L = M$. To summarise, any sequence $\{x_n\}$ in $A$, not equal to $c$ and converging to $c$, we have $\{f(x_n)\}$ converges to the same value. According to the sequentially criterion of limit of functions, $\lim\limits_{x\to c} f(x)$ exists. □

**17.** Assume $h : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and let $K = \{x \,|\, h(x) = 0\}$. Show that $K$ is a closed set.

*Proof.* Let $x$ be a limit point of $K$, that is there exists a sequence $\{x_n\} \subset K$ and $x_n \neq x$ such that $\{x_n\} \to x$. From the definition of $K$, we have $h(x_n) = 0$. It then follows from the continuity of $h$ that $h(x) = \lim_{n\to\infty} h(x_n) = 0$ and thus $x \in K$. Therefore $K$ is closed. □

**18.** Observe that if $a$ and $b$ are real numbers, then
$$\max\{a, b\} = \frac{(a + b) + |a - b|}{2}.$$

(i) Show that if $f_1, f_2, \dots, f_n$ are continuous functions, then
$$g(x) = \max\{f_1(x), f_2(x), \dots, f_n(x)\}$$
is a continuous function.

(ii) Let's explore whether the result in (i) extends to the infinite case. For each $n \in \mathbb{N}$, define $f_n$ on $\mathbb{R}$ by
$$f_n(x) = \begin{cases} 1 & \text{if } |x| > 1/n \\ n|x| & \text{if } |x| \leq 1/n. \end{cases}$$
Now explicitly compute $h(x) = \sup\{f_1(x), f_2(x), f_3(x), \dots\}$.

*Proof.* (i) Note that $\max\{f_1(x), f_2(x)\} = \dfrac{(f_1(x) + f_2(x)) + |f_1(x) - f_2(x)|}{2}$ is a continuous function of $x$ provided that $f_1$ and $f_2$ are continuous.

Assume $h(x) = \max\{f_1(x), f_2(x), \dots, f_{n-1}(x)\}$ is continuous. Then
$$g(x) = \max\{f_1(x), f_2(x), \dots, f_n(x)\} = \max\{h(x), f_n(x)\}$$
is also continuous whenever $f_n(x)$ is also continuous. By induction, $g(x)$ is continuous for any $n \in \mathbb{N}$.

(ii)
$$h(x) = \sup\{f_1(x), f_2(x), f_3(x), \dots\} = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$
Hence, $h(x)$ is not a continuous function. □

**19.** Let $F \subset \mathbb{R}$ be a nonempty closed set and define $g(x) = \inf\{|x - a| : a \in F\}$. Show that $g$ is continuous on all of $\mathbb{R}$ and $g(x) \neq 0$ for all $x \notin F$.

*Proof.* Let $x \in \mathbb{R}$ be fixed and $\epsilon > 0$ arbitrary. Now let $y \in V_{\epsilon/2}(x)$ also be arbitrary but fixed. By the definition of $g(x)$ as an infimum, there exists $a_1 \in F$ such that

$$|x - a_1| > g(x) + \frac{\epsilon}{2},$$

we also have

$$g(y) \leq |y - a|, \qquad \forall a \in F.$$

Then, combining the above two inequalities with the triangle inequality, we have

$$g(y) - g(x) < |y - a_1| - \left(|x - a_1| - \frac{\epsilon}{2}\right) \leq |(y - a_1) - (x - a_1)| + \frac{\epsilon}{2} = |y - x| + \frac{\epsilon}{2} < \epsilon$$

Similarly, there also exists $a_2 \in F$ such that

$$|y - a_2| > g(y) - \frac{\epsilon}{2},$$

$$g(x) \leq |x - a| \qquad \forall a \in F,$$

and that

$$g(x) - g(y) < |x - a_2| - \left(|y - a_2| - \frac{\epsilon}{2}\right) \leq |(x - a_2) - (y - a_2)| + \frac{\epsilon}{2} = |x - y| + \frac{\epsilon}{2} < \epsilon.$$

Therefore, if we choose $\delta = \frac{\epsilon}{2}$, then

$$|g(x) - g(y)| < \epsilon \qquad \forall |x - y| < \delta,$$

which means $g(x)$ is continuous on $\mathbb{R}$. (Notice that, $g$ is indeed uniformly continuous on $\mathbb{R}$, since the choice of $\delta$ is independent of $x$)

Suppose $g(x) = 0$ for some $x \in \mathbb{R}$. Then by the definition of $g(x)$, there exists a sequence $\{a_n\} \subset F$ such that

$$0 \leq |x - a_n| < \frac{1}{n},$$

which implies that $\{a_n\}$ converges to $x$. Then two cases: (i) if $x = a_n$ for some $n \in \mathbb{N}$, $x \in F$; or (ii) if $a_n \neq x$ for each $n$, it follows that $x$ is a limit point of $F$ and thus in $F$ since $F$ is closed. $\square$

**20.** Recall the theorem "A function that is continuous on a compact set $K$ is uniformly continuous on $K$." Provide a proof by the definition "$K \subset \mathbb{R}$ is compact if every open cover of $K$ has a finite subcover."

*Proof.* Given $\epsilon > 0$. For any $x \in K$, $f$ is continuous at $x$ implies that there exists $\delta_x < 0$ such that

$$|f(x) - f(y)| < \frac{\epsilon}{2}, \qquad \forall y \in V_{\delta_x}(x) \cap K$$

Notice that $\{V_{\delta_x/2}(x)\}_{x\in K}$ form an open cover of $K$. Since $K$ is compact, we have a finite subcover $\{V_{\delta_n/2}(x_n)\}_{n=1}^N$ covers $K$, where we have denoted $\delta_n := \delta_{x_n}$.

Now set

$$\delta = \min_{1\le n\le N} \left\{ \frac{\delta_n}{2} \right\}.$$

Note that $V_{\delta_n}(x_n) \supset V_{\delta_n/2}(x_n)$, thus $\{V_{\delta_n}(x_n)\}_{n=1}^N$ also covers $K$. For each $x \in K$, there exists $1 \le m \le N$ such that $x \in V_{\delta_m/2}(x_m)$. If $|x-y| < \delta$, then

$$|y - x_m| = |(y-x) + (x - x_m)| \le |y - x| + |x - x_m| < \delta + \frac{\delta_m}{2} \le \frac{\delta_m}{2} + \frac{\delta_m}{2} = \delta_m,$$

hence, $y \in V_{\delta_m}(x_m)$. Therefore, whenever $|x-y| < \delta$, we have

$$|f(x)-f(y)| = |(f(x)-f(x_m))-(f(y)-f(x_m))| \le |f(x)-f(x_m)|+|f(y)-f(x_m)| < \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon,$$

as desired. □

**21.** (i) Assume that $g$ is defined on an open interval $(a, c)$ and it is known to be uniformly continuous on $(a, b]$ and $[b, c)$, where $a < b < c$. Prove that $g$ is uniformly continuous on $(a, c)$.

(ii) Prove that $f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$.

(iii) Show that $f(x) = x^p$ with $p \in \mathbb{R}$ is uniformly continuous on $(0, \infty)$ if and only if $0 \le p \le 1$.

(iv) Assume $f(x)$ is a continuous function defined on $[0, \infty)$, and assume that $\lim_{x\to\infty} f(x) = L \in \mathbb{R}$. Show that $f(x)$ is uniformly continuous on $[0, \infty)$.

*Proof.* (i) Given $\epsilon > 0$. By $f(x)$ is uniformly continuous on $(a, b]$ and $[b, c)$, there exists $\delta_1, \delta_2 > 0$ such that

$$|f(x) - f(y)| < \frac{\epsilon}{2}, \qquad \forall |x-y| < \delta_1 \,\text{and}\, x, y \in (a, b],$$

$$|f(x) - f(y)| < \frac{\epsilon}{2}, \qquad \forall |x-y| < \delta_2 \,\text{and}\, x, y \in [b, c),$$

Now, take $\delta = \min\{\delta_1, \delta_2\}$. For $x, y \in (a, c)$, there are three cases, (i) both in $(a, b]$; (ii) both in $[b, c)$; or (iii) each interval contains one of $x, y$. For cases (i) and (ii), we have

$$|f(x) - f(y)| < \frac{\epsilon}{2} < \epsilon \qquad \forall |x-y| < \delta.$$

For case (iii), when $|x-y| < \delta$, we also have $|x-b| < \delta$ and $|y-b| < \delta$, thus

$$|f(x) - f(y)| = |f(x) - f(b) + f(b) - f(y)| \le |f(x) - f(b)| + |f(y) - f(b)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Summarizing all three cases yields that $f(x)$ is uniformly continuous on $(a, b)$.

(ii) Let $f(x) = \sqrt{x}$. When $x, y \geq 1$, we have

$$|f(x) - f(y)| = |\sqrt{x} - \sqrt{y}| = \frac{|x-y|}{\sqrt{x}+\sqrt{y}} \leq \frac{|x-y|}{2}.$$

For any $\epsilon > 0$, choose $\delta = 2\epsilon$ yields that

$$|f(x) - f(y)| < \epsilon \qquad \forall |x-y| < \delta,$$

which means $f(x)$ is uniformly continuous on $[1, \infty)$.

Since $[0, 1]$ is bounded and closed, thus compact. And $f(x)$ is continuous on the compact set $[0, 1]$ implies that it is uniformly continuous there.

By part (i), $f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$.

(iii) When $0 \leq p \leq 1$, the proof of $x^p$ is uniformly continuous on $[0, \infty)$ is similar to part (ii), with an application of the Mean Value Theorem

$$|f(x) - f(y)| = |x^p - y^p| = p\xi^{p-1}|x-y|,$$

where $\xi \in (x, y)$ for $x < y$. Note that $\xi^{p-1} \leq 1$ when $1 \leq x < y$. Thus

$$|f(x) - f(y)| = |x^p - y^p| = p\xi^{p-1}|x-y| \leq p|x-y|$$

The uniform continuity of $f(x) = x^p$ with $0 \leq p \leq 1$ now follows by the $\epsilon$-$\delta$ definition with choosing $\delta = \epsilon/p$ for $0 < p \leq 1$ and $\delta = 1$ for $p = 0$. Now $f(x) = x^p$ is continuous on $[0, 1]$ and hence uniformly continuous there. Thus, by part (i), $f(x) = x^p$ is uniformly continuous on $[0, \infty)$.

When $p > 1$, choose two sequences $x_n = n$ and $y_n = n + \frac{1}{n^{p-1}}$ for all $n \geq 1$. Now,

$$|x_n - y_n| = \frac{1}{n^{p-1}} \to 0 \qquad \text{as } n \to \infty,$$

but, by the Mean Value Theorem,

$$|f(x_n) - f(y_n)| = |x_n^p - y_n^p| = p\xi_n^{p-1}|x_n - y_n| \geq pn^{p-1}\frac{1}{n^{p-1}} = p$$

Thus $f(x) = x^p$ is not uniformly continuous on $(0, \infty)$ when $p > 1$.

When $p < 0$, choose two sequences $x_n = \frac{1}{n}$ and $y_n = \frac{1}{2n}$. Now,

$$|x_n - y_n| = \frac{1}{2n} \to 0 \qquad \text{as } n \to \infty.$$

But, by the Mean Value Theorem,

$$|f(x_n) - f(y_n)| = |x_n^p - y_n^p| = |p|\xi_n^{p-1}|x_n - y_n| \geq |p|\frac{1}{n^{p-1}}\frac{1}{2n} = \frac{|p|}{2n^p} \geq \frac{|p|}{2}.$$

Thus $f(x) = x^p$ is not uniformly continuous on $(0, \infty)$ when $p < 0$. $\square$

**22.** Give an example of each of the following, or provide a short argument for why the request is impossible.

(a) A continuous function defined on $[0, 1]$ with range $(0, 1)$.
(b) A continuous function defined on $(0, 1)$ with range $[0, 1]$.
(c) A continuous function defined on $(0, 1]$ with range $(0, 1)$.

*Solution.* (a) Not possible, since a continuous function preserves compactness.

(b) $f(x) = \dfrac{1 + \sin(4\pi x)}{2}$.

(c) $f(x) = \frac{1}{2}\left(1 + (1 - x)\sin\frac{1}{x}\right)$. □

**23** (Continuous Extension Theorem)**.** (i) Show that a uniformly continuous function preserves Cauchy sequences; that is, if $f : A \to \mathbb{R}$ is uniformly continuous and $\{x_n\} \subset A$ is a Cauchy sequence, then show $f(x_n)$ is a Cauchy sequence.

(ii) Let $g$ be a continuous function on the open interval $(a, b)$. Prove that $g$ is uniformly continuous on $(a, b)$ if and only if it is possible to define values $g(a)$ and $g(b)$ at the endpoints so that the extended function $g$ is continuous on $[a, b]$. (In the forward direction, first produce candidates for $g(a)$ and $g(b)$, and then show the extended $g$ is continuous.)

*Proof.* (i) Given any $\epsilon > 0$, by the uniform continuity of $f$ ion $A$ there exists a $\delta > 0$ such that

$$|f(x) - f(y)| < \epsilon \qquad \forall |x - y| < \delta \quad \text{and} \quad x, y \in A.$$

Since $\{x_n\}$ is a Cauchy sequence, there exists $N \in \mathbb{N}$ such that

$$|x_n - x_m| < \delta \qquad \forall n > m \geq N.$$

Therefore,

$$|f(x_n) - f(x_m)| < \epsilon \qquad \forall n > m \geq N,$$

which says that $\{f(x_n)\}$ is also a Cauchy sequence.

(ii) ($\Rightarrow$) Assume $g$ is uniformly continuous on $(a, b)$. We shall show that $\lim_{x\to a} g(x)$ exists and we shall define $g(a)$ to be this limit value. For any sequence $\{x_n\}$ converges to $a$, by the Cauchy Criterion, it is a Cauchy sequence. By part (i), $\{g(x_n)\}$ is also a Cauchy sequence, thus converges, say

$$\lim_{n\to\infty} g(x_n) = L.$$

If $\{y_n\}$ is also a sequence that converges to $a$. Then the sequence $\{z_n\}$ defined by

$$z_{2n-1} = x_n \qquad z_{2n} = y_n,$$

also converges to $a$ and thus the previous argument shows that $\{g(z_n)\}$ converges. Since the subsequence $\{g(z_{2n-1})\}$ converges to $L$, any subsequence will converge to the same value $L$, that is to say $g(y_n) = g(z_{2n})$ also converges to $L$. Therefore, for any sequence $\{y_n\}$ converges to $a$, we have $g(x_n) \to L$. By the sequential criterion of functional limit, we have

$$\lim_{x\to a} g(x) = L.$$

Similarly, the limit of $g$ as $x \to b$ also exists,

$$\lim_{x \to b} g(x) := R.$$

Now define

$$g(a) = L \qquad g(b) = R.$$

We then have a continuous function $g$ on $[a, b]$.

($\Leftarrow$) Assume $g(x)$ can be extended as a continuous function on $[a, b]$. Since $[a, b]$ is closed and bounded, thus compact, and therefore $g$ is uniformly continuous on $[a, b]$, which implies that $g$ is uniformly continuous on the subset $(a, b)$. □

**24.** Show that the following functions is not uniform continuous on (0,1).

$$(a) \quad f(x) = \sin \frac{1}{x}; \qquad (b) \quad g(x) = \ln x; \qquad (c) \quad h(x) = \frac{1}{1-x}.$$

*Proof.* (a) Let $x_n = \frac{1}{2n\pi}$ and $y_n = \frac{1}{2n\pi + \frac{\pi}{2}}$. We then have

$$|x_n - y_n| \to 0 \qquad \text{but} \qquad |f(x_n) - f(y_n)| = 1.$$

(b) Let $x_n = \frac{1}{n}$ and $y_n = \frac{1}{2n}$. We then have

$$|x_n - y_n| \to 0 \qquad \text{but} \qquad |g(x_n) - g(y_n)| = \ln 2.$$

(c) $x_n = 1 - \frac{1}{n}$ and $y_n = 1 - \frac{1}{n+1}$. We then have

$$|x_n - y_n| \to 0 \qquad \text{but} \qquad |h(x_n) - h(y_n)| = 1.$$ □

**25.** Let $f : [0, 1] \to \mathbb{R}$ be continuous with $f(0) = f(1)$.

(i) Show that there must exist $x, y \in [0, 1]$ satisfying $|x - y| = 1/2$ and $f(x) = f(y)$.

(ii) Show that for each $n \in \mathbb{N}$ there exist $x_n, y_n \in [0, 1]$ with $|x_n - y_n| = 1/n$ and $f(x_n) = f(y_n)$.

(iii) If $h \in (0, 1/2)$ is not of the form $1/n$, there does not necessarily exist $|x - y| = h$ satisfying $f(x) = f(y)$. Provide an example that illustrates this using $h = 2/5$.

**26.** Let $f$ be a continuous function on the closed interval $[0, 1]$ with range also contained in $[0, 1]$. Prove that $f$ must have a fixed point; that is, show $f(x) = x$ for at least one value of $x \in [0, 1]$.

*Proof.* Let $g(x) = f(x) - x$, then $g$ is continuous on $[0, 1]$. Now,

$$g(0) = f(0) - 0 \geq 0, \qquad g(1) = f(1) - 1 \leq 0.$$

If one of $g(0)$ and $g(1)$ is equal to 0 we are done. If not, we have $g(0) > 0$ and $g(1) < 0$, by the intermediate value theorem, there exists $c \in (0, 1)$ such taht $g(c) = 0$, that is $f(c) = c$. □

**27** (Inverse functions)**.** If a function $f : A \to \mathbb{R}$ is one-to-one, then we can define the inverse function $f^{-1}$ on the range of $f$ in the natural way: $f^{-1}(y) = x$ where $y = f(x)$. Show that if $f$ is continuous on an bounded interval $[a, b]$ and one-to-one, then $f^{-1}$ is also continuous.

*Proof.* Suppose, for a contradiction, that $f^{-1}$ is not continuous on $B := f(A) = f([0, 1])$. There exists a sequence $\{y_n\} \subset B$ such that

$$\lim_{n\to\infty} y_n = y \in B \quad \text{but} \quad \lim_{n\to\infty} f^{-1}(y_n) \neq f^{-1}(y).$$

Now, let

$$x = f^{-1}(y) \quad \text{and} \quad x_n = f^{-1}(y_n).$$

Now, we have

$$\lim_{n\to\infty} x_n \neq x,$$

thus there exists an $\epsilon_0 > 0$ and a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}$ such that

$$|x_{n_k} - x| \geq \epsilon_0 \qquad \forall k \in \mathbb{N}.$$

Since $\{x_{n_k}\}$ is a bounded sequence, by the Bolzano–Weierstrass theorem, it contains a convergent subsequence, $\{x_{n_{k_j}}\}$ that converges to some $x' \in A$. By the above inequality, we must have $x \neq x'$. The continuity of $f$ implies that

$$\lim_{j\to\infty} y_{n_{k_j}} = \lim_{j\to\infty} f(x_{n_{k_j}}) = f(x'),$$

On the other hand, as a subsequence of $\{y_n\} \to y$, we must also have

$$\lim_{j\to\infty} y_{n_{k_j}} = \lim_{n\to\infty} y_n = y = f(x).$$

The uniqueness of limits now implies that

$$f(x) = f(x'),$$

which is a contradiction with that $x \neq x'$ and $f$ is one-to-one. Therefore, $f^{-1}$ is a continuous function. $\square$

*Method II.* Recall that a function is continuous on $A$ if and only if the preimage of any open set under $f$ is still open. To show $f^{-1}$ is a continuous function now reduces to

$$(f^{-1})^{-1}(O) = f(O)$$

is open for any given open set $O \subset A := [a, b]$.

Let $y_0 \in f(O)$ be fixed. For any $y \in f(A \setminus O) = f(A) \setminus f(O)$, there exists a neighbourhood $V_y$ of $y$ and neighbourhood $U_y$ of $y_0$ such that $V_y \cap U_y = \emptyset$. Since $f$ is continuous, $O_y = f^{-1}(V_y)$ is an open set in $[a, b]$. Now, it is readily seen that $\{O,\, O_y \,|\, y \in f(A \setminus O)\}$ is an open cover of $[a, b]$. Since $[a, b]$ is bounded and closed, by the Heine–Borel theorem, $[a, b]$ is compact, and thus there exists a finite subcover $\{O,\, O_{y_n} \,|\, n = 1, \ldots, N\}$ for $[a, b]$. Now, $U := \bigcap_{n=1}^{N} U_{y_n}$ is a open subset of $O$ and $y_0 \in U$. [Why?] Thus for any point $y_0 \in f(O)$, we find a open subset of $f(O)$ that contains $y_0$, which means $f(O)$ is open and thus completes the proof. $\square$

**Remark.** A general result in topology says that any bijective continuous function from a compact space to a Hausdorff space has a continuous inverse on its image, thus is a homeomorphism. A Hausdorff space is a topological space with the property: for any two distinct points $x_1$ and $x_2$, there exist two disjoint open sets $U_1$ and $U_2$ such that $x_1 \in U_1$ and $x_2 \in U_2$. In particular, $\mathbb{R}$ is Hausdorff.

**28.** (i) Given a countable set $A = \{a_1, a_2, a_3, \dots\}$, define $f(a_n) = 1/n$ and $f(x) = 0$ for all $x \notin A$. Find $D_f$.

(ii) Is it possible for a function $f$ such that $D_f = \mathbb{I}$?

*Solution.* (i) $D_f = A$.

If $x \in A$, then $f(x) = \frac{1}{k}$ for some $k \in \mathbb{N}$. Given any $n \in \mathbb{N}$, $(x - \frac{1}{n}, x + \frac{1}{n})$ is uncountable and thus contains at least one point $x_n$ in $A^c$. We have $\{x_n\} \to x$, but $\{f(x_n)\} \to 0 \neq 1/k$. Hence $f$ is not continuous at $x$.

If $x \notin A$ we shall show that $f$ is continuous at $x$. There are two cases, (a) $x \notin \overline{A}$; or (b) $x$ is a limit point of $A$. In case (a), $x \in (\overline{A})^c$, thus there exists a neighbourhood of $x$, $V_\epsilon(x) \subset (\overline{A})^c$, and where $f(x) = 0$, thus $f(x)$ is continuous at $x$.

In case (b), for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $1/N < \epsilon$. Take $\delta = \min\{|x_n - x|\}_{n=1}^N$. Then, whenever $|y - x| < \delta$ we have either $f(y) = 0$ or $f(y) = 1/n$ for some $n > N$. Hence we have

$$|f(y) - f(x)| < \epsilon, \qquad \forall |y - x| < \delta.$$

That is $f$ is continuous at $x$.

(ii) No, since $\mathbb{I}$ is not $F_\sigma$ but $D_f$ must be an $F_\sigma$ set. $\square$

— End —