MAT 2006 Assignment 2. 118010350

1. proof. Since $\lim_{n\to\infty} x_n = l$, by definition.

$\forall \varepsilon>0$. $\exists N_1 \in \mathbb{N}$, s.t. $|x_n - l| < \varepsilon$, $\forall n \geq N_1$.

$|x_n - l| < \varepsilon$, $\forall n \geq N_1 \Rightarrow l-\varepsilon < x_n < l+\varepsilon$, $\forall n \geq N_1$

$x_n \leq y_n$, $\forall n \in \mathbb{N} \Rightarrow l-\varepsilon < x_n \leq y_n$, $\forall n \geq N_1$.

Since $\lim_{n\to\infty} z_n = l$, by definition.

$\forall \varepsilon>0$, $\exists N_2 \in \mathbb{N}$. s.t. $|z_n - l| < \varepsilon$, $\forall n \geq N_2$.

$|z_n - l| < \varepsilon$, $\forall n \geq N_2 \Rightarrow l-\varepsilon < z_n < l+\varepsilon$, $\forall n \geq N_2$.

$y_n \leq z_n$, $\forall n \in \mathbb{N} \Rightarrow y_n \leq z_n < l+\varepsilon$, $\forall n \geq N_2$.

Take $N = \max\{N_1, N_2\}$. then for $\forall \varepsilon > 0$.

$\exists N \in \mathbb{N}$. s.t $l-\varepsilon < x_n \leq y_n \leq z_n < l+\varepsilon$, $\forall n \geq N$.

$\Rightarrow l-\varepsilon < y_n < l+\varepsilon$. $\forall n \geq N$.

$\Rightarrow |y_n - l| < \varepsilon$, $\forall n \geq N$.

By definition of limit, $\lim_{n\to\infty} y_n = l$.

2. proof. (i) Since $\sqrt[n]{1} < \sqrt[n]{1+\frac{a}{n}} \leq \sqrt[n]{\frac{2n}{n}} = \sqrt[n]{2}$. for $\forall n \geq a > 0$.

$\lim_{n\to\infty} \sqrt[n]{1} = \lim_{n\to\infty} \sqrt[n]{2} = 1$.

By Squeeze Theorem, $\lim_{n\to\infty} \sqrt[n]{1+\frac{a}{n}} = 1$

(ii) Since $\frac{n^k}{n!} = \frac{n}{n} \cdot \frac{n}{n-1} \cdots \frac{n}{n-k+1} \cdot \frac{1}{(n-k)!}$

$\lim_{n\to\infty} \frac{n}{n-1} = 1$, $\lim_{n\to\infty} \frac{n}{n-2} = 1$, $\cdots$,

$\lim_{n\to\infty} \frac{n}{n-k+1} = \lim_{n\to\infty} (1+\frac{k-1}{n-k+1}) = 1$, $k \in \mathbb{N}$.

And $\lim_{n\to\infty} \frac{1}{(n-k)!} = 0$. then $\lim_{n\to\infty} \frac{n^k}{n!} = \overbrace{1 \cdot 1 \cdots 1}^{k} \cdot 0 = 0$.

(iii). Since $a>1$, then $\exists b>0$. s.t $a = b+1$.

$a^n = (b+1)^n = \sum_{i=0}^{n} \binom{n}{i} b^i > \binom{n}{k+1} b^{k+1}$, $k \in \mathbb{N}$.

$= \frac{n \cdots (n-k)}{(k+1)!} b^{k+1}$

$0 < \frac{n^k}{a^n} < \frac{n^k}{n \cdots (n-k)} \cdot \frac{(k+1)!}{b^{k+1}}$

$\lim_{n\to\infty} \frac{n}{n-1} = 1$, $\lim_{n\to\infty} \frac{n}{n-2} = 1$, $\cdots$

$$\lim_{n\to\infty} \frac{n}{n-k+1} = \lim_{n\to\infty} \left(1+\frac{k-1}{n-k+1}\right) = 1$$

Then $\lim_{n\to\infty} \frac{n^k}{n\cdots(n-k)} \cdot \frac{(k+1)!}{b^{k+1}} = 1\cdots1$

$$= \lim_{n\to\infty} \frac{n}{n}\cdot\frac{n}{n-1}\cdots\frac{n}{n-k+1}\cdot\frac{(k+1)!}{b^{k+1}}\cdot\frac{1}{n-k}$$

$$= \underbrace{1\cdots1}_{k} \cdot \frac{(k+1)!}{b^{k+1}} \cdot \lim_{n\to\infty} \frac{1}{n-k} = 0.$$

By Squeeze Theorem, $\lim_{n\to\infty} \frac{n^k}{a^n} = 0$.

(iv) ① when $a>0$, let $x_n = \frac{a^n}{n!}$, $x_{n+1} = \frac{a^{n+1}}{(n+1)!}$

$$\frac{x_{n+1}}{x_n} = \frac{a^{n+1}}{(n+1)!} \cdot \frac{n!}{a^n} = \frac{a}{n+1} \leq \frac{a}{a+1}. \text{ for } n \geq \lceil a \rceil$$

$$\Rightarrow x_{n+1} \leq \left(\frac{a}{a+1}\right) x_n \quad \forall n \geq \lceil a \rceil$$

$$\Rightarrow x_n \leq \left(\frac{a}{a+1}\right)^{n-\lceil a \rceil} x_{\lceil a \rceil}, \quad \forall n \geq \lceil a \rceil$$

Since $\frac{a}{a+1} < 1$, then $\lim_{n\to\infty} \left(\frac{a}{a+1}\right)^{n-\lceil a \rceil} x_{\lceil a \rceil} = 0$.

And $x_n > 0$. for $n \in \mathbb{N}$.

By Squeeze Theorem, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{a^n}{n!} = 0$.

② when $a=0$, $\frac{a^n}{n!} = 0$, $\Rightarrow \lim_{n\to\infty} \frac{a^n}{n!} = 0$.

③ when $a<0$, $\frac{a^n}{n!} = (-1)^n \cdot \frac{|a|^n}{n!}$

By ①, we have $\frac{|a|^n}{n!} > 0$, and $\frac{|a|^n}{n!} \downarrow 0$, $\forall n \geq \lceil a \rceil$.

By Alternative Series test, $\lim_{n\to\infty} (-1)^n \frac{|a|^n}{n!} = \lim_{n\to\infty} \frac{a^n}{n!} = 0$.

(V) Prove $\lim_{n\to\infty} \sqrt[n]{\frac{1}{n}} = 1$ first. $\Leftrightarrow$ Prove $\lim_{n\to\infty} \sqrt[n]{n} = 1$

Let $x_n = n^{\frac{1}{n}} - 1$, $x_n \geq 0$.

$$(x_n+1)^n = n \geq \frac{n(n-1)}{2} x_n^2$$

$$\Rightarrow x_n^2 \leq \frac{2}{n-1}, \quad \forall n \geq 2$$

$$\Rightarrow 0 \leq x_n \leq \sqrt{\frac{2}{n-1}}, \quad \forall n \geq 2.$$

Since $\lim_{n\to\infty} \sqrt{\frac{2}{n-1}} = 0$. then by Squeeze, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} \sqrt[n]{n} - 1 = 0$

$\Rightarrow \lim_{n\to\infty} \sqrt[n]{n} = 1$

Then, prove $\lim_{n\to\infty} \sqrt[n]{\frac{a^n}{n} + \frac{b^n}{n^2}} = b$.

Since $\sqrt[n]{\frac{b^n}{n^2}} < \sqrt[n]{\frac{a^n}{n} + \frac{b^n}{n^2}} \le \sqrt[n]{\frac{(n+1)b^n}{n^2}} \le \sqrt[n]{\frac{(2n)b^n}{n^2}}$, $b \ge a > 0$.

$\lim_{n\to\infty} \sqrt[n]{\frac{b^n}{n^2}} = \lim_{n\to\infty} b\left(\sqrt[n]{\frac{1}{n}}\right)^2 = \lim_{n\to\infty} b\left(\sqrt[n]{\frac{1}{n}}\right)\left(\sqrt[n]{\frac{1}{n}}\right) = b$.

$\lim_{n\to\infty} \sqrt[n]{\frac{(2n)b^n}{n^2}} = \lim_{n\to\infty} b \cdot \sqrt[n]{\frac{2}{n}} = \lim_{n\to\infty} b\left(\sqrt[n]{2}\right)\left(\sqrt[n]{\frac{1}{n}}\right) = b$.

By Squeeze Theorem, $\lim_{n\to\infty} \sqrt[n]{\frac{a^n}{n} + \frac{b^n}{n^2}} = b$.

(vi). Prove $\lim_{n\to\infty} \frac{\sqrt[3]{n^2}}{n+1}$ first.

Since $0 < \frac{\sqrt[3]{n^2}}{n+1} = \sqrt[3]{\frac{n^2}{(n+1)^3}} < \sqrt[3]{\frac{n^2}{n^3}}$, $\forall n \in \mathbb{N}$.

$\lim_{n\to\infty} \sqrt[3]{\frac{n^2}{n^3}} = \lim_{n\to\infty} \sqrt[3]{\frac{1}{n}} = 0$.

By Squeeze Theorem, $\lim_{n\to\infty} \frac{\sqrt[3]{n^2}}{n+1} = 0$.

Then, prove $\lim_{n\to\infty} \frac{\sqrt[3]{n^2} \sin n!}{n+1} = 0$.

Since $|\sin n!| \le 1$, $\forall n \in \mathbb{N}$. which is bounded.

Then $\lim_{n\to\infty} \frac{\sqrt[3]{n^2} \sin n!}{n+1} = 0$.

(vii) Since $\frac{n^2-1}{(n+1)^2} \le \frac{n^2+\cos n}{[n+(-1)^n]^2} \le \frac{n^2+1}{(n-1)^2}$, $\forall n \in \mathbb{N}$.

$\lim_{n\to\infty} \frac{n^2-1}{(n+1)^2} = \lim_{n\to\infty} \frac{(n+1)(n-1)}{(n+1)^2} = \lim_{n\to\infty} \frac{n-1}{n+1} = 1$.

$\lim_{n\to\infty} \frac{n^2+1}{(n-1)^2} = \lim_{n\to\infty} \frac{n^2+1}{n^2-2n+1} = \lim_{n\to\infty} \left(1 + \frac{2}{n+\frac{1}{n}-2}\right) = 1$.

By Squeeze Theorem, $\lim_{n\to\infty} \frac{n^2+\cos n}{[n+(-1)^n]^2} = 1$

3. (i) proof. Since $\{x_n\}$ is a convergent sequence.

Assume $\lim_{n\to\infty} x_n = a$. then $\forall \varepsilon > 0$.

$\Rightarrow N_1 \in \mathbb{N}$. s.t. $|x_n - a| < \frac{\varepsilon}{2}$, $\forall n \ge N_1$.

$$|y_n - a| = \left|\frac{x_1+\cdots+x_n}{n} - a\right| \le \frac{|(x_1-a)+\cdots+(x_n-a)|}{n}$$
$$\le \frac{|x_1-a|}{n} + \cdots + \frac{|x_n-a|}{n}$$
$$= \sum_{i=1}^{N_1-1} \frac{|x_i-a|}{n} + \sum_{i=N_1}^{n} \frac{|x_i-a|}{n}, \quad n \ge N_1.$$

Since $\sum_{i=1}^{N_1-1} \frac{|x_i-a|}{n} \le \frac{N_1-1}{n} \max_{1\le i\le N_1-1} \{|x_i-a|\}$.

Take N_2 as the smallest integer larger than $\frac{2(N_1-1)}{\varepsilon} \max_{1\le i\le N_1-1} \{|x_i-a|\}$

$$\sum_{i=1}^{N_1-1} \frac{|x_i-a|}{n} < \frac{\varepsilon}{2}, \quad \forall n \ge N_2$$

Since $|x_n - a| < \frac{\varepsilon}{2}$, $\forall n \ge N_1$, then $\frac{|x_n-a|}{n} < \frac{\varepsilon}{2n}$, $\forall n \ge N_1$

$$\sum_{i=N_1}^{n} \frac{|x_i-a|}{n} < \left(\frac{n-N_1+1}{n}\right)\cdot\frac{\varepsilon}{2} = \left(1-\frac{N_1}{n}+\frac{1}{n}\right)\frac{\varepsilon}{2}$$
$$\le \frac{\varepsilon}{2}, \quad \forall n \ge N_1$$

Take $N = \max\{N_1, N_2\}$, then for $\forall \varepsilon > 0$.

$\exists N \in \mathbb{N}$. s.t. $|y_n - a| \le \sum_{i=1}^{N_1-1} \frac{|x_i-a|}{n} + \sum_{i=N_1}^{n} \frac{|x_i-a|}{n}$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \quad \forall n \ge N.$$

Thus, $\lim_{n\to\infty} y_n = \lim_{n\to\infty} x_n = a$.

(ii) Suppose $x_n = (-1)^n$, then $y_n = \frac{\sum_{i=1}^{n}(-1)^n}{n}$

Since $\lim_{n\to\infty} x_n = 1$, if $n = 2k$, $k \in \mathbb{N}$,

$\lim_{n\to\infty} x_n = -1$, if $n = 2k-1$, $k \in \mathbb{N}$.

Then $\{x_n\}$ does not cvg

Since $\lim_{n\to\infty} y_n = 0$, if $n = 2k$, $k \in \mathbb{N}$.

$\lim_{n\to\infty} y_n = \lim_{n\to\infty} (-1)\cdot\frac{1}{n} = 0$, if $n = 2k-1$, $k \in \mathbb{N}$.

Since the odd and even terms both cvgs to 0.

Then $\{y_n\}$ cvgs to 0.

4. proof. Let $a_{n+1} = \sqrt{2+a_n}$, $a_1 = \sqrt{2}$.

① Prove that $\sqrt{2} \le a_n \le 2$, $\forall n \in \mathbb{N}$.

when $n = 1$, $\sqrt{2} \le a_1 = \sqrt{2} \le 2$. holds.

Suppose when $n = k$, holds, $\sqrt{2} \le a_k \le 2$.

Then when $n=k+1$, $a_{k+1} = \sqrt{2+a_k} \geq \sqrt{2}$

$a_{k+1} \leq \sqrt{2+2} = 2$. holds

By induction, $\sqrt{2} \leq a_n \leq 2$, $\forall n \in \mathbb{N}$.

② Prove $\{a_n\}$ is increasing, $a_{n+1} \geq a_n$, $\forall n \in \mathbb{N}$.

when $n=1$, $a_2 = \sqrt{2+\sqrt{2}} \geq a_1 = \sqrt{2}$ holds.

Suppose when $n=k$ holds, then $a_{k+1} \geq a_k$,

$n=k+1$, $a_{k+2} = \sqrt{2+a_{k+1}} \geq \sqrt{2+a_k} = a_{k+1}$ holds.

By induction, $a_{n+1} \geq a_n$, $\forall n \in \mathbb{N}$.

By MCT, $\{a_n\}$ cvgs. Assume $\lim_{n\to\infty} a_n = a$.

$a_{n+1} = \sqrt{2+a_n} \Rightarrow a_{n+1}^2 = a_n + 2$.

$\lim_{n\to\infty} a_{n+1}^2 = a^2 = \lim_{n\to\infty}(a_n+2) = a+2$.

$\Rightarrow a^2 = a+2 \Rightarrow a=2$ or -1.

Since $a_n > 0$, for $n \in \mathbb{N}$, then $\lim_{n\to\infty} a_n = 2$.

5. Proof. ① Prove $\sqrt{2} \leq x_n \leq 2$ $\forall n \in \mathbb{N}$.

when $n=1$, $x_1 = \sqrt{2}$, holds

Suppose when $n=k$ holds, then $\sqrt{2} \leq x_k \leq 2$.

when $n=k+1$, $x_{k+1} = \frac{x_k}{2} + \frac{1}{x_k} \leq 1 + \frac{1}{2} < 2$

$x_{k+1} = \frac{x_k}{2} + \frac{1}{x_k} \geq 2\sqrt{\frac{x_k}{2} \cdot \frac{1}{x_k}} = \sqrt{2}$. holds

By induction, $0 < x_n < \sqrt{2}$ $\forall n \in \mathbb{N}$.

② Prove $\{x_n\}$ is decreasing, $x_{n+1} \leq x_n$. $\forall n \in \mathbb{N}$.

$x_n - x_{n+1} = \frac{x_n}{2} - \frac{1}{x_n}$, $x_n \geq \sqrt{2} \Rightarrow x_n^2 \geq 2$.

$\Rightarrow \frac{x_n}{2} \geq \frac{1}{x_n} \Rightarrow \frac{x_n}{2} - \frac{1}{x_n} \geq 0$.

Then $x_n - x_{n+1} \geq 0 \Rightarrow x_{n+1} \leq x_n$. $\forall n \in \mathbb{N}$.

Assume $\lim_{n\to\infty} x_n = a$. then $\lim_{n\to\infty} x_{n+1} = a = \lim_{n\to\infty}\left(\frac{x_n}{2} + \frac{1}{x_n}\right)$

$= \frac{a}{2} + \frac{1}{a}$.

$\Rightarrow a = \frac{a}{2} + \frac{1}{a} \Rightarrow a = \sqrt{2} \Rightarrow \lim_{n\to\infty} x_n = \sqrt{2}$.

6. Proof. ① Prove that $\sup E$ and $\inf E$ exist.

$E = \{ s \in \mathbb{R} \mid s$ is the limit of some subsequence $\{x_{n_k}\}\}$

By BW, $\{x_n\}$ is bounded $\Rightarrow$

There exists a convergent $\{x_{n_k}\}$, i.e. $E \neq \phi$.

LUBP. GLBP $\Rightarrow$ $\sup E$ exists, $\inf E$ exists.

② Prove $\limsup_{n\to\infty} x_n = \sup E$, and $\liminf_{n\to\infty} x_n = \inf E$.

By definition, $\limsup_{n\to\infty} x_n = \lim_{m\to\infty} \sup_{n\geq m} x_n$.

Then for any subsequence $\{x_{n_k}\}$, $\sup_{n\geq m} x_n \geq x_{n_k}$, $\forall n_k \geq m$.

$\Rightarrow \lim_{m\to\infty} \sup_{n\geq m} x_n \geq \lim_{k\to\infty} x_{n_k}$. $\Rightarrow \limsup_{n\to\infty} x_n \geq \lim_{k\to\infty} x_{n_k} = S$.

$\Rightarrow \limsup_{n\to\infty} x_n$ is an U.B of E.

Since for $\forall \varepsilon > 0$, $\exists\, a \in \{x_n \mid n \geq m\}$,
s.t. $\sup_{n\geq m} x_n - \varepsilon < a$.

Take $\varepsilon = \frac{1}{n}$, then for $\forall \frac{1}{n}$, $\exists\, a_n \in \{x_n \mid n \geq m\}$,
s.t. $\sup_{n\geq m} x_n - \frac{1}{n} < a_n$.

$\Rightarrow \sup_{n\geq m} x_n - \frac{1}{n} < a_n < \sup_{n\geq m} x_n$.

Take limit, by squeeze Theorem, $\lim_{n\to\infty} a_n = \limsup_{n\to\infty} x_n$.

Since $\{a_n\}$ is a subsequence of $\{x_n\}$,
then $\limsup_{n\to\infty} x_n \in E$. $\Rightarrow \limsup_{n\to\infty} x_n = \max E = \sup E$.

Similarly, $\inf_{n\geq m} x_n \leq x_{n_k}$, $\forall n_k \geq m$. $\Rightarrow \liminf_{n\to\infty} x_n \leq \lim_{k\to\infty} x_{n_k} = S$.

$\Rightarrow \liminf_{n\to\infty} x_n$ is an LB of E.

For $\forall \frac{1}{n} > 0$, $\exists\, b_n \in \{x_n \mid n \geq m\}$, s.t. $\inf_{n\geq m} x_n + \frac{1}{n} > b_n$
$\Rightarrow \inf_{n\geq m} x_n < b_n < \inf_{n\geq m} x_n + \frac{1}{n}$

By squeeze Theorem, $\lim_{n\to\infty} b_n = \liminf_{n\to\infty} x_n$.

Then $\liminf_{n\to\infty} x_n \in E \Rightarrow \liminf_{n\to\infty} x_n = \min E = \inf E$.

7. (i). $x_n = (-1)^n$, $\limsup_{n\to\infty} x_n = 1$, $\liminf_{n\to\infty} x_n = -1$.

(ii). $x_n = (-1)^n \cdot n$. $\limsup_{n\to\infty} x_n$ and $\liminf_{n\to\infty} x_n$ does not exist.

(iii). $x_n = (-1)^n \frac{1}{n}$. $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = 0$.

8. (a). $\sup A = 1$, $\inf A = 0$, $\max A = 1$, $\min A$ does not exist.
(b). $\sup B = 1$, $\inf B = 0$, $\max B$ does not exist, $\min B = 0$.

9. proof. ① Prove "$\Leftarrow$".
Since $\inf_{n\geq m} x_n \leq x_m \leq \sup_{n\geq m} x_n$.
Take limit, by Squeeze Theorem,
$\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n \Rightarrow \{x_n\}$ cvgs.

② Prove "$\Rightarrow$"
$\sup_{n\geq m} x_n = \sup\{x_n \mid n\geq m\}$. is an L.U.B of set.

$\Rightarrow \forall \varepsilon > 0, \exists a \in \{x_n \mid n\geq m\}$ s.t $\sup_{n\geq m} x_n - \varepsilon < a$.
Take $\varepsilon = \frac{1}{n}$, $\exists a_n \in \{x_n \mid n\geq m\}$. s.t. $\sup_{n\geq m} x_n - \frac{1}{n} < a_n$.
Then $\sup_{n\geq m} x_n - \frac{1}{n} < a_n \leq \sup_{n\geq m} x_n$.

By Squeeze Theorem, $\lim_{n\to\infty} a_n = \limsup_{n\to\infty} x_n$.

Since $\{a_n\}$ is a subsequence of $\{x_n\}$,
then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} x_n \Rightarrow \lim_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$.

Similarly, for $\forall \frac{1}{n}$, $\exists b_n \in \{x_n \mid n\geq m\}$ s.t $\inf_{n\geq m} x_n + \frac{1}{n} > b_n$.
$\inf_{n\geq m} x_n \leq b_n < \inf_{n\geq m} x_n + \frac{1}{n}$
By Squeeze Theorem, $\lim_{n\to\infty} b_n = \liminf_{n\to\infty} x_n$.
Since $\{b_n\}$ is a subsequence of $\{x_n\}$.

then $\lim_{n\to\infty} b_n = \lim_{n\to\infty} x_n \Rightarrow \lim_{n\to\infty} x_n = \liminf_{n\to\infty} x_n$.

Thus, $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n$.

10. proof. $x_n \le y_n, \forall n \ge M. \Rightarrow \inf_{n\ge M} x_n \le \inf_{n\ge M} y_n$.

By order Limit Theorem,

$\lim_{M\to\infty} \inf_{n\ge M} x_n \le \lim_{M\to\infty} \inf_{n\ge M} y_n$

$\Rightarrow \liminf_{n\to\infty} x_n \le \liminf_{n\to\infty} y_n$.

$x_n \le y_n, \forall n \ge M \Rightarrow \sup_{n\ge M} x_n \le \sup_{n\ge M} y_n$.

By Order Limit Theorem,

$\lim_{M\to\infty} \sup_{n\ge M} x_n \le \lim_{M\to\infty} \sup_{n\ge M} y_n$.

$\Rightarrow \limsup_{n\to\infty} x_n \le \limsup_{n\to\infty} y_n$.

11. proof. ① Prove that $\limsup_{n\to\infty} \frac{x_n}{n}$, $\liminf_{n\to\infty} \frac{x_n}{n}$ exist.

$0 \le x_{n+m} \le x_n + x_m \Rightarrow 0 \le x_n \le x_1 + x_{n-1}$

$\le x_1 + x_1 + x_{n-2} \le \cdots \le n x_1$.

$\Rightarrow 0 \le \frac{x_n}{n} \le x_1. \Rightarrow \limsup_{n\to\infty} \frac{x_n}{n}$ and $\liminf_{n\to\infty} \frac{x_n}{n}$ exist.

② Prove that $\{\frac{x_n}{n}\}$ cvgs.

$\forall n \in \mathbb{N}. \exists m, q, r \in \mathbb{N}$. s.t $n = mq + r$, $0 < r \le m$.

$$0 \le \frac{x_n}{n} = \frac{x_{qm+r}}{qm+r} \le \frac{x_{qm} + x_r}{qm+r} \le \frac{q x_m + x_r}{qm+r}.$$

when m is fixed, q and r are varied,

Then $\limsup_{n\to\infty} \frac{x_n}{n} \le \limsup_{q\to\infty} \frac{q x_m + x_r}{qm+r}$

$$= \lim_{q\to\infty} \frac{q x_m + x_r}{qm+r} = \lim_{q\to\infty} \left(\frac{q x_m}{qm+r} + \frac{x_r}{qm+r} \right) = \frac{x_m}{m}.$$

For $n \to \infty$, it is also valid for $m \to \infty$.

$\Rightarrow \limsup_{n\to\infty} \frac{x_n}{n} \le \frac{x_m}{m}$, $\forall m \in \mathbb{N}$.

$$\Rightarrow \liminf_{m\to\infty}\left(\limsup_{n\to\infty}\frac{x_n}{n}\right) \le \liminf_{m\to\infty}\frac{x_m}{m}.$$

$$\Rightarrow \limsup_{n\to\infty}\frac{x_n}{n} \le \liminf_{n\to\infty}\frac{x_n}{n}, \text{ and } \liminf_{n\to\infty}\frac{x_n}{n} \le \limsup_{n\to\infty}\frac{x_n}{n}.$$

Thus, $\liminf_{n\to\infty}\frac{x_n}{n} = \limsup_{n\to\infty}\frac{x_n}{n} \Rightarrow \{\frac{x_n}{n}\}$ crgs.

12. proof. ① Prove $\lim_{n\to\infty}\frac{\frac{1}{2}x_1+\frac{1}{3}x_2+\cdots+\frac{1}{n+1}x_n}{n}=0$.

Let $y_n = \frac{\frac{1}{2}x_1+\frac{1}{3}x_2+\cdots+\frac{1}{n+1}x_n}{n}$

$\lim_{n\to\infty} x_n = A \Rightarrow \forall \varepsilon>0.\ \exists N\in\mathbb{N}.$ s.t $|x_n-A|<\varepsilon.\ \forall n\ge N.$

$\Rightarrow A-\varepsilon < x_n < A+\varepsilon,\ \forall n\ge N.$

$\Rightarrow |x_n| < \max\{|A-\varepsilon|, |A+\varepsilon|\}.$

$$|y_n| = \frac{|\frac{1}{2}x_1+\frac{1}{3}x_2+\cdots+\frac{1}{n+1}x_n|}{n}$$

$$\le \frac{|\frac{1}{2}x_1|}{n}+\frac{|\frac{1}{3}x_2|}{n}+\cdots+\frac{|\frac{1}{n+1}x_n|}{n}$$

$$= \frac{1}{n}\cdot\sum_{i=1}^{N-1}\left|\frac{1}{i+1}x_i\right| + \frac{1}{n}\cdot\sum_{i=N}^{n}\left|\frac{1}{i+1}x_i\right|$$

$$\le \left(\frac{N-1}{n}\right)\cdot\max_{1\le i\le N-1}\left\{\left|\frac{1}{i+1}x_i\right|\right\} + \frac{n-N+1}{n(n+1)}\max\{|A-\varepsilon|, |A+\varepsilon|\}.$$

By Limit Order Theorem, we have.

$$0\le \lim_{n\to\infty}|y_n| \le \lim_{n\to\infty}\left(\frac{N-1}{n}\right)\cdot\max_{1\le i\le N-1}\left\{\left|\frac{1}{i+1}x_i\right|\right\}$$

$$+\lim_{n\to\infty}\frac{n-N+1}{n(n+1)}\cdot\max\{|A-\varepsilon|, |A+\varepsilon|\} = 0.$$

$$\Rightarrow \lim_{n\to\infty}|y_n| = 0.\ \underset{(*)}{\Rightarrow}\ \lim_{n\to\infty} y_n = 0.$$

✓

Short proof for $(*)$. $\forall \varepsilon>0,\ \exists N\in\mathbb{N}$, s.t. $|y_n-0| = \big||y_n|-0\big| < \varepsilon.\ \forall n\ge N.$

② Prove $\lim_{n\to\infty}\frac{\frac{1}{2}x_1+\frac{2}{3}x_2+\cdots+\frac{n}{n+1}x_n}{n} = A.$

By Question 3. $\lim_{n\to\infty} \frac{x_1+x_2+\cdots+x_n}{n} = \lim_{n\to\infty} x_n = A$.

Then $\lim_{n\to\infty} \frac{\frac{1}{2}x_1+\frac{2}{3}x_2+\cdots+\frac{n}{n+1}x_n}{n}$

$= \lim_{n\to\infty} \frac{x_1+x_2+\cdots+x_n}{n} + \lim_{n\to\infty} \frac{\frac{1}{2}x_1+\frac{1}{3}x_2+\cdots+\frac{1}{n+1}x_n}{n} = A+0=A$.

13. proof Let $y_1 = x_1$, and $y_n = \frac{x_n}{x_{n-1}}$, $n>1$.

$\Rightarrow x_n = \frac{x_n}{x_{n-1}} \cdot \frac{x_{n-1}}{x_{n-2}} \cdots \frac{x_2}{x_1} \cdot x_1$

$= y_n \cdot y_{n-1} \cdots y_1$. $\forall n\in\mathbb{N}$.

$\Rightarrow \sqrt[n]{x_n} = \sqrt[n]{y_n \cdot y_{n-1} \cdots y_1} = e^{\frac{\ln y_1+\cdots+\ln y_n}{n}}$. $\forall n\in\mathbb{N}$.

Since $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = l < \infty$. $y_n = \frac{x_n}{x_{n-1}}$.

$\Rightarrow \lim_{n\to\infty} y_n = l < \infty$. $y_n > 0$. $\forall n\in\mathbb{N}$.

$\underset{(*)}{\Rightarrow} \lim_{n\to\infty} \ln y_n = \ln l$, (*) because $\ln x$ is continuous.

By Question 3, $\lim_{n\to\infty} \frac{\ln y_1+\cdots+\ln y_n}{n} = \ln l$.

$\Rightarrow \lim_{n\to\infty} \sqrt[n]{x_n} = \lim_{n\to\infty} e^{\frac{\ln y_1+\cdots+\ln y_n}{n}}$

$\underset{(*)}{=} e^{\lim_{n\to\infty}\frac{\ln y_1+\cdots+\ln y_n}{n}} = e^{\ln l} = l$.

(*) because e^x is continuous.

14. proof. Suppose $\limsup_{n\to\infty} \sqrt[n]{x_n}$, and $\limsup_{n\to\infty} \frac{x_{n+1}}{x_n}$ exist ($<\infty$).

Assume $\limsup_{n\to\infty} \frac{x_{n+1}}{x_n} = l$.

$\forall \varepsilon>0$. $\exists N\in\mathbb{N}$. s.t $\left| \sup_{n\ge m} \frac{x_{n+1}}{x_n} - l \right| < \varepsilon$. $\forall m\ge N$.

$\Rightarrow \sup_{n\ge m} \frac{x_{n+1}}{x_n} < l+\varepsilon$. $\forall m\ge N$.

$\Rightarrow \frac{x_{n+1}}{x_n} < l+\varepsilon$, $\forall n\ge N$.

$\Rightarrow x_{n+1} < (l+\varepsilon)\cdot x_n$. $\forall n\ge N$.

$\Rightarrow x_n \le (l+\varepsilon)^{n-N} \cdot x_N$, $\forall n > N$.

Then $\sqrt[n]{x_n} \le (l+\varepsilon)^{\frac{n-N}{n}} \cdot \sqrt[n]{x_N}$. $\forall n\ge N$.

By Question 10, $\limsup_{n\to\infty} \sqrt[n]{x_n} \le \limsup_{n\to\infty} (l+\varepsilon)^{\frac{n-N}{n}} \sqrt[n]{x_N}$

$\lim_{n\to\infty} (l+\varepsilon)^{\frac{n-N}{n}} = l+\varepsilon$, $\lim_{n\to\infty} \sqrt[n]{x_N} = 1$,

$\Rightarrow \lim_{n\to\infty} (l+\varepsilon)^{\frac{n-N}{n}} \sqrt[n]{x_N} = l+\varepsilon$.

By Question 9, $\limsup_{n\to\infty} (l+\varepsilon)^{\frac{n-N}{n}} \sqrt[n]{x_N} = l+\varepsilon$.

Then, $\limsup_{n\to\infty} \sqrt[n]{x_n} < l+\varepsilon$, for $\forall \varepsilon > 0$

Thus, $\limsup_{n\to\infty} \sqrt[n]{x_n} \le l = \limsup_{n\to\infty} \frac{x_{n+1}}{x_n}$

15. (i) proof. Suppose for contradiction that $\mathbb{N}$ is bdd above.

Let $a_n = n$, $n \in \mathbb{N}$. $(a_1 = 1, a_2 = 2, \cdots a_n = n, \cdots)$.

$\{a_n\}$ is bdd and increasing. MCT $\Rightarrow \{a_n\}$ cvgs

$\lim_{n\to\infty} a_n = \infty \Rightarrow \{a_n\}$ dvgs. Contradiction!

$\Rightarrow \mathbb{N}$ is not bdd above.

Archimedean Property is proved.

(ii) proof. Given a sequence of intervals $\{I_n\}$

$I_n = [a_n, b_n] = \{x \in \mathbb{R} \mid a_n \le x \le b_n\}$.

$I_n \supset I_{n+1}$, $n \in \mathbb{N}$. $\Rightarrow a_n \le a_{n+1}$, $\forall n \in \mathbb{N}$. $a_n < b_1$, $\forall n \in \mathbb{N}$.

and $b_n \ge b_{n+1}$, $\forall n \in \mathbb{N}$. $b_n > a_1$, $\forall n \in \mathbb{N}$.

MCT $\Rightarrow \{a_n\}, \{b_n\}$ cvg. Assume $\lim_{n\to\infty} a_n = l$.

If $\exists N_1 \in \mathbb{N}$, st $a_{N_1} > l$, then $a_n \ge a_{N_1} > l$, $\forall n \ge N_1$

$\lim_{n\to\infty} a_n > l$, $\Rightarrow a_n \le l$, $\forall n \in \mathbb{N}$

If $\exists N_2 \in \mathbb{N}$, st $b_{N_2} < l$, then $a_n < b_n \le b_{N_2}$, $\forall n \ge N_2$.

$\lim_{n\to\infty} a_n < l \Rightarrow b_n \ge l$, $\forall n \in \mathbb{N}$.

$\Rightarrow l \in I_n = [a_n, b_n]$, $\forall n \in \mathbb{N}$.

$\Rightarrow l \in \bigcap_{n=1}^{\infty} I_n$, $\bigcap_{n=1}^{\infty} I_n \neq \phi$.

16. proof. Suppose $A \subset \mathbb{R}$ and A is bounded above.

① If $\max A$ exists, then $\sup A = \max A$.

Thus least upper bound exists.

② If $\max A$ does not exist.

then A contains infinite elements.

W.L.O.G. Consider the case A contains infinite positive elements.

Suppose A is bdd above by $M > 0$.

Let $I_1 = [0, M]$, divide I_1 into $[0, \frac{M}{2}]$, $[\frac{M}{2}, M]$

let I_2 be one of two halfs which contains infinite elements.
(Suppose $I_2 = [\frac{M}{2}, M]$), divide I_2 into $[\frac{M}{2}, \frac{3}{4}M]$, $[\frac{3}{4}M, M]$.

....

$\Rightarrow$ I_k contains infinite elements in A.

$\Rightarrow$ Can choose I_{k+1} contains infinite elements in A.

Then $I_1 \supset I_2 \supset \cdots \supset I_k \supset I_{k+1} \supset \cdots$

N.I.P. $\Rightarrow \bigcap_{n=1}^{\infty} I_n \neq \phi$. Suppose $l \in \bigcap_{n=1}^{\infty} I_n$, $l \in I_n$, $\forall n \in \mathbb{N}$.

pick $a_n \in I_n$, s.t $a_{n+1} > a_n$, $\forall n \in \mathbb{N}$. W.T.S. $\lim_{n\to\infty} a_n = l$.

$a_n \in I_n$, $l \in I_n$, for $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$.

s.t $|a_n - l| \le \frac{M}{2^{n-1}} < \varepsilon$, $\forall n \ge N$. $\Rightarrow \lim_{n\to\infty} a_n = l$.

W.T.S l is the least upper bound.

$a_n < l$, $\forall n \in \mathbb{N}$, and $\forall a \in A$, $\exists N \in \mathbb{N}$, s.t $a_N > a$.

$\Rightarrow \forall a \in A$, $a < l$. $\Rightarrow l$ is an upper bound.

$\lim_{n\to\infty} a_n = l \Rightarrow \forall \varepsilon > 0$. $\exists N \in \mathbb{N}$, s.t $|a_n - l| < \varepsilon$, $\forall n \ge N$.

Suppose there exists an upper bound l', $l' < l$.

$l - l' = \delta_0$, $\exists \varepsilon_0 < \delta_0$, s.t $|a_n - l| < \varepsilon_0$ $\forall n \ge N_0$.

$\Rightarrow$ $a_n > l - \varepsilon_0$, $\forall n \ge N_0$.

$\Rightarrow$ $a_n - l' > l - l' - \varepsilon_0 \Rightarrow a_n - l' > \delta_0 - \varepsilon_0 > 0$, $\forall n \ge N_0$.

$\Rightarrow$ $a_n > l'$, $\forall n > N_0$.

l' is not an upper bound. Contradiction!!

$\Rightarrow$ l is the least upper bound.

(BW $\Rightarrow$ MCT)

17. Proof. Suppose $\{X_n\}$ is bdd and monotone. W.T.S $\{X_n\}$ cvgs.

B.W. $\Rightarrow$ $\{X_n\}$ contains a cvg subseq. $\{X_{n_k}\}$

W.L.O.G. Assume $\{X_n\}$ is increasing.

Suppose $\lim_{k\to\infty} X_{n_k} = X$, then $\forall \varepsilon > 0$, $\exists M \in \mathbb{N}$.

s.t. $|X_{n_k} - X| < \varepsilon$, $\forall k \ge M$.

$\Rightarrow$ $X - \varepsilon < X_{n_k} < X + \varepsilon$, $\forall k \ge M$.

when $m \ge n_M$, $X - \varepsilon < X_{n_M} \le X_m \le X_{n_m} < X + \varepsilon$.

$\Rightarrow$ $|X_m - X| < \varepsilon$, $\forall m \ge n_M$.

$\Rightarrow$ $\lim_{m\to\infty} X_m = X$.

$\Rightarrow$ $\{X_n\}$ cvgs to X.

(C.C. $\Rightarrow$ B.W.).

18. proof. Suppose $\{x_n\}$ is bdd. $\exists M>0$. s.t $|x_n|\le M$, $\forall n\in\mathbb{N}$.

Construct $I_1, I_2, \cdots I_n, \cdots$ Similarly.

let $I_0=[-M,M]$, divide I_1 into $[-M,0]$ $[0,M]$

let I_1 be one of two halfs which contains infinite elements

(Suppose $I_1=[0,M]$), divide I_1 into $[0,\frac{M}{2}]$ $[\frac{M}{2},M]$.

let I_2 be one of two halfs which contains infinite elements

...

$\Rightarrow$ I_k contains infinite elements in $\{x_n\}$.

$\Rightarrow$ Can choose I_{k+1} contains infinite elements in $\{x_n\}$.

$I_1\supset I_2\supset\cdots\supset I_k\supset I_{k+1}\supset\cdots$

N.I.P. $\Rightarrow \bigcap_{n=1}^{\infty} I_n \neq \phi$.

pick $x_{n_1}\in I_1$, $x_{n_2}\in I_2$, $\cdots$, $x_{n_k}\in I_k$, $\cdots$.

W.T.S $\{x_{n_k}\}$ is a cvg subsequence.

$x_{n_k}\in I_k$, $x_{n_{k+1}}\in I_k$, $x_{n_{k+2}}\in I_k$, $\cdots$.

$\Rightarrow \forall \varepsilon>0$. $\exists N\in\mathbb{N}$. s.t $|x_{n_{k_1}}-x_{n_{k_2}}|\le \frac{M}{2^{N-1}}\underset{(*)}{<}\varepsilon$, $\forall k_2\ge k_1\ge N$.

(*) because of A.P.

C.C. $\Rightarrow$ $\{x_{n_k}\}$ cvgs.

19. proof. ① $\sum_{n=1}^{\infty} a_n^2$, $\sum_{n=1}^{\infty} b_n^2$ cvg $\Rightarrow \sum_{n=1}^{\infty} a_n^2+b_n^2$ cvgs

$a_n^2+b_n^2\ge 2\sqrt{a_n^2b_n^2}=2|a_nb_n|$. (*).

$\Rightarrow 0\le |a_nb_n|\le a_n^2+b_n^2$

By Comparison Test, $\sum_{n=1}^{\infty}|a_nb_n|$ cvgs.

② (*) $\Rightarrow 0\le 2|a_nb_n|\le a_n^2+b_n^2$

$\Rightarrow \sum_{n=1}^{\infty} 2|a_nb_n|$ cvgs.

$\Rightarrow \sum_{n=1}^{\infty} 2a_nb_n$ cvgs.

$(a_n+b_n)^2=a_n^2+b_n^2+2a_nb_n$.

$\Rightarrow \sum_{n=1}^{\infty}(a_n+b_n)^2$ cvgs.

③ let $b_n=\frac{1}{n}$, $\sum_{n=1}^{\infty} b_n^2=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots$

$<\frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\frac{1}{3\cdot4}+\cdots$

$<\lim_{n\to\infty} 1-\frac{1}{n}=1$

$\Rightarrow \sum_{n=1}^{\infty} b_n^2$ cvgs

$\frac{|a_n|}{n} = |a_n \cdot b_n|$. By ①, $\sum_{n=1}^{\infty} \frac{|a_n|}{n}$ cvgs.

W.L.O.G. Assume $a>0$.

20. Proof. ① Show $\{a_n\}$ is decreasing

Suppose $\{a_n\}$ is increasing for contradiction

Let $b_n = n a_n$, $\lim_{n\to\infty} b_n = a > 0$.

$\frac{b_{n+1}}{b_n} = \frac{(n+1)a_{n+1}}{n \ a_n} > k > 1 \Rightarrow b_{n+1} > k \cdot b_n$.

$\Rightarrow b_n > (k)^{n-1} \cdot b_1, k>1$.

$\Rightarrow \{b_n\}$ dvgs as $n\to\infty$. Contradiction !!

Thus $\{a_n\}$ is decreasing.

② Show $\sum_{n=1}^{\infty} a_n$ dvgs.

Suppose $\sum_{n=1}^{\infty} a_n$ cvgs for contradiction.

Let $S_n = \sum_{k=1}^{n} a_k$, $\sum_{n=1}^{\infty} a_n$ cvgs $\Rightarrow \{S_n\}$ cvgs..

$\Rightarrow \lim_{n\to\infty} S_{2n} = \lim_{n\to\infty} S_n \Rightarrow \lim_{n\to\infty} S_{2n} - \lim_{n\to\infty} S_n = 0$.

$\Rightarrow \lim_{n\to\infty} S_{2n} - S_n = 0$.

$a>0 \Rightarrow \exists N_1 \in \mathbb{N}$, s.t $a_n > 0$, $\forall n \geq N_1$.

$\lim_{n\to\infty} S_{2n} - S_n = 0 \Rightarrow \forall \varepsilon > 0, \exists N_2 \in \mathbb{N}$, s.t $|S_{2n} - S_n| < \frac{\varepsilon}{2}$, $\forall n \geq N_2$.

$|S_{2n} - S_n| = \left|\sum_{k=1}^{2n} a_k - \sum_{k=1}^{n} a_k\right| = \left|\sum_{k=n+1}^{2n} a_k\right|$

Take $N = \max\{N_1, N_2\}$., then $2|S_{2n} - S_n| = 2\sum_{k=n+1}^{2n} a_k \geq 2n \cdot a_{2n}$.

$\Rightarrow 2n \cdot a_{2n} < \varepsilon$. $\forall n \geq N$.

$\Rightarrow |2n \cdot a_{2n} - 0| < \varepsilon$. $\forall n \geq N$.

By definition, $\lim_{n\to\infty} n a_n = 0$. Contradiction !!

Thus $\sum_{n=1}^{\infty} a_n$ dvgs.

(C.C.)

21. Proof (a) Suppose $n, m \in \mathbb{N}$. and $n > m$.

$S_n - S_m = (-1)^{m+2} a_{m+1} + \cdots + (-1)^{n+1} a_n$

$= (-1)^{m+2} (a_{m+1} + \cdots + (-1)^{n-m-1} a_n)$

$\Rightarrow |S_n - S_m| = |a_{m+1} + \cdots + (-1)^{n-m-1} a_n|$

Since $a_{m+1} \geq a_{m+2} \geq \cdots \geq a_n$..

then $a_{m+1}+\cdots+(-1)^{n-m-1}a_n=(a_{m+1}-a_{m+2})+\cdots+(a_{n-1}-a_n)\geq 0.$ ①

(or $\cdots=(a_{m+1}-a_{m+2})+\cdots+(a_{n-2}-a_{n-1})+a_n\geq 0$)

Since $a_{m+2}\geq a_{m+3}\geq\cdots\geq a_n$,

then $a_{m+1}+\cdots+(-1)^{n-m-1}a_n=a_{m+1}-(a_{m+2}-a_{m+3})-\cdots\leq a_{m+1}$ ②

By ① and ②, $0\leq|S_n-S_m|\leq a_{m+1}$.

$\{a_n\}\downarrow 0\Rightarrow\forall\varepsilon>0,\exists N\in\mathbb{N}$ s.t. $a_{m+1}<\varepsilon$, $m\geq N$.

$\Rightarrow|S_n-S_m|\leq a_{m+1}<\varepsilon$, $\forall n>m\geq N$.

$\Rightarrow\{S_n\}$ is a Cauchy Sequence. $\Rightarrow\{S_n\}$ cvgs

(N.I.P) (b). Let $I_1=[0,S_1]$, $I_2=[S_2,S_1]$, ...

$I_k=[S_{k-1},S_k]$ if k is odd.

$I_k=[S_k,S_{k-1}]$ if k is even.

$\Rightarrow I_1\supset I_2\supset\cdots\supset I_k\supset I_{k+1}\supset\cdots$

N.I.P $\Rightarrow\bigcap_{n=1}^{\infty}I_n\neq\emptyset$. Assume $l\in\bigcap_{n=1}^{\infty}I_n$.

W.T.S. $\lim_{n\to\infty}S_n=l$.

$\{a_n\}\downarrow 0\Rightarrow\forall\varepsilon>0,\exists N\in\mathbb{N}$. s.t $a_n<\varepsilon$, $\forall n\geq N$.

$S_n\in I_n$, $l\in I_n$, $\forall n\in\mathbb{N}$. $\Rightarrow|S_n-l|<a_n<\varepsilon$, $\forall n\geq N$.

$\Rightarrow\lim_{n\to\infty}S_n=l\Rightarrow\{S_n\}$ cvgs.

(M.C.T) (C) $\{S_{2n}\}\nearrow$ and bdd above by S_1.

$\{S_{2n+1}\}\searrow$ and bdd below by S_2.

MCT. $\Rightarrow\{S_{2n}\}$. $\{S_{2n+1}\}$ cvg.

Suppose $\lim_{n\to\infty}S_{2n}=l_1$, $\lim_{n\to\infty}S_{2n+1}=l_2$.

$\Rightarrow\forall\varepsilon>0,\exists N_1\in\mathbb{N}$. s.t $|S_{2n}-l_1|<\frac{\varepsilon}{3}$ $\forall n\geq N_1$

$\forall\varepsilon>0,\exists N_2\in\mathbb{N}$. s.t. $|S_{2n+1}-l_2|<\frac{\varepsilon}{3}$ $\forall n\geq N_2$.

$\{a_n\}\downarrow 0\Rightarrow\forall\varepsilon>0,\exists N_3\in\mathbb{N}$. s.t $a_{2n+1}<\frac{\varepsilon}{3}$ $\forall n\geq N_3$.

Take $N=\max\{N_1,N_2,N_3\}$

$\Rightarrow|l_1-l_2|=|l_1-S_{2n}+(S_{2n}-S_{2n+1})+S_{2n+1}-l_2|$

$\leq|l_1-S_{2n}|+|a_{2n+1}|+|S_{2n+1}-l_2|<\varepsilon$.

$\Rightarrow l_1=l_2\Rightarrow\{S_{2n}\}.\{S_{2n+1}\}$ cvg to same limit.

Odd and even term of $\{S_n\}$ cvg to same limit. (*)

$\Rightarrow\{S_n\}$ cvgs.

✓

Prove: (*) Pick $N=\max\{N_1,N_2\}$. $\forall\varepsilon>0$. $|S_{2n}-l|<\varepsilon$, $\forall n\geq N_1$.

$|S_{2n+1}-l|<\varepsilon$, $\forall n\geq N_2$.

$\Rightarrow|S_n-l|<\varepsilon$, $\forall n\geq N$. $\Rightarrow\{S_n\}$ cvgs.

22. (i). $\sum_{n=1}^{\infty}\left|\frac{n\cos\frac{n\pi}{3}}{2^n}\right| \le \sum_{n=1}^{\infty}\frac{n}{2^n}$. let $a_n = \frac{n}{2^n}$.

$\Rightarrow \frac{a_{n+1}}{a_n} = \frac{n+1}{2n} \Rightarrow \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$.

By Ratio Test. $\sum_{n=1}^{\infty}\frac{n}{2^n}$ cvgs.

By comparison Test. $\sum_{n=1}^{\infty}\frac{n\cdot\cos\frac{n\pi}{3}}{2^n}$ cvgs absolutely.

(ii). $\sum_{n=1}^{\infty}(-1)^n\cdot\frac{\sin^2 n}{n} \le \sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$.

$\{\frac{1}{n}\}\downarrow 0$. By Alternative Series Test, $\sum_{n=1}^{\infty}(-1)\frac{1}{n}$ cvgs.

By Comperison Test, $\sum_{n=1}^{\infty}(-1)^n\frac{\sin^2 n}{n}$ cvgs.

$\sum_{n=1}^{\infty}\frac{\sin^2 n}{n} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1-\cos 2n}{n}$

$= \frac{1}{2}\left(\sum_{n=1}^{\infty}\frac{1}{n} - \sum_{n=1}^{\infty}\frac{\cos(2n)}{n}\right)$.

$\sum_{n=1}^{\infty}\frac{1}{n}$ dvgs. check $\sum_{n=1}^{\infty}\frac{\cos(2n)}{n}$.

$\Rightarrow 2\sin(1)\sum_{n=1}^{N}\cos(2n) = \sum_{n=1}^{N}2\cos(2n)\sin 1$

$= \sum_{n=1}^{N}(-\sin(2n-1)+\sin(2n+1))$.

$= (-\sin 1+\sin 3)+(-\sin 3+\sin 5)+\cdots+(-\sin(2N-1)+\sin(2N+1))$

$= -\sin 1+\sin(2N+1)$.

let $t_n = \sum_{k=1}^{n}\cos(2k) = \frac{-\sin 1+\sin(2n+1)}{2\sin 1}$

By Dirichlet's test, $\sum_{n=1}^{\infty}\frac{\cos(2n)}{n}$ cvgs.

$\Rightarrow \sum_{n=1}^{\infty}\frac{\sin^2 n}{n}$ dvgs.

Thus $\sum_{n=1}^{\infty}(-1)^2\frac{\sin^2 n}{n}$ cvgs conditionally.

23. praf (i). $\sum_{k=m}^{n}S_k(y_k-y_{k+1}) = (x_1+\cdots+x_m)(y_m-y_{m+1})$

$+(x_1+\cdots+x_{m+1})(y_{m+1}-y_{m+2})$

$+\cdots$

$+(x_1+\cdots+x_n)(y_n-y_{n+1})$.

$= \sum_{k=m}^{n}x_k y_k + (x_1+\cdots+x_{m-1})y_m - (x_1+\cdots+x_n)y_{n+1}$

$= \sum_{k=m}^{n}x_k y_k + S_{m-1}\cdot y_m - S_n y_{n+1}$.

$$\Rightarrow \sum_{k=m}^{n} x_k y_k = S_n \cdot y_{n+1} - S_{m-1} \cdot y_m + \sum_{k=m}^{n} S_k (y_k - y_{k+1})$$

(ii). $\sum_{k=1}^{\infty} x_k$ cvgs $\Rightarrow$ $\{S_n\}$ cvgs.

$\exists M > 0$. s.t $|S_n| \leq M$, $\forall n \in \mathbb{N}$.

$$\sum_{k=m}^{\infty} |S_k (y_k - y_{k+1})| = \sum_{k=m}^{\infty} |S_k| (y_k - y_{k+1})$$

$$\leq M \cdot \sum_{k=m}^{\infty} (y_k - y_{k+1}) \leq M \cdot \left(y_m - \lim_{k\to\infty} y_k\right).$$

$y_1 \geq y_2 \geq \cdots \geq 0$, MCT $\Rightarrow$ $\{y_k\}$ cvgs.

$\Rightarrow$ $\sum_{k=m}^{\infty} |S_k (y_k - y_{k+1})|$ cvgs.

$\Rightarrow$ $\sum_{k=m}^{\infty} S_k (y_k - y_{k+1})$ cvgs absolutely.

$$\lim_{n\to\infty} \sum_{k=1}^{n} x_k y_k = \lim_{n\to\infty} S_n \cdot y_{n+1} - S_0 \cdot y_1 + \lim_{n\to\infty} \sum_{k=1}^{n} S_k (y_k - y_{k+1})$$

Since $\{S_n \cdot y_{n+1}\}$ cvgs. $\sum_{k=1}^{\infty} S_k (y_k - y_{k+1})$ cvgs.

Then $\sum_{k=1}^{\infty} x_k y_k$ cvgs. $\Rightarrow$ Abel's Test proved.

24. proof (i). Abel's Test: $\sum_{k=1}^{\infty} x_k$ cvgs. $\{y_k\}$ $y_1 \geq y_2 \geq \cdots \geq 0$.

Dirichelt's Test: $\{S_n\}$ bdd, $\{y_k\} \downarrow 0$.

$$\sum_{k=1}^{n} x_k y_k = S_n \cdot y_{n+1} - S_0 \cdot y_1 + \sum_{k=1}^{n} S_k (y_k - y_{k+1}).$$

$(S_0 = 0)$: $\quad = S_n \cdot y_{n+1} + \sum_{k=1}^{n} S_k (y_k - y_{k+1})$

$\{S_n\}$ bdd, $\{y_k\} \downarrow 0 \Rightarrow \sum_{k=1}^{\infty} |S_k (y_k - y_{k+1})| \leq M (y_1 - \lim_{k\to\infty} y_k)$

$\Rightarrow$ $\sum_{k=1}^{\infty} S_k (y_k - y_{k+1})$ cvgs absolutely.

$\Rightarrow$ $\{S_n \cdot y_{n+1}\}$ cvgs.

$\Rightarrow$ $\sum_{k=1}^{\infty} x_k y_k$ cvgs. $\Rightarrow$ Dirichelt's Test proved.

(ii) Alterating Series Test:

let $x_k = (-1)^{k+1}$ $\Rightarrow$ $0 \leq \sum_{k=1}^{\infty} x_k \leq 1$ $\Rightarrow$ $\sum_{k=1}^{\infty} x_k$ bdd.

$\{y_k\}$. $y_1 \geq y_2 \geq \cdots \geq 0$. $\lim_{k\to\infty} y_k = 0$.

$\Rightarrow$ $\sum_{k=1}^{\infty} x_k y_k = \sum_{k=1}^{\infty} (-1)^{k+1} y_k$ cvgs.