Stochastic Processes (Lecture 9)

DTMC: Recurrence, Positive Recurrence, Ergodic Theorem

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Recurrence and Transience

$$T_i = \min\{n \geq 1 | X_n = i\} = \text{first time to reach } i$$

Definition

A state i of a DTMC is recurrent if

$$\mathbb{P}\{T_i < \infty | X_0 = i\} = 1;$$

otherwise, state i is transient.

$$N_i = \text{number of visits to state } i = \sum_{n=1}^{\infty} 1_{\{X_n = i\}}.$$

Theorem

- *If state i is recurrent, then $\mathbb{E}[N_i | X_0 = i] = \infty$.*
- *If state i is transient, then $\mathbb{E}[N_i | X_0 = i] < \infty$.*

Stopping times

- Let $X = \{X_n : n = 0, 1, \ldots\}$ be a DTMC on space space S.
- A $\{0, 1, \ldots\} \cup \{\infty\}$-valued random variable is called a *stopping time of* the DTMC if the event $\{T = n\}$ depends only on X_0,X_1, …, X_n for $n = 0$, ….
- For a set $A \subset S$, the first passage time to A,

$$T_A = \inf\{n \geq 1 : X_n \in A\}.$$

- $T_i = T_{\{i\}}$.
- Last passage time

$$L_A = \text{ the last time to visit } A.$$

Strong Markov Property

- X is a DTMC on state space S with transition matrix P; T is a stopping time.

$$\mathbb{P}\Big(X_{T+1} = j | \{T < \infty\} \cap \{X_0 = i_0, X_i = i_1, \ldots, X_{T-1} = i_{T-1}, X_T = i\}\Big) = P_{ij} \quad (1)$$

for any $i, j, i_0, i_1, \ldots \in S$.

Geometric random variable (a review)

- Given a coin with probability of $p > 0$ leading a head, let N be the number of tosses needed to get the first head.
- $\mathbb{P}\{N = n\} = q^{n-1}p$ for $n = 1, 2, \ldots$
- $\mathbb{P}\{N \geq 1\} = 1$
- $\mathbb{P}\{N \geq 2\} = q$
- $\mathbb{P}\{N \geq n\} = q^{n-1}$
- $\mathbb{P}\{N < \infty\} = 1$
- $\mathbb{E}(N) = \sum_{n=1}^{\infty} nq^{n-1}p = 1/p$.
- $\mathbb{E}(N) = \sum_{n=1}^{\infty} \mathbb{P}\{N \geq n\} = \sum_{n=1}^{\infty} q^{n-1} = \frac{1}{1-q} = 1/p$

Transience and geometric random variable

- Define

$$T_i = \min\{n \geq 1 | X_n = i\} = \text{ first time to reach } i.$$

- Define

$$N_i = \text{number of visits to state } i = \sum_{n=1}^{\infty} 1_{\{X_n = i\}}.$$

- Prove that

$$\{N_i \geq 1\} = \{T_i < \infty\} \tag{2}$$

- Define

$$f_i = \mathbb{P}\{T_i < \infty | X_0 = i\}.$$

Prove that ($\mathbb{P}_i$ means $\mathbb{P}(\cdot | X_0 = i)$)

$$\mathbb{P}_i\{N_i \geq 2\} = f_i^2.$$

Transience

- In general,

$$\mathbb{P}_i\{N_i \geq n\} = (f_i)^n.$$

- Thus,

$$\mathbb{E}_i(N_i) = \sum_{n=1}^{\infty} (f_i)^n = \frac{f_i}{1 - f_i}.$$

Recurrence and Transience

Lemma

N_i is a "0-based geometric random variable"

Definition (review)

a state i of a DTMC is said to be recurrent if

$$\mathbb{P}\{T_i < \infty | X_0 = i\} = 1;$$

otherwise, state i is transient.

Theorem

- *If state i is transient, then $\mathbb{E}[N_i | X_0 = i] = \frac{f_i}{1-f_i} < \infty$.*
- *If state i is recurrent, then $\mathbb{E}[N_i | X_0 = i] = \infty$.*

Transience

Corollary

(a) State i is transient iff

$$\sum_{n=1}^{\infty} P_{ii}^{n} < \infty.$$

(b) If state i is transient,

$$\mathbb{P}_i(N_i < \infty) = 1.$$

“Solidarity” Property

Theorem

If states i and j communicate, then they are either both recurrent or both transient.

What if the DTMC is irreducible?

Theorem

If a finite DTMC is irreducible, then every state is recurrent.

What if the state space of the DTMC is infinite?

One-dimensional simple random walk

- $P_{00}^{2n} = \binom{2n}{n} p^n q^n$
- Stirling's formula: $n! \sim \sqrt{2\pi n}(n/e)^n$

Two-dimensional simple symmetric random walk

Positive recurrence

- A recurrent state i of a DTMC is said to be positive recurrent if

$$\mathbb{E}[T_i | X_0 = i] < \infty;$$

- Otherwise, the current state i is said to be null recurrent.

Theorem (SLLN)

Assume that state i is positive recurrent and $f : S \to \mathbb{R}$ is bounded.

$$\mathbb{P}_i \left\{ \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(X_k) = c \right\} = 1,$$

where

$$c = \frac{\mathbb{E}_i \Big(\sum_{k=0}^{T_i - 1} f(X_k) \Big)}{\mathbb{E}_i(T_i)}.$$

Existence of stationary distribution

- Assume state i is positive recurrent.
- Define $\pi = (\pi(j), j \in S)$ via

$$\pi(j) = \frac{\mathbb{E}_i\Big(\sum_{k=0}^{T_i - 1} 1_{\{X_k = j\}} \Big)}{\mathbb{E}_i(T_i)}, \quad j \in S. \tag{3}$$

- Then π is stationary distribution of the Markov chain.
- Proof.

Uniqueness of stationary distributions

Theorem

Assume the DTMC is irreducible. It is positive recurrent and only if there is a unique stationary distribution. The unique stationary distribution π is given by

$$\pi(j) = \frac{1}{\mathbb{E}_j(T_j)}.$$

What do we mean by "a positive recurrent DTMC"? Solidarity!

Ergodic Theorem

- Let $N_j(n) = \sum_{k=1}^{n} 1_{\{X_k=j\}}$ the number of visits to state j in $[1, n]$.

  $\frac{N_j(n)}{n}$ the fraction of "time" that the DTMC spends in state j.

Theorem

(a) Assume i is positive recurrent.

$$\mathbb{P}_i \left\{ \lim_{n\to\infty} \frac{1}{n} N_j(n) = \frac{\mathbb{E}_i\Big(\sum_{k=0}^{T_i-1} 1_{\{X_k=j\}} \Big)}{\mathbb{E}_i(T_i)} = \pi(j) \right\} = 1 \quad \textit{for } j \in S.$$

(b) Assume i is positive recurrent and $f : S \to \mathbb{R}$ is bounded. Define $\pi(f) = \sum_{i\in S} \pi(i) f(i)$.

$$\mathbb{P}_i \left\{ \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(X_k) = \pi(f) \right\} = 1.$$

One dimensional symmetric random walk

Reflected random walks

Positive recurrence criterion

- Let $N_i(n) = \sum_{k=1}^n 1_{\{X_k=i\}}$ be the number of times visiting state i in $[1, n]$. Then

$$\mathbb{E}_i(N_i(n)) = \sum_{k=1}^n \mathbb{E}_i 1_{\{X_k=i\}} = \sum_{k=1}^n \mathbb{P}_i\{X_k = i\} = \sum_{k=1}^n P_{ii}^k.$$

Theorem

State i is positive recurrent if and only if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n P_{ii}^k > 0.$$

- Proof.

Recall recurrence criterion

- Recall that state i is recurrent iff

$$\sum_{k=1}^{\infty} P_{ii}^k = \infty.$$

- State i is positive recurrent iff

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} P_{ii}^k > 0.$$

Solidarity of positive recurrence

Lemma 1

Assume states i and j communicate. State i is p.r. iff state j is p.r.

- Proof: there exist k_1 and k_2 such that $P_{ij}^{k_1} > 0$ and $P_{ji}^{k_2} > 0$.
- Assume j is p.r. Then $\lim_{n\to\infty}(1/n)\sum_{k=1}^{n} P_{jj}^{k} > 0$. Lemma follows from

$$P_{ii}^{k_1+k+k_2} \geq P_{ij}^{k_1} P_{jj}^{k} P_{ji}^{k_2},$$

$$\frac{1}{n}\sum_{k=1}^{n+k_1+k_2} P_{ii}^{k} = \frac{1}{n}\sum_{k=1}^{n} P_{ii}^{k_1+k+k_2} + \frac{1}{n}\sum_{k=1}^{k_1+k_2} P_{ii}^{k} > 0$$

when n is large enough.

Limiting behavior of transition matrix P

- Assume that the DTMC is irreducible.
- If it is positive recurrent, for every pair of states $i, j \in S$, (to be proved)

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (P^k)_{ji} = \frac{1}{\mathbb{E}_i(T_i)} > 0.$$

  Namely,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} P^k = P^{(\infty)},$$

  where $P_{ij}^{(\infty)} = \pi_j = 1/\mathbb{E}_j(T_j)$.
- If it is not positive recurrent, for every pair of states

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (P^k)_{ji} = 0.$$