Midterm
Oct 26th

For each question, please provide details or middle steps in addition to final answers.

1. (25 points) You are a shop floor manager in a manufacturing company, responsible for the machine maintenance. The machine can be graded as “Excellent”, “Workable” and “Failed”. The machine is inspected every morning. When the machine is in “Excellent” condition, with probability 0.1 it is re-graded as “Workable” because of possible operational mistakes and with probability 0.05 it is re-graded as “Failed” because of possible operational failure. When the machine is in “Workable” condition, it is re-graded as “Failed” with probability 0.3 and cannot be re-graded as “Excellent” again. When the machine is “Failed”, the machine is repaired which takes a day and becomes “Excellent” the next morning.

   (a) (10 points) Model the system as a DTMC. What is the one-step transition matrix?

   (b) (7 points) What is the long-run fraction of days you spend on repairing the machine?

   (c) (8 points) When the machine is in “Excellent” and “Workable” conditions, the machine can generate \$1000 and \$300 profit daily, respectively. However, when the machine is “Failed”, it costs \$500 to repair it. What is the long-run average daily profit generated by the machine?

Solution

(a) i. (1) Modle the system as a DTMC :
Excellent: $X_n = 1$; Workable: $X_n = 2$; Failed: $X_n = 3$
Do nothing: $U_n = 1$; Operational mistakes: $U_n = 2$; Operational failure: $U_n = 3$;
Repair: $U_n = 4$
DTMC: $X_{n+1} = f(X_n, U_{n+1}), (n >= 1)$

X_n	U_{n+1}	X_{n+1}
1	1	1
1	2	2
1	3	3
2	1	2
2	3	3
3	4	1

ii. one-step transient matrix P

$$\begin{bmatrix} 0.85 & 0.1 & 0.05 \\ 0 & 0.7 & 0.3 \\ 1 & 0 & 0 \end{bmatrix}$$

(b) Stationary distribution $\pi = [\pi_1, \pi_2, \pi_3]$, solve

$$\pi = \pi P, \pi_1 + \pi_2 + \pi_3 = 1$$

That is,

$$\begin{cases} \pi_1 = 0.85\pi_1 + \pi_3 \\ \pi_2 = 0.1\pi_1 + 0.7\pi_2 \\ \pi_1 + \pi_2 + \pi_3 = 1 \end{cases}$$

The answer is $\pi = [\frac{60}{89}, \frac{20}{89}, \frac{9}{89}]$
The long-run fraction of days I spend on repairing the machine: $1 \times \pi_3 = \frac{9}{89}$.

(c) Value function $g(X_n), X_n = 1, 2, 3$. The long-run average daily profit

$$\mathbb{E}(profit) = \pi g = \pi_1 g(1) + \pi_2 g(2) + \pi_3 g(3) = \frac{61500}{89}$$

2. (25 points) David buys fruits and vegetables wholesale and retails them at Davids Produce. One of the more difficult decisions is the amount of bananas to buy. Let us make some simplifying assumptions, and assume that David purchases bananas once a week at 10 cents per pound and retails them at 30 cents per pound during the week. Bananas that are more than a week old are too ripe and are sold for 5 cents per pound. Suppose that the demand for the good bananas has a continuous distribution that is uniformly distributed between 5 and 10. Assume that David buys 8 pounds of banana every week.

   (a) (10 points) What is the expectation and variance of the profit of David in a week?

   (b) (10 points) What is the 95% confidence interval of his cumulative profit in 100 weeks?(Leaving the answer as an expression is OK)

   (c) (5 points) What is the optimal order quantity each week to maximize the expected profit per week?

Solution.

(a) Sell price c_p, variable price c_v, salvage price c_s, buy amount q.

$$\begin{aligned}
profit &= c_p(\min(q, D)) - c_v q + c_s(q - D)^+ \\
&= c_p[q - (q - D)^+] - c_v q + c_s(q - D)^+ \\
&= (c_s - c_p)(q - D)^+ + (c_p - c_v)q \\
&= 25(q - 1)^+ + 160
\end{aligned}$$

expectation of the profit:

$$\begin{aligned}
\mathbb{E}(profit) &= c_p \mathbb{E}(\min q, D) - c_v q + c_s \mathbb{E}[(q - D)^+] \\
\mathbb{E}[\min q, D] &= \int_5^8 x\frac{1}{5}dx + \int_8^{10} 8\frac{1}{5}dx \\
&= \frac{1}{5}\frac{1}{2}x^2|_5^8 + \frac{8}{5} \times 2 \\
&= 7.1 \\
\mathbb{E}((q - D)^+) &= \int_5^8 (8 - x)\frac{1}{5}dx = 0.9 \\
\text{Thus, } \mathbb{E}(profit) &= 137.5
\end{aligned}$$

variance of the profit:

$$\begin{aligned}
Var(profit) &= 25^2\mathbb{E}[((q - D)^+)^2] - \mathbb{E}^2((q - D)^+) \\
\mathbb{E}[((q - D)^+)^2] &= \int_5^8 (8 - x)^2\frac{1}{5}dx = \frac{9}{5} \\
\text{Thus, } Var(profit) &= 618.75
\end{aligned}$$

(b) $\mu = 137.5$, $\sigma^2 = 618.75$, $n = 100$, X_i is the profit of ith week. law of large number:

$$\begin{aligned}
&\mathbb{P}(\frac{\sum_{i=1}^n X_i - n\mu}{\sqrt{n}\sigma} < x) = \Phi(x) \\
&-1.96 \leq \frac{\sum_{i=1}^n X_i - n\mu}{\sqrt{n}\sigma} \leq 1.96 \\
&13750 - 1.96 \times \sqrt{100} \times \sqrt{618.75} \leq \sum_{i=1}^n X_i \leq 13750 + 1.96 \times \sqrt{100} \times \sqrt{618.75}
\end{aligned}$$

The 95% confidence intersections $[13750 - 1.96 \times \sqrt{100} \times \sqrt{618.75}, 13750 + 1.96 \times \sqrt{100} \times \sqrt{618.75}]$

(c)

$$
\begin{aligned}
&h(p) = c_p(\min(q, D)) - c_v q + c_s(q-D)^+ \\
&F(x) = \mathbb{P}(D <= x) = \frac{1}{5}(x-5), 5 <= x <= 10 \\
&\text{x* is the smallest x satisfys} \\
&F(x^*) \geq \frac{c_p - c_v}{c_p - c_s} = \frac{30-10}{30-5} = 0.8 \\
&\text{optimal q is : } x^* = 9
\end{aligned}
$$

3. (15 points) Consider a symmetric random walk on 5-node circle $\{X_n\}$ as follows

$$P_{i,i+1} = P_{i,i-1} = 1/2, \text{ for } i = 2,3,4; P_{12} = P_{15} = 1/2; P_{51} = P_{54} = 1/2.$$

Let $T_i = \min\{n \geq 1 | X_n = i\}$ be the first time to reach state i.

(a) (10 points) For all $j \in \{1,2,3,4,5\}$, compute $\mathbb{P}(T_2 < T_4 | X_0 = j)$.

(b) (5 points) Compute $\mathbb{E}_1[T_4]$, the expected number of steps to reach 4 starting from 1.

Solution

(a) Suppose $q_i = \mathbb{P}(T_2 < T_4 | X_0 = i)$.

$$
\begin{cases}
q_1 = \frac{1}{2} \times 1 + \frac{1}{2} q_5 \\
q_5 = \frac{1}{2} \times 0 + \frac{1}{2} q_1 \\
q_3 = \frac{1}{2} \times 1 + \frac{1}{2} \times 0 \\
q_4 = \frac{1}{2} q_3 + \frac{1}{2} q_5 \\
q_2 = \frac{1}{2} q_3 + \frac{1}{2} q_1
\end{cases}
$$

$$q_1 = \frac{2}{3}, q_2 = \frac{7}{12}, q_3 = \frac{1}{2}, q_4 = \frac{5}{12}, q_5 = \frac{1}{3}$$

(b) Suppose $N_i = \mathbb{E}_i[T_4]$.

$$
\begin{cases}
N_1 = 1 + \frac{1}{2} N_5 + \frac{1}{2} N_2 \\
N_2 = 1 + \frac{1}{2} N_1 + \frac{1}{2} N_3 \\
N_3 = 1 + \frac{1}{2} \times 0 + \frac{1}{2} N_2 \\
N_4 = 0 \\
N_5 = 1 + \frac{1}{2} \times 0 + \frac{1}{2} N_1
\end{cases}
$$

$$N_1 = 6, N_2 = 6, N_3 = 4, N_4 = 5, N_5 = 4$$

The solution is : $\mathbb{E}_1[T_4] = N_1 = 6$

4. (30 points) Let $X = \{X_n : n = 0, 1, 2, \ldots\}$ be a discrete time Markov chain on state space $S = \{1,2,3,4\}$ with transition matrix

$$P = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & 0 & \frac{3}{4} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & 0 & \frac{3}{4} & 0 \end{pmatrix}.$$

(a) (2 points) Draw a transition diagram.

(b) (4 points) Find $\mathbb{P}\{X_2 = 4 | X_0 = 2\}$.

(c) (5 points) Find $\mathbb{P}\{X_2 = 2, X_4 = 4, X_5 = 1 | X_0 = 2\}$.

(d) (5 points) What is the period of each state?

(e) (5 points) Let $\pi = (\frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{4})$. Is π the unique stationary distribution of X? Explain your answer.

(f) (5 points) Let P^n be the nth power of P. Does $\lim_{n\to\infty} P^n_{1,4} = \frac{1}{4}$ hold? Explain your answer.

(g) (4 points) Let T_1 be the first $n \geq 1$ such that $X_n = 1$. Compute $\mathbb{E}(T_1|X_0 = 1)$. (If it takes you a long time to compute it, you are likely on a wrong track.)

Solution.

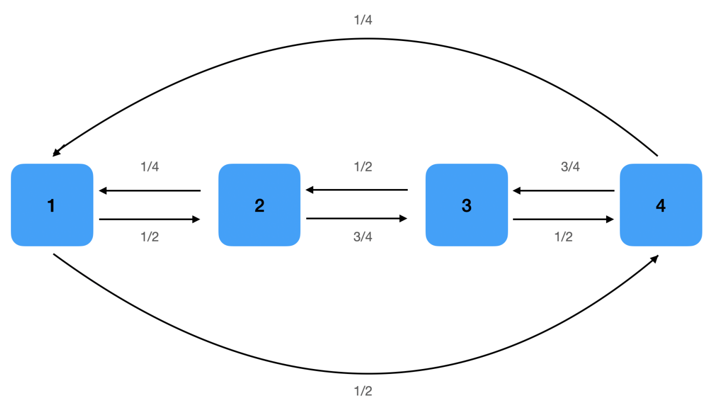

Figure 1: transition graph

(a)

(b) Note that

$$P^{(2n)} = \begin{pmatrix} \frac{1}{4} & 0 & \frac{3}{4} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & 0 & \frac{3}{4} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix},$$

hence, we have $\mathbb{P}(X_2 = 4|X_0 = 2) = P^2_{2,2} = 0.5$.

(c)

$$\begin{aligned} & \mathbb{P}\{X_2 = 2, X_4 = 4, X_5 = 1|X_0 = 2\} \\ = \; & \mathbb{P}\{X_5 = 1|X_0 = 2, X_2 = 2, X_4 = 4\}\mathbb{P}\{X_4 = 4|X_0 = 2, X_2 = 2\}\mathbb{P}\{X_2 = 2|X_0 = 2\} \\ = \; & \mathbb{P}\{X_5 = 1|X_4 = 4\}\mathbb{P}\{x_4 = 4|X_2 = 2\}\mathbb{P}\{X_2|X_0 = 2\} \\ = \; & P_{4,1}P^{(2)}_{2,4}P^{(2)}_{2,2} \\ = \; & \frac{1}{4} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{16}. \end{aligned}$$

(d) Note that

$$P^{(2n-1)} = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & 0 & \frac{3}{4} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & 0 & \frac{3}{4} & 0 \end{pmatrix};$$

hence, $d(1) = 2$ implied by

$$P^{(2n)}_{1,1} = \frac{1}{4} > 0, \text{ and } P^{(2n-1)}_{1,1} = 0.$$

(e) Since this finite DTMC is irreducible, it is positive recurrenet. Hence, it has a unique stationary distribution.

(f) We have computed that $P^{(2n)} \neq P^{(2n-1)}$, *i.e.*, $P_{1,4}^{(2n)} = 0 \neq \frac{1}{2} = P_{1,4}^{(2n-1)}$ for all n. Therefore, $\lim_{n\to\infty} P^n = \frac{1}{4}$ cannot hold.

(g) $\pi_1 = \dfrac{1}{E_1(T_1)} \Rightarrow E_i(T_1) = 8.$

5. (5 points) Construct a DTMC that has all three types of states (transient, positive recurrent and null recurrent), and the period of each state is 1.

Solution. We can modify the 2-d random walk so that it satisfies the requirements.

We add states T, P into the state space $\mathbb{S}_1$ to construct a state space $\mathbb{S}$, that is,

$$\mathbb{S} = \mathbb{S}_1 \cup \mathbb{S}_2 \cup \mathbb{T} = \{\dots,\ -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\dots\} \cup \{P\} \cup \{T\}.$$

- All of states $i \in \mathbb{S}_1$, state i can go to state $i-1$, state i itself, or state $i+1$, each one *w.p.* $\frac{1}{3}$.

- For state $T \in \mathbb{T}$, it can go to state 0, state T, or state A, each one *w.p.* $\frac{1}{3}$.

- For state $P \in \mathbb{S}_2$, it can only go to itself *w.p.* 1.