MAT2002 Ordinary Differential Equations
System of first order linear equations IV

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Overview

1. Homogeneous system with constant coefficients
   - The general case: $n \times n$ matrix

Outline

1. Homogeneous system with constant coefficients
   - The general case: $n \times n$ matrix

The general case: $n \times n$ matrix

Consider the system of the form

$$\mathbf{y}'(t) = \mathbf{A}\mathbf{y}(t), \qquad t \in I, \tag{1}$$

where $\mathbf{A} \in \mathbb{R}^{n \times n}$ is a constant matrix.
Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be a matrix with eigenvalues $\lambda_1, \ldots, \lambda_k$ where $k \in \mathbb{N}$, and each eigenvalue λ_i has an alg. mult. of $m_i \in \mathbb{N}$. This implies that the characteristic equation looks like

$$P_{\mathbf{A}}(\lambda) = det(\lambda I - \mathbf{A}) = (\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_k)^{m_k},$$

where $m_1 + \cdots + m_k = n$. Now suppose each eigenvalue λ_i has a geo. mult. of q_i, where for each $1 \le i \le k$, $1 \le q_i \le m_i$ (recall $1 \le$ geo. mult. $\le$ alg. mult).

Goal: Find m_i linearly independent solutions corresponding to the eigenvalue λ_i (for $i = 1, \cdots, k$).

Diagonalizable matrix

Recall:

Theorem

Let A be a square matrix with size n, then A is diagonalizable if and only if the algebraic multiplicity and geometric multiplicity are the same for each eigenvalue.

If $q_i = m_i$ for all $i = 1, \ldots, k$, then $\mathbf{A}$ is diagonalizable. There are n linearly independent eigenvectors $\xi_1, \cdots, \xi_n$ corresponding to eigenvalues $r_1, \cdots, r_n$, where $r_1, \cdots, r_n \in \{\lambda_1, \ldots, \lambda_k\}$, and
$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{\Lambda} = diag(r_1, \cdots, r_n), \mathbf{P} = [\xi_1, \cdots, \xi_n]$.
We define the new vector $\mathbf{x} := \mathbf{P}^{-1}\mathbf{y}$. Then,

$$\mathbf{x}'(t) = \mathbf{P}^{-1}\mathbf{y}'(t) = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{x}(t) \Rightarrow \boxed{\mathbf{x}'(t) = \mathbf{\Lambda}\mathbf{x}(t)}.$$

The solution for $\mathbf{x}$ can be solved easily, which is given by:

$$\mathbf{x}(t) = [c_1 e^{r_1 t}, \cdots, c_n e^{r_n t}]^T.$$

and $\mathbf{y}(t) = \mathbf{P}\mathbf{x}(t) = c_1 e^{r_1 t}\boldsymbol{\xi}_1 + \cdots + c_n e^{r_n t}\boldsymbol{\xi}_n$.
Indeed, $e^{r_1 t}\boldsymbol{\xi}_1, \cdots, e^{r_n t}\boldsymbol{\xi}_n$ is a fundamental set of solutions.

Example: geometric multiplicty q=algebraic multiplicity m

Example 12.1

For

$$\mathbf{y}' = \mathbf{A}\mathbf{y}, \quad \mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with eigenvalues and corresponding eigenvectors

$$r_1 = 3,\ \boldsymbol{\xi}_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix},\ r_2 = -1,\ \boldsymbol{\xi}_2 = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix},\ r_3 = 1,\ \boldsymbol{\xi}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

the general solution is

$$\mathbf{y}(t) = c_1 e^{3t} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + c_3 e^{t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Example: geometric multiplicty q=algebraic multiplicity m

Example 12.2

For

$$\mathbf{y}' = \mathbf{A}\mathbf{y}, \quad \mathbf{A} = \begin{pmatrix} -3 & -2 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with eigenvalues $r_1 = \overline{r_2}$ and corresponding eigenvectors $\mathbf{x}_1 = \overline{\mathbf{x}_2}$:

$$r_{1,2} = -1 \pm 2i, \ \xi_{1,2} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \pm i \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, r_3 = 1, \xi_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

The general complex solution is

$$\begin{aligned} \mathbf{y}(t) =& c_1 e^{(-1+2i)t} \left(\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right), \\ &+ c_2 e^{(-1-2i)t} \left(\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} - i \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right) + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \end{aligned}$$

Example: geometric multiplicty q=algebraic multiplicity m

Example 12.2

The general real solution is

$$\begin{aligned}\mathbf{y}(t) =& c_1 e^{-t}\left(\cos(2t)\begin{pmatrix}-1\\1\\0\end{pmatrix} - \sin(2t)\begin{pmatrix}0\\1\\0\end{pmatrix}\right)\\ &+ c_2 e^{-t}\left(\sin(2t)\begin{pmatrix}-1\\1\\0\end{pmatrix} + \cos(2t)\begin{pmatrix}0\\1\\0\end{pmatrix}\right) + c_3\begin{pmatrix}0\\0\\1\end{pmatrix}.\end{aligned}$$

Example: geometric multiplicty q=algebraic multiplicity m

Example 12.3

For

$$\mathbf{y}' = \mathbf{A}\mathbf{y}, \quad \mathbf{A} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

with eigenvalues and corresponding eigenvectors

$$r_1 = 2,\ \xi_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ r_2 = 2,\ \xi_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, r_3 = 2,\ \xi_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

the general solution is

$$\mathbf{y}(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 e^{2t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Non-diagonalizable matrix

However, if there is a repeated eigenvalue λ with geometric multiplicity **strictly less** than its algebraic multiplicity, $\mathbf{A}$ is not diagonalizable. In this case, the theory is more complicated.

In the following, we carry out a systematic way to find m_i linearly independent solutions corresponding to the eigenvalue λ_i (for $i = 1, \cdots, k$), where $m_1 + \cdots + m_k = n$.

The general case: $n \times n$ matrix

For all distinct eigenvalues $(\lambda_1, \cdots, \lambda_k)$, we will need to carry out the following process.

For $i = 1, \cdots, k$, do the following.

Let $\lambda = \lambda_i$ be the eigenvalue of $\mathbf{A}$, $m = m_i$ is algebraic multiplicity of λ, and $q = q_i$ is geometric multiplicity of λ. Then we want to find m linearly independent solutions corresponding to the eigenvalue λ.

Case 1: If geometric multiplicty q=algebraic multiplicity m for the eigenvalue λ, then suppose $\mathbf{r}_1, \cdots, \mathbf{r}_m$ are m linearly independent eigenvectors w.r.t λ $(\mathbf{A}\mathbf{r}_j = \lambda \mathbf{r}_j, j = 1, \cdots, m)$. $\left(\frac{d}{dt}\left(\mathbf{r}_j e^{\lambda t}\right) = \lambda \mathbf{r}_j e^{\lambda t}\right)$
Thus, we already have m linearly independent solutions $\mathbf{r}_1 e^{\lambda t}, \cdots, \mathbf{r}_m e^{\lambda t}$.

Case: geometric multiplicty q <algebraic multiplicity m

Case 2: If geometric multiplicty $q <$ algebraic multiplicity m for the eigenvalue λ. Then we will need to construct m linearly independent solutions. We look for the solutions of the following form:

$$\mathbf{y}(t) = \left(\mathbf{r}_0 + \mathbf{r}_1 t + \mathbf{r}_2 \frac{t^2}{2} + \cdots + \mathbf{r}_{m-1} \frac{t^{m-1}}{(m-1)!}\right) e^{\lambda t}$$

Then

$$\begin{aligned}
&\mathbf{y}'(t) - \mathbf{A}\mathbf{y}(t) \\
&= \left(\mathbf{r}_1 + \mathbf{r}_2 t + \cdots + \mathbf{r}_{m-1} \frac{t^{m-2}}{(m-2)!}\right) e^{\lambda t} \\
&+ \left(\lambda\mathbf{r}_0 + \lambda\mathbf{r}_1 t + \cdots + \lambda\mathbf{r}_{m-1} \frac{t^{m-1}}{(m-1)!}\right) e^{\lambda t} \\
&- \left(A\mathbf{r}_0 + A\mathbf{r}_1 t + \cdots + A\mathbf{r}_{m-1} \frac{t^{m-1}}{(m-1)!}\right) e^{\lambda t} \\
&= (\mathbf{r}_1 - (\mathbf{A} - \lambda\mathbf{I})\mathbf{r}_0)e^{\lambda t} + (\mathbf{r}_2 - (\mathbf{A} - \lambda\mathbf{I})\mathbf{r}_1)te^{\lambda t} + (\mathbf{r}_3 - (\mathbf{A} - \lambda\mathbf{I})\mathbf{r}_2)\frac{t^2}{2}e^{\lambda t} + \cdots \\
&+ (\mathbf{r}_{m-1} - (\mathbf{A} - \lambda\mathbf{I})\mathbf{r}_{m-2})\frac{t^{m-2}}{(m-2)!}e^{\lambda t} - (\mathbf{A} - \lambda\mathbf{I})\mathbf{r}_{m-1}\frac{t^{m-1}}{(m-1)!}e^{\lambda t}
\end{aligned}$$

Case: geometric multiplicty q <algebraic multiplicity m

In order to get $\mathbf{y}'(t) = A\mathbf{y}(t)$, we need

$$\begin{aligned}
\mathbf{r}_1 &= (\mathbf{A} - \lambda\mathbf{I})\mathbf{r}_0 \\
\mathbf{r}_2 &= (\mathbf{A} - \lambda\mathbf{I})\mathbf{r}_1 \\
\mathbf{r}_3 &= (\mathbf{A} - \lambda\mathbf{I})\mathbf{r}_2 \\
&\vdots \\
\mathbf{r}_{m-1} &= (\mathbf{A} - \lambda\mathbf{I})\mathbf{r}_{m-2} \\
(\mathbf{A} - \lambda\mathbf{I})\mathbf{r}_{m-1} &= 0.
\end{aligned}$$

Substituting the first $m-1$ equations into the last equation gives

$$(A - \lambda I)^m \mathbf{r}_0 = 0$$

$$\begin{aligned}
\mathbf{r}_1 &= (\mathbf{A} - \lambda\mathbf{I})\mathbf{r}_0 \\
\mathbf{r}_2 &= (\mathbf{A} - \lambda\mathbf{I})\mathbf{r}_1 \\
\mathbf{r}_3 &= (\mathbf{A} - \lambda\mathbf{I})\mathbf{r}_2 \\
&\vdots \\
\mathbf{r}_{m-1} &= (\mathbf{A} - \lambda\mathbf{I})\mathbf{r}_{m-2}.
\end{aligned}$$

Case: geometric multiplicty q <algebraic multiplicity m

Fact

If λ is an eigenvalue of $\mathbf{A}$ with algebraic multiplicity m, then

$$(\mathbf{A} - \lambda \mathbf{I})^m \mathbf{r} = \mathbf{0}$$

will have m linearly independent solutions $\mathbf{r}_0^{(1)}, \cdots, \mathbf{r}_0^{(m)}$.

Remark: $Null((\mathbf{A} - \lambda \mathbf{I})^m)$ is called the generalized eigenspace for eigenvalue λ with algebraic multiplicity m, $\dim(Null((\mathbf{A} - \lambda \mathbf{I})^m)) = m$. We skip the proof. (It can be proved by using the Jordan matrix form of $\mathbf{A}$)

And this fact have been proved in many Advanced linear algebra textbooks.

Case: geometric multiplicty q <algebraic multiplicity m

Start from $\mathbf{r}_0^{(1)}$, we have

$$
\begin{aligned}
\mathbf{r}_1^{(1)} &= (\mathbf{A} - \lambda \mathbf{I})\mathbf{r}_0^{(1)} \\
\mathbf{r}_2^{(1)} &= (\mathbf{A} - \lambda \mathbf{I})\mathbf{r}_1^{(1)} \\
\mathbf{r}_3^{(1)} &= (\mathbf{A} - \lambda \mathbf{I})\mathbf{r}_2^{(1)} \\
&\vdots \\
\mathbf{r}_{m-1}^{(1)} &= (\mathbf{A} - \lambda \mathbf{I})\mathbf{r}_{m-2}^{(1)}.
\end{aligned}
$$

We can construct the first solution

$$
\mathbf{y}^{(1)}(t) = \left(\mathbf{r}_0^{(1)} + \mathbf{r}_1^{(1)} t + \mathbf{r}_2^{(1)} \frac{t^2}{2} + \cdots + \mathbf{r}_{m-1}^{(1)} \frac{t^{m-1}}{(m-1)!} \right) e^{\lambda t}
$$

Case: geometric multiplicty q <algebraic multiplicity m

Start from $\mathbf{r}_0^{(2)}$, we have

$$
\begin{aligned}
\mathbf{r}_1^{(2)} &= (\mathbf{A} - \lambda \mathbf{I})\mathbf{r}_0^{(2)} \\
\mathbf{r}_2^{(2)} &= (\mathbf{A} - \lambda \mathbf{I})\mathbf{r}_1^{(2)} \\
\mathbf{r}_3^{(2)} &= (\mathbf{A} - \lambda \mathbf{I})\mathbf{r}_2^{(2)} \\
&\vdots \\
\mathbf{r}_{m-1}^{(2)} &= (\mathbf{A} - \lambda \mathbf{I})\mathbf{r}_{m-2}^{(2)}.
\end{aligned}
$$

We can construct the second solution

$$
\mathbf{y}^{(2)}(t) = \left(\mathbf{r}_0^{(2)} + \mathbf{r}_1^{(2)} t + \mathbf{r}_2^{(2)} \frac{t^2}{2} + \cdots + \mathbf{r}_{m-1}^{(2)} \frac{t^{m-1}}{(m-1)!} \right) e^{\lambda t}
$$

Case: geometric multiplicty q <algebraic multiplicity m

The process continue until we start from $\mathbf{r}_0^{(m)}$, then we have

$$\begin{aligned}
\mathbf{r}_1^{(m)} &= (\mathbf{A} - \lambda \mathbf{I})\mathbf{r}_0^{(m)} \\
\mathbf{r}_2^{(m)} &= (\mathbf{A} - \lambda \mathbf{I})\mathbf{r}_1^{(m)} \\
\mathbf{r}_3^{(m)} &= (\mathbf{A} - \lambda \mathbf{I})\mathbf{r}_2^{(m)} \\
&\vdots \\
\mathbf{r}_{m-1}^{(m)} &= (\mathbf{A} - \lambda \mathbf{I})\mathbf{r}_{m-2}^{(m)}.
\end{aligned}$$

We can construct the mth solution

$$\mathbf{y}^{(m)}(t) = \left(\mathbf{r}_0^{(m)} + \mathbf{r}_1^{(m)} t + \mathbf{r}_2^{(m)} \frac{t^2}{2} + \cdots + \mathbf{r}_{m-1}^{(m)} \frac{t^{m-1}}{(m-1)!}\right) e^{\lambda t}$$

Case: geometric multiplicty q <algebraic multiplicity m

Fact

$$\mathbf{y}^{(1)}(t), \cdots, \mathbf{y}^{(m)}(t)$$

are m linearly independent solutions corresponding to the eigenvalue λ.

Proof. Suppose

$$\alpha_1 \left(\mathbf{r}_0^{(1)} + \mathbf{r}_1^{(1)} t + \cdots + \mathbf{r}_{m-1}^{(1)} \frac{t^{m-1}}{(m-1)!} \right) e^{\lambda t} + \cdots + \alpha_m \left(\mathbf{r}_0^{(m)} + \mathbf{r}_1^{(m)} t + \cdots + \mathbf{r}_{m-1}^{(m)} \frac{t^{m-1}}{(m-1)!} \right) e^{\lambda t} = \mathbf{0}$$

Then $\alpha_1 \mathbf{r}_0^{(1)} + \cdots + \alpha_1 \mathbf{r}_0^{(m)} = \mathbf{0}$. Thus $\alpha_1 = \cdots = \alpha_n = 0$ since $\mathbf{r}_0^{(1)}, \cdots, \mathbf{r}_0^{(m)}$ are linearly independent.

In the following, we will use several examples to illustrate this fact.

3×3 matrix with geo. mult.=1< alg. mult. $= 3$

Example 12.4

For

$$\mathbf{y}' = \mathbf{A}\mathbf{y}, \quad \mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 2 \end{pmatrix}$$

the characteristic polynomial is

$$P_{\mathbf{A}}(\lambda) = \begin{vmatrix} \lambda - 1 & -1 & 0 \\ 0 & \lambda & -1 \\ 0 & -1 & \lambda - 2 \end{vmatrix} = (\lambda - 1)^3 \Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 1.$$

Furthermore,

$$\mathbf{A} - \mathbf{I} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix}, \; (\mathbf{A} - \mathbf{I})^2 = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; (\mathbf{A} - \mathbf{I})^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Note: $\dim(Null(\mathbf{A} - \mathbf{I})) = 1$.

3×3 matrix with geo. mult.$=1<$ alg. mult. $=3$

Example 12.4

The system

$$(\mathbf{A}-\mathbf{I})^3\mathbf{r}_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\mathbf{r}_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

has three linearly independent solutions:

$$\mathbf{r}_0^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{r}_0^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{r}_0^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Start from $\mathbf{r}_0^{(1)}$, we have

$$\mathbf{r}_1^{(1)} = (\mathbf{A}-\mathbf{I})\,\mathbf{r}_0^{(1)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\mathbf{r}_2^{(1)} = (\mathbf{A}-\mathbf{I})\,\mathbf{r}_1^{(1)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

3×3 matrix with geo. mult.=1< alg. mult. $= 3$

Example 12.4

The first solution is

$$\mathbf{y}_1(t) = e^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Start from $\mathbf{r}_0^{(2)}$, we have

$$\mathbf{r}_1^{(2)} = (\mathbf{A} - \mathbf{I})\,\mathbf{r}_0^{(2)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix},$$

$$\mathbf{r}_2^{(2)} = (\mathbf{A} - \mathbf{I})\,\mathbf{r}_1^{(2)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}.$$

The second solution is

$$\mathbf{y}_2(t) = e^t \left[\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} + \frac{t^2}{2} \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \right] = \begin{pmatrix} -\frac{t^2}{2} + t \\ 1 - t \\ -t \end{pmatrix} e^t.$$

3×3 matrix with geo. mult.=1< alg. mult. $= 3$

Example 12.4

Start from $\mathbf{r}_0^{(3)}$, we have

$$\mathbf{r}_1^{(3)} = (\mathbf{A} - \mathbf{I})\,\mathbf{r}_0^{(3)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},$$

$$\mathbf{r}_2^{(3)} = (\mathbf{A} - \mathbf{I})\,\mathbf{r}_1^{(3)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

The third solution is

$$\mathbf{y}_3(t) = e^t \left[\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \frac{t^2}{2} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right] = \begin{pmatrix} \frac{t^2}{2} \\ t \\ 1 + t \end{pmatrix} e^t.$$

3×3 matrix with geo. mult.$=1<$ alg. mult. $=3$

Example 12.4

The Wronskian is

$$W(\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3)[t] = \begin{vmatrix} 1 & t - \frac{t^2}{2} & \frac{t^2}{2} \\ 0 & 1-t & t \\ 0 & -t & 1+t \end{vmatrix} e^{3t} = e^{3t} \neq 0$$

$\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ is a fundamental set of solutions. The general solution is

$$\mathbf{y}(t) = c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + c_3\mathbf{y}_3 = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^t + c_2 \begin{pmatrix} -\frac{t^2}{2} + t \\ 1-t \\ -t \end{pmatrix} e^t + c_3 \begin{pmatrix} \frac{t^2}{2} \\ t \\ 1+t \end{pmatrix} e^t.$$

3×3 matrix with geo. mult.=2< alg. mult. $= 3$

Example 12.5

For

$$\mathbf{y}' = \mathbf{A}\mathbf{y}, \quad \mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

the characteristic polynomial is

$$P_{\mathbf{A}}(\lambda) = \begin{vmatrix} \lambda - 1 & -1 & 0 \\ 0 & \lambda & -1 \\ 0 & 0 & \lambda - 1 \end{vmatrix} = (\lambda - 1)^3 \Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 1.$$

Furthermore,

$$\mathbf{A} - \mathbf{I} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ (\mathbf{A} - \mathbf{I})^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ (\mathbf{A} - \mathbf{I})^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Note: $\dim(Null(\mathbf{A} - \mathbf{I})) = 2$.

3×3 matrix with geo. mult.=2< alg. mult. $= 3$

Example 12.5

The system

$$(\mathbf{A}-\mathbf{I})^3\,\mathbf{r}_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathbf{r}_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

has three linearly independent solutions:

$$\mathbf{r}_0^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{r}_0^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{r}_0^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Start from $\mathbf{r}_0^{(1)}$, we have

$$\mathbf{r}_1^{(1)} = (\mathbf{A}-\mathbf{I})\,\mathbf{r}_0^{(1)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\mathbf{r}_2^{(1)} = (\mathbf{A}-\mathbf{I})\,\mathbf{r}_1^{(1)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

3×3 matrix with geo. mult.=2< alg. mult. $= 3$

Example 12.5

The first solution is

$$\mathbf{y}_1(t) = e^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Start from $\mathbf{r}_0^{(2)}$, we have

$$\mathbf{r}_1^{(2)} = (\mathbf{A} - \mathbf{I})\,\mathbf{r}_0^{(2)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\mathbf{r}_2^{(2)} = (\mathbf{A} - \mathbf{I})\,\mathbf{r}_1^{(2)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The second solution is

$$\mathbf{y}_2(t) = e^t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

3×3 matrix with alg. mult. $= 3$

Example 12.5

Start from $\mathbf{r}_0^{(3)}$, we have

$$\mathbf{r}_1^{(3)} = (\mathbf{A} - \mathbf{I})\,\mathbf{r}_0^{(3)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

$$\mathbf{r}_2^{(3)} = (\mathbf{A} - \mathbf{I})\,\mathbf{r}_1^{(3)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The third solution is

$$\mathbf{y}_3(t) = e^t \left[\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right] = \begin{pmatrix} t \\ 0 \\ 1 \end{pmatrix} e^t.$$

3×3 matrix with alg. mult. $= 3$

Example 12.5

The Wronskian is

$$W(\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3)[t] = \begin{vmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} e^{3t} = e^{3t} \neq 0$$

$\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ is a fundamental set of solutions. The general solution is

$$\mathbf{y}(t) = c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + c_3\mathbf{y}_3 = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^t + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^t + c_3 \begin{pmatrix} t \\ 0 \\ 1 \end{pmatrix} e^t.$$

3×3 matrix with two distinct eigenvalues

Example 12.6

For

$$\mathbf{y}' = \mathbf{A}\mathbf{y}, \quad \mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 2 & 1 \\ -1 & 0 & 2 \end{pmatrix}$$

the characteristic polynomial is

$$P_{\mathbf{A}}(\lambda) = \begin{vmatrix} \lambda - 1 & 0 & 0 \\ 2 & \lambda - 2 & -1 \\ 1 & 0 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 2)^2 \Rightarrow \lambda_1 = 1, \lambda_2 = \lambda_3 = 2.$$

For $\lambda_1 = 1$

$$\mathbf{A} - \lambda_1 \mathbf{I} = \begin{pmatrix} 0 & 0 & 0 \\ -2 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix}$$

Choose the eigenvector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. We get one solution $\mathbf{y}_1(t) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^t$.

3×3 matrix with two distinct eigenvalues

Example 12.6

For $\lambda_1 = 2$

$$\mathbf{A} - \lambda_2 \mathbf{I} = \begin{pmatrix} -1 & 0 & 0 \\ -2 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}, \quad (\mathbf{A} - \lambda_2 \mathbf{I})^2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

The system

$$(\mathbf{A} - \lambda_2 \mathbf{I})^2 \mathbf{r}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \mathbf{r}_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

has two linearly independent solutions:

$$\mathbf{r}_0^{(1)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{r}_0^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Start from $\mathbf{r}_0^{(1)}$, we have

$$\mathbf{r}_1^{(1)} = (\mathbf{A} - \lambda_2 \mathbf{I})\, \mathbf{r}_0^{(1)} = \begin{pmatrix} -1 & 0 & 0 \\ -2 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

3×3 matrix with two distinct eigenvalues

Example 12.6

We can have one solution corresponding to λ_2:

$$\mathbf{y}_2(t) = e^{2t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Start from $\mathbf{r}_0^{(2)}$, we have

$$\mathbf{r}_1^{(2)} = (\mathbf{A} - \lambda_2 \mathbf{I})\, \mathbf{r}_0^{(2)} = \begin{pmatrix} -1 & 0 & 0 \\ -2 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

Another solution corresponding to λ_2 is

$$\mathbf{y}_3(t) = e^{2t} \left[\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right] = e^{2t} \begin{pmatrix} 0 \\ t \\ 1 \end{pmatrix}.$$

3×3 matrix with two distinct eigenvalues

Example 12.6

The Wronskian is

$$W(\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3)[t] = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & t \\ 1 & 0 & 1 \end{vmatrix} e^{5t} = e^{5t} \neq 0$$

$\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ is a fundamental set of solutions. The general solution is

$$\mathbf{y}(t) = c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + c_3\mathbf{y}_3 = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{2t} + c_3 \begin{pmatrix} 0 \\ t \\ 1 \end{pmatrix} e^{2t}.$$