# MAT2006: Elementary Real Analysis
# Assignment #1

Reference Solution

**1.** Given an $n \in \mathbb{N}$ being not a square number, (i.e., $n \neq 1, 4, 9, 25, \ldots$). Show that $\sqrt{n}$ is irrational.
**Hint.** The Fundamental Theorem of Arithmetic says that any natural number has a unique factorization

$$n = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k},$$

where $p_1, p_2, \ldots, p_k$ are prime numbers and $n_1, n_2, \ldots, n_k$ are natural numbers.

*Proof.* Suppose, by contradiction, that $n = (p/q)^2$, where $p, q \in \mathbb{N}$ and have no common factor. By the Fundamental Theorem of Arithmetic, we have

$$n = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k},$$

where $p_1, p_2, \ldots, p_k$ are prime numbers and $n_1, n_2, \ldots, n_k$ are natural numbers. Then

$$p^2 = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k} q^2,$$

which implies that $p^2$ is divisible by $p_1, p_2, \cdots p_k$, and so is $p$. Since $p$ and $q$ have no common factor, $q$ is not divisible by $p_1, p_2, \cdots p_k$, and $n_1, n_2, \ldots, n_k$ are all even numbers, which is a contradiction with the hypothesis that $n$ is not a square number. $\square$

**2.** Given any rational number $r$ and irrational number $i$, why $r + i$ and $ri$ are irrational? (For the latter one, we also assume $r \neq 0$.)

*Proof.* The set of rational numbers $\mathbb{R}$ is a field, which is closed under addition and multiplication.

If $(r + i)$ and $r$ are rational, so are $-r$ and $(r + i) + (-r) = i$. Thus, $r + i$ is irrational.

If $r \neq 0$ is rational, $1/r$ is rational, Suppose $ri$ is rational, then $i = ri * (1/r)$ is rational, which is a contradiction with the hypothesis $i$ is irrational. Thus $ri$ is irrational provided that $r \neq 0$. $\square$

**3.** The *direct (Cartesian) product* of two sets $X$ and $Y$ is the set

$$X \times Y = \{(x, y) \mid x \in X \text{ and } y \in Y\}.$$

A *relation* between $X$ and $Y$ is a subset $R$ of $X \times Y$, and $x$ and $y$ are said to be $R$-related if $(x, y) \in R$.

(i) Show that a function $f : X \to Y$ can be regarded as a relation. (**Note.** the corresponding $R$ is called the graph of $f$).

(ii) Show that the order $\leq$ of real numbers is a relation, and illustrate $R$ in the $\mathbb{R}^2$ plane.

(iii) Show that the "is a subset of" (or, "is contained in") $\subset$ among all subsets of a set $M$ is a relation.

*Proof.* (i) Let

$$R = \{(x, y) \in X \times Y \,|\, y = f(x), \quad \forall x \in X\}.$$

Then $R \subset X \times Y$, and the function $f$ is characterized by $y = f(x)$ if and only if $(x, y) \in R$, that is if and only if $x$ and $y$ are $R$-related.

(ii) Let

$$R = \{(x, y) \in \mathbb{R}^2 \,|\, x \leq y\}.$$

Then $R \subset \mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, and the inequality $\leq$ is characterized by $x \leq y$ if and only if $(x, y) \in R$, that is if and only if $x$ and $y$ are $R$-related.

(iii) Given a nonempty set $M$, let

$$R = \{(A, B) \,|\, A \subset B \subset M\}.$$

Then $R \subset P(M) \times P(M)$, and the inclusion $\subset$ is characterized by $A \subset B$ if and only if $(A, B) \in R$, that is if and only if $A$ and $B$ are $R$-related. $\square$

**4.** (i) Show that the relation in part (iii) of the last problem '$\subset$' is not a total order but a partial order. (Assume $M$ has at least two elements.)

(ii) Given two points $(x_1, y_1)$ and $(x_2, y_2)$ on the $\mathbb{R}^2$ plane, define a relation $\preceq$ by the following

$$(x_1, y_1) \preceq (x_2, y_2) \qquad \text{if and only if} \quad x_1 < x_2 \quad \text{or} \quad (x_1 = x_2,\, y_1 \leq y_2).$$

Show that $\preceq$ is a total order.

(iii) Given two points $(x_1, y_1)$ and $(x_2, y_2)$ on the $\mathbb{R}^2$ plane, define a relation $\prec$ by the following

$$(x_1, y_1) \prec (x_2, y_2) \qquad \text{if and only if} \quad x_1 \leq x_2 \quad \text{and} \quad y_1 \leq y_2.$$

Show that $\prec$ is a partial order.

*Proof.* (i) Let $A, B$ are both subsets of $M$. Then clearly, we have

(1) $A \subset A$;

(2) If $A \subset B$ and $B \subset A$ then $A = B$;

(3) If $A \subset B$ and $B \subset C$ then $A \subset C$.

Thus '$\subset$' is a partial order on the power set $P(M)$.

Choose $x_1, x_2 \in M$ and $x_1 \neq x_2$, we have $\{x_1\} \not\subset \{x_2\}$ and $\{x_2\} \not\subset \{x_1\}$, thus '$\subset$' is not a total order on the power set $P(M)$.

(ii) Given two points $(x_1, y_1)$ and $(x_2, y_2)$ on the $\mathbb{R}^2$ plane, we have

(1) $(x_1, y_1) \preceq (x_1, y_1)$ since $x_1 = x_1$ and $y_1 \leq y_1$.

(2) Assume $(x_1, y_1) \preceq (x_2, y_2)$ and $(x_2, y_2) \preceq (x_1, y_1)$. Then we have $x_1 \leq x_2$ and $x_2 \leq x_1$, and so $x_1 = x_2$. Moreover, we have $y_1 \leq y_2$ and $y_2 \leq y_1$, and so $y_1 = y_2$. Thus we get $(x_1, y_1) = (x_2, y_2)$.

(3) Assume $(x_1, y_1) \preceq (x_2, y_2)$ and $(x_2, y_2) \preceq (x_3, y_3)$. The former relation implies that $x_1 \leq x_2$ while the latter implies that $x_2 \leq x_3$. Thus, $x_1 \leq x_3$. If $x_1 < x_3$, then we have $(x_1, y_1) \preceq (x_3, y_3)$. If $x_1 = x_3$, we must have $x_1 = x_2 = x_3$, and in this case it also holds $y_1 \leq y_2$ and $y_2 \leq y_3$ and hence $y_1 \leq y_3$, which again implies that $(x_1, y_1) \preceq (x_3, y_3)$.

(4) Given $(x_1, y_1)$ and $(x_2, y_2)$, we have three cases $x_1 < x_2$, $x_2 \leq x_1$ or $x_1 = x_2$. If $x_1 < x_2$, it follows that $(x_1, y_1) \preceq (x_2, y_2)$. If $x_2 < x_1$, it follows that $(x_2, y_2) \preceq (x_1, y_1)$. For the third case $x_1 = x_2$, we divided it into two subcases $y_1 \leq y_2$ which implies $(x_1, y_1) \preceq (x_2, y_2)$; and $y_1 > y_2$ which implies $(x_2, y_2) \preceq (x_1, y_1)$. Thus, for all cases, we have either $(x_1, y_1) \preceq (x_2, y_2)$ or $(x_2, y_2) \preceq (x_1, y_1)$.

Hence, $\preceq$ is a total order on $\mathbb{R}^2$.

(iii) Given two points $(x_1, y_1)$ and $(x_2, y_2)$ on the $\mathbb{R}^2$ plane, we have

(1) $(x_1, y_1) \prec (x_1, y_1)$;

(2) Assume $(x_1, y_1) \prec (x_2, y_2)$ and $(x_2, y_2) \prec (x_1, y_1)$. Then we have $x_1 \leq x_2$, $y_1 \leq y_2$, $x_2 \leq x_1$ and $y_1 \leq y_2$, which implies that $(x_1, x_2) = (y_1, y_2)$.

(3) Assume $(x_1, y_1) \prec (x_2, y_2)$ and $(x_2, y_2) \prec (x_3, y_3)$. It follows from the former that $x_1 \leq x_2$, $y_1 \leq y_2$ and from the latter that $x_2 \leq x_3$ and $y_2 \leq y_3$. Hence, $x_1 \leq x_3$ and $y_1 \leq y_3$, which implies that $(x_1, y_1) \prec (x_3, y_3)$.

Therefore '$\prec$' is a partial order on $\mathbb{R}^2$. It is not a total order since neither $(0, 1) \prec (1, 0)$ nor $(1, 0) \prec (0, 1)$ holds. □

**5.** Let $A$ and $B$ be nonempty bounded above subsets of $\mathbb{R}$, and let $A + B$ be the set of all sums $a + b$ where $a \in A$ and $b \in B$. Show that $\sup(A + B) = \sup A + \sup B$.

*Proof.* For every $x \in A + B$, there exists $a \in A$ and $b \in B$ such that $x = a + b$. Since $\sup A$ is an upper bound of $A$, we have $a \leq \sup A$. Similarly, $b \leq \sup B$. Thus $x = a + b \leq \sup A + \sup B$, which indicates that $\sup A + \sup B$ is an upper bound of $A + B$.

If $M$ is an upper bound of $A + B$, then $a + b \leq M$ whenever $a \in A$ and $b \in B$. Hence $a \leq M - b$, which means $M - b$ is an upper bound of $A$, thus $\sup A \leq M - b$ for all $b \in B$, or equivalently, $b \leq M - \sup A$ for all $b \in B$, which implies that $M - \sup A$ is an upper bound of $B$, and hence $\sup B \leq M - \sup A$. That is $\sup A + \sup B \leq M$. Thus $\sup A + \sup B$ is the least upper bound of $A + B$, or, $\sup(A + B) = \sup A + \sup B$.

□

**6.** Find the sup, inf, max and min for the following sets

$$(a)\quad A=\left\{\frac{1}{n}\,|\,n\in\mathbb{N}\right\};\qquad (b)\quad B=\left\{1-\frac{1}{n}\,|\,n\in\mathbb{N}\right\}.$$

*Solution.*

$$(a)\qquad \sup A=\max A=1,\qquad \inf A=0,\qquad \min A\quad \text{does not exist.}$$

$$(b)\qquad \inf B=\min B=0,\qquad \sup B=1,\qquad \max B\quad \text{does not exist.}$$

□

**7.** If $A\subset B$ and $B$ is countable, then $A$ is either finite or countable.
**Hint.** Assume $A$ is infinite, find a 1–1 and onto function $f$ from $\mathbb{N}$ to $A$. Assume $g$ is a 1–1 and onto from $\mathbb{N}$ to $B$. Let $n_1=\min\{n\in\mathbb{N}\,|\,g(n)\in A\}$ and set $f(1)=g(n_1)$. Continue this construction of $f$.

*Proof.* Assume $A$ is infinite, find a 1–1 and onto function $f$ from $\mathbb{N}$ to $A$. Assume $g$ is a 1–1 and onto from $\mathbb{N}$ to $B$. Let $n_1=\min\{n\in\mathbb{N}\,|\,g(n)\in A\}$ and set $f(1)=g(n_1)$. Let $n_2=\min\{n\in\mathbb{N}\,|\,n>n_1 \text{ and } g(n)\in A\}$ and set $f(2)=g(n_2)$. In general, after choosing $n_k$ and setting $f(k)=g(n_k)$, we may let $n_{k+1}=\min\{n\in\mathbb{N}\,|\,n>n_k \text{ and } g(n)\in A\}$ and set $f(k+1)=g(n_{k+1})$. By induction, $f$ is a function from $\mathbb{N}$ to $A$.

If $p,q\in\mathbb{N}$ with $p\neq q$, assuming without loss of generality $p<q$, then $n_p<n_q$ and $g(n_p)\neq g(n_q)$ since $g$ is 1–1, and hence $f(p)\neq f(q)$. This means that $f$ is 1–1.

To show that $f$ is onto, let a $a\in A$ be arbitrary. Because $g$ is onto, $a=g(n')$ for some $n'\in\mathbb{N}$. This means $n'\in\{n\,|\,g(n)\in A\}$ and as we inductively remove the minimal element, $n'$ must eventually be the minimum by at least the $n'-1$st step. □

**8.** (i) If $A_1,A_2,\ldots,A_m$ are each countable sets, then the union $A_1\cup A_2\cup\cdots\cup A_m$ is countable. [The union of finite many of countable sets is countable.]
**Hint.** Use the induction argument. For the case $A_1\cup A_2$, let $B=A_2\setminus A_1$, then $A_1$ and $B$ are disjoint and $A_1\cup B=A_1\cup A_2$.

(ii) If $A_n$ is countable for each $n\in\mathbb{N}$, then $\bigcup_{n=1}^{\infty}A_n$ is countable. [The union of countable many of countable sets is countable.]
**Hint.** The induction argument doesnot apply here, (why?)

(iii) Show that the set of lattice points $\mathbb{N}^2=\mathbb{N}\times\mathbb{N}$ is countable.

*Proof.* (i) We fist show that $A_1\cup A_2$ is countable provided that $A_1$ and $A_2$ are both countable.

Because $A_1$ is countable, there exists a 1–1 and onto function $f:\mathbb{N}\to A$.

Set $B_2=A_2\setminus A_1$, then $A_1\cup A_2=A_1\cup B_2$.

If $B_2=\emptyset$, then $A_1\cup A_2=A_1$ which is countable by the hypothesis.

If $B_2=\{b_1,b_2,\cdots,b_m\}$ has $m$ elements, then define $h:\mathbb{N}\to A\cup B_2$ by

$$h(n)=\begin{cases} b_n, & \text{if } 1\le n\le m,\\ f(n-m) & \text{if } n>m.\end{cases}$$

It follows immediately from the fact that $f$ is 1–1 and onto that $h$ is also 1–1 and onto.

If $B_2$ is infinite, then $B_2$, as a subset of countable set $A_2$, must be countable, which is follows from the above Problem. Hence, there exists a 1–1 and onto function $g : \mathbb{N} \to B_2$, and we define a function $h : \mathbb{N} \to A_1 \cup B_2$

$$h(n) = \begin{cases} f((n+1)/2), & \text{if } n \text{ is odd,} \\ g(n/2) & \text{if } n \text{ is even.} \end{cases}$$

Again, the proof that $h$ is 1–1 and onto is derived directly from the fact that $f$ and $g$ are both bijections.

To prove that any finite union of countable sets is countable, we apply the induction argument. Now let's assume that the union of $m$ countable sets is countable, and show that the union of $m+1$ countable sets is countable.

Assume $A_1, A_2, A_3, \ldots$ are countable sets. Note that

$$A_1 \cup A_2 \cup A_3 \cup \cdots A_m \cup A_{m+1} = \big(A_1 \cup A_2 \cup A_3 \cup \cdots A_m\big) \cup A_{m+1}$$

Now, $A_1 \cup A_2 \cup A_3 \cup \cdots A_m$ is countable, and the above union is a union of two countable sets, which are countable by the previous step. This completes the proof.

(ii) Notice here the induction can not be used when we have infinite number of sets. It can only be used to prove facts that hold true for each value of $n \in \mathbb{N}$. We need another approach.

Let's first consider the case where the sets $\{A_n\}$ are disjoint. In order to achieve 1–1 correspondence between the set $\mathbb{N}$ and $\bigcup_{n=1}^{\infty} A_n$, we first label the elements in each countable set $A_n$ as

$$A_n = \{a_{n1}, a_{n2}, a_{n3}, \ldots\}.$$

Now arrange the elements of $\bigcup_{n=1}^{\infty} A_n$ in a two dimensional array

$$\begin{array}{llllll} A_1: & a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & \cdots \\ A_2: & a_{21} & a_{22} & a_{23} & a_{24} & \cdots \\ A_3: & a_{31} & a_{32} & a_{33} & \cdots \\ A_4: & a_{41} & a_{42} & \cdots \\ A_5: & a_{51} & \cdots \\ \vdots \end{array}$$

Arrange the elements of $\mathbb{N}$ in a similar two dimensional array

$$\begin{array}{lllll} 1 & 3 & 6 & 10 & \cdots \\ 2 & 5 & 9 & \cdots \\ 4 & 8 & \cdots \\ 7 & \cdots \\ \vdots \end{array}$$

This establishes a 1–1 and onto mapping $g : \mathbb{N} \to \bigcup_{n=1}^{\infty} A_n$ where $g(n)$ corresponds to the element $a_{ij}$ where $(i,j)$ is the row and column location of $n$ in the array for $\mathbb{N}$ given above.

If the sets $\{A_n\}$ are not disjoint then our mapping may not be 1–1. In this case we could again replace $A_n$ with $B_n = A_n \setminus (A_1 \cup A_2 \cup \cdots \cup A_{n-1})$.

(iii) Note that

$$\mathbb{N}^2 = \bigcup_{n=1}^{\infty} \{(m,n) \,|\, m \in \mathbb{N}\}$$

is a countable union of countable sets, and hence it is countable according to part (ii). $\square$

**9.** In the lecture notes, a sketch proof of the Schröder–Bernstein Theorem is given. Complete the proof with details.

*Proof.* Set $A_1 = X \setminus g(Y) = \{x \in X \,|\, x \notin g(Y)\}$. If $A_1 = \emptyset$, which means that $g$ is onto, and thus $Y \sim X$.

If $A_1 \neq \emptyset$, we inductively define a sequence of sets by letting $A_{n+1} = g(f(A_n))$. For convenience, we also denote $B_n = f(A_n)$. We shall show that $\{A_n \,|\, n \in \mathbb{N}\}$ is a pairwise disjoint collection of subsets of $X$, while $\{B_n \,|\, n \in \mathbb{N}\}$ is a similar collection in $Y$.

First, by the definition of $A_1$, if $x \in A_1$, then $x$ is not a image of any $y \in Y$, and thus $x \notin g(f(A_n)) = A_{n+1}$ for all $n \in \mathbb{N}$, that is $A_1$ and $A_{n+1}$ are disjoint for all $n \in \mathbb{N}$. In particular, $A_1 \cap A_2 = \emptyset$, thus $f(A_1) \cap f(A_2) = \emptyset$ because $f$ is 1–1. That is $B_1 \cap B_2 = \emptyset$.

To apply the induction argument, assume $A_1, A_2, \ldots, A_m$ are pairwise disjoint, and $A_1, A_2, \ldots, A_m$ are pairwise disjoint. Then $g(B_m) \cap g(B_n) =$ for all $n = 1, 2, \ldots, m-1$ due to the fact that $g$ is 1–1, which means that $A_{m+1}$ is disjoint with $A_2, A_3, \cdots, A_n$. Combining with the fact that $A_1$ is disjoint with $A_{m+1}$ and the induction hypothesis, we have that $A_1, A_2, \ldots, A_m, A_{m+1}$ are pairwise disjoint. And it then follows that $f(A_{m+1} \cap f(A_n) = \emptyset$ with $n = 1, 2, \ldots, m$ by using the fact $f$ is 1–1. Thus $B_1, B_2, \ldots, B_m, B_{m+1}$ are pair wise disjoint, which completes the proof of this part.

(b) Let $A = \bigcup_{n=1}^{\infty} A_n$ and $B = \bigcup_{n=1}^{\infty} B_n$. We shall show that $f$ maps $A$ onto $B$.

We first show that $f$ is indeed a function from $A$ to $B$. Since $f$ is a function from $X$ to $Y$, for any $x \in A = \bigcup A_n$, we have $x \in A_n$ for some $n \in \mathbb{N}$, and so that $f(x) \in f(A_n) = B_n \subset B$. Thus, $f$ is a function from $A$ to $B$.

To show that $f$ maps $A$ onto $B$ is to verify that for any $y \in B$ there exists $x \in A$ such that $y = f(x)$. Given any $y \in B = \bigcup_{n=1}^{\infty} B_n$, these exists $n \in \mathbb{N}$ such that $y \in B_n = f(A_n)$, which implies that there exists $x \in A_n \subset A$ such that $f(x) = y$. Thus $f$ is a function from $A$ onto $B$.

(c) Let $A' = X \setminus A$ and $B' = Y \setminus B$, we shall show that $g$ is a function from $B'$ onto $A'$.

We first show that $g$ is indeed a function from $B'$ to $A'$. For any $y \in B' \subset Y$, there exists $x \in X$ such that $x = g(y)$ since $g$ is a function from $Y$ to $X$. We claim that $x \in A'$. Suppose, for contradiction, $x \notin A'$, that is $x \in A = \bigcup_{n=1}^{\infty} A_n$. Note that $x \notin A_1 = Y \setminus g(Y)$ since $x = g(y)$ for some $y \in Y$. Thus $x \in A_n = g(B_{n-1})$ for some $n \geq 2$, which implies that there exists $y' \in B_n$ such that $x = g(y')$. But $x = g(y)$ and noting that $g$ is 1–1, we must have $y = y' \in B_n \subset B$, which is a contradiction with the assumption that $y \in B'$. Therefore, we have verified $x = g(y) \in A'$ and thus $g$ is a function from $B'$ to $A'$.

To show that $g$ maps $B'$ onto $A'$, we shall prove by contradiction. Suppose $g : B' \to A'$ is not onto, there exists $x \in A'$ such that $x \notin g(B')$. Since $x \in A'$, then $x \notin A_1 = X \setminus g(Y)$, and there exists $y \in Y$ such that $x = g(y)$. Combining this with the hypothesis $x \notin g(B')$, we must have $y \in B = \bigcup_{n=1}^{\infty} B_n$. Hence $y \in B_n$ for some $n \in \mathbb{N}$, which further implies that $x = g(y) \in g(B_n) = A_{n+1} \subset A$, a contradiction with the hypothesis $x \in A'$. Thus, $g: B' \to A'$ is onto.

(d) Since $g: B' \to A'$ is 1–1 and onto, we can define its inverse $g^{-1}: A' \to B'$ by

$$g^{-1}(x) = y \qquad \text{provided} \qquad x = g(y).$$

$g^{-1}: A' \to B'$ is a 1–1 and onto function follows immediately that $g$ is 1–1 and onto. Now define

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g^{-1}(x) & \text{if } x \in A'. \end{cases}$$

Then $h: A \to B$ is 1–1 and onto follows from (b) and that $g^{-1}: A' \to B'$ is a bijection. Thus, $A$ and $B$ have the same cardinality. □

**10.** Show the following equivalent of two sets (have the same cardinality) using the Schröder–Bernstein Theorem.

$$(a) \quad [0,1] \sim (0,1) \qquad (b) \quad [0,1] \sim [0,1] \times [0,1].$$

**Hint.** For (b), to find a 1–1 function from $[0,1] \times [0,1]$ to $[0,1]$, consider the decimal representation of real numbers.

*Proof.* (a) Let $f: [0,1] \to (0,1)$ be the function $f(x) = \frac{1+x}{3}$. Since $f'(x) = 1/3 > 0$ and $f(x)$ is strictly increasing, which implies that $f(x_1) < f(x_2)$ whenever $x_1 < x_2$, that is $f(x)$ is 1–1.

Let $g: (0,1) \to [0,1]$ be the identity function, $g(x) = x$. Clearly, $g(x)$ is 1–1.

According to the Schröder–Bernstein theorem, $[0,1] \sim (0,1)$.

(b) Let $f: [0,1] \to [0,1] \times [0,1]$ be the function $f(x) = (x, 0)$. Clearly $f(x)$ is a 1–1 function.

To obtain a 1–1 function $g: [0,1] \times [0,1] \to [0,1]$. Write $x \in (0,1]$ by their decimal representation

$$x = 0.x_1x_2x_3x_4\cdots,$$

where $x_n$ are digits taking values from $0, 1, 2, \cdots, 9$. If $x$ is a rational number with finite nonzero digits, we take its infinite representation. For example, we write

$$0.32 = 0.31999999\ldots,$$

and

$$1 = 0.999998\ldots.$$

At the same time, we still write

$$0 = 0.0000000\ldots.$$

Under this rule, the decimal representation of $x \in [0,1]$ is unique.

Then for a point $(x, y) \in [0, 1] \times [0, 1]$ with $y$ taking the same decimal representation

$$y = 0.y_1y_2y_3y_4\cdots$$

we let

$$g(x, y) = 0.x_1y_1x_2y_2x_3y_3\ldots.$$

It is clear that $0 \leq g(x, y) \leq 1$ whenever $0 \leq x, y \leq 1$, thus it is indeed a function from $[0, 1] \times [0, 1]$ to $[0, 1]$. To see it is also 1–1, noting that if $(x, y) \neq (x', y')$, then $x \neq x'$ or $y \neq y'$. Assume $x \neq x'$ without loss of generality. At leat one of the digits of $x_1$ and $x - 1'$ are not equal, thus $g(x, y) \neq g(x', y')$, which implies that $g$ is 1–1.

According to the Schröder–Bernstein theorem, $[0, 1] \sim [0, 1] \times [0, 1]$. □

**11.** Let $S$ be the set consisting of all sequences of 0's and 1's. Observe that $S$ is not a particular sequence, but rather a large set whose elements are sequences; namely,

$$S = \{(a_1, a_2, a_3, \ldots) \mid a_n = 0 \text{ or } 1\}$$

As an example, the sequence $(1, 0, 1, 0, 1, 0, 1, 0, \ldots)$ is an element of $S$, as is the sequence $(1, 1, 1, 1, 1, 1, \ldots)$. Show that $S$ is uncountable.
**Hint.** Consider Cantor's digitalization method.

*Proof.* The proof will have the same structure as that of Cantor's. So let us assume for contradiction that there exists a function $f : \mathbb{N} \to S$ that is 1–1 and onto. The 1–1 correspondence between $\mathbb{N}$ and $S$ can be represented by the following indexed array where

$$\begin{array}{ccccccccccc}
\mathbb{N} & & (0,1) & & & & & & & & \\
\hline
1 & \leftrightarrow & f(1) & = & 0.\boldsymbol{a_{11}} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & \cdots \\
2 & \leftrightarrow & f(2) & = & 0.a_{21} & \boldsymbol{a_{22}} & a_{23} & a_{24} & a_{25} & a_{26} & \cdots \\
3 & \leftrightarrow & f(3) & = & 0.a_{31} & a_{32} & \boldsymbol{a_{33}} & a_{34} & a_{35} & a_{36} & \cdots \\
4 & \leftrightarrow & f(4) & = & 0.a_{41} & a_{42} & a_{43} & \boldsymbol{a_{44}} & a_{45} & a_{46} & \cdots \\
5 & \leftrightarrow & f(5) & = & 0.a_{51} & a_{52} & a_{53} & a_{54} & \boldsymbol{a_{55}} & a_{56} & \cdots \\
6 & \leftrightarrow & f(6) & = & 0.a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & \boldsymbol{a_{66}} & \cdots \\
\vdots & & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

$a_{mn} = 1$ or $0$ for $m, n \in \mathbb{N}$. Now let us define a sequence $\{x_n\} = \{x_1, x_2, x_3, \ldots\} \in S$ via

$$x_n = \begin{cases} 0 & \text{if} \quad a_{nn} = 1 \\ 1 & \text{if} \quad a_{nn} = 0. \end{cases}$$

From this definition we can see that $f(1)$ is not the sequence $\{x_n\}$ because $a_{11}$ is not the same as $x_1$. Similarly, $f(2) \neq \{x_n\}$ since $a_{22} \neq x_2$. In general, $f(m) \neq \{x_n\}$ since $a_{mm} \neq x_m$ for all $m \in \mathbb{N}$. Because $f$ is onto, all sequences in $S$ should be in the range of $f$. However, the specific sequence $\{x_n\}$ we defined above is not equal to $f(m)$ for any $m \in \mathbb{N}$. This contradiction implies that the set $S$ is uncountable. □

**12.** Answer each of the following by establishing a 1–1 correspondence with a set of known cardinality.

(i) Is the set of all functions from $\{0,1\}$ to $\mathbb{N}$ countable or uncountable?

(ii) Is the set of all functions from $\mathbb{N}$ to $\{0,\, 1\}$ countable or uncountable?

*Solution.* (i) The set $A$ of functions from $\{0,1\}$ to $\mathbb{N}$ is countable. To see this, first observe that $A$ can be put into a 1–1 correspondence with the set of ordered pairs $\{(m,n) \mid m,n \in \mathbb{N}\}$. To be precise, if $f \in A$, then $f$ is a function from $\{0,1\}$ to $\mathbb{N}$, and we can match it up with the ordered pair $(m,n)$ where $m = f(0)$ and $n = f(1)$. $\{(m,n) \mid m,n \in \mathbb{N}\} = \mathbb{N}^2$ is countable has been shown in Problem 8 (iii).

(ii) This set is uncountable. A function from $\mathbb{N}$ to $\{0,1\}$ is in fact just a sequence consisting of 0's and 1's, so the set of such functions is precisely the set of binary sequences, which is uncountable as shown in Problem 11. $\square$

— End —