MAT 2006 Tutorial 4 10.12.

1. Proof. $a_{n+1} = \sqrt{2a_n}$, $a_1 = \sqrt{2}$. Apply MCT.

① Prove $0 < a_n \le 2$. $\forall n \in \mathbb{N}$.

when $n=1$, holds

Suppose $n=k$, holds. $0 < a_k \le 2$.

when $n=k+1$, $a_{k+1} = \sqrt{2a_k} > 0$.

$\sqrt{2\sqrt{2}}$ $\sqrt{2\sqrt{2\sqrt{2}}}$ ... $a_{k+1} = \sqrt{2a_k} \le \sqrt{2 \cdot 2} = 2$.

$2^{\frac{1}{2}+\frac{1}{4}}$ $2^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}}$ ... $\Rightarrow 0 < a_n \le 2$. $\forall n \in \mathbb{N}$.

② Prove $\{a_n\}$ is increasing. $a_{n+1} \ge a_n$.

when $n=1$, $a_1 \le a_2$.

Suppose $n=k$ holds. $a_k \le a_{k+1}$

when $n=k+1$, $a_{k+2} = \sqrt{2a_{k+1}} \ge \sqrt{2a_k} = a_{k+1}$

$\Rightarrow a_{n+1} \ge a_n$, $\forall n \in \mathbb{N}$. ($\sqrt{x}$ cts)

By MCT, $\{a_n\}$ cvgs.

Let $a = \lim_{n\to\infty} a_n$, $\lim_{n\to\infty} a_{n+1} = a = \lim_{n\to\infty} \sqrt{2a_n} = \sqrt{2a}$. ✓

$\Rightarrow a^2 = 2a \Rightarrow a(a-2) = 0$ Alternatively.

$\Rightarrow a = 2$ or 0. $a_{n+1}^2 = 2a_n$

$\forall n \in \mathbb{N}$. $a_n \ge a_1 \Rightarrow a_n \ge a_1 = \sqrt{2} > 0$ $\Rightarrow a^2 = 2a$.

$\Rightarrow a = 2$.

2. Proof. (i). $\limsup_{n\to\infty} a_n = \lim_{m\to\infty} \sup_{n\ge m} a_n$.

① W.T.S. $\lim_{m\to\infty} \sup_{n\ge m} (a_n + b_n) \le \lim_{m\to\infty} \sup_{n\ge m} a_n + \lim_{m\to\infty} \sup_{n\ge m} b_n$.

② It suffices to show $\sup_{n\ge m} (a_n + b_n) \le \sup_{n\ge m} a_n + \sup_{n\ge m} b_n$. $\forall m \in \mathbb{N}$.

Only need to show $a_n + b_n \le \sup_{n\ge m} a_n + \sup_{n\ge m} b_n$ $\forall n \ge m$.

✓

Show $\forall n \ge m$, $a_n \le \sup_{n\ge m} a_n$, $b_n \le \sup_{n\ge m} b_n$

$\Rightarrow a_n + b_n \le \sup_{n \ge m} a_n + \sup_{n \ge m} b_n \quad \forall n \ge m.$

$\Rightarrow$ ② $\cdots$ $\Rightarrow$ ① $\cdots$

(ii). $a_n = (-1)^n,\ b_n = (-1)^{n+1}. \Rightarrow a_n + b_n \equiv 0.$

$\Rightarrow \limsup_{n\to\infty} (a_n + b_n) = 0.$

$\limsup_{n\to\infty} a_n = \limsup_{n\to\infty} b_n = 1.$

3. proof. BW $\Rightarrow$ MCT. Suppose $\{x_n\}$ is bdd and monotone.

W.T.S. $\{x_n\}$ cvg.

BW. $\Rightarrow$ $\{x_n\}$ contains a convergent subseq. $\{x_{n_k}\}$

W.L.O.G. Assume $\{x_n\}$ is increasing.

(If $\{x_n\}$ is decreasing, apply the result to $\{-x_n\}$.)

Suppose $x = \lim_{k\to\infty} x_{n_k}$. $\forall \varepsilon > 0. \Rightarrow M \in \mathbb{N}$. s.t

$\forall k \ge M. \quad x - \varepsilon < x_{n_k} < x + \varepsilon. \quad m \le n_m$

when $m \ge n_M$. $x - \varepsilon < x_{n_M} \le x_m \le x_{n_m} < x + \varepsilon.$

$\Rightarrow |x_m - x| < \varepsilon.$

$\Rightarrow x_m \to x$ as $m \to \infty$.

4. proof.¹ Apply MCT $\forall n \in \mathbb{N}$. $y_{n+1} - y_n = \frac{p_{n+1}}{10^{n+1}} \ge 0$

$\Rightarrow y_{n+1} \ge y_n \Rightarrow \{y_n\} \nearrow.$

$p_0 \le y_n \le p_0 + \frac{9}{10} + \frac{9}{10^2} + \cdots + \frac{9}{10^n}.$

$\le p_0 + (\frac{9}{10} + \cdots + \frac{9}{10^n} + \cdots) = p_0 + 1.$

$\Rightarrow p_0 \le y_n \le p_0 + 1. \Rightarrow \{y_n\}$ bdd

$\{y_n\}$ is cvg

proof 2. $|y_m - y_n| = \frac{p_{m+1}}{10^{m+1}} + \cdots + \frac{p_n}{10^n}.$

$(m<n) \quad \le \frac{9}{10^{m+1}} + \cdots + \frac{9}{10^n} \le \frac{9}{10^{m+1}} + \cdots + \frac{9}{10^n} + \cdots = \frac{1}{10^m}$

$\forall \varepsilon > 0.\ \exists M \in \mathbb{N}.\ \text{s.t. } \frac{1}{10^M} < \varepsilon.$

$(m<n).\quad \forall n, m \geq M.\ |y_m - y_n| \leq \frac{1}{10^m} \leq \frac{1}{10^M} < \varepsilon.$

5. Proof. continued fraction ✓. $a_{n+1} = 2 + \frac{1}{a_n},\ a_1 = 2.$ $\left(= \sqrt{2}+1\right)$ ✗

Let $x = \lim\limits_{n\to\infty} a_n.$ $x = 2 + \frac{1}{x} \Rightarrow x = \sqrt{2}+1$

$x_1 \quad x_3 \quad x \quad x_2$

$2. \quad 2.4 \quad \sqrt{2}+1 \quad 2.5.$

Hypothesis: $\{x_{2n-1}\} \nearrow \sqrt{2}+1.$
$\{x_{2n}\} \searrow \sqrt{2}+1.$

$\{x_{2n-1}\} \rightarrow \approx 2.414 \leftarrow \{x_{2n}\}.$

Strategy: ① use induction to show $1 < x_n < 3.\ \forall n \in \mathbb{N}.$
② Use induction to show $\left.\begin{array}{l} x_{2n-1} < x_{2n+1}. \\ x_{2n} > x_{2n+2} \end{array}\right\} n \in \mathbb{N}.$

when $n=1$: $x_1 < x_3,\ x_2 > x_4.$

Suppose $n=k$, $x_{2k-1} < x_{2k+1},\ x_{2k} > x_{2k+2}.$

$x_{2k+1} = 2 + \frac{1}{x_{2k}}$
$x_{2k+3} = 2 + \frac{1}{x_{2k+2}}.$ $\Rightarrow x_{2k+1} < x_{2k+3}.$

$x_{2k+2} = 2 + \frac{1}{x_{2k+1}}$
$x_{2k+4} = 2 + \frac{1}{x_{2k+3}}$ $\Rightarrow x_{2k+2} > x_{2k+4}.$

③ $\{x_{2n-1}\} \rightarrow x.\ \{x_{2n}\} \rightarrow x'.$

$$\begin{cases} x_{2n+1} = 2 + \frac{1}{x_{2n}} \\ x_{2n+2} = 2 + \frac{1}{x_{2n+1}} \end{cases} \xRightarrow{\lim\limits_{n\to\infty}} \begin{cases} x = 2 + \frac{1}{x'} \\ x' = 2 + \frac{1}{x}. \end{cases}$$

$$\Rightarrow \begin{cases} xx' = 2x' + 1 \\ xx' = 2x + 1 \end{cases} \Rightarrow x = x' = \sqrt{2}+1.$$

$\Rightarrow \lim\limits_{n\to\infty} x_{2n-1} = \lim\limits_{n\to\infty} x_{2n} = \sqrt{2}+1. \Rightarrow x_n \rightarrow \sqrt{2}+1$

Proof 2. $x_{n+2} - x_n = \overset{>0}{(\cdots)}\,(x_n - (\sqrt{3}+1)) \Rightarrow \begin{cases} ① \text{ odd } \nearrow \\ ② \text{ even } \searrow \end{cases}$

Proof by contradiction

$\exists \varepsilon_0, \exists n_m$, s.t. $|x_{n_m} - x| \geq \varepsilon_0$, $\forall m \in \mathbb{N}$.