MAT2002 ODEs
Nonlinear Differential Equations and Stability II

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Overview

1. First-order linear systems
   - Real distinct eigenvalues with the same sign
   - Real distinct eigenvalues with opposite sign
   - Equal eigenvalues, two linearly independent eigenvectors
   - Complex eigenvalues, non-zero real part
   - Purely imaginary eigenvalues
   - Zero as one of the eigenvalues

Outline

1. First-order linear systems
   - Real distinct eigenvalues with the same sign
   - Real distinct eigenvalues with opposite sign
   - Equal eigenvalues, two linearly independent eigenvectors
   - Complex eigenvalues, non-zero real part
   - Purely imaginary eigenvalues
   - Zero as one of the eigenvalues

First-order linear systems

We now turn to first order linear systems of the form

$$\frac{d\mathbf{y}(t)}{dt} = \mathbf{A}\mathbf{y}(t),$$

where $\mathbf{A} \in \mathbb{R}^{2\times 2}$ is a constant matrix with real coefficients. For the upcoming analysis, we will make the following assumptions:

$$\mathbf{A} \text{ is non-singular} \Leftrightarrow \det \mathbf{A} \neq 0, \text{ and } 0 \text{ is not an eigenvalue of } \mathbf{A}.$$

Then, the only possible solution to

$$\mathbf{A}\mathbf{y} = \mathbf{0}$$

is the zero vector $\mathbf{y} = \mathbf{0}$. This implies that $\mathbf{0}$ is the **unique critical point.**

Compare to first order equations, the solution $\mathbf{y}$ to $\mathbf{y}' = \mathbf{A}\mathbf{y}(t)$ is a vector, and in this case $\mathbf{y} = (y_1, y_2)$. Without going to a three dimensional plot $(t, y_1(t), y_2(t))$, we can still obtain information on the behaviour of the solution $\mathbf{y}(t)$.

First-order linear systems

Definition 16.1

We call the (y_1, y_2) plane as the **phase plane**. A solution $\mathbf{y}(t) = (y_1(t), y_2(t))$ for $t \in I$ traces out a curve in the phase plane, which we call a **trajectory**. As it is impossible to draw all trajectories, **for a representative set of trajectories** (means that it indicate all possible behavior of different trajectories) we call a **phase portrait**.

The phase portrait will yield crucial information about the stability of the critical points - which are determined by the eigenvalues of the matrix $\mathbf{A}$. For a 2×2 matrix, we have the following three possibilities for eigenvalues:

A Real, distinct eigenvalues $r_1 \neq r_2$,

B Real, repeated eigenvalues $r_1 = r_2$.

C Complex conjugate pairs of eigenvalues $r_1 = \lambda + i\mu, r_2 = \overline{r_1}$,

We are mainly interested in the tendency of the trajectories when $t \to \infty$. Sometimes, we also look at the tendency of the trajectories when $t \to -\infty$ in order to get the full insights of the trajectories and phase portrait.

First-order linear systems-direction field

For the ODE system

$$\frac{d\mathbf{y}(t)}{dt} = \mathbf{f}(\mathbf{y}),$$

where $\mathbf{y}(t) = (y_1(t), y_2(t))^T, \mathbf{f} = (f_1(y_1, y_2), f_2(y_1, y_2))^T$. In the phase plane, for each point (y_1, y_2), we can draw a line segment with arrow in the direction of vector $(f_1(y_1, y_2), f_2(y_1, y_2))^T$, then we can get the direction field of the ODE system.

Case A(1): Real distinct eigenvalues with the same sign

Recall that if $r_1 \neq r_2$, then the eigenvectors $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2$ corresponding to r_1 and r_2 are linearly independent, and the general solution to $\mathbf{y}' = \mathbf{A}\mathbf{y}(t)$ is

$$\mathbf{y}(t) = c_1 \boldsymbol{\xi}_1 e^{r_1 t} + c_2 \boldsymbol{\xi}_2 e^{r_2 t}.$$

If both r_1 and r_2 are negative, then as $t \to \infty$, we have that $\mathbf{y}(t) \to \mathbf{0}$. In particular all solutions tend to the critical point. We now illustrate how this happens in the phase portrait.

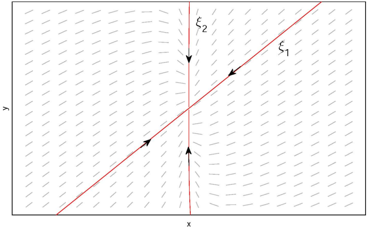
Subcase 1: $r_1 < r_2 < 0$

First, if $c_2 = 0$, which means the initial condition $\mathbf{y}(0) = \mathbf{y_0}$ is a constant multiple of $\boldsymbol{\xi}_1$, then $\mathbf{y}(t) = c_1 \boldsymbol{\xi}_1 e^{r_1 t}$ and thus the solution **always stays** on the line spanned by the vector $\boldsymbol{\xi}_1$.

Similarly, if $c_1 = 0$, which means $\mathbf{y}_0$ is a constant multiple of $\boldsymbol{\xi}_2$, then $\mathbf{y}(t)$ always stays on the line spanned by $\boldsymbol{\xi}_2$.

Case A(1): Real distinct eigenvalues with the same sign

This is illustrated in the following figure for the matrix $\mathbf{A} = \begin{pmatrix} -1 & 0 \\ -1 & -0.25 \end{pmatrix}$ with eigenvalues $r_1 = -1$, $r_2 = -0.25$ and eigenvectors $\boldsymbol{\xi}_1 = (3, 4)^T$ and $\boldsymbol{\xi}_2 = (0, 1)^T$.



Behaviour of solution to $\frac{d\mathbf{y}(t)}{dt} = \begin{pmatrix} -1 & 0 \\ -1 & -0.25 \end{pmatrix} \mathbf{y}(t)$, where eigenvalues are negative and distinct ($r_1 = -1$, $r_2 = -0.25$). The red lines indicate the lines spanned by the eigenvectors $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$, which need not be perpendicular.

Case A(1): Real distinct eigenvalues with the same sign

What about $c_1 \neq 0, c_2 \neq 0$(the initial condition $\mathbf{y}_0$ does not lie on the lines spanned by $\boldsymbol{\xi}_1$ or $\boldsymbol{\xi}_2$)? How does $\mathbf{y}(t)$ approaches the critical point in this case? We rewrite the expression for the general solution into

$$\mathbf{y}(t) = e^{r_2 t}(c_1\boldsymbol{\xi}_1 e^{(r_1 - r_2)t} + c_2\boldsymbol{\xi}_2) \ if \ r_1 < r_2 < 0.$$

If $r_1 < r_2$, then as $t \to \infty$, the term $c_1\boldsymbol{\xi}_1 e^{(r_1 - r_2)t}$ is negligible. Therefore, the trajectories **tend towards** the line spanned by $\boldsymbol{\xi}_2$. Indeed, the trajectories tend to tangent with the $\boldsymbol{\xi}_2$ at the critical point as $t \to \infty$.

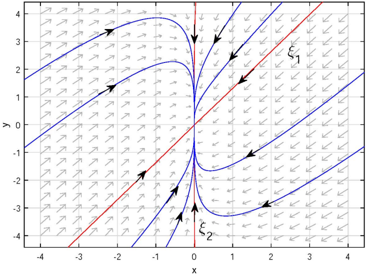
$$\mathbf{y}(t) = e^{r_1 t}(c_1\boldsymbol{\xi}_1 + c_2\boldsymbol{\xi}_2 e^{(r_2 - r_1)t}) \ if \ r_1 < r_2 < 0.$$

Also note that as $t \to -\infty$ (running backwards in time), the term $c_2\boldsymbol{\xi}_2 e^{(r_2 - r_1)t}$ is negligible, hence, the trajectories would have nearly the same slope as $\boldsymbol{\xi}_1$ as $t \to -\infty$.

For the above cases, the eigenvalues are negative, we have $\lim\limits_{t\to\infty} \mathbf{y}(t) = \mathbf{0}$, we call the critical point $\mathbf{0}$ a **nodal sink**, since all trajectories point towards $\mathbf{0}$.

Case A(1): Real distinct eigenvalues with the same sign

For $\frac{d\mathbf{y}(t)}{dt} = \begin{pmatrix} -1 & 0 \\ -1 & -0.25 \end{pmatrix} \mathbf{y}$, this leads to the following figure.



Phase portrait for $\frac{d\mathbf{y}(t)}{dt} = \begin{pmatrix} -1 & 0 \\ -1 & -0.25 \end{pmatrix} \mathbf{y}$, where eigenvalues are negative and distinct ($r_1 = -1$, $r_2 = -0.25$).. The trajectories point towards $\mathbf{0}$ (**nodal sink**). The red lines indicate the lines spanned by the eigenvectors eigenvector $\boldsymbol{\xi}_1 = (3, 4)^T$ and $\boldsymbol{\xi}_2 = (0, 1)^T$, which need not be perpendicular.

Case A(1): Real distinct eigenvalues with the same sign

Example 16.2

For

$$\frac{d\mathbf{y}(t)}{dt} = \begin{pmatrix} -7/2 & 5/2 \\ 5/2 & -7/2 \end{pmatrix} \mathbf{y}(t) =: \mathbf{A}\mathbf{y}(t)$$

determine the critical points and their stability. Draw the phase portrait.

(1) $\mathbf{x}$ is a critical point if $\mathbf{A}\mathbf{x} = \mathbf{0}$. Since $\det \mathbf{A} = 6 \neq 0$, the matrix $\mathbf{A}$ is invertible and so $\mathbf{0}$ is the only critical point.

(2) Computing the eigenvalues, we find that

$$\det(\mathbf{A} - r\mathbf{I}) = (r+6)(r+1) = 0$$

and so $r_1 = -6$ and $r_2 = -1$. There are real distinct eigenvalues and thus $\mathbf{0}$ is an asym. stable node.

Case A(1): Real distinct eigenvalues with the same sign

Example 16.2 continue

(3) Computing the eigenvectors yields

$$\boldsymbol{\xi}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \boldsymbol{\xi}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

then the general solution is

$$\mathbf{y}(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-6t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} = e^{-t}\left(c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-5t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right).$$

For large $t > 0$, $\mathbf{y}(t) \to \mathbf{0}$ with trajectories parallel to $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, this yields the phase portrait in Fig. 3. For large $t < 0$, the dominating term is $c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-5t}$ and so the trajectories are parallel to $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Case A(1): Real distinct eigenvalues with the same sign

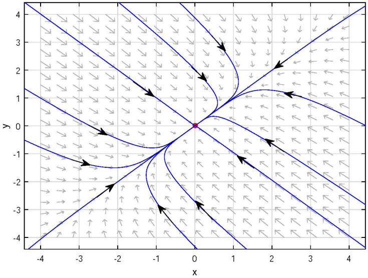


Fig. 3. Phase portrait for $\frac{d\mathbf{y}(t)}{dt} = \begin{pmatrix} -7/2 & 5/2 \\ 5/2 & -7/2 \end{pmatrix} \mathbf{y}(t)$.

Case A(1): Real distinct eigenvalues with the same sign

Subcase 2: $0 < r_2 < r_1$ $\mathbf{y}(t) = c_1\boldsymbol{\xi}_1 e^{r_1 t} + c_2\boldsymbol{\xi}_2 e^{r_2 t}$. In this case, we get the similar phase portrait, but the direction of motion is reversed.

(a) If $c_1 = 0$, and $c_2 \neq 0$, then $\mathbf{y}(t) = c_2\boldsymbol{\xi}_2 e^{r_2 t} \to \mathbf{0}$ as $t \to -\infty$.

(b) If $c_2 = 0$, and $c_1 \neq 0$, then $\mathbf{y}(t) = c_1\boldsymbol{\xi}_1 e^{r_1 t} \to \mathbf{0}$ as $t \to -\infty$.

(c) If $c_1 \neq 0, c_2 \neq 0$, rewriting

$$\mathbf{y}(t) = e^{r_1 t}(c_1\boldsymbol{\xi}_1 + c_2\boldsymbol{\xi}_2 e^{(r_2 - r_1)t})$$

for $t \to \infty$, with $r_2 - r_1 < 0$, the term $c_2\boldsymbol{\xi}_2 e^{(r_2 - r_1)t}$ is negligible compared to the other term $c_1\boldsymbol{\xi}_1$. Hence, the trajectories tend to parallel to the line spanned by $\boldsymbol{\xi}_1$ as $t \to \infty$.

(d) If $c_1 \neq 0, c_2 \neq 0$, rewriting

$$\mathbf{y}(t) = e^{r_2 t}(c_1\boldsymbol{\xi}_1 e^{(r_1 - r_2)t} + c_2\boldsymbol{\xi}_2),$$

for $t \to -\infty$, with $r_1 - r_2 > 0$, the term $c_1\boldsymbol{\xi}_1 e^{(r_1 - r_2)t}$ is negligible compared to the other term $c_2\boldsymbol{\xi}_2$. Hence, the trajectories tend to tangent with the line spanned by $\boldsymbol{\xi}_2$ as $t \to -\infty$.

In all above three case, the trajectories will move away from the critical point $\mathbf{0}$ when $t \to \infty$.

Case A(1): Real distinct eigenvalues with the same sign

Hence, If $(c_1, c_2) \neq (0, 0)$, then trajectories will move away from the critical point $\mathbf{0}$ see Fig. 4. In this case we call $\mathbf{0}$ a **nodal source**.

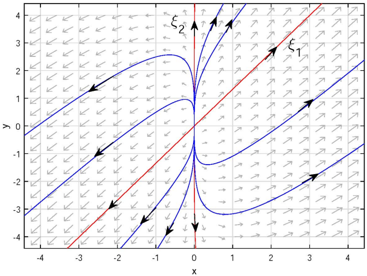


Fig. 4. Phase portrait for $\frac{d\mathbf{y}(t)}{dt} = \begin{pmatrix} 1 & 0 \\ 1 & 0.25 \end{pmatrix} \mathbf{y}$, where eigenvalues are negative and distinct ($r_1 = 1$, $r_2 = 0.25$). The trajectories point away from $\mathbf{0}$ (**nodal source**). The red lines indicate the lines spanned by the eigenvectors $\boldsymbol{\xi}_1 = (3, 4)^T$ and $\boldsymbol{\xi}_2 = (0, 1)^T$.

Case A(1): Real distinct eigenvalues with the same sign

Summary:

When $r_1 \neq r_2$ and r_1, r_2 have the same sign, then the critical point is called the **Node**. If $r_1 < 0, r_2 < 0$, the critical point is the **nodal sink**, when $r_1 > 0, r_2 > 0$, the critical point is the **nodal source**.

Case A(2): Real distinct eigenvalues with opposite sign

Without loss of generality, suppose $r_2 < 0 < r_1$. Then, from the expression for the general solution

$$\mathbf{y}(t) = c_1\boldsymbol{\xi}_1 e^{r_1 t} + c_2\boldsymbol{\xi}_2 e^{r_2 t},$$

we observe the following:

(a) If $c_1 = 0$, and $c_2 \neq 0$, then $\mathbf{y}(t) = c_2\boldsymbol{\xi}_2 e^{r_2 t} \to \mathbf{0}$ as $t \to \infty$.

(b) If $c_2 = 0$, and $c_1 \neq 0$, then $\mathbf{y}(t) = c_1\boldsymbol{\xi}_1 e^{r_1 t} \to \mathbf{0}$ as $t \to -\infty$.

(c) If $c_1 \neq 0, c_2 \neq 0$, rewriting

$$\mathbf{y}(t) = e^{r_1 t}(c_1\boldsymbol{\xi}_1 + c_2\boldsymbol{\xi}_2 e^{(r_2 - r_1)t})$$

for $t \to \infty$, with $r_2 - r_1 < 0$, the term $c_2\boldsymbol{\xi}_2 e^{(r_2-r_1)t}$ is negligible compared to the other term $c_1\boldsymbol{\xi}_1$. Hence, the trajectories approach the line spanned by $\boldsymbol{\xi}_1$ as $t \to \infty$.

Case A(2): Real distinct eigenvalues with opposite sign

(d) If $c_1 \neq 0, c_2 \neq 0$, rewriting

$$\mathbf{y}(t) = e^{r_2 t}(c_1 \boldsymbol{\xi}_1 e^{(r_1 - r_2)t} + c_2 \boldsymbol{\xi}_2),$$

for $t \to -\infty$, with $r_1 - r_2 > 0$, the term $c_1 \boldsymbol{\xi}_1 e^{(r_1 - r_2)t}$ is negligible compared to the other term $c_2 \boldsymbol{\xi}_2$. Hence, the trajectories are parallel to the line spanned by $\boldsymbol{\xi}_2$ as $t \to -\infty$.

These four observations yields the phase portrait in Fig. 5 for the matrix $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ -2 & -2 \end{pmatrix}$ with eigenvalues $r_1 = 2, r_2 = -1$ and eigenvectors $\boldsymbol{\xi}_1 = (-2, 1)^T$ (lower red line), $\boldsymbol{\xi}_2 = (-1, 2)^T$ (higher red line).

Case A(2): Real distinct eigenvalues with opposite sign

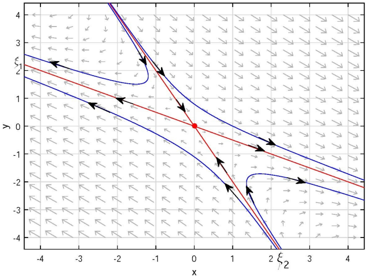


Fig. 5. Phase portrait for $\frac{d\mathbf{y}(t)}{dt} = \begin{pmatrix} 3 & 2 \\ -2 & -2 \end{pmatrix} \mathbf{y}(t)$. Critical point is a **saddle point**. The red lines indicate the lines spanned by the eigenvectors $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$, which need not be perpendicular.

For the case $r_1 < 0 < r_2$ we obtain the same portrait, but the arrows are reversed.
The critical point is called **saddle point** where two eigenvalues have different sign.

Case B(1): Equal eigenvalues, two linearly independent eigenvectors

Assume we have a repeated eigenvalue $r_1 = r_2 = r$, with two linearly independent eigenvectors $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$. Then the expression for the general solution is

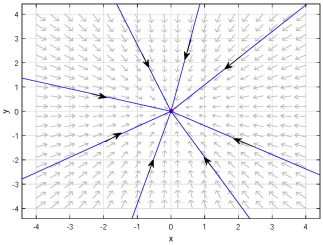
$$\mathbf{y}(t) = (c_1\boldsymbol{\xi}_1 + c_2\boldsymbol{\xi}_2)e^{rt}.$$

If $r < 0$ then $\mathbf{y} \to \mathbf{0}$ as $t \to \infty$ independently of the sign of c_1 and c_2 since $c_1\boldsymbol{\xi}_1 + c_2\boldsymbol{\xi}_2$ represent a line spanned by $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$. This means that every trajectory is a **straight line** through the critical point $\mathbf{0}$, see Fig. 6 (a).

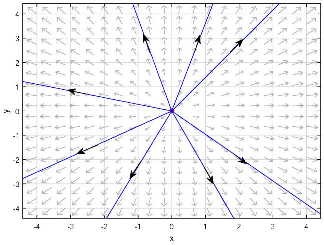
Similarly, if $r > 0$, then $\mathbf{y}(t) \to \mathbf{0}$ as $t \to -\infty$ independently of the sign of c_1 and c_2, see Fig. 6 (b).The trajectories are also straight lines through the critical point.

In these cases, we call the critical point a **proper node** or **star point**.

Case B(1): Equal eigenvalues, two linearly independent eigenvectors



(a) Phase portrait for $\frac{d\mathbf{y}(t)}{dt} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{y}(t)$.



(b) Phase portrait for $\frac{d\mathbf{y}(t)}{dt} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{y}(t)$.

Fig. 6. (a) stable proper node , (b) unstable proper node.

Remark: Fig. 6 (a) is also the **nodal sink** for the proper node. And Fig. 6 (b) is also the **nodal source** for the proper node.

Case B(2): Equal eigenvalues, one linearly independent eigenvector

Let $r_1 = r_2 = r$ be the repeated eigenvalue, and $\boldsymbol{\xi}$ the associated eigenvector. Then, the general solution is

$$\begin{aligned}\mathbf{y}(t) &= c_1\boldsymbol{\xi}e^{rt} + c_2(\boldsymbol{\xi}t + \boldsymbol{\eta})e^{rt} \\ &= e^{rt}((c_1\boldsymbol{\xi} + c_2\boldsymbol{\eta}) + tc_2\boldsymbol{\xi}) =: e^{rt}\mathbf{z}(t),\end{aligned}$$

where we recall that η is a **generalized eigenvector** to the eigenvalue r, i.e.,

$$(\mathbf{A} - r\mathbf{I})\boldsymbol{\xi} = \mathbf{0}, \qquad (\mathbf{A} - r\mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}.$$

First consider $r < 0$.
To sketch the trajectories, note for fixed c_1 and c_2, the vector function $\mathbf{z}(t) = (c_1\boldsymbol{\xi} + c_2\boldsymbol{\eta}) + c_2\boldsymbol{\xi}t$ is a straight line through the point $c_1\boldsymbol{\xi} + c_2\boldsymbol{\eta}$ (initial position corresponding to $t = 0$) in the direction of $c_2\boldsymbol{\xi}$ (parallel to $\boldsymbol{\xi}$), which is the direction of **increasing** t. (Note that the direction of increasing t is different for $c_2 > 0$ and for $c_2 < 0$). Writing the solution $\mathbf{y}(t)$ as $\mathbf{y}(t) = e^{rt}\mathbf{z}(t)$ allows us to interpret that $\mathbf{z}(t)$ determines the **direction** of the trajectory and e^{rt} determine its the magnitude.

Case B(2): Equal eigenvalues, one linearly independent eigenvector

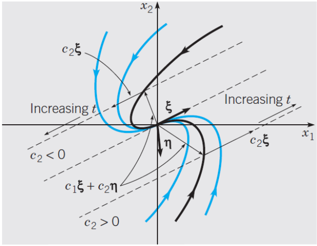
$$\mathbf{y}(t) = (c_1\boldsymbol{\xi} + c_2\boldsymbol{\eta} + c_2\boldsymbol{\xi}t)e^{rt}.$$

If $c_2 = 0$, then the trajectory is just the line spanned by ξ. However, if $c_2 \neq 0$, then the dominating term is $c_2 t\boldsymbol{\xi}e^{rt}$ for large (positive/negative) values of t, and we expect the trajectories to be tangent to the line spanned by $\boldsymbol{\xi}$ as $t \to \infty$ and parallel to the line spanned by $\boldsymbol{\xi}$ as $t \to -\infty$ since $r < 0$.

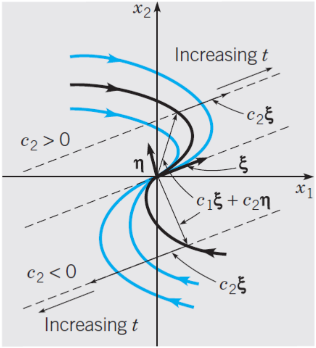
Now we draw the lines $c_1\boldsymbol{\xi} + c_2\boldsymbol{\eta} + tc_2\boldsymbol{\xi}$ passing through the point $c_1\boldsymbol{\xi} + c_2\boldsymbol{\eta}$ (initial position corresponding to $t = 0$) in the direction of $c_2\boldsymbol{\xi}$ (parallel to $\boldsymbol{\xi}$), which is the direction of **increasing** t. Note that the direction of increasing t is different for $c_2 > 0$ and for $c_2 < 0$. As t increases, the direction of the trajectories follow the direction of increasing t.

Due to $r < 0$, the magnitude is shrinking exponentially, the trajectories tend to origin as $t \to \infty$. We expect that the trajectories to travel along the direction of increasing t. Thus, we expect that the trajectories (except the line spanned by $\boldsymbol{\xi}$) will do a **sharp turn** and tangent to the line spanned by $\boldsymbol{\xi}$ at the origin. The orientation of the trajectories depends on the relative positions of $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$, two possible situations are shown in the following figure 7.

Case B(2): Equal eigenvalues, one linearly independent eigenvector



(a) Phase portrait for improper node.



(b) Phase portrait for improper node.

Fig. 7. Asymtotically stable improper node.

Case B(2): Equal eigenvalues, one linearly independent eigenvector

Next consider $r > 0$. For the case $r > 0$, we have the similar phase portrait, but the direction of the trajectories are reversed, i.e., the origin is unstable and every trajectories is leaving the origin.

In these cases, where the geo. mult. of the repeated eigenvalue is equal to one, we call the critical point an **improper node** or a **degenerate node**.

Case B(2): Equal eigenvalues, one linearly independent eigenvector

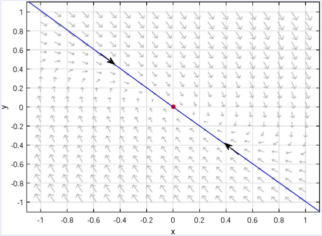
Example 16.3

For $\mathbf{y}' = \mathbf{A}\mathbf{y}(t)$, where $\mathbf{A} = \begin{pmatrix} 1 & 4 \\ -4 & -7 \end{pmatrix}$ with a repeated eigenvalue $r = -3$ and eigenvector $\boldsymbol{\xi} = (-1, 1)^T$, where along the blue line the trajectories move towards the critical point $\mathbf{0}$ if $r < 0$. The one of the generalized eigenvector $\eta = (-\frac{1}{8}, -\frac{1}{8})^T$.

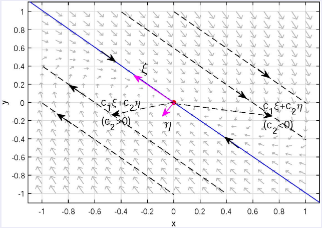
To draw the phase portrait, the first thing to draw is the line given by $c_1\boldsymbol{\xi}e^{rt} = (-c_1e^{-3t}, c_1e^{-3t})^T$, this can be done easily. Next, we draw the lines $c_1\boldsymbol{\xi} + c_2\boldsymbol{\eta} + tc_2\boldsymbol{\xi}$ passing through the point $c_1\boldsymbol{\xi} + c_2\boldsymbol{\eta}$ (initial position corresponding to $t = 0$) in the direction of $c_2\boldsymbol{\xi}$ (parallel to $\boldsymbol{\xi}$), which is the direction of **increasing** t. Note that the direction of increasing t is different for $c_2 > 0$ and for $c_2 < 0$. This is given in Fig. 8.

Case B(2): Equal eigenvalues, one linearly independent eigenvector

Example 16.3 continue



(a) The line given by $c_1\boldsymbol{\xi}e^{rt} = (-c_1e^{-3t}, c_1e^{-3t})^T$.



(b) The dotted lines given by $c_1\boldsymbol{\xi}+c_2\boldsymbol{\eta}+tc_2\boldsymbol{\xi} = (-c_1-\frac{1}{8}c_2-c_2t, c_1-\frac{1}{8}c_2+c_2t)^T$.

Fig. 8. Phase portrait for $\frac{d\mathbf{y}(t)}{dt} = \begin{pmatrix} 1 & 4 \\ -4 & -7 \end{pmatrix}\mathbf{y}(t)$. (a): the line given by $c_1\boldsymbol{\xi}e^{rt} = (-c_1e^{-3t}, c_1e^{-3t})^T$. (b): the dotted lines given by $c_1\boldsymbol{\xi}+c_2\boldsymbol{\eta}+tc_2\boldsymbol{\xi} = (-c_1-\frac{1}{8}c_2-c_2t, c_1-\frac{1}{8}c_2+c_2t)^T$.

Case B(2): Equal eigenvalues, one linearly independent eigenvector

Example 16.3 continue

From the expression

$$\mathbf{y}(t) = (c_1\boldsymbol{\xi} + c_2\boldsymbol{\eta} + c_2 t\boldsymbol{\xi})e^{rt},$$

the dominating term is $c_2 t\boldsymbol{\xi}e^{rt}$ for large (positive/negative) values of t, and so we expect the trajectories to be parallel to the line spanned by $\boldsymbol{\xi}$. As t increases, the direction of the trajectories follow the direction of increasing t, but due to $r < 0$, the magnitude is shrinking exponentially. So we expect that the trajectory to travel along the direction of increasing t, but do a **sharp turn** to go back to the origin. This is reflected in Fig. 9.

Case B(2): Equal eigenvalues, one linearly independent eigenvector

Example 16.3 continue

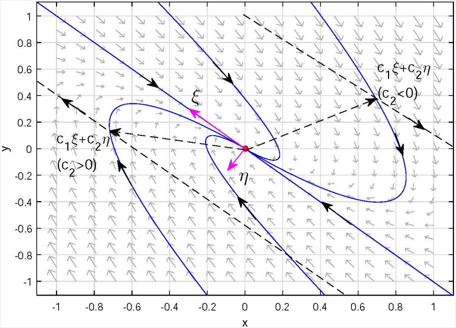


Fig. 9. Phase portrait for $\frac{d\mathbf{y}(t)}{dt} = \begin{pmatrix} 1 & 4 \\ -4 & -7 \end{pmatrix} \mathbf{y}(t)$. Trajectories are marked by blue curves. Most of the trajectories (curved trajectories) make a **sharp turn** and tangent to the line spanned by ξ at the origin. The dotted lines given by $c_1\boldsymbol{\xi} + c_2\boldsymbol{\eta} + tc_2\boldsymbol{\xi}$

Case B(2): Equal eigenvalues, one linearly independent eigenvector

For the case $r > 0$, we have the similar phase portrait, but the direction of the trajectories are reversed, i.e., the origin is unstable and every trajectories is leaving the origin. In these cases, where the geo. mult. of the repeated eigenvalue is equal to one, we call the critical point an **improper node** or a **degenerate node**.

An example with $r_1 = r_2 > 0$ is given by Fig. 10, where the matrix $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix}$ with repeated eigenvalue $r = 1$ and eigenvector $\boldsymbol{\xi} = (-1, 2)^T$, the generalized eigenvector $\boldsymbol{\eta} = (0, -1)^T$. This gives an unstable critical point at the origin.

Case B(2): Equal eigenvalues, one linearly independent eigenvector

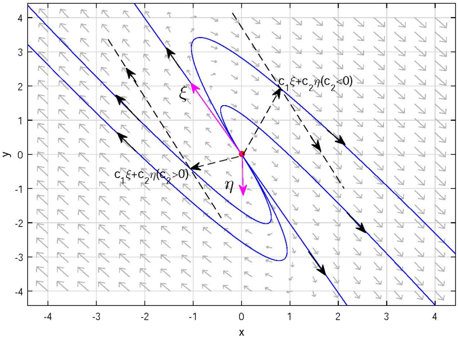
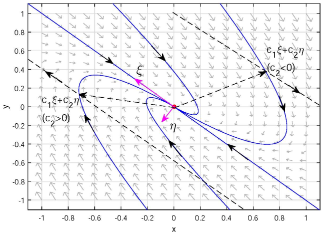
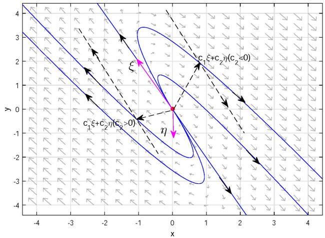


Fig. 10. Phase portrait for $\frac{d\mathbf{y}(t)}{dt} = \begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix} \mathbf{y}(t)$. Trajectories are marked by blue curves. Most of the trajectories (curved trajectories) make a **sharp turn** and tangent to the line spanned by $\boldsymbol{\xi}$ at the origin. The dotted lines given by $c_1\boldsymbol{\xi} + c_2\boldsymbol{\eta} + tc_2\boldsymbol{\xi}$.

Case B: Equal eigenvalues



(a) Phase portrait for $\frac{d\mathbf{y}(t)}{dt} = \begin{pmatrix} 1 & 4 \\ -4 & -7 \end{pmatrix} \mathbf{y}(t)$



(b) Phase portrait for $\frac{d\mathbf{y}(t)}{dt} = \begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix} \mathbf{y}(t)$

Fig. 11. (a) **nodal sink** for the improper node. (b) **nodal source** for the improper node.

Remark: Fig. 11(a) is also the **nodal sink** for the improper node. And Fig. 11(b) is also the **nodal source** for the improper node.

Case C(1): Complex eigenvalues, non-zero real part

Suppose $r_1 = \lambda + i\mu$, for $\lambda, \mu \in \mathbb{R}$ with $\lambda \neq 0$, and so $r_2 = \lambda - i\mu$. Denoting the corresponding eigenvectors to be $\boldsymbol{\xi}_1 = \mathbf{u} + i\mathbf{v}$ with $\boldsymbol{\xi}_2 = \mathbf{u} - i\mathbf{v}$, and set

$$\mathbf{y}_1(t) = e^{\lambda t}(\mathbf{u}\cos(\mu t) - \mathbf{v}\sin(\mu t)) = e^{\lambda t}\mathbf{z}_1(t),$$
$$\mathbf{y}_2(t) = e^{\lambda t}(\mathbf{u}\sin(\mu t) + \mathbf{v}\cos(\mu t)) = e^{\lambda t}\mathbf{z}_2(t),$$

we have the general solution

$$\mathbf{y}(t) = e^{\lambda t}(c_1\mathbf{z}_1(t) + c_2\mathbf{z}_2(t)),$$

for $c_1, c_2 \in \mathbb{R}$ arbitrary. Note that $\mathbf{z}_1$ and $\mathbf{z}_2$ are functions of cosine and sine, and therefore are periodic bounded functions in t. We expect that

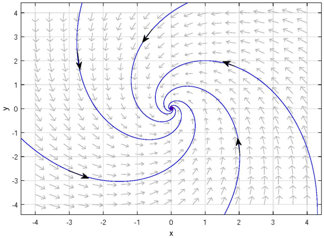
(1) if $\lambda < 0$, then $\mathbf{y}(t) \to \mathbf{0}$ as $t \to \infty$;

(1) if $\lambda > 0$, then $\mathbf{y}(t) \to \mathbf{0}$ as $t \to -\infty$.

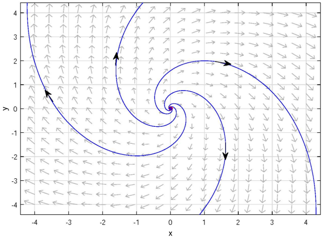
So, for $\lambda < 0$, we expect the trajectories tend to $\mathbf{0}$ like a spiral.

Case C(1): Complex eigenvalues, non-zero real part

Example, see Fig. 12 (a) for $\mathbf{A} = \begin{pmatrix} -1 & -1 \\ 2 & -1 \end{pmatrix}$ with $r_{1(2)} = -1 \pm \sqrt{2}i$. For $\lambda > 0$, we have the same phase portrait, but we spiral outwards, see Fig. 12 (b) for $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}$ with $r_{1(2)} = 1 \pm \sqrt{2}i$.



(a) Phase portrait for $\frac{d\mathbf{y}(t)}{dt} = \begin{pmatrix} -1 & -1 \\ 2 & -1 \end{pmatrix} \mathbf{y}(t)$



(b) Fig. 22. Phase portrait for $\frac{d\mathbf{y}(t)}{dt} = \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix} \mathbf{y}(t)$

Fig. 12. (a) critical point is a **spiral sink**. (b) critical point is a **spiral source**.

Case C(1): Complex eigenvalues, non-zero real part

One way to determine whether the trajectories spiral "clockwise" or "anticlockwise" is to look at the transformation of a point by the matrix $\mathbf{A}$. For example

$$\frac{d\mathbf{y}(t)}{dt} = \begin{pmatrix} -1 & -1 \\ 2 & -1 \end{pmatrix} \mathbf{y}(t)$$

has a matrix $\mathbf{A}$ with eigenvalues $-1 \pm \sqrt{2}i$. Applying the matrix $\mathbf{A}$ to the point $\mathbf{x} = (0,1)^T$ yields

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

The vector $(-1,-1)^T$ provides a direction which the trajectories will be traveling. Therefore, if a trajectory starts at $(0,1)^T$, it will move in a direction $(-1,-1)^T$ and so the trajectories spiral in an anti-clockwise direction when t increases.

We call the critical point $\mathbf{0}$ a **spiral point** in the case where the eigenvalues of the matrix $\mathbf{A}$ are complex conjugate pairs with non-zero real part. If $\lambda < 0$ we have a **spiral sink** and if $\lambda > 0$ we have a **spiral source**.

Case C(2): Purely imaginary eigenvalues

We now consider the case where the eigenvalues of the matrix $\mathbf{A}$ are purely imaginary, i.e., $r_1 = i\mu, r_2 = -i\mu$ for $\mu \in \mathbb{R}$. In this case the general solution is

$$\begin{aligned}\mathbf{y}(t) &= c_1\mathbf{z}_1(t) + c_2\mathbf{z}_2(t)\\ &= c_1(\mathbf{u}\cos(\mu t) - \mathbf{v}\sin(\mu t)) + c_2(\mathbf{u}\sin(\mu t) + \mathbf{v}\cos(\mu t)),\end{aligned}$$

where $\boldsymbol{\xi}_1 = \mathbf{u} + i\mathbf{v}$ is the eigenvector corresponding to r_1. Due to the periodic nature of $\mathbf{z}_1$ and $\mathbf{z}_2$ we expect the trajectories to encircle the critical point, but neither approach nor move away as $t \to \infty$.

Case C(2): Purely imaginary eigenvalues

This can also be seen from rewriting the general solution:

$$\begin{aligned}\mathbf{y}(t) &= \sqrt{c_1^2+c_2^2}\left(\mathbf{u}\cos(\mu t)\frac{c_1}{\sqrt{c_1^2+c_2^2}} + \mathbf{u}\sin(\mu t)\frac{c_2}{\sqrt{c_1^2+c_2^2}}\right)\\ &\quad + \sqrt{c_1^2+c_2^2}\left(\mathbf{v}\cos(\mu t)\frac{c_2}{\sqrt{c_1^2+c_2^2}} - \mathbf{v}\sin(\mu t)\frac{c_1}{\sqrt{c_1^2+c_2^2}}\right)\\ &= \sqrt{c_1^2+c_2^2}(\mathbf{u}\sin(\theta+\mu t)+\mathbf{v}\cos(\theta+\mu t)),\end{aligned}$$

where $\theta \in [0, 2\pi]$ is a constant such that $\sin(\theta) = \frac{c_1}{\sqrt{c_1^2+c_2^2}}$ and $\cos(\theta) = \frac{c_2}{\sqrt{c_1^2+c_2^2}}$. The last line shows that the trajectory $\{\mathbf{y}(t)\}_{t\in I}$ can be seen as a ellipse centered at the origin with a fixed distance that is not changing in time.

Case C(2): Purely imaginary eigenvalues

For example, the matrix $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ -5 & -2 \end{pmatrix}$ with eigenvalues $r_1 = i, r_2 = -i$. the phase protrait for $\frac{d\mathbf{y}(t)}{dt} = \mathbf{A}\mathbf{y}(t)$ is shown in Fig. 13.
Again, the direction of the trajectories "clockwise" or "anticlockwise" can be determined by testing one point with the matrix $\mathbf{A}$. For example starting from the point $\mathbf{x} = (0, 1)^T$ we have

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Therefore, if a trajectory starts at $(0, 1)^T$, it will move in a direction $(1, -2)^T$ and so the trajectories move in the clockwise direction when t increases.
From this we expect the trajectories to move clockwise. For the case where the eigenvalues of $\mathbf{A}$ are purely imaginary, we call the critical point a **center**.

Case C(2): Purely imaginary eigenvalues

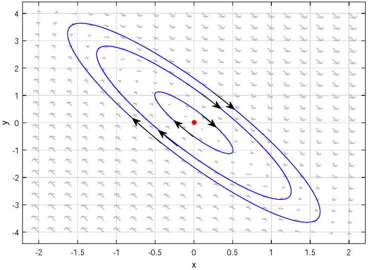


Fig. 13. Phase portrait for $\frac{d\mathbf{y}(t)}{dt} = \begin{pmatrix} 2 & 1 \\ -5 & -2 \end{pmatrix} \mathbf{y}(t)$. Each trajectory is an elliptic curve.

Summary

Summary. The behaviour of trajectories for the system $\frac{d\mathbf{y}(t)}{dt} = \mathbf{A}\mathbf{y}(t)$ where the origin $\mathbf{0}$ is a critical point depends heavily on the non-zero eigenvalues r_1, r_2. One of the following three situations can occur:

- All trajectories approach $\mathbf{0}$ as $t \to \infty$, then $\mathbf{0}$ is either a **nodal sink** or a **spiral sink**.
- All trajectories remains bounded (contained in a bounded set in the phase space) but do not approach $\mathbf{0}$ as $t \to \infty$. Then $\mathbf{0}$ is a **center**.
- Some trajectories (possibly all) except the trajectory $\mathbf{y}_*(t) = \mathbf{0}$ for all t, becomes unbounded as $t \to \infty$. Then $\mathbf{0}$ is either a **nodal source**, a **spiral source** or a **saddle point**.

Note that due to the **uniqueness**, through each point (y_1, y_2) of the phase plane, there is **only** one trajectory passing through that point. This implies that trajectories **do not cross each other**.

Definition of Stability

Definition 16.4

(Stability). Let $\mathbf{y}_*$ be a critical point of the autonomous system

$$\frac{d\mathbf{y}(t)}{dt} = \mathbf{f}(\mathbf{y}(t)),$$

i.e., $\mathbf{f}(\mathbf{y}_*) = \mathbf{0}$. We say that

(1) $\mathbf{y}_*$ is **stable** if for any $\epsilon > 0$, there exists a $\delta > 0$ (depending on y_* and ϵ) such that any solution $\mathbf{y} = \phi(t)$ to $\frac{d\mathbf{y}(t)}{dt} = \mathbf{f}(\mathbf{y}(t))$ and $\mathbf{y}(t_0) = \phi(t_0)$ satisfies

$$\text{if} \quad \|\phi(\mathrm{t}_0) - \mathbf{y}_*\| < \delta \quad \text{then} \quad \|\phi(\mathrm{t}) - \mathbf{y}_*\| < \epsilon \quad \forall \mathrm{t} \geq \mathrm{t}_0,$$

where t_0 is some real number.

1. $\mathbf{y}_*$ is **unstable** if it is not stable.
2. $\mathbf{y}_*$ is **asymptotically stable** if there exists $\delta > 0$ (depending only on $\mathbf{y}_*$) such that

$$\text{if} \quad \|\phi(\mathrm{t}_0) - \mathbf{y}_*\| < \delta \quad \text{then} \quad \phi(\mathrm{t}) \to \mathbf{y}_* \text{ as } \mathrm{t} \to \infty.$$

asymptotically stable must be **stable**, **stable** may not be **asymptotically stable**.

Understanding the stability by physics-Oscillating Pendulum

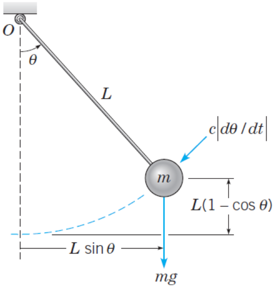


Fig. 14. An oscillating pendulum.

The concepts of asymptotic stability, stability, and instability can be easily visualized in terms of an oscillating pendulum. Consider the configuration shown in Figure 14, in which an object with mass m is attached to one end of a rigid rod of length L, the rod is weightless. The other end of the rod is supported (fixed) at the origin O, and the rod is free to rotate in the entire plane of the paper.

Understanding the stability by Oscillating Pendulum

The position of the pendulum is described by the angle θ between the rod and the central vertical direction, with the anticlockwise direction taken as positive. The gravitational force mg acts downward, while the air resistance force (damping force) is always opposite to the direction of motion.
Now first, we assume that both θ and $d\theta/dt$ are positive, the tangential velocity v of the object is given as $v = L\frac{d\theta}{dt}$ ($\frac{d\theta}{dt}$ is the angular velocity), and the acceleration $a = \frac{dv}{dt}$ can be computed as $a = L\frac{d^2\theta}{dt^2}$. If we assume that the damping force proportional to the velocity ($F_a = cv = cL\frac{d\theta}{dt}$, $c > 0$ is positive constant) and is in the opposite direction to the motion of the pendulum.
Then Newton's second law gives

$$-mg\sin\theta - cL\frac{d\theta}{dt} = ma = mL\frac{d^2\theta}{dt^2}$$

Indeed, the above equation is valid for the other three possible sign combinations for θ and $d\theta/dt$, you can take this as an exercise. Thus, the above equation is always valid no matter what sign is for θ and $d\theta/dt$.

Understanding the stability by Oscillating Pendulum

Thus, we now have the **damped pendulum equation**

$$\frac{d^2\theta}{dt^2} + \gamma \frac{d\theta}{dt} + w^2 \sin\theta = 0, \tag{1}$$

where $w^2 = \frac{g}{L}$, $\gamma = \frac{c}{mL} > 0$ is a damping factor taking into account friction forces.
Let $x = \theta$ and $y = d\theta/dt$, then

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -\omega^2 \sin x - \gamma y. \tag{2}$$

Since γ and ω^2 are constants, the system (2) is an autonomous system of the form (1).

Understanding the stability by Oscillating Pendulum

The critical points of Eqs. (2) are found by solving the equations

$$y = 0, \qquad -\omega^2 \sin x - \gamma y = 0.$$

We obtain $y = 0$ and $x = \pm n\pi$, where n is an integer. These points correspond to two physical equilibrium positions, one with the object directly below the point of support ($\theta = 0$) and the other with the object directly above the point of support ($\theta = \pi$). Our intuition suggests that the first is stable and the second is unstable.

More precisely, if the object is slightly displaced from the lower equilibrium position, it will oscillate back and forth with gradually decreasing amplitude, eventually converging to the equilibrium position as the initial potential energy is dissipated by the damping force. This type of motion illustrates **asymptotic stability** and is shown in Figure 15a.

Understanding the stability by Oscillating Pendulum

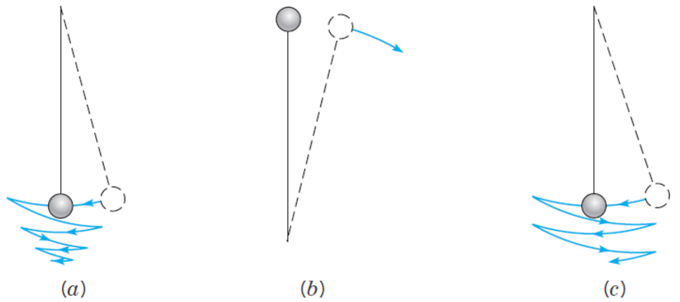


Figure 15. Qualitative motion of a pendulum. (a)With air resistance. (b)With or without air resistance. (c)Without air resistance.

Understanding the stability by Oscillating Pendulum

On the other hand, if the object is slightly displaced from the upper equilibrium position, it will fall very fast due to the gravity, and will ultimately converge to the lower equilibrium position. This type of motion illustrates **instability**. See Figure 15b. In practice, it is impossible to maintain the pendulum in its upward equilibrium position since a slight perturbation will cause the object to fall.

Finally, consider the ideal situation in which the damping coefficient c (or γ) is zero. In this case, if the object is displaced slightly from its lower equilibrium position, it will oscillate ∞ times with constant amplitude around the equilibrium position. Since there is no dissipation in the system, the object will remain near the equilibrium position but will not approach it asymptotically. This type of motion is stable but not asymptotically **stable**, as indicated in Figure 15c. In general, this motion is impossible to achieve in reality, because a slight air resistance or friction will eventually cause the pendulum to converge to its lower equilibrium position.

Stability classification of critical point

Note that asymptotic stability is a stronger property than stability. Furthermore, the stability property means that the trajectories do not have to tend towards $\mathbf{y}_*$, they just have to remain close by.

We can now classify for $\frac{d\mathbf{y}}{dt} = \mathbf{Ay}(\mathbf{A} \in \mathbb{R}^{2\times 2})$ the stability of the critical point $\mathbf{0}$:

Eigenvalues	Type	Stability
$r_1 > r_2 > 0$	Node	Unstable
$r_1 > 0 > r_2$	Saddle	Unstable
$r_1 < r_2 < 0$	Node	Asym.stable
$r_1 = r_2 < 0$	Proper/improper node	Asym.stable
$r_1 = r_2 > 0$	Proper/improper node	Unstable
$r_1 = \lambda + i\mu$	Spiral	Unstable ($\lambda > 0$), Asym.stable ($\lambda < 0$)
$r_1 = i\mu$	Center	Stable

Table: Type and stability of the critical point based on the eigenvalues

Stability classification of critical point

Important observation: for $\frac{d\mathbf{y}}{dt} = \mathbf{Ay}(\mathbf{A} \in \mathbb{R}^{2\times 2})$.

(a) The critical point $\mathbf{y} = \mathbf{0}$ for the above linear ODE system is unstable if $\mathbf{A}$ has (at least) one eigenvalue with positive real part.

(b) The critical point $\mathbf{y} = \mathbf{0}$ for the above linear ODE system is asymptotically stable stable if all eigenvalues of of $\mathbf{A}$ have negative real parts.

(c) If the real part of complex eigenvalues are zero, we have purely imaginary eigenvalues, the critical point $\mathbf{y} = \mathbf{0}$ for the above linear ODE system is stable but not asymptotically stable.

Zero as one of the eigenvalues

For $\frac{d\mathbf{y}}{dt} = \mathbf{Ay}(\mathbf{A} \in \mathbb{R}^{2\times 2})$. The above discussion is about the case that $\mathbf{A}$ is non-singular. If $\mathbf{A}$ is singular, what happens?

What if 0 is an eigenvalue of $\mathbf{A}$? That is $r_1 = 0$ and $r_2 \neq 0$. Then note that the corresponding eigenvector $\boldsymbol{\xi}$ to the eigenvalue 0 satisfies

$$\mathbf{A}\xi = \mathbf{0},$$

and so every point on the straight line $\{t\boldsymbol{\xi} : t \in \mathbb{R}\}$ is a critical point.
For example, consider the matrix

$$\mathbf{A} = \begin{pmatrix} -1 & 0 \\ -1 & 0 \end{pmatrix} \quad \textit{with } r_1 = 0, \quad r_2 = -1, \quad \boldsymbol{\xi}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \boldsymbol{\xi}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Thus, all points on the vertical axis is a critical point. The phase portrait is plotted in Fig. 16. The solution of the ODE is

$$y_1(t) = c_1 e^{-t}, \quad y_2(t) = c_1 e^{-t} + c_2$$

If $c_1 = 0$, then the trajectory is the point on the vertical axis. Otherwise if $c_1 \neq 0$, then the trajectory is the straight line pointing to the vertical axis.

Zero as one of the eigenvalues

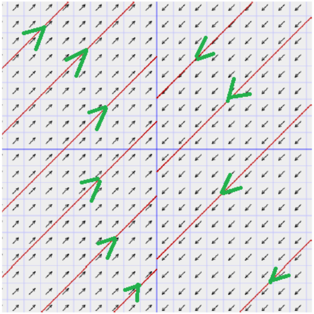


Fig. 16. Phase portrait for $\frac{d\mathbf{y}(t)}{dt} = \begin{pmatrix} -1 & 0 \\ -1 & 0 \end{pmatrix} \mathbf{y}(t)$.

Note that once a trajectory hits the vertical axis, it stops and does not appear on the other side of the axis.

Remark: These critical points are not the isolated critical points. A critical point is an isolated critical point if there is a circle around it and there is no other critical point inside the circle.