MAT3253 Homework 3

Due date: 19 Feb.

Question 1. (Brown & Churchill Ex.1.11.7) Determine the accumulation points of each of the following sets:

a. $z_n = i^n$ $(n = 1, 2, \ldots)$ b. $z_n = i^n/n$ $(n = 1, 2, \ldots)$

c. $0 \le arg(z) < \pi/2$ $(z \ne 0)$ d. $z_n = (-1)^n(1+i)\frac{n-1}{n}$ $(n = 1, 2, \ldots)$.

Question 2. Compute the following powers. Express your answer in polar form.

(a) $(-1+i)^{2021}$

(b) i^{3253}

Question 3.

(a) Find all cube roots of -1.

(b) Find all roots of $z^4 + 16 = 0$.

Question 4. (Brown & Churchill 2.1.1) For each of the functions below, describe the domain of definition that is understood. In part (b) "Arg" stands for the principal argument.

a. $f(z) = \frac{1}{z^2+1}$ b. $f(z) = Arg(1/z)$

c. $f(z) = \frac{z}{z+\bar{z}}$ d. $f(z) = \frac{1}{1-|z|^2}$.

Question 5. (Bak & Newman 2.3) By Cauchy-Riemann equations, determine which of the following polynomials are analytic.

(a) $P(x+iy) = x^3 - 3xy^2 - x + i(3x^2y - y^3 - y)$

(b) $P(x+iy) = 2xy + i(y^2 - x^2)$

Question 6. (Bak & Newman 3.2)

(a) Show that $f(z) = x^2 + iy^2$ is complex differentiable at all points on the line $y = x$.

(b) Show that it is nowhere analytic.

Question 7. (Bak & Newman 3.3) Prove that the composition of differentiable functions is differentiable. That is, if f is differentiable at z, and if g is differentiable at $f(z)$, then $g \circ f$ is differentiable at z.

(Hint: begin by noting

$$g(f(z+h)) - g(f(z)) = [g'(f(z)) + \epsilon][f(z+h) - f(z)]$$

where $\epsilon \to 0$ as $h \to 0$.)

Question 8. (Bak & Newman 3.4) Suppose that g is a continuous "$\sqrt{z}$" (i.e., $g^2(z) = z$) in some neighborhood of z. Verify that $g'(z) = 1/(2\sqrt{z})$.

(Hint: use

$$1 = \frac{g^2(z) - g^2(z_0)}{z - z_0}$$

to evaluate

$$\lim_{z \to z_0} \frac{g(z) - g(z_0)}{z - z_0}.)$$

Question 9. (Bak & Newman 3.5) Suppose f is analytic in a region and $f' \equiv 0$ there. Show that f is constant.

Question 10 (Bak & Newman 2.2a) Suppose $f(z)$ is real-valued and differentiable for all real z. Show that $f'(z)$ is also real-valued for real z.

Question 11 Suppose $f(z)$ is an analytic function with domain D and $D_1 \subset D$ is a domain in which $f(z)$ is nonzero. Show that $1/f(z)$ is analytic in D_1 with derivative $-f'(z)/f(z)^2$.