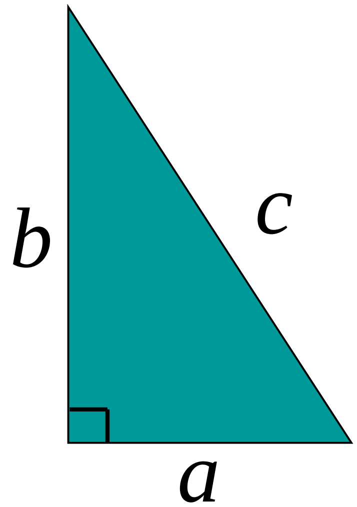
Propositional Logic

Content

1. Mathematical proof (what and why)
2. Logic, basic operators
3. Using simple operators to construct any operator
4. Logical equivalence, DeMorgan's law
5. Conditional statement (if, if and only if)
6. Arguments

Pythagorean theorem

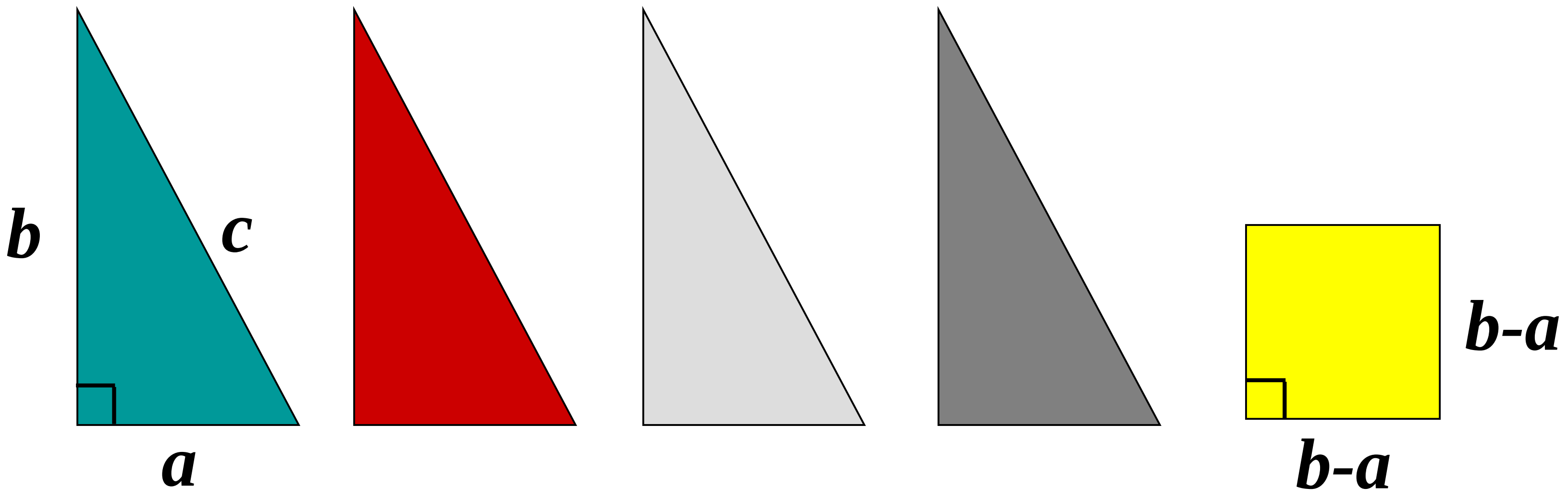


$$a^2 + b^2 = c^2$$

Familiar?

Obvious?

Good Proof



We will show that these five pieces can be rearranged into:

(i) a c×c square, and

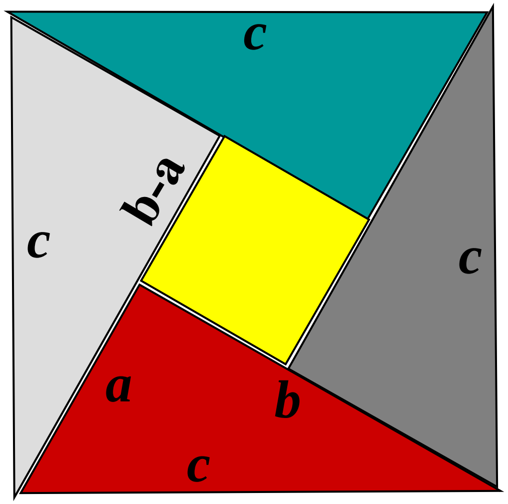
(ii) an a×a & a b×b square

So we can conclude that $c^2 = a^2 + b^2$

Good Proof

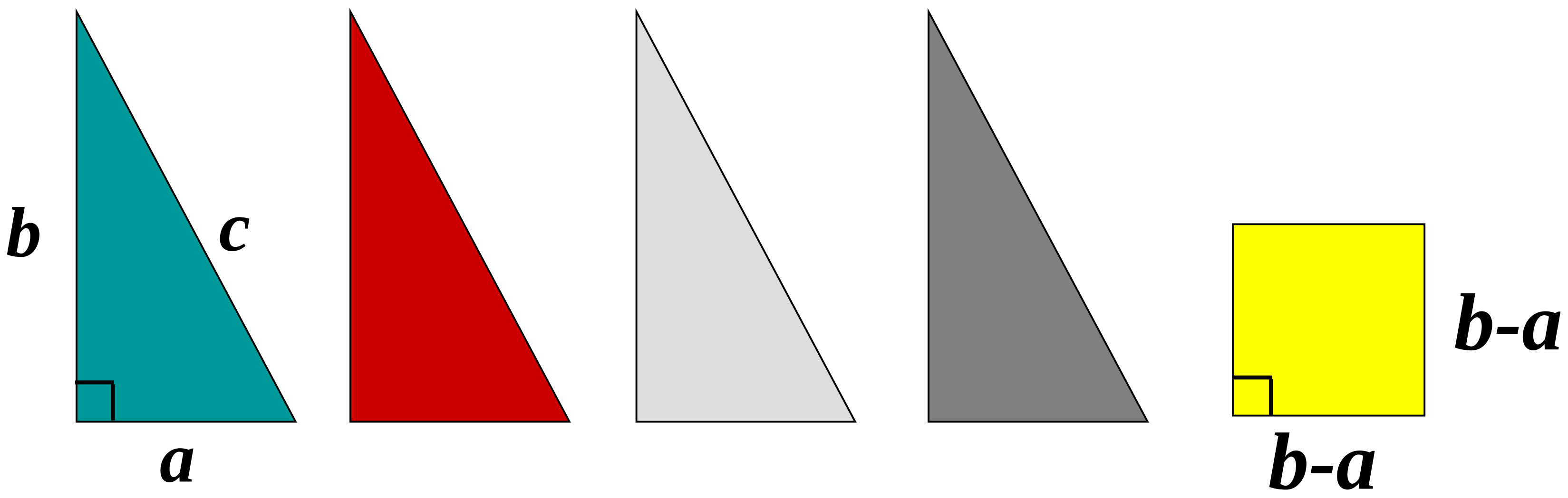
The five pieces can be rearranged into:

(i) a $c \times c$ square

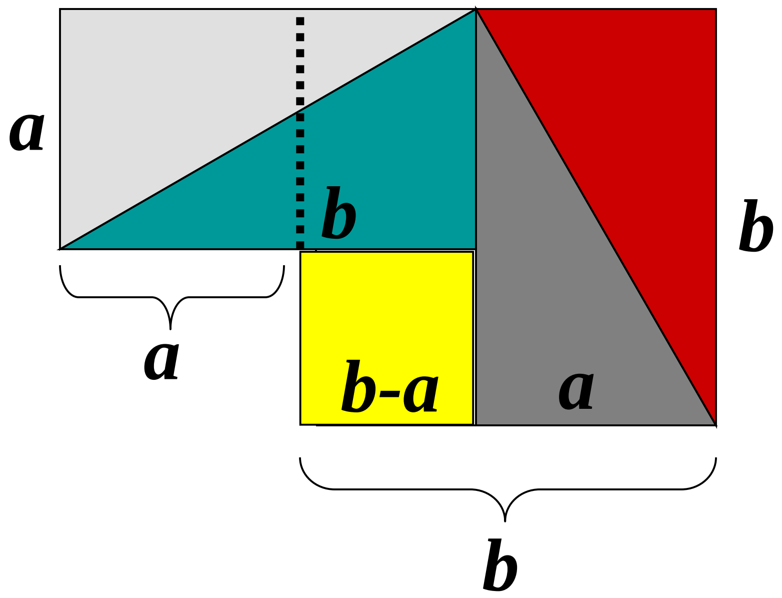


Good Proof

How to rearrange them into an axa square and a bxb square?



Good Proof

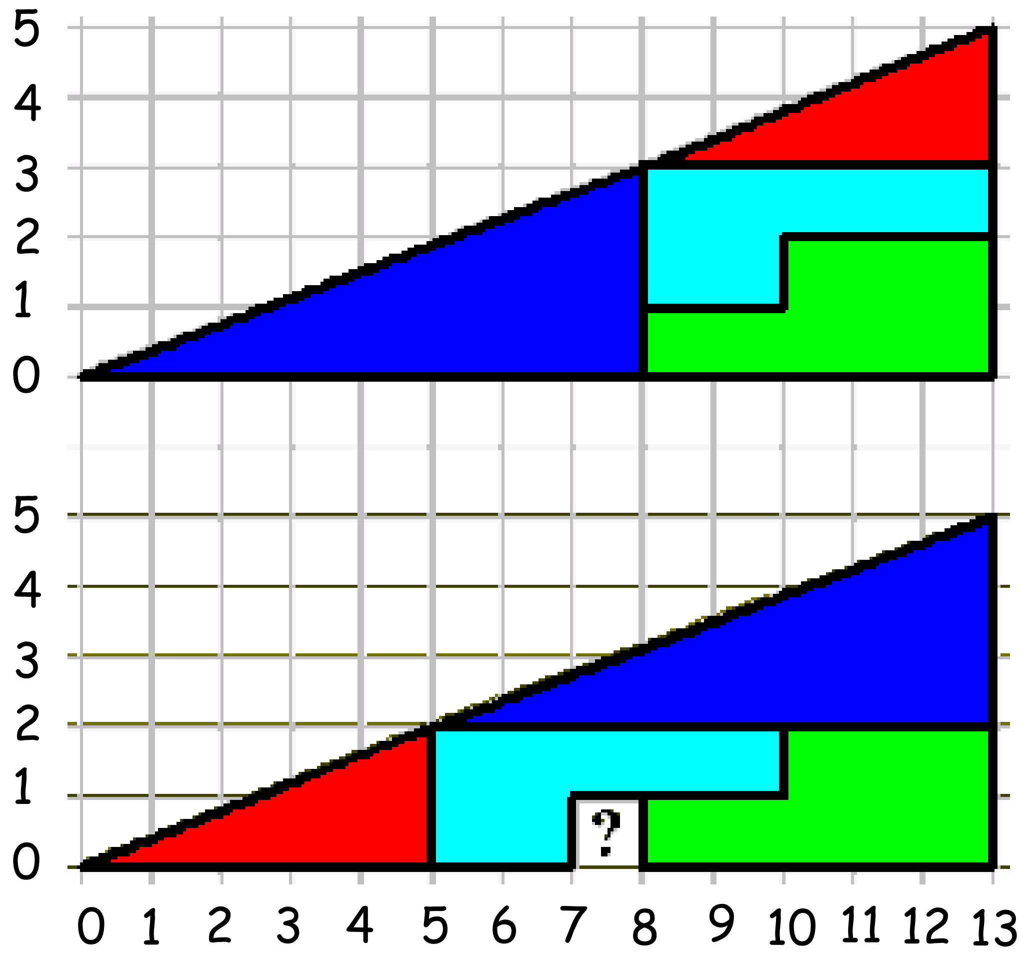


74 proofs in http://www.cut-the-knot.org/pythagoras/index.shtml

Bad Proof

A similar rearrangement technique shows that 65=63…

What's wrong with the proof?



2:5≠3:8

$\Theta_1=\tan^{-1}(3/8)=0.36^c$
$\Theta_2=\tan^{-1}(2/5)=0.38^c$

The "hypotenuse" of the big triangle is actually not a straight line

Mathematical Proof

To prove mathematical theorems, we need a more rigorous system.

The standard procedure for proving mathematical theorems was invented by Euclid in around 300BC. First he started with five **axioms** (the truth of these statements are taken for granted). Axioms are self-evident truths. Then he used **logic** to deduce the truth of other statements, called **propositions**. Important propositions are called **theorems,** and a **corollary** is a proposition that follows from a theorem. A **lemma** is a preliminary proposition useful for later proving theorems.

1. It is possible to draw a straight line from any point to any other point.
2. It is possible to produce a finite straight line continuously in a straight line.
3. It is possible to describe a circle with any center and any radius.
4. It is true that all right angles are equal to one another.
5. ("Parallel postulate") It is true that, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, intersect on that side on which are the angles less than the two right angles.

Euclid's proof of Pythagorean theorem http://en.wikipedia.org/wiki/Pythagorean_theorem

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Now, we have seen the need of a rigorous proof system.

We will proceed to define the basic logic system.

Statement (Proposition)

A *Statement* is a sentence that is either **True** or **False**

Examples:

$2 + 2 = 4$ True

$3 \times 3 = 8$ False

787009911 is a prime

Today is Tuesday.

Non-examples:

$x+y>0$

$x^2+y^2=z^2$

They are true for some values of x and y
but are false for some other values of x and y.

Logic Operators

Logic operators are used to construct new statements from old statements.

There are three main logic operators: NOT, AND, OR.

$$\neg \ ::= \text{NOT}$$

¬ P is true if and only if P is false

P	$\neg P$
T	F
F	T

$$\neg P = \overline{P}$$

Sometimes ~ P is also used for ¬ P

Logic Operators

Logic operators are used to construct new statements from old statements.

There are three main logic operators: NOT, AND, OR; these are called respectively, **negation**, **conjunction**, and **disjunction**. An expression which evaluates to either true or false is called a **Boolean expression**. Sometimes T and F are represented respectively as 1 and 0.

$$\wedge ::= \text{AND} \qquad \vee ::= \text{OR}$$

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Compound Statement

p = "it is hot" q = "it is sunny"

It is hot and sunny $p \wedge q$

It is not hot but sunny $\neg p \wedge q$

It is neither hot nor sunny $\neg p \wedge \neg q$

We can also define logic operators on three or more statements, e.g.

$$\overline{p \wedge q} \vee r$$

More Logical Operators

We can define more logical operators as we need.

coffee "or" tea

$\bigoplus$ exclusive-or

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

majority

P	Q	R	M(P,Q,R)
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	F
F	F	T	F
F	F	F	F

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We can define as many new operators as we like. But we will see how to construct any operator from AND, OR, NOT.

Formula for Exclusive-Or

Idea 0: Guess and check

$$p \oplus q \equiv (p \vee q) \wedge \neg(p \wedge q)$$

p	q	$p \oplus q$	$p \vee q$	$\neg(p \wedge q)$	
T	T	F	T	F	F
T	F	T	T	T	T
F	T	T	T	T	T
F	F	F	F	T	F

Logical equivalence: Two statements have the same truth table

As you will see, there are many different ways to write the same logical formula.

One can always use a truth table to check whether two statements are equivalent.

DeMorgan's Laws

Logical equivalence: Two statements have the same truth table

Statement: Tom is in the football team and the basketball team.

Negation: Tom is not in the football team or not in the basketball team.

De Morgan's Law
$$\boxed{\neg(p \wedge q) \equiv \neg p \vee \neg q}$$

Why the negation of the above statement is not the following
"Tom is not in the football team <u>and</u> not in the basketball team"?

The definition of the negation is that at least "p" or "q" is false, but it does not always have to be both false.

(e.g. Tom is in the football team but not in the basketball team).

DeMorgan's Laws

Logical equivalence: Two statements have the same truth table

Statement: The number 783477841 is divisible by 7 or 11.

Negation: The number 783477841 is not divisible by 7 and not divisible by 11.

De Morgan's Law

$$\boxed{\neg(p \vee q) \equiv \neg p \wedge \neg q}$$

Again, the negation of the above statement is not
"The number 783477841 is not divisible by 7 <u>or</u> not divisible by 11".

In either case, we "flip" the inside operator from OR to AND or from AND to OR.

DeMorgan's Laws

Logical equivalence: Two statements have the same truth table

De Morgan's Law

$$\neg(p \land q) \equiv \neg p \lor \neg q$$

p	q	$\neg(p \land q)$	$\neg p \lor \neg q$
T	T	F	F
T	F	T	T
F	T	T	T
F	F	T	T

De Morgan's Law

$$\neg(p \lor q) \equiv \neg p \land \neg q$$

Exclusive-Or

Is there a more systematic way to construct such a formula?

p	q	p ⊕ q
T	T	F
T	F	T
F	T	T
F	F	F

Idea 1: Look at each true row

Find a formula so that it is only true when having exactly the same input: the second row gives a T value when p is true but q is false, i.e. $(p \wedge \neg q)$; likewise do this for all rows with T value and OR them together

$$(p \wedge \neg q) \vee (\neg p \wedge q)$$

This sub-formula is true only when the input is the second row

And the formula is true exactly when the input is the second row or the third row.

Analogous to ordinary numbers, $\wedge$ may be viewed as product, and $\vee$ may be viewed as sum. The above approach is often called **sum-of-products**.

Exclusive-Or

Converting a truth table to Boolean expressions: **product-of-sums**

p	q	p ⊕ q
T	T	F
T	F	T
F	T	T
F	F	F

Idea 2: Look at the false rows

Find a formula so that it is only false when having exactly the same input. The second row gives a F value when both p and q are true, i.e. . ¬ (p ∧ q); likewise, do this for all rows with F value and AND them together, giving:

$$\neg(p \land q) \land \neg(\neg p \land \neg q).$$

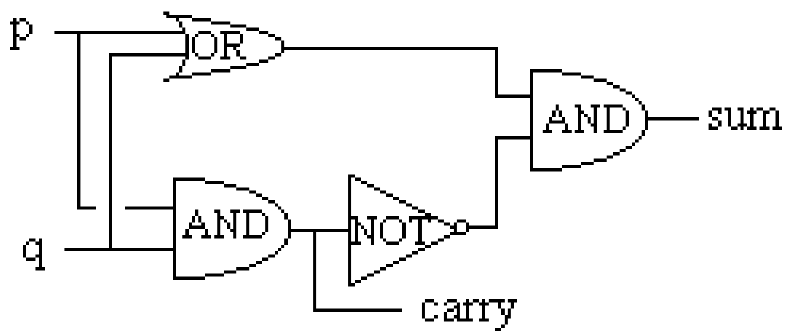
By DeMorgan's Law, this becomes

$$(\neg p \lor \neg q) \land (p \lor q),$$

which is a product-of-sums

Writing Logical Formula for a Truth Table

Digital logic:



p	q	sum	carry
1	1	0	1
1	0	1	0
0	1	1	0
0	0	0	0

Given a digital circuit, we can construct the truth table.

Now, suppose we are given only the truth table, how can we construct a digital circuit (i.e. formula) using only simple gates (such as AND, OR, NOT) that has the same function? From the previous slide, the truth table above corresponds to the Boolean expression

$$(\neg p \vee \neg q) \wedge (p \vee q) = \neg(p \wedge q) \wedge (p \vee q),$$

where the top part of the circuit corresponds to the second term on the right, and the bottom part corresponds to the first term.

Writing Logical Formula for a Truth Table

Use idea 1 or idea 2.

Idea 1: Look at the true rows and take the "or".

	p	q	r	output
$p \wedge q \wedge r$	T	T	T	F
$p \wedge q \wedge \neg r$	T	T	F	T
$p \wedge \neg q \wedge r$	T	F	T	T
$p \wedge \neg q \wedge \neg r$	T	F	F	F
$\neg p \wedge q \wedge r$	F	T	T	T
$\neg p \wedge q \wedge \neg r$	F	T	F	T
$\neg p \wedge \neg q \wedge r$	F	F	T	T
$\neg p \wedge \neg q \wedge \neg r$	F	F	F	F

$$(p \wedge q \wedge \neg r)$$
$$\vee (p \wedge \neg q \wedge r)$$
$$\vee (\neg p \wedge q \wedge r)$$
$$\vee (\neg p \wedge q \wedge \neg r)$$
$$\vee (\neg p \wedge \neg q \wedge r)$$

The formula is true exactly when the input is one of the true rows.

Writing Logical Formula for a Truth Table

Idea 2: Look at the false rows, negate and take the "and".

	p	q	r	output	
$p \wedge q \wedge r$	T	T	T	F	$\neg(p \wedge q \wedge r)$
$p \wedge q \wedge \neg r$	T	T	F	T	
$p \wedge \neg q \wedge r$	T	F	T	T	
$p \wedge \neg q \wedge \neg r$	T	F	F	F	$\wedge \neg(p \wedge \neg q \wedge \neg r)$
$\neg p \wedge q \wedge r$	F	T	T	T	
$\neg p \wedge q \wedge \neg r$	F	T	F	T	
$\neg p \wedge \neg q \wedge r$	F	F	T	T	
$\neg p \wedge \neg q \wedge \neg r$	F	F	F	F	$\wedge \neg(\neg p \wedge \neg q \wedge \neg r)$

The formula is true exactly when the input is not one of the false rows.

Logical rules

There are many different ways to write the same logical formula. As we have seen, one can always write a formula using only AND, OR, NOT.

1. *Commutative laws:*	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
2. *Associative laws:*	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
3. *Distributive laws:*	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
4. *Identity laws:*	$p \wedge \mathbf{t} \equiv p$	$p \vee \mathbf{c} \equiv p$
5. *Negation laws:*	$p \vee \sim p \equiv \mathbf{t}$	$p \wedge \sim p \equiv \mathbf{c}$
6. *Double negative law:*	$\sim(\sim p) \equiv p$	
7. *Idempotent laws:*	$p \wedge p \equiv p$	$p \vee p \equiv p$
8. *Universal bound laws:*	$p \vee \mathbf{t} \equiv \mathbf{t}$	$p \wedge \mathbf{c} \equiv \mathbf{c}$
9. *De Morgan's laws:*	$\sim(p \wedge q) \equiv \sim p \vee \sim q$	$\sim(p \vee q) \equiv \sim p \wedge \sim q$
10. *Absorption laws:*	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
11. *Negations of* **t** *and* **c**:	$\sim \mathbf{t} \equiv \mathbf{c}$	$\sim \mathbf{c} \equiv \mathbf{t}$

"t" means "tautology", i.e. always true;
"c" means "contradiction", i.e. always false.

Simplifying Statement

We can use logical rules to simplify a logical formula.

$$\neg(\neg p \wedge q) \wedge (p \vee q)$$

$$\equiv (\neg\neg p \vee \neg q) \wedge (p \vee q)$$ DeMorgan

$$\equiv (p \vee \neg q) \wedge (p \vee q)$$

$$\equiv p \vee (\neg q \wedge q)$$ Distributive law

$$\equiv p \vee \text{False}$$

$$\equiv p$$

The DeMorgan's Law allows us to always "move the NOT inside".

Tautology, Contradiction

A **tautology** is a statement that is always true.

$$p \vee \neg p$$

$$(p \wedge q) \vee (p \wedge \neg q) \vee (\neg p \wedge q) \vee (\neg p \wedge \neg q)$$

A **contradiction** is a statement that is always false. (negation of a tautology)

$$p \wedge \neg p$$

$$(\neg p \vee \neg q) \wedge (\neg p \vee q) \wedge (p \vee \neg q) \wedge (p \vee q)$$

In general it is "difficult" to tell whether a statement is a contradiction.

It is one of the most important problems in CS – the satisfiability problem

(e.g. whether the variables of a given Boolean expression can be consistently replaced by the values TRUE or FALSE in such a way that the formula evaluates to TRUE, cf NP Problems)

Checkpoint

Key points to know.

1. Write a logical formula from a truth table.
2. Check logical equivalence of two logical formulas.
3. DeMorgan’s rule and other simple logical rules (e.g. distributive).
4. Use simple logical rules to simplify a logical formula.

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Conditional Statement

If p then q

$$p \longrightarrow q$$

p implies q

p is called the hypothesis; q is called the conclusion

The department says: "If your GPA is 4.0, then you will have full scholarship."

When is the above sentence false?

- It is false when your GPA is 4.0 but you don't receive full scholarship.
- But it is not false if your GPA is below 4.0.

Another example: "If it is yellow typhoon sign today, then there will be no class."

When is the above sentence false?

Logic Operator

$$\rightarrow \; ::= \text{IMPLIES}$$

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Convention: if we don't say anything wrong, then it is not false, and thus true.

Make sure you understand the definition of IF.

The IF operation is very important in mathematical proofs.

Logical Equivalence

$$p \rightarrow q \equiv ?$$

If you see a question in the above form, there are usually 3 ways to deal with it.

(1) Truth table

(2) Use logical rules

(3) Intuition

If-Then as Or

$$p \rightarrow q \equiv ?$$

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Idea 2: Look at the false rows, negate and take the "and".

$$\neg(P \wedge \neg Q)$$
$$\equiv \neg P \vee Q$$

- If you don't give me all your money, then I will kill you.
- Either you give me all your money or I will kill you (or both 😂).

- If you talk to her, then you can never talk to me.
- Either you don't talk to her or you can never talk to me. (or both 😂).

Negation of If-Then

$$\neg(p \rightarrow q) \equiv ?$$

- If you eat an apple everyday, then you have no toothache.
- You eat an apple everyday but you have toothache.

- If my computer is not working, then I cannot finish my homework.
- My computer is not working but I can finish my homework.

$$\neg(P \rightarrow Q)$$

$$\equiv \neg(\neg P \vee Q) \qquad \boxed{\text{previous slide}}$$

$$\equiv \neg\neg P \wedge \neg Q \qquad \boxed{\text{DeMorgan}}$$

$$\equiv P \wedge \neg Q$$

Contrapositive

The contrapositive of "if p then q" is "if ¬q then ¬p".

Statement: If you are a CS year 2 student, then you are taking CSC 3001.

Contrapositive: If you are not taking CSC 3001, then you are not a CS year 2 student.

Statement: If you drive, then you don't drink.

Contrapositive: If you drink, then you don't drive.

Fact: A conditional statement is logically equivalent to its contrapositive.

Proofs

Statement: If P, then Q

Contrapositive: If ¬Q, then ¬P.

P	Q	$P \to Q$	$\neg Q$	$\neg P$	$\neg Q \to \neg P$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

In words, the only way the above statements are false is when P true and Q false.

Contrapositive

Statement: If P, then Q

Contrapositive: If $\neg$Q, then $\neg$P.

Or we can see it using logical rules:

$$P \rightarrow Q \equiv \neg P \vee Q \equiv Q \vee \neg P \equiv \neg Q \rightarrow \neg P$$

Contrapositive is useful in mathematical proofs, often used in indirect proofs or *reductio ad absurdum* (reduce to absurdity); e.g. to prove

Statement: If x^2 is an even number, then x is an even number.

You could instead prove:

Contrapositive: If x is an odd number, then x^2 is an odd number.

This is equivalent and is easier to prove.

If, Only-If

- You succeed if you work hard.
- You succeed only if you work hard.

R if S means "**if S then R**" or equivalently "**S implies R**"

We also say S is a sufficient condition for R.

R only if S means "**if R then S**" or equivalently "**R implies S**"

We also say S is a necessary condition for R.

You will succeed if and only if you work hard.

P if and only if (iff) Q means P and Q are logically equivalent.

That is, P implies Q and Q implies P.

Necessary AND Sufficient Condition

$$\leftrightarrow \ ::= \mathrm{IFF}$$

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Note: $P \longleftrightarrow Q$ is equivalent to $(P \longrightarrow Q) \wedge (Q \longrightarrow P)$

Note: $P \longleftrightarrow Q$ is equivalent to $(P \longrightarrow Q) \wedge (\neg P \longrightarrow \neg Q)$

Is the statement "x is an even number if and only if x^2 is an even number" true?

Math vs English

Parent: if you don't $\underbrace{\text{clean your room}}_{C}$, then you can't $\underbrace{\text{watch a DVD}}_{D}$.

This sentence says $\neg C \rightarrow \neg D$

In real life it also means $C \rightarrow D$

So $C \leftrightarrow D$

Mathematician: if an integer x greater than 2 is not an odd number, then x is not a prime number.

This sentence says $\neg O \rightarrow \neg P$

But of course it doesn't mean $O \rightarrow P$

Necessary, Sufficient Condition

Mathematician: if an integer x greater than 2 is not an odd number, then x is not a prime number.

This sentence says $\neg O \rightarrow \neg P$

But of course it doesn't mean $O \rightarrow P$

Being an odd number > 2 is a necessary condition for this number to be prime.

Being a prime number > 2 is a sufficient condition for this number to be odd.

Checkpoint

- Conditional Statements
  - The meaning of IF and its logical forms
  - Contrapositive
  - If, only if, if and only if

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Argument

An argument is a sequence of statements.

All statements but the final one are called assumptions or hypothesis.

The final statement is called the conclusion.

An argument is valid if:

whenever all the assumptions are true, then the conclusion is true.

If today is Wednesday, then yesterday was Tuesday.

Today is Wednesday.

$\therefore$ Yesterday was Tuesday.

Informally, an argument is valid if the conclusion follows from the assumptions.

Argument

An argument is a sequence of statements.

All statements but the final one are called assumptions or hypothesis.

The final statement is called the conclusion.

An argument is valid if:

> whenever all the assumptions are true, then the conclusion is true.

1. It is possible to draw a straight line from any point to any other point.
2. It is possible to produce a finite straight line continuously in a straight line.
3. It is possible to describe a circle with any center and any radius.
4. It is true that all right angles are equal to one another.
5. ("Parallel postulate") It is true that, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, intersect on that side on which are the angles less than the two right angles.

$\therefore$ Pythagorean's theorem

This is the formal way to prove theorems from axioms.

Modus Ponens

Rule:

If p then q.
p
∴ q

If typhoon, then class cancelled.
Typhoon.
∴ Class cancelled.

$\overbrace{\hspace{6em}}^{\text{assumptions}}$ conclusion

p	q	$p \rightarrow q$	p	q
T	T	T	T	T
T	F	F	T	F
F	T	T	F	T
F	F	T	F	F

Modus ponens is Latin meaning "method of affirming".

Modus Tollens

Rule:

If p then q.
~q
∴ ~p

If typhoon, then class cancelled.
Class not cancelled.
∴ No typhoon.

p	q	$p \rightarrow q$	~q	~p
T	T	T	F	F
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

Modus tollens is Latin meaning "method of denying".

Equivalence

A student is trying to prove that propositions P, Q, and R are all true.
She proceeds as follows.

First, she proves three facts:

- P implies Q
- Q implies R
- R implies P.

Then she concludes,

``Thus P, Q, and R are all true.''

Proposed argument:

assumption

$$\frac{(P \rightarrow Q),\ (Q \rightarrow R),\ (R \rightarrow P)}{P \wedge Q \wedge R}$$

conclusion

Is it valid?

Valid Argument?

$$\frac{(P \rightarrow Q),\ (Q \rightarrow R),\ (R \rightarrow P)}{P \wedge Q \wedge R}$$

Is it valid?

assumptions

conclusion

P	Q	R
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

$P \rightarrow Q$	$Q \rightarrow R$	$R \rightarrow P$
T	T	T
T	F	T
F	T	T
F	T	T
T	T	F
T	F	T
T	T	F
T	T	T

$P \wedge Q \wedge R$	OK?
T	yes
F	yes
F	yes
F	yes
F	yes
F	yes
F	yes
F	no

To prove an argument is not valid, we just need to find a counter-example.

Valid Arguments?

If p then q.
q
∴ p

		$\overbrace{\quad p \to q \quad\quad q \quad}^{\text{assumptions}}$		conclusion
p	q	p→q	q	p
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	F

Assumptions are true, but not the conclusion.

If you are a fish, then you drink water.
You drink water.
You are a fish.

Valid Arguments?

p	q	$\overbrace{p \to q \quad \sim p}^{\text{assumptions}}$		$\overset{\text{conclusion}}{\sim q}$
T	T	T	F	F
T	F	F	F	T
F	T	T	T	F
F	F	T	T	T

If p then q.
~p
$\therefore$ ~q

If you are a fish, then you drink water.

You are not a fish.

You do not drink water.

Exercises

$$\begin{array}{l} \quad p \\ \therefore\; p \vee q \end{array} \quad \checkmark$$

$$\begin{array}{l} \quad p \wedge q \\ \therefore\; p \end{array} \quad \checkmark$$

$$\begin{array}{l} \quad p \vee q \\ \quad \neg q \\ \therefore\; p \end{array} \quad \checkmark$$

$$\begin{array}{l} \quad p \\ \therefore\; p \wedge q \end{array} \quad \times$$

$$\begin{array}{l} \quad p \vee q \\ \therefore\; p \end{array} \quad \times$$

$$\begin{array}{l} \quad p \longrightarrow q \\ \quad q \longrightarrow r \\ \therefore\; p \longrightarrow r \end{array} \quad \checkmark$$

More Exercises

$$\begin{array}{l} \neg p \to q \\ \neg q \\ \therefore\ p \end{array}$$

$$\begin{array}{l} \neg p \to \neg q \\ \therefore\ p \to q \end{array}$$

$$\begin{array}{l} \neg p \to \neg q \\ \therefore\ q \to p \end{array}$$

$$1 = -1$$

$\therefore$ Today is Tuesday.

Valid argument $\not\longrightarrow$ True conclusion

True conclusion $\longrightarrow$ Valid argument

Assumption may not be true.

Contradiction

$$\neg p \longrightarrow c$$
$$\therefore\ p$$

To see this argument is valid, you need to show:

If assumption is true, then conclusion is true.

p	$\sim p$	c	$\sim p \rightarrow$ c	p
T	F	F	T	T
F	T	F	F	

Truth-tellers and Liars

Truth-tellers always tell the truth.

Liars always lie.

A says: B is a truth-teller.

B says: A and I are of opposite type.

Suppose A is a truth-teller.

Then B is a truth-teller (because what A says is true).

Then A is a liar (because what B says is true)

A contradiction.

So A must be a liar.

So B must be a liar (because what A says is false).

No contradiction.

Quick Summary

- **Arguments**
  - definition of a valid argument
  - method of affirming, denying, contradiction

Key points:

(1) Make sure you understand conditional statements and contrapositive.

(2) Make sure you can check whether an argument is valid.

Valid Argument Forms

Modus Ponens	$p \to q$ p $\therefore q$	**Elimination**	**a.** $p \vee q$ $\sim q$ $\therefore p$ **b.** $p \vee q$ $\sim p$ $\therefore q$
Modus Tollens	$p \to q$ $\sim q$ $\therefore \sim p$	**Transitivity**	$p \to q$ $q \to r$ $\therefore p \to r$
Generalization	**a.** p $\therefore p \vee q$ **b.** q $\therefore p \vee q$	**Proof by Division into Cases**	$p \vee q$ $p \to r$ $q \to r$ $\therefore r$
Specialization	**a.** $p \wedge q$ $\therefore p$ **b.** $p \wedge q$ $\therefore q$		
Conjunction	p q $\therefore p \wedge q$	**Contradiction Rule**	$\sim p \to c$ $\therefore p$

Which is true?

Which is false?

"The sentence below is false."

"The sentence above is true."