Stochastic Processes

Lecture 15: Poisson processes (III)

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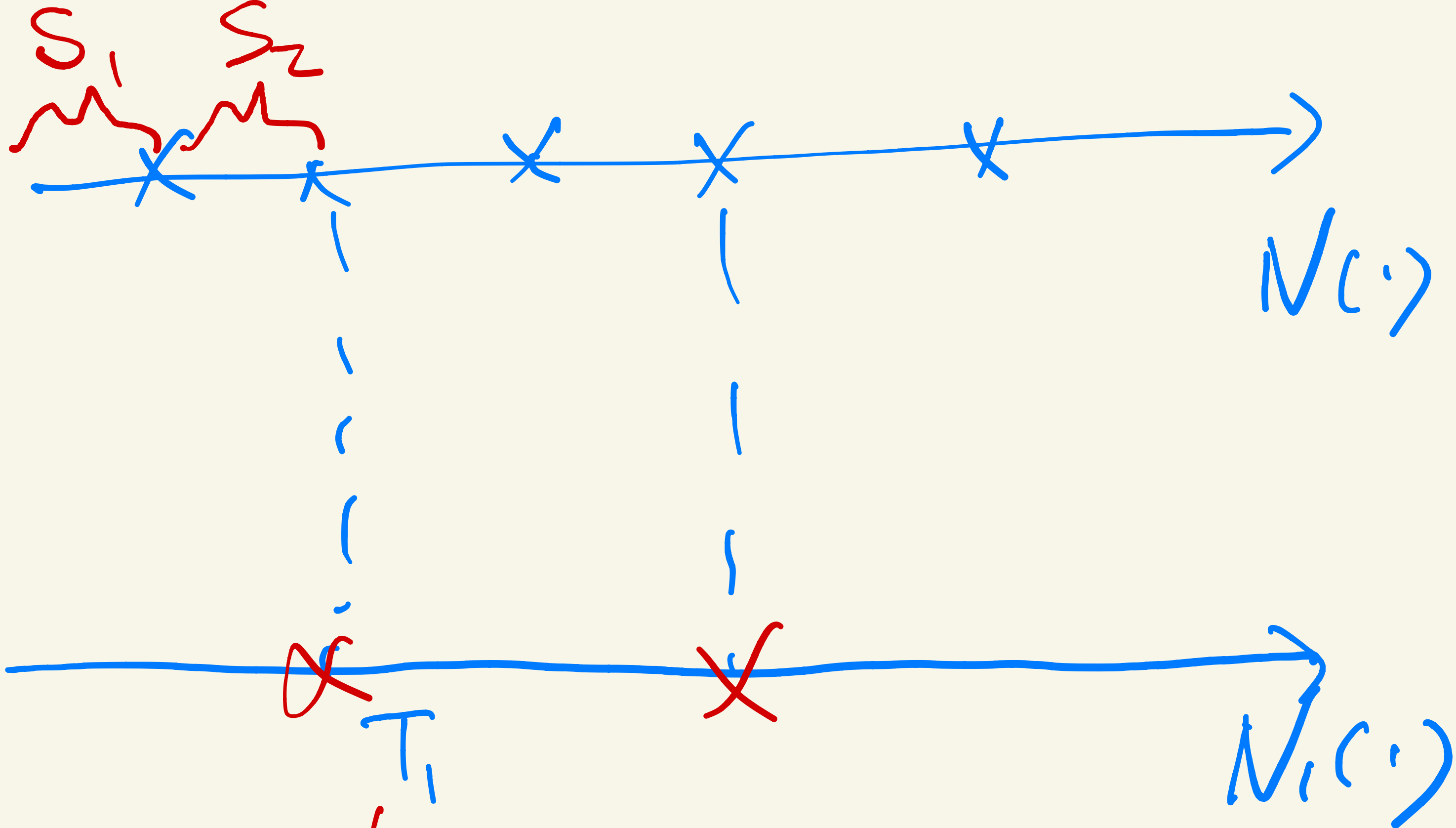
Thinning

- Let $N = \{N(t), t \geq 0\}$ be a Poisson process with rate λ.
- Each arrival flips a coin with probability of p getting a head.
- $N_1(t)$ is the number of heads in $(0, t]$,
- $N_2(t)$ is the number of tails in $(0, t]$.
- $N_i = \{N_i(t), t \geq 0\}$ is a Poisson process with rate λ_i, $i = 1, 2$, where $\lambda_1 = \lambda p$ and $\lambda_2 = \lambda(1 - p)$.
- Furthermore N_1 and N_2 are independent.



Poisson? X

$$T_1 = S_1 + S_2 = \text{Gamma}(2, \lambda)$$

$\neq$ Exponential!

$$N_1(t) \sim \mathcal{P}(\lambda p t)$$

$$= \sum_{k=1}^{N(t)} \mathbb{1}\{X_k = 1\} = \sum_{k=1}^{N(t)} Y_k$$

m.g.f. $\qquad Y_k \sim \mathrm{Ber}(p)$

$$\mathbb{E}\, e^{sN_1(t)} = \mathbb{E}\, e^{s\sum_{k=1}^{N(t)} Y_k}$$

$$= \sum_{n=1}^{\infty} \mathbb{P}(N(t)=n) \cdot \underline{\mathbb{E}\, e^{s\sum_{k=1}^{n} Y_k}}$$

$$= \sum_{n=1}^{\infty} e^{-\lambda t} \cdot \frac{(\lambda t)^n}{n!} \cdot \left(\mathbb{E} e^{sY_1}\right)^n$$

$$= \sum_{n=1}^{\infty} e^{-\lambda t} \cdot \frac{(\lambda t)^n}{n!} \cdot (p(e^s-1)+1)^n$$

$$= \sum_{n=1}^{\infty} e^{-\lambda t} \cdot \frac{[\lambda t(p(e^s-1)+1)]^n}{n!}$$

$$= e^{-\lambda t} \cdot e^{\lambda t(p(e^s-1)+1)} = \underline{e^{\lambda p t(e^s-1)}}$$

$[s_1, t_1]$ $[s_2, t_2]$ $[s_3, t_3]$

$N(s_1, t_1] \perp N(s_2, t_2] \perp N(s_3, t_3]$

Definition

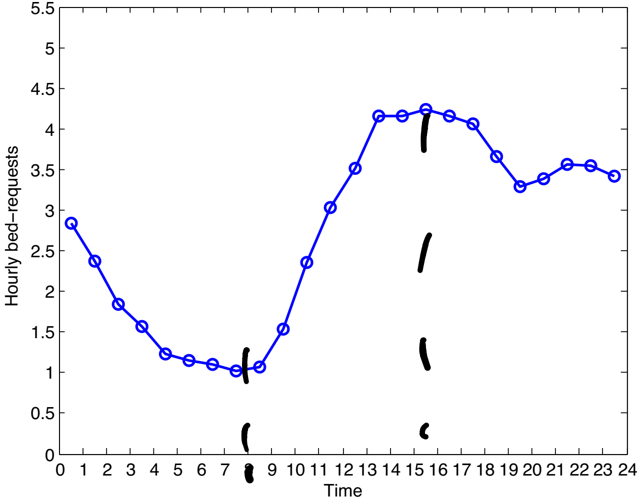
A stochastic process $N = \{N(t), t \geq 0\}$ is said to be a (time-nonhomogeneous) Poisson process with rate function $\{\lambda(t), t \geq 0\}$ if (a) it has independent increments, (b) $N(s,t] \sim \text{Poisson}(\int_s^t \lambda(u)\, du)$ for any $0 \leq s < t$, (c) $N(0) = 0$.

$\lambda(t) = \lambda$

$\int_s^t \lambda(u)\, du = \int_s^t \lambda\, du = \lambda(t-s)$

$N(t) \sim \mathcal{P}\left(\int_0^t \lambda(s)\, ds\right)$

Bed-request patterns



Similar patterns observed in Armony et al. (2015), Griffin et al. (2011), Powell et al. (2012)

Simulating a non-homogeneous Poisson process

$G(t)$ = # of arrivals before t.

$N(t)$ = # of arrivals before $\Lambda(t)$

- time-change: G is a rate-1 Poisson process

$$N(t) = G\left(\underbrace{\int_0^t \lambda(u)\,du}\right) = \Lambda(t), \quad t \geq 0.$$

- N is a Poisson process with rate function $\{\lambda(t), t \geq 0\}$.
- accept-reject: Suppose that $\lambda(t) \leq \Lambda$ for all $t \geq 0$. Let G be a Poisson process with rate Λ. At each arrival time t of G, flip a coin with probability of $\lambda(t)/\Lambda$ getting a head.

$$N(t) = \# \text{ of heads in } (0, t].$$

- Is N what we expect to be?

$G(\cdot)$ $\quad Exp(\Lambda) = \frac{1}{\Lambda} exp(1)$

Simulate Unit rate Poisson $N(t) = G(\Lambda(t))$

$T_1, T_2, \cdots, T_n, \cdots$

$U_1, U_2, \cdots, U_n, \cdots \qquad U_i \sim \mathrm{Unif}(0,1)$

$T_1 = -\log U_1, \quad T_2 = T_1 + (-\log U_2), \cdots$

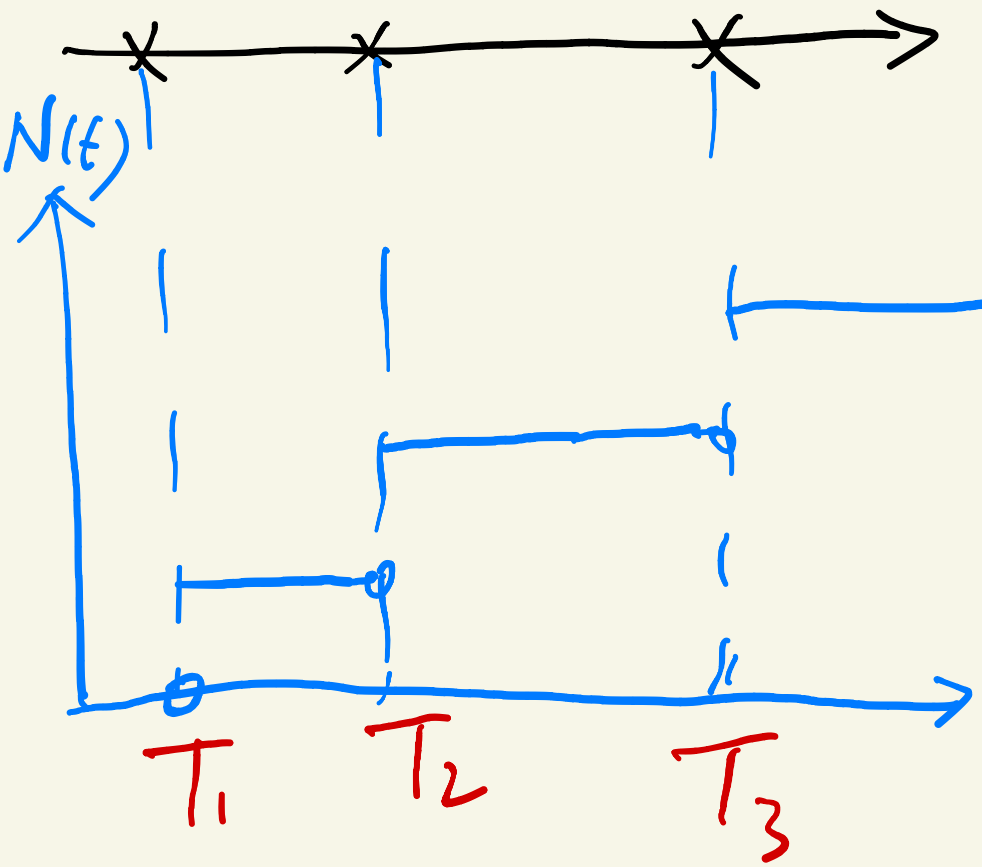
$$N(s_1, t_1) \perp N(s_2, t_2)$$

$$= G(\Lambda(t_1)) - G(\Lambda(s_1)) \perp = G(\Lambda(t_2)) - G(\Lambda(s_2))$$

$$(\Lambda(s_1), \Lambda(t_1)) \perp (\Lambda(s_2), \Lambda(t_2))$$

$$N(t) = G(\Lambda(t)) \sim \mathcal{P}(\Lambda(t))$$

$$= \mathcal{P}\left(\int_0^t \lambda(s)\, ds\right)$$



$\lambda(t) = t$

$\Lambda(t) = \frac{1}{2} t^2$

$G(\cdot)$ 2, 3, 8.2

$t \to \frac{1}{2}t^2 = 2$

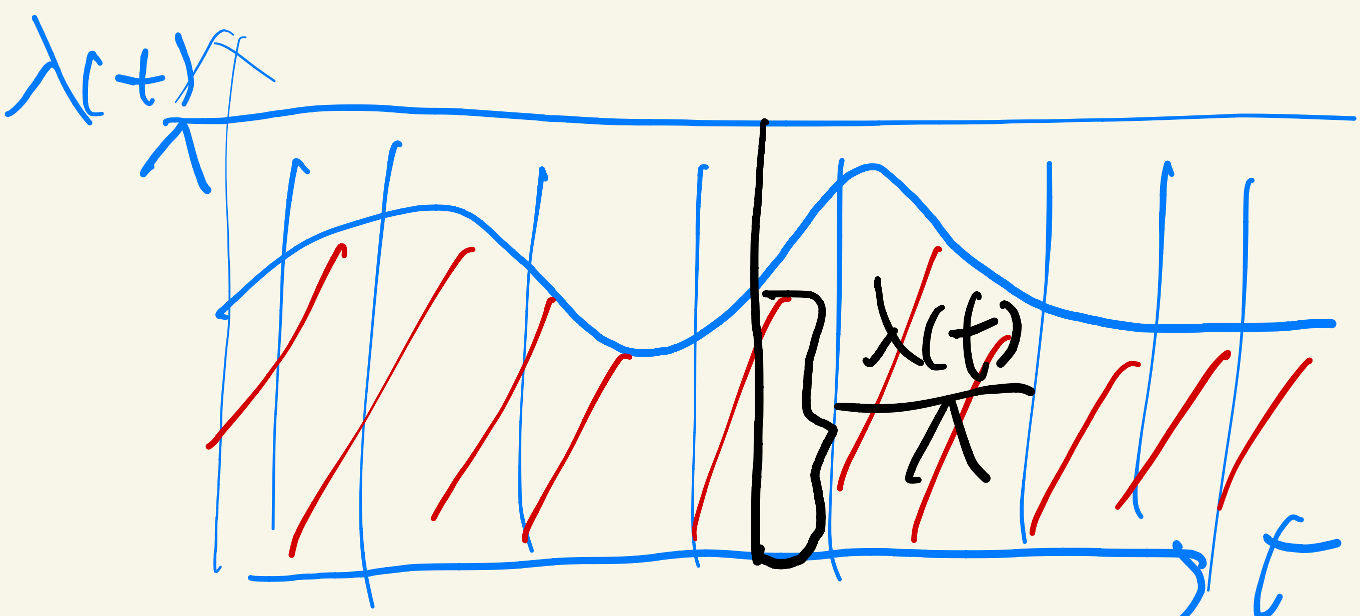
$t = 2$

$N(\cdot)$ 2, $\sqrt{6}$, ...

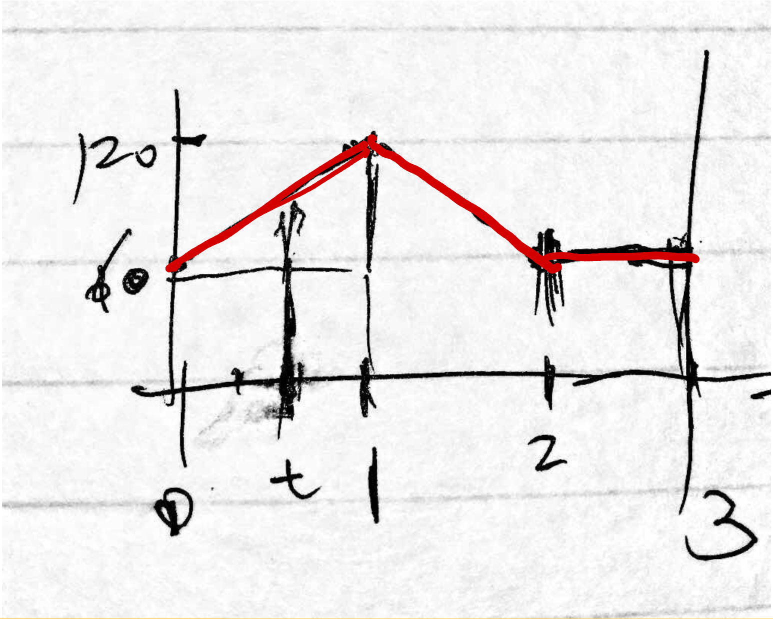
$$N(t) = G(\Lambda(t))$$

$G(\cdot)$ s_1, s_2, ..., s_n, ...

$N(\cdot)$ $\Lambda^{-1}(s_1)$, $\Lambda^{-1}(s_2)$, ..., $\Lambda^{-1}(s_n)$, ...

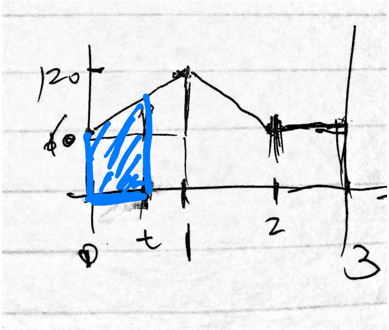


An example



Rate function $\lambda(t)$

- $\lambda(t) = 60 + 60t$ for $0 \leq t \leq 1$
- $\lambda(t) = 120 - 60(t - 1)$ for $1 \leq t \leq 2$
- $\lambda(t) = 60$ for $t \cancel{\leq} \geq 2$.



$$\Lambda(s) = (60+30s)s = t$$

$$30s^2 + 60s = t$$

$$30(s+1)^2 = t+30$$

$$s+1 = \sqrt{\frac{t}{30}+1}$$

$$s = \sqrt{1+\frac{t}{30}} - 1 = \Lambda^{-1}(t).$$

- $\Lambda(s) = \frac{60+(60+60s)}{2}s$ for $0 \leq s \leq 1$.
- $\Lambda(1) = 90$.
- $\Lambda(s) = 90 + \frac{60+(120+60(2-s))}{2}(s-1)$ for $1 \leq s \leq 2$.
- $\Lambda(2) = 180$.
- $\Lambda(s) = 180 + 60(s-2)$ for $2 \leq s \leq 3$.

$\Lambda^{-1}(t)$

- For $0 \leq t \leq 90$,

$$\Lambda^{-1}(t) = \sqrt{1 + t/30} - 1.$$

- For $90 \leq t \leq 180$,

$$30s^2 - 90s + t - 120 = 0,$$
$$s = 3 - \sqrt{9 - (t + 60)/30}$$

- For $180 \leq t$,

$$s = 2 + \frac{t - 180}{60}.$$

Example, $A = 120$

- For $0 \leq t \leq 1$

$$\frac{\lambda(t)}{A} = \frac{60 + 60t}{120} = .5 + .5t.$$

Given $N(s)$, $N(s+t) \perp \{N(r): r \le s\}$

THEOREM

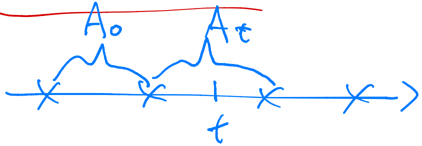
Let $\{N(t)\}_{t\geq 0}$ be a Poisson process of rate λ. Then, for any $s \geq 0$, $\{N(s+t) - N(s)\}_{t\geq 0}$ is also a Poisson process of rate λ, independent of $\{N(r) : r \leq s\}$.

Comapre with Markov Property for DTMC.

Renewal Process

Interarrival time Erlang, Hyper-expon, uniform, ...

Inspection Paradox



$E[A_t] > E A_0.$