MAT2006 Tutorial #6

1. Decide which of the following sets are compact.

$$
\begin{array}{ll}
(a) & \mathbb{N} \\
(b) & \mathbb{Q} \cap [0,1] \\
(c) & \text{the Cantor set} \\
(d) & \left\{ \displaystyle\sum_{k=1}^{n} \frac{1}{k^2} \quad \middle| \quad n \in \mathbb{N} \right\} \\
(e) & \{1, 1/2, 2/3, 3/4, 4/5, \dots\}
\end{array}
$$

2. Recall that compactness is defined using "every open cover of A has a finite subcover then A is compact." and recall that HB says "$A \subset \mathbb{R}$ is compact iff A is bounded and closed"

Show that

(a) from a system of closed intervals covering a closed interval it is not always possible to choose a finite subsystem covering the interval;

(b) from a system of open intervals covering an open interval it is not always possible to choose a finite subsystem covering the interval;

(c) from a system of closed intervals covering an open interval it is not always possible to choose a finite subsystem covering the interval

3. Let $f(x)$ be a function defined on a bounded interval I. Assume for each $x \in I$, there exists a neighborhood of $V_\epsilon(x)$ such that $f(x)$ is bounded on $V_\epsilon(x) \cap I$.

(i) If $I = [a, b]$, show that $f(x)$ is bounded.

(ii) If $I = (a, b)$, is $f(x)$ bounded on I?

4. Prove that very bounded infinite set has a limit point using two alternative approaches:

(i) Bolzano–Weierstrass Theorem (or equivalently, a set is sequentially compact iff it is bounded and closed.)

(ii) Heine-Borel Theorem.

— End —