Sheet 7 118010350

A7.1 (a) Since $f(x) = 100(x_2 - x_1^2)^2 + (1-x_1)^2$, then we compute the gradient and the Hessian of $f(x)$.

$$\nabla f(x) = \begin{bmatrix} 400x_1^3 - 400x_1x_2 + 2x_1 - 2 \\ 200x_2 - 200x_1^2 \end{bmatrix}$$

$$\nabla^2 f(x) = \begin{bmatrix} 1200x_1^2 - 400x_2 + 2 & -400x_1 \\ -400x_1 & 200 \end{bmatrix}$$

The report of globalized Newton method is as follows:

① The number of iterations is 23;

② The final objective function value is $5.7424 \times 10^{-26} \approx 0$.

③ The points converges to $x^* = [1;1]^T$.

The approach select both the Newton direction and the gradient direction.

(b). According to the graph, a quadratic covergence can be observed

⇒ Graph is attached at the end.

(c). The report of gradient method is as follows:

① The number of iterations is 6699.

② The final objective function value is $1.1742 \times 10^{-12} \approx 0$.

③ The points converges to $x^* = [1;1]^T$.

Plot the graph $(\|x^k - x^*\|)_k$ versus k, at first, a slow linear convergence can be observed, then flollows a quick linear convergence.

Comparison: The Newton method uses less iterations to approach the optimal, and the convergence rate is fast; the gradient method uses much more iterations to approach the optimal, and the convergence rate is slow.

A7.2. ① The LP relaxation (S) of the original problem is,

$$\begin{aligned} \text{maximize} \quad & 2x + y \\ \text{subject to} \quad & -3x + 2y \le 5 \\ & -x - 2y \le -2 \\ & 5x + 2y \le 17 \end{aligned}$$

In this case, the optimal solution is $x = 1.5$, $y = 4.75$, the optimal value is 7.75.

Consider four subproblems, $x \le 1, y \le 4 (\rightarrow S_1)$; $x \le 1, y \ge 5 (\rightarrow S_2)$; $x \ge 2, y \le 4 (\rightarrow S_3)$; $x \ge 2, y \ge 5 (\rightarrow S_4)$.

② Subproblem (S_1): In this case, the optimal solution is $x = 1$, $y = 4$, the optimal value is 6, which is the lower bound (Stop).

③ Subproblem (S_2): In this case, it is infeasible (Stop).

④ Subproblem (S_3): In this case, the optimal solution is $x = 2$, $y = 3.5$, the optimal value is 7.5.

Consider two subproblems, $x \ge 2, y \le 3 (\rightarrow S_5)$; $x \ge 2, y = 4 (\rightarrow S_6)$.

⑤ Subproblem (S_5): In this case, the optimal solution is $x = 2.2$, $y = 3$, the optimal value is 7.4.

Consider two subproblems, $x = 2, y \le 3 (\rightarrow S_7)$; $x \ge 3, y \le 3 (\rightarrow S_8)$

⑥ Subproblem (S_7): In this case, the optimal solution is $x = 2$, $y = 3$, the optimal value is 7, which is the lower bound (Stop).

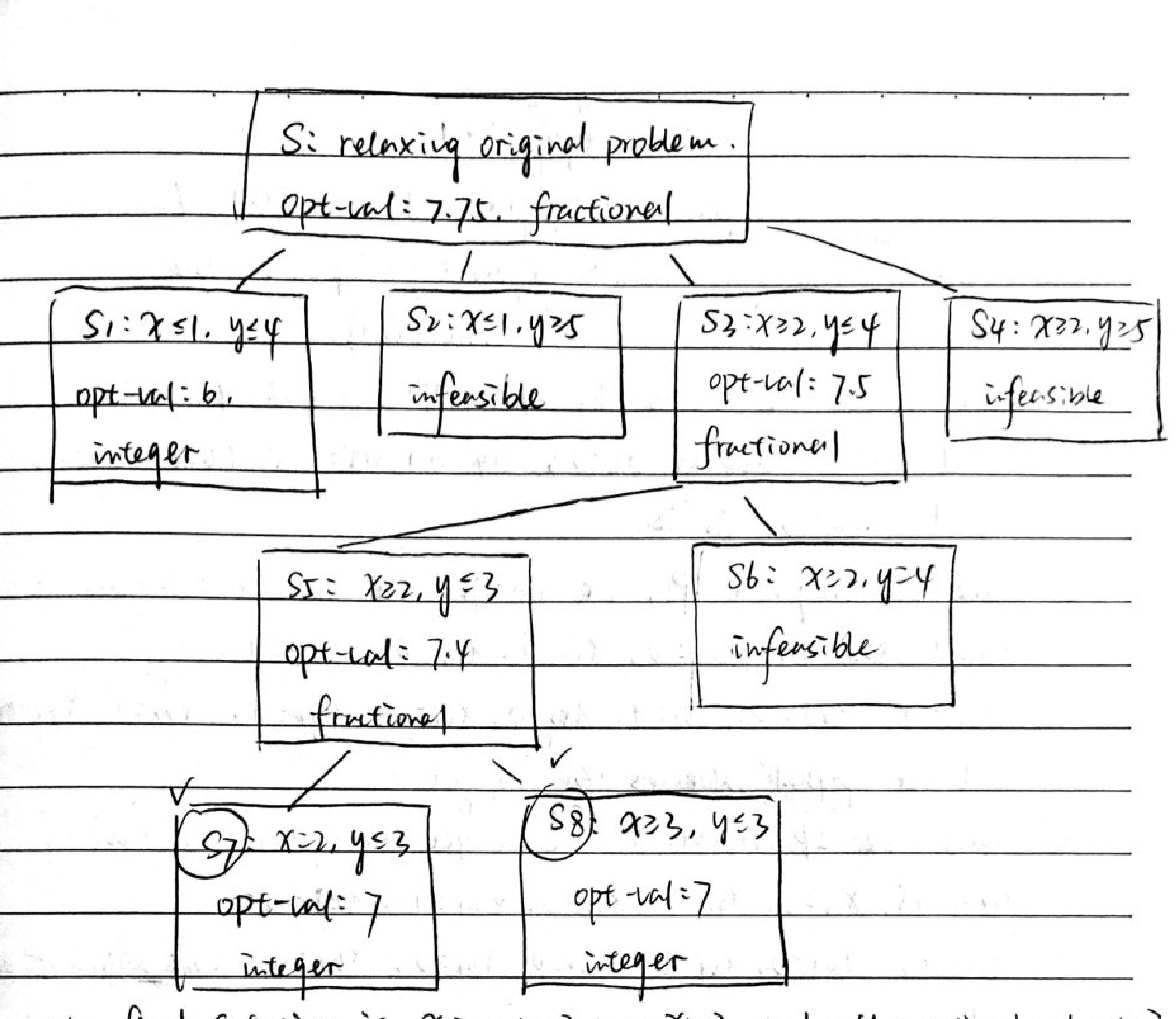
⑦ Subproblem (S_8): In this case, the optimal solution is $x = 3$, $y = 1$, the optimal value is 7. (Stop).

⑧ Subproblem (S_6): In this case, it is infeasible (Stop).

⑨ Subproblem (S_4): In this case, it is infeasible (Stop).

(Done).

Then we can draw the branch-and-bound tree for this linear problem.



The final solution is $x = 2$, $y = 3$ or $x = 3$, $y = 1$, the optimal value is 7.

A7.3. (a). We introduce variables x_{ij} to denote whether item i is placed in knapsack j, then the problem can be formulated as follows:

$$\begin{aligned} \text{maximize} \quad & \sum_{i=1}^{n} \sum_{j=1}^{m} v_i \cdot x_{ij} \\ \text{subject to} \quad & \sum_{j=1}^{m} x_{ij} \le 1, \quad \forall i \in [1, n] \\ & \sum_{i=1}^{n} a_i x_{ij} \le c_j, \quad \forall j \in [1, m] \\ & x_{ij} \in \{0, 1\}. \end{aligned}$$

(b). We can know that $n = 7$ and $m = 2$.

Then we can formulate the corresponding IP.

$$\begin{aligned} \text{maximize} \quad & \sum_{i=1}^{7} \sum_{j=1}^{2} v_i x_{ij} \\ \text{subject to} \quad & \sum_{j=1}^{2} x_{ij} \le 1, \quad \forall i = 1, \dots, 7. \\ & \sum_{i=1}^{7} a_i \cdot x_{ij} \le c_j, \quad \forall j = 1, 2. \\ & x_{ij} \in \{0, 1\}. \end{aligned}$$

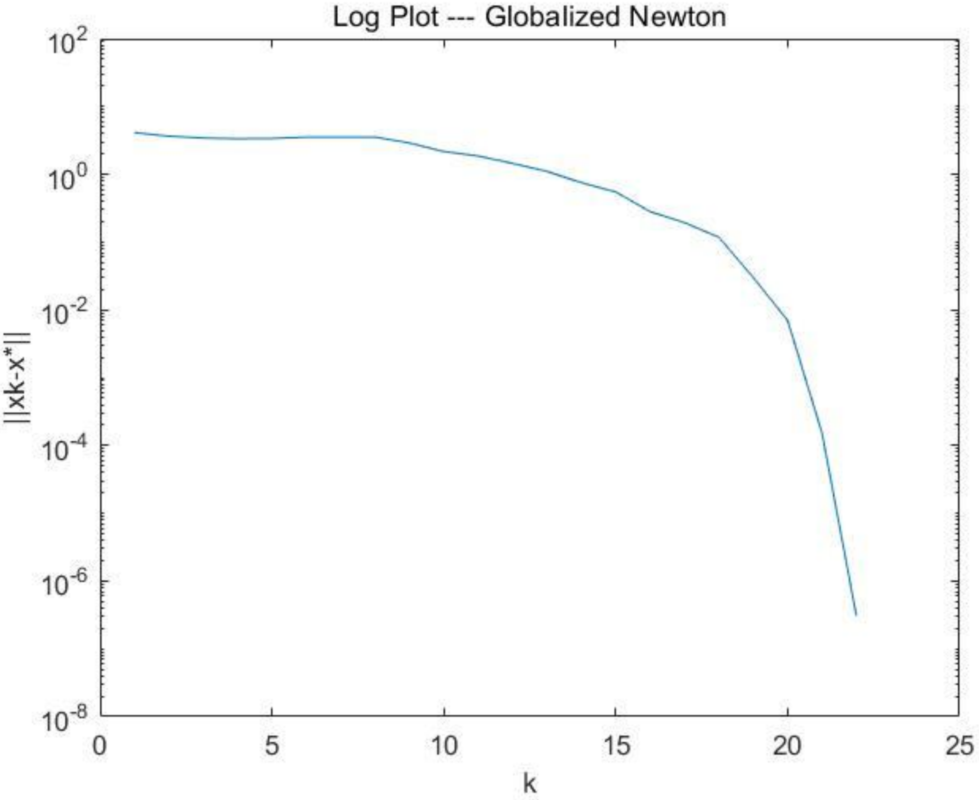
where $v_1 = 2$, $v_2 = 1$, $v_3 = 3$, $v_4 = 2$, $v_5 = 1$, $v_6 = 4$, $v_7 = 2$ and $a_1 = 2$, $a_2 = 0.5$, $a_3 = 0.5$, $a_4 = 0.1$, $a_5 = 0.5$, $a_6 = 1$, $a_7 = 1.5$ and $c_1 = 3$, $c_2 = 2$.

① Solve the original IP, we can get the optimal solution is $x_{11} = 0$, $x_{12} = 0$, $x_{21} = 0$, $x_{22} = 1$, $x_{31} = 0$, $x_{32} = 1$, $x_{41} = 1$, $x_{42} = 0$, $x_{51} = 1$, $x_{52} = 0$, $x_{61} = 0$, $x_{62} = 1$, $x_{71} = 1$, $x_{72} = 0$ and the optimal value is 13.

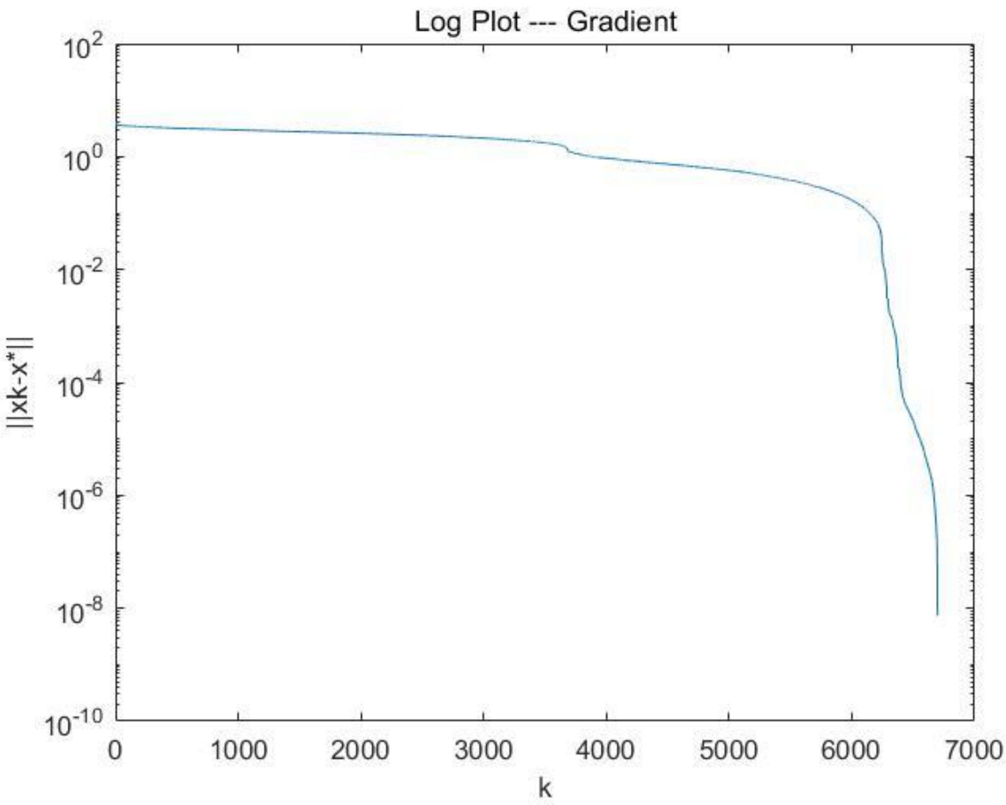
② Solve the LP relaxation, we can get the optimal solution is $x_{11} = 0.45$, $x_{12} = 0$, $x_{21} = 1$, $x_{22} = 0$, $x_{31} = 1$, $x_{32} = 0$, $x_{41} = 1$, $x_{42} = 0$, $x_{51} = 1$, $x_{52} = 0$, $x_{61} = 0$, $x_{62} = 1$, $x_{71} = 0.33$, $x_{72} = 0.67$ and the optimal value is 13.9.

The integrality gap is $13.9 - 13 = 0.9$.

A7.1(b)


Log Plot --- Globalized Newton
10^2
10^0
10^{-2}
10^{-4}
10^{-6}
10^{-8}
||xk-x*||
0
5
10
15
20
25
k

A7.1(c)


Log Plot --- Gradient
10^2
10^0
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
||xk-x*||
0
1000
2000
3000
4000
5000
6000
7000
k