MAT 3007 – Optimization

Sensitivity Analysis

Lecture 10 *June 30th*

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Repetition

Recap: Duality Theory

- Construct the dual problem.
- Weak duality theorem/strong duality theorem.
- Complementarity conditions
- Interpret the dual problem in applications:
  - The production planning problem
  - The multi-firm alliance problem
  - The alternative systems problem
  - The maximum flow problem

Sensitivity Analysis

One important question when studying LP is as follows:

- How do the optimal solution and the optimal value change when the input changes?

This type of problems is called sensitivity analysis.

- We first study this question from a local perspective and then continue with global discussions.

Local Sensitivity

Consider the standard LP:

$$\begin{aligned} \text{minimize}_x \quad & c^\top x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0. \end{aligned}$$

We denote the associated optimal value by V.

- If A and c are fixed, V can be viewed as a function of b: $V(b)$.

Theorem: Differentiability of the Optimal Value Function

If the dual has a unique optimal solution y^*, then $\nabla V(b) = y^*$.

- If the dual optimal solution is not unique (or is unbounded or infeasible), then the gradient is not well-defined.
- If one changes b_i by a small amount Δb_i, then the change of the objective value will be $\Delta b_i y_i^*$

Explanation

We know that the optimal value V is also the optimal value of the dual problem:

$$\begin{aligned} \text{maximize}_y \quad & b^\top y \\ \text{s.t.} \quad & A^\top y \leq c, \end{aligned}$$

i.e., $V(b) = b^\top y^*$.

$\rightsquigarrow$ If we change b by a small amount Δb, such that the optimal sol. does not change, then the change of V must be $\Delta b^\top y^*$.

Similarly, if A and b are fixed, V can be viewed as a function of c.

Theorem: Differentiability of $V(c)$

If the primal prob. has a unique optimal sol. x^*, then $\nabla V(c) = x^*$.

If one changes c_i by a small amount Δc_i, then the change of the objective value will be $\Delta c_i x_i^*$.

$\rightsquigarrow$ Reason: If we change c by a small amount Δc, such that the optimal solution does not change, then the change of V must be $\Delta c^\top x^*$.

Local Sensitivity

The latter results also hold for inequality constraints (or maximization problems):

$$\begin{aligned} \text{maximize}_x \quad & c^\top x \\ \text{s.t.} \quad & Ax \le b \\ & x \ge 0. \end{aligned}$$

We have:

1. If the dual has a unique optimal sol. y^*, then $\nabla V(b) = y^*$.
2. If the primal has a unique optimal sol. x^*, then $\nabla V(c) = x^*$.

- To see why this must be true, one can add a slack variable and transform it back to the standard form. We can then use the earlier result.

$$\begin{array}{rrrl} \text{maximize} & x_1 & +2x_2 & \\ \text{subject to} & x_1 & & \leq 100 \\ & & 2x_2 & \leq 200 \\ & x_1 & +x_2 & \leq 150 \\ & x_1, & x_2 & \geq 0 \end{array}$$

The optimal solution is $x^* = (50, 100)$ with optimal value 250.

The dual problem is

$$\begin{array}{rrrrl} \text{minimize} & 100y_1 & +200y_2 & +150y_3 & \\ \text{subject to} & y_1 & & +y_3 & \geq 1 \\ & & 2y_2 & +y_3 & \geq 2 \\ & y_1, & y_2, & y_3 & \geq 0 \end{array}$$

The optimal solution is $y^* = (0, 0.5, 1)$ with optimal value 250.

Example: Continued - I

$$
\begin{array}{rrrl}
\text{maximize} & x_1 & +2x_2 & \\
\text{subject to} & x_1 & & \leq 100 \\
 & & 2x_2 & \leq 200 \\
 & x_1 & +x_2 & \leq 150 \\
 & x_1, & x_2 & \geq 0
\end{array}
$$

The optimal solution is $x^* = (50, 100)$ with optimal value 250. The dual optimal solution is $y^* = (0, 0.5, 1)$.

Q1: What is the optimal value if we have 202 units of resource 2?

- It will change by $\Delta b_2 y_2^* = 1$. Therefore, the optimal value would be 251 ($\rightsquigarrow$ check with CVX: $\checkmark$).

$$\begin{array}{lrrl} \text{maximize} & x_1 & +2x_2 & \\ \text{subject to} & x_1 & & \leq 100 \\ & & 2x_2 & \leq 200 \\ & x_1 & +x_2 & \leq 150 \\ & x_1, & x_2 & \geq 0 \end{array}$$

The optimal solution is $x^* = (50, 100)$ with optimal value 250. The dual optimal solution is $y^* = (0, 0.5, 1)$.

Q2: What is the optimal value if we have 99 units of resource 1?

- It will change by $\Delta b_1 y_1^* = 0$. Therefore, the optimal value would be unchanged ($\rightsquigarrow$ check with CVX: $\checkmark$).

$$\begin{array}{rrrl} \text{maximize} & x_1 & +2x_2 & \\ \text{subject to} & x_1 & & \leq 100 \\ & & 2x_2 & \leq 200 \\ & x_1 & +x_2 & \leq 150 \\ & x_1, & x_2 & \geq 0 \end{array}$$

The optimal solution is $x^* = (50, 100)$ with optimal value 250. The dual optimal solution is $y^* = (0, 0.5, 1)$.

Q3: What is the opt. value if the profit of product 1 becomes 1.02?

- It will increase by $\Delta c_1 x_1^* = 1$. Therefore, the optimal value would be 251 ($\rightsquigarrow$ check with CVX: $\checkmark$).

$$
\begin{array}{rrrl}
\text{maximize} & x_1 & +2x_2 & \\
\text{subject to} & x_1 & & \leq 100 \\
 & & 2x_2 & \leq 200 \\
 & x_1 & +x_2 & \leq 150 \\
 & x_1, & x_2 & \geq 0
\end{array}
$$

The optimal solution is $x^* = (50, 100)$ with optimal value 250. The dual optimal solution is $y^* = (0, 0.5, 1)$

Q4: What is the opt. value if the profit of product 2 becomes 1.97?

- It will decrease by $\Delta c_2 x_2^* = -3$. Therefore, the optimal value would be 247 ($\rightsquigarrow$ check with CVX: ✓).

$$\begin{aligned} \text{maximize}_x \quad & c^\top x \\ \text{s.t.} \quad & Ax \le b \\ & x \ge 0 \end{aligned}$$

At an optimal x^* suppose we have $a_i^\top x^* < b_i$. What happens if we change b_i?

- By the complementarity conditions, the corresponding dual variable y_i^* must be 0.
- Therefore, changing the right-hand-side of an inactive constraint by a small amount will not affect the optimal value (also the optimal solution).
- Intuition: If the stock of a resource is not critical, then increasing or reducing the stock by a small amount does not matter.

Change of the Optimal Value Function:

- $\nabla V(b) = y^*$, where y^* is the (unique) optimal dual solution.

We call y^* the shadow prices of b.

- In the production example, the shadow price of a resource corresponds to the increment of profit if there is one unit more of that resource (locally).
- Therefore, it can be viewed as the unit value or unit fair price for that resource.
- Remember we came up with the same explanation when discussing its dual problem!

The above analysis is only local, meaning that it can only deal with small changes!

- Basically, it is valid as long as the optimal basis does not change.

$\rightsquigarrow$ It may not be true otherwise.

Example: In the production planning problem, if the amount of resource 1 reduces to 0, then the optimal solution will be $(0, 100)$ with optimal value 200 (reduced by 50). This difference would be different from $\Delta b_1 y_1^* = 0$.

- We want to study what ranges of changes belong to small changes.
- This is part of the global sensitivity analysis.

Global Sensitivity

We now study what will happen if:

1. b changes to $b + \Delta b$
2. c changes to $c + \Delta c$

Recall the simplex tableau:

$c^\top - c_B^\top A_B^{-1} A$	$-c_B^\top {A_B}^{-1} b$
$A_B^{-1} A$	$A_B^{-1} b$

At the Optimum:

- The reduced costs satisfy $c^\top - c_B^\top A_B^{-1} A \geq 0$.
- $A_B^{-1} b$ and $(A_B^{-1})^\top c_B$ are the basic part of the optimal primal solution and the optimal dual solution, respectively.

Changing b

Suppose b becomes $\tilde{b} = b + \Delta b$. Now, the new basic solution corresponding to the original optimal basis is:

$$\tilde{x}_B = A_B^{-1}(b + \Delta b) = x^* + A_B^{-1}\Delta b.$$

Note that the reduced costs $c^\top - c_B^\top A_B^{-1} A$ do not depend on b!

- If $\tilde{x}_B \geq 0$, then B is still the optimal basis and the new optimal solution is $(\tilde{x}_B, 0)$ with the new optimal value:

  $$V(\tilde{b}) = V^* + c_B^\top A_B^{-1}\Delta b = V^* + (y^*)^\top \Delta b,$$

  where y^* is the optimal dual solution (this explains the local theorem).
- If the original basis is still optimal, then the local sensitivity analysis holds.

We now study when the change only occurs in one component of b:

- What ranges of changes qualify for a small change?
- When does the local sensitivity analysis hold?

Assume $\Delta b = \lambda e_i$ (e_i is a vector with 1 at position i). Then, we need to have:

$$x^* + \lambda A_B^{-1} e_i \geq 0$$

so that the optimal basis remains the same.

$\rightsquigarrow$ We can find the range of λ by solving these inequalities!

Consider the production example:

$$\begin{array}{rrrl}
\text{maximize} & x_1 & +2x_2 & \\
\text{subject to} & x_1 & & \leq 100 \\
& & 2x_2 & \leq 200 \\
& x_1 & +x_2 & \leq 150 \\
& x_1, & x_2 & \geq 0
\end{array}$$

The optimal basis is $\{1, 2, 3\}$ and we have $x^* = (50, 100, 50, 0, 0)^\top$.

- How much can we change the third right-hand-side coefficient (150) such that the optimal basis remains the same?

The final simplex tableau is

B						
	0	0	0	1/2	1	250
1	1	0	0	-1/2	1	50
3	0	0	1	1/2	-1	50
2	0	1	0	1/2	0	100

Thus $A_B^{-1} = \begin{bmatrix} 0 & -0.5 & 1 \\ 0 & 0.5 & 0 \\ 1 & 0.5 & -1 \end{bmatrix}$. If b changes to $b + \lambda \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, then

$$\tilde{x}_B = x_B^* + \lambda A_B^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 50 \\ 100 \\ 50 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

In order for this to be positive, we need $-50 \leq \lambda \leq 50$.

Now suppose c changes to $\tilde{c} = c + \Delta c$.

- In order for the basic solution to be still optimal, we need to guarantee that the reduced costs are nonnegative!
- We only need to consider the non-basic part since the basic part must still be 0:

$$\tilde{c}_N^\top - \tilde{c}_B^\top A_B^{-1} A_N \geq 0.$$

Note that this basis still provides a basic feasible solution since the feasibility does not depend on c.

We now assume $\Delta c = \lambda e_j$. We discuss two cases: $j \in B$ and $j \in N$. We study how to find ranges for λ such that the original basis is still optimal (and thus we can apply the local sensitivity analysis).

In this case, the reduced costs are:

$$\begin{aligned} & c_N^\top - (c_B^\top + \lambda e_j^\top) A_B^{-1} A_N \\ & \quad = \quad c_N^\top - c_B^\top A_B^{-1} A_N - \lambda e_j^\top A_B^{-1} A_N. \end{aligned}$$

Note that $c_N^\top - c_B^\top A_B^{-1} A_N$ are the reduced costs for the original problem. We denote it by $r_N^\top$.

Therefore, in order to maintain the optimality of the current basis, we need to have:

$$r_N^\top - \lambda e_j^\top A_B^{-1} A_N \geq 0. \tag{1}$$

- We can solve the range of λ from (1).
- This is a set of inequalities.

Case 2: $j \in N$

In this case, the reduced costs are:

$$c_N^\top + \lambda e_j^\top - c_B^\top A_B^{-1} A_N = r_N^\top + \lambda e_j^\top$$

Therefore, in order to maintain the optimality of the current basis, we need to have:

$$r_N + \lambda e_j \geq 0. \tag{2}$$

- We can solve the range of λ from (2).

Consider the same production example:

$$\begin{array}{rrrl} \text{maximize} & x_1 & +2x_2 & \\ \text{subject to} & x_1 & & \le 100 \\ & & 2x_2 & \le 200 \\ & x_1 & +x_2 & \le 150 \\ & x_1, & x_2 & \ge 0 \end{array}$$

The final simplex tableau is:

B						
B	0	0	0	1/2	1	250
1	1	0	0	-1/2	1	50
3	0	0	1	1/2	-1	50
2	0	1	0	1/2	0	100

How much can we change the first objective coefficient so that we can use the local sensitivity analysis?

Example: Continued

We have

$$A_B^{-1} = \begin{bmatrix} 0 & -0.5 & 1 \\ 1 & 0.5 & -1 \\ 0 & 0.5 & 0 \end{bmatrix}; \quad A_N = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad r_N = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}.$$

Assume we change the profit 1 from 1 to $1 + \lambda$ (i.e., $-1 - \lambda$ in the standard form). Then, we need:

$$r_N - \lambda A_N^\top (A_B^{-1})^\top \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} - \lambda \begin{bmatrix} 0.5 \\ -1 \end{bmatrix} \geq 0$$

$\rightsquigarrow -1 \leq \lambda \leq 1.$

- If the profit coefficient of the first product is between 0 and 2, we can use the local sensitivity theorem to compute the opt. value using x^*.

What if the Change is Outside the Range?

If we change c so much such that the reduced cost of the current solution contains negative components, then:

- We can continue with the simplex tableau until it reaches optimal solution.

If the change of b is so much that the solution corresponding to the original optimal basis B is no longer feasible, then:

- We may need to solve the problem from the start.
- However, we can also have a dual perspective: the objective coefficients of the dual problem have changed. We can then use the method that deals with changes in the objective coefficients.

Changing A

If the change appears in a non-basic column, say in A_j, then the original optimal solution is still feasible.

The only change occurs in the reduced costs of jth variable.

- Recompute $\bar{c}_j$. If it is still nonnegative, then the original optimal solution stays optimal. Otherwise, update the tableau for the jth column as well as the reduced cost and continue from there.

If the change appears in a basic column, then nearly all numbers in the tableau will change. In general, there is not a simple way to deal with it.

Other Changes

Adding a Variable (the rest are kept the same):

- The original BFS is still a BFS, the reduced costs are unchanged.
- We only need to check the reduced cost corresponding to the new variable.
- If it is nonnegative, then the original optimal solution is still optimal; otherwise continue the simplex method from there.

Adding a Constraint:

- If the original optimal solution satisfies the constraint, then it is still optimal.
- If not, then the best way to deal with it is to interpret it as adding a dual variable. Then use the simplex tableau for the dual problem to continue calculations.

Questions?