MAT2002 Ordinary Differential Equations
Existence and Uniqueness Theorem for first-order ODE

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Overview

Outline

Motivation of Existence and uniqueness

While ODEs which are first order and linear have been completely solved. For first-order nonlinear ODE, a general method is still missing.

The mathematical theory we want to develop consists of the following:

- Under what the **conditions**, the general first order ODE

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0$$

  has a solution.
- If there is a solution, is it the only one?

Together these two questions form the issue of **existence and uniqueness** of solutions to first order ODEs. Let us first discuss why these are important properties to study.

Existence of solutions

Existence of solutions. An ODE is often derived as a **model** of some real-world physical phenomenon.

It is important to stress that models are only an **approximation** of the true phenomenon, since in the real-world there are many processes are too complicated to model.

Therefore to obtain a tractable model, certain simplified assumptions have to be made. For example, for the falling object problem discussed in the first lecture, we have assumed that the air resistance force is proportional to the velocity of the object, this is actually a simple assumption, which is only the approximation of the real situation.

Once a model has been proposed, one should first check if a solution to the model exists. If no solution exists, then the model is **not consistent** with reality and **modifications** should be made.

Uniqueness of solutions

If the model has at least one solution, we need to ask **is it the only solution**?

- If there is only one solution to the model, then we have **completely determined** the behaviour of the solution.
- If there are multiple solutions to the model, then we have to ask which solution is the one that is observed in reality.

Existence and Uniqueness for first order linear ODEs

Let's first state the theoretical results for the 1st-order linear ODE.

Theorem 4.1

(Existence and Uniqueness for first order linear ODEs).
Suppose functions p and q are **continuous** *on* $(\alpha, \beta) \subset \mathbb{R}$ *(α, β are some real numbers). Then, for any* $t_0 \in (\alpha, \beta), y_0 \in \mathbb{R}$*, there* **exists** *a* **unique** *function* $y(t)$ *satisfying*

$$\frac{dy}{dt} = p(t)y + q(t), \quad \forall t \in (\alpha, \beta),$$
$$y(t_0) = y_0.$$

And the solution is defined throughout the interval (α, β)*.*

Existence and Uniqueness for first order linear ODEs

Proof.

Repeating the ideas from the method of integrating factors, we first look at the function

$$\mu(t) = \exp\left(- \int p(t)dt \right). \tag{1}$$

Since p is continuous in (α, β) one can show that $\mu(t)$ is also continuous and nonzero (due to the exponential) for $t \in (\alpha, \beta)$. Therefore the reciprocal $1/\mu(t)$ makes sense and the integral $\int \mu(t)q(t)dt$ is well-defined and differentiable. In particular the formula for the general solution

$$y(t) = \frac{1}{\mu(t)}\left[\int \mu(t)q(t)dt + c \right]. \tag{2}$$

is well-defined for $t \in (\alpha, \beta)$. This shows **existence**. □

Existence and Uniqueness for first order linear ODEs

Proof.

For **uniqueness** we have to uniquely determine the constant of integration c. Since the equation (1) defines the integrating factor $\mu(t)$ up to a multiplicative factor that depends on the lower limit of the integration, in choosing this lower limit to be t_0, that is

$$\mu(t) := \exp\left(\int_{t_0}^{t} -p(s)ds\right) \Rightarrow \mu(t_0) = 1.$$

Then, modifying (2) to

$$y(t) = \frac{1}{\mu(t)}\left[\int_{t_0}^{t} \mu(s)q(s)ds + c\right],$$

to satisfy the initial condition $y(t_0) = y_0$ we must have $c = y_0$. Therefore the unique solution to the IVP is

$$\boxed{y(t) = \frac{1}{\mu(t)}\left[\int_{t_0}^{t} \mu(s)q(s)ds + y_0\right]}.$$

□

Existence and Uniqueness for first order linear ODEs

Remark:

- The above theorem states that the unique solution to the IVP

$$y' = p(t)y + q(t), \quad y(t_0) = y_0$$

exists throughout any interval (α, β) containing $t = t_0$ if the functions p and q are **continuous** in (α, β). The solution **globally** exists in the interval (α, β) in which p and q are continuous.

Existence and Uniqueness for first order linear ODEs

Example 4-1

For the IVP

$$\frac{dy}{dt} = -\frac{2}{t}y + 4t^2, \quad y(1) = 2,$$

find an interval for which a unique solution exists.

$$p(t) = -\frac{2}{t}, \quad q(t) = 4t^2.$$

From this, $q(t)$ is continuous for all $t \in \mathbb{R}$, but $p(t)$ is continuous only in $\mathbb{R} \setminus \{0\}$. Since the interval $(0, \infty)$ contains $t_0 = 1$, a unique solution to the IVP exists only for $t \in (0, \infty)$.

Consequently, if we change the initial condition to $y(-1) = 2$, then a unique solution exists to the IVP for $t \in (-\infty, 0)$.

Existence and Uniqueness for first order linear ODEs

Example 4-2

Consider the IVP

$$t\frac{dy}{dt} = 2y, \quad y(t_0) = y_0,$$

Looking at the standard form: $\frac{dy}{dt} = \frac{2}{t}y$, $p(t) = \frac{2}{t}$, $p(t)$ is not continuous at $t = 0$.

By using the above Existence and Uniqueness theorem, one has

- If $t_0 > 0$, then there is a unique solution to the IVP in the interval $(0, +\infty)$ since $p(t)$ is continuous in $(0, +\infty)$ and $t_0 \in (0, +\infty)$.
- If $t_0 < 0$, then there is a unique solution to the IVP in the interval $(-\infty, 0)$ since $p(t)$ is continuous in $(-\infty, 0)$ and $t_0 \in (0, +\infty)$.

However, if $t_0 = 0$, we cannot find an interval containing t_0 and in which $p(t)$ is continuous, Thus, the condition for Existence and Uniqueness theorem is not satisfied. Indeed, in this case, there could be no solution or infinity solutions to the IVP depending on the value of y_0.

Existence and Uniqueness for first order linear ODEs

Example 4-2

The ODE is separable.

$$\frac{1}{y}\frac{dy}{dt} = \frac{2}{t}$$

This gives $\ln|y(t)| = 2\ln|t| + c_0, c_0 \in \mathbb{R}$. Thus, the general solution is $|y(t)| = kt^2, k \in \mathbb{R}$.

Note: the general solution all satisfy: $y(0) = 0$. Thus, if $(t_0, y_0) = (0, 0)$, then there are infinity solutions to the IVP.

However, if $t_0 = 0, y_0 \neq 0$, then there is no solution to the IVP since $(0, y_0)(y_0 \neq 0)$ cannot satisfy the general solution $y(t) = kt^2, k \in \mathbb{R}$.

Existence and Uniqueness for first order non-linear ODEs

Question: What about the non-linear ODEs?

Existence and Uniqueness for first order non-linear ODEs

Theorem 4.2

Consider the initial value problem (IVP)

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0.$$

Let R be a closed rectangle

$$R = \{(t, y) | |t - t_0| \le a, \quad |y - y_0| \le b\} (a > 0, b > 0).$$

Assume that both $f(t, y)$ and $\frac{\partial f}{\partial y}$ are continuous on R.
Then the IVP has a unique solution $y = y(t)$ defined on the interval $(t_0 - h, t_0 + h)$, where $h = \min\left(\frac{b}{M}, a\right)$ and $M = \max_{(t,y)\in R} |f(t, y)|$.

Existence and Uniqueness for first order non-linear ODEs

Remark:

- Under the assumption of the theorem, the solution only exists in a small interval $(t_0 - h, t_0 + h) \subset [t_0 - a, t_0 + a]$ since $h = \min\left(\frac{b}{M}, a\right)$ depends on the size of the region R. And h also depends on the values of the function $f(t, y)$ in the region R ($M = \max_{(t,y)\in R} |f(t, y)|$). The solution only **locally** exists in the interval $[t_0 - a, t_0 + a]$.

We will show the proof later on, now we look at some examples first.

Existence and Uniqueness for first order non-linear ODEs

Example (Size of the rectangle R)

For non-linear ODEs, existence and uniqueness theorem requires that f and $\frac{\partial f}{\partial y}$ to be continuous in a rectangle R which contains the point (t_0, y_0).
Consider the IVP

$$2(y-1)\frac{dy}{dt} = 3t^2 + 4t + 2, \quad y(t_0) = y_0.$$

The standard form is

$$\frac{dy}{dt} = \frac{3t^2 + 4t + 2}{2(y-1)}, \quad y(t_0) = y_0.$$

Then we observe that

$$f(t,y) = \frac{3t^2 + 4t + 2}{2(y-1)}, \quad \frac{\partial f}{\partial y}(t,y) = -\frac{3t^2 + 4t + 2}{2(y-1)^2}.$$

are continuous everywhere except on the line $y = 1$.

Existence and Uniqueness for first order non-linear ODEs

Example (Size of the rectangle R)

If our initial poinit (t_0, y_0) does not intersect the line $y = 1$, then we can always draw a rectangle R around the point (t_0, y_0) for which f and $\frac{\partial f}{\partial y}$ are continuous in R. Then the existence and uniqueness theorem says that there is a unique solution to IVP in some interval containing $t = t_0$.

However, even though the rectangle can be stretched infinitely far in both the positive and the negative t directions where f and $\frac{\partial f}{\partial y}$ is continuous on this rectangle, this does not necessarily mean that the solution exists for all $t \in (-\infty, +\infty)$.

Let's consider the IVP

$$\frac{dy}{dt} = \frac{3t^2 + 4t + 2}{2(y-1)}, \quad y(0) = -1.$$

The general (implicit) solution is

$$y^2(t) - 2y(t) = t^3 + 2t^2 + 2t + c.$$

Existence and Uniqueness for first order non-linear ODEs

Example-Continue

Thus

$$y(t) = 1 \pm \sqrt{t^3 + 2t^2 + 2t + 1 + c},$$

and the solution is valid when $g(t) = t^3 + 2t^2 + 2t + 1 + c$ is non-negative. Using the initial condition: $y(0) = -1$, then $1 \pm \sqrt{1+c} = -1$, $c = 3$. $y(t) = 1 - \sqrt{t^3 + 2t^2 + 2t + 4} = 1 - \sqrt{(t^2+2)(t+2)}$. When the solution is defined only when $t \in [-2, +\infty)$. Therefore one has to **be careful** about the interval of definition for the IVP.

Remark: If the initial condition is $y(0) = 1$ (the point (t_0, y_0) lies on the line $y = 1$), then we cannot find a rectangle R containing the point (t_0, y_0) such that both f and $\frac{\partial f}{\partial y}$ are continuous on R. The assumption of the existence and uniqueness theorem is not satisfied. We may expect no solution or multiples solutions.

Indeed, if we substitute the condition $y(0) = 1$ into the general solution form, one can get $c = -1$, and $y(t) = 1 \pm \sqrt{t^3 + 2t^2 + 2t} = 1 \pm \sqrt{t(t^2 + 2t + 2)}$ are both solutions, the interval of definition for IVP is $[0, +\infty)$.

Existence and Uniqueness for first order non-linear ODEs

Example (Non-uniqueness)

We now give another example where multiple solution exists if the assumptions of existence and uniqueness theorem are not satisfied. Consider the IVP

$$\frac{dy}{dt} = y^{1/3}, \quad y(0) = 0, \text{for t} \geq 0.$$

The function $f(t, y) = y^{1/3}$ is continuous everywhere in $[0, \infty) \times (-\infty, +\infty)$, but the partial derivative $\frac{\partial f}{\partial y} = \frac{1}{3} y^{-2/3}$ does not exist at $y = 0$, and so it is not continuous at $y = 0$. However, the point (t_0, y_0)=$(0, 0)$ lies on the line $y = 0$ in the (t, y) plane. We cannot find a rectangle R containing the point (t_0, y_0) such that f and $\frac{\partial f}{\partial y}$ are continuous on R.

Existence and Uniqueness for first order non-linear ODEs

Example (Non-uniqueness)

Since the ODE is separable, we obtain as a particular solution

$$y(t) = \left[\frac{2}{3}t\right]^{3/2} \quad \textit{for } t \geq 0.$$

But note that the function $y_1(t) \equiv 0$ is also another solution, so is the function $y_2(t) = -\left[\frac{2}{3}t\right]^{3/2}$, and for arbitrary positive t_* the family of functions

$$y_*(t) = \begin{cases} 0 & \textit{if } 0 \leq t \leq t_*, \\ \pm\left[\frac{2}{3}(t - t_*)\right]^{3/2} & \textit{if } t \geq t_*, \end{cases}$$

also solves the IVP. In particular we have found an **infinite** family of solutions to the IVP.

Existence and Uniqueness for first order non-linear ODEs

Example (Non-uniqueness)

It is **important** to notice that this **does not** contradict the existence and uniqueness Theorem, since the condition "$\frac{\partial f}{\partial y}$ is continuous in R" is not satisfied and so the theorem is **not applicable** to this IVP. Nevertheless, if we consider another initial condition (t_0, y_0) such that $y_0 \neq 0$, then the existence and uniqueness theorem guarantees there is a unique solution to the IVP with $y(t_0) = y_0$.

Existence and Uniqueness for first order non-linear ODEs

Example (Blow up-non global existence)

(Blow up). Consider the IVP

$$\frac{dy}{dt} = y^2, \quad y(0) = 1.$$

$f(t, y) = y^2, \frac{\partial f}{\partial y} = 2y$ are continuous in $(-\infty, +\infty) \times (-\infty, +\infty)$. Using the fact that the ODE is a separable equation we obtain the following general solution $y(t) = -\frac{1}{t+c}, \quad c \in \mathbb{R}$.
For the initial condition $y(0) = 1$, we compute $c = -1$ and so $y(t) = \frac{1}{1-t}$, which is the unique solution for the IVP.

Note that $y(t) \to \infty$ as $t \to 1$ (a behaviour which we call **blow up** as the solution becomes unbounded). Thus, the solution to the IVP only exists in the interval $I = (-\infty, 1)$.
Although $f(t, y) = y^2, \frac{\partial f}{\partial y} = 2y$ are continuous in $(-\infty, +\infty) \times (-\infty, +\infty)$, the solution to the IVP only exists locally.

Outline

Knowledge from mathematical analysis/calculus

Next we will show the proof of the existence and uniqueness theorem for first order non-linear ODEs. But before that, we need to review/introduce some fundamental mathematical knowledge from mathematical analysis/calculus.

Knowledge from mathematical analysis/calculus

Definition 4.3 (Uniform convergence of sequence of functions)

Suppose that $\{\phi_n(t)\}_{n=1}^{\infty} : E \to \mathbb{R}$ ($E \subseteq \mathbb{R}$) is a sequence of real functions, the sequence of functions $\{\phi_n(t)\}_{n=1}^{\infty}$ is uniformly convergent on E with limit $\phi(t)$ ($\phi(t) : E \to \mathbb{R}$ is a real function defined on E) if

$\forall \varepsilon > 0$, $\exists$ a positive integer M (**does not depend on** t) such that

$$|\phi_n(t) - \phi(t)| < \varepsilon$$

for all $t \in E$ and $n \geq M$.

Remark: If the sequence of functions $\{\phi_n(t)\}_{n=1}^{\infty}$ is uniformly convergent on E with limit $\phi(t)$, then the sequence of functions $\{\phi_n(t)\}_{n=1}^{\infty}$ is point wise convergent to $\phi(t)$ on E.

Knowledge from mathematical analysis/calculus

Example 4.4

Given a sequence of functions $\left\{\frac{t}{1+n^2t^2}\right\}_{n=1}^{\infty}$ defined on $t \in (0, \infty)$, then it can be easily verified that, for every $\varepsilon > 0$, there exists a natural number $M = [\frac{1}{2\varepsilon}] + 1$ such that

$$\left|\frac{t}{1+n^2t^2} - 0\right| < \frac{1}{2n} < \varepsilon$$

for all $t \in (0, \infty)$ and $n \geq M$. Thus, $\{\frac{t}{1+n^2t^2}\}_{n=1}^{\infty}$ is uniformly convergent to the function 0.

Knowledge from mathematical analysis/calculus

The following theorem provides a method to judge a sequence of functions are uniformly convergent or not. These results can be found in standard mathematical analysis/calculus textbooks, the proof are omitted.

Theorem 4.5 (Weierstrass M-test)

Let $\{\phi_n(t)\}_{n=1}^{\infty} : E \to \mathbb{R}$ ($E \subseteq \mathbb{R}$) be a sequence of functions and $\{M_n\}_{n=1}^{\infty}$ be a sequence of positive numbers. If

- $|\phi_n(t)| \leq M_n$ *for all* $t \in E$ *and* $n \in \{1, \cdots\}$,
- $\sum\limits_{n=1}^{\infty} M_n$ *converges.*

Then the partial sum $\Phi_n(t) = \sum\limits_{k=1}^{n} \phi_k(t)$ *is uniformly convergent on* E.

Knowledge from mathematical analysis/calculus

Theorem 4.6 (Property of uniform convergence of functions)

Suppose that $\{\phi_n(t)\}_{n=1}^{\infty} : [a, b] \to \mathbb{R}$*(*$a, b$ *are given real numbers) are given integrable functions.*
If $\{\phi_n(t)\}_{n=1}^{\infty}$ *is uniformly convergent to the function* $\phi(t) : [a, b] \to \mathbb{R}$*, then the operation of taking limit and integration can be exchanged, i.e.*

$$\lim_{n\to\infty} \int_a^b \phi_n(t)dt = \int_a^b \lim_{n\to\infty} \phi_n(s)ds = \int_a^b \phi(t)dt$$

Outline

1. Introduction to Existence and Uniqueness
2. Knowledge from mathematical analysis/calculus
3. Proof for Existence and uniqueness theorem of the first-order non-linear ODE

Existence and uniqueness of the solution

Theorem 4.7 (Existence and uniqueness of the solution)

Consider the initial value problem (IVP)

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0. - - - - - - - - (*)$$

Let R be a closed rectangle

$$R = \{(t, y) || t - t_0| \leq a, \quad |y - y_0| \leq b\}(a > 0, b > 0).$$

(a, b are given positive constants)
Assume that $f(t, y)$ and $\frac{\partial f}{\partial y}$ are continuous on R.
Then the IVP has a unique solution $y = y(t)$ on the interval $(t_0 - h, t_0 + h)$ where $h = \min\left(\frac{b}{M}, a\right)$ and $M = \max\limits_{(t,y) \in R} |f(t, y)|$.

Existence and uniqueness of the solution

Proof.

Integrating (*) from t_0 to t, one has
$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))ds. - - - - - - - - (**)$
Indeed, it can be easily shown that (*) and (**) are equivalent.
One method of showing that the integral equation (**) has a unique solution is known as the **Picard's iterative method**. First, we define the following successive iterations $\{\phi_n(t)\}_{n=0}^{\infty}$.

$$\phi_0(t) = y_0.$$

$$\phi_1(t) = y_0 + \int_{t_0}^{t} f(s, \phi_0(s))ds.$$

$$\vdots$$

$$\phi_{n+1}(t) = y_0 + \int_{t_0}^{t} f(s, \phi_n(s))ds$$

□

Existence and uniqueness of the solution

Proof.

We will use the following four steps to prove the theorem

- Show all $\{\phi_n(t)\}_{n=0}^{\infty}$ satisfy $|\phi_n(t) - y_0| \le b$, $\forall t \in (t_0 - h, t_0 + h)$ (We need $(t, \phi_n(t)) \in R$ for $t \in (t_0 - h, t_0 + h)$ in order to show $\{\phi_n(t)\}_{n=0}^{\infty}$ is uniformly convergent)
- Show the sequence $\{\phi_n(t)\}_{n=0}^{\infty}$ is uniformly convergent.
- Show the limit function of $\{\phi_n(t)\}_{n=0}^{\infty}$ is the solution of the integration equation (**).
- Show the uniqueness of the solution.

$\square$

Existence and uniqueness of the solution

Step 1: All $\{\phi_n(t)\}_{n=0}^{\infty}$ satisfy $|\phi_n(t) - y_0| \leq b$, $\forall t \in (t_0 - h, t_0 + h)$.

First, it is easy to check that $\phi_0(t) = y_0, t \in (t_0 - h, t_0 + h)$ satisfies $|\phi_0(t) - y_0| = 0 \leq b$.
Next suppose that $\phi_n(t)$ satisfies $|\phi_n(t) - y_0| \leq b$ for $(t_0 - h, t_0 + h)$, then

$$\phi_{n+1}(t) = y_0 + \int_{t_0}^{t} f(s, \phi_n(s))ds.$$

Now we look for $f(s, \phi_n(s))$ when s is between t_0 and $t(t \in (t_0 - h, t_0 + h))$, we have $|s - t_0| \leq |t - t_0| < h \leq a$ and $|\phi_n(s) - y_0| \leq b$. Thus, $(s, \phi_n(s)) \in R$ and $|f(s, \phi_n(s))| \leq M$ when s is between t_0 and $t(|t - t_0| < h)$.

Existence and uniqueness of the solution

Thus,

$$
\begin{aligned}
|\phi_{n+1}(t) - y_0| =& \left| \int_{t_0}^{t} f(s, \phi_n(s)) ds \right| \\
\leq& M|t - t_0| \quad (|f(s, \phi_n(s))| \leq M) \\
\leq& Mh \\
\leq& b
\end{aligned}
$$

since $h = \min\left(\frac{b}{M}, a\right)$. Therefore, by mathematical induction, all $\{\phi_n(t)\}_{n=0}^{\infty}$ satisfy $|\phi_n(t) - y_0| \leq b$, $\forall t \in (t_0 - h, t_0 + h)$.

Existence and uniqueness of the solution

Step 2: The sequence $\{\phi_n(t)\}_{n=0}^{\infty}$ from the Picard's iteration is uniformly convergent on the interval $(t_0 - h, t_0 + h)$.
We could use Weierstrass M-test theorem to show $\{\phi_n(t)\}_{n=0}^{\infty}$ is uniformly convergent:
Firstly, since $\frac{\partial f}{\partial y}$ is continuous on R, for $\forall\ (t, y_1), (t, y_2) \in R$, there exists y_3 between y_1 and y_2 s.t.

$$f(t, y_1) - f(t, y_2) = \left[\frac{\partial}{\partial y} f(t, y_3)\right](y_1 - y_2)$$

Obviously, $\frac{\partial f}{\partial y}$ is bounded on R, there exists $K > 0$ s.t.

$$\left|\frac{\partial f}{\partial y}(t, y)\right| \leq K \qquad \text{for } \forall t, y \in R$$

Hence we derive:

$$|f(t, y_1) - f(t, y_2)| = \left|\frac{\partial}{\partial y} f(t, y_3)\right| |y_1 - y_2| \leq K |y_1 - y_2|$$

for $\forall\ (t, y_1)$ and $(t, y_2) \in R$.

Existence and uniqueness of the solution

Secondly, we set $y_1 = \phi_n(t)$ and $y_2 = \phi_{n-1}(t)$ to obtain:

$$|f(t, \phi_n(t)) - f(t, \phi_{n-1}(t))| \leq K|\phi_n(t) - \phi_{n-1}(t)|$$

Then we use induction to show that

$$|\phi_n(t) - \phi_{n-1}(t)| \leq \frac{MK^{n-1}|t - t_0|^n}{n!}$$

For $n = 1$, we observe that

$$|\phi_1(t) - \phi_0(t)| = |\int_{t_0}^{t} f(s, \phi_0(s))ds| \leq M|t - t_0|$$

Hence for $n = p$, we assume that

$$|\phi_p(t) - \phi_{p-1}(t)| \leq \frac{MK^{p-1}|t - t_0|^p}{p!}$$

Existence and uniqueness of the solution

Then for $n = p + 1$, we derive:

$$\begin{aligned}
|\phi_{p+1}(t) - \phi_p(t)| &= \left| \int_{t_0}^{t} \{f(s, \phi_p(s)) - f(s, \phi_{p-1}(s))\}\, ds \right| \\
&\leq \int_{t_0}^{t} |f(s, \phi_p(s)) - f(s, \phi_{p-1}(s))|\, ds \qquad \text{when } t_0 + h > t \geq t_0 \\
&\leq K \int_{t_0}^{t} |\phi_p(s) - \phi_{p-1}(s)|\, ds \\
&\leq K \int_{t_0}^{t} \left(\frac{MK^{p-1}(s - t_0)^p}{p!} \right) ds \\
&\leq \frac{MK^p |t - t_0|^{p+1}}{(p+1)!}
\end{aligned}$$

When $t_0 - h < t < t_0$, similarly, the above inequality can be proved.
We could write $\phi_n(t)$ into the form:

$$\phi_n(t) = \phi_0(t) + [\phi_1(t) - \phi_0(t)] + \cdots + [\phi_n(t) - \phi_{n-1}(t)] = \phi_0(t) + \sum_{i=1}^{n} [\phi_i(t) - \phi_{i-1}(t)]$$

Existence and uniqueness of the solution

And

$$|\phi_i(t) - \phi_{i-1}(t)| \le \frac{MK^{i-1}|t-t_0|^i}{i!} \le \frac{MK^{i-1}h^i}{i!}, \quad i = 1, 2, \cdots$$

when $t \in (t_0 - h, t_0 + h)$.
And

$$\sum_{i=1}^{\infty} \frac{MK^{i-1}h^i}{i!} \le \frac{M}{K} \sum_{i=1}^{\infty} \frac{(Kh)^i}{i!} = \frac{M}{K}(e^{Kh} - 1)$$

By Weierstrass M-test theorem, $\sum_{i=1}^{n}[\phi_i(t) - \phi_{i-1}(t)]$ is uniform convergent on the interval $(t_0 - h, t_0 + h)$. Thus,

$$\phi_n(t) = \phi_0(t) + \sum_{i=1}^{n}[\phi_i(t) - \phi_{i-1}(t)]$$

is also uniform convergent on the interval $(t_0 - h, t_0 + h)$.

Existence and uniqueness of the solution

Proof.

Step 3: Let $\phi(t) = \lim_{n\to\infty} \phi_n(t)$, then $\phi(t)$ satisfies the integration equation (**), hence $\phi(t)$ is the solution of IVP. Since $\phi(t) = \lim_{n\to\infty} \phi_n(t)$ and $\phi_n(t)$ is uniformly convergent, thus, taking the limit for both sides of the following relation, one has

$$\begin{aligned}
\phi(t) = & \lim_{n\to\infty} \phi_{n+1}(t) \\
& = y_0 + \lim_{n\to\infty} \int_{t_0}^{t} f(s, \phi_n(s)) ds \\
& = \int_{t_0}^{t} \lim_{n\to\infty} f(s, \phi_n(s)) ds \text{ (using uniform convergenece of } \phi_\mathrm{n}(\mathrm{t}) \text{ and } \mathrm{f}(\mathrm{t}, \phi_\mathrm{n}(\mathrm{t}))\text{)} \\
& = \int_{t_0}^{t} f(s, \lim_{n\to\infty} \phi_n(s)) ds \text{ (using the continuity of the function f)} \\
& = \int_{t_0}^{t} f(s, \phi(s)) ds
\end{aligned}$$

Therefore, $\phi(t)$ satisfies the integration equation (**), hence $\phi(t)$ is the solution of IVP.

Existence and uniqueness of the solution

Proof.

Step 4: The solution of the integration equation (**) is unique.
Suppose that $\phi(t)$ and $\omega(t)$ $(t \in (t_0 - h, t_0 + h))$ are two solution of (**), then

$$\phi(t) = y_0 + \int_{t_0}^{t} f(s, \phi(s))ds, \quad \omega(t) = y_0 + \int_{t_0}^{t} f(s, \omega(s))ds$$

and

$$\begin{aligned} |\phi(t) - \omega(t)| =& \left| \int_{t_0}^{t} f(s, \phi(s)) - f(s, \omega(s))ds \right| \\ \leq& \int_{t_0}^{t} |f(s, \phi(s)) - f(s, \omega(s))| \, ds, \textit{when} \quad t_0 \leq t < t_0 + h \\ \leq& K \int_{t_0}^{t} |\phi(s) - \omega(s)| ds \end{aligned}$$

□

Existence and uniqueness of the solution

Proof.

Now let $g(t) = \int_{t_0}^{t} |\phi(s) - \omega(s)| ds$, then $\frac{dg}{dt} = |\phi(t) - \omega(t)|$ and $g(t_0) = 0$. Therefore,

$$\frac{dg}{dt} \leq Kg(t), \quad g(t_0) = 0.$$

Both side multiplying e^{-Kt}, one has

$$e^{-Kt}\left(\frac{dg}{dt} - Kg(t)\right) \leq 0.$$

Thus,

$$\frac{d\left(e^{-Kt}g(t)\right)}{dt} \leq 0 - - - - - - - - - - - - -(***)$$

□

Existence and uniqueness of the solution

Proof.

Integrating (***) from t_0 to t, $t \in [t_0, t_0 + h)$ on both sides of the above inequality, one has

$$e^{-Kt}g(t) - e^{-Kt_0}g(t_0) \leq 0, \quad e^{-Kt}g(t) \leq 0$$

Thus, $g(t) \leq 0$ for all $t \in [t_0, t_0 + h)$.
Therefore,

$$|\phi(t) - \omega(t)| = \frac{dg}{dt} \leq Kg(t) \leq 0 \quad \text{for all } t \in [t_0, t_0 + h].$$

Thus, $\phi(t) = \omega(t)$ for all $t \in [t_0, t_0 + h)$.
Similarly, $\phi(t) = \omega(t)$ for all $t \in (t_0 - h, t_0]$.
Therefore, $\phi(t) = \omega(t)$ for all $t \in (t_0 - h, t_0 + h)$. The two solutions are the same. □

Existence for first order non-linear ODEs

Remark: If $f(t, y)$ is continuous on the rectangle R containing (t_0, y_0). Then it can be shown that $\frac{dy}{dt} = f(t, y), y(t_0) = y_0$ has a solution, but uniqueness of the solution is not guaranteed.

Recall: Example (Non-uniqueness)

$$\frac{dy}{dt} = y^{1/3}, \quad y(0) = 0.$$

$f(t, y) = y^{1/3}$ is continuous, while $\frac{\partial f}{\partial y}(t, y) = \frac{1}{3} y^{-\frac{2}{3}}$ is not continuous at $y = 0$. Thus, the IVP has at least one solution, but the uniqueness is not guaranteed. Indeed, it has ∞ solutions.

Existence for first order non-linear ODEs

Theorem 4.8 (Existence of the solution)

Consider the initial value problem (IVP)

$$\frac{dy}{dt} = f(t,y), \quad y(t_0) = y_0. - - - - - - - - (*)$$

Let R be a closed rectangle

$$R = \{(t,y) | |t - t_0| \leq a, \quad |y - y_0| \leq b\} (a > 0, b > 0).$$

Assume that $f(t,y)$ is continuous on R. Then the IVP has a solution $y = y(t)$ on the interval $(t_0 - h, t_0 + h)$ where $h = \min\left(\frac{b}{M}, a\right)$ and $M = \max\limits_{(t,y)\in R} |f(t,y)|$.

The proof of this theorem can be done by using the Picard iterative method and the Arzela-Ascoli Theorem (knowledge from mathematical analysis/real analysis). The proof is omitted in this course.