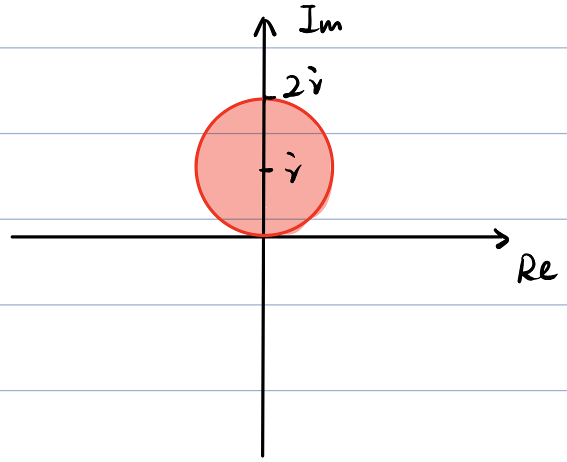
MAT 3253 Homework 2 118010350

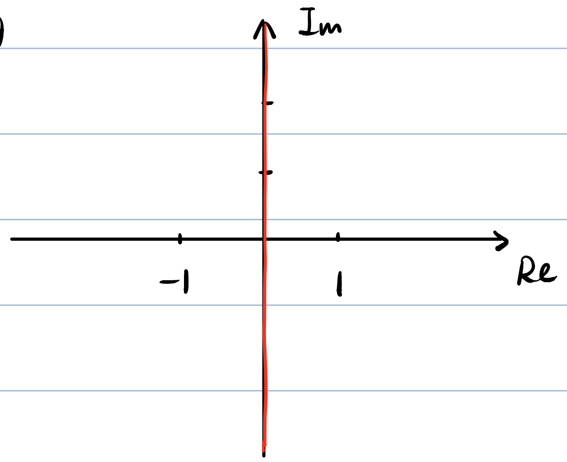
1. (a).



$|z-i| \le 1$

Not region.

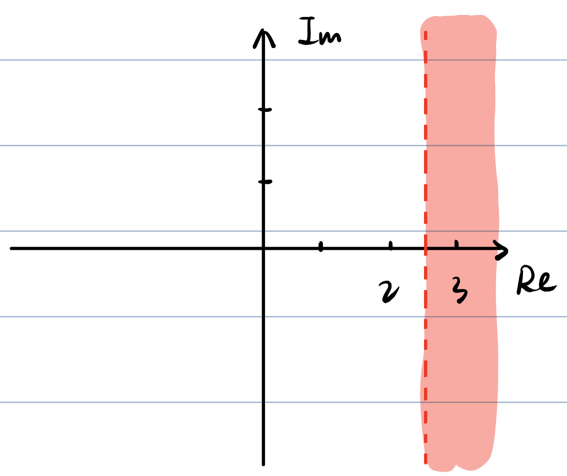
(b)



$\left|\frac{z-1}{z+1}\right| = 1$

Not region.

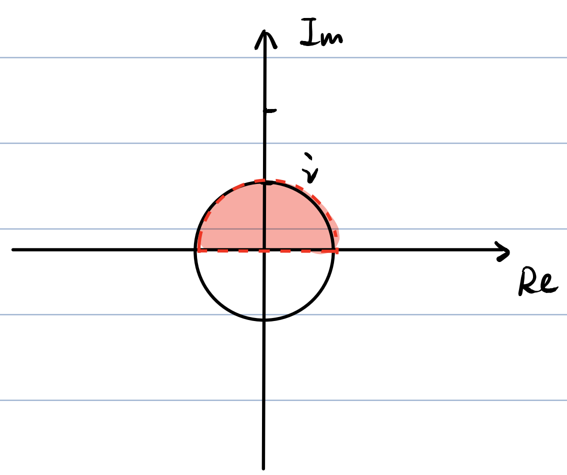
(c)



$|z-2| > |z-3|$

Region.

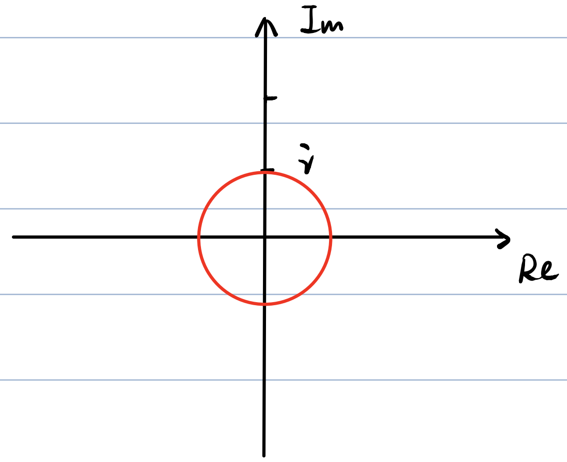
(d)



$|z|<1$ and $Im(z)>0$

Region.

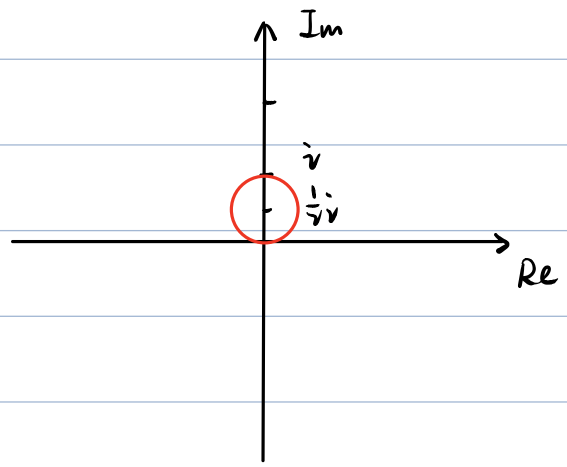
(e)



$\frac{1}{z} = \bar{z}$

Not region.

(f)



$|z|^2 = \operatorname{Im} z$

Not region.

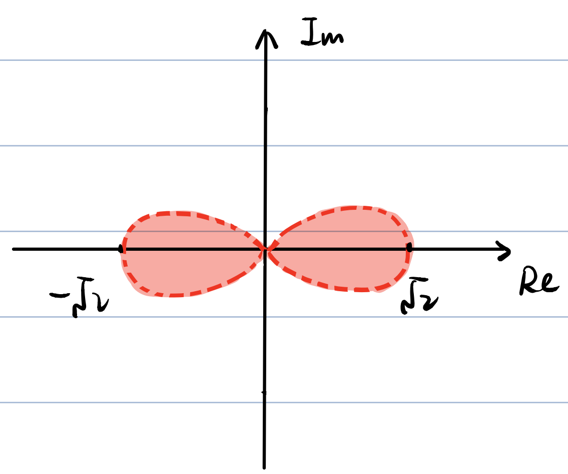
(g) Let $z = r(\cos\theta + i\sin\theta)$, $z^2 = r^2(\cos 2\theta + i\sin 2\theta)$

$|z^2 - 1| < 1 \Rightarrow |r^2(\cos 2\theta + i\sin 2\theta) - 1| < 1$

$\Rightarrow r^4\cos^2 2\theta - 2r^2\cos 2\theta + 1 + r^4\sin^2 2\theta < 1$

$\Rightarrow 2r^2\cos 2\theta > r^4$

$\Rightarrow r^2 < 2\cos 2\theta$



$|z^2 - 1| < 1$

Region.

2. proof. Since $\frac{z-1}{z+1} = \frac{(z-1)(\bar{z}+1)}{(z+1)(\bar{z}+1)}$

$$= \frac{z\cdot\bar{z} + z - \bar{z} - 1}{|z+1|^2}$$

$$= \frac{2\,Im(z)\cdot i}{|z+1|^2}.$$

and $|z+1|^2 \in \mathbb{R}$, then $Re(\frac{z-1}{z+1}) = 0$.

thus $Arg(\frac{z-1}{z+1}) = \pm\frac{\pi}{2}$.

If $Im(z) > 0$, then $Im(\frac{z-1}{z+1}) > 0 \Rightarrow Arg(\frac{z-1}{z+1}) = \frac{\pi}{2}$.

If $Im(z) < 0$, then $Im(\frac{z-1}{z+1}) < 0 \Rightarrow Arg(\frac{z-1}{z+1}) = -\frac{\pi}{2}$

3. proof. Let $a_k = \sum_{n=1}^{k} z_n$, $b_k = \sum_{n=1}^{k} w_n$, $k \in \mathbb{N}$.

Since $\sum_{n=1}^{\infty} z_n = S$, $\sum_{n=1}^{\infty} w_n = T$, then,

By definition, $(a_k)_{k=1}^{\infty} \to S$, $(b_k)_{k=1}^{\infty} \to T$

For $\forall \varepsilon > 0$, $\exists N_1 \in \mathbb{N}$, s.t. $|a_k - S| < \varepsilon/2$, $\forall k \geq N_1$

$\exists N_2 \in \mathbb{N}$. s.t. $|b_k - T| < \varepsilon/2$, $\forall k \geq N_2$.

then take $N = \max\{N_1, N_2\}$, for $\forall k \geq N$.

$$|a_k + b_k - S - T| \leq |a_k - S| + |b_k - T| < \varepsilon$$

Thus $(a_k + b_k)_{k=1}^{\infty} \to S+T$.

By definition, $\sum_{n=1}^{\infty}(z_n + w_n) = S+T$.

4. (a) proof. Let $S = 1 + z + z^2 + \cdots + z^n$,

then $z\cdot S = z + z^2 + z^3 + \cdots + z^{n+1}$.

$\Rightarrow (1-z)S = 1 - z^{n+1}$

$\Rightarrow S = \frac{1-z^{n+1}}{1-z}$

thus, $1 + z + z^2 + \cdots + z^n = \frac{1-z^{n+1}}{1-z}$

(b) proof. By the result of (a), $\sum_{k=0}^{n} z^k = \frac{1-z^{n+1}}{1-z}$.

Based on fact $|z| < 1$, W.T.S. $(z^{n+1})_{n=0}^{\infty} \to 0$.

Let $z = r(\cos\theta + i\sin\theta)$, then $r<1$.

$$\Rightarrow z^{n+1} = r^{n+1}(\cos(n+1)\theta + i\sin(n+1)\theta),$$

$$\Rightarrow |z^{n+1} - 0| = |r^{n+1}(\cos(n+1)\theta + i\sin(n+1)\theta)|$$

$$= |r^{n+1}| = r^{n+1} \to 0 \text{ as } n \to \infty$$

By definition, $(z^{n+1})_{n=0}^{\infty} \to 0$

then $\lim\limits_{n\to\infty} \sum\limits_{k=0}^{n} z^k = \lim\limits_{n\to\infty} \frac{1-z^{n+1}}{1-z} = \frac{1}{1-z}$

thus $\sum\limits_{k=0}^{\infty} z^k = \frac{1}{1-z}$.

(c) By the result of (b), $\sum\limits_{k=0}^{\infty} z^k = \frac{1}{1-z}$.

Let $z = r(\cos\theta + i\sin\theta)$, and $0 \le r < 1$, $\theta \in \mathbb{R}$.

then $\sum\limits_{k=0}^{\infty} r^k(\cos k\theta + i\sin k\theta) = \frac{1}{1 - r(\cos\theta + i\sin\theta)}$

Since $\operatorname{Re}\left(\sum\limits_{k=0}^{\infty} r^k(\cos k\theta + i\sin k\theta)\right) = \sum\limits_{k=0}^{\infty} r^k \cos k\theta$,

$$\frac{1}{1 - r(\cos\theta + i\sin\theta)} = \frac{1}{(1 - r\cos\theta) - r\sin\theta\, i}$$

$$= \frac{1 - r\cos\theta + r\sin\theta\, i}{[(1-r\cos\theta) - r\sin\theta\, i][(1-r\cos\theta) + r\sin\theta\, i]}$$

$$= \frac{1 - r\cos\theta + r\sin\theta\, i}{1 + r^2 - 2r\cos\theta}$$

then $\operatorname{Re}\left(\frac{1}{1 - r(\cos\theta + i\sin\theta)}\right) = \frac{1 - r\cos\theta}{1 + r^2 - 2r\cos\theta}$

thus $\sum\limits_{k=0}^{\infty} r^k \cos(k\theta) = \frac{1 - r\cos\theta}{1 + r^2 - 2r\cos\theta}$