MAT3253 Complex Variables
Lecture Notes

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Contents

This is a set of notes for MAT3253 Complex variables.

References

[BakNewman] J. Bak and D. J. Newman, *Complex Analysis*, 3rd edition, Springer, New York, 2010.

[BrownChurchill] J. W. Brown and R. V. Churchill *Complex Variables and Applications*, 8th edition, McGraw-Hill, New York, 2009.

1 Lecture 1 (Complex numbers)

Summary:

- Construction of complex field using pairs of real numbers.
- Construction of complex field using 2×2 matrices.

Complex numbers were first invented to solve algebraic equations. As a vector space, the set of complex numbers is an extension of the real numbers of dimension 2. It is also equipped with a multiplication operator that extends the multiplication of real numbers.

We start with the axioms of a complex field.

Definition 1.1. A number system $(F, +, \cdot)$ is called a *field* if

1. (closed) $a + b \in F$ for all $a, b \in F$.
2. (associative) $(a + b) + c = a + (b + c)$, for all $a, b, c \in F$.
3. (commutative) $a + b = b + a$, for all $a, b \in F$.
4. (existence of zero) $\exists 0 \in F$ such that $0 + a = a + 0 = a$, for all $a \in F$.
5. (additive inverse) for all $a \in F$, $\exists a' \in F$ such that $a + a' = 0$.
6. (closed) $a \cdot b \in F$ for all $a, b \in F$.
7. (associative) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$, for all $a, b, c \in F$.

8. (commutative) $a \cdot b = b \cdot a$, for all $a, b \in F$.

9. (existence of one) $\exists 1 \in F$ such that $1 \cdot a = a \cdot 1 = a$, for all $a \in F$.

10. (multiplicative inverse) for all $a \in F \setminus \{0\}$, $\exists a'' \in F$ such that $a \cdot a'' = 1$.

11. (distributive) $a \cdot (b + c) = a \cdot b + a \cdot c$, for all $a, b, c \in F$.

A subset K of a field F is called a *subfield* of F if the elements in K satisfy the all axioms of a field, and F is called an *extension* of K. Examples of fields include the rational numbers $\mathbb{Q}$ and the real numbers $\mathbb{R}$. A field F containing $\mathbb{R}$ as a subfield and a special element I that satisfies $I^2 + 1 = 0$ is called a *complex field*. Complex field is denoted by $\mathbb{C}$.

Using this terminology, we say that the complex field is obtained by extending $\mathbb{R}$ so that the equation $x^2 + 1 = 0$ has a solution.

Remark. The definition of complex field above is not 100% accurate. To be precise, we should say that $\mathbb{C}$ is *generated* by the real numbers in $\mathbb{R}$ and the special number I. The term "generated" roughly means that all numbers in $\mathbb{C}$ can be obtained from "mixing" real numbers and the number I by addition, subtraction, multiplication and division.

Remark. In general, a field needs not be infinite. (None of the axioms require that there are infinitely many elements in a field.) We can construct number systems consisting of finitely many elements satisfying the axioms of field. To construct an example of a field of size 3, we can label the elements by 0, 1, 2, and define the addition and multiplication by the following tables

$+$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$\cdot$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

The addition and multiplication are addition and multiplication modulo 3.

The special number I in a complex number is usually called the *imaginary unit*. However, the calculations with complex numbers is very concrete and not imaginary. We provide two constructions of complex field below. In the first construction a complex number is a pair of real numbers. In the second one a complex number is a 2×2 matrix over the real numbers.

Construction of complex field (I)

Let

$$F_1 \triangleq \{(a,b) : \; a,b \in \mathbb{R}\}. \tag{1.1}$$

A "complex number" is thus regarded as a point on a plane, called the *complex plane* or *Argand plane*. The addition and multiplication operators are defined by

$$(a,b) + (c,d) \triangleq (a+b, \; c+d),$$
$$(a,b) \cdot (c,d) \triangleq (ac-bd, \; ad+bc).$$

The additive and multiplicative identities are $(0,0)$ and $(1,0)$, respectively. The real numbers are embedded in F_1 by $x \mapsto (x,0)$. Real-number calculation can be carried out in F_1. By identifying x_1 with $(x_1,0)$ and x_2 with $(x_2,0)$, the sum and product of x_1 and x_2 are respectively

$$(x_1,0) + (x_2,0) = (x_1+x_2,0), \text{ and}$$
$$(x_1,0) \cdot (x_2,0) = (x_1x_2,0).$$

The special number I in this representation is $I = (0,1)$. We can check that

$$I^2 = (0,1) \cdot (0,1) = ((0)(0) - (1)(1), \; (0)(1) + (1)(0)) = (1,0).$$

F_1 is therefore a complex field.

Construction of complex field (II)

Let

$$F_2 \triangleq \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} : \; a,b \in \mathbb{R} \right\}. \tag{1.2}$$

Addition and multiplication are performed using the usual matrix addition and multiplication. The additive and multiplicative identities are the zero matrix and identity matrix, respectively. The "imaginary unit" I is $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. The field F_2 contains $\mathbb{R}$ as a subfield because the subset

$$\left\{ \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} : \; x \in \mathbb{R} \right\} \tag{1.3}$$

can be identified with the set of real numbers. Real numbers are represented as diagonal matrices with equal diagonal entries. The matrix addition and multiplication reduces to real-number addition and multiplication when restricted to matrices in (1.3),

$$\begin{bmatrix} x_1 & 0 \\ 0 & x_1 \end{bmatrix} + \begin{bmatrix} x_2 & 0 \\ 0 & x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 & 0 \\ 0 & x_1 + x_2 \end{bmatrix}$$
$$\begin{bmatrix} x_1 & 0 \\ 0 & x_1 \end{bmatrix} \cdot \begin{bmatrix} x_2 & 0 \\ 0 & x_2 \end{bmatrix} = \begin{bmatrix} x_1 x_2 & 0 \\ 0 & x_1 x_2 \end{bmatrix}.$$

We can use F_2 as a numerical model for calculating complex numbers.

The two constructions are essentially the same (meaning that F_1 and F_2 are isomorphic). The first construction emphasizes that a complex number is a pair of real numbers. The second construction emphasizes that complex multiplication is the same as multiplying by matrix in a special form. One can check that

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} ac - bd & -(ad + bc) \\ ad + bc & ac - bd \end{bmatrix}.$$

We will write $a + bi$ as a notation for (a, b) or $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. Using the $a + bi$ notation, the multiplication of complex numbers can be written as

$$(a + bi)(c + di) = ac - bd + i(ad + bc).$$

The complex numbers as a vector space has dimension 2 over $\mathbb{R}$. We can pick 1 and i as a basis. Using the first construction method, a complex number (x, y) can be written as

$$(x, y) = x(1, 0) + y(0, 1),$$

with $(1, 0)$ and $(0, 1)$ serving as the standard basis vectors. If we using the second construction method, we can use $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ as a basis,

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

2 Lecture 2 (Basic notions and operations in $\mathbb{C}$)

Summary

- Complex conjugate, modulus
- Complex division
- Polar form of complex numbers, DeMoivre formula

Definition 2.1. For a complex number $z = a + bi$ in $\mathbb{C}$, define the *real* and *imaginary part* of z as

$$\mathrm{Re}(z) \triangleq a \quad \text{and} \quad \mathrm{Im}(z) \triangleq b.$$

Define the *complex conjugate* of z by

$$\bar{z} \triangleq z^* \triangleq a - bi.$$

The *modulus* of z is defined as

$$|z| = \sqrt{a^2 + b^2}.$$

The modulus of z is also called the *absolute value* or the *radius*.

Geometrically, the complex conjugate of z is the reflection of z along the real axis. The modulus is the distance between the origin and the point z in the complex plane. The next proposition says that the complex conjugate, as a mapping from $\mathbb{C}$ to $\mathbb{C}$, is compatible with complex addition and multiplication.

Proposition 2.2.
(i) $(z^*)^* = z$ *for any* $z \in \mathbb{C}$.
(ii) Given any two complex numbers z_1 *and* z_2 *in* $\mathbb{C}$*,*

$$(z_1 + z_2)^* = z_1^* + z_2^* \quad \textit{and} \quad (z_1 z_2)^* = z_1^* z_2^*.$$

Part (ii) in Prop. 2.2 says that the reflection of the sum (resp. product) of two complex numbers is the same as the sum (resp. product) of the two points obtained by reflection. The proof is simple and is omitted. Using Prop. 2.2, we can show that the modulus is a multiplicative function.

Proposition 2.3. *For any two complex numbers $z_1, z_2 \in \mathbb{C}$, $|z_1 z_2| = |z_1|\,|z_2|$.*

Proof. Use the fact that $|z|^2 = z\bar{z}$ for any $z \in \mathbb{C}$, and complex multiplication is commutative

$$|z_1 z_2|^2 = (z_1 z_2)(z_1 z_2)^* = z_1 z_2 z_1^* z_2^* = z_1 z_1^* z_2 z_2^* = |z_1|^2\,|z_2|^2.$$

□

The relationship between the real part, imaginary part and complex conjugate are

$$\operatorname{Re}(z) = \frac{z + z^*}{2} \quad \text{and} \quad \operatorname{Im}(z) = \frac{z - z^*}{2i}. \tag{2.1}$$

We can use complex conjugate to perform division in complex numbers. Suppose we want to divide $z_1 = a + bi$ by $z_2 = c + di$, where c and d are not zero. We multiply and divide by the conjugate of z_2,

$$\frac{z_1}{z_2} = \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di} = \frac{ac + bd}{c^2 + d^2} + i\frac{bc - ad}{c^2 + d^2}. \tag{2.2}$$

We can also do complex division using the 2×2 representation of complex numbers. Division is the same as taking matrix inverse,

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & -d \\ d & c \end{bmatrix}^{-1} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & d \\ -d & c \end{bmatrix} \frac{1}{c^2 + d^2} = \frac{1}{c^2 + d^2} \begin{bmatrix} ac + bd & ad - bc \\ bc - ad & ac + bd \end{bmatrix}.$$

The answer is the same as in (2.2). We note that $c^2 + d^2$ is the determinant of the matrix $\begin{bmatrix} c & -d \\ d & c \end{bmatrix}$ and is the same as the square of the absolute value of $c + di$.

Definition 2.4. The *argument* of a nonzero complex number z is defined as the angle from the positive real axis to the straight line from 0 to z. We write $\arg(z)$ to denote the argument function. The argument of $z = 0$ is not defined. The argument of a nonzero complex number is defined only up to integral multiples of 2π.

Definition 2.5. The points in the complex plane with modulus equal to 1 is called the *unit circle*.

A complex number $z = x + iy$ can be written in polar form

$$z = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta),$$

where r is the modulus of z and θ is an argument of z. Note that $\cos\theta + i\sin\theta$ lies on the unit circle for any θ. There are more than one way to write a complex number in polar form, because we can always add $2\pi k$ to θ, for any integer k, and get the same point on the complex plane.

Using the polar form, complex multiplication can calculated in terms of the modulus and the argument.

Proposition 2.6. *Given $z_1 = r_1\cos\theta_1 + ir_1\sin\theta_1$ and $z_2 = r_2\cos\theta_2 + ir_2\sin\theta_2$ in polar form, their product can be computed by*

$$z_1z_2 = r_1r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)].$$

Proof. The proof follows from the definition of complex multiplication and basic trigonometric identities,

$$\begin{aligned} z_1z_2 &= r_1r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= r_1r_2\big[\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)\big] \\ &= r_1r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]. \end{aligned}$$

□

The polar form suggests that the operation of complex multiplication can be decomposed into two parts. Using the second construction of complex numbers (1.2), the 2×2 matrix corresponding to a complex number $a+bi$ can be factorized as

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = r\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

where $r = \sqrt{a^2+b^2}$ and $\theta = \arg(a+bi)$. The matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is a rotation matrix. The matrix-vector product

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}$$

is the point obtained by rotating (x, y) counter-clockwise by angle θ. The geometric meaning of multiplication by $a + bi = r(\cos\theta + i \sin\theta)$ is thus, (i) first rotate by θ counter-clockwise, then (ii) scale up (or down) by a factor of r.

Using the polar form, complex conjugate and complex division are computed by

$$(r\cos\theta + ir\sin\theta)^* = (r\cos(-\theta) + ir\sin(-\theta))$$
$$(r_1\cos\theta_1 + ir_1\sin\theta_1)/(r_2\cos\theta_2 + ir_2\sin\theta_2) = (r_1/r_2)(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)),$$

provided that $r_2 \neq 0$.

Theorem 2.7 (DeMoivre formula)**.** *For any $n \in \mathbb{Z}$ and $\theta \in \mathbb{R}$, we have*

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta). \tag{2.3}$$

Proof. The formula is obviously true when $n = 1$. When $n = 2$, it follows directly from Prop. 2.6,

$$(\cos\theta + i\sin\theta)^2 = \cos(\theta + \theta) + i\sin(\theta + \theta) = \cos(2\theta) + i\sin(2\theta).$$

We apply mathematical induction to establish (2.3) for all positive integers n and for all real numbers θ.

For negative n, we first note that

$$\begin{aligned}(\cos\theta + i\sin\theta)^{-1} &= \frac{1}{\cos\theta + i\sin\theta}\\ &= \frac{1}{\cos\theta + i\sin\theta} \cdot \frac{\cos\theta - i\sin\theta}{\cos\theta - i\sin\theta}\\ &= \cos\theta - i\sin\theta\\ &= \cos(-\theta) + i\sin(-\theta).\end{aligned}$$

Hence for positive integer m, we have

$$\begin{aligned}(\cos\theta + i\sin\theta)^{-m} &= ((\cos\theta + i\sin\theta)^{-1})^m\\ &= (\cos(-\theta) + i\sin(-\theta))^m\\ &= \cos(-m\theta) + i\sin(-m\theta).\end{aligned}$$

□

Using the matrix representation of complex numbers, the DeMoivre's formula can be stated as

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^n = \begin{bmatrix} \cos(n\theta) & -\sin(n\theta) \\ \sin(n\theta) & \cos(n\theta) \end{bmatrix}.$$

Geometrically speaking, this says that rotating n times by an angle θ is the same as rotating once by angle $n\theta$.

Example 2.1. Compute $(-1+i\sqrt{3})^8$.

The complex number $-1+i\sqrt{3}$ in polar form is

$$2(-1/2+i\sqrt{3}/2) = 2(\cos(2\pi/3)+i\sin(2\pi/3)).$$

Hence, with the use of DeMoivre's formula, we get

$$(-1+i\sqrt{3})^8 = 2^8(\cos(8\cdot 2\pi/3)+i\sin(8\cdot 2\pi/3)) = 256(\cos(4\pi/3)+i\sin(4\pi/3)).$$

In Cartesian form, the answer is $128(-1-i\sqrt{3})$.

Example 2.2. Compute $(-1+i)^{20}$.

Express $-1+i$ in polar form $\sqrt{2}(\cos(3\pi/4)+i\sin(3\pi/4))$. By DeMoivre's formula,

$$\begin{aligned}(-1+i)^{20} &= 20^{20/2}(\cos(20\cdot 3\pi/4)+i\sin(20\cdot 3\pi/4)) \\ &= 1024(\cos\pi + i\sin\pi) \\ &= -1024.\end{aligned}$$

Example 2.3. Express $\sin(5\theta)$ as a polynomial in $\sin(\theta)$.

By DeMoivre's formula,

$$\begin{aligned}&(\cos(5\theta)+i\sin(5\theta)) = (\cos\theta+i\sin\theta)^5 \\ &= \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta + 5\cos^4\theta\sin\theta + i\sin^5\theta.\end{aligned}$$

Equating the imaginary parts, we obtain

$$\begin{aligned}\sin(5\theta) &= 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta \\ &= 5(1-\sin^2\theta)^2\sin\theta - 10(1-\sin^2\theta)\theta\sin^3\theta + \sin^5\theta \\ &= 5\sin\theta - 10\sin^3\theta + 5\sin^5\theta - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta \\ &= 5\sin\theta - 20\sin^3\theta + 16\sin^5\theta.\end{aligned}$$

3 Lecture 3 (n-th roots of complex number)

Summary

- Complex division
- Principal argument
- Extracting the n-th roots of a complex number

Dividing a complex number $a + ib$ by $c + di$, where a, b, c, and d are arbitrary real numbers, means finding a complex number $w = x + iy$ such that $(x + iy)(c + di) = a + bi$. The problem can be reduced to a system of linear equations. By equating real and imaginary partsin

$$(x + iy)(c + di) = (cx - dy) + i(xd + yc) = a + bi,$$

we obtain

$$\begin{bmatrix} c & -d \\ d & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}.$$

The solution can be obtained by multiplying both sides by the inverse of $\begin{bmatrix} c & -d \\ d & c \end{bmatrix}$,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{c^2 + d^2} \begin{bmatrix} c & d \\ -d & c \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

A faster method is to apply the following trick using complex conjugate

$$\frac{a + bi}{c + di} = \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di} = \frac{(ac + bd) + i(bc - ad)}{c^2 + d^2}.$$

Definition 3.1. Given a nonzero complex number z, the *principal argument* of z is the *unique* angle θ_0 (in radian) in $(-\pi, \pi]$ such that $z = |z|(\cos(\theta_0) + i\sin(\theta_0))$.

We follow the notation in [BrownChurchill] and denote the principal argument of z by $\mathrm{Arg}(z)$. In general, the argument function is multi-valued; it can be equal to $\mathrm{Arg}(z) + 2k\pi$ for any integer k.

Example 3.1. Compute the square roots of $4+i$.

Write $4+i$ as $\sqrt{17}(\cos\phi + i\sin\phi)$, where $\phi = \tan^{-1}(1/4)$. A complex number w is called a square root of $4+i$ if $w^2 = 4+i$. By DeMoivre's formula, the modulus of w must be $\sqrt{\sqrt{17}}$. If we denote the argument of w by θ, then $2\theta = \phi + 2k\pi$ for some integer k. This gives

$$\theta = \frac{\phi}{2} + k\pi,$$

and we can take $k = 0, 1$ (because adding an integral multiple of 2π to the argument gives the same complex number.) We can write the answer as

$$\sqrt{4+i} = (17)^{1/4}(\cos(\phi/2 + k\pi) + i\sin(\phi/2 + k\pi)), \qquad \text{for } k = 0, 1,$$

or

$$\sqrt{4+i} = \pm(17)^{1/4}(\cos(\phi/2) + i\sin(\phi/2)).$$

Example 3.2. Compute the cube roots of unit.

Method 1. It amounts to solving $z^3 - 1 = 0$. After factorizing the polynomial into

$$(z-1)(z^2+z+1) = 0,$$

the solutions are 1, and the two roots of z^2+z+1, namely $-\frac{1}{2} \pm i\sqrt{\frac{3}{2}}$.

Method 2. We find all complex numbers with unit modulus and argument θ such that $3\theta = 0$. There are three possible values for θ, and they are 0, $2\pi/3$ and $-2\pi/3$. The cube roots of unity are

$$1,\ \cos(2\pi/3) + i\sin(2\pi/3),\ \cos(2\pi/3) - i\sin(2\pi/3).$$

Example 3.3. Compute the cube roots of i.

The principal argument of i is $\pi/2$. We want to find the values of θ such that

$$3\theta = \frac{\pi}{2} + 2\pi k, \qquad \text{for } k \in \mathbb{Z}.$$

There are three choices for θ, namely, $\pi/6 + 2\pi k/3$, for $k = 0, 1, 2$. The cube roots of i are

$$\cos(\frac{\pi}{6} + \frac{2\pi k}{3}) + i\sin(\frac{\pi}{6} + \frac{2\pi k}{3}).$$

for $k = 0, 1, 2$.

In general, there are n solutions when taking the n-th root of a nonzero number. The method is the same as in the above examples. If we plot the n solutions in the complex plane, they form a regular n-gon with the origin as the center.

4 Lecture 4 (Complex plane as metric space and topological space)

Summary

- Point at infinity
- Concepts from metric space

In a 3-dimensional space, identify the points in the horizontal plane as the complex numbers. A point in the 3-D space has coordinates (ξ, η, ζ). A complex number $x + iy$ is thus located at $(x, y, 0)$. Put a sphere of radius 1 on the x-y plane, touching the x-y plane at the origin. The equation of the sphere is

$$\xi^2 + \eta^2 + (\zeta - \frac{1}{2})^2 = (\frac{1}{2})^2.$$

If we draw a straight line connecting $(0, 0, 1)$ and a point $(x, y, 0)$ on the horizontal plane, there is a unique intersection point on the sphere. This gives a one-to-one correspondence between the points on the complex plane and the points on the sphere, except the north pole. This mapping is called the *stereographical projection*. The totality of all complex numbers can be represented as the points on a punctured sphere. The picture can be completed by adjoining an extra point to the complex numbers.

Definition 4.1. The *extended complex numbers* as a set is defined as $\mathbb{C} \cup \{\infty\}$, where ∞ is a symbol called the *point at infinity*. The symbol ∞ corresponds to the north pole in the stereographic projection. The sphere in the stereographical projection is called the *Riemann sphere.*

Remark. The importance of the Riemann sphere is that it is a compact set, and compact set has nice topological properties.

In this lecture we study complex numbers as points on a metric space, with the metric induced by the complex absolute value; the distance between two complex numbers z_1 and z_2 is $|z_1 - z_2|$. We readily check that the triangular inequality is satisfied.

Proposition 4.2 (Triangle inequality)**.**

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

for any two complex numbers z_1 and z_2.

Proof. Take the square of the left-hand side,

$$\begin{aligned} |z_1 + z_2|^2 &= (z_1 + z_2)(z_1^* + z_2^*) \\ &= |z_1|^2 + 2\mathrm{Re}(z_1 z_2^*) + |z_2|^2 \\ &\le |z_1|^2 + 2|\mathrm{Re}(z_1 z_2^*)| + |z_2|^2. \end{aligned}$$

It is sufficient to prove

$$|\mathrm{Re}(z_1 z_2^*)| \le |z_1|\,|z_2|, \tag{4.1}$$

because it will immediately give $|z_1 + z_2|^2 \le (|z_1|^2 + |z_2|)^2$.

To prove (4.1), suppose $z_1 = a + bi$ and $z_2 = c + di$, and write $\mathrm{Re}(z_1 z_2^*) = ac + bd$. We want to prove $(ac + bd)^2 \le (a^2 + b^2)(c^2 + d^2)$. This inequality holds for any real numbers a, b, c and d because

$$(a^2 + b^2)(c^2 + d^2) - (ac + bd)^2 = (ad - bc)^2 \ge 0.$$

□

Notation: a *sequence* of complex numbers $z_1, z_2, z_3, \dots$ is denoted by $(z_k)_{k=1}^\infty$ or $\{z_k\}$.

Definition 4.3. Given a complex sequence $(z_n)_{n=1}^\infty$, we say that z_n *converges* to w if

$$\forall \epsilon > 0\ \exists N,\ s.t.\ |z_n - w| < \epsilon,\ \forall n \ge N.$$

It is equivalent to requiring that $(|z_n - w|)_{n=1}^\infty$ as a real sequence is converging to 0 as $n \to \infty$. We write $z_n \to w$ if z_n converges to w, and

$$\lim_{n\to\infty} z_n = w.$$

Example 4.1.

$$\frac{1}{n} + \frac{i}{n^2} \to 0.$$

$$(0.5)^n(\cos n + i \sin n) \to 0.$$

Example 4.2. Compute $\lim_{n\to\infty} \frac{n}{n+i}$.

We can first make a guess that the limit should be 1, because when n is large, adding i to n has negligible effect. To make the argument rigorous, we write

$$\frac{n}{n+i} = \frac{n+i-i}{n+i} = 1 - \frac{i}{n+i}.$$

Then

$$\left|\frac{n}{n+i} - 1\right| = \left|\frac{i}{n+i}\right| = \frac{1}{\sqrt{n^2+1}} \to 0, \qquad \text{as } n \to \infty.$$

Therefore $n/(n+i) \to 1$ as $n \to \infty$.

Example 4.3. The sequence $((2i)^n)_{n=1}^{\infty}$ does not converge to any complex number in $\mathbb{C}$. However, if we look at the projection of $(2i)^n$ on the Riemann sphere, it is converging to the point at infinity ∞. Hence we can say that $(2i)^n \to \infty$.

More generally, we say that a sequence of complex numbers $(z_n)_{n=1}^{\infty}$ *converges to the point at infinity* if $z_n^{-1} \to 0$.

Definition 4.4. A sequence $(z_k)_{k=1}^{\infty}$ is called a *Cauchy sequence* if for all $\epsilon > 0$, there exists an integer N such that

$$|z_m - z_n| \leq \epsilon \quad \text{whenever } m, n \geq N.$$

The basic property of Cauchy sequence for real numbers extends to the complex case.

Theorem 4.5. *A complex sequence $(z_k)_{k=1}^{\infty}$ converges if and only if $(z_k)_{k=1}^{\infty}$ is Cauchy.*

We have the following relationship between convergence of complex sequence and real sequences.

Theorem 4.6. *A complex sequence $(z_k)_{k=1}^{\infty}$ converges if and only if both $(\mathrm{Re}(z_k))_{k=1}^{\infty}$ and $(\mathrm{Im}(z_k))_{k=1}^{\infty}$ converge.*

The notion of infinite series for complex numbers is the same as in calculus.

Definition 4.7. An infinite series of complex numbers $\sum_{k=1}^{\infty} z_k$ *converges* if the sequence of partial sums

$$\Big(\sum_{k=1}^{n} z_k\Big)_{n=1}^{\infty}$$

is convergent.

Proposition 4.8. *Given a sequence of complex numbers $(z_k)_{k=1}^{\infty}$, if the real infinite series $\sum_{k=1}^{\infty} |z_k|$ converges, then $\sum_{k=1}^{\infty} z_k$ also converges.*

The proof can be done by consider the real and imaginary parts of z_k, and reduced to the real case.

Definition 4.9. We say that a series $(z_k)_{k=1}^{\infty}$ is *absolutely convergent* if $\sum_{k=1}^{\infty} |z_k|$ is convergent.

Example 4.4. (Complex geometric series) Evaluate $\sum_{k=1}^{\infty}(0.5i)^k$.

We can check that this is absolutely convergent. Because $|0.5i| = 0.5$,

$$\sum_{k=1}^{\infty} |0.5i|^k = \sum_{k=1}^{\infty} (0.5)^k$$

is a geometric series with common ratio strictly less than 1, and hence is convergent.

For any finite n, we have

$$\sum_{k=1}^{n} (0.5i)^k = \frac{(0.5i)^{n+1} - 0.5i}{0.5i - 1}.$$

We take limit as $n \to \infty$,

$$\sum_{k=1}^{\infty} (0.5i)^k = \lim_{n\to\infty} \frac{(0.5i)^{n+1} - 0.5i}{0.5i - 1} = \frac{-0.5i}{0.5i - 1} = \frac{-1+2i}{5}.$$

Definition 4.10. An *open disc* centered at z_0 with radius r is defined as

$$D(z_o; r) \triangleq \{z \in \mathbb{C} : |z - z_0| < r\}.$$

A *circle* centered at z_0 with radius r is

$$C(z_o; r) \triangleq \{z \in \mathbb{C} : |z - z_0| \leq r\}.$$

A set S in the complex plane is said to be *open* if for any $z \in S$, we can find $\delta > 0$ such that $D(z; \delta_0) \subseteq S$.

It can be shown that an open disc is indeed open.

Definition 4.11. The *boundary* of a set S, denoted by ∂S, is defined as

$$\{z \in \mathbb{C} : \forall \delta > 0,\ D(z; \delta) \cap S \neq \emptyset \text{ and } D(z; \delta) \cap S^c \neq \emptyset\}.$$

A set is said to be a *closed set* if the complement is open.

A set is *bounded* if it is contained in $D(0; M)$ for some larger M.

A set is *compact* if it is closed and bounded.

5 Lecture 5 (Complex function)

Summary

- Domain of a function
- Continuous function
- Complex differential function

In complex analysis, a *domain/region* is an open and connected set in $\mathbb{C}$.

In MAT3253, we can understand "connected" as "path-connected", i.e., any two points in the set are connected by a path. The following is a useful fact for two-dimensional region.

Proposition 5.1. *Suppose R is an open set, and A and B are points in R that are connected by a path, then there exists a polygonal path from A to B with finitely many linear parts.*

The above proposition means that when we consider connectedness, it is sufficient to consider piece-wise linear paths.

Definition 5.2. A function $f : \mathbb{C} \to \mathbb{C}$ is said to be *continuous at z_0* if for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$|z - z_0| < \delta \;\Rightarrow\; |f(z) - f(z_0)| < \epsilon.$$

A function f is said to be *continuous* in a domain D if f is continuous at every point in D.

By consider the real and imaginary part separately, we can prove the following

Theorem 5.3. *A complex function f is continuous if and only if the real and imaginary parts are continuous.*

Example 5.1. Show that $f(z) = 1/z$ is continuous in the domain $\mathbb{C} \setminus \{0\})$.

Suppose $z = x + iy$ and $z \neq 0$.

$$\frac{1}{z} = \frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2}$$

The real part of $f(z)$ is $x/(x^2 + y^2)$ and the imaginary part is $-y/(x^2 + y^2)$. Both of them are continuous functions in the domain $\mathbb{C} \setminus \{0\})$. Hence $f(z)$ is continuous by the previous theorem.

A complex function $f(z)$ can be interpreted as a two-dimensional vector field,

$$f(x+iy) = u(x,y) + iv(x,y).$$

When we say that $f(x+iy)$ is *real differentiable*, we mean that the vector-valued function $(u(x,y), v(x,y))$ is differentiable as in multivariable calculus. By the definition of differentiability, if $(u(x,y), v(x,y)$ is differentiable at a point (x_0, y_0), we can approximate the effect of a small change in x and y by linear function,

$$\begin{bmatrix} u(x_0+\Delta x, y_0+\Delta y) \\ v(x_0+\Delta x, y_0+\Delta y) \end{bmatrix} \approx \begin{bmatrix} u(x_0,y_0) \\ v(x_0,y_0) \end{bmatrix} + \begin{bmatrix} u_x(x_0,y_0) & u_y(x_0,y_0) \\ v_x(x_0,y_0) & v_y(x_0,y_0) \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}. \tag{5.1}$$

The entries in the 2×2 matrix are the partial derivatives of u and v evaluated at (x_0, y_0). The symbol "≈" means that the higher-order terms are converging to zero faster than the linear term. More precisely, it means that the limit

$$\lim_{(\Delta x, \Delta y)\to(0,0)} \frac{\|\text{Difference between L.H.S and R.H.S. of (5.1)}\|}{\sqrt{\Delta x^2 + \Delta y^2}} = 0.$$

Example 5.2. The function $f(z)$ defined by $x^2 + i(x+y)$ is real differentiable. Partial derivatives of the real part $u(x,y) = x^2$ and imaginary part $v(x,y) = x+y$ exist, and we have

$$\begin{bmatrix} (x+\Delta x)^2 \\ x+\Delta x+y+\Delta y \end{bmatrix} \approx \begin{bmatrix} x^2 \\ x+y \end{bmatrix} + \begin{bmatrix} 2x & 0 \\ y & x \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} \tag{5.2}$$

for any x and y.

Definition 5.4. A complex function f is said to be *complex differentiable at z_0* if the limit

$$\lim_{\Delta z\to 0} \frac{f(z_0+\Delta) - f(z_0)}{\Delta z} \tag{5.3}$$

exists. This is equivalent to requiring that

$$f(z_0+\Delta z) \approx f(z_0) + w_0 \Delta z,$$

were w_0 is a complex constant and is the limit in (5.3). The limit in (5.3) is denoted by $f'(z_0)$.

We can understand this by interpreting complex multiplication as matrix multiplication. If a function f is complex differentiable at a point z_0, then the 2×2 matrix in (5.1) must be in the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, so that we can realize the matrix multiplication in (5.1) by complex multiplication.

Example 5.3. Consider the function

$$f(z) = z^3 = (x + iy)^3 = (x^3 - 3xy^2) + i(3x^2y - y^3).$$

The real and imaginary parts are $u(x, y) = x^3 - 3xy^2$ and $v(x, y) = 3x^2y - y^3$, respectively. Suppose we fix a base point $(x_0, y_0) = (1, 1)$. The linear approximation in (5.1) at $(1, 1)$ can be written as

$$\begin{bmatrix} u(1 + \Delta x, 1 + \Delta y) \\ v(1 + \Delta x, 1 + \Delta y) \end{bmatrix} \approx \begin{bmatrix} -2 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 & -6 \\ 6 & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}.$$

For general $z = x + iy$, the 2×2 derivative matrix is

$$\begin{bmatrix} 3x^2 - 3y^2 & -6xy \\ 6xy & 3x^2 - 3y^2 \end{bmatrix}.$$

We can use complex arithemtic to realize the linear approximation

$$f(z + \Delta x) = f(z) + (3x^2 - 3y^2 + i(6xy)) \cdot \Delta z,$$

and the complex derivative turns out to be equal to $(3x^2 - 3y^2 + i(6xy)) = 3z^2$.

The function in Example 5.2 is real differentiable everywhere but not complex differentiable in general. In fact it is complex differentiable only at $(x, y) = (0, 0)$. However, the function in Example 5.3 is complex differentiable at all points in $\mathbb{C}$, and the complex derivative is $3z^2$.

6 Lecture 6 (Analytic functions)

Summary

- Definition of analytic function
- Cauchy-Riemann equation

We recall two basic results from multivariable calculus.

Theorem 6.1. *Suppose $\vec{f}(x,y) = (u(x,y), v(x,y))$ be a two-dimensional vector field.*

1. *A necessary condition for $\vec{f}$ to be real differentiable at a point (x_0, y_0) is that all partial derivatives u_x, u_y, v_x and v_y exists in a neighborhood of (x_0, y_0).*
2. *A sufficient condition for $\vec{f}$ to be real differentiable at (x_0, y_0) is (i) partial derivatives u_x, u_y, v_x and v_y exists in a neighborhood of (x_0, y_0), and (ii) the partial derivatives u_x, u_y, v_x are continuous at (x_0, y_0).*

We first derive an important necessary condition for complex differentiability.

Theorem 6.2 (Cauchy-Riemann equations)**.** *Suppose $f(x+iy) = u(x,y) + iv(x,y)$ is complex differentiable at z_0 (see Definition 5.4), where z_0 is in the domain of $f(z)$. Then*

$$u_x = v_y, \quad \textit{and} \quad v_y = -v_x.$$

Proof. The limit in computing complex derivative does not depend on how we approach the point z_0. We can approach z_0 horizontally or vertically, and the results must be the same if the function is complex differentiable.

Let $\Delta z = \Delta x$ and take $\Delta x \to 0$.

$$\begin{aligned}
&\lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) + iv(x_0 + \Delta x, y_0) - u(x_0, y_0) - iv(x_0, y_0)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x} \\
&= \frac{\partial u}{\partial x}(x_0, y_0) + i \frac{\partial v}{\partial x}(x_0, y_0).
\end{aligned}$$

Next suppose $\Delta z = i\Delta y$ and take $\Delta y \to 0$.

$$\begin{aligned}
&\lim_{\Delta y \to 0} \frac{u(x_0, y_0 + \Delta y) + iv(x_0, y_0 + \Delta y) - u(x_0, y_0) - iv(x_0, y_0)}{i\Delta y} \\
&= \lim_{\Delta y \to 0} \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{i\Delta y} + i \lim_{\Delta y \to 0} \frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{i\Delta y} \\
&= -i \frac{\partial u}{\partial x}(x_0, y_0) + \frac{\partial v}{\partial x}(x_0, y_0).
\end{aligned}$$

By equating real and imaginary parts of the two limits, we get $u_x(x_0, y_0) = v_y(x_0, y_0)$ and $u_y(x_0, y_0) = -v_x(x_0, y_0)$. $\square$

Similar to Theorem 6.1, we have the following sufficient condition for complex differentiability.

Theorem 6.3. *A complex function f is complex differentiable at z_0 if*

1. *The partial derivatives u_x, u_y, v_x, v_y exists in a neighborhood of z_0.*
2. *Cauchy-Riemann equations are satisfied at z_0.*
3. *u_x, u_y, v_x, v_y are continuous at z_0.*

If the above conditions hold, the complex derivative of f is given by $u_x + iv_x = v_y - iu_y$.

Example 6.1. The function $f(z) = az + b$, for any $a, b \in \mathbb{C}$ is complex differentiable at any $z \in \mathbb{C}$. The complex derivative is $f'(z) = a$.

Example 6.2. The conjugate function $f(z) = z^*$ is not complex differentiable anywhere. It is because $u_x = 1$ and $v_y = -1$, and $u_x \neq v_y$ at any point in $\mathbb{C}$.

Example 6.3. Consider the square function $f(z) = z^2$. The real and imaginary parts are $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$, respectively. We check that the partial derivatives

$$u_x = 2x,\ u_y = -2y,\ v_x = 2y,\ v_y = 2x$$

exist and are continuous at every point in $\mathbb{C}$. This check conditions 1 and 3 in Theorem 6.3. Furthermore, the Cauchy-Riemann equalities are satisfied everywhere, because

$$u_x = v_y = 2x, \quad \text{and} \quad u_y = -v_x = -2y.$$

By Thoeorem 6.3, $f(z) = z^2$ is differentiable, and the complex derivative is $u_x + iv_x = 2z$.

Example 6.4. The function $f(z) = |z|^2 = x^2 + y^2$ has zero imaginary part. As a real-valued function it is real differentiable. However it is complex differentiable only at $z = 0$. We see this by computing the partial derivatives

$$\begin{aligned} u_x &= 2x, & v_x &= 0, \\ u_y &= 2y, & v_y &= 0. \end{aligned}$$

The Cauchy-Riemann equations are satisfied only at $z = 0$. Therefore it is not complex differentible if $z \neq 0$. By Theorem 6.3, it is indeed complex differentiable at $z = 0$.

Example 6.5. The function $f(z) = 1/z$ is defined in the domain $\mathbb{C} \setminus \{0\}$. It is complex differentiable everywhere in the domain because, for $z \neq 0$,

$$\begin{aligned}\frac{\frac{1}{z+h} - \frac{1}{z}}{h} &= \frac{1}{h}\Big(\frac{z-(z+h)}{(z+h)z}\Big) \\ &= -\frac{1}{z(z+h)}.\end{aligned}$$

When $h \to 0$, the limit of $-\frac{1}{z(z+h)}$ is $-1/z^2$. Therefore $f(z) = 1/z$ is complex differentiable for $z \in \mathbb{C} \setminus \{0\}$, and the complex derivative is $-1/z^2$.

The function in Example 6.4 is complex differentiable only at one time, and is considered as pathological. The main theorems in complex analysis usually require that the function is complex differentiable in a domain (the interior is nonempty).

Definition 6.4. A function f is said to be *analytic*/*holomorphic*/*regular* at a point z_0 if there is a neighborhood of z_0 such that f is complex differentiable at every point in the neighborhood. A function is said to be *entire* if it is complex differentiable at every point in $\mathbb{C}$.

For example, the function in Example 6.1 and 6.3 is entire. The function $1/z$ in Example 6.5 is analytic in the domain of definition. Example 6.2 and 6.4 are not analytic anywhere.