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CSC3001 Discrete Mathematics
Practice **Exercise** 4 Sep 2020

Induction

1. Prove that $\sum_{i=1}^{n} i^2 = n(2n+1)(n+1)/6$ for all $n \geq 1$.
2. Prove that $\sum_{i=1}^{n} i^3 = n^2(n+1)^2/4$ for all $n \geq 1$.
3. Prove that $\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$.
4. Use induction to prove that the following equation holds for all $n \geq 2$:

$$(1 - \frac{1}{2})(1 - \frac{1}{3}) \ldots (1 - \frac{1}{n}) = \frac{1}{n}.$$

5. Prove that $\sum_{k=0}^{n} \binom{k}{r} = \binom{n+1}{r+1}$, where $1 \leq r \leq n$.
6. Consider the Fibonacci sequence (i.e. $F_0 = 0, F_1 = 1, F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$). Prove that $F_1 + F_2 + F_3 + F_4 + \cdots + F_n = F_{n+2} - 1$.
7. If $n \in \mathcal{N}$ andF_n is the n^{th} Fibonacci number. Prove that

$$\binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \binom{n-3}{3} + \cdots + \binom{0}{n} = F_{n+1}.$$

(For example, $\binom{6}{0} + \binom{5}{1} + \binom{4}{2} + \binom{3}{3} + \binom{2}{4} + \binom{1}{5} + \binom{0}{6} = 1 + 5 + 6 + 1 + 0 + 0 + 0 = 13 = F_{6+1}$.)

8. Prove that $n^3 + 2n$ is divisible by 3 for every positive integer n.
9. Use induction to prove that all element in the matrix

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^n$$

are $\leq 2n$ for all positive integer n.

10. Let $S(n)$ be the statement: for any n non negative real numbers $x_1, x_2, \ldots, x_n$,

$$\frac{x_1 + x_2 + \ldots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \ldots x_n}$$

(a) Show that $S(1)$ and $S(2)$ is true.
(b) Show that if $S(k)$ is true, $S(2k)$ is true for any positive integer k.
(c) Show that if $S(k+1)$ is true, $S(k)$ is true for any positive integer k.

6. Proof. Consider the Fibonacci sequence F_n.

① $P(n)$: $F_1+F_2+\cdots+F_n = F_{n+2}-1$. $(n=1,2,\cdots)$.

② $P(1)$: $F_1=1$, $F_2=1$, $F_3=2$, $F_1=F_3-1$.

$P(1)$ is true.

③ Suppose $P(k)$ is true, $k=1,2,\cdots$.

Then $F_1+\cdots+F_k = F_{k+2}-1$.

$F_1+\cdots+F_k+F_{k+1} = (F_{k+2}+F_{k+1})-1 = F_{k+3}-1$

$P(k+1)$ is true.

By induction, $P(n)$ is true for $n=1,2,\cdots$.

7. Proof. Consider F_n as the n^{th} Fibonacci number.

① $P(n)$: $\binom{n}{0}+\binom{n-1}{1}+\cdots+\binom{0}{n} = F_{n+1}$. $(n=0,1,2\cdots)$.

② $P(0)$: $\binom{0}{0}=1=F_1$. $P(0)$ is true.

③ Suppose $P(t)$ is true, for $\forall\, t\in\{1,2,\cdots k\}$.

Then $\binom{k-1}{0}+\binom{k-2}{1}+\cdots+\binom{0}{k-1}=F_k$.

$\binom{k}{0}+\binom{k-1}{1}+\cdots+\binom{1}{k-1}+\binom{0}{k}=F_{k+1}$

Since $\binom{m}{n}=\binom{m-1}{n-1}+\binom{m-1}{n}$, then

$\binom{k-1}{0}+\binom{k-1}{1}=\binom{k}{1}$, $\binom{k-2}{1}+\binom{k-2}{2}=\binom{k-1}{2}$, $\cdots$, $\binom{0}{k-1}+\binom{0}{k}=\binom{1}{k}$

and $F_k+F_{k+1}=F_{k+2}$, then

$F_{k+2}=\binom{k}{0}+\binom{k}{1}+\binom{k-1}{2}+\cdots+\binom{1}{k}$

$=\binom{k+1}{0}+\binom{k}{1}+\binom{k-1}{2}+\cdots+\binom{1}{k}+\binom{0}{k+1}$

$P(t+1)$ is true for $t\in\{1,2,\cdots k\}$.

By induction, $P(n)$ is true for $n=0,1,2,\cdots$.

8. Proof. ① $P(n)$: n^3+2n is divisible by 3 $(n=1,2,\cdots)$.

② $P(1)$: $n^3+2n=1^3+2\cdot1=3=3\cdot1$, $P(1)$ is true.

② Suppose $P(k)$ is true, $k=1,2,\cdots$.

then $k^3+2k=3\cdot p$. $p\in\mathbb{Z}$.

then $(k+1)^3+2(k+1)=k^3+3k^2+3k+1+2k+2$.

$$=k^3+2k+3(k^2+k+1)$$

$$=3\cdot p+3(k^2+k+1).$$

we can get that $(k+1)^3+2(k+1)$ is divisible by 3.

Thus $P(k+1)$ is true.

By induction, $P(n)$ is true for $n=1,2,\cdots$.

9. Proof. ① $P(n)$: elements in $\begin{pmatrix}1&2\\0&1\end{pmatrix}^n$ are $\begin{pmatrix}1&2n\\0&1\end{pmatrix}$. $(n=1,2,\cdots)$.

② $P(1)$: $\begin{pmatrix}1&2\\0&1\end{pmatrix}^1=\begin{pmatrix}1&2\\0&1\end{pmatrix}$, $1,2,0,1\le 2$.

$P(1)$ is true.

③ Suppose $P(k)$ is true, $k=1,2,\cdots$.

Assume $\begin{pmatrix}1&2\\0&1\end{pmatrix}^k=\begin{pmatrix}1&2k\\0&1\end{pmatrix}$, then we have

$$\begin{pmatrix}1&2\\0&1\end{pmatrix}^{k+1}=\begin{pmatrix}1&2k\\0&1\end{pmatrix}\begin{pmatrix}1&2\\0&1\end{pmatrix}=\begin{pmatrix}1&2(k+1)\\0&1\end{pmatrix}$$

$P(k+1)$ is true.

By induction, $P(n)$ is true for $n=1,2,\cdots$.

Thus, $\begin{pmatrix}1&2\\0&1\end{pmatrix}^n=\begin{pmatrix}1&2n\\0&1\end{pmatrix}$, and all elements are $\le 2n$.

10. Proof. (a). $S(1)$: $x_1\ge x_1 \Rightarrow \frac{x_1}{1}\ge x_1$. $S(1)$ is true.

$S(2)$: $x_1+x_2\ge 2\sqrt{x_1x_2} \Rightarrow \frac{x_1+x_2}{2}\ge\sqrt{x_1x_2}$. $S(2)$ is true.

(b). Suppose $S(k)$ is true, then $\frac{x_1+\cdots+x_k}{k}\ge\sqrt[k]{x_1\cdots x_k}$.

and $\frac{x_{k+1}+\cdots+x_{2k}}{k}\ge\sqrt[k]{x_{k+1}\cdots x_{2k}}$.

$$\Rightarrow \frac{x_1+\cdots+x_k}{k}+\frac{x_{k+1}+\cdots+x_{2k}}{k}\ge\sqrt[k]{x_1\cdots x_k}\cdot\sqrt[k]{x_{k+1}\cdots x_{2k}}$$

$$\Rightarrow \frac{x_1+\cdots+x_{2k}}{k} \geq \sqrt[k]{x_1\cdots x_{k+1}}\ \sqrt[k]{x_{k+1}\cdots x_{2k}} \geq 2\sqrt[2k]{x_1\cdots x_{2k}}$$

$$\Rightarrow \frac{x_1+\cdots+x_{2k}}{2k} \geq \sqrt[2k]{x_1\cdots x_{2k}}$$. $S(2k)$ is true.

(c) Suppose $S(k+1)$ is true, then $\frac{x_1+\cdots+x_{k+1}}{k+1} \geq \sqrt[k+1]{x_1\cdots x_{k+1}}$

Let $x_t = x_k + x_{k+1}$, then $\frac{x_1+\cdots+x_t}{k+1} \geq \sqrt[k+1]{x_1\cdots x_t}$

① If $x_1\cdots x_{k+1} \leq 1$, then we can get

$$\frac{x_1+\cdots+x_t}{k} > \frac{x_1+\cdots+x_t}{k+1} \geq \sqrt[k+1]{x_1\cdots x_t} \geq \sqrt[k]{x_1\cdots x_t}.$$

$$\Rightarrow \frac{x_1+\cdots+x_k}{k} \geq \sqrt[k]{x_1\cdots x_k} \qquad (\text{Let } x_k = x_t)$$

② If $x_1\cdots x_{k+1} > 1$, then we can get

$$\frac{x_1+\cdots+x_t}{k+1} \geq \sqrt[k+1]{x_1\cdots x_t} > 1$$

and $$\left(\frac{x_1+\cdots+x_t}{k+1}\right)^{1+\frac{1}{k}} \geq \sqrt[k]{x_1\cdots x_t}$$

and $$\frac{x_1+\cdots+x_t}{k} \geq \left(\frac{x_1+\cdots+x_t}{k+1}\right) \cdot \left(\frac{x_1+\cdots+x_t}{k+1}\right)^{\frac{1}{k}}$$

$$\Rightarrow \frac{x_1+\cdots+x_k}{k} \geq \sqrt[k]{x_1\cdots x_k} \qquad (\text{Let } x_k = x_t)$$

$S(k)$ is true.