MAT 3283 Lecture 21

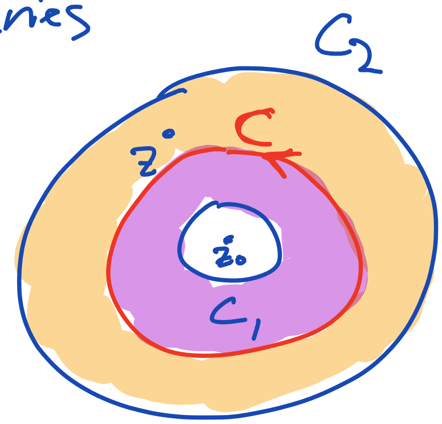
Laurent series $f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^{-n}$

Residue of f at $0 \triangleq b_1$

Uniqueness of Laurent series

$$a_n = \frac{1}{2\pi i}\int_{C_2} \frac{f(w)}{(w-z_0)^{n+1}}\, dw$$

$$b_n = \frac{1}{2\pi i}\int_{C_1} f(w)(w-z_0)^{n-1}\, dw$$



b_2 $\quad$ b_1 $\quad$ $a_0 \updownarrow$ $\quad$ $a_1 \updownarrow$

$$\frac{1}{2\pi i}\int_{\substack{C_1 \\ C}} f(w)(w-z_0)\, dz \qquad \frac{1}{2\pi i}\int_{\substack{C_1 \\ C}} f(w)\, dw \qquad \frac{1}{2\pi i}\int_{\substack{C_2 \\ C}} \frac{f(w)}{w-z_0}\, dw \qquad \frac{1}{2\pi i}\int_{\substack{C_2 \\ C}} \frac{f(w)}{(w-z_0)^2}\, dw \quad \cdots$$

Suppose $f(z) = \sum_{n=0}^{\infty} a_n' z^n + \sum_{n=1}^{\infty} b_n' z^{-n}$ is another Laurent expansion.

$(z_0 = 0)$

$$\int_C f(z)\, dz = \underbrace{\int_C \sum_{n=2}^{\infty} b_n' z^{-n}\, dz}_{\substack{\text{also has an} \\ \text{anti-derivative} \\ \text{integral} = 0}} + \int_C \frac{b_1'}{z}\, dz + \underbrace{\int_C \sum_{n=0}^{\infty} a_n' z^n\, dz}_{\substack{\text{analytic} \\ \text{integral} = 0}}$$

anti-derivative $\sum_n \frac{a_n' z^{n+1}}{n+1}$

$$\int_C f(z)\,dz = b_1' \int_C \frac{1}{z}\,dz$$
$$= 2\pi i\, b_1'$$
$$b_1' = \frac{1}{2\pi i}\int_C f(z)\,dz$$

$$\int \frac{f(z)}{z}\,dz = \underbrace{\int_C \cdots dz}_{=0} + \int_C \frac{a_0}{z}\,dz + \underbrace{\int_C \cdots dz}_{=0}$$

$$a_0 = \frac{1}{2\pi i}\int_C \frac{f(z)}{z}\,dz$$

In general

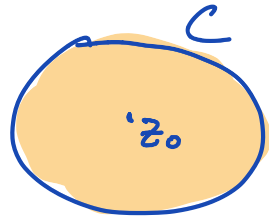
for $n \geq 0$, $\quad a_n = \dfrac{1}{2\pi i}\displaystyle\int_C \frac{f(z)}{z^{n+1}}\,dz$

for $n \geq 1$, $\quad b_n = \dfrac{1}{2\pi i}\displaystyle\int_C f(z)\, z^{n-1}\,dz$

Def The <u>residue</u> of f at $z = z_0$ is the cofficient of $\frac{b_1}{z}$ in the Laurent expansion at $z = z_0$

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$
$$+ \frac{b_1}{z - z_0} + \sum_{n=2}^{\infty} b_n / (z - z_0)^n$$

$$b_1 = \frac{1}{2\pi i}\int_C f(z)\,dz$$



Notation: $\operatorname*{Res}_{z=z_0}(f)$ $\quad$ $\mathrm{Res}(f;z_0)$

$\mathrm{Res}_{z_0}(f)$ $\quad$ $\mathrm{Res}_f(z_0)$

<u>Simple pole</u> $\quad f(z) = \dfrac{b_1}{z-z_0} + a_0 + a_1(z-z_0) + \dots$

$$(z-z_0)f(z) = b_1 + a_0(z-z_0) + a_1(z-z_0)^2 + \dots$$

$$\boxed{\lim_{z\to z_0}(z-z_0)f(z) = b_1}$$

<u>Double pole</u>

$$f(z) = \frac{b_2}{(z-z_0)^2} + \frac{b_1}{z-z_0} + a_0 + a_1(z-z_0) + \dots$$

$$(z-z_0)^2 f(z) = b_2 + b_1(z-z_0) + \dots$$

$$\frac{d}{dz}(z-z_0)^2 f(z) = b_1 + 2a_0(z-z_1)$$

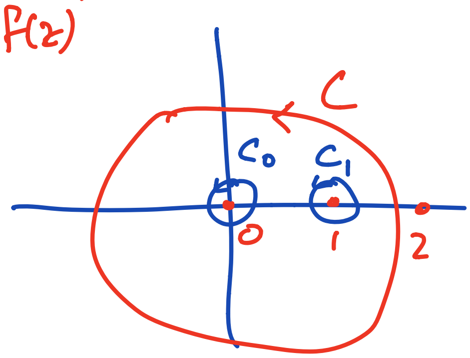
$$\boxed{b_1 = \lim_{z\to z_0}\frac{d}{dz}\left[(z-z_0)^2 f(z)\right]}$$

pole of order m

$$\boxed{b_1 = \lim_{z\to z_0}\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}(z-z_0)^m f(z)}$$

Example

Calculate $\int_C \underbrace{\frac{1}{z(z-1)(z-2)}}_{f(z)} dz$



$$\int_C \frac{1}{z(z-1)(z-2)} dz$$

$$= \int_{C_1} + \int_{C_2}$$

$$= 2\pi i \operatorname{Res}(f, 0) + 2\pi i \operatorname{Res}(f, 1)$$

$$\operatorname{Res}(f, 0) = \lim_{z \to 0} z \frac{1}{z(z-1)(z-2)} = \frac{1}{2}$$

$$\operatorname{Res}(f, 1) = \lim_{z \to 1} \cancel{(z-1)} \frac{1}{z\cancel{(z-1)}(z-2)}$$

$$= \left. \frac{1}{z(z-2)} \right|_{z=1}$$

$$= -1$$

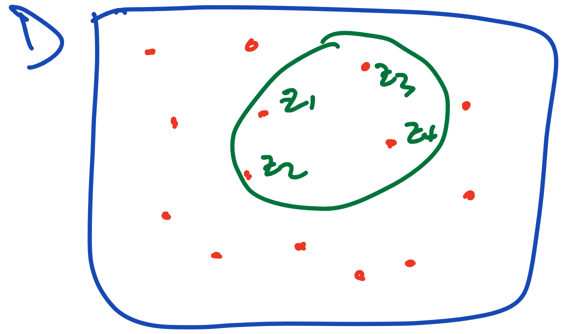
$$\int_C \frac{1}{z(z-1)(z-2)} dz = 2\pi i \left(\frac{1}{2} - 1 \right) = \underline{\underline{-\pi i}}$$

Residue theorem

Suppose f is analytic in a domain D except some isolated singularities.

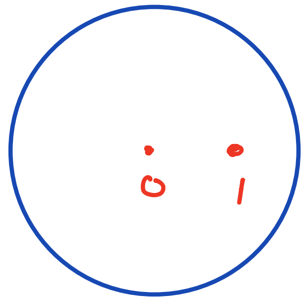
Suppose C is simple closed curve with singular points $z_1, z_2 \sim z_k$ inside C.

$$\int_C f(z)\,dz = 2\pi i \sum_{j=1}^{k} \operatorname{Res}(f ; z_j)$$



Example

$$\int_{|z|=2} \underbrace{\frac{1}{z(z-1)^2}}_{f(z)}\,dz$$



$$\operatorname{Res}(f ; 0) = \left.\frac{1}{(z-1)^2}\right|_{z=0}$$

$$= 1$$

$$\operatorname{Res}(f ; 1) = \lim_{z\to 1} \frac{d}{dz} (z-1)^2 f(z)$$

$$= \lim_{z\to 1} \frac{d}{dz}\frac{1}{z}$$

$$= \lim_{z\to 1}\left(-\frac{1}{z^2}\right)$$

$$= -1$$

$$\int_C f(z)\,dz = 2\pi i\,(1 + (-1)) = \underline{\underline{0}}$$