MAT2002 Ordinary Differential Equations
First-order equations I

Dongdong He

The Chinese University of Hong Kong (Shenzhen)

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Overview

Motivation

Given an ODE, our goal is to determine the existence and uniqueness of the solution. If the solution exist, we also want to find the explicit solution of the ODE.

In this chapter, we will study the first-order ODE, and we will start with first-order linear ODE, which can be solved explicitly.

Outline

Example

Example 2.1

We begin our study with two examples of first order linear ODEs.
For given real constants a, b, t_0, y_0, solve

$$\begin{cases} \frac{dy}{dt} = ay + b, \\ y(t_0) = y_0. \end{cases}$$

This is a linear and autonomous ODE.
Let us consider the case $a = 0$. Then the ODE becomes

$$y' = b, \quad y(t_0) = y_0$$

Integrating yields the general solution

$$y(t) = bt + c, \quad c \in \mathbb{R},$$

and the initial condition gives the particular solution

$$\boxed{y(t) = y_0 + b(t - t_0)}.$$

Example

Example 2.1 continue

For the case $a \neq 0$, we rearrange the ODE into another form:

$$y' = ay + b = a(y + b/a) \Rightarrow \frac{1}{y + \frac{b}{a}} \frac{dy}{dt} = a.$$

If there exists a function $H(y)$ such that $\frac{dH}{dy} = (y + b/a)^{-1}$, then the ODE becomes (via the Chain rule)

$$\frac{dH}{dy} \frac{dy}{dt} = \frac{d}{dt} H(y(t)) = a.$$

It turns out that $H(y) = \ln |y + b/a|$, and so we have
$\ln |y(t) + b/a| = at + c, \quad c \in \mathbb{R}.$
Taking exponential then leads to the general solution

$$\boxed{y(t) = \kappa \exp(at) - \frac{b}{a}}, \quad \kappa = \begin{cases} \exp(c), & \text{if } y(t) + \frac{b}{a} > 0, \\ -\exp(c), & \text{if } y(t) + \frac{b}{a} < 0, \end{cases}$$

Example

Example 2.1 continue

Also $y(t) = -\frac{b}{a}$ is the equilibrium solution for the ODE. Thus, the general solution is

$$y(t) = \kappa \exp(at) - \frac{b}{a}, \quad \kappa \in \mathbb{R}.$$

Using the initial condition $y(t_0) = y_0$ we obtain the particular solution

$$\boxed{y(t) = (y_0 + b/a)\exp(a(t - t_0)) - \frac{b}{a}}.$$

In summary we find that

$$y(t) = \begin{cases} y_0 + b(t - t_0), & \text{for } a = 0, \\ (y_0 + b/a)\exp(a(t - t_0)) - \frac{b}{a}, & \text{for } a \neq 0. \end{cases}$$

This example shows that the explicit formula for the solution can **depend** on the values of the given coefficients. **Always** keep this in mind before starting to solve the ODE.

Example 2: General solution to $\frac{dy}{dt} = p(t)y$

For a given function $p(t)$, find the general solution to

$$\frac{dy}{dt} = p(t)y.$$

Note that $y(t) \equiv 0$ is one solution! Supposed that $y(t_*) \neq 0$ for some $t_* \in I$, then we can rearrange the ODE into the form

$$\frac{1}{y}\frac{dy}{dt} = p(t) \Rightarrow \frac{d}{dt}\ln|y(t)| = p(t).$$

$$\ln|y(t)| = \int p(t)dt + c.$$

Here $\int p(t)dt$ is the indefinite integral (antiderivative) of $p(t)$ (the value of indefinite integral (antiderivative) is not unique, indefinite integral is unique up to an arbitary constant, antiderivatives for the same function could have an arbitary constant difference).

Remark

Once we take the exponential we find that

$$|y(t)| = \exp\left(\int p(t)dt \right) \exp(c).$$

Setting

$$\kappa = \begin{cases} \exp(c), & \mathit{if}\ \ y(t) > 0, \\ -\exp(c), & \mathit{if}\ \ y(t) < 0, \end{cases} \Rightarrow y(t) = \kappa \exp\left(\int p(t)dt \right).$$

The sign of the constant κ does not matter once we use the initial condition to determine the solution of the IVP.

Outline

In the above Example 2, we obtain that the general solution to the ODE

$$\frac{dy}{dt} = p(t)y$$

is

$$y(t) = \kappa \exp\left(\int p(t)dt \right),$$

for some arbitrary constant κ.

Method of integrating factors

We now study the IVP of the general linear first order ODE:

$$\begin{cases} \frac{dy}{dt} = p(t)y + q(t), \\ y(t_0) = y_0, \end{cases} \tag{1}$$

for some given functions $p(t)$, $q(t)$ and constants t_0 and y_0.

One example is the equation for the motion of the falling object: $mv' = mg - \gamma v$, where we set $y = v$, $p = -\gamma/m$ and $q = g$. The method we use is called the **method of integrating factors**.

Method of integrating factors

Idea: Multiply the ODE (1) by a function $\mu(t)$, leading to

$$\mu(t)\frac{dy}{dt} - \mu(t)p(t)y(t) = \mu(t)q(t). \tag{2}$$

Suppose

$$\boxed{\mu(t)\frac{dy}{dt} - \mu(t)p(t)y(t) = \frac{d}{dt}(\mu(t)y(t))}, \tag{3}$$

then, the multiplied ODE (2) becomes

$$\frac{d}{dt}(\mu(t)y(t)) = \mu(t)q(t) \Rightarrow \boxed{\mu(t)y(t) = \int \mu(t)q(t)dt + c}, \quad c \in \mathbb{R}. \tag{4}$$

If in addition, $\mu(t)$ is **<u>non-zero</u>**, we can divide by $\mu(t)$ and end up with the general solution

$$\boxed{y(t) = \frac{1}{\mu(t)}\left[\int \mu(t)q(t)dt + c\right].} \tag{5}$$

Method of integrating factors

Definition 2.2

If such a function $\mu(t)$ exists satisfying (3), then we call $\mu(t)$ the **integrating factor**.

But does such a function $\mu(t)$ exists? If it doesn't then this is a useless method. What is the equation satisfied by μ? From (3) we see that

$$\begin{aligned}\mu(t)y'(t) - \mu(t)p(t)y(t) =& \frac{d}{dt}(\mu y) = \mu'(t)y(t) + \mu(t)y'(t) \\ \Rightarrow y(t)\left(\frac{d\mu}{dt} + p(t)\mu(t)\right) =& 0.\end{aligned}$$

The above equation is satisfied if $y(t) = 0$ or $\mu'(t) + p(t)\mu(t) = 0$. The first case $y(t) = 0$ is not desirable, since if the initial condition y_0 is non-zero, we have a contradiction.

Method of integrating factors

Therefore, we consider the second case and obtain the equation

$$\boxed{\frac{d\mu}{dt} = -p(t)\mu} \tag{6}$$

as the ODE for μ. But this type of equation has been encountered before, $\mu(t) \equiv 0$ is the solution of the above ODE but without any interest. When $\mu(t) \neq 0$

$\boxed{\frac{1}{\mu}\frac{d\mu}{dt} = -p(t)}$ Thus, $\boxed{\ln|\mu(t)| = -\int p(t)dt + c}$

Choosing the arbitary constant c to be zero, then one can get a simplest integrating factor:

$$\mu(t) = \exp\left(-\int p(t)dt\right). \tag{7}$$

<u>**Take note of the minus sign!**</u>

Method of integrating factors

This implies that we take the integrating factor $\mu(t)$ to be

$$\boxed{\mu(t) = \exp\left(-\int p(t)dt\right)}, \tag{8}$$

and the general solution $y(t)$ to the ODE $y' = p(t)y + q(t)$ is given as

$$\boxed{y(t) = e^{\int p(t)dt}\left[\int e^{-\int p(t)dt} q(t)dt + c\right]}. \tag{9}$$

The particular solution and the constant c can be computed with the initial condition $y(t_0) = y_0$.

Example: method of integrating factors

Example 2.3

Derive the solution to the ODE

$$\begin{cases} t\frac{dy}{dt} + 2y = 4t^2, \\ y(1) = 2, \end{cases} \tag{10}$$

Step 1. Write the ODE in the form $y' = p(t)y + q(t)$ and identify p and q:

$$t\frac{dy}{dt} + 2y = 4t^2 \Rightarrow \frac{dy}{dt} = -\frac{2}{t}y + 4t$$

$$\Rightarrow p(t) = -\frac{2}{t}, \quad q(t) = 4t.$$

Example: method of integrating factors

Example 2.3 continue

Step 2. Compute the integrating factor $\mu(t)$:

$$\mu(t) = \exp\left(-\int p(t)dt\right) = \exp\left(\int_1^t \frac{2}{t}dt\right) = t^2$$

Step 3. Plug into the formula (9) to get the general solution

$$y(t) = \frac{1}{t^2}\left[\int t^2 \times 4t \; dt + c\right] = t^2 + \frac{c}{t^2}.$$

Step 4. To satisfy the initial condition, one has $c = 1$. The solution is $y(t) = t^2 + \frac{1}{t^2}$.

Step 5. Interval of definition $t \in I = [0, \infty)$. Since $t^2 + \frac{1}{t^2}$ has two branches, only the right branch passes through $(1, 2)$. The right branch is the solution for the IVP.

Example: method of integrating factors

Example 2.3 continue

This solution is shown by the black curve in the following Figure.

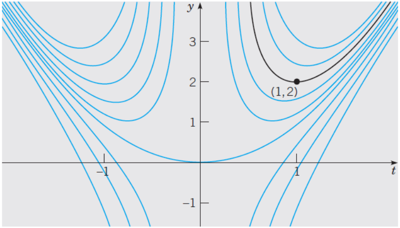

FIGURE 2.1.3 Integral curves of the differential equation $ty' + 2y = 4t^2$; the black curve passes through the point $(1, 2)$.

Note that $y(t) = t^2 + \frac{1}{t^2}$ becomes unbounded and is asymptotic to the positive y-axis as $t \to 0$ from the right. This is the effect of the infinite discontinuity in the coefficient $p(t)$ at the origin. The function $y = t^2 + \frac{1}{t^2}$ for $t < 0$ is not part of the solution of this initial value problem.

Remark

Remark 1

The general solution $y(t) = t^2 + \frac{c}{t^2}$, for $c \neq 0$, is not defined at the point $t = 0$. So far in the course, we have not really discussed the interval of definition $I \subset \mathbb{R}$. In this case, the general solution is defined only for $t \in (-\infty, 0) \cup (0, \infty) = \mathbb{R} \setminus \{0\}$. The graph of $y(t)$ is sketched, we see that the graph has two parts, one to the left of the y-axis and one to the right of the y-axis. Which part we take depends on the initial condition.

Remark 2

If we consider an initial condition $y(t_0) = y_0$, where $t_0 > 0$, then we choose the right part - since we can determine the arbitrary constant c in the general solution only in the interval $(0, \infty)$. In this case the interval of definition is $I = (0, \infty)$. Similarly, if $t_0 < 0$, then we choose the left part as the solution, with $I = (-\infty, 0)$. This example states that the solution $y(t)$ to ODEs may not be defined for all values of $t \in \mathbb{R}$, and the initial condition plays a role in determining the interval of definition. If $t_0 = 0$, then $y(t) = t^2$ is the unique solution if $y_0 = 0$, and there is no solution if $y_0 \neq 0$.

Example: method of integrating factors

Example 2.4

Derive the solution to the ODE

$$\begin{cases} 2\frac{dy}{dt} + ty = 2, \\ y(0) = 1, \end{cases} \tag{11}$$

Step 1. Write the ODE in the form $y' = p(t)y + q(t)$ and identify p and q:

$$\frac{dy}{dt} = -\frac{t}{2}y + 1$$

$$\Rightarrow p(t) = -\frac{t}{2}, \quad q(t) = 1.$$

Example: method of integrating factors

Example 2.4 continue

Step 2. Compute the integrating factor $\mu(t)$:

$$\mu(t) = \exp\left(-\int p(t)dt\right) = \exp\left(\int \frac{t}{2}dt\right) = \exp\left(\frac{t^2}{4}\right)$$

Step 3. Plug into the formula (9) and set the lower limit to be 0, one can get the general solution

$$y(t) = \frac{1}{\exp\left(\frac{t^2}{4}\right)}\left[\int_0^t \exp\left(\frac{t^2}{4}\right)\, dt + c\right].$$

Step 4. To satisfy the initial condition, one has $c = 1$. The solution is

$$y(t) = \frac{1}{\exp\left(\frac{t^2}{4}\right)}\left[\int_0^t \exp\left(\frac{t^2}{4}\right)\, dt + 1\right], \quad t \in (-\infty, +\infty).$$

Example: method of integrating factors

Example 2.4 continue

The integral on the right side cannot be evaluated in terms of the usual elementary functions, so we leave the integral unevaluated. The main purpose of this example is to illustrate that sometimes the solution must be left in terms of an integral.