MAT3253 Homework 12

Due date: 23 Apr.

Question 1. (Brown&Churchill Ex.62.2) Derive the Laurent series representation

$$\frac{e^z}{(z+1)^2} = \frac{1}{e}\sum_{n=0}^{\infty}\frac{(z+1)^n}{(n+2)!} + \frac{1}{e}\frac{1}{(z+1)} + \frac{1}{e}\frac{1}{(z+1)^2}$$

for $0 < |z+1| < \infty$.

Question 2. (Bak&Newman Chapter 9 Ex.9) Classify the singularities of

(a). $\dfrac{1}{z^4+z^2}$;

(b). $\cot z$;

(c). $\csc z$;

(d). $\dfrac{\exp(1/z^2)}{z-1}$.

Question 3. (Bak&Newman Chapter 9 Ex.12) Find the Laurent expansion of $f(z) = \frac{1}{z(z-1)(z-2)}$ (in powers of z) for

(a). $0 < |z| < 1$;

(b). $1 < |z| < 2$;

(c). $|z| > 2$.

Question 4. (Brown&Churchill Ex.62.9) Suppose that a series

$$\sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

converges to an analytic function $X(z)$ in some annulus $R_1 < |z| < R_2$. That sum $X(z)$ is called the z-transform of $x[n]$ $(n = 0, \pm1, \pm2, \ldots)$. Use the integral expression for for the coefficients in a Laurent series given below to show that if the annulus contains the unit circle $|z| = 1$, then the inverse z-transform of $X(z)$ can be written

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{i\theta}) e^{in\theta} \, d\theta,$$

for $n = 0, \pm 1, \pm 2, \ldots$.

(If $f(z)$ is analytic in an annulus $R_1 < |z| < R_2$ then it can be expanded as a Laurent series $\sum_{n=-\infty}^{\infty} a_n z^n$, and the coefficient a_n can be computed by a complex integral

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(w)}{w^{n+1}} \, dw,$$

where C can be any circle inside the annulus, with counter-clockwise orientation.)

Question 5. (Brown&Churchill Ex.62.10)

(a) Let z be any complex number, and let C denote the unit circle

$$w = e^{i\phi} \quad (-\pi \leq \phi \leq \pi)$$

in the w plane. Then use that contour integral given in the last question for the coefficients in a Laurent series, to show that

$$\exp\left[\frac{z}{2}\left(w - \frac{1}{w}\right)\right] = \sum_{n=-\infty}^{\infty} J_n(z) w^n,$$

where

$$J_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(-i(n\phi - z\sin\phi)) \, d\phi$$

for $n = 0, \pm 1, \pm 2, \ldots$.

(b) Use the property of integrals for even and odd functions, show that the coefficients in part (a) here can be written

$$J_n(z) = \frac{1}{\pi} \int_0^{\pi} \cos(n\phi - z\sin\phi) \, d\phi$$

for $n = 0, \pm 1, \pm 2, \ldots$.

Question 6. (Brown&Churchill Ex.62.8) The *Euler numbers* are the numbers E_n $(n = 0, 1, 2, \ldots)$ in the Maclaurin series representation

$$\frac{1}{\cosh z} = \sum_{n=0}^{\infty} \frac{E_n}{n!} z^n \qquad (|z| < \pi/2).$$

Point out why this representation is valid in the indicated disk and why

$$E_{2n+1} = 0$$

for $n = 0, 1, 2, 3, \ldots$

Then show that

$$E_0 = 1, \quad E_2 = -1, \quad E_4 = 5, \quad E_6 = -61.$$