Cook's measure:

$$D_i = \frac{(\underline{\hat{\beta}}_{(i)} - \underline{\hat{\beta}})^T X^T X (\underline{\hat{\beta}}_{(i)} - \underline{\hat{\beta}})}{p \cdot MS_{Res}} \quad (1)$$

$$= \frac{1}{p} \cdot r_i^2 \cdot \frac{h_{ii}}{1-h_{ii}} \quad (2)$$

$\boxed{\text{Recall}}$:

☆ $X_{(i)}^T X_{(i)} = X^T X - \underline{x}_i \cdot \underline{x}_i^T \quad (3)$

$\underline{x}_i^T$ is the i-th row of $\underbrace{X_{n \times p}}$

$\to 1 \times p$

$\underline{x}_i = (\underline{x}_i^T)^T$ is the i-th Column of $X^T_{n \times p}$

$\to p \times 1$

☆ Apply matrix inversion lemma:

$$\left(X_{(i)}^T X_{(i)}\right)^{-1} = (X^TX)^{-1} + \frac{(X^TX)^{-1} \underline{x}_i \underline{x}_i^T (X^TX)^{-1}}{1-h_{ii}} \quad ④$$

$$h_{ii} = \underline{x}_i^T (X^TX)^{-1} \underline{x}_i \quad \text{scalar}$$

☆ $$\boxed{X_{(i)}^T \underline{y}_{(i)}} = \boxed{X^T \underline{y} - \underline{x}_i y_i} \quad ⑤$$

Outer-product view.

$$\hat{\underline{\beta}}_{(i)} = \hat{\underline{\beta}} - (X^TX)^{-1} \underline{x}_i y_i + \frac{(X^TX)^{-1} \underline{x}_i \underline{x}_i^T (X^TX)^{-1} (X^T\underline{y} - \underline{x}_i y_i)}{1-h_{ii}}$$

$$\Rightarrow \hat{\underline{\beta}} - \hat{\underline{\beta}}_{(i)} = \frac{(X^TX)^{-1} \underline{x}_i e_i}{1-h_{ii}} \quad ⑥$$

$$X(\hat{\beta}-\hat{\beta}_{(i)}) = \frac{X(X^T X)^{-1} \underline{x}_i e_i}{1-h_{ii}} \quad ⑦$$

$$\frac{(\hat{\beta}-\hat{\beta}_{(i)})^T X^T X (\hat{\beta}-\hat{\beta}_{(i)})}{}$$

$$= \frac{\underline{x}_i^T \cancel{(X^T X)^{-1}} \cancel{X^T} X (X^T X)^{-1} \underline{x}_i \cdot e_i^2}{(1-h_{ii})^2}$$

$$= \frac{\boxed{\underline{x}_i^T (X^T X)^{-1} \underline{x}_i} \cdot e_i^2}{(1-h_{ii})^2}$$

$$= \frac{h_{ii}}{(1-h_{ii})^2} \cdot e_i^2 \quad ⑧$$

Finally, $D_i = \dfrac{h_{ii}}{(1-h_{ii})^2} \cdot e_i^2 \cdot \dfrac{1}{p \cdot MS_{Res}}$

(8-b) $$r_i = \left(\frac{e_i}{\sqrt{MS_{Res}(1-h_{ii})}} \right)^2 \cdot \frac{h_{ii}}{1-h_{ii}} \cdot \frac{1}{p}$$

DFFITs measure :

$$DFFITS_i = \frac{\boxed{\hat{y}_i - \hat{y}_{(i)}}}{\sqrt{S_{(i)}^2 h_{ii}}} \qquad (9)$$

$$\downarrow$$

$$= t_i \sqrt{\frac{h_{ii}}{1-h_{ii}}} \qquad (10)$$

$$\boxed{\hat{\underline{\beta}} - \hat{\underline{\beta}}_{(i)} = \frac{(X^T X)^{-1} \underline{x}_i e_i}{1-h_{ii}}}$$

$$\hat{y}_i - \hat{y}_{(i)} = \underline{x}_i^T (\hat{\underline{\beta}} - \hat{\underline{\beta}}_{(i)})$$

$$= \frac{\boxed{\underline{x}_i^T (X^T X)^{-1} \underline{x}_i} \cdot e_i}{1-h_{ii}} = \frac{h_{ii}}{1-h_{ii}} e_i$$

(11)

Inserting (11) to (9), we get

$$\frac{\hat{y}_i - \hat{y}_{(i)}}{\sqrt{S_{(i)}^2 h_{ii}}} = \frac{h_{ii} \cdot e_i}{(1-h_{ii})\sqrt{S_{(i)}^2 h_{ii}}}$$

$$= \boxed{\frac{e_i}{\sqrt{S_{(i)}^2 (1-h_{ii})}}} \cdot \frac{\sqrt{h_{ii}}}{\sqrt{1-h_{ii}}}$$

R_studentized residual

$$= t_i \cdot \sqrt{\frac{h_{ii}}{1-h_{ii}}} \qquad (12)$$

Connection:

a. Ignore $\frac{1}{p}$ in (8-b), then

b. $D_i \approx (DFFITS_i)^2$, if $t_i \approx r_i$