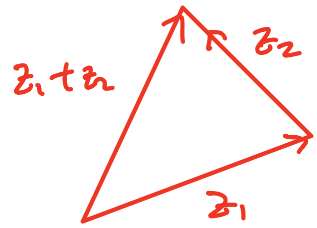
MAT 3253 Lecture 4

Tutorial 1

$$|z_1 + z_2| \le |z_1| + |z_2| \quad \forall z_1, z_2 \in \mathbb{C}$$

triangle inequality



Riemann sphere / spherical representation

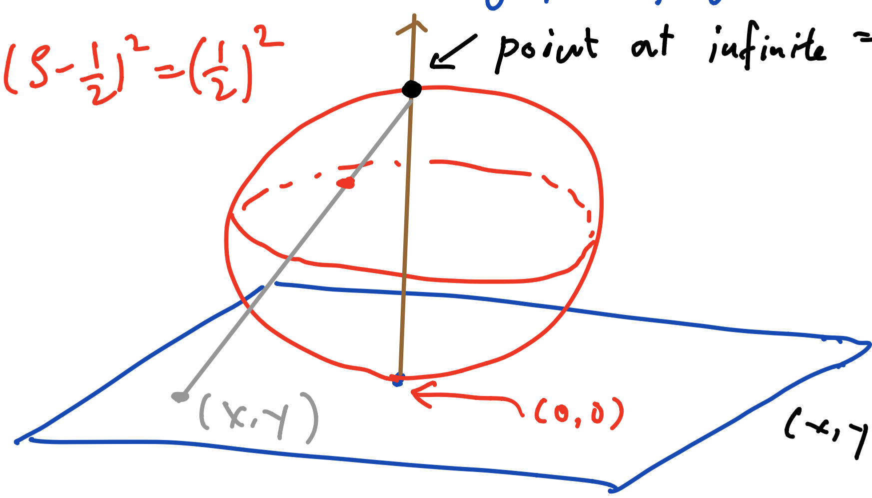
stereographic projection

$\xi^2 + \eta^2 + (\zeta - \frac{1}{2})^2 = (\frac{1}{2})^2$

point at infinite $= (0,0,1)$

$\mathbb{R}^3$

(ξ, η, ζ)



$(x,y) \leftrightarrow (x,y,0)$

$\mathbb{C} \cup \{\infty\}$ extended complex number

symbol

set of all complex numbers

Do not say $z < \infty$ $\quad z \in \mathbb{C}$

Extended real number $\quad \mathbb{R} \cup \{\infty, -\infty\}$

$a \in \mathbb{R} \qquad a < \infty$

$-\infty < a$

Sequence $z_1, z_2, z_3, \dots \in \mathbb{C}$

$(z_k)_{k=1}^{\infty}$, $\{z_k\}$

e.g. $z_n = \frac{1}{n} + \frac{i}{n^2} \longrightarrow 0$

$z_n = (0.5)^n (\cos n + i \sin n) \longrightarrow 0$

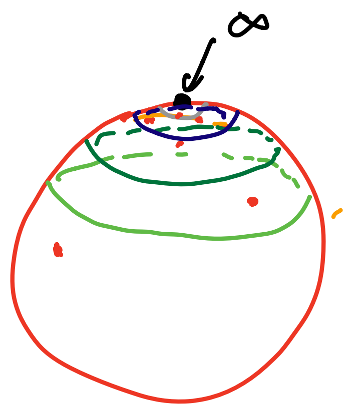
<u>Def</u> $z_n \to w$ if $\forall \varepsilon\ \exists N\ |z_n - w| < \varepsilon\ \forall n \geq N$.

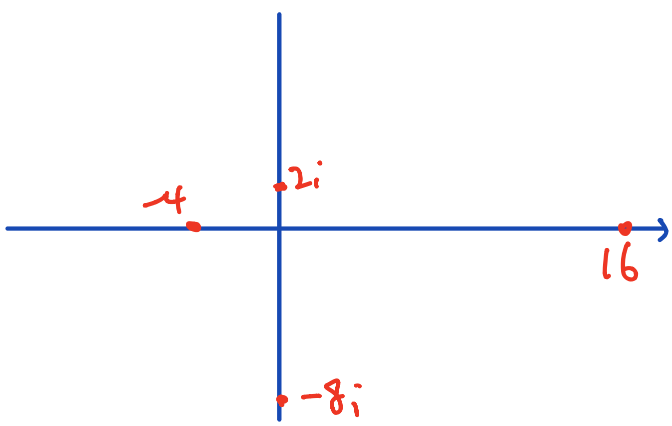
$z_n \to w$ iff $|z_n - w| \to 0$ as $n \to \infty$

e.g. $z_n = (2i)^n \quad \forall n$

$\{z_n\}$ not convergent in $\mathbb{C}$

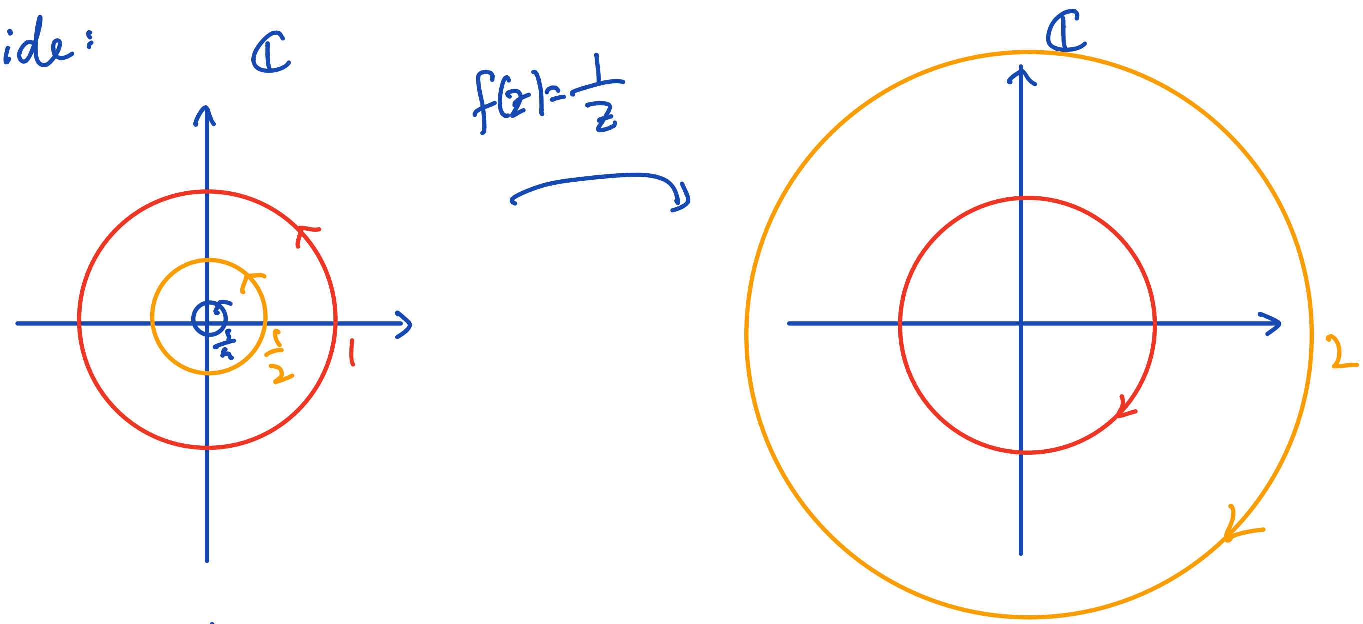
$z_n \to \infty$ in $\mathbb{C} \cup \{\infty\}$





$\frac{1}{z_n} = \frac{1}{(2i)^n} \longrightarrow 0 \qquad z_n \to \infty$

Aside:



small circles $\to 0$

$$0 \xrightarrow{\frac{1}{z}} \infty$$

e.g. $z_n = \dfrac{n}{n+i}$ $\qquad\qquad$ $\dfrac{n}{n+i} = \dfrac{1}{1 + i/n}$

$$\frac{n}{n+i} = \frac{n+i-i}{n+i} = 1 - \frac{i}{n+i}$$

$$\left| \frac{n}{n+i} - 1 \right| = \left| \frac{i}{n+i} \right| = \frac{1}{\sqrt{n^2+1}} \to 0$$

$\therefore \quad \dfrac{n}{n+i} \to 1 \quad$ as $n \to \infty$

Cauchy sequence

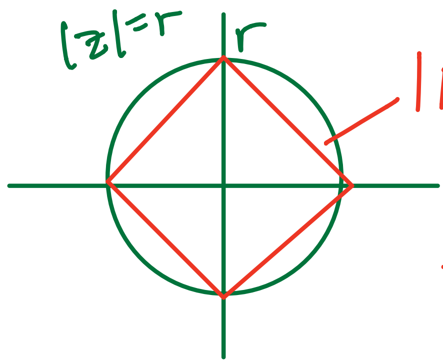
<u>Def</u> $(z_k)_{k=1}^{\infty}$ is a <u>Cauchy sequence</u> if

$\forall \varepsilon > 0$, $\exists N$ s.t. $|z_m - z_n| < \varepsilon$ whenever $m, n \geq N$.

Theorem $(z_k)_{k=1}^{\infty}$ converges iff $(z_k)_{k=1}^{\infty}$ is Cauchy.

Theorem $\{z_k\}$ is convergent iff $\{\mathrm{Re}\, z_k\}$ is convergent and $\{\mathrm{Im}\, z_k\}$ is convergent.

$$|z| \leq |\mathrm{Re}(z)| + |\mathrm{Im}(z)|$$



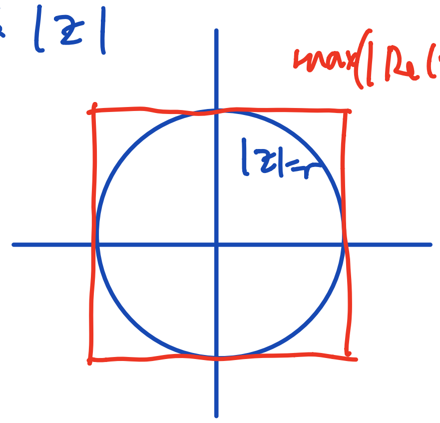
$|\mathrm{Re}(z)| + |\mathrm{Im}(z)| = r$

If $|\mathrm{Re}(z)| + |\mathrm{Im}(z)| \leq r$

$\Rightarrow \quad |z| \leq r$

$$\max\big(|\mathrm{Re}(z)|, |\mathrm{Im}(z)|\big) \leq |z|$$

$\max(|\mathrm{Re}(z)|, |\mathrm{Im}(z)|) = r$



$|z| \leq r$

$\Rightarrow \max\big(|\mathrm{Re}(z)|, |\mathrm{Im}(z)|\big) \leq r$

Series

Def $\sum_{k=1}^{\infty} z_k$ converges if $\left(\sum_{k=1}^{n} z_k\right)_{n=1}^{\infty}$ is convergent.

partial sum

Prop If $\sum_{k=1}^{\infty} |z_k|$ converges, then $\sum_{k=1}^{\infty} z_k$ is convergent.

Proof idea: $\sum_{k=1}^{\infty} |Re(z_k)|$ is convergent

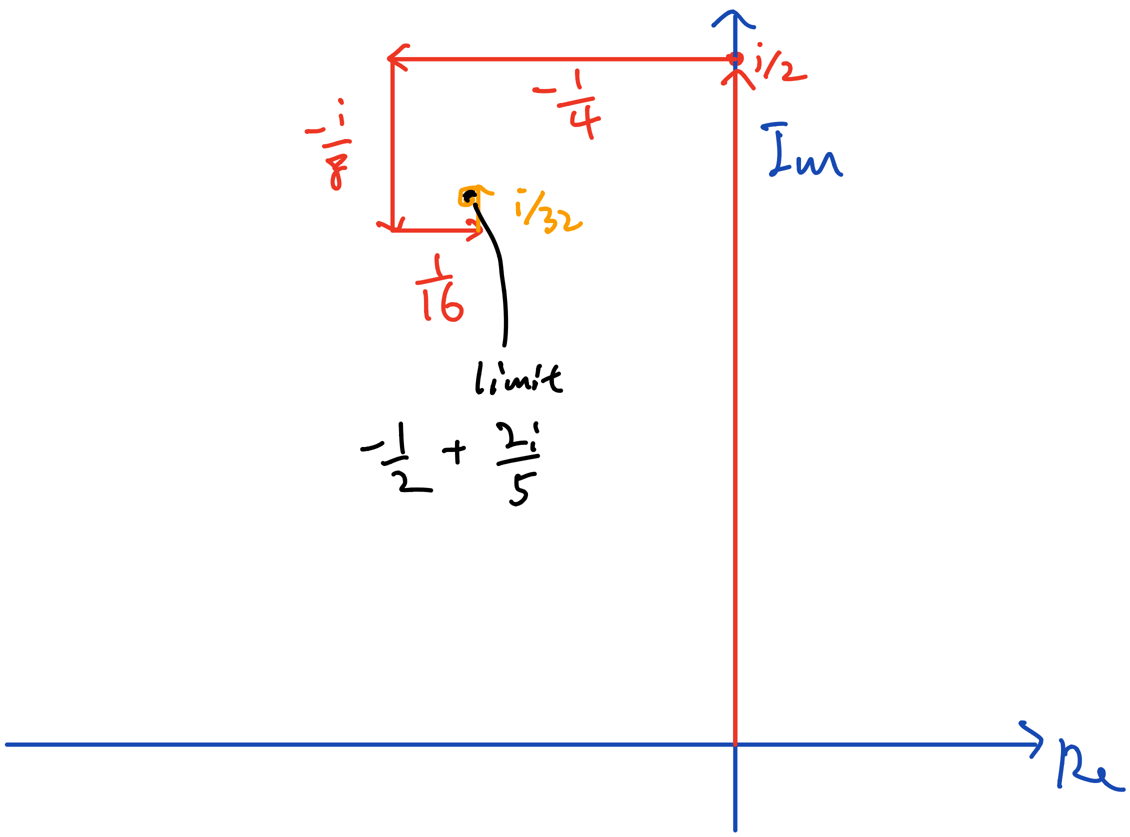
$\sum_{k=1}^{\infty} |Im(z_k)|$ is convergent.

e.g. $\sum_{k=1}^{\infty} (0.5i)^k$ (complex geometric series)

$$\sum_{k=1}^{\infty} |0.5i|^k = \sum_{k=1}^{\infty} \frac{1}{2^k} = 1$$

$$\sum_{k=1}^{\infty} (0.5i)^k = \frac{0.5i}{1-0.5i} = \frac{i}{2-i} = \frac{i}{2-i} \cdot \frac{2+i}{2+i}$$

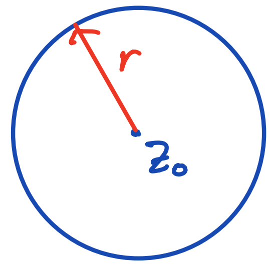
$$= \frac{-1+2i}{5}$$



Definitions

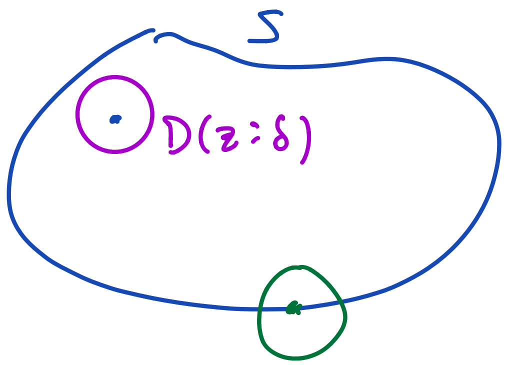
Open disc $D(z_0 ; r) \triangleq \{ z \in \mathbb{C} : |z - z_0| < r \}$

Circle $C(z_0 ; r) \triangleq \{ z \in \mathbb{C} : |z - z_0| = r \}$



Open set S means

$\forall z \in S \quad \exists \delta > 0 \quad$ s.t. $\quad D(z ; \delta) \subset S$



* Open disc is open

∂S boundary of S

$$= \left\{ z \in \mathbb{C} : \forall \delta > 0 \quad \begin{array}{l} D(z ; \delta) \cap S \neq \phi \\ D(z ; \delta) \cap S^c \neq \phi \end{array} \right\}$$

Def

Close set is the complement of open set.

A set S is closed if $S = S \cup \partial S$.

S is bounded if $S \subseteq D(0 ; M)$ for some M,

S is compact if S is closed and bounded.