Stochastic Processes

Lecture 24: From random walks to martingales

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Conditional expectation: I

- Suppose that (X, Y) is uniform on

$$(0,0),(1,0),(0,1),(2,0),(1,1),(0,2).$$

- Distribution of Y

$$\mathbb{P}\{Y=0\}=\frac{3}{6}, \quad \mathbb{P}\{Y=1\}=\frac{2}{6}, \quad \mathbb{P}\{Y=2\}=\frac{1}{6}.$$

- Find $\mathbb{E}(X^2|Y)$

$$\begin{aligned}
\mathbb{E}(X^2|Y=0) &= 0^2\frac{1}{3}+1^2\frac{1}{3}+2^2\frac{1}{3}=\frac{5}{3}, \\
\mathbb{E}(X^2|Y=1) &= 0^2\frac{1}{2}+1^2\frac{1}{2}=\frac{1}{2}, \\
\mathbb{E}(X^2|Y=2) &= 0^2\frac{1}{1}=0.
\end{aligned}$$

Thus,

$$\mathbb{E}(X^2|Y)=\frac{5}{3}1\{Y=0\}+\frac{1}{2}1\{Y=1\}.$$

Conditional expectation: II

- $\mathbb{E}(X^2)$

$$\mathbb{E}(X^2) = 0^2\frac{3}{6} + 1^2\frac{2}{6} + 2\frac{1}{6} = 1.$$

- $\mathbb{E}\Big[\mathbb{E}(X^2|Y)\Big]$

$$\mathbb{E}\Big[\mathbb{E}(X^2|Y)\Big] = \frac{5}{3}\frac{3}{6} + \frac{1}{2}\frac{2}{6} + 0\frac{1}{6} = 1.$$

- In general,

$$\mathbb{E}\Big[\mathbb{E}(f(X)|Y)\Big] = \mathbb{E}f(X), \tag{1}$$
$$\mathbb{E}(f(X)g(Y)|Y) = g(Y)\mathbb{E}(f(X)|Y). \tag{2}$$

Conditional expectation: III

- Suppose that (X, Y) is continuous uniform on the set

$$\{(x, y) : x, y \geq 0, x + y \leq 2\}.$$

- Given $Y = y$, X is uniform on $[0, 2 - y]$. Thus, for $y \in [0, 2]$

$$\mathbb{E}(X^2 | Y = y) = \frac{1}{3}(2 - y)^2.$$

- Therefore,

$$\mathbb{E}(X^2 | Y) = \frac{1}{3}(2 - Y)^2.$$

- Y can be a random vector

Random walk

- Assume that $\{Y_n : n = 1, 2, \ldots\}$ is an iid sequence with $\mathbb{E}(Y_1) = 0$.
- Define a random walk $\{X_n : n \geq 0\}$ via $X_0 = 0$,

$$X_n = Y_1 + \ldots + Y_n \equiv g(Y_1, \ldots, Y_n) \quad n \geq 1.$$

- $X_{n+1} = X_n + Y_{n+1}$
- Conditional expectation:

$$\begin{aligned}
\mathbb{E}(X_{n+1}|Y_1, \ldots Y_n) &= \mathbb{E}(X_n + Y_{n+1}|Y_1, \ldots, Y_n) \\
&= \mathbb{E}(X_n|Y_1, \ldots, Y_n) + \mathbb{E}(Y_{n+1}|Y_1, \ldots, Y_n) \\
&= g(Y_1, \ldots, Y_n) + \mathbb{E}(Y_{n+1}) \\
&= X_n.
\end{aligned}$$

Martingale

- $\{X_n : n = 1, 2, \ldots\}$ is said to be a martingale with respect to $\{Y_n : n = 1, 2, \ldots\}$ if
- X_n can determined from $Y_1, \ldots, Y_n$, i.e., there exists a (deterministic) function g_n such that

$$X_n = g_n(Y_1, \ldots, Y_n),$$

- $\mathbb{E}(|X_n|) < \infty$ for each n,
- and the martingale property holds

$$\mathbb{E}(X_{n+1}|Y_1, \ldots, Y_n) = X_n \quad n \geq 1.$$

2nd martingale for a random walk

- $X_n = Y_1 + \ldots + Y_n$ with $\mathbb{E}(Y_1) = 0$
- Let $\sigma^2 = \mathbb{E}(Y_1^2)$. Define

$$Z_n = X_n^2 - \sigma^2 n.$$

- $\{Z_n : n = 1, \ldots, \}$ is a martingale with respect to $\{Y_n : n \geq 1\}$.
- Proof

Ward martingale (the 3rd martingale)

- Assume $\mathbb{E}(e^{\theta Y_1}) = 1$
- Define

$$Z_n = e^{\theta X_n}.$$

- Then

$$\begin{aligned}\mathbb{E}(Z_{n+1}|Y_1,\ldots,Y_n) &= \mathbb{E}\Big(e^{\theta X_n}e^{\theta Y_{n+1}}|Y_1,\ldots Y_n\Big), \\ &= e^{\theta X_n}\mathbb{E}\Big(e^{\theta Y_{n+1}}\Big) \\ &= Z_n\end{aligned}$$

Stopping time

- A random variable T taking values on $\{0, 1, 2, \ldots, \} \cup \{\infty\}$ is a stopping time with respect to $\{Y_n : n \geq 1\}$ if $\{T = n\}$ is determined by $Y_1, \ldots, Y_n$ for each n or equivalently $\{T \leq n\}$ is determined by $Y_1, \ldots, Y_n$ for each n.

Doob's optional sampling theorem

Theorem (Optional stopping theorem)

Let $M = \{M_n\}$ be a martingale with respect to $\{Y_n : n = 1, 2, \ldots\}$ and T be a stopping time. Suppose one of the following conditions holds.

1. $T \leq k$ *for some* k,
2. $T < \infty$ *and* $|M_n| \leq C$ *whenever* $n \leq T$.

Then $\mathbb{E}M_T = \mathbb{E}M_1$.

Proof when 1 holds.

$$\begin{aligned} M_T - M_1 &= (M_T - M_{T-1}) + \ldots + (M_2 - M_1) \\ &= \sum_{n=1}^{T-1} (M_{n+1} - M_n) \\ &= \sum_{n=1}^{k-1} (M_{n+1} - M_n) 1_{\{n<T\}} \end{aligned}$$

Proof

Proof when 2 holds.

$$|\mathbb{E}M_T - \mathbb{E}M_1| = |\mathbb{E}M_T - \mathbb{E}M_{T\wedge n}| \leq \mathbb{E}\,|M_T - M_{T\wedge n}| \leq 2C\mathbb{P}(T > n).$$

□

Simple, symmetric random walk

- Fix $a, b > 0$.
- Let $T_{-a,b}$ be the first hitting time to either $-a$ or b, i.e.,

$$T_{-a,b} = \inf\{n \geq 0 : X_n = -a \quad \text{or} \quad X_n = b\}.$$

- Define T_b

$$T_b = \inf\{n \geq 0 : \quad X_n = b\}.$$

- Then

$$T_{-a,b} = T_{-a} \wedge T_b.$$

Hitting probabilities

- Use the first martingale to prove

$$\mathbb{P}\{T_{-a} < T_b\} = \frac{b}{a+b}.$$

- Use the 2nd martingale to prove (in homework)

$$\mathbb{E}(T_{a,-b}) = ab.$$

Simple, non-symmetric random walk

- $P_{i,i+1} = p$ and $P_{i,i-1} = q$.
- Define

$$M_n = \left(\frac{q}{p}\right)^{X_n}.$$

- M is a martingale.

Theorem

$$\mathbb{P}\{T_{-a} < T_b\} = \frac{1 - (q/p)^b}{(q/p)^{-a} - (q/p)^b}.$$

- Assume $q > p$. As $a \to \infty$,

$$\mathbb{P}\{T_b < \infty\} = (p/q)^b. \tag{3}$$

Extreme probabilities

Theorem

For a simple random walk. Assume $q > p$.

$$\mathbb{P}\left\{\sup_{n\geq 0} X_n \geq b\right\} = (p/q)^b.$$

More martingales

For a DTMC X,

$$(P^n f)(i) = \sum_{j \in S} P_{ij}^{(n)} f_j = \mathbb{E}_i[f(X_n)].$$

Theorem

Let $X = \{X_n : n \geq 0\}$ be a stochastic process with values in S and let P be a stochastic matrix. Then the following are equivalent.

1. *X is a DTMC with transition matrix P*
2. *For all bounded functions $f : S \to \mathbb{R}$, the following process is a martingale:*

$$M_n^f = f(X_n) - f(X_0) - \sum_{m=0}^{n-1} (P - I) f(X_m).$$

Filtration

- Let $Y = \{Y_n : n = 0, 1 \ldots ,\}$ be a DTMC.
- Fix an integer $n \geq 0$.

$$A_{i_0,i_1,\ldots,i_n} = \{Y_0 = i_0, \ldots, Y_n = i_n\}$$

  is said be an elementary event.
- $\mathcal{F}_n$ is the collection of countable union of elementary events.
- For example,

$$A_{i_0,i_1,\ldots,i_n} \cup A_{j_0,j_1,\ldots,j_n} \in \mathcal{F}_n$$

- The sequence $\{\mathcal{F}_n : n = 0, 1, 2, \ldots\}$ is said to be the *filtration* of the stochastic process $Y = \{Y_n : n \geq 0\}$.
- $\mathcal{F}_n$ is the "state of knowledge" or information up to time n.

Martingale

- A (discrete-time) stochastic process $M = \{M_n : n \geq 0\}$ is said to be *adapted* to Y if for each n, M_n "depends only on" $Y_0, \ldots, Y_n$, i.e.,

$$\{M_n \leq a\} \in \mathcal{F}_n \quad \text{for each } a \in \mathbb{R}.$$

- The stochastic process M is said to be *integrable* if $\mathbb{E}\,|M_n| < \infty$ for each $n \geq 0$.

Definition

An adapted integrable process $M = \{M_n : n \geq 0\}$ is called a *martingale* if

$$\mathbb{E}[(M_{n+1} - M_n)1_A] = 0 \quad \text{for each } A \in \mathcal{F}_n \text{ and each } n \geq 0. \tag{4}$$

If suffices to verify (4) for A to be elementary events.

Conditional expectation

- For a given r.v. X and an event A,

$$\mathbb{E}(X|A) = \frac{\mathbb{E}(X 1_A)}{\mathbb{P}(A)}$$

  if $\mathbb{P}(A) > 0$. When $P(A) = 0$, define $\mathbb{E}(X|A) = 0$.

- Conditioning $\mathcal{F}_n$:

$$\mathbb{E}(X|\mathcal{F}_n) = \sum_{i_0,\ldots,i_n} \mathbb{E}[X|Y_0 = i_0, \ldots, Y_n = i_n] 1_{\{Y_0=i_0,\ldots,Y_n=i_n\}}.$$

- $\mathbb{E}(X|\mathcal{F}_n)$ is random, depending only on $Y_0, \ldots, Y_n$.
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$$\mathbb{E}[\mathbb{E}(X|\mathcal{F}_n)] = \mathbb{E}(X).$$