MAT2002 Ordinary Differential Equations
Laplace Transform

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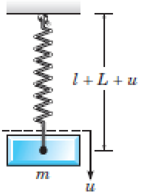
Overview

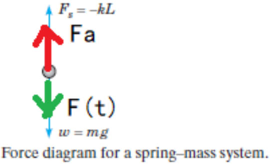
1. Laplace Transform

Motivation

Recall: In previous example of second-order linear ODEs.



(a) Spring-mass system



(b) Force analysis

Motivation

Force analysis

(1) the weight $Fg = mg$ acting downwards;

(2) a restoring force from the spring pulling the mass upwards $F_s = -k(L+u)$;

(3) a resistance force F_a (air resistance/friction) that acts in the opposite of the motion and is proportional to the speed u'. This is usually referred to as **viscous damping** and $F_a = -\gamma u'$ with constant $\gamma > 0$ (**damping constant**);

(4) an external force $F(t)$ that acting on the object (could be upward or downward).

Using Newton's second law, we arrive at the ODE

$$\begin{aligned} mu''(t) &= mg - k(L+u(t)) - \gamma u'(t) + F(t) \\ \Rightarrow mu''(t) &+ \gamma u'(t) + ku(t) = F(t) \end{aligned}$$

Question: When $F(t)$ is a discontinuous function, how to solve the ODE?

Motivation

Many practical engineering problems involve mechanical systems acted on by discontinuous force. For such problems the methods described in the previous chapters are often rather awkward to use.

Another method that is especially well suited to these problems is based on the Laplace transform.

Outline

Review of Improper Integrals

An improper integral over an unbounded interval is defined as a limit of integrals over finite intervals;

$$\int_a^\infty f(t)dt = \lim_{A\to\infty} \int_a^A f(t)dt, \tag{1}$$

where A is a positive real number. If the integral from a to A exists for each $A > a$, and if the limit as $A \to \infty$ exists, then the improper integral **converges**. Otherwise it **diverges**.

Example

Let $f(t) = e^{ct}, t \geq 0$, where c is a real nonzero constant. Then

$$\int_0^\infty e^{ct}dt = \lim_{A\to\infty} \int_0^A e^{ct}dt = \lim_{A\to\infty} \frac{e^{ct}}{c}\Big|_0^A = \lim_{A\to\infty} \frac{1}{c}(e^{cA} - 1).$$

If $c < 0$, the improper integral converges to the value $-1/c$ and it diverges if $c > 0$. If $c = 0$, the integrand $f(t)$ is the constant function with value 1. In this case

$$\lim_{A\to\infty} \int_0^A 1dt = \lim_{A\to\infty} (A - 0) = \infty,$$

so the integral diverges.

Review

Before discussing the possible existence of $\int_a^\infty f(t)dt$, it is helpful to define certain terms. A function f is said to be **piecewise continuous** if it is continuous there except for a finite number of jump discontinuities. If f is piecewise continuous on $\alpha \leq t \leq \beta$ for every $\beta > \alpha$, then f is said to be piecewise continuous on $t \geq \alpha$. An example of a piecewise continuous function is shown in Fig.1

Review

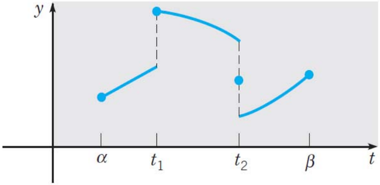


Fig. 1. A piecewise continuous function $y = f(t)$.

The integral of a piecewise continuous function on a finite interval is just the sum of the integrals on the subintervals created by the partition points. For instance, for the function $f(t)$ shown in Fig. 1, we have

$$\int_{\alpha}^{\beta} f(t)dt = \int_{\alpha}^{t_1} f(t)dt + \int_{t_1}^{t_2} f(t)dt + \int_{t_2}^{\beta} f(t)dt. \tag{2}$$

Review of Improper Integrals

Theorem 14.1

If f is piecewise continuous for $t \geq a$, if $|f(t)| \leq g(t)$ when $t \geq M$ for some positive constant M, and if $\int_M^\infty g(t)dt$ converges, then $\int_a^\infty f(t)dt$ also converges. On the other hand, if $f(t) \geq g(t) \geq 0$ for $t \geq M$, and if $\int_M^\infty g(t)dt$ diverges, then $\int_a^\infty f(t)dt$ also diverges.

This result can be found in any standard mathematical analysis/calculus textbook.

Definition for Laplace Transform

Definition 14.2

Suppose that $f(t)$ is a real function defined on $[0, +\infty)$ and

1. f is piecewise continuous on the interval $0 \leq t \leq A$ for any positive A.
2. $|f(t)| \leq Ke^{at}$ when $t \geq M$. In this inequality, K, a, and M are real constants $(K > 0, M > 0)$. ($f(t)$ is in the "exponential" order.)

Then the **Laplace transform** $\mathcal{L}\{f(t)\}(s) = F(s)$ defined as

$$\mathcal{L}\{f(t)\}(s) = F(s) = \int_0^\infty e^{-st} f(t) dt, \tag{3}$$

exists for $s > a$.

Definition for Laplace Transform

Proof.
To establish this definition, we must show that the integral in Eq. (3) converges for $s > a$. Splitting the improper integral into two parts, we have

$$\int_0^\infty e^{-st} f(t)dt = \int_0^M e^{-st} f(t)dt + \int_M^\infty e^{-st} f(t)dt. \tag{4}$$

The first integral on the right side of Eq. (4) exists by hypothesis (1) of the theorem; hence the existence of $F(s)$ depends on the convergence of the second integral. By hypothesis (2) we have, for $t \geq M$,

$$|e^{-st} f(t)| \leq Ke^{-st} e^{at} = Ke^{(a-s)t}.$$

By the comparison theorem, $F(s)$ exists provided that $\int_M^\infty e^{(a-s)t} dt$ converges.

Example of Laplace Transform

Example Let $f(t) = 1$, $t \geq 0$. Then,

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st} dt = -\lim_{A\to\infty} \frac{e^{-st}}{s}\Big|_0^A = \frac{1}{s}, \quad s > 0.$$

Example Let $f(t) = e^{at}$, $t \geq 0$. Then,

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at} dt = \int_0^\infty e^{-(s-a)t} dt = \frac{1}{s-a}, \quad s > a.$$

Example of Laplace Transform

Example Let $f(t) = \sin at, t \geq 0$. Then

$$\mathcal{L}\{\sin at\} = F(s) = \int_0^\infty e^{-st} \sin at dt, \quad s > 0.$$

Since

$$F(s) = \lim_{A\to\infty} \int_0^A e^{-st} \sin at dt,$$

upon integrating by parts, we obtain

$$F(s) = \lim_{A\to\infty} \Big[-\frac{e^{-st}\cos at}{a}\Big|_0^A - \frac{s}{a}\int_0^A e^{-st}\cos at dt\Big] = \frac{1}{a} - \frac{s}{a}\int_0^\infty e^{-st}\cos at dt.$$

A second integration by parts then yields

$$F(s) = \frac{1}{a} - \frac{s^2}{a^2}\int_0^\infty e^{-st}\sin at dt = \frac{1}{a} - \frac{s^2}{a^2}F(s).$$

Hence, solving for $F(s)$, we have

$$F(s) = \frac{a}{s^2 + a^2} \quad s > 0.$$

Definition for Laplace Transform

Definition 14.3

If the **Laplace transform** $\mathcal{L}\{f(t)\} = F(s)$ is defined as

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)dt, \tag{5}$$

then the **inverse Laplace transform** for $F(s) = \mathcal{L}\{f(t)\}$ is defined as

$$\mathcal{L}^{-1}\{F(s)\} = f(t), \tag{6}$$

Thus,

$$\mathcal{L}^{-1}\{F(s)\} = f(t) \Leftrightarrow \mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)dt.$$

Example of Laplace Transform

Example Let $f(t) = 1$, $t \geq 0$. Then,

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st} dt = -\lim_{A\to\infty} \frac{e^{-st}}{s}\Big|_0^A = \frac{1}{s}, \quad s > 0. \quad \mathcal{L}^{-1}\{\frac{1}{s}\} = 1.$$

Example Let $f(t) = e^{at}$, $t \geq 0$. Then,

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at} dt = \int_0^\infty e^{-(s-a)t} dt = \frac{1}{s-a}, \quad s > a. \quad \mathcal{L}^{-1}\{\frac{1}{s-a}\} = e^{at}.$$

Example of Laplace Transform

Example Let $f(t) = \sin at, t \geq 0$. Then

$$\mathcal{L}\{\sin at\} = F(s) = \int_0^\infty e^{-st} \sin at dt, \quad s > 0.$$

Since

$$F(s) = \lim_{A\to\infty} \int_0^A e^{-st} \sin at dt,$$

upon integrating by parts, we obtain

$$F(s) = \lim_{A\to\infty} \Big[-\frac{e^{-st}\cos at}{a}\Big|_0^A - \frac{s}{a}\int_0^A e^{-st}\cos at dt\Big] = \frac{1}{a} - \frac{s}{a}\int_0^\infty e^{-st}\cos at dt.$$

A second integration by parts then yields

$$F(s) = \frac{1}{a} - \frac{s^2}{a^2}\int_0^\infty e^{-st}\sin at dt = \frac{1}{a} - \frac{s^2}{a^2}F(s).$$

Hence, solving for $F(s)$, we have

$$F(s) = \frac{a}{s^2+a^2} \quad s > 0. \quad \mathcal{L}^{-1}\{\frac{a}{s^2+a^2}\} = sin(at).$$

Laplace Transform Table

Table for Elementary Laplace Transforms

	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1.	1	$\frac{1}{s}, \ s > 0$
2.	e^{at}	$\frac{1}{s-a}, \ s > a$
3.	$t^n, \ n > 0$ integer	$\frac{n!}{s^{n+1}}, \ s > 0$
4.	$t^p, \ p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}, \ s > 0$
5.	$\sin at$	$\frac{a}{s^2+a^2}, \ s > 0$
6.	$\cos at$	$\frac{s}{s^2+a^2}, \ s > 0$
7.	$\sinh at$	$\frac{a}{s^2-a^2}, \ s > \lvert a\rvert$
8.	$\cosh at$	$\frac{s}{s^2-a^2}, \ s > \lvert a\rvert$

Laplace Transform Table

Table for Elementary Laplace Transforms

	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
9.	$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$
10.	$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$
11.	$t^n e^{at}, \quad n > 0$ integer	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
12.	$u_c(t)$	$\frac{e^{-cs}}{s}, s > 0$
13.	$u_c(t)f(t-c)$	$e^{-cs}F(s)$
14.	$e^{ct}f(t)$	$F(s-c)$
15.	$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right), \quad c > 0$
16.	$\int_0^t f(t-\tau)g(\tau)d\tau$	$F(s)G(s)$
17.	$\delta(t-c)$	e^{-cs}

Property of Laplace Transform

Theorem 14.4

If $\mathcal{L}\{f_1(t)\}(s)$ and $\mathcal{L}\{f_2(t)\}(s)$ are defined for $s > a$, and c_1 and c_2 are any numbers, then

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1\mathcal{L}\{f_1(t)\} + c_2\mathcal{L}\{f_2(t)\}. \qquad \square \tag{7}$$

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = \int_0^\infty e^{-st}[c_1 f_1(t) + c_2 f_2(t)]dt = c_1\int_0^\infty e^{-st} f_1(t)dt + c_2\int_0^\infty e^{-st} f_2(t)dt.$$

Example Find the Laplace transform of $f(t) = 5e^{-2t} - 3\sin 4t, \quad t \geq 0$. Then, we obtain

$$\begin{aligned}\mathcal{L}\{f(t)\} =& 5\mathcal{L}\{e^{-2t}\} - 3\mathcal{L}\{\sin 4t\} \\ =& \frac{5}{s+2} - 3\frac{4}{s^2+16}, \quad s > 0. \\ =& \frac{5}{s+2} - \frac{12}{s^2+16}, \quad s > 0.\end{aligned}$$

Property of Laplace Transform

Theorem 14.5

Suppose that f is continuous and f' is piecewise continuous on any interval $0 \leq t \leq A$. Suppose further that there exist constants K, a, and M such that $|f(t)| \leq Ke^{at}$ for $t \geq M$. Then $\mathcal{L}\{f'(t)\}$ exists for $s > a$, and moreover,

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0). \tag{8}$$

Property of Laplace Transform

Proof.

$$\int_0^{\infty} e^{-st} f'(t)dt = limit_{A\to\infty} \int_0^{A} e^{-st} f'(t)dt.$$

If f' has points of discontinuity (denoted by $t_1, t_2, \ldots, t_k$) in the interval $0 \le t \le A$, Then the above integral can be rewritten as

$$\int_0^{A} e^{-st} f'(t)dt = \int_0^{t_1} e^{-st} f'(t)dt + \int_{t_1}^{t_2} e^{-st} f'(t)dt + \ldots + \int_{t_k}^{A} e^{-st} f'(t)dt.$$

Using integration by parts, one has

$$\int_0^{A} e^{-st} f'(t)dt = e^{-st} f(t)\big|_0^{t_1} + e^{-st} f(t)\big|_{t_1}^{t_2} + \ldots + e^{-st} f(t)\big|_{t_k}^{A}$$
$$+s\Big[\int_0^{t_1} e^{-st} f(t)dt + \int_{t_1}^{t_2} e^{-st} f(t)dt + \ldots + \int_{t_k}^{A} e^{-st} f(t)dt\Big].$$

Since f is continuous, after some calculation, we obtain

$$\int_0^{A} e^{-st} f'(t)dt = e^{-sA} f(A) - f(0) + s\int_0^{A} e^{-st} f(t)dt. \tag{9}$$

Property of Laplace Transform

Let $A \to \infty$, $limit_{A\to\infty} \int_0^A e^{-st} f(t)dt = \mathcal{L}\{f(t)\}$. Further, for $A \geq M$, we have $|f(A)| \leq Ke^{aA}$; consequently, $|e^{-sA} f(A)| \leq Ke^{-(s-a)A}$. Hence $e^{-sA} f(A) \to 0$ as $A \to \infty$ whenever $s > a$. Thus the right side of Eq.(9) has the limit $s\mathcal{L}\{f(t)\} - f(0)$. Consequently, the left side of Eq.(9) also has a limit $\mathcal{L}\{f'(t)\}$. Therefore, for $s > a$, we conclude that

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0), s > a.$$

Corollary

Suppose that $f, f', \cdots, f^{(n-1)}$ are continuous and $f^{(n)}$ is piecewise continuous on any interval $0 \leq t \leq A$. Suppose further that there exist constants K, a and M such that $|f^{(i)}(t)| \leq Ke^{at}$, $i = 0, 1, \cdots, n-1$ for $t \geq M$. Then $\mathcal{L}\{f^{(n)}(t)\}$ exists for $s > a$, and

$$\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0). \quad \square \tag{10}$$

where $F(s) = \mathcal{L}\{f(t)\}$.

One more property

Theorem 14.6

If $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a \geq 0$, and if c is a constant, then

$$\mathcal{L}\{e^{ct}f(t)\} = F(s-c), \quad s > a + c. \tag{11}$$

Conversely, if $f(t) = \mathcal{L}^{-1}\{F(s)\}$, then

$$e^{ct}f(t) = \mathcal{L}^{-1}\{F(s-c)\}. \tag{12}$$

According to Theorem 14.6, multiplication of $f(t)$ by e^{ct} results in a translation of the transform $F(s)$ a distance c in the positive s direction, and conversely. To prove this theorem, we evaluate $\mathcal{L}\{e^{ct}f(t)\}$. Thus

$$\mathcal{L}\{e^{ct}f(t)\} = \int_0^\infty e^{-st}e^{ct}f(t)dt = \int_0^\infty e^{-(s-c)t}f(t)dt = F(s-c),$$

which is Eq. (11). The restriction $s > a + c$ follows from $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a \geq 0$. Equation (12) is obtained by taking the inverse Laplace transform of Eq. (11), and the proof is complete.

One more property

Example: Let $f(t) = e^{ct}\sin at, t \geq 0$ Since

$$\mathcal{L}\{\sin at\} = F(s) = \int_0^\infty e^{-st}\sin at dt = \frac{a}{s^2+a^2}, \quad s > 0.$$

$$\mathcal{L}\{e^{ct}\sin at\} = \int_0^\infty e^{-(s-c)t}\sin at dt = F(s-c) = \frac{a}{(s-c)^2+a^2}, \quad s > c.$$

$$\mathcal{L}^{-1}\{\frac{a}{(s-c)^2+a^2}\} = e^{ct}\sin at \quad s > c.$$

One more property

Example Find the inverse transform of

$$G(s) = \frac{1}{s^2 - 4s + 5}.$$

By completing the square in the denominator, we can write

$$G(s) = \frac{1}{(s-2)^2 + 1} = F(s-2),$$

where $F(s) = (s^2+1)^{-1}$. Since $\mathcal{L}^{-1}\{F(s)\} = \sin t$, it follows from Theorem 14.6 that

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = e^{2t} \sin t.$$

Solution of Initial Value Problems

The method by using the Laplace transform to solve a differential equation:

1. Use the Laplace transform to transform an initial value problem for an unknown function f in the t-domain to an algebraic problem for F in the s-domain.
2. Solve this algebraic problem to find F.
3. Recover the desired function f from its transform F. This last step is known as "inverting the transform," or taking the inverse Laplace transform.

Solution of Initial Value Problems

Consider the following second order linear equation with constant coefficients

$$ay'' + by' + cy = f(t). \tag{13}$$

Assuming that the solution $y = \phi(t)$ satisfies the conditions of the above Corollary for $n = 2$, we can take the transform of Eq. (13) and thereby obtain

$$a[s^2 Y(s) - sy(0) - y'(0)] + b[sY(s) - y(0)] + cY(s) = F(s), \tag{14}$$

where $F(s)$ is the transform of $f(t)$. By solving Eq. (14) for $Y(s)$, we find that

$$Y(s) = \frac{(as + b)y(0) + ay'(0)}{as^2 + bs + c} + \frac{F(s)}{as^2 + bs + c}. \tag{15}$$

The problem is then solved, provided that we can find the function $y = \phi(t)$ whose Laplace transform is $Y(s)$, i.e., $\phi(t) = \mathcal{L}^{-1}\{Y(s)\}$.

Solution of Initial Value Problems

Example 14.7

Find the solution to the IVP

$$y'' - y' - 2y = 0, \quad y(0) = 1, y'(0) = 0. \tag{16}$$

We assume that the IVP has a solution $y = \phi(t)$, which with its first two derivatives satisfies the conditions of the above Corollary. Then, taking the Laplace transform of the differential equation, we have

$$s^2 Y(s) - sy(0) - y'(0) - (sY(s) - y(0)) - Y(s) = 0,$$

Substituting for $y(0)$ and $y'(0)$ from the initial conditions and solving for $Y(s)$, we obtain

$$Y(s) = \frac{s-1}{s^2 - s - 2}. \tag{17}$$

Using partial fractions, we can write $Y(s)$ in the form

$$Y(s) = \frac{s-1}{s^2 - s - 2} = \frac{1}{3}\frac{1}{s-2} + \frac{2}{3}\frac{1}{s+1}. \tag{18}$$

$$y(t) = \mathcal{L}^{-1}(\frac{1}{3}\frac{1}{s-2}) + \mathcal{L}^{-1}(\frac{2}{3}\frac{1}{s+1}) = \frac{1}{3}\mathcal{L}^{-1}(\frac{1}{s-2}) + \frac{2}{3}\mathcal{L}^{-1}(\frac{1}{s+1}) = \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}. \tag{19}$$

Solution of Initial Value Problems

Example 14.8

Find the solution to the IVP

$$y'' + y = \sin 2t, \quad y(0) = 2, y'(0) = 1. \tag{20}$$

We assume that the IVP has a solution $y = \phi(t)$, which with its first two derivatives satisfies the conditions of the above Corollary. Then, taking the Laplace transform of the differential equation, we have

$$s^2 Y(s) - sy(0) - y'(0) + Y(s) = 2/(s^2 + 4),$$

where the transform of $\sin 2t$ has been obtained. Substituting for $y(0)$ and $y'(0)$ from the initial conditions and solving for $Y(s)$, we obtain

$$Y(s) = \frac{2s^3 + s^2 + 8s + 6}{(s^2 + 1)(s^2 + 4)}. \tag{21}$$

Using partial fractions, we can write $Y(s)$ in the form

$$Y(s) = \frac{as + b}{s^2 + 1} + \frac{cs + d}{s^2 + 4} = \frac{(as + b)(s^2 + 4) + (cs + d)(s^2 + 1)}{(s^2 + 1)(s^2 + 4)}. \tag{22}$$

Solution of Initial Value Problems

Example 14.8 continue

By expanding the numerator on the right side of Eq. (22) and equating it to the numerator in Eq. (21), we find that

$$2s^3 + s^2 + 8s + 6 = (a + c)s^3 + (b + d)s^2 + (4a + c)s + (4b + d)$$

for all s. Then, comparing coefficients of like powers of s, we have

$$a + c = 2, \quad b + d = 1,$$
$$4a + c = 8, \quad 4b + d = 6.$$

Consequently, $a = 2, c = 0, b = 5/3$, and $d = -2/3$, from which it follows that

$$Y(s) = \frac{2s}{s^2 + 1} + \frac{5/3}{s^2 + 1} - \frac{2/3}{s^2 + 4}. \tag{23}$$

The solution of the given initial value problem is

$$y = \phi(t) = 2\cos t + \frac{5}{3}\sin t - \frac{1}{3}\sin 2t. \tag{24}$$

Higher Order IVP-Laplace transform method

The Laplace transform method can be used with linear differential equations of higher order than second order, as long as the coefficients in the equation are constants. Below we show how we can solve a fourth order equation

Solution of Initial Value Problems

Example 14.9

Find the solution of the initial value problem

$$y^{(4)} - y = 0, \tag{25}$$

$$y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 0, \quad y'''(0) = 0. \tag{26}$$

In this problem we need to assume that the solution $y = \phi(t)$ satisfies the conditions of Corollary for $n = 4$. The Laplace transform of the differential equation (25) is

$$s^4 Y(s) - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0) - Y(s) = 0.$$

Then, using the initial conditions (26) and solving for $Y(s)$, we have

$$Y(s) = \frac{s^2}{s^4 - 1}. \tag{27}$$

A partial fraction expansion of $Y(s)$ is

$$Y(s) = \frac{as + b}{s^2 - 1} + \frac{cs + d}{s^2 + 1}.$$

Solution of Initial Value Problems

Example 14.9 continue

And it follows that

$$(as+b)(s^2+1)+(cs+d)(s^2-1)=s^2 \tag{28}$$

for all s. By setting $s=1$ and $s=-1$, respectively, in Eq. (28), we obtain the pair of equations

$$2(a+b)=1, \quad 2(-a+b)=1,$$

and therefore $a=0$ and $b=1/2$. If we set $s=0$ in Eq. (28), then $b-d=0$, so $d=1/2$. Finally, equating the coefficients of the cubic terms on each side of Eq. (28), we find that $a+c=0$, so $c=0$. Thus

$$Y(s)=\frac{1/2}{s^2-1}+\frac{1/2}{s^2+1}, \tag{29}$$

the solution of the initial value problem (25), (26) is

$$y=\phi(t)=\frac{\sinh t+\sin t}{2}. \tag{30}$$

System of linear ODE-Laplace transform method

The Laplace transform method can be used for the system of linear differential equations, as long as the coefficients in the equation are constants.

Solution of Initial Value Problems

Example 14.10

Find the solution of the initial value problem

$$x' = x + 2y, \quad y' = 2x + y \tag{31}$$

$$x(0) = 0, \quad y(0) = 2. \tag{32}$$

The Laplace transform of the differential equation (31) is

$$sX(s) - x(0) = X(s) + 2Y(s), \quad sY(s) - y(0) = 2X(s) + Y(s)$$

Thus

$$sX(s) = X(s) + 2Y(s), \quad sY(s) - 2 = 2X(s) + Y(s).$$

Solution of Initial Value Problems

Example 14.10 continue

Then, solving for $X(s)$, $Y(s)$, we have

$$X(s) = 2\frac{2}{(s-1)^2-4} = 2\frac{2}{(s-1)^2-2^2} = F_1(s-1). \tag{33}$$

where $F_1(s) = \frac{4}{s^2-2^2}, \mathcal{L}^{-1}(2\frac{2}{s^2-2^2}) = 2\sinh 2t$, and

$$Y(s) = \frac{2(s-1)}{(s-1)^2-4} = F_2(s-1). \tag{34}$$

where $F_2(s) = 2\frac{s}{s^2-2^2}, \mathcal{L}^{-1}(2\frac{s}{s^2-2^2}) = 2\cosh 2t$.

Thus,

$$x(t) = \mathcal{L}^{-1}(X(s)) = 2e^t\sinh(2t) = e^{3t} - e^{-t},$$
$$y(t) = \mathcal{L}^{-1}(Y(s)) = 2e^t\cosh(2t) = e^{3t} + e^t$$

.

Step functions

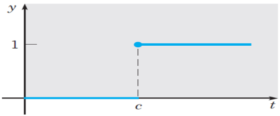


Figure 1. Graph of $y = u_c(t)$.

Definition 14.11

The **unit step function** *or* **Heaviside function** *is defined by*

$$u_c(t) = \begin{cases} 0, & t < c, \\ 1, & t \geq c, \end{cases} \quad c \geq 0. \qquad \square \tag{35}$$

The Laplace transform of u_c for $c \geq 0$ is easily determined:

$$\mathcal{L}\{u_c(t)\} = \int_0^\infty e^{-st} u_c(t) dt = \int_c^\infty e^{-st} dt = \frac{e^{-cs}}{s}, \quad s > 0. \tag{36}$$

Shifting theorems and Step functions

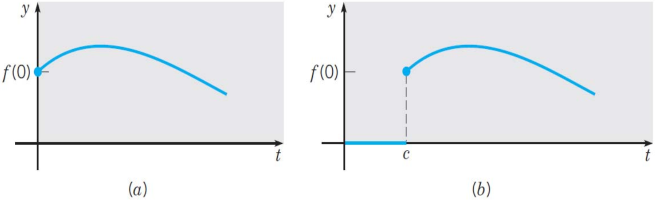


Figure 2. A translation of the given function.(a) $y = f(t)$; (b) $y = u_c(t)f(t-c)$.

For a given function f defined for $t \geq 0$, we will often want to consider the related function g defined by

$$y = g(t) = \begin{cases} 0, & t < c, \\ f(t-c), & t \geq c, \end{cases} = u_c(t)f(t-c)$$

which represents a translation of f a distance c in the positive t direction;see Figure 3.

Shifting theorems and Step functions

Theorem 14.12

If $F(s) = \mathcal{L}\{f(t)\}$ *exists for* $s > a \geq 0$, *and if* c *is a positive constant, then*

$$\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\} = e^{-cs}F(s), \quad s > a. \tag{37}$$

Conversely, if $f(t) = \mathcal{L}^{-1}\{F(s)\}$, *then*

$$u_c(t)f(t-c) = \mathcal{L}^{-1}\{e^{-cs}F(s)\}. \tag{38}$$

To prove Theorem 14.12, it is sufficient to compute the transform of $u_c(t)f(t-c)$:

$$\mathcal{L}\{u_c(t)f(t-c)\} = \int_0^\infty e^{-st}u_c(t)f(t-c)dt = \int_c^\infty e^{-st}f(t-c)dt.$$

Introducing a new integration variable $\xi = t - c$, we have

$$\mathcal{L}\{u_c(t)f(t-c)\} = \int_0^\infty e^{-(\xi+c)s}f(\xi)d\xi = e^{-cs}\int_0^\infty e^{-s\xi}f(\xi)d\xi = e^{-cs}F(s).$$

Thus Eq. (37) is established; Eq. (38) follows by taking the inverse Laplace transform of both sides of Eq. (37).

Shifting theorems and Step functions

Example If the function f is defined by

$$f(t) = \begin{cases} \sin t, & 0 \le t < \pi/4, \\ \sin t + \cos(t - \pi/4), & t \ge \pi/4, \end{cases}$$

find $\mathcal{L}\{f(t)\}$. The graph of $y = f(t)$ is shown in Figure 4.

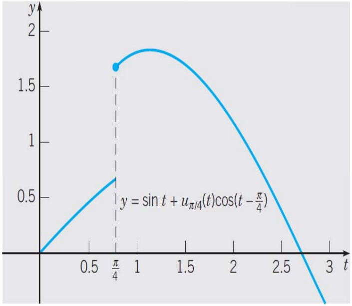


Figure 3. Graph of the function in Example 3.

Shifting theorems and Step functions

Note that $f(t) = \sin t + g(t)$, where

$$g(t) = \begin{cases} 0, & 0 \leq t < \pi/4, \\ \cos(t - \pi/4), & t \geq \pi/4, \end{cases}$$

Thus

$$g(t) = u_{\pi/4}(t) \cos(t - \pi/4)$$

and

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin t\} + \mathcal{L}\{u_{\pi/4}(t) \cos(t - \pi/4)\} = \mathcal{L}\{\sin t\} + e^{-\pi s/4} \mathcal{L}\{\cos t\}.$$

Introducing the transforms of $\sin t$ and $\cos t$, we obtain

$$\mathcal{L}\{f(t)\} = \frac{1}{s^2 + 1} + e^{-\pi s/4} \frac{s}{s^2 + 1} = \frac{1 + s e^{-\pi s/4}}{s^2 + 1}.$$

You should compare this method with the calculation of $\mathcal{L}\{f(t)\}$ directly from the definition.

Shifting theorems and Step functions

Example Find the inverse transform of

$$F(s) = \frac{1 - e^{-2s}}{s^2}.$$

From the linearity of the inverse transform, we have

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\Big\{\frac{1}{s^2}\Big\} - \mathcal{L}^{-1}\Big\{\frac{e^{-2s}}{s^2}\Big\} = t - u_2(t)(t-2).$$

The function f may also be written as

$$f(t) = \begin{cases} t, & 0 \le t < 2, \\ 2, & t \ge 2. \end{cases}$$

Shifting theorems and Step functions

Example 14.13

Find the solution of the differential equation

$$2y'' + y' + 2y = g(t), \quad y(0) = 0, \quad y'(0) = 0. \tag{39}$$

where

$$g(t) = u_5(t) - u_{20}(t) = \begin{cases} 1, & 5 \le t < 20, \\ 0, & 0 \le t < 5 \quad \text{and} \quad t \ge 20. \end{cases} \tag{40}$$

The Laplace transform of Eq.(39) is

$$\begin{aligned} 2s^2 Y(s) - 2sy(0) - 2y'(0) + sY(s) - y(0) + 2Y(s) &= \mathcal{L}\{u_5(t)\} - \mathcal{L}\{u_{20}(t)\} \\ &= (e^{-5s} - e^{-20s})/s. \end{aligned}$$

Using the initial values and solving for $Y(s)$, we obtain

$$Y(s) = \frac{e^{-5s} - e^{-20s}}{s(2s^2 + s + 2)}. \tag{41}$$

Example 14.13 continue

To find $y = \phi(t)$, it is convenient to write $Y(s)$ as

$$Y(s) = (e^{-5s} - e^{-20s})H(s), \quad H(s) = \frac{1}{s(2s^2 + s + 2)}. \tag{42}$$

Then, if $h(t) = \mathcal{L}^{-1}\{H(s)\}$, we have

$$y = \phi(t) = u_5(t)h(t-5) - u_{20}(t)h(t-20). \tag{43}$$

Finally, to determine $h(t)$, we use the partial fraction expansion of $H(s)$:

$$H(s) = \frac{a}{s} + \frac{bs + c}{2s^2 + s + 2}. \tag{44}$$

Example 14.13 continue

Upon determining the coefficients, we find that $a = 1/2$, $b = -1$, and $c = -1/2$. Thus

$$\begin{aligned} H(s) &= \frac{1/2}{s} - \frac{s+1/2}{2s^2+s+2} = \frac{1/2}{s} - \frac{1}{2}\frac{(s+1/4)+1/4}{(s+\frac{1}{4})^2+\frac{15}{16}} \\ &= \frac{1/2}{s} - \frac{1}{2}\Big[\frac{s+1/4}{(s+\frac{1}{4})^2+(\frac{\sqrt{15}}{4})^2} + \frac{1}{\sqrt{15}}\frac{\sqrt{15}/4}{(s+\frac{1}{4})^2+(\frac{\sqrt{15}}{4})^2}\Big]. \end{aligned} \tag{45}$$

By Theorem 14.6

$$\mathcal{L}^{-1}\Big\{\frac{s+1/4}{(s+\frac{1}{4})^2+(\frac{\sqrt{15}}{4})^2}\Big\} = e^{-t/4}\cos\frac{\sqrt{15}}{4}t, \tag{46}$$

and

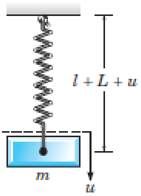
$$\mathcal{L}^{-1}\Big\{\frac{\sqrt{15}/4}{(s+\frac{1}{4})^2+(\frac{\sqrt{15}}{4})^2}\Big\} = e^{-t/4}\sin\frac{\sqrt{15}}{4}t, \tag{47}$$

Thus, we obtain

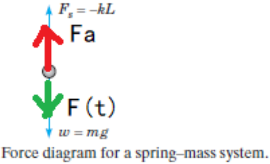
$$h(t) = \frac{1}{2} - \frac{1}{2}\big[e^{-t/4}\cos(\sqrt{15}t/4) + (\sqrt{15}/15)e^{-t/4}\sin(\sqrt{15}t/4)\big]. \tag{48}$$

Unit impulses and the Dirac delta functions

Recall: In previous example,



(a) Spring-mass system



(b) Force analysis

Unit impulses and the Dirac delta functions

Force analysis

(1) the weight $Fg = mg$ acting downwards;

(2) a restoring force from the spring pulling the mass upwards $F_s = -k(L+u)$;

(3) a resistance force F_a (air resistance/friction) that acts in the opposite of the motion and is proportional to the speed u'. This is usually referred to as **viscous damping** and $F_a = -\gamma u'$ with constant $\gamma > 0$ (**damping constant**);

(4) an external force $F(t)$ that acting on the object (could be upward or downward).

Using Newton's second law, we arrive at the ODE

$$\begin{aligned} mu''(t) &= mg - k(L + u(t)) - \gamma u'(t) + F(t) \\ \Rightarrow mu''(t) &+ \gamma u'(t) + ku(t) = F(t) \end{aligned}$$

Unit impulses and the Dirac delta functions

In some applications it is necessary to deal with phenomena of an impulsive nature for example, forces that act over very short time intervals. Such problems often lead to differential equations of the form

$$ay'' + by' + cy = g(t), \tag{49}$$

where $g(t)$ is nonzero during a short interval $t_0 - \tau < t < t_0 + \tau$ for some $\tau > 0$, and is otherwise zero.

The integral $I(\tau)$, defined by

$$I(\tau) = \int_{-\infty}^{\infty} g(t)dt, \tag{50}$$

is a measure of the strength of the forcing function. In a mechanical system, where $g(t)$ is a force which is nonzero only in the interval $(t_0 - \tau, t_0 + \tau)$, $I(\tau)$ is the total impulse of the force $g(t)$ over the time interval $(t_0 - \tau, t_0 + \tau)$.

Unit impulses and the Dirac delta functions

In particular, let's suppose that t_0 is zero and that $g(t)$ is given by

$$g(t) = d_\tau(t) = \begin{cases} \dfrac{1}{2\tau}, & -\tau < t < \tau, \\ 0, & t \leq -\tau, \text{ or } t \geq \tau, \end{cases} \tag{51}$$

where $\tau > 0$ is a small constant. In this case, it follows immediately that, $I(\tau) = 1$ independent of the value of τ, as long as $\tau \neq 0$.

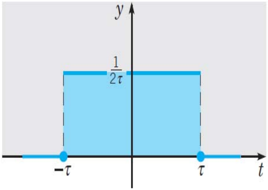
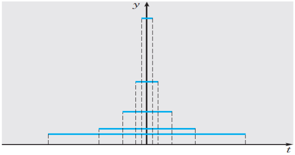


Fig 6. Graph of $y = d_\tau(t)$.

Unit impulses and the Dirac delta functions

In the limit as $\tau \to 0$, we get an idealized unit impulse function δ, which imposes an impulse of magnitude one at $t = 0$ but is zero for all t other than zero. This function, which is not an ordinary function studied in elementary calculus, is called Dirac **delta function** .



Graph of $y = d_\tau(t)$ as $\tau \to 0^+$.

Unit impulses and the Dirac delta functions

The delta function is defined to have the following properties:

$$\delta(t) = 0, \;\; t \neq 0, \quad \text{and} \int_{-\infty}^{\infty} \delta(t)dt = 1. \tag{52}$$

A unit impulse at an arbitrary point $t = t_0$ is given by $\delta(t - t_0)$, which satisfies

$$\delta(t - t_0) = 0, \;\; t \neq t_0, \quad \text{and} \int_{-\infty}^{\infty} \delta(t - t_0)dt = 1. \tag{53}$$

Remark: Delta function is not a usual function, it is a generalized function. Dirac delta function $\delta(t - t_0)$ satisfies the following property: for any integrable function $f(t)$ over $(-\infty, +\infty)$,

$$\lim_{\tau \to 0^+} \int_{-\infty}^{\infty} d_\tau(t - t_0)f(t)dt = \int_{-\infty}^{\infty} \delta(t - t_0)f(t)dt. \tag{54}$$

Unit impulses and the Dirac delta functions

Theorem 14.14

Suppose that $f(t)$ is integrable on $(-\infty, +\infty)$ and continuous at t_0. Then

$$\int_{-\infty}^{\infty} \delta(t - t_0) f(t) dt = f(t_0). \qquad \square \tag{55}$$

For $t_0 > 0$, the Laplace transform of $\delta(t - t_0)$ is defined by

$$\begin{aligned}
\mathcal{L}\{\delta(t - t_0)\} &= \lim_{\tau \to 0+} \mathcal{L}\{d_\tau(t - t_0)\} \\
&= \lim_{\tau \to 0+} \int_0^\infty e^{-st} d_\tau(t - t_0) dt \\
&= \lim_{\tau \to 0+} \int_{t_0 - \tau}^{t_0 + \tau} e^{-st} d_\tau(t - t_0) dt \\
&= \lim_{\tau \to 0+} \frac{1}{2\tau} \int_{t_0 - \tau}^{t_0 + \tau} e^{-st} dt \\
&= \lim_{\tau \to 0+} \frac{1}{2s\tau} e^{-st_0} (e^{s\tau} - e^{-s\tau}) = e^{-st_0}
\end{aligned}$$

Unit impulses and the Dirac delta functions

Proof.

$$\int_{-\infty}^{\infty} \delta(t-t_0)f(t)dt = \lim_{\tau\to 0^+}\int_{-\infty}^{\infty} d_\tau(t-t_0)f(t)dt. \tag{56}$$

Using the definition of $d_\tau(t)$ and the mean value theorem for integrals, we find that

$$\int_{-\infty}^{\infty} d_\tau(t-t_0)f(t)dt = \frac{1}{2\tau}\int_{t_0-\tau}^{t_0+\tau} f(t)dt = \frac{1}{2\tau}\cdot 2\tau\cdot f(t^*) = f(t^*),$$

where $t_0-\tau < t^* < t_0+\tau$. Hence $t^*\to t_0$ as $\tau\to 0^+$, and it follows from Eq. (56) that

$$\int_{-\infty}^{\infty} \delta(t-t_0)f(t)dt = f(t_0). \tag{57}$$

Unit impulses and the Dirac delta functions

Example 14.15

Find the solution of the initial value problem

$$2y'' + y' + 2y = \delta(t-5), \tag{58}$$

$$y(0) = 0, \quad y'(0) = 0. \tag{59}$$

This initial value problem arises from the study of the electric circuit or mechanical oscillator (can be regarded as the damped oscillator subject to the unit impulse $\delta(t-5)$). The only difference is in the forcing term. To solve the given problem, we take the Laplace transform of the differential equation and use the initial conditions, obtaining

$$(2s^2 + s + 2)Y(s) = e^{-5s}.$$

Thus

$$Y(s) = \frac{e^{-5s}}{2s^2 + s + 2} = \frac{e^{-5s}}{2} \frac{1}{(s+\frac{1}{4})^2 + \frac{15}{16}}. \tag{60}$$

Unit impulses and the Dirac delta functions

Example 14.15 continue

By Theorem 14.6

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+\frac{1}{4})^2+\frac{15}{16}}\right\}=\frac{4}{\sqrt{15}}e^{-t/4}\sin\frac{\sqrt{15}}{4}t. \tag{61}$$

Hence, by Theorem 14.12, we have

$$y=\mathcal{L}^{-1}\{Y(s)\}=\frac{2}{\sqrt{15}}u_5(t)e^{-(t-5)/4}\sin\frac{\sqrt{15}}{4}(t-5), \tag{62}$$

which is the formal solution of the given problem. It is also possible to write y in the form

$$y=\begin{cases} 0, & t<5, \\ \frac{2}{\sqrt{15}}e^{-(t-5)/4}\sin\frac{\sqrt{15}}{4}(t-5). & t\geq 5, \end{cases} \tag{63}$$

The convolution theorem

Definition 14.16

If $f(t)$ and $g(t)$ are defined on $[0,\infty)$, then the convolution $f * g$ is the function defined by

$$(f*g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau = \int_0^t f(\tau)g(t-\tau)d\tau. \quad \square \tag{64}$$

Example $f = \cos(t)$

$$\begin{aligned}(f*1)(t) &= \int_0^t \cos(t-\tau)d\tau = -sin(t-\tau)\Big|_{\tau=0}^{\tau=t} \\ &= -\sin 0 + \sin t \\ &= \sin t.\end{aligned}$$

Example $f = e^{-t}, g(t) = \sin(t)$

$$(f*g)(t) = \int_0^t e^{-\tau}\sin(t-\tau)d\tau.$$

The convolution theorem

Property

$$f * g = g * f \qquad \text{commutative law} \tag{65}$$

$$f * (g_1 + g_2) = f * g_1 + f * g_2 \qquad \text{distributive law} \tag{66}$$

$$(f * g) * h = f * (g * h) \qquad \text{associative law} \tag{67}$$

$$f * 0 = 0 * f = 0 \tag{68}$$

The convolution theorem

Theorem 14.17

If $F(s) = \mathcal{L}\{f(t)\}$ *and* $G(s) = \mathcal{L}\{g(t)\}$ *both exist for* $s > a \geq 0$, *then*

$$\mathcal{L}\{(f * g)(t)\} = F(s)G(s), \quad s > a. \qquad \square \tag{69}$$

Proof.

$$\begin{aligned} F(s)G(s) &= \int_0^{+\infty} e^{-s\xi} f(\xi) d\xi \int_0^{+\infty} e^{-s\tau} g(\tau) d\tau \\ &= \int_0^{+\infty} g(\tau) \Big[\int_0^{+\infty} e^{-s(\xi+\tau)} f(\xi) d\xi \Big] d\tau. \end{aligned}$$

Let $\xi + \tau = t$, then we have

$$\begin{aligned} F(s)G(s) &= \int_0^{+\infty} g(\tau) \Big[\int_\tau^{+\infty} e^{-st} f(t-\tau) dt \Big] d\tau \\ &= \int_0^{+\infty} e^{-st} \Big[\int_0^t f(t-\tau) g(\tau) d\tau \Big] dt \\ &= \int_0^{+\infty} e^{-st} (f * g)(t) dt \\ &= \mathcal{L}\{(f * g)(t)\}. \qquad \square \end{aligned}$$

The convolution theorem

Theorem 14.18

If $\mathcal{L}^{-1}\{F(s)\} = f(t)$ *and* $\mathcal{L}^{-1}\{G(s)\} = g(t)$*, then*

$$\mathcal{L}^{-1}\{F(s)G(s)\} = (f * g)(t). \qquad \square \tag{70}$$

Example $f = e^{-t}, g(t) = \sin(t)$

$$(f * g)(t) = \int_0^t e^{-\tau} \sin(t - \tau) d\tau.$$

Since $\mathcal{L}\left(e^{-t}\right)(s) = \frac{1}{s+1}, \mathcal{L}\left(\sin(t)\right)(s) = \frac{1}{s^2+1}$

$$\mathcal{L}\left((f * g)(t)\right)(s) = \mathcal{L}\left(e^{-t}\right)(s)\mathcal{L}\left(\sin t\right)(s) = \frac{1}{(s+1)(s^2+1)}.$$

The convolution theorem

Example 14.19

Find the solution of the initial value problem

$$y'' + 4y = g(t), \tag{71}$$

$$y(0) = 3, \qquad y'(0) = -1. \tag{72}$$

where the forcing function g is given.
By taking the Laplace transform of the differential equation and using the initial conditions, we obtain

$$s^2 Y(s) - 3s + 1 + 4Y(s) = G(s)$$

The convolution theorem

Example 14.19 continue

Thus,

$$Y(s) = \frac{3s-1}{s^2+4} + \frac{G(s)}{s^2+4}. \tag{73}$$

Observe that the first and second terms on the right side of Eq. (73) contain the dependence of $Y(s)$ on the initial conditions and forcing function, respectively. It is convenient to write $Y(s)$ in the form

$$Y(s) = 3\frac{s}{s^2+4} - \frac{1}{2}\frac{2}{s^2+4} + \frac{1}{2}\frac{2}{s^2+4}G(s). \tag{74}$$

Then, using lines 5 and 6 of the Table for elementary Laplace transform and Theorem 14.17, we obtain

$$y = 3\cos 2t - \frac{1}{2}\sin 2t + \frac{1}{2}\int_0^t \sin 2(t-\tau)g(\tau)d\tau. \tag{75}$$

If a specific forcing function g is given, then the integral in Eq. (75) can be evaluated (by numerical computations, if necessary).

Appendix: Definition for inverse Laplace Transform

Question: If two functions have the same Laplace transform, are the two function the same?

Theorem 14.20

If $f(t)$ and $g(t)$ are continuous on $[0, +\infty)$ of exponential order and $\mathcal{L}\{f(t)\} = \mathcal{L}\{g(t)\}$, then $f = g$.

The proof is based on Weierstrass Approximation Theorem of continuous functions by polynomials, which is put in the appendix.

Appendix: Definition for inverse Laplace Transform

Theorem 14.21

If $f(t)$ and $g(t)$ are continuous on $[0, +\infty)$ of exponential order and $\mathcal{L}\{f(t)\} = \mathcal{L}\{g(t)\}$, then $f = g$.

The proof is based on Weierstrass Approximation Theorem of continuous functions by polynomials.

Theorem 14.22

Theorem (Weierstrass Approximation). If $f(t)$ is a continuous function on a closed interval $[a, b]$, then for every $\varepsilon > 0$ there exists a polynomial $q_\varepsilon(t)$ such that

$$\max_{t\in[a,b]} |f(t) - q_\varepsilon(t)| < \varepsilon.$$

The proof of this theorem can be found in any mathematical analysis textbook.

Appendix: Definition for inverse Laplace Transform

Theorem 14.23

If $f(t)$ and $g(t)$ are continuous on $[0, +\infty)$ of exponential order s_0 and $\mathcal{L}\{f(t)\}(s) = \mathcal{L}\{g(t)\}(s)$, then $f = g$.

Proof. Let $u = f - g$, then $\mathcal{L}\{u\}(s) = \mathcal{L}\{f - g\}(s) = 0$ for all $s > s_0$, and u is of exponential order s_0. There exists two positive constants k, T such that

$$|u(t)| < ke^{s_0 t}, t > T.$$

Evaluate $\mathcal{L}\{u\}(s)$ at $\hat{s} = s_1 + n + 1$, where s_1 is any real number such that $s_1 > s_0$ and n is any non-negative integer, one has

$$\mathcal{L}\{u\}(\hat{s}) = \int_0^\infty e^{(-s_1-n-1)t} u(t)dt = \int_0^\infty e^{-s_1 t} e^{-(n+1)t} u(t)dt$$

Let $e^{-t} = y$, then $dy = -e^{-t}dt$, then

$$\mathcal{L}\{u\}(\hat{s}) = \int_0^\infty e^{(-s_1-n-1)t} u(t)dt = \int_0^1 y^n y^{s_1} u(-ln(y))dy$$

Appendix: Definition for inverse Laplace Transform

Now introduce $v(y) = y^{s_1} u(-ln(y)) = e^{-s_1 t} u(t)$,

$$\lim_{y \to 0+} |v(y)| = \lim_{t \to \infty} e^{-s_1 t} |u(t)| < \lim_{t \to \infty} k e^{-(s_1 - s_0)t} = 0.$$

Thus, the function v does not diverge at $y = 0$. Since $\mathcal{L}\{u\}(\hat{s}) = 0$ for all $\hat{s} > s_0$. Thus,

$$\int_0^1 y^n v(y) dy = 0, \quad n = 0, 1, \cdots$$

Thus, for any polynomial $p(y)$, one has

$$\int_0^1 p(y) v(y) dy = 0.$$

Now,

$$\int_0^1 v^2(y) dy = \int_0^1 (v(y) - p(y)) v(y) dy + \int_0^1 p(y) v(y) dy.$$

Appendix: Definition for inverse Laplace Transform

$$
\begin{aligned}
\int_0^1 v^2(y)dy &= \int_0^1 (v(y) - p(y))v(y)dy + \int_0^1 p(y)v(y)dy \\
&= \int_0^1 (v(y - p(y)))v(y)dy \\
&\leq \int_0^1 |p(y) - v(y)||v(y)|dy \\
&\leq \max_{y\in[0,1]} |v(y)| \int_0^1 |p(y) - v(y)|dy
\end{aligned}
$$

this is true for any polynomial $p(y)$. For any $\varepsilon > 0$, there exists a polynomial $p_\varepsilon(y)$, s.t.

$$
\max_{y\in[0,1]} |v(y) - p(y)| \leq \varepsilon
$$

Appendix: Definition for inverse Laplace Transform

Thus,

$$\begin{aligned}\int_0^1 v^2(y)dy &\leq \max_{y\in[0,1]} |v(y)| \int_0^1 |p(y)-v(y)|dy \\ &\leq \max_{y\in[0,1]} |v(y)|\varepsilon\end{aligned}$$

Therefore,

$$\int_0^1 v^2(y)dy = 0.$$

$v(y)$ is continuous, thus, $v(y) \equiv 0$. And $u(t) \equiv 0$, $f(t) = g(t)$.

Appendix: Definition for inverse Laplace Transform

Remark: If $f(t), g(t)$ are piecewise continuous on the interval $0 \leq t \leq A$ for any positive A, and $\mathcal{L}\{f\} = \mathcal{L}\{g\}$, then $f(t)$ and $g(t)$ are equal except at a set of discrete points, where jump discontinuity happens, these two functions can be considered as the same (indeed, they are almost equal everywhere). The proof can be found in "R. Churchill. Operational Mathematics. McGraw-Hill, New york, 1958. Second Edition."