Chapter 9

Interval Estimation

9.1 Introduction

Definition 9.1.1: An ***Interval Estimate*** of a real-valued parameter θ is any pair of functions, $L(x_1,\dots,x_n)$ and $U(x_1,\dots,x_n)$, of a sample that satisfy $L(\mathbf{x}) \le U(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{X}$. If $\mathbf{X} = \mathbf{x}$ is observed, the inference $L(\mathbf{x}) \le \theta \le U(\mathbf{x})$ is made. The random interval $[L(\mathbf{X}), U(\mathbf{X})]$ is called an ***Interval Estimator***.

Note: An interval estimator replaces our point estimator by an interval.

Example 9.1.2: (Interval Estimator)
Let $X_1,\dots,X_4$ from a $\mathrm{n}(\mu,1)$. A possible interval estimator μ is $\left[\bar{X}-1, \bar{X}+1\right]$. This means that we will assert that μ is in this interval.

Example 9.1.3: (Continuation of Example 9.1.2)
Note that in this case $P(\bar{X}=\mu)=0$. Now consider the interval estimator $[\bar{X}-1, \bar{X}+1]$. Find $P\left(\mu \in \left[\bar{X}-1, \bar{X}+1\right]\right)$.

Note: What we get in return for giving up precision is a measure of "guarantee" of capturing the parameter of interest.

Definition 9.1.4: For an interval estimator $[L(\mathbf{X}), U(\mathbf{X})]$ of a parameter θ, the ***Coverage Probability*** of $[L(\mathbf{X}), U(\mathbf{X})]$ is the probability that the random interval $[L(\mathbf{X}), U(\mathbf{X})]$ covers the true parameter, θ. In symbols, it is denoted by either $P_\theta\left(\theta \in [L(\mathbf{X}), U(\mathbf{X})]\right)$ or $P\left(\theta \in [L(\mathbf{X}), U(\mathbf{X})]|\theta\right)$.

Definition 9.1.5: For an interval estimator $[L(\mathbf{X}), U(\mathbf{X})]$ of a parameter θ, the ***Confidence Coefficient*** of $[L(\mathbf{X}), U(\mathbf{X})]$ is the infimum of the coverage probabilities, $\inf_\theta P_\theta(\theta \in [L(\mathbf{X}), U(\mathbf{X})])$.

Remark:

1. Interval estimators are random quantities not parameters so that the probability in $P_\theta\left(\theta \in [L(\mathbf{X}), U(\mathbf{X})]\right)$ is not a statement about the probability of θ but the probability of the functions of $\mathbf{X}$.
2. Interval estimator with a measure of confidence is usually referred to as confidence intervals.
3. In general, we will be working on confidence sets rather than simple intervals where no closed form is available for these sets.
4. A confidence set with a confidence coefficient equal to $1-\alpha$ is called $1-\alpha$ confidence set.

Example 9.1.6: (Scale Uniform Interval Estimator)

Let $X_1, \ldots, X_n$ be a random sample from uniform$(0, \theta)$. Let $Y = X_{(n)} = \max(X_1, \ldots, X_n)$. Consider the following interval estimator for θ:

- Candidate 1: $[aY, bY]$, $1 \leq a < b$.
- Candidate 2: $[Y + c, Y + d]$, $0 \leq c < d$.

where a, b, c, d are specified constants. Find the coverage probabilities of each interval estimators.

9.2 Methods of Finding Interval Estimators

9.2.1 Inverting a Test Statistic

Example 9.2.1: (Inverting a Normal Test)

Let $X_1, \dots, X_n$ be iid $\mathrm{n}(\mu, \sigma^2)$ and consider testing $H_0 : \mu = \mu_0$ versus $H_1 : \mu \neq \mu_0$. For a fixed α, a most powerful unbiased test rejects H_0 when $\left\{\mathbf{x} : |\bar{x} - \mu_0| > z_{\alpha/2}\sigma/\sqrt{n}\right\}$. Note that H_0 is accepted for sample points with $|\bar{x} - \mu_0| \leq z_{\alpha/2}\sigma/\sqrt{n}$ or, equivalently,

$$\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \leq \mu_0 \leq \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

Since the test has size α, it means that

$$P(H_0 \text{ is rejected } |\mu = \mu_0) = \alpha$$

or, stated in another way,

$$P(H_0 \text{ is accepted}|\mu = \mu_0) = 1 - \alpha$$

which implies

$$P\left(\bar{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \leq \mu_0 \leq \bar{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \middle| \mu = \mu_0\right) = 1 - \alpha$$

This probability statement is true for every μ_0. Hence, it holds that

$$P\left(\bar{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) = 1 - \alpha$$

The interval $\left[\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right]$ obtained by inverting the acceptance region of the level α test, is a $1 - \alpha$ confidence interval.

Correspondence between Tests and Confidence Sets

The acceptance region of the hypothesis test is

$$A(\mu_0) = \left\{\mathbf{x} : \mu_0 - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \leq \bar{x} \leq \mu_0 + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right\}$$

and the confidence interval is given by

$$C(\mathbf{x}) = \left\{\mu : \bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right\}$$

These sets are connected to each other by the tautology:

$$\mathbf{x} \in A(\mu_0) \Longleftrightarrow \mu_0 \in C(\mathbf{x})$$

<u>Theorem 9.2.2</u>: For each $\theta_0 \in \Theta$, let $A(\theta_0)$ be the acceptance region of a level α test of $H_0 : \theta = \theta_0$. For each $\mathbf{x} \in \mathcal{X}$, define a set $C(\mathbf{x})$ in the parameter space by

$$C(\mathbf{x}) = \{\theta_0 : \mathbf{x} \in A(\theta_0)\}.$$

Then the random set $C(\mathbf{X})$ is a $1-\alpha$ confidence set. Conversely, let $C(\mathbf{X})$ be a $1-\alpha$ confidence set. For any $\theta_0 \in \Theta$, define

$$A(\theta_0) = \{\mathbf{x} : \theta_0 \in C(\mathbf{x})\}.$$

Then $A(\theta_0)$ is the acceptance region of a level α test of $H_0 : \theta = \theta_0$.

<u>Note</u>:

1. All of techniques we have for obtaining tests can be immediately used to construct confidence intervals.
2. There is no guarantee that the confidence set obtained by test inversion will be an interval.
3. Given $H_0 : \theta = \theta_0$, the alternative hypothesis H_1 will dictate the form of the acceptance region $A(\theta_0)$, which will further determine the shape of $C(\mathbf{x})$.
4. In most cases, one-sided tests give one-sided intervals, two-sided tests give two-sided intervals, strange-shaped acceptance regions give strange-shaped confidence sets.
5. The properties of inverted tests also carry over (sometimes suitably modified) to the confidence set.
6. Since we can confine attention to sufficient statistics when looking for a good test, it follows that we can also confine attention to sufficient statistics when looking for good confidence sets.

<u>Example 9.2.3</u>: (Inverting an LRT)

Suppose we want a confidence interval for the mean λ, of an exponential(λ) population. We can obtain such an interval by inverting a level α test of $H_0 : \lambda = \lambda_0$ versus $H_1 : \lambda \neq \lambda_0$. Let $X_1, \ldots, X_n$ be a random sample. For fixed λ_0, the acceptance region of the LRT is given by

$$A(\lambda_0) = \left\{ \mathbf{x} : \left(\frac{\sum x_i}{\lambda_0} \right)^n e^{-\sum x_i/\lambda_0} \geq k^* \right\}, \tag{9.2.2}$$

where k^* is a constant chosen to satisfy $P_{\lambda_0}(\mathbf{X} \in A(\lambda_0)) = 1 - \alpha$.
Inverting this acceptance region gives the $1 - \alpha$ confidence set

$$C(\mathbf{x}) = \left\{ \lambda : \left(\frac{\sum x_i}{\lambda} \right)^n e^{-\sum x_i/\lambda} \geq k^* \right\}.$$

This confidence set depends on $\mathbf{x}$ only through $\sum x_i$. So the confidence interval can be expressed in the form

$$C\left(\sum x_i\right) = \left\{ \lambda : L\left(\sum x_i\right) \leq \lambda \leq U\left(\sum x_i\right) \right\} \tag{9.2.3}$$

where L and U are functions determined by the constraints that the set (9.2.2) has probability $1 - \alpha$ and

$$\left(\frac{\sum x_i}{L\left(\sum x_i\right)} \right)^n e^{-\sum x_i / L(\sum x_i)} = \left(\frac{\sum x_i}{U\left(\sum x_i\right)} \right)^n e^{-\sum x_i / U(\sum x_i)} \tag{9.2.4}$$

If we set

$$a = \frac{\sum x_i}{L\left(\sum x_i\right)} \quad \text{and} \quad b = \frac{\sum x_i}{U\left(\sum x_i\right)},$$

then the confidence interval (9.2.3) becomes $\left\{ \lambda : \frac{1}{a} \sum x_i \leq \lambda \leq \frac{1}{b} \sum x_i \right\}$, where a and b satisfy

$$P_\lambda\left(\frac{1}{a} \sum X_i \leq \lambda \leq \frac{1}{b} \sum X_i \right) = P\left(b \leq \frac{\sum X_i}{\lambda} \leq a \right)$$

and, from (9.2.4), $a^2 e^{-a} = b^2 e^{-b}$. Note that $\sum X_i \sim \text{gamma}(n, \lambda)$ and $\sum X_i/\lambda \sim \text{gamma}(n, 1)$.

Example 9.2.4: (Normal One-sided Confidence Bound)

Let $X_1, \ldots, X_n$ be a random sample a $\mathrm{n}(\mu, \sigma^2)$ population. Consider constructing a $1-\alpha$ upper confidence bound for μ, i.e., a confidence interval of the form $C(\mathbf{x}) = (-\infty, U(\mathbf{x})]$. We will construct the one-sided confidence interval by inverting one-sided tests of $H_0 : \mu = \mu_0$ versus $H_1 : \mu \neq \mu_0$. The size α LRT of H_0 versus H_1 rejects H_0 if

$$\frac{\bar{X} - \mu_0}{S/\sqrt{n}} < -t_{n-1,\alpha}$$

and the acceptance region for this test is

$$A(\mu_0) = \left\{ \mathbf{x} : \bar{x} \geq \mu_0 - t_{n-1,\alpha} \frac{s}{\sqrt{n}} \right\}$$

and

$$\mathbf{x} \in A(\mu_0) \Longleftrightarrow \mu_0 \in C(\mathbf{x}) = \left\{ \mu_0 : \mu_0 \leq \bar{x} + t_{n-1,\alpha} \frac{s}{\sqrt{n}} \right\}.$$

By Theorem 9.2.2, the random set $C(\mathbf{X}) = (-\infty, \bar{X} + t_{n-1,\alpha} S/\sqrt{n}]$ is a $1-\alpha$ one-sided confidence interval for μ.

Example 9.2.5: (Binomial One-sided Confidence Bound)

Observe $X_1, \ldots, X_n$, where $X_i \sim \text{Bernoulli}(p)$. Consider to obtain a one-sided confidence interval of the form $(L(x_1, \ldots, x_n), 1]$, where

$$P_p\Big(p \in \big(L(x_1, \ldots, x_n), 1\big]\Big) \geq 1 - \alpha.$$

9.2.2 Pivotal Quantities

Definition 9.2.6: A random variable $Q(\mathbf{X}, \theta) = Q(X_1, \ldots, X_n, \theta)$ is a ***Pivotal Quantity*** (or pivot) if the distribution of $Q(\mathbf{X}, \theta)$ is independent of all parameters. That is, if $\mathbf{X} \sim F(\mathbf{x}|\theta)$, then $Q(\mathbf{X}, \theta)$ has the same distribution for all values of θ.

Example 9.2.7: (Location-Scale Pivots)
Let $X_1, \ldots, X_n$ be a random sample from the indicated pdfs, and let $\bar{X}$ and S be the sample mean and standard deviation:

- Pivot for location family with pdf $f(x-\mu) : \bar{X} - \mu$
- Pivot for scale family with pdf $\frac{1}{\sigma} f\left(\frac{1}{\sigma}\right)$: $\frac{\bar{X}}{\sigma}$
- Pivot for location and scale family with pdf $\frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right)$: $\frac{\bar{X}-\mu}{S}$

Note: In general, *differences* are pivotal for location parameters, while *ratio* (or products) are pivotal for scale problems.

Example 9.2.8: (Gamma Pivot)

Suppose that $X_1, \ldots, X_n$ are iid exponential(λ). Then $T = \sum_{i=1}^n X_i$ is a sufficient statistic for λ and $T \sim$ gamma(n, λ), which is a scale family. Hence a pivot that may be used is

$$Q_1(T, \lambda) = \frac{T}{\lambda} \sim \text{gamma}(n, 1)$$

or

$$Q_2(T, \lambda) = \frac{T}{2\lambda} \sim \text{gamma}(n, 2) = \chi^2_{2n}$$

How to find a pivot for a general pdf or pmf?

If the pdf of a statistic T can be expressed in the form

$$f(t|\theta) = g\big(Q(t, \theta)\big)\left|\frac{\partial}{\partial t}Q(t, \theta)\right|$$

for some function g and some monotone function Q (monotone in t for each θ). Then Theorem 2.1.5 (transformation technique) can be used to show that $Q(T, \theta)$ is a pivot.

How to use a pivot to construct a confidence set?

Given a pivot $Q(\mathbf{X}, \theta)$, we find numbers a and b such that

$$P_\theta\left(a \le Q(\mathbf{X}, \theta) \le b\right) \ge 1 - \alpha$$

The acceptance region for a level α test for $H_0 : \theta = \theta_0$ is given by

$$A(\theta_0) = \{\mathbf{x} : a \le Q(\mathbf{x}, \theta_0) \le b\}$$

Using Theorem 9.2.2, we invert these tests to obtain

$$C(\mathbf{x}) = \{\theta_0 : a \le Q(\mathbf{x}, \theta_0) \le b\},$$

and $C(\mathbf{X})$ is a $1 - \alpha$ confidence set for θ. If θ is a real-valued parameter and if, for each $\mathbf{x} \in \mathcal{X}$, $Q(\mathbf{x}, \theta)$ is a monotone function of θ, then $C(\mathbf{x})$ will be an interval.

- If $Q(\mathbf{x}, \theta)$ is an increasing function of θ, then $C(\mathbf{x})$ has the form

$$L(\mathbf{x}, a) \le \theta \le U(\mathbf{x}, b).$$

- If $Q(\mathbf{x}, \theta)$ is a decreasing function of θ (which is typical), then $C(\mathbf{x})$ has the form

$$L(\mathbf{x}, b) \le \theta \le U(\mathbf{x}, a).$$

Example 9.2.9: (Continuation of Example 9.2.8)

Recall the pivot for λ is $Q(T, \lambda) = 2T/\lambda \sim \chi^2_{2n}$. Choose a and b such that

$$P_\lambda\left(a \le \frac{2T}{\lambda} \le b\right) = P_\lambda(a \le Q(T, \lambda) \le b) = P(a \le \chi^2_{2n} \le b) = 1 - \alpha.$$

Inverting the set $A(\lambda) = \left\{t : a \le \frac{2t}{\lambda} \le b\right\}$ gives $C(t) = \left\{\lambda : \frac{2t}{b} \le \lambda \le \frac{2t}{a}\right\}$, which is a $1 - \alpha$ confidence interval.

<u>Example 9.2.10</u>: (Normal Pivotal Interval)
Let $X_1, \ldots, X_n$ be a random sample from a $\mathrm{n}(\mu, \sigma^2)$ population. If σ^2 is known, $\dfrac{\bar{X}-\mu}{\sigma/\sqrt{n}}$ is a pivot and can be used to construct a $1-\alpha$ confidence interval for μ below:

$$\left\{\mu : \bar{x} - a\frac{\sigma}{\sqrt{n}} \le \mu \le \bar{x} + a\frac{\sigma}{\sqrt{n}}\right\}$$

where a satisfy

$$P\left(-a \le \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \le a\right) = P(-a \le Z \le a) = 1-\alpha, \quad (Z \text{ is standard normal}).$$

If σ^2 is also unknown, we can use the location-scale pivot $\dfrac{\bar{X}-\mu}{S/\sqrt{n}}$, which has Student's t distribution such that

$$P\left(-a \le \frac{\bar{X}-\mu}{S/\sqrt{n}} \le a\right) = P(-a \le T_{n-1} \le a).$$

Thus, if we take $a = t_{n-1,\alpha/2}$, a $1-\alpha$ confidence interval for μ can be obtained by

$$\left\{\mu : \bar{x} - t_{n-1,\alpha/2}\frac{s}{\sqrt{n}} \le \mu \le \bar{x} + t_{n-1,\alpha/2}\frac{s}{\sqrt{n}}\right\}.$$

Moreover, if we also want an interval estimate for σ, we can utilize a pivot $\dfrac{(n-1)S^2}{\sigma^2}$, because $\dfrac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$, to a $1-\alpha$ confidence interval

$$\left\{\sigma^2 : \frac{(n-1)s^2}{b} \le \sigma^2 \le \frac{(n-1)s^2}{a}\right\}$$

where a and b satisfy

$$P\left(a \le \frac{(n-1)S^2}{\sigma^2} \le b\right) = P(a \le \chi^2_{n-1} \le b) = 1-\alpha.$$

One choice of a and b that will produce the required interval is $a = \chi^2_{n-1,1-\alpha/2}$ and $b = \chi^2_{n-1,\alpha/2}$.

<u>Note</u>: For constructing a confidence set for k parameters simultaneously, we can use the Bonferroni Inequality. In this case, we construct a $1-\dfrac{\alpha}{k}$ confidence interval for each individual parameter and combine them into a $1-\alpha$ confidence set for these k parameters simultaneously.

9.3 Methods of Evaluating Interval Estimators

Two Important Considerations

1. Size of a Confidence Set: Length of an Interval or Volume of a Set
2. Confidence Coefficient: Infimum of Coverage Probability

9.3.1 Size and Coverage Probability

Example 9.3.1: (Optimizing Length)
Let $X_1, \dots, X_n$ be iid $\mathrm{n}(\mu, \sigma^2)$, where σ is known. From the pivot method proposed in Section 9.2.2 and the fact that $Z = \dfrac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim \mathrm{n}(0, 1)$, any a and b that satisfy

$$P(a \leq Z \leq b) = 1 - \alpha$$

will give the $1 - \alpha$ confidence interval

$$\left\{\mu : \bar{x} - b\frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{x} - a\frac{\sigma}{\sqrt{n}}\right\}$$

Which choice of a and b will minimize the length of the confidence interval while maintaining $1 - \alpha$ coverage?

Definition: A pdf $f(x)$ is ***unimodal*** if there exists x^* such that $f(x)$ is nondecreasing $x < x^*$ and $f(x)$ is nonincreasing for $x \geq x^*$.

Theorem 9.3.2: Let $f(x)$ be a unimodal pdf. If the interval $[a, b]$ satisfies

i. $\int_a^b f(x)dx = 1 - \alpha$,

ii. $f(a) = f(b) > 0$, and

iii. $a \leq x^* \leq b$, where x^* is mode of $f(x)$,

then $[a, b]$ is the shortest among all intervals that satisfy (i).

Example 9.3.3: (Optimizing Expected Length)
Recall that intervals for normal distribution can be based on the pivot $\dfrac{\bar{X} - \mu}{S/\sqrt{n}}$. Consider the $1 - \alpha$ confidence interval of the form

$$\bar{x} - b\frac{s}{\sqrt{n}} \leq \mu \leq \bar{x} - a\frac{s}{\sqrt{n}}$$

The interval with $a = -t_{n-1,\alpha/2}$ and $b = t_{n-1,\alpha/2}$ is the shortest length $1 - \alpha$ confidence interval. The interval length is a fucntion of s with general form

$$\text{Length}(s) = (b - a)\frac{S}{\sqrt{n}}.$$

Note: The following example illustrates how to get the shortest pivotal interval where Theorem 9.3.2 cannot be used directly.

Example 9.3.4: (Shortest Pivotal Interval)

Let $X \sim \text{gamma}(k, \beta)$. A pivotal quantity is given by $Y = \dfrac{X}{\beta}$ where $Y \sim \text{gamma}(k, 1)$. Find a and b such that

$$P(a \leq Y \leq b) = 1 - \alpha.$$

The interval for β is of the form

$$\frac{x}{b} \leq \beta \leq \frac{x}{a}$$

The length of this interval is

$$L = \left(\frac{1}{a} - \frac{1}{b}\right) x,$$

which is not proportional to $b - a$. Hence, we cannot use condition (ii) of Theorem 9.3.2 to find the shortest pivotal interval.