Sheet 2 118010350

A2.1 The linear program is equivalent to

minimize $-x_1-2x_2-3x_3-8x_4$

subject to $x_1-x_2+x_3+s_1=2$

$x_3-x_4+s_2=1$

$2x_2+3x_3+4x_4+s_3=8$

$x_1,x_2,x_3,x_4,s_1,s_2,s_3\geq 0$

$$A=\begin{bmatrix}1&-1&1&0&1&0&0\\0&0&1&-1&0&1&0\\0&2&3&4&0&0&1\end{bmatrix},\quad b=\begin{bmatrix}2\\1\\8\end{bmatrix}$$

indices 1 2 3 4 5 6 7

① choose basic indices, $B=\{1,2,3\}$.

basic solution, $x=[3.5\ 2.5\ 1\ 0\ 0\ 0\ 0]^T$

objective function, $f=-11.5$

obtain $\bar{c}_6=-2.5$, $d=[2.5\ 1.5\ 1\ 0\ 0\ 1\ 0]^T$, $\theta^*=1$

② choose basic indices, $B=\{1,2,6\}$

basic solution $x=[0\ 4\ 0\ 0\ 0\ 1\ 0]^T$

objective function, $f=-14$.

obtain $\bar{c}_4=-2$, $d=[-2\ -2\ 0\ 1\ 0\ 0\ 1\ 0]^T$, $\theta^*=2$

③ choose basic indices, $B=\{1,4,6\}$

basic solution $x=[2\ 0\ 0\ 2\ 0\ 3\ 0]^T$

objective function, $f=-18$.

Since we cannot obtain any $\bar{c}_i<0$,

thus the final solution is $x_1=2$, $x_2=0$, $x_3=0$, $x_4=2$,

and objective function, $f=x_1+2x_2+3x_3+8x_4=18$

Maxleaf

A2.2 The original linear program is equivalent to

$$\begin{aligned} \text{minimize } & x_1 - x_2 + x_3 \\ \text{subject to } & -2x_1 + x_2 - x_3 - s_1 = 1 \\ & x_1 - x_2 - x_3 + s_2 = 4 \\ & x_1, x_2, x_3, s_1, s_2 \geq 0 \end{aligned}$$

$$A = \begin{bmatrix} -2 & 1 & -1 & -1 & 0 \\ 1 & -1 & -1 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

① Phase-I: Construct the auxiliary problem.

$$\begin{aligned} \text{minimize } & y_1 + y_2 \\ \text{subject to } & -2x_1 + x_2 - x_3 - s_1 + y_1 = 1 \\ & x_1 - x_2 - x_3 + s_2 + y_2 = 4 \\ & x_1, x_2, x_3, s_1, s_2, y_1, y_2 \geq 0 \end{aligned}$$

The initial tableau:

| B | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 1 | 0 | 2 | 1 | -1 | 0 | 0 | -5 |
| 6 | -2 | 1 | -1 | -1 | 0 | 1 | 0 | 1 |
| 7 | 1 | -1 | -1 | 0 | $\boxed{1}$ | 0 | 1 | 4 |

Step 1:

| B | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 2 | -1 | 1 | 1 | 0 | 0 | 1 | -1 |
| 6 | -2 | $\boxed{1}$ | -1 | -1 | 0 | 1 | 0 | 1 |
| 5 | 1 | -1 | -1 | 0 | 1 | 0 | 1 | 4 |

Step 2:

| B | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 |
| 2 | -2 | 1 | -1 | -1 | 0 | 1 | 0 | 1 |
| 5 | -1 | 0 | -2 | -1 | 1 | 1 | 1 | 5 |

The optimal value for auxiliary problem is 0,

$x = [0\ 1\ 0\ 0\ 5]$, $B = \{2, 5\}$.

Phase-II: The new simplex tableau:

| B | 3 | 0 | 3 | 2 | 0 | 1 |
|---|---|---|---|---|---|---|
| 2 | -2 | 1 | -1 | -1 | 0 | 1 |
| 5 | -1 | 0 | -2 | -1 | 1 | 5 |

Since all the reduced cost $\geq 0$, then the final solution is $x_1 = 0$, $x_2 = x_4 = 1$, $x_3 = 0$.
and objective function, $f = x_1 - x_2 + x_3 = -1$

A2.3 According to the linear program:

$$A = \begin{bmatrix} 1 & 3 & 0 & 4 & 1 \\ 1 & 3 & 0 & -3 & 1 \\ -1 & -4 & 3 & 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

Phase-I: Construct the auxiliary problem.

minimize $y_1 + y_2 + y_3$

Subject to
$$\begin{aligned} x_1 + 3x_2 \quad + 4x_4 + x_5 + y_1 &= 2 \\ x_1 + 3x_2 \quad - 3x_4 + x_5 + y_2 &= 2 \\ -x_1 - 4x_2 + 3x_3 \quad + y_3 &= 1 \\ x_1, x_2, x_3, x_4, x_5, y_1, y_2, y_3 &\geq 0 \end{aligned}$$

The initial tableau:

| B | -1 | -2 | -3 | -1 | -2 | 0 | 0 | 0 | -5 |
|---|---|---|---|---|---|---|---|---|---|
| 6 | $\boxed{1}$ | 3 | 0 | 4 | 1 | 1 | 0 | 0 | 2 |
| 7 | 1 | 3 | 0 | -3 | 1 | 0 | 1 | 0 | 2 |
| 8 | -1 | -4 | 3 | 0 | 0 | 0 | 0 | 1 | 1 |

Step 1:

| B | 0 | 1 | -3 | 3 | -1 | 1 | 0 | 0 | -3 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 3 | 0 | 4 | 1 | 1 | 0 | 0 | 2 |
| 7 | 0 | 0 | 0 | -7 | 0 | -1 | 1 | 0 | 0 |
| 8 | 0 | -1 | $\boxed{3}$ | 4 | 1 | 1 | 0 | 1 | 3 |

step 2:

| B | 0 | 0 | 0 | 7 | 0 | 2 | 0 | 1 | 0 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 3 | 0 | 4 | 1 | 1 | 0 | 0 | 2 |
| 7 | 0 | 0 | 0 | -7 | 0 | -1 | 1 | 0 | 0 |
| 3 | 0 | $-\frac{1}{3}$ | 1 | $\frac{4}{3}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | 0 | $\frac{1}{4}$ | $\frac{3}{4}$ |

The optimal value for the anxiliary problem is 0,

and $X = [2\ 0\ \frac{3}{4}\ 0\ 0]^T$, $B = \{1, 3, 7\}$.

replace the original B with $B = \{1, 3, 4\}$

Phase-II: The new simplex tableau:

| B | 0 | -3 | 0 | 0 | -4 | -4 |
|---|---|---|---|---|---|---|
| 1 | 1 | $\boxed{3}$ | 0 | 0 | 1 | 2 |
| 4 | 0 | 0 | 0 | 1 | 0 | 0 |
| 3 | 0 | $-\frac{1}{3}$ | 1 | 0 | $\frac{1}{4}$ | $\frac{3}{4}$ |

Step 1:

| B | 1 | 0 | 0 | 0 | -3 | -2 |
|---|---|---|---|---|---|---|
| 2 | $\frac{1}{3}$ | 1 | 0 | 0 | $\boxed{\frac{1}{3}}$ | $\frac{2}{3}$ |
| 4 | 0 | 0 | 0 | 1 | 0 | 0 |
| 3 | $\frac{1}{9}$ | 0 | 1 | 0 | $\frac{13}{36}$ | $\frac{35}{36}$ |

Step 2:

| B | 4 | 9 | 0 | 0 | 0 | 4 |
|---|---|---|---|---|---|---|
| 5 | 1 | 3 | 0 | 0 | 1 | 2 |
| 4 | 0 | 0 | 0 | 1 | 0 | 0 |
| 3 | $\frac{1}{36}$ | $-\frac{13}{12}$ | 1 | 0 | 0 | $\frac{1}{4}$ |

Since all the reduced cost $\geq 0$, then the

final solution is $X_1 = 0$, $X_2 = 0$, $X_3 = \frac{1}{4}$, $X_4 = 0$, $X_5 = 2$.

and objective function, $f = 2X_1 + 3X_2 + X_4 - 2X_5 = -4$.

A2.4. The simplex tableau:

| B | | | | | | | |
|---|---|---|---|---|---|---|---|
| B | 0 | 0 | $\delta$ | 0 | -2 | $\eta$ | 0 |
| 4 | 0 | 0 | $\alpha$ | 1 | 1 | -2 | $\beta$ |
| 1 | 1 | 0 | -2 | 0 | 1 | -2 | 10 |
| 2 | 0 | 1 | -1 | 0 | -1 | 1 | 1 |
| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | |

(a) We need $\beta \geq 0$ for feasibility, and $\eta \in \mathbb{R}$.

If $\delta \geq 0$, then $\alpha \in \mathbb{R}$;

If $\delta < 0$, then we set $\alpha > 0$, and the LP is bounded

or we set $\alpha \leq 0$, and the LP is unbounded.

(b) No exact need for feasibility, then $\beta \in \mathbb{R}$, and $\eta \in \mathbb{R}$.

Since the LP is unbounded, we need $\delta < 0$ and $\alpha \leq 0$.

(c) We need $\beta \geq 0$ for feasibility, and $\eta \in \mathbb{R}$.

Since the basis change to $B = \{4, 5, 2\}$,

then $\delta \geq 0$, $\beta > 10$, and $\alpha \in \mathbb{R}$

(d) We need $\beta \geq 0$ for feasibility.

Since we reach the optimal solution after one interation, then

Situation 1: Set $A_5$ as pivot column, need $\delta \geq 0$.

If $\beta \leq 10$, then $2\alpha + \delta \geq 0$ and $\eta - 4 \geq 0$.

If $\beta > 10$, then $\delta - 4 \geq 0$, $\eta - 4 \geq 0$, and $\alpha \in \mathbb{R}$.

Situation 2: Set $A_3$ as pivot column, need $\delta < 0$ and $\alpha > 0$.

then $-\frac{\delta}{\alpha} - 2 \geq 0$ and $-\frac{2\delta}{\alpha} + \eta \geq 0$.

(e). Here we need a degenerate BFS, then $\beta = 0$.

For simplicity, set $\delta \geq 0$ and $\alpha \in \mathbb{R}$

Since we need to avoid reach $A_6$ after several interations,

then $\eta - 4 \geq 0$

Another option is $\delta<0$ and $\alpha>0$.

then $-\frac{\delta}{\alpha}-2\geq 0$ and $-\frac{2\delta}{\alpha}+y\geq 0$

or $-\frac{\delta}{\alpha}-2<0$ and $-\frac{2\delta}{\alpha}+y-4\geq 0$.

A25. (a). Suppose $x^*$ is not the unique optimal solution.

Let $y\in\{x\in\mathbb{R}^n : Ax=b, x\geq 0\}$, $d=y-x^*$,

and $c^Tx^*\geq c^Ty$, given $y\neq x^*$.

Then we have $Ad=Ay-Ax=0$,

that is $0=[A_B\ A_N]\begin{bmatrix} d_B \\ d_N \end{bmatrix}=A_Bd_B+A_Nd_N$

$$=A_Bd_B+\sum_{i\in N}A_id_i$$

Then $d_B=-\sum_{i\in N}A_B^{-1}A_id_i$

$$c^Td=[c_B^T\ c_N^T]\begin{bmatrix} d_B \\ d_N \end{bmatrix}=c_B^Td_B+\sum_{i\in N}c_id_i$$

$$=\sum_{i\in N}(c_i-c_B^TA_B^{-1}A_i)d_i=\sum_{i\in N}\bar{c}_id_i$$

Since $c^Tx^*\geq c^Ty$, then $c^Td=\sum_{i\in N}\bar{c}_id_i=\sum_{i\in N}\bar{c}_iy_i\leq 0$

Given that $\bar{c}_i>0$, $\forall i\in N$ and since

$x^*$, $y$ are in different basis,

Thus $y_i$ are not all 0, for $i\in N$.

Then there exists $l\in N$, s.t. $y_l>0$.

However, it must satisfies that $\sum_{i\in N}\bar{c}_iy_i\leq 0$

Then there exists $m\in N$ s.t. $y_m<0$.

Thus we get the contradiction, for that

$y$ is not a valid solution.

Therefore, $x^*$ is the unique optimal solution.

(b) Suppose there exists $\bar{c}_i < 0$, for some $i \in N$.

Let $d = \begin{bmatrix} d_B \\ d_N \end{bmatrix}$ be the moving direction for $x^*$

where $d_i = 1$, and $d_{i'} = 0$ for all other non-basic indices.

Since we still need $(x^* + \theta d)$ is feasible,

then $A(x^* + \theta d) = Ax^*$, $Ad = 0$.

Thus, $0 = [A_B \ A_N] \begin{bmatrix} d_B \\ d_N \end{bmatrix} = A_B d_B + A_i$.

$d_B = -A_B^{-1} A_i$

we get $d = [-A_B^{-1} A_i \ 0 \ 0 \cdots \overset{i}{1} \cdots 0]^T$

Thus we find the new feasible solution, $(x^* + \theta d)$.

and the objective function value is changed

by $c^T \cdot \theta d = (c_i - c_B^T A_B^{-1} A_i) \cdot \theta$

$= \bar{c}_i \cdot \theta < 0$

Since $x^*$ is not degenerate, then we can

find $\theta = \min\limits_{\{i \in B : d_i < 0\}} -\frac{x_i}{d_i}$ and $\theta > 0$.

Thus if we choose $\theta > 0$, then $\bar{c}_i \cdot \theta < 0$.

Thus we get the contradiction, for that $x^*$ is not the optimal solution.

Therefore, the reduced cost $\bar{c}_i \geq 0$, for $\forall i \in N$.