STA4030: Categorical Data Analysis
Multicategory Logit Models

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Agenda

1. 9.1 Multicategory Logit Models
2. 9.2 Cumulative Logit Models
3. 9.3 Adjacent-Category Logit Models

9.1 Multicategory Logit Models

9.1.1 Model description

Let J = number of categories for y

$$\pi_j = p(y=j), \quad \sum_{j=1}^{J} \pi_j = 1.$$

The baseline-Category logits:

$$\log(\frac{\pi_j}{\pi_J}), \;\; j = 1 \ldots J-1.$$

The baseline-category logit model:

$$\log(\frac{\pi_j}{\pi_J}) = \alpha_j + \beta_j x, \;\; j = 1, \ldots, J-1. \tag{9.1}$$

9.1 Multicategory Logit Models

If $J = 2$, Model (9.1) reduces to

$$\log\left(\frac{\pi_1}{1-\pi_1}\right) = \text{logit}(\pi_1) = \alpha_1 + \beta_1 x.$$

From Model (9.1),

$$\begin{aligned} \log\left(\frac{\pi_a}{\pi_b}\right) &= \log\left(\frac{\pi_a/\pi_J}{\pi_b/\pi_J}\right) \\ &= \log\left(\frac{\pi_a}{\pi_J}\right) - \log\left(\frac{\pi_b}{\pi_J}\right) \\ &= (\alpha_a - \alpha_b) + (\beta_a - \beta_b)x. \end{aligned} \qquad (9.2)$$

9.1 Multicategory Logit Models

Example 9.1 Alligator Food Choice

Table 9.1 comes from a study of factors influencing the primary food choice of alligators (Agresti, 2007, p174). For 59 alligators sampled in Lake George, Florida, the table shows

- primary food type: Fish (F), Invertebrate (I), and Other (O).
- alligator length — varied between 1.24 and 3.89 meters

Table 9.1: Alligator size (meters) and primary food choice for 59 Florida Alligators

1.24 I	1.30 I	1.30 I	1.32 F	1.32 F	1.40 F	1.42 I	1.42 F
1.45 I	1.45 O	1.47 I	1.47 F	1.50 I	1.52 I	1.55 I	1.60 I
1.63 I	1.65 O	1.65 I	1.65 F	1.65 F	1.68 F	1.70 I	1.73 O
1.78 I	1.78 I	1.78 O	1.80 I	1.80 F	1.85 F	1.88 I	1.93 I
1.98 I	2.03 F	2.03 F	2.16 F	2.26 F	2.31 F	2.31 F	2.36 F
2.36 F	2.39 F	2.41 F	2.44 F	2.46 F	2.56 O	2.67 F	2.72 I
2.79 F	2.84 F	3.25 O	3.28 O	3.33 F	3.56 F	3.58 F	3.66 F
3.68 O	3.71 F	3.89 F					

9.1 Multicategory Logit Models

Let Y = primary food choice and x = alligator length. For Model (9.1) with J = 3, Table 7.2 shows some output (from PROC LOGISTIC in SAS), with "other" as the baseline category.

The ML prediction equations are

$$\log(\hat{\pi}_1/\hat{\pi}_3) = 1.618 - 0.110x,$$

$$\log(\hat{\pi}_2/\hat{\pi}_3) = 5.697 - 2.465x.$$

9.1 Multicategory Logit Models

Table 7.2 Computer Output for Baseline-Category Logit Model with Alligator Data

Testing Global Null Hypothesis: BETA = 0

Test	Chi-Square	DF	Pf > ChiSq
Likelihood Ratio	16.8006	2	0.0002
Score	12.5702	2	0.0019
Wald	8.9360	2	0.0115

Analysis of Maximum Likelihood Estimates

Parameter	choice	DF	Estimate	Standard Error	Wald Chi-Square	Pr > ChiSq
Intercept	F	1	1.6177	1.3073	1.5314	0.2159
Intercept	I	1	5.6974	1.7938	10.0881	0.0015
length	F	1	-0.1101	0.5171	0.0453	0.8314
length	I	1	-2.4654	0.8997	7.5101	0.0061

Odds Ratio Estimates

Effect	choice	Point Estimate	95% Wald Confidence Limits	
length	F	0.896	0.325	2.468
length	I	0.085	0.015	0.496

9.1 Multicategory Logit Models

From Table 9.2, the estimated log odds that response is "fish" rather "invertebrate" equals

$$\log\left(\frac{\hat{\pi}_1}{\hat{\pi}_2}\right) = (\hat{\alpha}_1 - \hat{\alpha}_2) + (\hat{\beta}_1 - \hat{\beta}_2)x,$$
$$= (1.618 - 5.697) + [-0.110 - (-2465)]x,$$
$$= -4.08 + 2.355x.$$

Larger alligators are more likely to select Fish rather than Invertebrates.

Conditional on the event that the outcome was one of these two categories (Fish and Invertebrate),

$$\log \theta_{x+1,x} = 2.355, \quad \hat{\theta}_{x+1,x} = e^{2.355} = 10.5.$$

9.1 Multicategory Logit Models

i.e. For alligators of length $x+1$ meters, the estimated odds that primary food type is “fish” rather than “invertebrate” equal 10.5 times the estimated odds at length x meters.

We can test the hypothesis that primary food choice is independent of alligator length is $H_0 : \beta_1 = \beta_2 = 0$

$$G^2 = -2\log(\ell_0/\ell_1) = 16.8, \quad df = 2.$$

P-value = 0.0002 provides strong evidence of a length effect.

9.1 Multicategory Logit Models

9.1.2 Estimating Probabilities π_j

From Model (9.1),

$$\log\left(\frac{\pi_j}{\pi_J}\right) = \alpha_j + \beta_j x, \quad j = 1, \ldots, J-1,$$

$$\pi_j = \pi_J \exp(\alpha_j + \beta_j x), \quad 1 = \sum_j \pi_j = \pi_J \sum_j \exp(\alpha_j + \beta_j x).$$

So $$\pi_J = \frac{1}{\sum_j \exp(\alpha_j + \beta_j x)},$$

$$\pi_j = \frac{\exp(\alpha_j + \beta_j x)}{\sum_j \exp(\alpha_j + \beta_j x)}, \quad j = 1, \ldots, J-1,$$

where $\alpha_J = \beta_J = 0$ (baseline category).

9.1 Multicategory Logit Models

We must have $\sum_j \hat{\pi}_j = 1$

$$\hat{\pi}_1 = \frac{e^{1.62-0.11x}}{1+e^{1.62-0.11x}+e^{5.70-2.47x}},$$

$$\hat{\pi}_2 = \frac{e^{5.70-2.47x}}{1+e^{1.62-0.11x}+e^{5.70-2.47x}},$$

$$\hat{\pi}_3 = \frac{1}{1+e^{1.62-0.11x}+e^{5.70-2.47x}}.$$

9.1 Multicategory Logit Models

Table 7.3 Parameter Estimates and Standard Errors (in parentheses) for Baseline-category Logit Model

	Food Choice Categories for Logit	
Parameter	(Fish/Other)	(Invertebrate/Other)
Intercept	1.618	5.697
Length	−0.110 (0.517)	−2.465 (0.900)

E.g. for an alligator of the maximum observed length of x = 3.89 meters, the estimated probability that primary food choice is "other" equals

$$\hat{\pi}_3 = 1/[1 + e^{1.62-0.11(3.89)} + e^{5.70-2.47(3.89)}] = 0.23.$$

9.1 Multicategory Logit Models

Likewise, $\hat{\pi}_1 = 0.76$ and $\hat{\pi}_2 = 0.005$. Very large alligators prefer to eat fish. Figure 7.1 shows the three estimated response probabilities as a function of alligator length.

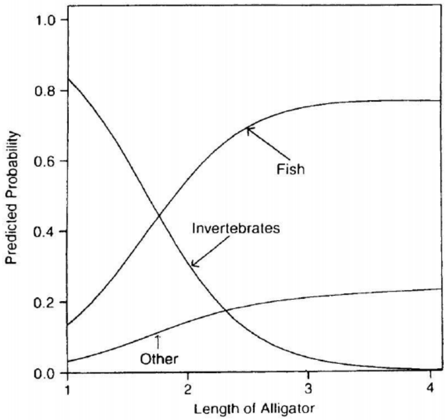

Figure 7.1 Estimated probabilities for primary food choice.

9.1 Multicategory Logit Models

- Example 9.2 Belief in Afterlife (explanatory variables are categorical)

The data are from a General Social Survey.

Y = belief in afterlife (Yes, Undecided, No)
X_1 = gender (1 = females, 0 = males)
X_2 = race (1 = whites, 0 = blacks)

Let “no” be the baseline category for Y, the model is

$$\log(\frac{\pi_j}{\pi_3}) = \alpha_j + \beta_j^G x_1 + \beta_j^R x_2, \quad j = 1, 2,$$

where G and R identity the gender and race parameters.

9.1 Multicategory Logit Models

Table 7.4 Belief in Afterlife by Gender and Race

Race	Gender	Belief in Afterlife: Yes	Undecided	No
White	Female	371	49	74
	Male	250	45	71
Black	Female	64	9	15
	Male	25	5	13

Source: General Social Survey.

Table 7.5 Parameter Estimates and Standard Errors (in parentheses) for Baseline-category Logit Model Fitted to Table 7.4

Parameter	Belief Categories for logit: (Yes/No)	(Undecided/No)
Intercept	0.883 (0.243)	−0.758 (0.361)
Gender ($F = 1$)	0.419 (0.171)	0.105 (0.246)
Race ($W = 1$)	0.342 (0.237)	0.271 (0.354)

9.1 Multicategory Logit Models

In Table 7.5, the effect parameters represent log odds ratios with the baseline category.

e.g. $\beta_1^G = \log \theta_{(y=1,y=3)G|R}$

Since $\hat{\beta}_1^G = 0.419$, then $\hat{\theta}_{y(yes,no)G|R} = e^{0.419} = 1.5.$

i.e. for females the estimated odds of response "yes" rather than "no" on life after death are 1.5 times those for males, controlling for race.

Since $\hat{\beta}_1^R = \log \hat{\theta}_{(y=1,y=3)R|G} = 0.342$, then $\hat{\theta}_{(y=1,y=3)R|G} = 1.4.$

i.e. for whites, the estimated odds of response "yes" rather than "no" on life after death are 1.4 times those for blacks, controlling for gender.

9.1 Multicategory Logit Models

Similar interpretations are applied to

$$\beta_2^G = \log \theta_{(y=2,y=3)G|R} \text{ and } \beta_2^R = \log \theta_{(y=2,y=3)R|G}.$$

Test of gender effect: $H_0 : \beta_1^G = \beta_2^G = 0$

$\mathrm{G}^2 = 7.2$, df = 2, P-value = 0.03 shows evidence of gender effect.

Test of race effect: $H_0 : \beta_1^R = \beta_2^R = 0$

$\mathrm{G}^2 = 2.8$, df = 2, P-value = 0.59. The insignificant race effect may be due to great imbalance in sample sizes.

9.1 Multicategory Logit Models

Table 7.6 Estimated Probabilities for Belief in Afterlife

		Belief in Afterlife		
Race	Gender	Yes	Undecided	No
White	Female	0.76	0.10	0.15
	Male	0.68	0.12	0.20
Black	Female	0.71	0.10	0.19
	Male	0.62	0.12	0.26

Table 7.6 displays estimated probabilities for the three response categories.

9.1 Multicategory Logit Models

For white females:

$$\hat{P}_{(y=1|x_1=1,x_2=1)} = \frac{e^{0.83+0.419+0.342}}{1+e^{0.883+0.419+0.342}+e^{-0.758+0.105+0.271}}$$
$$= 0.76.$$

For black females:

$$\hat{P}_{(y=2|x_1=1,x_2=0)} = \frac{e^{-0.758+0.105}}{1+e^{0.883+0.419}+e^{-0.758+0.105}} = 0.10.$$

For black males:

$$\hat{P}_{(y=3|x_1=0,x_2=0)} = \frac{1}{1+e^{0.883}+e^{-0.758}} = 0.26.$$

For white males:

$$\hat{P}_{(y=3|x_1=0,x_2=1)} = \frac{1}{1+e^{0.883+0.342}+e^{-0.758+0.271}} = 0.20.$$

9.2 Cumulative Logit Models

- Cumulative Logit Models:

For an ordinal response, the logits can utilize the ordering. This results in models that have simpler interpretation and greater power than baseline category logit models. Let

$$P(y \leq j) = \pi_1 + \cdots + \pi_j, \quad j = 1, \ldots, J.$$

The cumulative probabilities reflect the ordering, with

$$P(y \leq 1) \leq P(y \leq 2) \leq \cdots \leq P(y \leq J) = 1.$$

The logits of the cumulative probabilities are

$$\text{logit}[P(y \leq j)] = \log\left[\frac{P(y \leq j)}{1 - P(y \leq j)}\right] = \log\left[\frac{\pi_1 + \cdots + \pi_j}{\pi_{j+1} + \cdots + \pi_J}\right].$$

9.2 Cumulative Logit Models

These are called cumulative logits

e.g. $J = 3$,

$$\mathrm{logit}[P(y \le 1)] = \log \frac{\pi_1}{\pi_2 + \pi_3}, \quad \mathrm{logit}[P(y \le 2)] = \log \frac{\pi_1 + \pi_2}{\pi_3}.$$

The cumulative logit model:

$$\mathrm{logit}[P(y \le j)] = \alpha_j + \beta x, \quad j = 1, \ldots, J-1. \tag{9.3}$$

This model assumes that the effect of x is identical for all $(J-1)$ cumulative logits. When this model fits well, it requires a single parameter rather than $(J-1)$ parameters to describe the effect of x.

9.2 Cumulative Logit Models

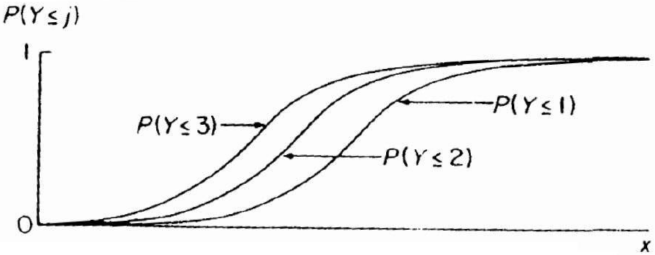

Figure 7.2 Depicition of cumulative probabilities in proportional odds model.

Figure 7.2 shows curves $P(y \leq j)$ with $J = 4$ and a quantitative x $(\beta > 0)$. The curve for $P(y \leq j)$ looks like a logistic regression curve for a binary response with two outcomes $(y \leq j)$ and $(y > j)$. The common effect β for each j implies that the three curves have the same shape, the size of $|\beta|$ determines how quickly the curves climb or drop.

9.2 Cumulative Logit Models

Figure 7.3 shows curves for $P(y = j) = P(y \leq j) - P(y \leq j - 1)$.

When $\beta > 0$, as x increases, y is more likely to fall at the low end of the ordinal scale.

When $\beta < 0$, as x increases, the curves in Figure 7.2 descend, the labels in Figure 7.3 reverse order. As x increase, y is more likely to fall at the high end of the scale.

When $\beta = 0$, the graph has a horizontal line for each $P(y \leq j)$, and x and y are independent.

9.2 Cumulative Logit Models

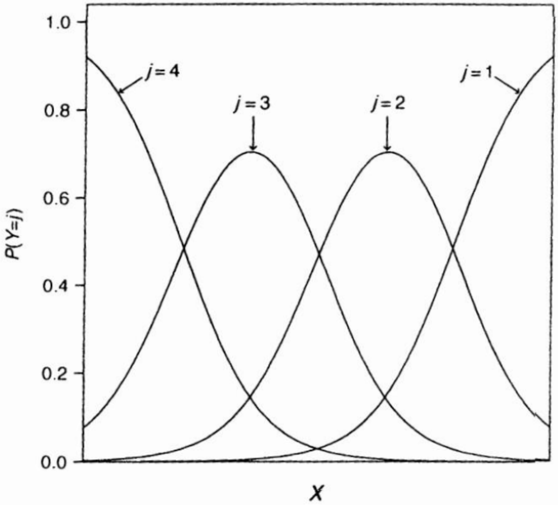

Figure 7.3 Depicitoin of category probabilities in propotional odds model. At any particular x value, the probabilities sum to 1.

9.2 Cumulative Logit Models

- Odds ratio interpretation

For two values x and $x+1$, an odds ratio comparing the cumulative probabilities is

$$\frac{P(y \le j|X = x+1)}{1 - P(y \le j|X = x+1)} \Bigg/ \frac{P(y \le j|X = x)}{1 - P(y \le j|X = x)},$$

$$= \frac{P(y \le j|X = x+1)/P(y > j|X = x+1)}{P(y \le j|X = x)/P(y > j|X = x)}.$$

9.2 Cumulative Logit Models

From Model (9.3) (the cumulative logit models),

$$\mathrm{logit}[P(y \leq j|X = x+1)] - \mathrm{logit}[P(y \leq j|X = x)] = \beta,$$

or

$$\log \frac{P(y \leq j|X = x+1)/P(y > j|X = x+1)}{P(y \leq j|X = x)/P(y > j|X = x)} = \beta,$$

or

$$\theta_{x,x+1} = \frac{P(y \leq j|X = x+1)/P(y > j|X = x+1)}{P(y \leq j|X = x)/P(y > j|X = x)} = e^{\beta}.$$

The odds of response below any given category multiply by e^{β} for each unit increase in x.

9.2 Cumulative Logit Models

For the log odds ratio β, the same proportionality constant (β) applies for each cumulative probability $P(y \leq j)$. This property is called the proportional odds assumption of the model:

$$\text{logit}[P(y \leq j)] = \alpha_j + \beta x, \quad j = 1, \ldots, J-1,$$

where x can be quantitative, categorical, or both types.

- Example 9.3 Political Ideology and Party Affiliation

The data, from a Social Survey, relate political ideology to political party affiliation in Table 9.7.

9.2 Cumulative Logit Models

Table 7.7 Political Ideology by Gender and political Party

Gender	Political Party	Political Ideology: Very Liberal	Slightly Liberal	Moderate	Slightly Conservative	Very Conservative
Female	Democratic	44	47	118	23	32
	Republican	18	28	86	39	48
Male	Democratic	36	34	53	18	23
	Republican	12	18	62	45	51

Source: General Social Survey.

Let

$$X = \text{Political Party } (1 = \text{Democrats}, 0 = \text{Republication}),$$
$$Y = j, \quad j = 1 \text{ (Very Liberal)}, \ldots, 5 \text{ (Very Conservative)}.$$

9.2 Cumulative Logit Models

Table 7.8 Computer Output (SAS) for Cumulative Logit Model with Political Ideology Data

Analysis of Maximum Likelihood Estimates

Parameter	DF	Estimate	Standard Error	Wald Chi-Square	Pr > ChiSq
Intercept 1	1	−2.4690	0.1318	350.8122	<.0001
Intercept 2	1	−1.4745	0.1091	182.7151	<.0001
Intercept 3	1	0.2371	0.0948	6.2497	.0124
Intercept 4	1	1.0695	0.1046	104.6082	<.0001
party	1	0.9745	0.1291	57.0182	<.0001

Odds Ratio Estimates

Effect	Point Estimate	95% Wald Confidence Limits	
party	2.650	2.058	3.412

Testing Global Null Hypothesis: BETA = 0

Test	Chi-Square	DF	Pr > ChiSq
Likelihood Ratio	58.6451	1	<.0001
Score	57.2448	1	<.0001
Wald	57.0182	1	<.0001

Deviance and Pearson Goodness-of-Fit Statistics

Criterion	Value	DF	Value/DF	Pr > ChiSq
Deviance	3.6877	3	1.2292	0.2972
Pearson	3.6629	3	1.2210	0.3002

9.2 Cumulative Logit Models

$\hat{\beta} = 0.975(SE = 0.129)$. For any fixed j, the estimated odds that a Democrat's response is in the liberal direction rather than the conservative direction (i.e. $y \leq j$ rather $y > j$) equal $e^{0.975} = 2.65$ times the estimated odds for Republicans. So, Democrats tend to be more liberal than Republicans.

From the cumulative logit model (9.3),

$$P(y \leq j) = \frac{\exp(\alpha_j + \beta x)}{1 + \exp(\alpha_j + \beta x)},$$

So for Democrats,

$$\hat{P}(y \leq 1) = \frac{\exp[-2.469 + 0.975(1)]}{1 + \exp[-2.469 + 0.975(1)]} = 0.18.$$

Similarly, $\hat{P}(y \leq 2) = 0.38, \ \ \hat{P}(y \leq 3) = 0.77, \ \ \hat{P}(y \leq 4) = 0.89.$

9.2 Cumulative Logit Models

Since

$$P(y=j) = P(y \le j) - P(y \le j-1),$$

then

$$\hat{P}(y=2) = P(y \le 2) - P(y \le 1) = 0.38 - 0.18 = 0.2,$$
$$\hat{P}(y=3) = P(y \le 3) - P(y \le 2) = 0.77 - 0.38 = 0.39,$$
$$\hat{P}(y=4) = P(y \le 4) - P(y \le 3) = 0.89 - 0.77 = 0.12,$$
$$\hat{P}(y=5) = P(y \le 5) - P(y \le 4) = 1 - 0.89 = 0.11,$$
$$\hat{P}(y=1) = \hat{P}(y \le 1) = 0.18, \quad \sum_{j=1}^{5} \hat{\pi}_j = 1.$$

9.2 Cumulative Logit Models

Test independence: $H_0 : \beta = 0$

Table 7.8 reports $G^2 = 58.6$ (likelihood ratio test), with df = 1, P-value < 0.0001 provides a strong evidence of political party.

A 95% confidence interval for β is:

$$0.975 \pm 1.96 \times 0.129 \text{ or } (0.72, 1.23).$$

A 95% confidence interval for odds ratio of $P(y \leq j)$ is:

$$[\exp(0.72), \exp(1.23)] \text{ or } (2.1, 3.4).$$

The odds of being at the liberal end of the political ideology scale is at least twice as high for Democrats as for Republicans.

9.2 Cumulative Logit Models

- Goodness-of-fit Test:

The Pearson χ^2 and deviance G^2 statistics compare ML fitted cell counts under the model to the observed cell counts. Table 7.8 reports $X^2 = 3.7, G^2 = 3.7, df = 3$, and P-value = 0.3. Thus, the model fits adequately.

- Remark:

A more complex model is

$$\text{logit}[P(y \leq j)] = \alpha_j + \beta_j x, \quad j = 1, \ldots, J-1,$$

which implies that curves for different $P(y \leq j)$ cross at certain x values. This is inappropriate because it violates the order of $P(y \leq j)$. e.g. $P(y \leq j) \leq P(y \leq j+1)$ at any x. Proportional odds solve this problem.

9.2 Cumulative Logit Models

- Example 9.4 Modeling Mental Health

Y – Mental impairment (ordinal)
1 = well
2 = mild symptom formation
3 = moderate symptom formation
4 = impairment

X_1 – life event index (a composite measure of the number and severity of important life events, such as birth of child, new job, divorce, death in family within 3 years)

X_2 – socioeconomic status (SES) (1 = high, 0 = low)

9.2 Cumulative Logit Models

Table 7.9 Mental Impairment by SES and Life Events

Subject	Mental Impairment	SES	Life Events	Subject	Mental Impairment	SES	Life Events
1	Well	1	1	21	Mild	1	9
2	Well	1	9	22	Mild	0	3
3	Well	1	4	23	Mild	1	3
4	Well	1	3	24	Mild	1	1
5	Well	0	2	25	Moderate	0	0
6	Well	1	0	26	Moderate	1	4
7	Well	0	1	27	Moderate	0	3
8	Well	1	3	28	Moderate	0	9
9	Well	1	3	29	Moderate	1	6
10	Well	1	7	30	Moderate	0	4
11	Well	0	1	31	Moderate	0	3
12	Well	0	2	32	Impaired	1	8
13	Mild	1	5	33	Impaired	1	2
14	Mild	0	6	34	Impaired	1	7
15	Mild	1	3	35	Impaired	0	5
16	Mild	0	1	36	Impaired	0	4
17	Mild	1	8	37	Impaired	0	4
18	Mild	1	2	38	Impaired	1	8
19	Mild	0	5	39	Impaired	0	8
20	Mild	1	5	40	Impaired	0	9

9.2 Cumulative Logit Models

Table 7.10 Output for Fitting Cumulative Logit Model to Table 6.9

Score Test for the Proportional Odds Assumption

Chi-Square	DF	Pr > ChiSq
2.3255	4	0.6761

Parameter	Estimate	Std Error	Like Ratio 95% Conf Limits		Chi-Square	Pr > ChiSq
Intercept1	−0.2819	0.6423	−1.5615	0.9839	0.19	0.6607
Intercept2	1.2128	0.6607	−0.0507	2.5656	3.37	0.0664
Intercept3	2.2094	0.7210	0.8590	3.7123	9.39	0.0022
life	−0.3189	0.1210	−0.5718	−0.0920	6.95	0.0084
ses	1.1112	0.6109	−0.0641	2.3471	3.31	0.0689

The cumulative logit model is:

$$\text{logit}[P(y \leq j)] = \alpha_j + \beta_1 x_1 + \beta_2 x_2.$$

9.2 Cumulative Logit Models

$\hat{\beta}_1 = -0.319$ and $\hat{\beta}_2 = 1.111$ suggest that at $P(y \leq j)$ starting at the "well" end of the scale decreases as life events increases and increases at the higher level of SES.

$$\text{e.g. } \log \frac{P(y \leq j | x_1, x_2 = 1)/P(y > j | x_1, x_2 = 1)}{P(y \leq j | x_1, x_2 = 0)/P(y > j | x_1, x_2 = 0)} = \beta_2,$$

$$\hat{\theta}_{x_2 y(x_1)} = e^{\hat{\beta}_2} = e^{1.111} = 3.0.$$

Given life events score (x_1), at the high SES level $(x_2 = 1)$ the estimated odds of mental impairment below any fixed level $(y \leq j)$ are 3.0 times the estimated odds at the low SES level $(x_2 = 0)$.

9.2 Cumulative Logit Models

The estimated probabilities and interpretation:

$$P(y \le j) = \frac{\exp(\alpha_j + \beta_1 x_1 + \beta_2 x_2)}{1 + \exp(\alpha_j + \beta_1 x_1 + \beta_2 x_2)},$$

$$P(y = 1) = P(y \le 1),$$

$$P(y = j) = P(y \le j) - P(y \le j-1), \quad j = 2, 3, 4.$$

When controlling x_1, e.g. x_1 = 4.3 (the mean life events),

at $x_2 = 1$ (high SES)

$$\hat{P}(y = 1) = \frac{\exp[-0.282 - 0.319(4.3) + 1.111(1)]}{1 + \exp[-0.282 - 0.319(4.3) + 1.111(1)]} = 0.37,$$

at $x_2 = 0$ (low SES)

$$\hat{P}(y = 1) = \frac{\exp[-0.282 - 0.319(4.3) + 1.111(0)]}{1 + \exp[-0.282 - 0.319(4.3) + 1.111(0)]} = 0.16.$$

9.2 Cumulative Logit Models

When controlling x_2, e.g. $x_2 = 1$,

at $x_1 = 2.0$ (low quartile for life events)

$$\hat{P}(y = 1) = \frac{\exp[-0.282 - 0.319(2.0) + 1.111(1)]}{1 + \exp[-0.282 - 0.319(2.0) + 1.111(1)]} = 0.55,$$

at $x_1 = 6.5$ (upper quartile for life events)

$$\hat{P}(y = 1) = \frac{\exp[-0.282 - 0.319(6.5) + 1.111(1)]}{1 + \exp[-0.282 - 0.319(6.5) + 1.111(1)]} = 0.22.$$

9.3 Adjacent-Category Logit Models

- Adjacent-Category Logit Models:

Adjacent-Category logits:

$$\log\left(\frac{\pi_{j+1}}{\pi_j}\right), j = 1, \ldots, J-1.$$

The adjacent-category logit model:

$$\log\left(\frac{\pi_{j+1}}{\pi_j}\right) = \alpha_j + \beta_j x, \quad j = 1, \ldots, J-1. \tag{9.4}$$

A simpler proportional odds version is:

$$\log\left(\frac{\pi_{j+1}}{\pi_j}\right) = \alpha_j + \beta x, \quad j = 1, \ldots, J-1. \tag{9.5}$$

The adjacent-category logits, like the baseline-category logits, determine the logits for all pairs of response categories.

9.3 Adjacent-Category Logit Models

From Model (9.5),

$$\log \frac{(\pi_{j+1}/\pi_j)_{x+1}}{(\pi_{j+1}/\pi_j)_x} = \beta,$$

and

$$\log \frac{(\pi_a/\pi_b)_{x+1}}{(\pi_a/\pi_b)_x} = \beta(a-b).$$

The effect depends on the distance between categories.
So, this model recognizes the ordering of the response scale.

9.3 Adjacent-Category Logit Models

- Example 9.5 Political Ideology Revisited

Reanalyze Example 9.3 using adjacent categories logit model of proportional odds form: $\log(\frac{\pi_{j+1}}{\pi_j}) = \alpha_j + \beta x, \;\; j = 1, \ldots, J-1.$

$Y = j$ (very liberal, slightly liberal, moderate, slightly conservative, very conservative), $j = 1, 2, 3, 4, 5.$

$$X = \begin{cases} 1, & \text{Republican} \\ 0, & \text{Democrats} \end{cases}$$

Software reports $\hat{\beta} = 0.435$, and $\exp(\hat{\beta}) = 1.54.$

9.3 Adjacent-Category Logit Models

The estimated odds that a Republican's ideology classification is in category $(j+1)$ instead of j are 1.54 times the estimated odds for Democrats. e.g. the estimated odds of "slightly conservative" instead of "moderate" ideology are 54% higher for Republicans than for Democrats.

Since

$$\log \frac{(\pi_a/\pi_b)_{x+1}}{(\pi_a/\pi_b)_x} = \beta(a-b) \text{ or } \frac{odds_{(x+1)}}{odds_{(x)}} = \exp[\beta(a-b)],$$

the estimated odds that a Republican's ideology is "very conservative" (category 5) instead of "very liberal" (category 1) are $\exp[0.435(5-1)] = (1.54)^4 = 5.7$ times those for Democrats.

(End of Chapter 9)