Stochastic Processes

Lecture 12: Period, Limiting behavior, Finite state DTMC

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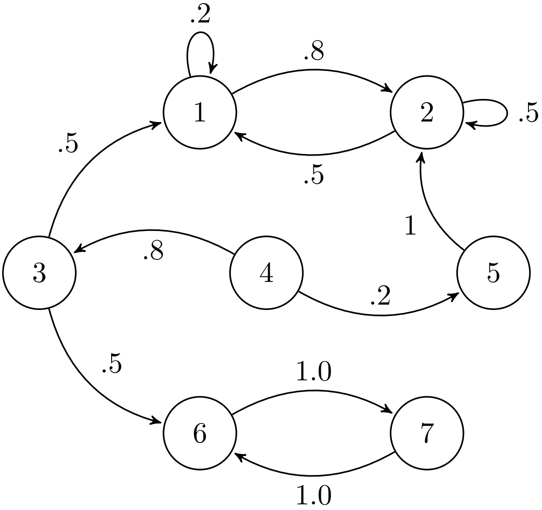
Consider the following DTMC.



Limiting distribution?

- $\lim_{n\to\infty} P^n$ does not exist. $\lim_{n\to\infty}(P^n + P^{n+1})/2$ exists.

$$\begin{pmatrix} 5/13 & 8/13 & 0 & 0 & 0 & 0 & 0 \\ 5/13 & 8/13 & 0 & 0 & 0 & 0 & 0 \\ (1/2)(5/13) & (1/2)(8/13) & 0 & 0 & 0 & (1/2)(.5) & (1/2)(.5) \\ (.6)(5/13) & (.6)(8/13) & 0 & 0 & 0 & (.4)(.5) & (.4)(.5) \\ 5/13 & 8/13 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & .5 & .5 \\ 0 & 0 & 0 & 0 & 0 & .5 & .5 \end{pmatrix}$$

Periodicity

Definition

The *period* of state i of a DTMC is $d(i) = \gcd\{n : P_{ii}^n > 0\}$.

Theorem (Solidarity property)

If state i and j communicate, then $d(i) = d(j)$.

- Assume $P_{ij}^{k_1} > 0$ and $P_{ji}^{k_2} > 0$. For $k \geq 0$,

$$P_{ii}^{k+k_1+k_2} \geq P_{ij}^{k_1} P_{jj}^{k} P_{ji}^{k_2}$$

- Take $k = 0$, $P_{ii}^{k_1+k_2} > 0$, which implies $d(i) \mid k_1 + k_2$.
- Whenever $P_{jj}^k > 0$, $P_{ii}^{k+k_1+k_2} > 0$, thus, $d(i) \mid k + k_1 + k_2$, which implies $d(i) \mid k$. Thus, $d(i) \leq d(j)$.

Periodicity and limit

Definition

An irreducible DTMC is *aperiodic* if $d = 1$. Otherwise, it's *periodic*.

Theorem

If an irreducible DTMC is aperiodic, then

$$\lim_{n\to\infty} P^n = P^{(\infty)}$$

exists, where $P_{ij}^{(\infty)} = 1/\mathbb{E}_j(T_j)$. *Therefore, when the DTMC is positive recurrent, every row of the limiting matrix* $P^{(\infty)}$ *is equal to the DTMC's stationary distribution* π.

The Theorem is false if the DTMC is periodic, but...

Limit of periodic DTMC

Recall

THEOREM

If an irreducible DTMC is periodic, then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} P^n = P^{(\infty)}$$

exists. ...

Limiting distribution and Stationary distribution

Random walks on circles

- R.w. on a circle of three points.
- R.w. on a circle of four points.

Communicating classes

Definition

(a) A set $C \subset S$ is said to be a communicating class if $i \in C$ and $i \leftrightarrow j$ imply $j \in C$.

(b) A communicating class is said to be *closed* if $i \in C$ and $i \to j$ imply $j \in C$.

Theorem

Let C be a communicating class. Then either all states in C are transient or all are recurrent.

Theorem

Every recurrent class is closed.

Corollary

For a finite state DTMC, there exists at least one (closed) recurrent class.

Corollary

For a finite state DTMC, at least one state is positive recurrent.

Proof.

Let C be a (closed) recurrent class. Restricting on C, the DTMC is a finite-state, irreducible DTMC. Suppose that every state is not positive recurrent. Then for each $i, j \in S$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (P^k)_{ji} = 0.$$

Note that for each n and state state i, $\frac{1}{n} \sum_{i\in S} \sum_{k=1}^{n} (P^k)_{ji} = 1$. Taking $n \to \infty$ on both sides, one has

$$1 = \lim_{n\to\infty} \frac{1}{n} \sum_{i\in S} \sum_{k=1}^{n} (P^k)_{ji} = \sum_{i\in S} \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} P_{ji}^{k} = \sum_{i\in S} 0 = 0.$$

□

Finite state DTMC

Corollary

For a DTMC having finitely many states, there is no state that is null recurrent.