MAT3253 Tutorial 2

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1 Analytic Function

Definition 1. *A domain is a connected open set in the complex plane, denoted by D.*

Definition 2. *A complex function of a complex variable $f(z)$ is analytic in a domain if $f'(z)$ exists everywhere on this domain.*

Definition 3. *$f(z)$ is analytic at a point z is it is analytic in a neighborhood of that point.*

1.1 $f(z)$ is analytic on D, then:

- Write $f(z) = u(x, y) + iv(x, y)$
- $f(z)$ is continuous on D.
  - u, v is continuous on D.
- The partial derivatives of u, v, u_x, u_y, v_x, v_y all exist on D, and they satisfy the Cauchy-Riemann equation: $u_x = v_y; u_y = -v_x$
- With the fact that $f'(z)$ is also analytic on D, we further have
  - The partial derivatives of all orders of u, v exist and are continuous on D.
  - As a consequence, the mixed partial derivatives are equal.
  - As a consequence, u, v and all its partial derivatives are also differentiable.

1.2 Exercises

1. The Cauchy Riemann Theorem is formally stated as follows:

   Theorem 1. ***If*** *$f(z) = u(x, y) + iv(x, y)$ is analytic on D,* ***then*** *u_x, u_y, v_x, v_y all exist and are continuous on D,moreover, they satisfy $u_x = v_y; u_y = -v_x$*

   Show the converse is also true.

2. Prove the product, sum, quotient of two analytic functions defined on the same domain D is again analytic on D, provided of course in the quotient case the denominator never vanishes on D.

3. Prove if $f(z)$ is analytic at a point z_0, $g(w)$ is analytic at the point $w_0 = g((z_0)$, then the $g(f(z))$ is analytic at z_0.