1. (a) $\frac{1}{6+2i}=\frac{6-2i}{(6+2i)(6-2i)}=\frac{6-2i}{40}=\frac{3}{20}-\frac{1}{20}i$

(b) $\frac{(2+i)(3+2i)}{1-i}=\frac{4+7i}{1-i}=\frac{(4+7i)(1+i)}{(1-i)(1+i)}=\frac{-3+11i}{2}=-\frac{3}{2}+\frac{11}{2}i$

(c) $\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^4=\left(\cos\frac{2}{3}\pi+i\sin\frac{2}{3}\pi\right)^4$

$=\cos\left(4\cdot\frac{2}{3}\pi\right)+i\sin\left(4\cdot\frac{2}{3}\pi\right)$

$=\cos\frac{2}{3}\pi+i\sin\frac{2}{3}\pi$

$=-\frac{1}{2}+\frac{\sqrt{3}}{2}i$

(d) $i^2=-1,\ i^3=-i,\ i^4=1,\ i^5=i$

2. proof. Suppose $P(x)=a_0+a_1x+a_2x^2+\cdots+a_nx^n$,

where $a_k\in\mathbb{R}$, for $\forall k=0,1,\cdots,n$.

① "$\Rightarrow$" Since $P(z)=0$, that is $a_0+a_1z+\cdots+a_nz^n=0$

then $\overline{a_0+a_1z+\cdots+a_nz^n}=0$,

$\overline{a_0}+\overline{a_1z}+\cdots+\overline{a_nz^n}=0$,

$\overline{a_0}+\overline{a_1}\cdot\overline{z}+\cdots+\overline{a_n}\cdot\overline{z}^n=0$

Since $a_0=\overline{a_0},\ a_1=\overline{a_1},\ \cdots\ a_n=\overline{a_n}$,

then $a_0+a_1\cdot\overline{z}+\cdots+a_n\overline{z}^n=0$. $\Rightarrow P(\overline{z})=0$.

② "$\Leftarrow$" Since $P(\overline{z})=0$, that is $a_0+a_1\overline{z}+\cdots+a_n\overline{z}^n=0$

and $a_0=\overline{a_0},\ a_1=\overline{a_1},\ \cdots\ a_n=\overline{a_n}$.

then $\overline{a_0}+\overline{a_1}\cdot\overline{z}+\cdots+\overline{a_n}\cdot\overline{z}^n=0$,

$\overline{a_0}+\overline{a_1z}+\cdots+\overline{a_nz^n}=0$,

$\overline{a_0+a_1z+\cdots+a_nz^n}=0$,

$a_0+a_1z+\cdots+a_nz^n=0\Rightarrow P(z)=0$

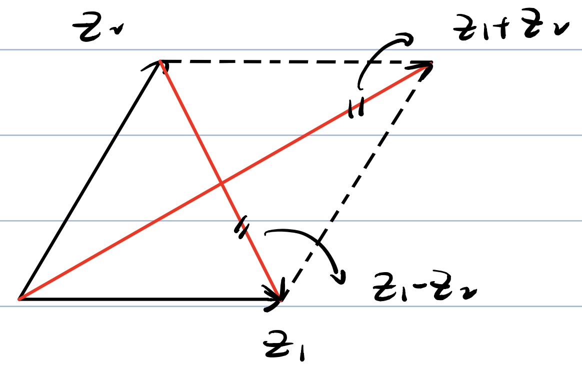
Therefore, $P(z)=0$ iff $P(\overline{z})=0$.

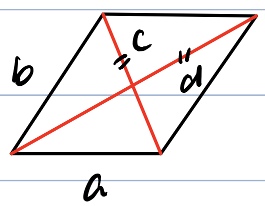
3. proof. Assume $z_1 = a+bi$, $z_2 = c+di$, then

$$|z_1+z_2|^2 + |z_1-z_2|^2 = (a+c)^2 + (b+d)^2 + (a-c)^2 + (b-d)^2$$
$$= 2(a^2+c^2) + 2(b^2+d^2)$$

$$2(|z_1|^2 + |z_2|^2) = 2(a^2+b^2+c^2+d^2)$$
$$= 2(a^2+c^2) + 2(b^2+d^2)$$

Thus, $|z_1+z_2|^2 + |z_1-z_2|^2 = 2(|z_1|^2+|z_2|^2)$





The equation $|z_1+z_2|^2 + |z_2-z_2|^2 = 2(|z_1|^2+|z_2|^2)$ means that, in the parallelogram, we have
$$2(a^2+b^2) = c^2+d^2$$

4. Proof. Assume $z = a+bi$, then $iz = -b+ai$

(a) $Re(iz) = -b$, $Im(z) = b \Rightarrow Re(iz) = -Im(z)$

(b) $Im(iz) = a$, $Re(z) = a \Rightarrow Im(iz) = Re(z)$

5. Since $(x,y)(x,y) + (x,y) + (1,0)$
$$= (x^2-y^2, 2xy) + (x,y) + (1,0)$$
$$= (x^2-y^2+x+1, 2xy+y),$$

then let $(x^2-y^2+x+1, 2xy+y) = (0,0)$

we get $\begin{cases} x^2-y^2+x+1 = 0 & ① \\ 2xy+y = 0 & ② \end{cases}$

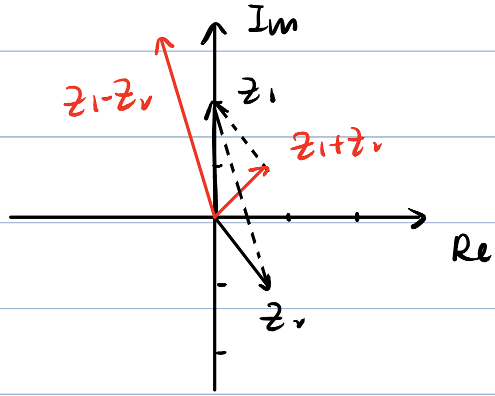
② $\Rightarrow$ $y(2x+1)=0$. $\Rightarrow$ $y=0$ or $x=-\frac{1}{2}$.

If $y=0$, then $x^2+x+1=0$, no real solution.

If $x=-\frac{1}{2}$, then $y=\pm\sqrt{x^2+x+1}=\pm\frac{\sqrt{3}}{2}$

Thus, $(x,y)=(-\frac{1}{2},\pm\frac{\sqrt{3}}{2})$

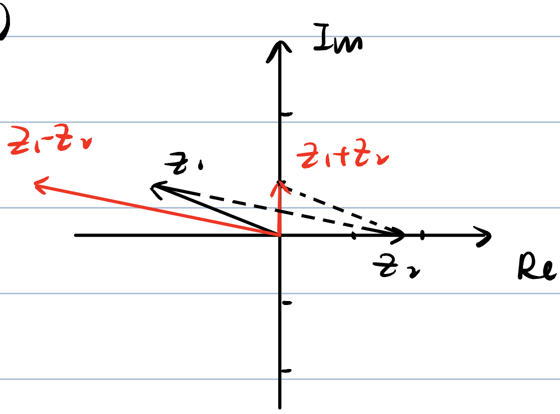
6. (a)



$z_1+z_2=\frac{2}{3}+i$

$z_1-z_2=-\frac{2}{3}+3i$

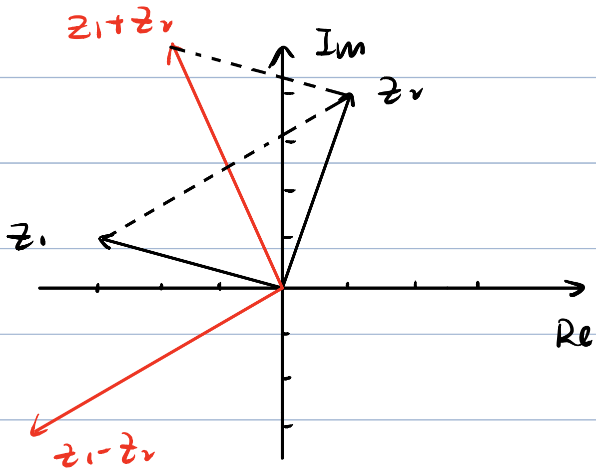
(b)



$z_1+z_2=i$

$z_1-z_2=-2\sqrt{3}+i$

(c)



$z_1+z_2=-2+5i$

$z_1-z_2=-4-3i$

(d) $z_1+z_2=2x_1$, z_1+z_2 lies on real axis.

$z_1-z_2=2y_1i$, z_1-z_2 lies on imaginary axis.