MAT2002 Ordinary Differential Equations
First-order equations II

Dongdong He

The Chinese University of Hong Kong (Shenzhen)

January 21, 2021

Overview

Outline

Separable equations

The theory of first order linear ODEs is complete with the method of integrating factors. We now turn to a subclass of ODEs that can be **non-linear**.

Example 3.1

Solve the following first order non-linear, non-autonomous ODE

$$\begin{cases} \frac{dy}{dt} = \frac{\sin(t)}{1-y^2}, \\ y(t_0) = y_0. \end{cases}$$

Idea: Bring the "y" to the LHS. Rearranging the ODE gives

$$(1-y^2)\frac{dy}{dt} = \sin(t).$$

Separable equations

Recognise that the LHS can be expressed as $\frac{d}{dt}H(y(t))$ by the Chain rule. In fact $H(y) = y - \frac{1}{3}y^3$. Hence, the general solution is

$$y(t) - \frac{1}{3}y(t)^3 = -\cos(t) + c, \quad c \in \mathbb{R}.$$

Using the initial condition, the particular solution is

$$y(t) - \frac{1}{3}y(t)^3 = \cos(t_0) - \cos(t) + y_0 - \frac{1}{3}y_0^3.$$

One thing to observe is that **there is no explicit expression** for $y(t)$ (due to the non-linear function $y(t)^3$). We call this an **implicit solution** to the ODE. This is a typical characteristic of non-linear ODEs.

Separable equations

Definition 3.1

(Separable equation). A first order ODE $y' = f(t, y)$ is **separable** if it can be written in the form

$$M(t) + N(y)\frac{dy}{dt} = 0 \tag{1}$$

for some functions M and N.

The key to solve separable equations is to recognise that $N(y)\frac{dy}{dt}$ can be written as $\frac{d}{dt}(n(y(t)))$ by the Chain rule if the anti-derivative n of N exists. Suppose there exist functions m and n such that

$$m' = M, \quad n' = N.$$

Then (1) can be written as

$$\frac{d}{dt}m(t) + \frac{d}{dt}n(y(t)) = 0.$$

Separable equations

Integrating yields the general (implicit) solution

$$\boxed{m(t) + n(y(t)) = c}, \quad c \in \mathbb{R} \tag{2}$$

For the initial data $y(t_0) = y_0$ we compute to find the particular (implicit) solution

$$\boxed{n(y(t)) - n(y_0) = m(t_0) - m(t)}. \tag{3}$$

Separable equations

Example 3.2

Let us return to the ODE $y' = p(t)y$ which has been discussed. This is a separable equation with

$$y' = p(t)y \Rightarrow -p(t) + \frac{1}{y}\frac{dy}{dt} = 0 \Rightarrow M(t) = -p(t), \quad N(y) = \frac{1}{y}.$$

Hence, by the formula (2) the general solution is

$$\begin{aligned} & -\int p(t)dt + \ln|y(t)| = c \\ \Rightarrow & y(t) = \pm \exp\left(\int p(t)dt\right)\exp(c) \\ \Rightarrow & y(t) = \kappa \exp\left(\int p(t)dt\right), \quad \kappa = \begin{cases} \exp(c), & \textit{if } y(t) > 0, \\ -\exp(c), & \textit{if } y(t) < 0, \end{cases} \end{aligned}$$

Separable equations

Example 3.3

Show that

$$\frac{dy}{dt} = \frac{t^2}{1 - y^2},$$

is separable, and then find its general solution.
The above equation can be rewritten as

$$-t^2 + (1 - y^2)\frac{dy}{dt} = 0,$$

which is equivalent to

$$\frac{d(-\frac{1}{3}t^3)}{dt} + \frac{d(y - \frac{1}{3}y^3)}{dt} = 0.$$

Thus,

$$-\frac{1}{3}t^3 + y - \frac{1}{3}y^3 = c.$$

Outline

Nonlinear to Linear: Bernoulli equation

Let n be a real number, $n \neq 0, 1$, and $p(t), q(t)$ be given functions. The **Bernoulli equation** is a first order non-linear ODE of the form

$$\boxed{\frac{dy}{dt} + p(t)y = q(t)y^n}. \tag{4}$$

As this is a non-linear ODE we cannot use integrating factors. Moreover, it doesn't seem like the equation is separable. Let us move the nonlinearity y^n to the derivative by multiplying the whole equation with y^{-n}:

$$y^{-n}\frac{dy}{dt} + p(t)y^{1-n} = q(t). \tag{5}$$

Now recognise that

$$\frac{d}{dt}\left(y^{1-n}\right) = (1-n)y^{-n}\frac{dy}{dt}.$$

Nonlinear to Linear: Bernoulli equation

So (5) can be simplified to

$$\frac{d}{dt}y^{1-n} + (1-n)p(t)y^{1-n} = (1-n)q(t). \tag{6}$$

Then, considering a **new variable** $v(t) = y^{1-n}(t)$, (6) becomes

$$\boxed{\frac{dv}{dt} + P(t)v = Q(t)}, \quad P(t) = (1-n)p(t), \quad Q(t) = (1-n)q(t), \tag{7}$$

which is a **linear** ODE for the variable v, and we can use integrating factors to solve. Let $\mu(t)$ be the integrating factor for (7), then the general solution is

$$v(t) = \frac{1}{\mu(t)}\left[\int Q(t)\mu(t)dt + c\right]$$

$$\Rightarrow \boxed{y(t) = \left(\frac{1}{\mu(t)}\left[\int Q(t)\mu(t)dt + c\right]\right)^{\frac{1}{1-n}}}.$$

Nonlinear to Linear: Bernoulli equation

The take-away message is that sometimes we can **transform** a non-linear ODE to a linear ODE, and using integrating factors to obtain the solution. Always try to look for suitable transformations!

Homogeneous equations

Definition 3.2

(Homogeneous first order equation). A first order ODE $\frac{dy}{dt} = f(t, y)$ is called **homogeneous** if the function f only depends on the **ratio** $\frac{y}{t}$. That is, we can express

$$f(t, y) = F(y/t) \text{ for some function } F.$$

Homogeneous equations

So how do we solve an ODE of the form $\frac{dy}{dt} = F(y/t)$? The answer is to use a transformation. Define a new variable $v = y/t \Leftrightarrow y = vt$. Then, the RHS of the ODE becomes just $F(v)$. For the LHS, by the product rule

$$y(t) = tv(t) \Rightarrow \frac{dy}{dt} = t\frac{dv}{dt} + v$$

$$\Rightarrow \boxed{t\frac{dv}{dt} + v(t) = F(v)}.$$

Note that the initial condition $y(t_0) = y_0$ also transforms:

$$y(t_0) = y_0 \Rightarrow \boxed{t_0 v(t_0) = y_0},$$

and it is important to see that if $y_0 \neq 0$ then we cannot choose $t_0 = 0$, otherwise we get a contradiction.

Homogeneous equations

The transformed ODE in the variable v is now

$$\frac{dv}{dt} = \frac{F(v) - v}{t} \Rightarrow \boxed{\frac{1}{F(v) - v}\frac{dv}{dt} = \frac{1}{t}},$$

which is a **separable equation**!

Homogeneous equations

Example 3.4

Consider the ODE

$$\frac{dy}{dt} = \frac{y-4t}{t-y} = f(t,y).$$

Dividing numerator and denominator by t leads to

$$f(t,y) = \frac{y-4t}{t-y} = \frac{y/t-4}{1-y/t} = F(y/t), \text{ where } F(s) = \frac{s-4}{1-s}.$$

Using a transformation $y = tv$ we find that v satisfied

$$\frac{1}{F(v)-v}\frac{dv}{dt} = \frac{1}{t} \Rightarrow \frac{1-v}{(v-2)(v+2)}\frac{dv}{dt} = \frac{1}{t}.$$

Using partial fractions the coefficient can be simplified to

$$\frac{1-v}{(v-2)(v+2)} = -\frac{1}{4}\frac{1}{v-2} - \frac{3}{4}\frac{1}{v+2}$$

Homogeneous equations

Example 3.4

Then, integrating gives the general solution

$$-\frac{1}{4}\ln|v-2|-\frac{3}{4}\ln|v+2|=\ln|t|+c$$
$$\Rightarrow -\frac{1}{4}\ln|y(t)/t-2|-\frac{3}{4}\ln|y(t)/t+2|=\ln|t|+c.$$

This gives

$$|y(t)/t-2|^{-1/4}|y(t)/t+2|^{-3/4}=e^c|t|, c\in\mathbb{R}.$$

It can be rewritten as

$$t|y(t)/t-2|^{1/4}|y(t)/t+2|^{3/4}=k, k\in\mathbb{R}.$$

(This form includes two particular solutions: $y(t)/t-2=0$ and $y(t)/t+2=0$).

Outline

Exact equations

We recall that an ODE is **separable** if it can be expressed in the form

$$M(t) + N(y)\frac{dy}{dt} = 0,$$

for some functions M and N such that their anti-derivative exist. What if M and N depend on both t and y? That is, we encounter an ODE of the form

$$M(t, y) + N(t, y)\frac{dy}{dt} = 0. \tag{8}$$

Example 3.5

The ODE

$$2t + y^2 + 2ty\frac{dy}{dt} = 0$$

is a non-linear, non-autonomous ODE with $M(t, y) = 2t + y^2$ and $N(t, y) = 2ty$.

Exact equations

Idea: If the LHS of $M(t,y) + N(t,y)\frac{dy}{dt} = 0$ can be written as $\frac{d\Psi(t,y(t))}{dt}$ for some function $\Psi(t,y)$, then the ODE can be solved simply by integration. And the requirement for $\frac{d\Psi(t,y(t))}{dt} = M(t,y) + N(t,y)\frac{dy}{dt}$ implies

$$\frac{\partial \Psi}{\partial y}(t,y) = N(t,y), \quad \frac{\partial \Psi}{\partial t}(t,y) = M(t,y)$$

since $\frac{d\Psi(t,y(t))}{dt} = \frac{\partial \Psi}{\partial t}(t,y) + \frac{\partial \Psi}{\partial y}(t,y)\frac{dy}{dt}$.

Exact equations

Now suppose there is a function $\Psi(t,y)$ such that

$$\boxed{\frac{\partial \Psi}{\partial y}(t,y) = N(t,y), \quad \frac{\partial \Psi}{\partial t}(t,y) = M(t,y).} \tag{9}$$

Then the ODE (8) can be expressed as

$$M(t,y) + N(t,y)\frac{dy}{dt} = 0 \Rightarrow \frac{\partial \Psi}{\partial t}(t,y) + \frac{\partial \Psi}{\partial y}(t,y)\frac{dy}{dt} = \frac{d}{dt}\Psi(t,y(t)) = 0,$$

Then, integrating gives the general (implicit) solution

$$\boxed{\Psi(t,y(t)) = c}, \quad c \in \mathbb{R}, \tag{10}$$

and if the initial condition is $y(t_0) = y_0$, then the particular (implicit) solution is

$$\boxed{\Psi(t,y(t)) = \Psi(t_0,y_0)}. \tag{11}$$

Exact equations

Example 3.6

Back to the block $(2t + y^2) + (2ty)\frac{dy}{dt} = 0$. If a function $\Psi(t, y)$ exists, then

$$\frac{\partial \Psi}{\partial t} = 2t + y^2, \quad \frac{\partial \Psi}{\partial y} = 2ty.$$

One possible choice is

$$\Psi(t, y) = t^2 + ty^2.$$

Then, the general (implicit) solution to the ODE is $\Psi(t, y) = c, \ c \in \mathbb{R}$, namely,

$$t^2 + ty^2(t) = c, \quad c \in \mathbb{R}.$$

Exact equations

Definition 3.3

(Exact equation). A first order ODE $M(t,y) + N(t,y)\frac{dy}{dt} = 0$ is an **exact equation** if there exists a function $\Psi(t,y)$ such that

$$\boxed{\frac{\partial \Psi}{\partial t}(t,y) = M(t,y), \quad \frac{\partial \Psi}{\partial y}(t,y) = N(t,y)}. \tag{12}$$

The general solution $y(t)$ to the ODE is given implicitly as $\Psi(t, y(t)) = c, \ c \in \mathbb{R}$.

Questions: 1) How to **determine** an ODE of the form $M(t,y) + N(t,y)\frac{dy}{dt} = 0$ is **exact**? 2) If it is an exact equation, how to find the function $\Psi(t,y)$?

Exact equations

Theorem 3.4

For fixed constants $\alpha, \beta, \gamma, \delta$ with $(\alpha, \beta) \subset I$, suppose $M, N, M_y = \frac{\partial M}{\partial y}$ and $N_t = \frac{\partial N}{\partial t}$ are **continuous** *in the rectangle $R := (\alpha, \beta) \times (\gamma, \delta)$, then*

$$\boxed{M(t,y) + N(t,y)\frac{dy}{dt} = 0 \text{ is exact } \Leftrightarrow M_y(t,y) = N_t(t,y) \text{ for each } (t,y) \in R}. \tag{13}$$

Moreover, let the function $\Psi(t,y)$ be defined as

$$\boxed{\Psi(t,y) = \int_{t_0}^{t} M(s,y)ds + \int_{y_0}^{y} N(t,r)dr - \int_{y_0}^{y} \frac{\partial}{\partial r}\int_{t_0}^{t} M(s,r)ds\, dr}, \tag{14}$$

for constants $t_0 \in (\alpha, \beta), y_0 \in (\gamma, \delta)$. Then,

$$\Psi_t(t,y) = M(t,y), \Psi_y(t,y) = N(t,y).$$

Exact equations

The **proof** of the theorem has two parts.
(1) Let us first show ($\Rightarrow$) of (13). If $M(t,y) + N(t,y)\frac{dy}{dt} = 0$ is exact, by the symmetry of second order derivatives, necessarily it holds that

$$\frac{\partial}{\partial t}N = \frac{\partial}{\partial t}\frac{\partial \Psi}{\partial y} = \frac{\partial}{\partial y}\frac{\partial \Psi}{\partial t} = \frac{\partial}{\partial y}M \Rightarrow M_y = N_t.$$

(2) For the reverse direction ($\Leftarrow$) of (13), suppose $M_y = N_t$ holds and let us construct the function Ψ. Since $\Psi_t = M$, integrating from $t_0 \in (\alpha, \beta)$ to $t > t_0$ gives

$$\frac{\partial \Psi}{\partial t} = M \Rightarrow \boxed{\Psi(t,y) = \int_{t_0}^{t} M(s,y)ds + h(y)}. \tag{15}$$

with some function $h(y)$ acting as the constant of integration. What are the conditions on h so that $\frac{\partial \Psi}{\partial y} = N$?

Exact equations

Now differentiate the formula for Ψ with respect to y and assume that:

$$N(t,y) = \frac{\partial \Psi}{\partial y}(t,y) = \frac{\partial}{\partial y} Q(t,y) + \frac{dh}{dy}, \quad Q(t,y) := \int_{t_0}^{t} M(s,y)ds$$

$$\Rightarrow \boxed{\frac{dh}{dy} = N(t,y) - \frac{\partial}{\partial y} Q(t,y)}. \tag{16}$$

We now claim that, under the condition $M_y = N_t$, the RHS of (16) does not depend on t. Indeed, just differentiating the RHS with respect to t gives

$$\frac{\partial}{\partial t}\left(N(t,y) - \frac{\partial}{\partial y} \int_{t_0}^{t} M(s,y)ds \right) = N_t(t,y) - \frac{\partial}{\partial y}\frac{\partial}{\partial t} \int_{t_0}^{t} M(s,y)ds$$
$$= N_t(t,y) - \frac{\partial}{\partial y} M(t,y) = (N_t - M_y)(t,y) = 0.$$

In the above, we used the formula

$$\frac{\partial}{\partial t} \int_{t_0}^{t} M(s,y)ds = M(t,y).$$

Exact equations

Under the hypothesis $M_y = N_t$, it turns out that the RHS of (16) depends only on y. So we have an equation of the form $h'(y) = f(y)$ for some function f. Integrating from y_0 to y gives

$$h(y) = \int_{y_0}^{y} \left(N(t,r) - \frac{\partial}{\partial r} \int_{t_0}^{t} M(s,r) ds \right) dr + b, \quad b \in \mathbb{R}.$$

Plugging this into (15) and as discussed before we can choose $b = 0$ then yields the formula (14).

It is an exercise to check that $\Psi(t,y)$ satisfied $\Psi_t = M$ and $\Psi_y = N$.

Exact equations

Example 3.7

Solve the ODE

$$(y\cos(t) + 2te^y) + (\sin(t) + t^2e^y - 1)\frac{dy}{dt} = 0.$$

Set

$$M(t,y) = y\cos(t) + 2te^y, \quad N(t,y) = \sin(t) + t^2e^y - 1,$$

and computing the partial derivatives gives

$$M_y = \cos(t) + 2te^y, \quad N_t = \cos(t) + 2te^y \Rightarrow \textit{ ODE is exact}!.$$

By Theorem 3.4 there exists a function $\Psi(t,y)$ such that

$$\Psi_t = M = y\cos(t) + 2te^y, \qquad \Psi_y = N = \sin(t) + t^2e^y - 1.$$

Exact equations: continue

Example 3.7

Integrating Ψ_t with respect to t gives

$$\Psi(t,y) = \int M(t,y)dt + h(y) = y\sin(t) + t^2e^y + h(y).$$

Differentiating with respect to y shows that

$$\frac{\partial \Psi}{\partial y} = \sin(t) + t^2e^y + h'(y) = N(t,y).$$

Comparing gives the relation

$$h'(y) = -1 \Rightarrow h(y) = -y \Rightarrow \Psi(t,y) = y\sin(t) + t^2e^y - y.$$

Therefore, the general (implicit) solution to the ODE is

$$y(t)\sin(t) + t^2e^{y(t)} - y(t) = c, \quad c \in \mathbb{R}.$$

Exact equations

Remark 1

What about separable equations $M(t) + N(y)\frac{dy}{dt} = 0$? The condition $M_y = N_t$ holds trivially since $M_y = 0 = N_t$. Then, from the formula (14), the function $\Psi(t, y)$ reads as

$$\begin{aligned}\Psi(t,y) &= \int_{t_0}^{t} M(s)ds + \int_{y_0}^{y} N(r)dr - \int_{y_0}^{y} \frac{\partial}{\partial r}\int_{t_0}^{t} M(s)dsdr \\ &= \int_{t_0}^{t} M(s)ds + \int_{y_0}^{y} N(r)dr \\ &= m(t) + n(y) + \text{ constant},\end{aligned}$$

which agrees with (2). Note that since M depends only on s,

$$\frac{\partial}{\partial r}\int_{t_0}^{t} M(s)ds = 0.$$

Outline

Non-exact equations

Example 3.8

The non-linear ODE $(3ty + y^2) + (t^2 + ty)\frac{dy}{dt} = 0$ is **not exact**! Since for $M(t, y) = 3ty + y^2$ and $N(t, y) = t^2 + ty$, the particle derivatives are

$$M_y = 3t + 2y \neq N_t = 2t + y.$$

If there was a function $\Psi(t, y)$ such that $\Psi_t = M$ and $\Psi_y = N$, then integrating $\Psi_t = M$ with respect to t leads to

$$\Psi(t, y) = \int 3ty + y^2 dt + h(y) = \frac{3}{2}t^2 y + ty^2 + h(y),$$

for some function $h(y)$. Then, differentiating the above express with respect to y leads to

$$\Psi_y = \frac{3}{2}t^2 + y^2 + h'(y)$$

and compare with $N(t, y) = t^2 + ty$ there is no possibility to satisfy the relation $\Psi_y = N$.

Exact equations with integrating factor

Question: How to solve a non-exact ODE?

Exact equations with integrating factor

Question: How to solve a non-exact ODE?

Idea: Similar to the way we treated the first order linear ODEs, consider multiplying with a "integrating factor μ" and hope things are better. We obtain after multiplying a new ODE

$$\mu M(t,y) + \mu N(t,y)\frac{dy}{dt} = 0. \tag{17}$$

If (17) is an exact equation, then by previous Theorem necessarily following relation must be satisfied:

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial t}(\mu N). \tag{18}$$

Exact equations with integrating factor

Let's first investigate two cases.

Case 1. μ is just a function of t, i.e., $\mu = \mu(t)$. Then (18) simplifies to

$$N(t,y)\frac{d\mu}{dt} + \mu(t)N_t(t,y) = \mu(t)M_y(t,y). \tag{19}$$

If $N(t,y) \neq 0$ for $(t,y) \in (\alpha,\beta) \times (\gamma,\delta) = R$, then we obtain an ODE for μ:

$$\frac{d\mu}{dt} = \mu(t)\left(\frac{M_y - N_t}{N}\right)(t,y) =: \mu(t)K(t,y). \tag{20}$$

Further suppose the factor $K(t,y)$ **depends only on** t, then (20) is a first order **linear** ODE in $\mu(t)$ which can be solved by the method of integrating factors.

Exact equations with integrating factor

Case 2. μ is just a function of y, i.e., $\mu = \mu(y)$. Then (18) simplifies to

$$M(t,y)\frac{d\mu}{dy} + \mu(y)M_y(t,y) = \mu(y)N_t(t,y). \tag{21}$$

If $M(t,y) \neq 0$ for $(t,y) \in (\alpha,\beta) \times (\gamma,\delta) = R$, then we obtain an ODE for μ:

$$\frac{d\mu}{dy} = \mu(y)\left(\frac{N_t - M_y}{M}\right)(t,y) =: \mu(y)H(t,y). \tag{22}$$

Further suppose the factor $H(t,y)$ **depends only on** y, then (20) is a first order **linear** ODE in $\mu(y)$ (where the independent variable is now y), and again can be solved by the method of integrating factors.

Exact equations with integrating factor

Take away message. If we encounter an ODE $M(t,y)+N(t,y)\frac{dy}{dt}=0$ that is not an exact equation, that is $M_y \neq N_t$, then try to compute

(1) $K(t,y)=\frac{M_y-N_t}{N}(t,y)$; or

(2) $H(t,y)=\frac{N_t-M_y}{M}(t,y)$.

(1) If K is only a function of t then solving for the integrating factor $\mu(t)$ that satisfies

$$\frac{d\mu}{dt}=\mu(t)K(t),$$

and multiplying with the non-exact ODE, the new ODE $\mu(t)M(t,y)+\mu(t)N(t,y)\frac{dy}{dt}=0$ becomes an **exact** equation.

Exact equations with integrating factor

(2) **Similarly**, if H is only a function of y, then solving for the integrating factor $\mu(y)$ that satisfies

$$\frac{d\mu}{dy} = \mu(y)H(y),$$

and multiplying with the non-exact ODE, the new ODE $\mu(y)M(t,y) + \mu(y)N(t,y)\frac{dy}{dt} = 0$ becomes an **exact** equation.

Exact equations with integrating factor

Example 3.9

Returning to the ODE $(3ty + y^2) + (t^2 + ty)\frac{dy}{dt} = 0$, which is not an exact equation. Computing

$$M_y = 3t + 2y, \quad N_t = 2t + y, \quad K = \frac{M_y - N_t}{N} = \frac{t + y}{t^2 + ty} = \frac{1}{t},$$
$$H = \frac{N_t - M_y}{M} = \frac{-t - y}{3ty + y^2}.$$

We see that K is only a function of t but H is not just a function of y. So we expect the integrating factor μ to be a function of t only, which solves the ODE

$$\frac{d\mu}{dt} = \frac{\mu(t)}{t} \Rightarrow \mu(t) = ct, \quad c \in \mathbb{R}.$$

Exact equations with integrating factor

Example 3.9

Multiplying this integrating factor (take $c = 1$) with the ODE yields

$$t(3ty + y^2) + t(t^2 + ty)\frac{dy}{dt} = 0,$$

which is now an exact equation with function $\Psi(t, y)$ given as

$$\Psi(t, y) = t^3 y + \frac{1}{2}t^2 y^2.$$

So the general (implicit) solution to the ODE is

$$t^3 y(t) + \frac{1}{2}t^2 y^2(t) = c, \quad c \in \mathbb{R}.$$

Exact equations with integrating factor

So far for non-exact ODEs of the form $M(t,y) + N(t,y)\frac{dy}{dt} = 0$, the suggestion is to check whether $K(t,y)$ is only a function of t or $H(t,y)$ is only a function of y. If either one is true then we can apply the method of integrating factors to obtain an exact equation. But **what if neither is true**?

The key requirement in the analysis of exact equations is the relation

$$\frac{\partial}{\partial t}(\mu N) = \frac{\partial}{\partial y}(\mu M).$$

If $\mu = \mu(t,y)$ is a function of t and y, computing using the product rule and chain rule yields

$$\boxed{M(t,y)\mu_y - N(t,y)\mu_t = \mu(t,y)(N_t - M_y)(t,y)}. \tag{23}$$

The above is a **Partial Differential Equation** (PDE) since it involves the partial derivatives of μ with respect to t and y.

Exact equations with integrating factor

In general the analysis for PDEs is much more involved than ODEs, in particular a PDE may not have a solution (**non-existence**) and even if a solution exists, there may be many (often infinitely many) of them (**non-uniqueness**). So the general situation seems to be not tractable.

Under certain assumption, we can deal with some special cases.

Exact equations with integrating factor

Looking back at transformation methods and how we dealt with homogeneous equations, we can use similar methods to treat the case if μ is a function of $z = ty$. Using the chain rule

$$\frac{\partial}{\partial t}\mu(z) = \mu'(z)\frac{\partial z}{\partial t} = y\mu'(ty),$$

$$\frac{\partial}{\partial y}\mu(z) = \mu'(z)\frac{\partial z}{\partial y} = t\mu'(ty).$$

Then, in (23) we now have the relation

$$\begin{aligned}(tM(t,y) - yN(t,y))\mu'(z) &= \mu(z)(N_t - M_y)(t,y)\\ \Rightarrow \mu'(z) &= \mu(z)\left(\frac{N_t - M_y}{tM - yN}\right)(t,y) = \mu(z)L(t,y),\end{aligned}$$

where $L(t,y) = \left(\frac{N_t - M_y}{tM - yN}\right)(t,y)$.

Exact equations with integrating factor

If the factor L is a function only of $z = ty, i.e., L = L(z) = L(ty)$, then we can deduce an integrating factor μ as a function of $z = ty$. Repeating our procedure this would then yield an exact equation.

Exercise: Show that if L is only a function of z, then the new ODE $\mu(ty)M(t, y) + \mu(ty)N(t, y)\frac{dy}{dt} = 0$ is an exact equation.

Outline

Summary for ODEs we have learnt

Linear vs Nonlinear ODEs - a comparison

Type	Method	Explicit/Implicit solution
$y' = p(t)y + q(t)$	Integrating factor	$y(t) = \mu(t)^{-1}(\int \mu(t)q(t)dt + c)$
$M(t) + N(y)y' = 0$	Separable equation	$m(t) + n(y(t)) = c$
$y' + p(t)y = q(t)y^n$	$v := y^{1-n}$	$y(t) = (\mu^{-1}(\int Q(t)\mu(t)dt + c)^{1/(1-n)}$
$y' = F(y/t)$	$v = y/t$	$1/(F(v) - v)\frac{dv}{dt} = \frac{1}{t}$
$M(t, y) + N(t, y)y' = 0$	Exact equation	$\Psi(t, y(t)) = c$