STA4001 | Stochastic Processes | Spring 2021

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Homework 3

Due: Feb 23

1. Consider a perishable product with lifetime of two periods. We say a product is less fresh if it has remaining lifetime one period, and fresh if it has remaining lifetime two periods. At the beginning of period n, an order for fresh product placed at the end of period $n-1$ arrives. After the order arrives, i.i.d. demand D_n is realized and satisfied first by less fresh inventory, then (if possible) by fresh ones. Then unsold less fresh inventory is expired and has to be disposed. Suppose that each left over (not expired) item has holding cost $h = \$.1$. Let X_n be the ending inventory level for period n (note that there is only less fresh inventory at the end of each period). Let the selling price $c_p = \$1.0$, variable cost $c_v = \$.25$, and disposal cost $c_d = \$0.5$. Suppose the manager uses an $S = 4$ order up to policy, i.e., orders enough to bring the total inventory level up to S at the beginning of next period.

   (a) Verify that $\{X_n\}$ is a DTMC by writing X_{n+1} as a function of X_n and D_{n+1}.

   (b) Suppose D_n follows Poisson distribution with rate 1, compute $\mathbb{P}\{X_1 = 1|X_0 = 3\}$ and $\mathbb{P}\{X_2 = 1|X_0 = 3\}$.

   (c) If customer has the option to choose, he/she will take fresh product first. Answer the above questions (a), (b) when demand is satisfied first by fresh inventory, then (if possible) by less fresh ones.

2. A store sells a particular nonperishable product. The demand for the product each day is 1 item with probability 1/6, 2 items with probability 3/6, and 3 items with probability 2/6. Assume that the daily demands are independent and identically distributed. The store uses an $(s, S) = (2, 4)$ policy to manage its inventory: each evening if the remaining stock is less than or equal to s items, the store orders enough to bring the total stock up to S items next morning. These items reach the store before the beginning of the following day. Assume that any demand is lost when the item is out of stock. Assume that each item sells at \$150, the variable cost per item is $c_v = \$50$, the fixed cost for each order is $c_f = \$100$, and the holding cost is \$5 for each item held overnight (for accounting).

   (a) Let X_n be the inventory level at the end of day n. Then $X = \{X_n : n = 0, 1, 2, \ldots,\}$ is a discrete time Markov chain. Find the state space S and the transition matrix of the DTMC.

   (b) Compute

   $$\mathbb{P}\{X_1 = 3|X_0 = 2\}, \quad \mathbb{P}\{X_2 = 3|X_0 = 2\}, \quad \mathbb{P}\{X_4 = 3|X_0 = 2\}, \quad \text{and} \quad \mathbb{P}\{X_8 = 3|X_0 = 2\}.$$

(You may use software like `Python` for the last two expressions.)

(c) Find

$$\mathbb{P}\{X_1 = 3, X_4 = 2 | X_0 = 2\}, \quad \text{and} \quad \mathbb{P}\{X_4 = 2, X_8 = 3 | X_0 = 2\}.$$

3. There are 3 black balls and 3 white balls in our hands and we distribute these 6 balls equally into two boxes (i.e. 3 balls in each box). We say that the system is in state $i \in \{0, 1, 2, 3\}$, if the first box contains i white ball(s). At each step, we draw one ball from each box and exchange them. Let X_n denote the state of the system after the n-th step.

   (a) Explain why X_n is a Markov chain and calculate the transition probability matrix.

   (b) Calculate $\mathbb{E}(X_2 \mid X_0 = 1)$ and $\mathrm{Var}(X_2 \mid X_0 = 1)$.

   (c) Assume that $\mathbb{P}\{X_0 = 1\} = 1/3$, $\mathbb{P}\{X_0 = 2\} = 1/3$, and $\mathbb{P}\{X_0 = 3\} = 1/3$. Compute

$$\mathbb{P}\{X_1 = 2 | X_2 = 1, X_3 = 2\}.$$

4. Consider the DTMC in Problem 3. Assume that

$$C(0) = \$0, \quad C(1) = \$3, \quad C(2) = \$1, \quad \text{and } C(3) = \$2.$$

   (a) Compute $\mathbb{E}[C(X_1) + C(X_2) | X_0 = 1]$. Namely, starting $X_0 = 1$, compute the expected total cost in two time periods given that each time the DTMC visits state i, it collects $\$C(i)$.

   (b) Compute $\mathbb{E}[\sum_{n=1}^{10} C(X_n) | X_0 = 1]$. You may use **Python** or other programming languages.

   (c) Assume that $X_0 = 1$. Find the expected total number of visits to state 2 by the DTMC X in 10 periods: 1, 2, ..., 10.

5. Consider one communication link from node A to node B. Each data packet over this link needs one timeslot to be transmitted. Let A_n be the number of packets that arrive at node A during timeslot n. Assume that $\{A_n : n = 0, 1, 2, \ldots,\}$ is an iid sequence with distribution

$$\begin{aligned}
\mathbb{P}\{A_1 = 0\} &= 3/5, \\
\mathbb{P}\{A_1 = 1\} &= 1/5, \\
\mathbb{P}\{A_1 = 2\} &= 1/5. \qquad (1)
\end{aligned}$$

   To make the counting of packets easier, assume the following sequence of events in each timeslot: packets arrive during the timeslot (waiting in a common buffer of infinite size if necessary), packets in transmission leave the system at the end of this timeslot if they reach the destination, a new packet starts transmission over a link at the beginning of next timeslot (equal to the end of current timeslot) if the link is free and there is a packet in the buffer to transmit. Let X_n be the number of packets in system at the beginning of time slot n **before any packets are transmitted**.

(a) Prove that $X = \{X_n : n = 0, 1, \ldots\}$ is a DTMC.

(b) Specify the state space and the transition matrix (if the state space is infinite, then specify all nonzero transition probabilities).

6. Consider two parallel communications links from node A to node B. One link is fast so that each data packet over this link needs one timeslots to be transmitted. The other link is slow so that each data packet over this link needs two timeslots to be transmitted. Let A_n be the number of packets that arrive at node A during timeslot n. Assume that $\{A_n : n = 1, 2, \ldots,\}$ is an iid sequence with distribution given in (1). To make the counting of packets easier, assume the following sequence of events in each timeslot: packets arrive during the timeslot (waiting in a common buffer of infinite size if necessary), packets in transmission leave the system at the end of this timeslot if they reach the destination, a new packet starts transmission over a link at the beginning of next timeslot (equal to the end of current timeslot) if the link is free and there is a packet in the buffer to transmit. When both links are free and there is only one packet to be transmitted, the fast link is always used for the transmission. Let X_n be the number of packets in system at the beginning of time slot n before any packets are transmitted.

   (a) Assume $X_0 = 0$. Compute

   $$\mathbb{P}\{X_3 = 1 | X_1 = 2, X_2 = 2\}$$

   (b) Assume $X_0 = 0$. Compute

   $$\mathbb{P}\{X_3 = 1 | X_1 = 1, X_2 = 2\}$$

   (c) Is $X = \{X_n : n = 0, 1, 2, \ldots\}$ a DTMC? Please justify your argument.

   (d) If X is not a DTMC, construct a DTMC that models the communication system. Provide state space and transition probabilities.