Stochastic Processes

Lecture 25: From random walks to martingales II

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Doob's optional sampling theorem

Theorem (Optional stopping theorem)

Let $M = \{M_n\}$ be a martingale and T be a stopping time. Suppose that at least one of the following conditions holds.

1. $T \leq k$ *for some* k,
2. $T < \infty$ *and* $|M_n| \leq C$ *whenever* $n \leq T$.

Then $\mathbb{E}M_T = \mathbb{E}M_1$.

Proof when 1 holds.

Assume 1 holds. Then

$$\begin{aligned} M_T - M_1 &= (M_T - M_{T-1}) + \ldots + (M_1 - M_1) \\ &= \sum_{n=1}^{T-1} (M_{n+1} - M_n) \\ &= \sum_{n=1}^{k-1} (M_{n+1} - M_n) 1_{\{n<T\}} \end{aligned}$$

Proof

- Therefore,

$$\mathbb{E}[M_T - M_1] = \sum_{n=1}^{k-1} \mathbb{E}[(M_{n+1} - M_n)1_{\{n<T\}}]$$

- $\{n < T\} = 1 - \{T \leq n\}$, which can be determined by $Y_1, \ldots, Y_n$.
- Thus,

$$\begin{aligned}
\mathbb{E}[M_{n+1}1_{\{n<T\}}] &= \mathbb{E}\Big(\mathbb{E}[M_{n+1}1_{\{n<T\}}|Y_1, \ldots Y_n]\Big) \\
&= \mathbb{E}\Big(1_{\{n<T\}}\mathbb{E}[M_{n+1}|Y_1, \ldots Y_n]\Big) \\
&= \mathbb{E}\Big(M_n 1_{\{n<T\}}\Big).
\end{aligned}$$

Proof

Proof when 2 holds.

$$|\mathbb{E}M_T - \mathbb{E}M_1| = |\mathbb{E}M_T - \mathbb{E}M_{T\wedge n}| \le \mathbb{E}\,|M_T - M_{T\wedge n}| \le 2C\mathbb{P}(T > n).$$

$\square$

Simple, symmetric random walk

- Fix $a, b > 0$.
- Let $T_{-a,b}$ be the first hitting time to either $-a$ or b, i.e.,

$$T_{-a,b} = \inf\{n \geq 0 : X_n = -a \quad \text{or} \quad X_n = b\}.$$

- Define T_b

$$T_b = \inf\{n \geq 0 : \quad X_n = b\}.$$

- Then

$$T_{-a,b} = T_{-a} \wedge T_b.$$

Hitting probabilities and hitting times

- Use the first martingale to prove

$$\mathbb{P}\{T_{-a} < T_b\} = \frac{b}{a+b}.$$

- Use the 2nd martingale to prove (in homework)

$$\mathbb{E}(T_{-a,b}) = ab.$$

A few facts

- If S and T are two stopping times with respect to $\{Y_n : n \geq 1\}$, then $\min(S, T)$ is also a stopping time.
- Dominated (Bounded) convergence theorem: If $\lim_{n\to\infty} Y_n = Y$ and $|Y_n| \leq C$ for some constant C, then

$$\lim_{n\to\infty} \mathbb{E}(Y_n) = \mathbb{E}(\lim_{n\to\infty} Y_n).$$

- Monotone convergence theorem: If $0 \leq Y_1 \leq Y_2 \leq \ldots \leq Y_n \leq \ldots$, then

$$\lim_{n\to\infty} \mathbb{E}(Y_n) = \mathbb{E}(\lim_{n\to\infty} Y_n).$$

Simple, non-symmetric random walk

- $P_{i,i+1} = p$ and $P_{i,i-1} = q$.
- Define

$$M_n = \left(\frac{q}{p}\right)^{X_n}.$$

- M is a martingale.

Theorem

$$\mathbb{P}\{T_{-a} < T_b\} = \frac{1 - (q/p)^b}{(q/p)^{-a} - (q/p)^b}.$$

- Assume $q > p$. As $a \to \infty$,

$$\mathbb{P}\{T_b < \infty\} = (p/q)^b. \tag{1}$$

Extreme probabilities

Theorem

For a simple random walk. Assume $q > p$.

$$\mathbb{P}\left\{\sup_{n\geq 0} X_n \geq b\right\} = (p/q)^b.$$

- When $q < p$,

$$\mathbb{P}\left\{\inf_{n\geq 0} X_n \leq -a\right\} = (q/p)^a.$$

Brownian motion

Definition

A continuous-time stochastic process $B = \{B(t) : t \geq 0\}$ is said to be a (μ_B, σ_B^2)-Brownian motion if

- $B(0) = 0$ and almost every sample path is continuous
- $\{B(t) : t \geq 0\}$ has stationary, independent increments
- $B(t)$ is normally distributed with mean $\mu_B t$ and variance $\sigma_B^2 t$ for every $t > 0$

A $(0, 1)$-Brownian motion is called a standard Brownian motion.

Theorem

For a standard Brownian motion B, define $T_b = \inf\{t \geq 0 : B(t) = b\}$ the first time hitting b, and $T_{-a,b} = T_{-a} \wedge T_b$ the first time hitting either $-a$ or b. Then,

$$\mathbb{P}\{T_{-a} < T_b\} = \frac{b}{a+b},$$
$$\mathbb{E}[T_{-a,b}] = ab.$$

Proof.

- $\{B(t), t \geq 0\}$ is a martingale;
- $\{B^2(t) - t, t \geq 0\}$ is a martingale;
- $\{e^{\theta B(t) - \frac{1}{2}\theta^2 t}, t \geq 0\}$ is a martingale for each $\theta \in \mathbb{R}$.

□

Proof

- $\{B(t), t \geq 0\}$ is a martingale; namely,

$$\mathbb{E}[B(t+s)|B(t_1), \ldots, B(t_{n-1}), B(t)] = B(t)$$

  for any $n \geq 1$ and any $t_1 < t_1 < t_{n-1} < t_n = t$.

- It suffices to prove that

$$\begin{aligned}
&\mathbb{E}\Big[B(t+s) - B(t)|B(t_1), \ldots, B(t_{n-1}), B(t)\Big] \\
&= \mathbb{E}[B(t+s) - B(t)] \\
&= 0.
\end{aligned}$$

A Poisson sample path with $\lambda = 1$

Let $\{E(t) : t \geq 0\}$ be a Poisson process with rate λ

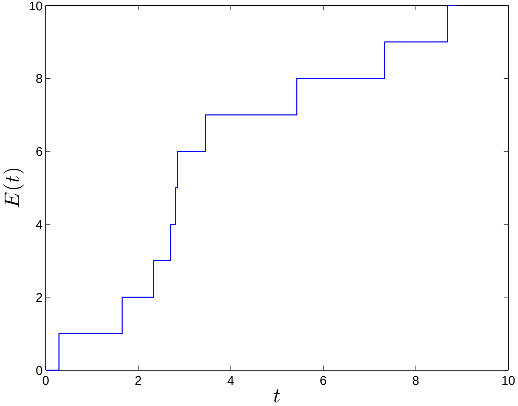

FIGURE: A Poisson sample path with rate $\lambda = 1$

The centered sample path with $\lambda = 1$

Then, $\{E(t) - \lambda t : t \geq 0\}$ is the centered process

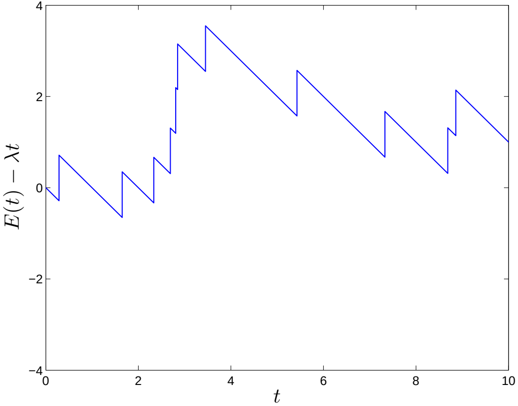

FIGURE: The sample path of the centered process with $\lambda = 1$

A Poisson sample path with $\lambda = 100$

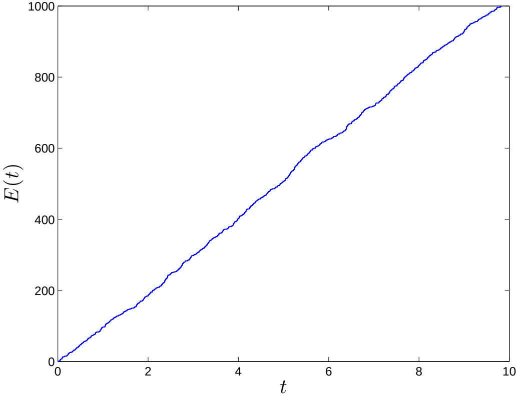

FIGURE: A Poisson sample path with rate $\lambda = 100$

The centered sample path with $\lambda = 100$

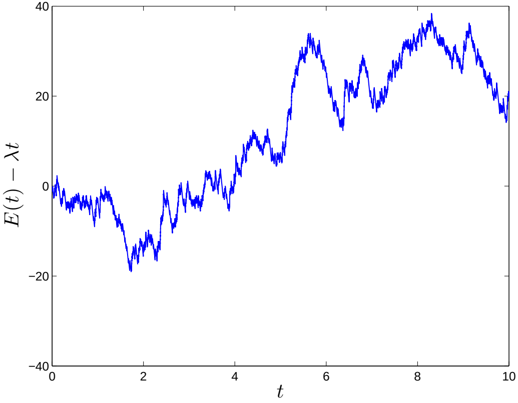

FIGURE: The sample path of the centered process with $\lambda = 100$

A Poisson sample path with $\lambda = 10,000$

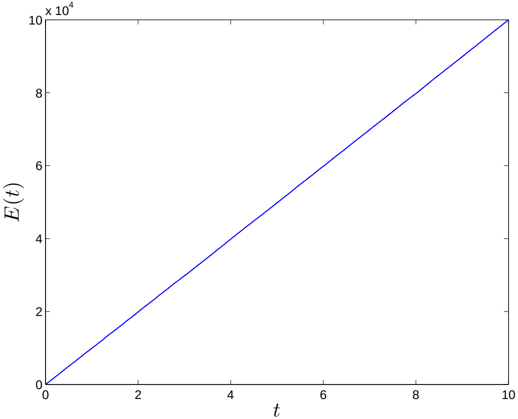

FIGURE: A Poisson sample path with rate $\lambda = 10,000$

The centered sample path with $\lambda = 10,000$

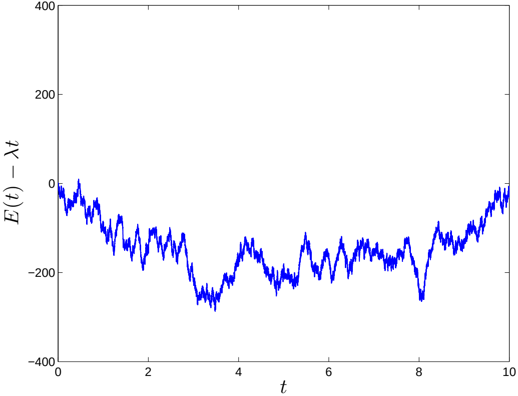

FIGURE: The sample path of the centered process with $\lambda = 10,000$

A functional central limit theorem

Let $E^{(\lambda)}$ be a Poisson process with rate λ. Define

$$\tilde{E}_\lambda(t) = \frac{E^{(\lambda)}(t) - \lambda t}{\sqrt{\lambda}}$$

Theorem

As $\lambda \to \infty$,

$$\tilde{E}_\lambda \Longrightarrow B.$$

Donsker's theorem implies that the process $\tilde{E}_\lambda$ is close to a standard Brownian motion when λ is large

Donsker's theorem

- $\{\xi(n), n = 1, 2, \ldots\}$ is an iid sequence with $\mathbb{E}[\xi(n)] = 0$ and $\text{var}(\xi(n)) = \sigma^2$.
- Define random walk $S = \{S_n : n = 1, 2, \ldots,\}$

$$S_n = \sum_{i=1}^{n} \xi(i).$$

- CLT $\frac{S_n}{\sqrt{n}} \Longrightarrow N(0, \sigma^2)$.
- Define

$$\hat{S}^n(t) = \frac{S_{\lfloor nt \rfloor}}{\sqrt{n}} \quad t \geq 0.$$

Theorem (Donsker's theorem)

As $n \to \infty$,

$$\tilde{S}^n \Longrightarrow (0, \sigma^2) - \textit{Brownian motion.}$$

Diffusion process

- Geometric Brownian motion $X = \{X(t), t \geq 0\}$, where

$$X(t) = e^{\sigma B(t) + \mu t}.$$

- $X = \{X(t), t \geq 0\}$ satisfies a stochastic differential equation (SDE)

$$dX(t) = b(X(t))dt + \sigma(X(t))dB(t),$$

which is equivalent to

$$X(t) = X(0) + \int_0^t b(X(u))du + \int_0^t \sigma(X(u))dB(u).$$

Ito's formula

Assume f is a C^2 function. Then

$$f(X(t)) - f(X(0)) = \int_0^t Gf(X(u))du + \int_0^t f'(X(u))\sigma(X(u))dB(u),$$

where

$$Gf(x) = \frac{1}{2}\sigma^2(x)f''(x) + b(x)f'(x).$$

For each $f \in C_b^2$,

$$f(X(t)) - f(X(0)) - \int_0^t Gf(X(u))du$$

is a martingale.