Stochastic Processes

Lecture 20: CTMC: an example, limiting distribution, and uniformization

Hailun Zhang@SDS of CUHK-Shenzhen

April 12, 2021

Three lines, two homogeneous agents

- 3 *phone lines*, 2 *agents*
- Calls *arrive* according to a Poisson process with rate $\lambda = 2$ calls per minute.
- *processing times* are iid exponential with rate $\mu = 1$ call per minute.
- Times that callers are willing to hold (i.e., their *patience times*) are iid exponential with mean 4 minutes.

The Corresponding CTMC

State Space $= \{0, 1, 2, 3\}$

- $X(t)$ is the number of calls in system at time t.
- Rate diagram

Stationary Distribution

The stationary distribution $\pi = [\pi_0, \pi_1, \pi_2, \pi_3]$ satisfies

$$\pi G = 0 \qquad \text{and} \qquad \pi_0 + \pi_1 + \pi_2 + \pi_3 = 1.$$

- Using the "cut-method", one can easily compute π to obtain

$$\pi = \left(\frac{9}{61}, \frac{18}{61}, \frac{18}{61}, \frac{16}{61}\right).$$

Some Performance Measures

What is the

- long-run fraction of time that both servers are free?

$$\pi_0 =$$

- long-run fraction of time that Bob (one of the servers) is busy?

$$\pi_2 + \pi_3 + \frac{1}{2}\pi_1 (\text{ Really? })$$

- the average utilization for per server?

$$\frac{0\pi_0 + 1\pi_1 + 2(\pi_2 + \pi_3)}{2} = \frac{43}{61}.$$

- long-run fraction of *arrivals* that get a busy signal?

$$\pi_3 = \frac{16}{61} \qquad \text{(Why???)}$$

Some Performance Measures

- What is the average queue size? $(1)\pi_3 = 16/61$.
- What is the rate at which the processed calls depart from the servers?

$$2(\text{average utilization})\mu = \frac{86}{61}.$$

- What is the long-run fraction of calls that abandoned without service, among those who enter into system?

$$\text{abandoment rate: } \lambda(1-\pi_3) - \frac{86}{61} = \frac{90}{61} - \frac{86}{61} = \frac{4}{61},$$
$$\text{abandonment fraction: } \frac{4/61}{90/61} = \frac{2}{45}.$$

- What is the average time to listen to the music, among those who have listened to the music?

  Those who have listened to music must have seen both servers busy upon arrival. So the amount of time to listen to music is $\min(U, V)$, where U is the time for a call in process to complete the service and V is the patience time. Clearly, $U \sim \exp(2\mu) = \exp(2)$ and $V \sim \exp(1/4)$. Thus

$$\mathbb{E}(\min(U, V)) = \frac{1}{2\mu + 1/4} = 4/9.$$

- What is the average time to listen to the music, among those who have completed services?

$$\frac{\pi_2 \mathbb{E}\big(U 1_{\{U<V\}}\big) + (\pi_1 + \pi_0)0}{\pi_2 \mathbb{P}(U < V) + \pi_1 + \pi_0},$$

  where

$$\mathbb{E}\big(U 1_{\{U<V\}}\big) = \int_0^\infty \left(\frac{1}{4}\exp(-x/4)\int_0^x 2y\exp(-2y)dy\right)dx$$

Eigen values and vectors

- CTMC X on $S = \{1, 2, 3\}$ with generator

$$G = \begin{pmatrix} -2 & 1 & 1 \\ 2 & -5 & 3 \\ 2 & 2 & -4 \end{pmatrix}.$$

- Stationary distribution $\pi G = 0$:

$$\pi = (1/2, 3/14, 2/7) = (0.500000.214290.28571).$$

- Eigen values and eigen vectors

$$G \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0, G \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = (-4) \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, G \begin{pmatrix} 1/8 \\ -13/8 \\ 1 \end{pmatrix} = (-7) \begin{pmatrix} 1/8 \\ -13/8 \\ 1 \end{pmatrix}$$

Eigen values and vectors II

$$\begin{pmatrix}-2 & 1 & 1\\ 2 & -5 & 3\\ 2 & 2 & -4\end{pmatrix}\begin{pmatrix}1 & -1 & 1/8\\ 1 & 1 & -13/8\\ 1 & 1 & 1\end{pmatrix}=\begin{pmatrix}1 & -1 & 1/8\\ 1 & 1 & -13/8\\ 1 & 1 & 1\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & -4 & 0\\ 0 & 0 & -7\end{pmatrix}$$

$$G=\begin{pmatrix}1 & -1 & 1/8\\ 1 & 1 & -13/8\\ 1 & 1 & 1\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & -4 & 0\\ 0 & 0 & -7\end{pmatrix}\begin{pmatrix}1 & -1 & 1/8\\ 1 & 1 & -13/8\\ 1 & 1 & 1\end{pmatrix}^{-1}$$

$$G^n=\begin{pmatrix}1 & -1 & 1/8\\ 1 & 1 & -13/8\\ 1 & 1 & 1\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & (-4)^n & 0\\ 0 & 0 & (-7)^n\end{pmatrix}\begin{pmatrix}1 & -1 & 1/8\\ 1 & 1 & -13/8\\ 1 & 1 & 1\end{pmatrix}^{-1}$$

Limiting distribution

$$P(t) = \texttt{expm}(tG) = \sum_{n=0}^{\infty} \frac{t^n G^n}{n!}$$

$$= \begin{pmatrix} 1 & -1 & 1/8 \\ 1 & 1 & -13/8 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-4t} & 0 \\ 0 & 0 & e^{-7t} \end{pmatrix} \begin{pmatrix} 1 & -1 & 1/8 \\ 1 & 1 & -13/8 \\ 1 & 1 & 1 \end{pmatrix}^{-1}$$

$$\to \begin{pmatrix} 1 & -1 & 1/8 \\ 1 & 1 & -13/8 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1/8 \\ 1 & 1 & -13/8 \\ 1 & 1 & 1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 1/2 & 3/14 & 2/7 \\ 1/2 & 3/14 & 2/7 \\ 1/2 & 3/14 & 2/7 \end{pmatrix}$$

- Spectral gap: eigen values $\lambda_1 = 0$, $\lambda_2 = -4$, $\lambda_4 = -7$, where $\lambda_1 > \lambda_2 > \lambda_3$ and

$$\text{spectral gap of } G = \lambda_1 - \lambda_2 = 4.$$

Limiting distribution

Theorem

(a) Assume CTMC is irreducible and positive recurrent. Then

$$P(t) \to \Pi, \quad \text{as } t \to \infty, \tag{1}$$

where each row of Π *is given by the (unique) stationary distribution.*
(b) Assume CTMC is irreducible on a finite state space.

$$||P(t) - \Pi|| \leq ce^{-\kappa t} \quad t \geq 0, \tag{2}$$

where κ *is the spectral gap of* G.

(Discrete time) jump chain

- Consider a CTMC $X = \{X(t), t \geq 0\}$ with generator

$$G = \begin{pmatrix} -1 & 1 & 0 \\ 2 & -4 & 2 \\ 0 & 1 & -1 \end{pmatrix}$$

- Stationary distribution $\pi = \pi_X$
- Jump chain $Y = \{Y_n : n = 0, 1, \ldots,\}$

$$J = \begin{pmatrix} 0 & 1 & 0 \\ .5 & 0 & .5 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\pi_Y$$

- Relationship between π_X and π_Y

Uniformization

- Sample path
- The holding rates at states 1 and 3 are smaller than that at state 2.
- How to "boost" the holding rates so that we get a "uniform" CTMC?
- The magic lies in the "jump matrix".
- Consider a CTMC $\tilde{X}$ with uniform holding time rates $\lambda = 4$ and "jump matrix"

$$J_{\text{Uniformization}} = \begin{pmatrix} 3/4 & 1/4 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1/4 & 3/4 \end{pmatrix}.$$

- $N(t)$ is a Poisson process with rate λ.
- $\tilde{X}(t) = Z_{N(t)}$, where $Z = \{Z_n : n \geq 0\}$ is the "uniform jump chain".
- $X \overset{d}{=} \tilde{X}$. (Proof needed.)
- For each $t \geq 0$

$$\begin{aligned} P_{ij}(t) &= \mathbb{P}\{Z_{N(t)} = j | Z_0 = i\} \\ &= \sum_{n=0}^{\infty} \mathbb{P}\{Z_n = j | Z_0 = i\} \mathbb{P}\{N(t) = n\} \\ &= \sum_{n=0}^{\infty} (J_{\mathrm{U}})^n_{ij} \frac{(\lambda t)^n}{n!} e^{-\lambda t} \end{aligned}$$

$$P(t) = \sum_{n=0}^{\infty} (J_{\mathrm{U}})^n \frac{(\lambda t)^n}{n!} e^{-\lambda t}.$$

- Stationary distribution of Z π_Z

- A CTMC is uniformizable if

$$\sup_{i \in S} |G_{ii}| < \infty. \tag{3}$$

- Under condition (3), the CTMC is regular.
- How to uniformize the following CTMC?

$$G = \begin{pmatrix} -4 & 2 & 2 \\ 1 & -2 & 1 \\ 2 & 1 & -3 \end{pmatrix}$$