MAT2002 ODEs
Nonlinear Differential Equations and Stability IV

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Overview

Outline

Introduction: Liapunov's method

We now present a method to infer stability information about the "even" critical points of the undamped pendulum. The approach we discuss now is called **Liapunov's method**, sometimes known as the **direct method**, since this approach needs no knowledge of the solution to the system of equations, and conclusions about stability/instability of a critical point can be obtained.

Application Liapunov's method to the undamped pendulum

For the undamped pendulum, the original equation is

$$\theta'' + \frac{g}{L}\sin\theta = 0,$$

where we set $w = \sqrt{g/L}$ for convenience. Introducing the variables $y_1 = \theta, y_2 = \theta'$ we obtain the first order system

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} y_2 \\ -\frac{g}{L}\sin y_1 \end{pmatrix}. \tag{1}$$

From physics there are two energies associated to the pendulum:

(a) Potential energy given by $mgL(1-\cos y_1) = mgL(1-\cos\theta)$;

(b) Kinetic energy given by $\frac{1}{2}mL^2y_2^2 = \frac{1}{2}mL^2(\theta')^2$.

Application Liapunov's method to the undamped pendulum

Let us make some observations

(i) The critical points to (1) are $(\pm n\pi, 0)$ for $n \in \mathbb{Z}$. We have previously studied the stability and type of the "odd" critical points which are unstable saddle points.

(ii) The potential energy is minimal (equal to zero) when $y_1 = \pm 2m\pi$ for $m \in \mathbb{Z}$, while the maximum potential energy (equal to $2mgL$) is achieved at $y_1 = \pm(2m+1)\pi$ for $m \in \mathbb{Z}$.

(iii) The total energy (the sum of the potential and kinetic energies) is

$$V(y_1, y_2) = mgL(1 - \cos y_1) + \frac{1}{2} mL^2 y_2^2$$

is conserved, i.e.,

$$\frac{d}{dt} V(y_1(t), y_2(t)) = 0.$$

And so, on trajectories $(y_1(t), y_2(t))_{t \in I}$ for an open interval $I \subset \mathbb{R}$, the total energy $V(y_1, y_2)$ remains unchanged.

Application Liapunov's method to the undamped pendulum

The last point is the crucial part of Liapunov's method. Note that at $y_1 = \pm 2m\pi$, $y_2 = 0$, both the potential and kinetic energies are zero, and so the total energy is zero at the "even" critical points. Hence, if we start with a trajectory $(y_1(t), y_2(t))_{t \in I}$ with initial condition (z_1, z_2), i.e., $y_1(t_0) = z_1$, $y_2(t_0) = z_2$, that is "close" to the "even" critical points, then by conservation of total energy we can infer that

$$V(y_1(t), y_2(t)) = V(z_1, z_2) \quad \forall t \in I,$$

and so the total energy for $t > t_0$ will remain small.

For example, pick (z_1, z_2) close to (0,0), and for small values of y_1, we can Taylor expand $\cos(\cdot)$ to obtain

$$\begin{aligned} V(y_1(t), y_2(t)) &= mgL(1 - \cos(y_1(t))) + \frac{1}{2} mL^2 (y_2(t))^2 \\ &\approx \frac{1}{2} mgL (y_1(t))^2 + \frac{1}{2} mL^2 (y_2(t))^2, \end{aligned}$$

Application Liapunov's method to the undamped pendulum

The conservation of total energy gives

$$V(z_1, z_2) = V(y_1(t), y_2(t)) \approx \frac{1}{2} mgL(y_1(t))^2 + \frac{1}{2} mL^2 (y_2(t))^2.$$

Roughly speaking, the trajectories $(y_1(t), y_2(t))_{t \in I}$ can be approximated by the equation

$$\boxed{\frac{y_1^2}{2\frac{V(z_1,z_2)}{mgL}} + \frac{y_2^2}{2\frac{V(z_1,z_2)}{mL^2}} = 1}.$$

This is the equation for an ellipse enclosing the critical point (0,0) where the major and minor axes are determined by the initial energy V (z_1, z_2). In particular, the smaller the initial energy $V(z_1, z_2)$, the smaller the ellipse. Nevertheless this shows that (0,0) is a stable critical point (not asym.stable like in the damped pendulum).

Application Liapunov's method to the undamped pendulum

What about the critical point $(2\pi, 0)$?
Pick (z_1, z_2) close to $(2\pi,0)$, and for small values of $y_1 - 2\pi$, we can Taylor expand $\cos(\cdot)$ to obtain

$$\begin{aligned} V(z_1, z_2) = V(y_1(t), y_2(t)) &= mgL(1 - \cos(y_1(t) - 2\pi)) + \frac{1}{2} mL^2 (y_2(t))^2 \\ &\approx \frac{1}{2} mgL(y_1(t) - 2\pi)^2 + \frac{1}{2} mL^2 (y_2(t))^2, \end{aligned}$$

Roughly speaking, the trajectories $(y_1(t), y_2(t))_{t\in I}$ can be approximated by the equation

$$\boxed{\frac{(y_1 - 2\pi)^2}{2\frac{V(z_1,z_2)}{mgL}} + \frac{y_2^2}{2\frac{V(z_1,z_2)}{mL^2}} = 1}.$$

This is the equation for an ellipse enclosing the critical point $(2\pi,0)$ where the major and minor axes are determined by the initial energy V (z_1, z_2).
The same arguments can be used to show that the "even" critical points of the undamped pendulum are all stable centers. The following figure shows the phase portrait for $w = 1$.

Application Liapunov's method to the undamped pendulum

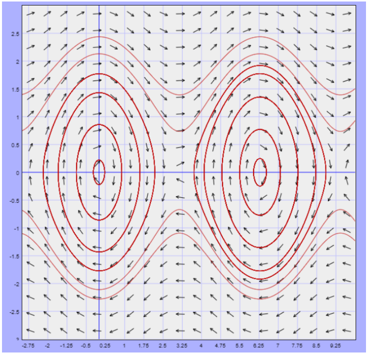


Fig. 5. Phase portrait for the undamped pendulum with $w = 1$.

Liapunov's method: general theory

In the undamped pendulum example, the function V plays a significant role in helping us determine the stability of some critical points. Let us now consider a nonlinear autonomous system

$$y_1' = F_1(y_1, y_2), \quad y_2' = F_2(y_1, y_2) \textit{ for } t \in I,$$

with a critical point (0; 0), i.e., $F_1(0,0) = F_2(0,0) = 0$. Denote by $D \subset \mathbb{R}^2$ a region containing (0,0), and a trajectory by $(y_1(t), y_2(t))_{t \in I}$.

Liapunov's method: general theory

Definition 18.1

(Positive/negative definite). Let V: $\mathbb{R}^2 \to \mathbb{R}$ be a function such that $V(z_1, z_2) < \infty$ for all $(z_1, z_2) \in D$. We say

(a) V is <u>**positive definite**</u> on D if $V(0,0) = 0$ and $V(z_1, z_2) > 0$ for all $(z_1, z_2) \in D \setminus \{(0,0)\}$;

(b) V is <u>**negative definite**</u> on D if $V(0,0) = 0$ and $V(z_1, z_2) < 0$ for all $(z_1, z_2) \in D \setminus \{(0,0)\}$;

(c) V is <u>**positive semidefinite**</u> on D if $V(0,0) = 0$ and $V(z_1, z_2) \geq 0$ for all $(z_1, z_2) \in D$;

(d) V is <u>**negative semidefinite**</u> on D if $V(0,0) = 0$ and $V(z_1, z_2) \leq 0$ for all $(z_1, z_2) \in D$.

Liapunov's method: general theory

Note that in all of the above definitions, we always have the condition $V(0,0) = 0$.

Example 18.2

The function

$$V(x,y) = \sin(x^2 + y^2),$$

on the region

$$D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < \pi/2\},$$

which is a circle centre at the origin with radius strictly less than $\pi/2$. Then, it is easy to check that $V(0,0) = 0$ and $V(x,y) > 0$ for $(x,y) \in D \setminus \{(0,0)\}$. Hence, V is positive definite.

Liapunov's method: general theory

Example 18.3

The function

$$V(x,y) = (x+y)^2$$

on the region $D = \mathbb{R}^2$ satisfies $V(0,0) = 0$. But $V(-y,y) = 0$ and so V is zero also on the line $y = x$. This V is positive semidefinite.

Returning to the nonlinear system

$$y_1' = F_1(y_1,y_2), \qquad y_2' = F_2(y_1,y_2) \text{ for } t \in I,$$

and let V be a function of (y_1,y_2). Then,

$$\boxed{\frac{d}{dt}V(y_1(t),y_2(t)) = \frac{\partial V}{\partial y_1}y_1' + \frac{\partial V}{\partial y_2}y_2' = \left(\frac{\partial V}{\partial y_1}F_1 + \frac{\partial V}{\partial y_2}F_2\right)(y_1,y_2) =: W(y_1,y_2)}.$$

We now state two theorems - the first is about stability and the second is about instability.

Liapunov's method: general theory

Theorem 18.4

(Liapunov's stability theorem). *Consider the autonomous system*

$$y_1' = F_1(y_1, y_2), \quad y_2' = F_2(y_1, y_2) \text{ for } t \in I,$$

with an isolated critical point $(0, 0)$*. Suppose there is a function* V *that is continuous with continuous derivatives and is* **positive definite** *on a region* D*. If*

(a) *the function* $W(y_1, y_2) = \left(\frac{\partial V}{\partial y_1} F_1 + \frac{\partial V}{\partial y_2} F_2\right)(y_1, y_2) = \frac{d}{dt} V(y_1(t), y_2(t))$ *is* **negative semidefinite** *on* D*, then* $(0, 0)$ *is* **stable***.*

(b) *the function* $W(y_1, y_2)$ *is* **negative definite** *on* D*, then* $(0, 0)$ *is* **asym. stable***.*

Remark: We can not determine the type of the critical point by Liapunov's method. We need to look at the associated linear system together to do that.

Liapunov's method: general theory

Let's apply this to the undamped pendulum: Recall we have the total energy

$$V(y_1, y_2) = mgL(1 - \cos y_1) + \frac{1}{2} mL^2 y_2^2.$$

Consider the region D given as

$$D := (-\pi/2, \pi/2) \times \mathbb{R},$$

then V is positive definite in D with $V(0,0) = 0$. We saw that

$$\frac{d}{dt} V(y_1(t), y_2(t)) = 0 = W(y_1, y_2).$$

Since the zero function is negative semi-definite on D, we obtain from Thm. 18.11 that the critical point $(0,0)$ is stable.

Liapunov's method: general theory

For the critical point $(2\pi, 0)$ we transform the system to

$$\begin{pmatrix} w_1' \\ w_2' \end{pmatrix} = \begin{pmatrix} w_2 \\ -\frac{g}{L}\sin w_1 \end{pmatrix}, \qquad \text{for } w_1 = y_1 - 2\pi, \quad w_2 = y_2.$$

The same function

$$V(w_1, w_2) = mgL(1 - \cos w_1) + \frac{1}{2} mL^2 w_2^2$$

satisfies

$$\frac{d}{dt} V(w_1(t), w_2(t)) = 0 = W(w_1(t), w_2(t)),$$

and V is positive definite on the region $D := (-\pi/2, \pi/2) \times \mathbb{R}$. This corresponds to the region $(3\pi/5, 5\pi/2) \times \mathbb{R}$ for the original variables (y_1, y_2). Therefore, by Thm. 18.11, $(2\pi, 0)$ is a stable critical point.

Liapunov's method: general theory

For instability we have the following theorem.

Theorem 18.5

(Liapunov's instability theorem). *Consider the autonomous system*

$$y_1' = F_1(y_1, y_2), \qquad y_2' = F_2(y_1, y_2) \quad \text{for } t \in I,$$

with an isolated critical point $(0,0)$. *Suppose there is a function* $V = V(x,y)$ *that is continuous with continuous derivatives and* $V(0,0) = 0$. *Suppose in* **every neighbourhood** *of* $(0,0)$ *there is at least one point* (z_{1*}, z_{2*}) *such that* $V(z_{1*}, z_{2*})$ *is positive (resp. negative).*
If there is a region D *with* $(0,0) \in D$ *and* $W(y_1, y_2)$ *is positive (resp. negative) definite in* D, *then the origin* $(0,0)$ *is an unstable critical point.*

The proof of this theorem is out the scope of this course.

Liapunov's method: general theory

For the instability theorem there is an additional condition to check, namely in **every neighbourhood** of $(0,0)$ there is at least one point (z_{1*}, z_{2*}) such that $V(z_{1*}, z_{2*})$ is positive (resp. negative). We demonstrate this with an example involving the critical point $(\pi, 0)$ of the undamped pendulum. Recall the equations are

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} y_2 \\ -\frac{g}{L}\sin y_1 \end{pmatrix},$$

and setting $z_1 = y_1 - \pi, z_2 = y_2$ yields

$$\begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = \begin{pmatrix} z_2 \\ \frac{g}{L}\sin z_1 \end{pmatrix},$$

so that the critical point $(y_1, y_2) = (\pi, 0)$ is now the critical point $(z_1, z_2) = (0,0)$. Looking at the total energy (now called U)

$$U(z_1, z_2) = mgL(1 - \cos(z_1 + \pi)) + \frac{1}{2}mL^2 z_2^2 = mgL(1 + \cos z_1) + \frac{1}{2}mL^2 z_2^2,$$

we see that $U(0,0) = 2mgL \neq 0$. Therefore we cannot use U as the function V and apply Thm. 18.5.

Liapunov's method: general theory

In addition, we can compute

$$\frac{d}{dt}U(z_1(t), z_2(t)) = 0,$$

and Thm. 18.5 requires W to be positive or negative definite (not semidefinite). Thus we need another function. The idea is to try

$$V(z_1, z_2) = z_2 \sin z_1.$$

Then, $V(0,0) = 0$ and

$$\frac{d}{dt}V(z_1(t), z_2(t)) = \frac{g}{L}\sin^2 z_1(t) + z_2(t)^2 \cos z_1(t) =: W(z_1(t), z_2(t)).$$

So for $z_1 \in (-\pi/4, \pi/4)$ and $z_2 \in \mathbb{R}$, the function $W(z_1, z_2)$ is positive definite in $D := (-\pi/4, \pi/4) \times \mathbb{R}$. The only thing remaining is to see if there are points in every neighbourhood of the origin where the function V is positive. Note that V is always positive in the region on D where $z_1, z_2 > 0$ or $z_1, z_2 < 0$. Hence, this condition is always satisfied and by Thm. 18.5 the critical point $(z_1, z_2) = (0, 0)$ is unstable.

Liapunov's method: general theory

Definition 18.6

(Liapunov function). The function V in Thm. 18.11 and 18.5 is known as a Liapunov function.

Remark 1

In general, there is no method to construct Liapunov functions, often a lucky guess is needed or intitution from physics.

Quadratic Liapunov functions

We now study system of equations that allows us to construct Liapunov functions with quadratic form. i.e., $V(x, y)$ looks like $ax^2 + bxy + cy^2$. First let's give a theorem.

Theorem 18.7

The function

$$V(x, y) = ax^2 + bxy + cy^2$$

for constants a, b, c satisfies the following properties

(a) *V is positive definite if and only if $a > 0$ and $4ac - b^2 > 0$.*

(b) *V is negative definite if and only if $a < 0$ and $4ac - b^2 > 0$.*

Quadratic Liapunov functions

Example 18.8

Consider the system

$$y_1' = -y_1 - y_1 y_2^2 = F_1(y_1, y_2), \qquad y_2' = -y_2 - y_1^2 y_2 = F_2(y_1, y_2).$$

Then, $F_1(0,0) = F_2(0,0) = 0$ and so $(0,0)$ is a critical point. If V is a Liapunov function then

$$\frac{d}{dt} V(y_1(t), y_2(t)) = \frac{\partial V}{\partial y_1}(-y_1 - y_1 y_2^2) + \frac{\partial V}{\partial y_2}(-y_2 - y_1^2 y_2).$$

We now assume V is of the form $V(x,y) = ax^2 + bxy + cy^2$. Then

$$\frac{\partial V}{\partial y_1} = 2ay_1 + by_2, \quad \frac{\partial V}{\partial y_2} = by_1 + 2cy_2,$$

so that

$$\frac{d}{dt} V(y_1(t), y_2(t)) = -\left[2a(y_1^2 + y_1^2 y_2^2) + b(2y_1 y_2 + y_1 y_2^3 + y_1^3 y_2) + 2c(y_2^2 + y_1^2 y_2^2)\right].$$

Quadratic Liapunov functions

Example 18.9. continue

Looking at the above expression, we should set $b = 0$ to remove the cubic terms (which can be positive or negative for different values of y_1 and y_2). Then, choosing for example $a = c = 0.5$, we obtain

$$\frac{dV}{dt} = -(y_1^2 + 2y_1^2 y_2^2 + y_2^2) =: W(y_1, y_2).$$

Now, it is easy to check that $W(0,0) = 0$ and $W(x,y) < 0$ for all $(x,y) \neq (0,0)$. This shows that W is negative definite on $D = \mathbb{R}^2$. By Thm. 18.11 we have that $(0,0)$ is an asym. stable critical point.

Quadratic Liapunov functions

Example 18.9. continue

If we use that method of locally linear systems, writing

$$\mathbf{f}(\mathbf{y}) = \begin{pmatrix} -y_1 - y_1 y_2^2 \\ -y_2 - y_1^2 y_2 \end{pmatrix},$$

we can show that $\frac{d\mathbf{y}(t)}{dt} = \mathbf{f}(\mathbf{y}(t))$ is locally linear near the critical point $(0,0)$. This is due to the fact that the entries of $\mathbf{f}$ are twice continuously differentiable functions. Computing the Jacobian matrix:

$$\mathbf{A} = D\mathbf{f}(\mathbf{x})|_{\mathbf{x}=\mathbf{0}} = \begin{pmatrix} -1 - y_2^2 & -2y_1 y_2 \\ -2y_1 y_2 & -1 - y_1^2 \end{pmatrix}\Bigg|_{(0,0)} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

we see that the eigenvalues of $\mathbf{A}$ are $r_1 = r_2 = -1$. By previous theorem, we deduce that the critical point $\mathbf{0}$ is asymptotically stable, which is consistent with our previous analysis with Liapunov's method.

Quadratic Liapunov functions

We present one more example involving stability.

Example 18.9

Consider

$$y_1' = -y_1^3 + 2y_1y_2^2 = F_1(y_1, y_2), \qquad y_2' = -2y_1^2y_2 - y_2^3 = F_2(y_1, y_2).$$

Note that $F_1(0,0) = F_2(0,0) = 0$ and so $(0,0)$ is a critical point. Assuming V is of the form $V(x,y) = ax^2 + bxy + cy^2$, computing

$$\begin{aligned}\frac{d}{dt}V(y_1(t), y_2(t)) &= (2ay_1 + by_2)(2y_1y_2^2 - y_1^3) + (by_1 + 2cy_2)(-2y_1^2y_2 - y_2^3)\\ &= 4ay_1^2y_2^2 + 2by_1y_2^3 - 2ay_1^4 - by_1^3y_2 - 2by_1^3y_2 - 4cy_1^2y_2^2 - by_1y_2^3 - 2cy_2^4.\end{aligned}$$

We again set $b = 0$ to remove the cubic terms, and choose $a = c = 1$, so that

$$\frac{dV}{dt} = -2y_1^4 - 2y_1^4 = W(y_1, y_2).$$

It is clear that W is negative definite on $D = \mathbb{R}^2$, and by Thm. 18.11 $(0,0)$ is an asymtotically stable critical point.

Quadratic Liapunov functions

The last example is about instability.

Example 18.10

Consider
$$x' = 2x^3 - y^3, \quad y' = 2xy^2 + 4x^2y + 2y^3,$$
where $(0,0)$ is a critical point. Consider $V(x,y) = ax^2 + cy^2$, then
$$\frac{d}{dt}V(x(t),y(t)) = 4ax^4 + 4cy^4 + 4cxy^3 - 2axy^3 + 8cx^2y^2.$$
Choosing $4c = 2a$ to remove the term involving xy^3, for example $a = 1, c = 0.5$, leads to
$$\frac{dV}{dt} = 4x^4 + 2y^4 + 4x^2y^2 = W(x,y).$$
It is clear that $W(0,0) = 0$ and $W(x,y)$ is positive for all $(x,y) \neq (0,0)$. So W is positive definite on $D = \mathbb{R}^2$. However, to apply Thm. 18.5 we still need to check that for every neighbourhood of $(0,0)$ there is a point (x_*, y_*) where the function $V(x,y) = x^2 + \frac{1}{2}y^2$ is positive at (x_*, y_*). Since $V(x,y)$ is strictly positive for $(x,y) \neq (0,0)$, this is satisfied. Thus, by Thm. 18.5 $(0,0)$ is an unstable critical point.

Appendix: Proof for Liapunov's stability theorem

Theorem 18.11

(Liapunov's stability theorem). *Consider the autonomous system*

$$y_1' = F_1(y_1, y_2), \quad y_2' = F_2(y_1, y_2) \text{ for } t \in I,$$

with an isolated critical point $(0, 0)$*. Suppose there is a function* V *that is continuous with continuous derivatives and is* **positive definite** *on a region* D*. If*

(a) *the function* $W(y_1, y_2) = \left(\frac{\partial V}{\partial y_1} F_1 + \frac{\partial V}{\partial y_2} F_2\right)(y_1, y_2) = \frac{d}{dt} V(y_1(t), y_2(t))$ *is* **negative semidefinite** *on* D*, then* $(0, 0)$ *is* **stable**.

(b) *the function* $W(y_1, y_2)$ *is* **negative definite** *on* D*, then* $(0, 0)$ *is* **asym. stable**.

Appendix: Proof for Liapunov's stability theorem

Recall the definition:

Definition 18.12

(Stability). Let $\mathbf{y}_*$ be a critical point of the autonomous system

$$\frac{d\mathbf{y}(t)}{dt} = \mathbf{f}(\mathbf{y}(t)),$$

i.e., $\mathbf{f}(\mathbf{y}_*) = \mathbf{0}$. We say that

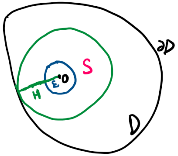
(1) $\mathbf{y}_*$ is **stable** if for any $\epsilon > 0$, there exists a $\delta > 0$ (depending on y_* and ϵ) such that any solution $\mathbf{y} = \phi(t)$ to $\frac{d\mathbf{y}(t)}{dt} = \mathbf{f}(\mathbf{y}(t))$ satisfies

$$\text{if} \quad \|\phi(\mathrm{t}_0) - \mathbf{y}_*\| < \delta \quad \text{then} \quad \|\phi(\mathrm{t}) - \mathbf{y}_*\| < \epsilon \quad \forall \mathrm{t} \geq \mathrm{t}_0,$$

where t_0 is some real number.

1. $\mathbf{y}_*$ is **unstable** if it is not stable.
2. $\mathbf{y}_*$ is **asymptotically stable** if it is **stable** and there exists $\delta_0 > 0$ (depending only on $\mathbf{y}_*$) such that
$$\text{if} \quad \|\phi(\mathrm{t}_0) - \mathbf{y}_*\| < \delta_0 \quad \text{then} \quad \phi(\mathrm{t}) \to \mathbf{y}_* \text{ as } \mathrm{t} \to \infty.$$

Appendix: Proof for Liapunov's stability theorem



$S = \{\vec{x} \mid \varepsilon \leq \|\vec{x}\| \leq H\}$ closed bounded region.

Proof. Only show (a). Assume that $H = \min_{\mathbf{x} \in \partial D} \|\mathbf{x}\|$ (∂D means the boundary of D, $\mathbf{x} = (x, y)$). Take any $\varepsilon (0 < \varepsilon < H)$, since V is positive definite in the region D, then the set $S = \{\mathbf{x} | \varepsilon \leq \|\mathbf{x}\| \leq H\}$ is a bounded closed set, thus there will be a minimum value for V on this set S, denote $l = \min_{\mathbf{x} \in S} V(\mathbf{x})$.

Since $V(\mathbf{x})$ is a continuous function, and $V(\mathbf{0}) = 0 < l$, there exists a $\delta (0 < \delta < \varepsilon)$ such that $V(\mathbf{x}) < l$ when $\|\mathbf{x}\| \leq \delta$.

Taking the initial point $\mathbf{x}_0$ with $\|\mathbf{x}_0\| \leq \delta$, and assuming $\mathbf{x}(t, t_0, \mathbf{x}_0)$ is the solution with starting point $\mathbf{x}_0$ and starting time t_0, we can show that for $t \geq t_0$, we always have $\|\mathbf{x}(t, t_0, \mathbf{x}_0)\| < \varepsilon$.

Appendix: Proof for Liapunov's stability theorem

Otherwise, if there is a time t_1 such that $\varepsilon \leq \|\mathbf{x}(t_1, t_0, \mathbf{x}_0)\| \leq H$, then $V(\mathbf{x}(t_1, t_0, \mathbf{x}_0)) \geq l$, but

$$V(\mathbf{x}(t_1, t_0, \mathbf{x}_0)) - V(\mathbf{x}_0) = \int_{t_0}^{t_1} \frac{dV}{dt} dt \leq 0, \quad \text{since} \frac{dV}{dt} \leq 0.$$

Thus

$$l \leq V(\mathbf{x}(t_1, t_0, \mathbf{x}_0)) \leq V(\mathbf{x}_0) < l.$$

Contradiction.