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Homework 5

Due: March 9, 2021

1. Same as Problem 5 of Assignment 3. Consider one communication link from node A to node B. Each data packet over this link needs one timeslot to be transmitted. Let A_n be the number of packets that arrive at node A during timeslot n. Assume that $\{A_n : n = 0, 1, 2, \dots, \}$ is an iid sequence with distribution

$$\begin{aligned} \mathbb{P}\{A_1 = 0\} &= 3/5, \\ \mathbb{P}\{A_1 = 1\} &= 1/5, \\ \mathbb{P}\{A_1 = 2\} &= 1/5. \end{aligned} \tag{1}$$

To make the counting of packets easier, assume the following sequence of events in each timeslot: packets arrive during the timeslot (waiting in a common buffer of infinite size if necessary), packets in transmission leave the system at the end of this timeslot if they reach the destination, a new packet starts transmission over a link at the beginning of next timeslot (equal to the end of current timeslot) if the link is free and there is a packet in the buffer to transmit. Let X_n be the number of packets in system at the beginning of time slot n. In Problem 5 of Assignment 3, $X = \{X_n : n = 0, 1, \dots\}$ was proved to be a DTMC.

   (a) Find a stationary distribution of the DTMC.

   (b) Is the stationary distribution unique? why?

   (c) What is the long-run fraction of packets that have to wait at least one timeslot before entering transmission?

   (d) What is the long-run expected number of packets in the system?

   (e) Find $T_{2,2}$, the expected number of timeslots needed for the DTMC to first return to state 2 when initially starting from state 2.

2. Assume $X = \{X_n : n \in \mathbb{Z}_+\}$ is a DTMC on state space $S = \{1, 2, 3\}$ with transition matrix

$$P = \begin{pmatrix} 0 & .3 & .7 \\ .6 & 0 & .4 \\ .4 & .6 & 0 \end{pmatrix}$$

When $X_n = i$ and $X_{n+1} = j$, a cost $c(i,j)$ of leaving state i and going to state j is incurred. Assume that

$$c = \begin{pmatrix} 0 & 2 & 5 \\ 10 & 0 & 1 \\ 4 & 10 & 0 \end{pmatrix}.$$

Find the long-run average cost per unit time.

3. Assume X is a DTMC on state space S. Suppose that state i is recurrent and $i \to j$. Prove that i and j communicate. Furthermore, compute

$$f_{ji} = \mathbb{P}_j\{T_i < \infty\}.$$

(Hint: By definition, $P_{ij}^k > 0$ for some k. Then use the "k-step analysis", which is similar to the first-step analysis.) As a consequence, state j is also recurrent.

4. Let X be a Markov chain with state space $\{a, b, c, d, e, f\}$ and transition matrix given by

$$P = \begin{bmatrix} 0.3 & 0.5 & 0 & 0 & 0 & 0.2 \\ 0 & 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 1.0 & 0 & 0 & 0 \\ 0 & 0.3 & 0 & 0 & 0 & 0.7 \\ 0.1 & 0 & 0.1 & 0 & 0.8 & 0 \\ 0 & 1.0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   (a) Draw the state diagram.
   (b) List positive recurrent states.
   (c) List the irreducible closed set(s).
   (d) List the transient states.
   (e) Calculate the $\lim_{n\to\infty} P^n$ matrix.
   (f) For each state state i in the state space, find $T_{i,i}$ the expected number of visits needed to come back to state i for the first time when the DTMC starts from state i.

5. Consider the following transition matrix:

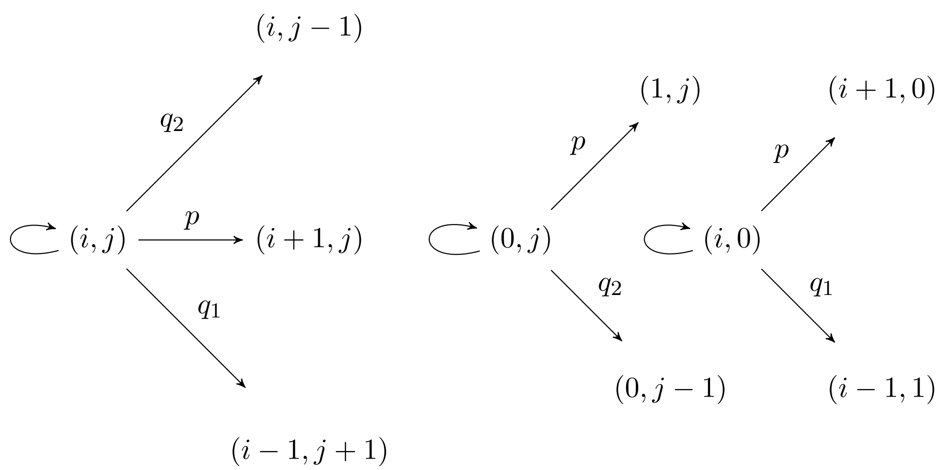
$$P = \begin{pmatrix} 0 & 0.5 & 0 & 0.5 \\ 0.6 & 0 & 0.4 & 0 \\ 0 & 0.7 & 0 & 0.3 \\ 0.8 & 0 & 0.2 & 0 \end{pmatrix} \qquad (2)$$

(a) Is the Markov chain periodic? Give the period of each state.

(b) Is $(\pi_1, \pi_2, \pi_3, \pi_4) = (33/96, 27/96, 15/96, 21/96)$ the stationary distribution of the Markov Chain?

(c) Is $P_{11}^{100} = \pi_1$? Is $P_{11}^{101} = \pi_1$? Give an expression for π_1 in terms of P_{11}^{100} and P_{11}^{101}.

6. (**Optional**) Consider a DTMC on state space $\mathbb{Z}_+^2$ with the following transition diagram with $p \in (0,1)$, $q_i \in (0,1)$, $i = 1, 2$. There are three situations: First, the current state is (i,j), where $i, j > 0$. Second, the current state is $(0,j)$, where $j > 0$. Third, the current state is $(i,0)$, where $i > 0$.



(a) Is the DTMC irreducible?

(b) Under which condition that the DTMC is positive recurrent?

(c) When the DTMC is positive recurrent, find its stationary distribution. Is it unique? why?

(d) Identify a condition under which the DTMC is transient? Prove your assertion.

(e) Identify a condition under which the DTMC is null recurrent. You are not expected to provide a proof.