Tutorial #1

1. Show that $\sqrt{3}$ is not a rational number.

2. (i) Prove the second part of the De Morgan law.

$$(A \cup B)^c = A^c \cap B^c$$

(ii) Show how induction can be used to conclude that

$$(A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n)^c = A_1^c \cap A_2^c \cap A_3^c \cap \cdots \cap A_n^c$$

for any finite $n \in \mathbb{N}$.

3. Let A and B be two nonempty subsets of $\mathbb{R}$, both bounded above.
(i) Show that

$$\sup(A \cup B) = \max\{\sup A, \sup B\}.$$

(ii) Extended the above formula to

$$\sup(A_1 \cup A_2 \cup \cdots \cup A_n) = \max\{\sup A_1, \sup A_2, \ldots, \sup A_n\}.$$

(iii) Does the above formula extend to the infinite case?

$$\sup\left(\bigcup_{n=1}^{\infty} A_n\right)$$

Considering two cases: (a) $A_n = [0, 1 - \frac{1}{n}]$ and (b) $A_n = [0, n]$.
(iv) If $A \subset B \subset \mathbb{R}$, A is nonempty, B is bounded above, show that

$$\sup A \leq \sup B.$$

4. Dedekind's Cut Property. If A and B are nonempty, disjoint sets with $A \cup B = \mathbb{R}$ and $a < b$ for all $a \in A$ and $b \in B$, then there exists $c \in \mathbb{R}$ such that $x \leq c$ whenever $x \in A$ and $x \geq c$ whenever $x \in B$.
The pair (A, B) is called a Dedekind's cut.

In this course, we take the Least Upper Bound Property as the Axiom of Completeness (Aoc), namely, every nonempty bounded subset of $\mathbb{R}$ there exists a least upper bound.

Question: Show that the Least Upper Bound Property and Dedekind's Cut Property are equivalent to each other.

Hence, one may choose Dedekind's Cut Property as the AoC, see [Zorich]. Indeed, there are several other equivalent properties can serve as AoC, [Tao] uses another one, namely any Cauchy's sequence is convergent in $\mathbb{R}$. We shall discuss them later.

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