Chapter 1. Preliminaries *

Reference Books

[Zorich] V.A. Zorich, *Mathematical Analysis I*, 2nd ed., Springer, Berlin, Heidelberg, 2016.

[Abbott] Stephen Abbott, *Understanding Analysis*, 2nd ed., Springer, New York, Heidelberg, 2015.

[Tao] Terence Tao, *Analysis I, II*, 3rd ed., Hindustan Book Agency, New Delhi, 2015.

[Rudin] Walter Rudin, *Principles of Mathematical Analysis*, 3rd ed., McGraw-Hill, New York, 1976.

1 What is analysis and why do Analysis?

See [Tao-I] Chapter 1.

2 Naive Set Theory

Recommend Reading: [Zorich] Section 1.2; [Tao] Chapter 3; [Rudin] Chapters 1 and 2.

Set theory and mathematical logic are foundations of almost every branch of mathematics. We shall use the concept of set to introduce the real number system later, and for the sake of brevity, we shall only discuss the *naive set theory* (vs. *axiomatic set theory*).

*Lecture notes for CUHKSZ course MAT2006: Elementary Real Analysis.

2.1 Definitions

Definition 1 (Set, informal)**.** We define a *set* A to be any unordered collection of objects, e.g., $\{3, 8, 5, 2\}$ is a set. If x is an object, we say that x is an element of A, denoted by $x \in A$, if x lies in the collection; otherwise we write $x \notin A$.

Definition 2 (Empty set)**.** The set which contains no element is called the *empty set* and denoted by $\emptyset$. If a set contains at least one element, it is *nonempty.*

Definition 3 (Subset and equality of sets)**.** If A and B are sets, and if every element of A is an element of B, we say that A is a *subset* of B, and write $A \subset B$ or $B \supset A$. If, in addition, there is an element of B which is not in A, then A is said to be *proper subset* of B.

If $A \subset B$ and $B \subset A$, then we say A and B are equal, and write $A = B$. Otherwise $A \neq B$.

Remark. (i) A set A is also an object, that is if A is a set, then $\{A\}$ is also a set. We have $A \in \{A\}$. Be careful: $A \neq \{A\}$ or $A \not\subset \{A\}$.

(ii) The set $\{\emptyset\}$ has one element, and the set $\{\emptyset, \{\emptyset\}\}$ has two elements. The two statements $\{\emptyset\} \in \{\emptyset, \{\emptyset\}\}$ and $\{\emptyset\} \subset \{\emptyset, \{\emptyset\}\}$ are both true. (Why?)

(iii) The definition of set in Definition 1 is intuitive enough, but informal, it doesnot clarify which collections of objects are considered to be sets. Indeed, historically, Cantor's[1] original definition of sets (the naive set theory) has challenged by the Russell's paradox, wether the collection of all sets is a set or not. If yes, then Bernard Russell has a paradox; see [Tao] Section 3.2. To remove this ambiguity, the *axiomatic set theory* gives a rigorous definition of sets based on several axioms.

Example 2.1. (i) There are many acceptable ways to assert the contents of a set. The set of natural numbers was defined by listing the elements: $\mathbb{N} = \{1, 2, 3, \dots\}$.

(ii) Sets can also be described in words. For instance, we can define the set E to be the collection of even natural numbers.

(iii) Sometimes it is more efficient to provide a kind of rule or algorithm for determining the elements of a set. As an example, let

$$S = \{x \in Q \,|\, x^2 < 2\}.$$

[1] Georg Cantor (1845–1918) was a German mathematician. He created set theory, which has become a fundamental theory in mathematics. Cantor established the importance of one-to-one correspondence between the members of two sets, defined infinite and well-ordered sets, and proved that the real numbers are more numerous than the natural numbers.

2.2 Elementary Operations on Sets

Let A and B be subsets of M.

a. The *union* of A and B is the set

$$A \cup B = \{x \in M \mid x \in A \text{ or } x \in B\}.$$

b. The *intersection* of A and B is the set

$$A \cap B = \{x \in M \mid x \in A \text{ and } x \in B\}.$$

c. The *difference* between A and B is the set

$$A \setminus B = \{x \in M \mid x \in A \text{ and } x \notin B\}.$$

The difference between the set M and one of its subsets A is usually called the *complement* of A in M and denoted $C_M A$, or simply CA when the set in which the complement of A is being taken is clear from the context. (Also denoted by A^c sometimes).

Exercise 1 (De Morgan's law)**.**

$$(A \cup B)^c = A^c \cap B^c$$
$$(A \cap B)^c = A^c \cup B^c$$

d. The *direct (Cartesian[2]) product* of A and B is the set

$$A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}.$$

Example 2.2. Using the previously defined sets in Example 2.1 to illustrate the operations of intersection and union, we notice that

$$\mathbb{N} \cup E = \mathbb{N}, \qquad \mathbb{N} \cap E = E, \qquad \mathbb{N} \cap S = \{1\}, \qquad E \cap S = \emptyset.$$

We say that the sets E and S are *disjoint*.

Example 2.3 (Union or intersection of infinite collection of sets)**.** Let

$$A_1 = \{1, 2, 3, \dots\} = \mathbb{N},$$
$$A_2 = \{2, 3, 4 \dots\},$$
$$A_3 = \{3, 4, 5 \dots\},$$

[2] R. Descartes (1596–1650) – outstanding French philosopher, mathematician and physicist who made fundamental contributions to scientific thought and knowledge.

and, in general, for each $n \in \mathbb{N}$, define the set

$$A_n = \{n, n+1, n+2, \dots\}.$$

The result is a nested chain of sets

$$A_1 \supset A_2 \supset A_3 \supset \cdots,$$

where each successive set is a subset of all the previous ones. Because of the nested property of this particular collection of sets, it is not too hard to see that

$$\bigcup_{n=1}^{\infty} A_n := A_1 \cup A_2 \cup A_3 \cup \cdots = A_1.$$

Example 2.4. (a) The set $\mathbb{R}^2 := \mathbb{R} \times \mathbb{R}$ is the real plane, and in general, $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \mathbb{R}$ (n copies of $\mathbb{R}$) is the real n-dimensional space.

(b) The set $\mathbb{Z}^2 := \mathbb{Z} \times \mathbb{Z}$ consists of all the integer lattice points (m, n) on the plane.

(c) The set $[0,1] \times [0,1]$ is a square in $\mathbb{R}^2$.

2.3 Functions

Let X and Y be certain sets. We say that there is a *mapping*, or a *function*, defined on X with values in Y if, by virtue of some rule f, to each element $x \in X$ there corresponds an element $y \in Y$. We write

$$f: \quad X \to Y.$$

Notations:

- X: *domain* of definition
- $x \in X$: argument or dependent variable
- $f(x) \in Y$: image of x
- $\{y \in Y \mid f(x) = y \text{ for some } x \in X\}$: *range* of f.

Example 2.5 (Dirichlet [3], 1829)**.**

$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

The domain of g is all of $\mathbb{R}$, and the range is the set $\{0, 1\}$.

[3] Johann Dirichlet (1805–1859) was a German mathematician who made deep contributions to number theory (including creating the field of analytic number theory), and to the theory of Fourier series and other topics in mathematical analysis; he is credited with being one of the first mathematicians to give the modern formal definition of a function.

If $A \subset X$ and $f: X \to Y$ is a function, we denote by $f|_A$ the function $\phi: A \to Y$ that agrees with f on A. More precisely, $f|_A(x) := f(x)$ if $x \in A$. The function $f|_A$ is called the *restriction* of f to A, and the function $f: X \to Y$ is called an *extension* or a *continuation* of ϕ to X.

The *image* of a set $A \subset X$ under the mapping $f: X \to Y$ is defined as the set

$$f(A) = \{y \in Y \mid \exists x \in X, \text{ s.t. } y = f(x)\}.$$

The set

$$f^{-1}(B) = \{x \in X \mid f(x) \in B\}.$$

consisting of the elements of X whose images belong to B is called the *pre-image* of the set $B \subset Y$.

A mapping $f: X \to Y$ is said to be

- *surjective* (a mapping of X onto Y) if $f(X) = Y$.
- *injective* (or an *embedding* or an *injection*) if $f(x_1) \neq f(x_2)$ for any $x_1 \neq x_2$.
- *bijective* (or a *one-to-one correspondence*) if it is both surjective and injective.

If the mapping $f: X \to Y$ is bijective, there naturally arise a mapping

$$f^{-1}: \quad Y \to X$$

defined as follows: if $f(x) = y$, then $f^{-1}(y) = x$. This mapping is called the *inverse* of the original mapping f.

3 The cardinality of a set

The set X is said to be *equipollent* to the set Y if there exists a bijective mapping of X onto Y, that is, a point $y \in Y$ is assigned to each $x \in X$, the elements of Y assigned to different elements of X are different, and every point of Y is assigned to some point of X.

The equipollent relation, denoted by $\sim$, between sets is an *equivalence* relation,

- reflexivity: $X \sim X$
- symmetry: if $X \sim Y$, then $Y \sim X$
- transitivity: if $X \sim Y$ and $Y \sim Z$, then $X \sim Z$.

Examples of equivalence relations: (i) similar triangles; (ii) similar matrices; (iii) matrices with same determinate; etc.

If a binary relation $\sim$ is an equivalence relation defined for elements in a set S. Then the *equivalence class* of $a \in S$ under $\sim$, denoted by $[a]$, is defined as $[a] = \{b \in S \mid b \sim a\}$

Return to the equipollent relation. The class to which a set X belongs is called the *cardinality* of X, and also the *cardinal* or *cardinal number* of X. It is denoted $\operatorname{card} X$. If $X \sim Y$, we write $\operatorname{card} X = \operatorname{card} Y$.

The possibility of being equipollent to a proper subset of itself is a characteristic of infinite sets that Dedekind even suggested taking as the definition of an infinite set. Thus a set is called finite (in the sense of Dedekind) if it is not equipollent to any proper subset of itself; otherwise, it is called infinite.

4 Logic and Proofs

[Abbott] pp. 8-10.

"Writing rigorous mathematical proofs is a skill best learned by doing, and there is plenty of on-the-job training just ahead. As Hardy indicates, there is an artistic quality to mathematics of this type, which may or may not come easily, but that is not to say that anything especially mysterious is happening. A proof is an essay of sorts. It is a set of carefully crafted directions, which, when followed, should leave the reader absolutely convinced of the truth of the proposition in question. To achieve this, the steps in a proof must follow logically from previous steps or be justified by some other agreed-upon set of facts. In addition to being valid, these steps must also fit coherently together to form a cogent argument. Mathematics has a specialized vocabulary, to be sure, but that does not exempt a good proof from being written in grammatically correct English."

Theorem 1. *There is no rational number whose square is 2.*

A rational number is any number that can be expressed in the form p/q, where p and q are integers. Thus, what the theorem asserts is that no matter how p and q are chosen, it is never the case that $(p/q)^2 = 2$. The line of attack is indirect, using a type of argument referred to as a *proof by contradiction*. The idea is to assume that there is a rational number whose square is 2 and then proceed along logical lines until we reach a conclusion that is unacceptable. At this point, we will be forced to retrace our steps and reject the erroneous assumption that some rational number squared is equal to 2. In short, we will prove that the theorem is true by demonstrating that it cannot be false.

Proof. Assume, for contradiction, that there exist integers p and q satisfying

$$\left(\frac{p}{q}\right)^2 = 2. \tag{4.1}$$

We may also assume that p and q have no common factor, because, if they had one, we could simply cancel it out and rewrite the fraction in lowest terms. Now, equation (4.1) implies

$$p^2 = 2q^2. \tag{4.2}$$

From this, we can see that the integer p^2 is an even number (it is divisible by 2), and hence p must be even as well because the square of an odd number is odd. This allows us to write $p = 2r$, where r is also an integer. If we substitute $2r$ for p in equation (4.2), then a little algebra yields the relationship

$$2r^2 = q^2,$$

which implies that q^2 is even, and hence q must also be even. Thus, we have shown that p and q are both even (i.e., divisible by 2) when they were originally assumed to have no common factor. From this logical impasse, we can only conclude that equation (4.1) cannot hold for any integers p and q, and thus the theorem is proved. $\square$

The one proof we have seen at this point uses an indirect strategy called *proof by contradiction*. This powerful technique will be employed a number of times in our upcoming work. Nevertheless, most proofs are direct. (It also bears mentioning that using an indirect proof when a direct proof is available is generally considered bad form.) A direct proof begins from some valid statement, most often taken from the theorem's hypothesis, and then proceeds through rigorously logical deductions to a demonstration of the theorem's conclusion. As we saw in Theorem 1, an indirect proof always begins by negating what it is we would like to prove. This is not always as easy to do as it may sound. The argument then proceeds until (hopefully) a logical contradiction with some other accepted fact is uncovered. Many times, this accepted fact is part of the hypothesis of the theorem. When the contradiction is with the theorem's hypothesis, we technically have what is called a *contrapositive* proof. The next proposition illustrates a number of the issues just discussed and introduces a few more.

Theorem 2. *Two real numbers a and b are equal if and only if for every real number $\epsilon > 0$ it follows that $|a - b| < \epsilon$.*

Proof. There are two key phrases in the statement of this proposition that warrant special attention. One is "for every," which will be addressed in a moment. The other is "if and only if." To say "if and only if" in mathematics is an economical way of stating that the proposition is true in two directions. In the forward direction, we must prove the statement:

($\Rightarrow$) *If* $a = b$*, then for every real number* $\epsilon > 0$ *it follows that* $|a - b| < \epsilon$*.*

We must also prove the converse statement:

($\Leftarrow$) *If for every real number* $\epsilon > 0$ *it follows that* $|a - b| < \epsilon$*, then we must have* $a = b$*.*

For the proof of the first statement, there is really not much to say. If $a = b$, then $|a - b| = 0$, and so certainly $|a - b| < \epsilon$ no matter what $\epsilon > 0$ is chosen.

For the second statement, we give a proof by contradiction. The conclusion of the proposition in this direction states that $a = b$, so we assume that $a \neq b$. Heading off in search of a contradiction brings us to a consideration of the phrase "for every $\epsilon > 0$." Some equivalent ways to state the hypothesis would be to say that "for all possible choices of $\epsilon > 0$ or "no matter how $\epsilon > 0$ is selected, it is always the case that $|a - b| < \epsilon$." But assuming $a \neq b$ (as we are doing at the moment), the choice of

$$\epsilon_0 = |a - b| > 0$$

poses a serious problem. We are assuming that $|a - b| < \epsilon$ is true for every $\epsilon > 0$, so this must certainly be true of the particular ϵ_0 just defined. However, the statements

$$|a - b| < \epsilon_0 \qquad \text{and} \qquad |a - b| = \epsilon_0$$

cannot both be true. This contradiction means that our initial assumption that $a \neq b$ is unacceptable. Therefore, $a = b$, and the indirect proof is complete. □

One of the most fundamental skills required for reading and writing analysis proofs is the ability to confidently manipulate the quantifying phrases "for al" and "there exists." Significantly more attention will be given to this issue in many upcoming discussions.

5 Induction

One final trick of the trade, which will arise with some frequency, is the use of *induction* arguments. Induction is used in conjunction with the natural numbers $\mathbb{N}$ (or sometimes with the set $\mathbb{N} \cup \{0\}$). The fundamental principle behind induction is that if S is some subset of $\mathbb{N}$ with the property that

(i) S contains 1 and

(ii) whenever S contains a natural number n, it also contains $n + 1$,

then it must be that $S = \mathbb{N}$. As the next example illustrates, this principle can be used to define sequences of objects as well as to prove facts about them.

Example 5.1. Let $x_1 = 1$, and for each $n \in \mathbb{N}$ define

$$x_{n+1} = \frac{1}{2}x_n + 1.$$

Using this rule, we can compute $x_2 = 3/2$, $x_3 = 7/4$, and it is immediately apparent how this leads to a definition of x_n for all $n \in \mathbb{N}$. The sequence just defined appears at the outset to be increasing. For the terms computed, we have $x_1 \leq x_2 \leq x_3$. Let's use induction to prove that this trend continues; that is, let's show

$$x_n \leq x_{n+1} \qquad \forall n \in \mathbb{N}. \tag{5.1}$$

For $n = 1$, $x_1 = 1$ and $x_2 = 3/2$, so that $x_1 \leq x_2$ is clear. Now, we want to show that

if we have $x_n \leq x_{n+1}$, then it follows that $x_{n+1} \leq x_{n+2}$.

Think of S as the set of natural numbers for which the claim in equation (5.1) is true. We have shown that $i \in S$. We are now interested in showing that if $n \in S$, then $n + 1 \in S$ as well. Starting from the induction hypothesis $x_n \leq x_{n+1}$, we can multiply across the inequality by $1/2$ and add 1 to get

$$\frac{1}{2}x_n + 1 \leq \frac{1}{2}x_{n+1} + 1,$$

which is precisely the desired conclusion $x_{n+1} \leq x_{n+2}$. By induction, the claim is proved for all $n \in \mathbb{N}$.

Any discussion about why induction is a valid argumentative technique immediately opens up a box of questions about how we understand the natural numbers. A more atheistic and mathematically satisfying approach to $\mathbb{N}$ is possible from the point of view of axiomatic set theory; see for example Chapter 2 of [Tao-I].