MAT 3007 – Optimization

Interpreting Dual Problems

Lecture 09 *June 29th*

Andre Milzarek iDDA / CUHK-SZ

Repetition & Logistics

What have we learned so far about LP duality?

- Given any linear program, we can construct its dual.
- Duality theorems (suppose the primal is a min. problem).

Weak Duality Theorem

If x and y are feasible points of the primal and dual problem, then:

$$b^\top y \le c^\top x.$$

- Primal (dual) feasible points can be used to construct bounds for the objective value of the dual (primal) problem.
- If one problem is unbounded, the other one must be infeasible.
- Optimality Conditions: If x is primal feasible, y is dual feasible, and $c^\top x = b^\top y$, then they are both optimal.

Strong Duality Theorem

If a linear program has an optimal solution, so does its dual, and the optimal value of the primal and dual problems are equal.

- In the proof of the strong duality theorem, we showed that the simplex method can actually find the dual optimal solution when it finishes.
- Complementarity conditions.

All possible states of a pair of primal/dual linear programs:

P | D	Finite Optimum	Unbounded	Infeasible
Finite Optimum	✓		
Unbounded			✓
Infeasible		✓	✓

Remark & Logistics

Remark/Consequence:

- If the objective function of an LP ($\rightsquigarrow$ min) is lower bounded, then the problem possesses an optimal solution.

Logistics:

- Exercise sheet 3 is online since Friday; it is due on Saturday, July 4th, 11:00am.
- The midterm project will be posted asap. The tentative project period is $\approx$ two weeks; the group size will be three members.

Agenda:

- Dual solutions via simplex tableau.
- Interpretation of dual problems.
- Sensitivity analysis.

Duality via Simplex Tableau

If the primal optimal solution is obtained from using the simplex tableau and the initial problem was constructed from adding slack variables, then one can find the optimal dual solution $(A_B^{-1})^\top c_B$ from the simplex tableau when it finishes.

When the initial tableau is constructed from adding slack variables (thus it is naturally a canonical form), we can write the initial tableau as follows:

$c^\top$	0_m	0
A	I_m	b

$c^\top$	0_m	0
A	I_m	b

Suppose after some iterations, the simplex method reaches an optimal solution with basis B. Then the tableau becomes:

$c^\top - c_B^\top A_B^{-1} A$	$-c_B^\top A_B^{-1}$	$-c_B^\top A_B^{-1} b$
$A_B^{-1} A$	A_B^{-1}	$A_B^{-1} b$

Therefore, the final reduced costs corresponding to the original identity matrix part is the optimal dual solution.

Consider the production planning problem:

$$
\begin{array}{rrrrrrl}
\text{minimize} & -x_1 & -2x_2 & & & & \\
\text{subject to} & x_1 & & +s_1 & & & = 100 \\
& & 2x_2 & & +s_2 & & = 200 \\
& x_1 & +x_2 & & & +s_3 & = 150 \\
& x_1, & x_2, & s_1, & s_2, & s_3 & \geq 0
\end{array}
$$

Dual:

$$
\begin{array}{rrrrl}
\text{maximize} & 100y_1 & +200y_2 & +150y_3 & \\
\text{subject to} & y_1 & & +y_3 & \leq -1 \\
& & 2y_2 & +y_3 & \leq -2 \\
& y_1, & y_2, & y_3 & \leq 0
\end{array}
$$

The initial tableau for the primal:

B	-1	-2	0	0	0	0
3	1	0	1	0	0	100
4	0	2	0	1	0	200
5	1	1	0	0	1	150

Final tableau:

B	0	0	0	1/2	1	250
1	1	0	0	-1/2	1	50
3	0	0	1	1/2	-1	50
2	0	1	0	1/2	0	100

What is the dual optimal solution?

- $(y_1, y_2, y_3) = (0, -1/2, -1)$, with objective value -250.

Discussion

If the problem is not derived from adding slack variables (therefore the initial top row is not in the form $(c^\top, 0)$), then this method may not give the right answer.

- In that case, one can just compute $(A_B^{-1})^\top c_B$.

Interpreting Dual Problems

We now discuss several examples and provide interpretations of the corresponding dual problems:

- The production planning problem.
- The multi-firm alliance problem.
- The alternative systems problem.
- The maximum flow problem.

Production Planning

Production Planning Problem

Recall the production planning problem:

$$
\begin{array}{rccl}
\text{maximize} & x_1 & +2x_2 & \\
\text{subject to} & x_1 & & \leq 100 \\
& & 2x_2 & \leq 200 \\
& x_1 & +x_2 & \leq 150 \\
& x_1, & x_2 & \geq 0.
\end{array}
$$

The objective corresponds to profits; the constraints are resource constraints.

Dual of the Production Planning Problem

We consider thee dual and associate the three constraints with dual variables: p_1, p_2 and p_3.

$$\begin{array}{rrrrl} \text{minimize} & 100p_1 & +200p_2 & +150p_3 & \\ \text{subject to} & p_1 & & +p_3 & \geq 1 \\ & & 2p_2 & +p_3 & \geq 2 \\ & p_1, & p_2, & p_3 & \geq 0 \end{array}$$

What is the meaning of this optimization problem?

- Suppose we want to buy all resources of the company at unit prices p_1, p_2 and p_3. What prices should we offer?
- One unit of resource 1 and resource 3 can produce one unit of product 1 which is worth \$1. Therefore, if $p_1 + p_3 < 1$, the company rather produces by itself (than agree to sell).
- Similarly, it wouldn't sell if $2p_2 + p_3 < 2$.
- Other than that, the buyer wants to buy the resources for the lowest price and the dual problem finds that price.

In this example, we interpret the dual variables as fair prices for each resource.

- This is a very common type of interpretation for dual variables.
- Next, we look at an extension of the production planning problem and see how the dual problem can help solving important problem-related questions.

Multi-Firm Alliances

Multi-Firm Alliance Problem

Suppose there is a collection of firms $1, 2, \dots, m$, each making the same set of products.

- Firm i has access to resources b_i with consumption matrix A and profit c for each product (A and c are the same across all companies).

$\rightsquigarrow$ Each firm is trying to solve a production problem:

$$\begin{aligned} \text{maximize}_x \quad & c^\top x \\ \text{s.t.} \quad & Ax \le b_i \\ & x \ge 0. \end{aligned}$$

We denote the optimal profit for company i by V^i.

Multi-Firm Alliance Problem

Now suppose the firms can form an alliance so that they can pool their resources and produce together.

Assume a subset of firms S form an alliance. To maximize the profit of the alliance S, we solve the following LP:

$$\begin{aligned} \text{maximize}_x \quad & c^\top x \\ \text{s.t.} \quad & Ax \le \textstyle\sum_{i \in S} b_i \\ & x \ge 0. \end{aligned}$$

The optimal profit for the alliance is denoted by V^S.

We call the alliance involving all firms the grand alliance. In a grand alliance, all resources are pooled together. We denote the optimal profit for the grand alliance by V^*.

Multi-Firm Alliance Problem

One important question before forming an alliance is:

- How to allocate the profit of the alliance to each of its members so that each member/sub-alliance has the incentive to stay in that alliance?

Let us consider the grand alliance. Our question is whether the firms have incentive to form a grand alliance and what allocation rule we should use?

What constraints do we have?

1. The total allocation equals the total profit.
2. The allocated profits to any sub-alliance should be greater than what the sub-alliance could get if they separate from the grand alliance (no subset of players would want to leave the alliance).

Multi-Firm Alliance Problem

In order to make the grand alliance stable, an allocation $(z_1, \ldots, z_m)$ must satisfy:

$$\begin{aligned} \sum_{i=1}^{m} z_i &= V^* \\ \sum_{i \in S} z_i &\geq V^S \qquad \forall\ S \subseteq \{1, \ldots, m\}. \end{aligned}$$

We call the allocation $(z_1, \ldots, z_m)$ the core of the grand alliance.

- It is not trivial to find such z_i's or to even prove its existence.
- It involves checking 2^m inequalities.

⇝ However, we can use duality to find such a solution easily!

Solution: Consider the dual of the grand alliance production problem:

$$\begin{aligned} \text{minimize}_y \quad & \left(\textstyle\sum_{i=1}^m b_i\right)^\top y \\ \text{s.t.} \quad & A^\top y \geq c \\ & y \geq 0. \end{aligned}$$

⇝ We denote the optimal solution by y^*.

Theorem: Recovering the Core

Set $z_i = b_i^\top y^*$. Then $(z_1, ..., z_m)$ is the core of the grand alliance.

- y^* can be viewed as the prices for the resources and we just allocate according to the available resources of each firm.

Alternative Systems

Consider a set of linear inequalities:

$$A^\top y \le c.$$

Question: (when) does this system have a solution?

- To verify existence, we only need to find a solution (we call it a certificate).
- To prove non-existence, can we also have such a certificate?

Answer: $\rightsquigarrow$ yes!

- If we can find a vector x satisfying

$$Ax = 0, \quad x \ge 0, \quad c^\top x < 0$$

then there must be no solution to the system $A^\top y \le c$.

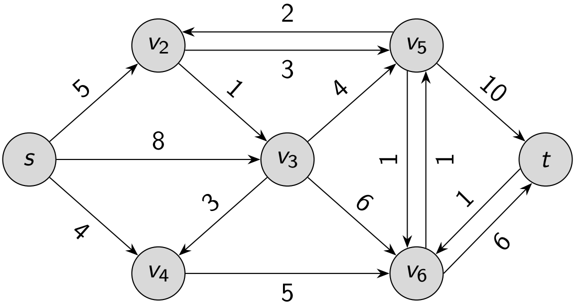
One can construct many more pairs of such alternative systems.

- Typically, it is hard to directly prove that a system is not feasible.
- LP duality provides an alternative approach via considering the dual and checking boundedness.

Maximum Flow Problems

The maximum flow problem can be described as follows:

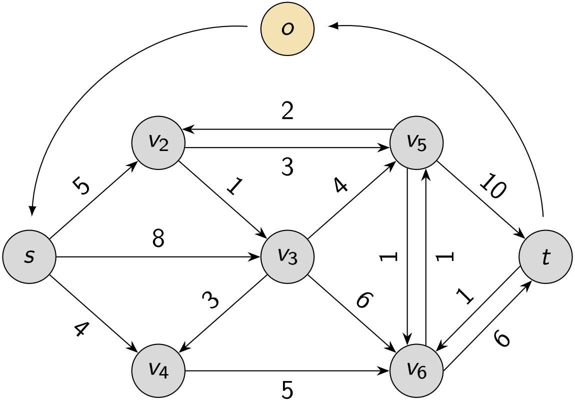
- Given a directed, weighted graph $G = (V, E)$ and a pair of nodes s and t (V: set of nodes; E: set of edges).
- One can think this as a traffic network.
- There is an edge capacity c_{ij} on each edge.
- Question: What is the largest amount of flow one can send from s to t subject to the capacity constraints?



Additional Transformation

Assume there is an imaginary node o, with edges (o, s) and (t, o). There is no capacity constraint on those two edges.

- The problem becomes a closed system. One wants to maximize the flow from o to s, which we denote by Δ.



LP Formulation

Using this transformation, we can write down the LP formulation. Let x_{ij} denote the amount of flow across edge (i,j).

$$
\begin{aligned}
\max_{x,\Delta} \quad & \Delta \\
\text{s.t.} \quad & \textstyle\sum_{j:(j,i)\in E} x_{ji} - \sum_{j:(i,j)\in E} x_{ij} = 0, && \forall\ i \neq s,t \\
& \textstyle\sum_{j:(j,s)\in E} x_{js} - \sum_{j:(s,j)\in E} x_{sj} + \Delta = 0 \\
& \textstyle\sum_{j:(j,t)\in E} x_{jt} - \sum_{j:(t,j)\in E} x_{tj} - \Delta = 0 \\
& x_{ij} \le w_{ij}, && \forall\ (i,j) \in E \\
& x_{ij} \ge 0, && \forall\ (i,j) \in E.
\end{aligned}
$$

- The first constraint is the flow balancing constraints for all nodes other than s and t.
- The second (third, resp.) constraint is the flow balancing constraints for node s (t, resp.).

We construct the dual problem:

$$\begin{aligned} \text{minimize} \quad & \sum_{(i,j)\in E} c_{ij} z_{ij} \\ \text{subject to} \quad & z_{ij} \geq y_i - y_j, \qquad && \forall\, (i,j) \in E \\ & y_s - y_t = 1 \\ & z_{ij} \geq 0 \end{aligned}$$

What does the dual problem mean?

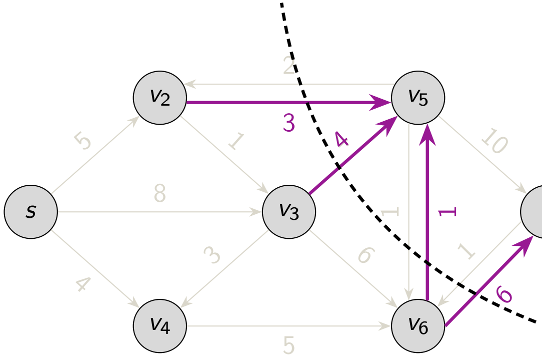
First assume all y_i's are either 0 or 1. Then:

- We assign a label (0 or 1) to each node; 1 to s and 0 to t.
- If i has a larger label than j for $(i,j) \in E$, there is a cost c_{ij}.

The dual problem is equivalent to finding a subset S of vertices containing s but not t, that minimizes the weight of a cut, i.e.

$$\sum_{i \in S, j \notin S} w_{ij}$$

- This is called the min-cut problem:



Theorem: Max-Flow & Min-Cut

The maximum flow of the network is equal to the smallest cut size of any subset S of vertices.

Corollary

If the value of a flow is equal to the value of some cut, then both are optimal.

- One can view the min-cut as the bottleneck of the network.
- The maximum flow that can be sent through this network is equal to the tightest bottleneck of this network.

This is one classical example of dual problems: maximum flow versus minimum cut.

Sensitivity Analysis

One important question when studying LP is as follows:

- How do the optimal solution and the optimal value change when the input changes?

This type of problems is called sensitivity analysis.

- We first study this question from a local perspective and then continue with global discussions.

Local Sensitivity

Consider the standard LP:

$$\begin{aligned} \text{minimize}_x \quad & c^\top x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0. \end{aligned}$$

We denote the associated optimal value by V.

- If A and c are fixed, V can be viewed as a function of b: $V(b)$.

Theorem: Differentiability of the Optimal Value Function

If the dual has a unique optimal solution y^*, then $\nabla V(b) = y^*$.

- If the dual optimal solution is not unique (or is unbounded or infeasible), then the gradient is not well-defined.
- If one changes b_i by a small amount Δb_i, then the change of the objective value will be $\Delta b_i y_i^*$

We know that the optimal value V is also the optimal value of the dual problem:

$$\begin{aligned} \text{maximize}_y \quad & b^\top y \\ \text{s.t.} \quad & A^\top y \le c, \end{aligned}$$

i.e., $V(b) = b^\top y^*$.

⇝ If we change b by a small amount Δb, such that the optimal sol. does not change, then the change of V must be $\Delta b^\top y^*$.

Similarly, if A and b are fixed, V can be viewed as a function of c.

Theorem: Differentiability of $V(c)$

If the primal prob. has a unique optimal sol. x^*, then $\nabla V(c) = x^*$.

If one changes c_i by a small amount Δc_i, then the change of the objective value will be $\Delta c_i x_i^*$.

- $\rightsquigarrow$ Reason: If we change c by a small amount Δc, such that the optimal solution does not change, then the change of V must be $\Delta c^\top x^*$.

The latter results also hold for inequality constraints (or maximization problems):

$$\begin{aligned} \text{maximize}_x \quad & c^\top x \\ \text{s.t.} \quad & Ax \leq b \\ & x \geq 0. \end{aligned}$$

We have:

1. If the dual has a unique optimal sol. y^*, then $\nabla V(b) = y^*$.
2. If the primal has a unique optimal sol. x^*, then $\nabla V(c) = x^*$.

- To see why this must be true, one can add a slack variable and transform it back to the standard form. We can then use the earlier result.

Questions?