Stochastic Processes

Lecture 17: Continous time Markov chains II

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Three machines, John & Jay repair

- On times are i.i.d. exponentially distributed with mean 6 hours.
- John's repair times are i.i.d. exponentially distributed with mean 2 hour.
- Jay's repair times are i.i.d. exponentially distributed with mean 1 hour.

Review: jump matrix and holding time rates

- $S = \{1, 2, 3\}$
- Jump matrix

$$J = \begin{pmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix}.$$

- Holding time rates

$$\lambda(1) = 3, \quad \lambda(2) = 2, \quad \lambda(3) = 1.$$

- Jump chain: $Y = \{Y_n : n = 0, 1, 2 \ldots\}$ is a DTMC with transition matrix J;

An alternative construction: competing clocks

- Competing clock parameters

$$\lambda_{ij} = \lambda(i) J_{ij}, \quad j \neq i.$$

- $u = \{u(n) : n = 1, 2, \dots\}$ is a i.i.d. random vectors; for each n, $u(n) = \left(u_1(n), \dots, u_{|S|}(n)\right)'$ is a vector of i.i.d. $\exp(1)$ random variables.
- $\sigma_0 = 0$, $X(\sigma_0) \in S$; σ_n is the nth jump times.
- $X(t) = X(\sigma_n) = i \in S$ for $\sigma_n \leq t < \sigma_{n+1}$, where the next jump time

$$\sigma_{n+1} = \sigma_n + \min_{j \neq i} \frac{1}{\lambda_{ij}} u_j(n+1). \tag{1}$$

- The new state that the Markov chain X jumps to is

$$X(\sigma_{n+1}) = j,$$

where j reaches minimum in (1).

Justification

- Holding times

$$\min_{j\neq i}\frac{1}{\lambda_{ij}}u_j(n+1)\sim\exp\Big(\sum_{j\neq i}\lambda_{ij}\Big)=\exp(\lambda_i)$$

- Jump probabilities

$$\mathbb{P}\Big\{\frac{1}{\lambda_{ij}}u_j(n+1)<\min_{k\neq i,j}\frac{1}{\lambda_{ik}}u_k(n+1)\Big\}$$
$$=$$

Review: generator matrix G

- Let

$$G = \begin{pmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix}.$$

- off-diagonals are non-negative: $G_{ij} = \lambda_{ij}$
- diagonals are strictly negative: $G_{ii} = -\sum_{j \neq i} G_{ij}$
- row sums are zero.

Review: three equivalent forms of input for a CTMC

- State $S = \{1, 2, 3\}$
- Jump matrix J plus holding time rates λ_1, λ_2, λ_3.
- Rate diagram, competing clock parameters: λ_{ij} for $i \neq j$.
- Generator

$$G = \begin{pmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix} = \begin{pmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{pmatrix}.$$

Chapman-Kolmogorov equation

- Note

$$
\begin{aligned}
&P_{ij}(t+s) = \mathbb{P}\{X(t+s) = j | X(0) = i\} \\
&= \sum_{k \in S} \mathbb{P}\{X(t+s) = j, X(t) = k | X(0) = i\} \\
&= \sum_{k \in S} \mathbb{P}\{X(t+s) = j | X(t) = k, X(0) = i\} \mathbb{P}\{X(t) = k | X(0) = i\} \\
&= \sum_{k \in S} P_{kj}(s) P_{ik}(t).
\end{aligned}
$$

- Thus,

$$P(t+s) = P(t)P(s), \quad P(2t) = (P(t))^2$$

Computing $P(t)$

- Suppose

$$P(0.1) = \begin{pmatrix} 0.7486327 & 0.1607327 & 0.0906346 \\ 0.0783127 & 0.8310527 & 0.0906346 \\ 0.0041073 & 0.0865273 & 0.9093654 \end{pmatrix} \tag{2}$$

- Compute

$$\begin{aligned} &\mathbb{P}\{X(.4) = 3, X(.2) = 1, X(.1) = 3 | X(0) = 2\} \\ &= (P(0.1))^2_{1,3} P_{3,1}(0.1) P_{2,3}(.1) \\ &= (0.164840)(0.0041073)(0.0906346). \end{aligned}$$

How to obtain (2)?

- From

$$P(t+s) = P(t)P(s),$$

one has (careful if infinite state space)

$$P'(s) = P'(0+)P(s), \quad s \geq 0,$$

where $P'(0+)$ exists and

$$P'(0+) = G. \tag{3}$$

- Solving $P'(s) = GP(s)$, one has unique solution

$$P(s) = e^{sG} = \sum_{k=0}^{\infty} \frac{s^k G^k}{k!} = \texttt{expm}(sG).$$

An example

- Assume $X = \{X(t), t \geq 0\}$ is a CTMC on state space $S = \{1, 2, 3\}$ with generator

$$G = \begin{pmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

- Find

$$\mathbb{P}\{X(5.4) = 3, X(2.1) = 1 | X(0) = 2)\} = P_{2,1}(2.7)P_{1,3}(3.3),$$

-

$$P(2.7) = e^{2.7G} = \begin{pmatrix} 0.12614 & 0.37612 & 0.49774 \\ 0.12612 & 0.37614 & 0.49774 \\ 0.12387 & 0.37387 & 0.50226 \end{pmatrix}, \quad P(3.3) = e^{3.3G}.$$

Proof of (3): a heuristic argument

- For $i \neq j$,

$$
\begin{aligned}
P'_{ij}(0+) &= \lim_{t\downarrow 0} \frac{P_{ij}(t) - 0}{t} \\
. &= \lim_{t\downarrow 0} \frac{\mathbb{P}\{X(t) = j | X(0) = i\}}{t} \\
&\approx \lim_{t\downarrow 0} \frac{\mathbb{P}\{u(1)/\lambda(i) < t\} J_{ij}}{t} \\
&= \lim_{t\downarrow 0} \frac{\left(1 - e^{-\lambda(i)t}\right) J_{ij}}{t} = \lambda(i) J_{ij} \\
&= G_{ij},
\end{aligned}
$$

where the third $\approx$ follows from the fact that within $[0, t]$, the probability of having at least two jumps is $o(t^2)$.

- For $i = j$

Proof of (3): Norris Theorem 2.8.2 (b)

- As $h \downarrow 0$,

$$\begin{aligned}\mathbb{P}_i(X(h) = i) \geq \mathbb{P}_i(\sigma_1 > h) = e^{-\lambda_i h} = 1 - \lambda_i h + o(h) \\ = 1 + G_{ii} h + o(h)\end{aligned}$$

- For $j \neq i$,

$$\begin{aligned}&\mathbb{P}_i(X(h) = j) \geq \mathbb{P}_i(\sigma_1 \leq h, Y_1 = j, (1/\lambda_j)u(2) > h) \\ &= (1 - e^{-\lambda_i h}) J_{ij} e^{-\lambda_j h} \\ &= \lambda_{ij} h + o(h) = G_{ij} h + o(h).\end{aligned}$$

- Thus, for every $j \in S$,

$$\mathbb{P}_i(X(h) = j) \geq \delta_{ij} + G_{ij} h + o(h)$$

Proof (cont')

- Since

$$\sum_j \mathbb{P}_i(X(h) = j) = 1$$

and

$$\sum_j (\delta_{ij} + G_{ij}h) = 1,$$

- one has, for every $j \in S$,

$$\mathbb{P}_i(X(h) = j) = \delta_{ij} + G_{ij}h + o(h).$$

- Therefore

$$\lim_{h \downarrow 0} \frac{\mathbb{P}_i(X(h) = j) - \mathbb{P}_i(X(0) = j)}{h} = G_{ij},$$

proving $P'(0+) = G$.