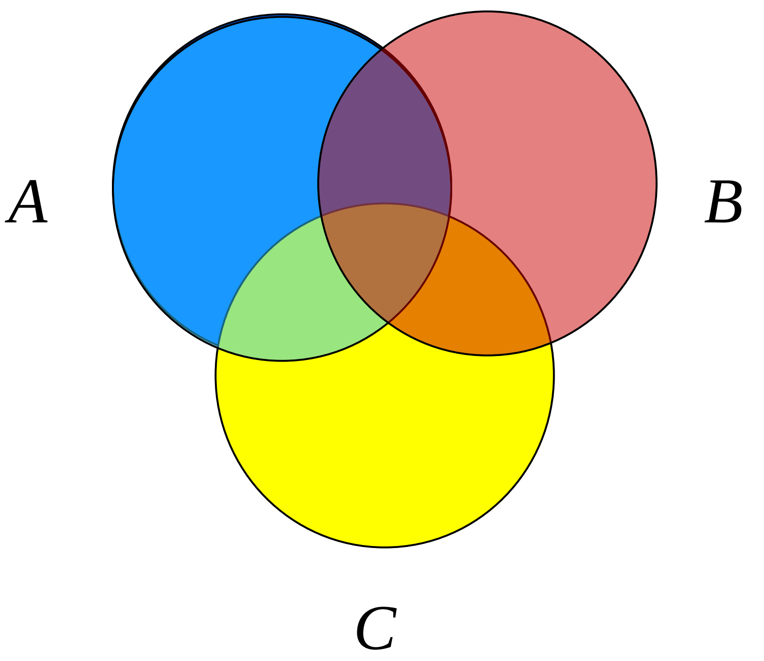
Sets



This Lecture

We will introduce some basic set theory in this session.

- Basic Definitions
- Operations on Sets
- Set Identities
- Russell's Paradox

Defining Sets

Definition: A **set** is an unordered collection of distinct objects.

The objects in a set are called the **elements** or **members** of the set S, and we say S **contains** its elements. The power set of a set S is the set (or class) of all subsets of S, and is sometimes denoted by 2^S.

e.g. $S = \{2, 3, 5, 7\} = \{3, 5, 7, 2\}$

$S = \{\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}\}$

$pow(\{a,b\}) = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}$ (power set of $\{a,b\}$)

Which of the following are sets?

- $S = \{2, 3, 5, 3, 7\}$ NO
- $S = \{\{a\},a\}$ YES

Classical Sets

The following are some well-known examples of sets.

Z: the set of all integers

$\mathbf{Z}^{+}$: the set of all positive integers

$\mathbf{Z}^{-}$: the set of all negative integers

N: the set of all nonnegative integers

R: the set of all real numbers

Q: the set of all rational numbers

C: the set of all complex numbers

Defining Sets by Properties

It is inconvenient, and sometimes impossible, to define a set by listing all its elements.

Alternatively, we use the notation $\{x \in A \mid P(x)\}$ to define the set as the set of elements, *x*, in *A* such that *x* satisfies property *P*.

e.g. $S = \{x \in \mathbb{R} \mid -2 < x < 5\}$

$S = \{x \mid x \text{ is a prime and } x < 70{,}000{,}000\}$

Definition: The **size** or **cardinality** of a set S, denoted by **|S|**, is defined as the number of elements contained in S.

Membership

The most basic question in set theory is whether an element is in a set.

$x \in A$ x is an element of A

x is in A

$x \notin A$ x is not an element of A

x is not in A

Definition:

- $A \subseteq B$ (A is a subset of B) $\Longleftrightarrow$ For any $x \in A$ we have $x \in B$.
- $A = B$ (A is equal to B) $\Longleftrightarrow$ $A \subseteq B$ and $B \subseteq A$.
- $A \subset B$ (A is a proper subset of B) $\Longleftrightarrow$ $A \subseteq B$ and $A \neq B$.

This Lecture

- Basic Definitions
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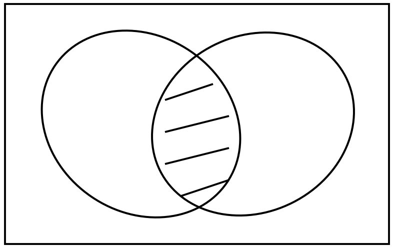
Basic Operations on Sets

Let A, B be two subsets of a *universal* set U

(depending on the context U could be R, Z, or other sets).

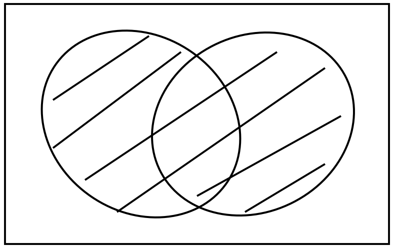
intersection: $A \cap B = \{x \in U \mid x \in A \text{ and } x \in B\}$



Defintion: Two sets are said to be **disjoint** if their intersection is an empty set.

e.g. Let A be the set of odd numbers, and B be the set of even numbers. Then A and B are disjoint.

union: $A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\}$



Fact: $|A \cup B| = |A| + |B| - |A \cap B|$

Basic Operations on Sets

- **Definition**

Unions and Intersections of an Indexed Collection of Sets

Given sets $A_0,\ A_1,\ A_2, \ldots$ that are subsets of a universal set U and given a nonnegative integer n,

$$\bigcup_{i=0}^{n} A_i = \{x \in U \mid x \in A_i \text{ for at least one } i = 0, 1, 2, \ldots, n\}$$

$$\bigcup_{i=0}^{\infty} A_i = \{x \in U \mid x \in A_i \text{ for at least one nonnegative integer } i\}$$

$$\bigcap_{i=0}^{n} A_i = \{x \in U \mid x \in A_i \text{ for all } i = 0, 1, 2, \ldots, n\}$$

$$\bigcap_{i=0}^{\infty} A_i = \{x \in U \mid x \in A_i \text{ for all nonnegative integers } i\}.$$

Basic Operations on Sets

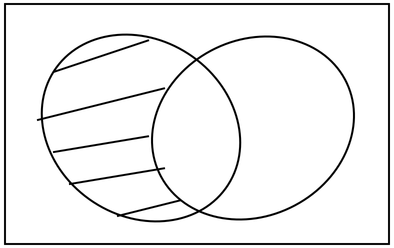
For each positive integer i, let $A_i = \left\{x \in \mathbf{R} \mid -\frac{1}{i} < x < \frac{1}{i}\right\} = A_i = \left(-\frac{1}{i}, \frac{1}{i}\right)$.

a. Find $A_1 \cup A_2 \cup A_3$ and $A_1 \cap A_2 \cap A_3$.

b. Find $\bigcup_{i=1}^{\infty} A_i$ and $\bigcap_{i=1}^{\infty} A_i$.

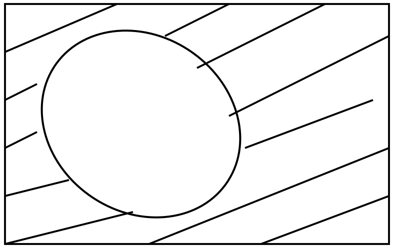
Basic Operations on Sets

difference: $A - B = \{x \in U \mid x \in A \text{ and } x \notin B\}$



Fact: $|A - B| = |A| - |A \cap B|$

complement: $\overline{A} = A^c = \{x \in U \mid x \notin A\}$



e.g. Let U = Z and A be the set of odd numbers.

Then $\overline{A}$ is the set of even numbers.

Fact: If $A \subseteq B$, then $\overline{B} \subseteq \overline{A}$

Examples

$A = \{1, 3, 6, 8, 10\}$ $B = \{2, 4, 6, 7, 10\}$

$A \cap B = \{6, 10\}$, $A \cup B = \{1, 2, 3, 4, 6, 7, 8, 10\}$ $A\text{-}B = \{1, 3, 8\}$

Let $U = \{ x \in Z \mid 1 \leq x \leq 100\}$.

$A = \{ x \in U \mid x \text{ is divisible by } 3\}$, $B = \{ x \in U \mid x \text{ is divisible by } 5\}$

$A \cap B = \{ x \in U \mid x \text{ is divisible by } 15\}$

$A \cup B = \{ x \in U \mid x \text{ is divisible by 3 or is divisible by 5 (or both)}\}$

$A - B = \{ x \in U \mid x \text{ is divisible by 3 but is not divisible by } 5 \}$

Exercise: compute $|A|$, $|B|$, $|A \cap B|$, $|A \cup B|$, $|A - B|$.

Partitions of Sets

Two sets are disjoint if their intersection is empty.

A collection of nonempty sets $\{A_1, A_2, \ldots, A_n\}$ is a partition of a set A if and only if

$$A = A_1 \cup A_2 \cup \ldots \cup A_n$$

$A_1, A_2, \ldots, A_n$ are mutually disjoint (or pairwise disjoint).

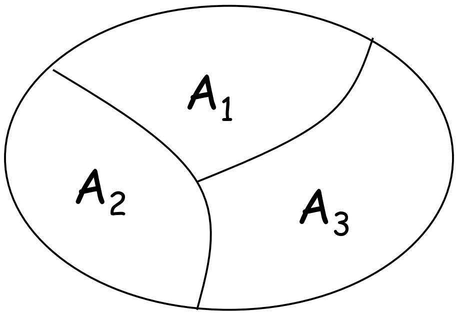
e.g. Let A be the set of integers.

$A_1 = \{x \in A \mid x = 3k+1 \text{ for some integer } k\}$

$A_2 = \{x \in A \mid x = 3k+2 \text{ for some integer } k\}$

$A_3 = \{x \in A \mid x = 3k \text{ for some integer } k\}$

Then $\{A_1, A_2, A_3\}$ is a partition of A



Partitions of Sets

e.g. $A = \{x \in \mathbb{Z} \mid x \text{ is divisible by } 6\}$.

$A_1 = \{x \in \mathbb{Z} \mid x \text{ is divisible by } 2\}$.

$A_2 = \{x \in \mathbb{Z} \mid x \text{ is divisible by } 3\}$.

Then $\{A_1, A_2\}$ is not a partition of A, because

- $A_1 \cap A_2 \neq \emptyset$
- $A \subset A_1 \cup A_2$

e.g. $A = \mathbb{Z}$.

$A_1 = \{x \in \mathbb{Z} \mid x < 0\}$.

$A_2 = \{x \in \mathbb{Z} \mid x > 0\}$.

Then $\{A_1, A_2\}$ is not a partition of A, because

$A \supset A_1 \cup A_2$ as 0 is contained in A.

Cartesian Products

Definition: Given two sets A and B, the **Cartesian product** A x B is the set of all **ordered** pairs (a,b), where a is in A and b is in B. That is,

$$A \times B = \{(a,b) \mid a \in A, b \in B\}$$

Ordered pairs means the ordering is **important**, e.g. $(1,2) \neq (2,1)$

e.g. Let A be the set of letters, i.e. {a,b,c,…,x,y,z}.

Let B be the set of digits, i.e. {0,1,…,9}.

AxA is just the set of strings with two letters.

BxB is just the set of strings with two digits.

AxB is the set of strings where the first character is a letter and the second character is a digit.

Cartesian Products

The definition can be generalized to any number of sets, e.g.

$$A \times B \times C = \{(a,b,c) \mid a \in A \text{ and } b \in B \text{ and } c \in C\}$$

Using the above examples, AxAxA is the set of strings with three letters.

e.g. the set of the vectors in $\mathbb{R}^3$ is the set $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$.

Fact: If $|A| = n$ and $|B| = m$, then $|A \times B| = nm$.

Fact: If $|A| = n$ and $|B| = m$ and $|C| = l$, then $|A \times B \times C| = nml$.

Fact: $|A_1 \times A_2 \times \ldots \times A_k| = |A_1| \times |A_2| \times \ldots \times |A_k|$.

Exercises

1. Let A be the set of prime numbers, and let B be the set of even numbers. What is $A \cap B$ and $|A \cap B|$?

2. Is $|A \cup B| > |A| > |A \cap B|$ always true?

3. Let A be the set of all n-bit binary strings, A_i be the set of all n-bit binary strings with i ones. Is $(A_1, A_2, \ldots, A_i, \ldots, A_n)$ a partition of A?

This Lecture

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Set Identities

Let A, B, C be subsets of a universal set U.

Commutative Law: (a) $A \cup B = B \cup A$ and (b) $A \cap B = B \cap A$

Associative Law: (a) $(A \cup B) \cup C = A \cup (B \cup C)$

(b) $(A \cap B) \cap C = A \cap (B \cap C)$

Distributive Law: (a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Identity Law: (a) $A \cup \emptyset = A$ and (b) $A \cap U = A$

Complement Law: (a) $A \cup A^c = U$ and (b) $A \cap A^c = \emptyset$

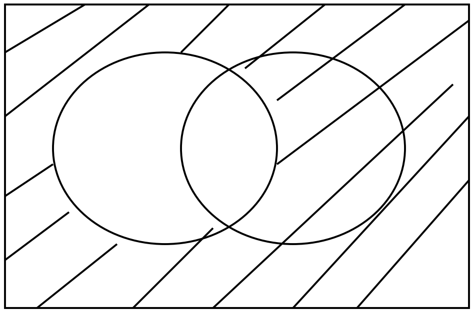
De Morgan's Law: (a) $(A \cup B)^c = A^c \cap B^c$ and (b) $(A \cap B)^c = A^c \cup B^c$

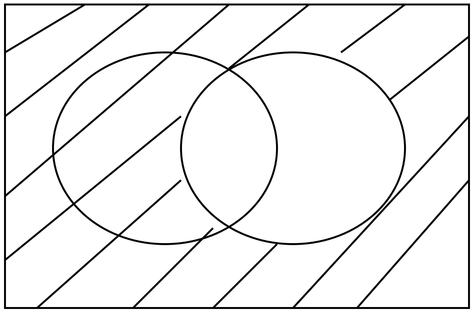
Set difference Law: $A - B = A \cap B^c$

Venn Diagram

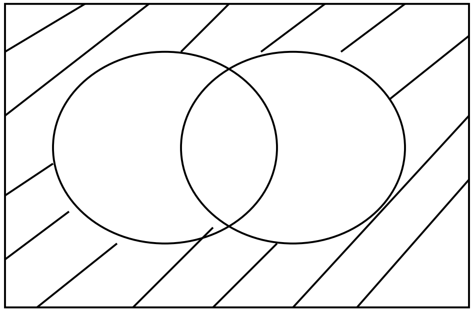
De Morgan's Law: $\overline{A \cup B} = \overline{A} \cap \overline{B}$

$\overline{A}$





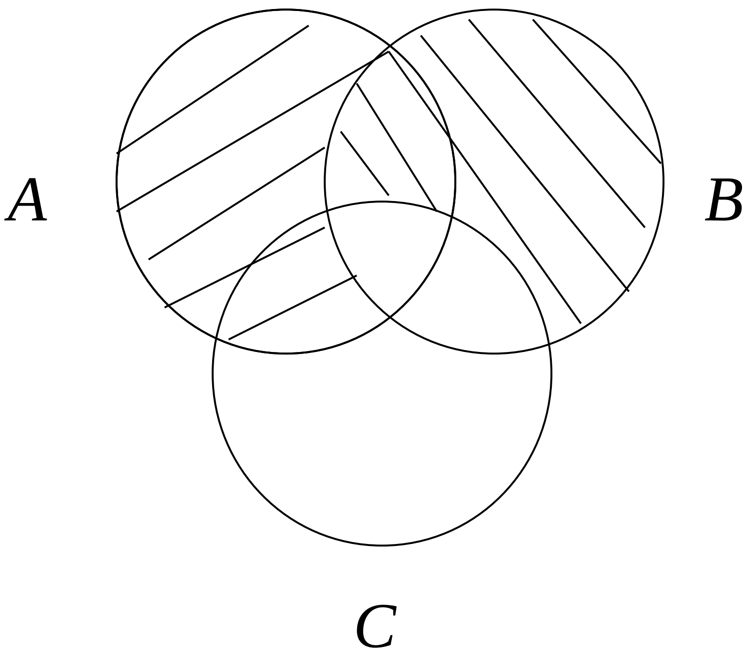
$\overline{B}$



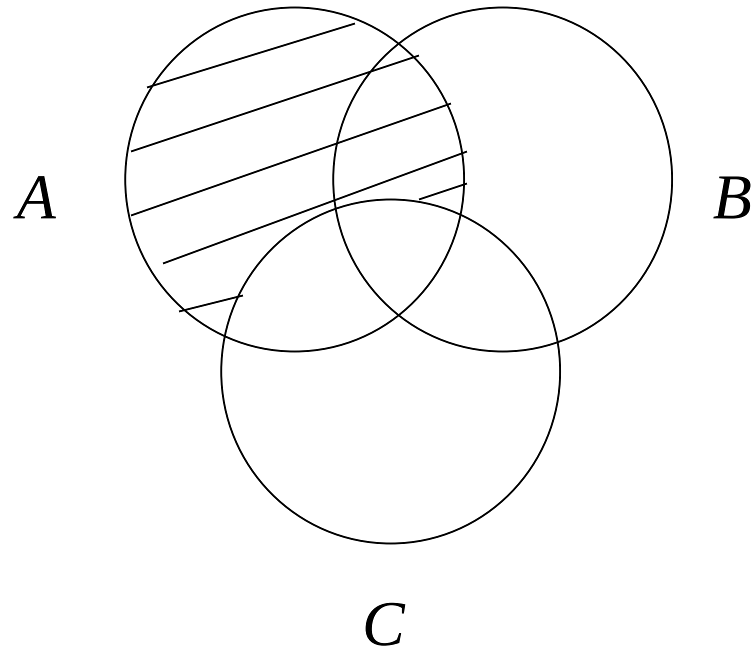
$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

Disproof

$$(A - B) \cup (B - C) = A - C?$$



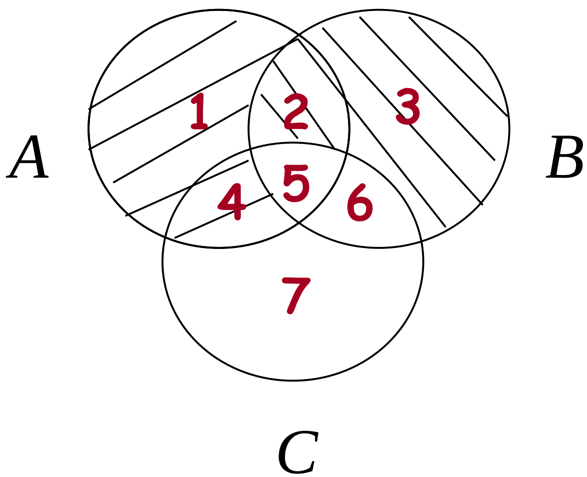
L.H.S

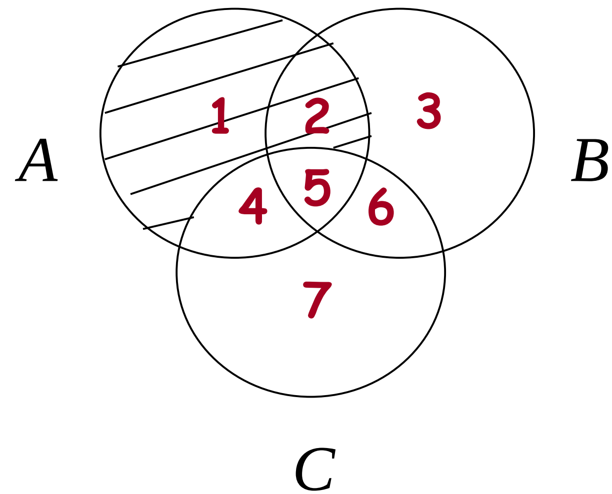


R.H.S

Disproof

$$(A - B) \cup (B - C) = A - C?$$





We can easily construct a **counter-example** to the equality, by putting a number in each region in the figure.

Let A = {1,2,4,5}, B = {2,3,5,6}, C = {4,5,6,7}.

Then we see that L.H.S = {1,2,3,4} and R.H.S = {1,2}.

Algebraic Proof

$$\overline{((A \cup C) \cap (B \cup C))} = (\overline{A} \cup \overline{B}) \cap \overline{C}?$$

$$\overline{((A \cup C) \cap (B \cup C))}$$

$$= \overline{(A \cup C)} \cup \overline{(B \cup C)}$$ by DeMorgan's law

$$= (\overline{A} \cap \overline{C}) \cup \overline{(B \cup C)}$$ by DeMorgan's law

$$= (\overline{A} \cap \overline{C}) \cup (\overline{B} \cap \overline{C})$$ by DeMorgan's law

$$= (\overline{A} \cup \overline{B}) \cap \overline{C}$$ by Distributive law

Proof by Definition

How to prove $(A \cap B) \times C = (A \times C) \cap (B \times C)$?

1. LHS $\subseteq$ RHS.

Since $(A \cap B) \times C \subseteq A \times C$, $(A \cap B) \times C \subseteq B \times C$

2. RHS $\subseteq$ LHS.

$(x,y) \in (A \times C) \cap (B \times C) \Rightarrow (x,y) \in A \times C$ and $(x,y) \in B \times C$

So $x \in A$ and $x \in B \Rightarrow x \in A \cap B$

Hence $(x,y) \in (A \cap B) \times C$.

Therefore, we complete the proof.

Exercises

$$A - (A \cap B) = A - B?$$

$$(A \cup B) - C = (A - C) \cup (B - C)?$$

$$\overline{(A \cup B \cup C)} = \overline{A} \cap \overline{B} \cap \overline{C}?$$

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Russell's Paradox

$$\text{Let } W ::= \{S \in \text{Sets} \mid S \notin S\}$$

Call a set an **ordinary set** if it does not contain itself as an element, and a call it an **extraordinary set** if it contains itself as an element. Thus, W is the set of all ordinary sets.

Is W in W? or Is W ordinary or extraordinary?

If W is in W, then W contains itself.

But the W property implies that W is not in W. So W is not in W.

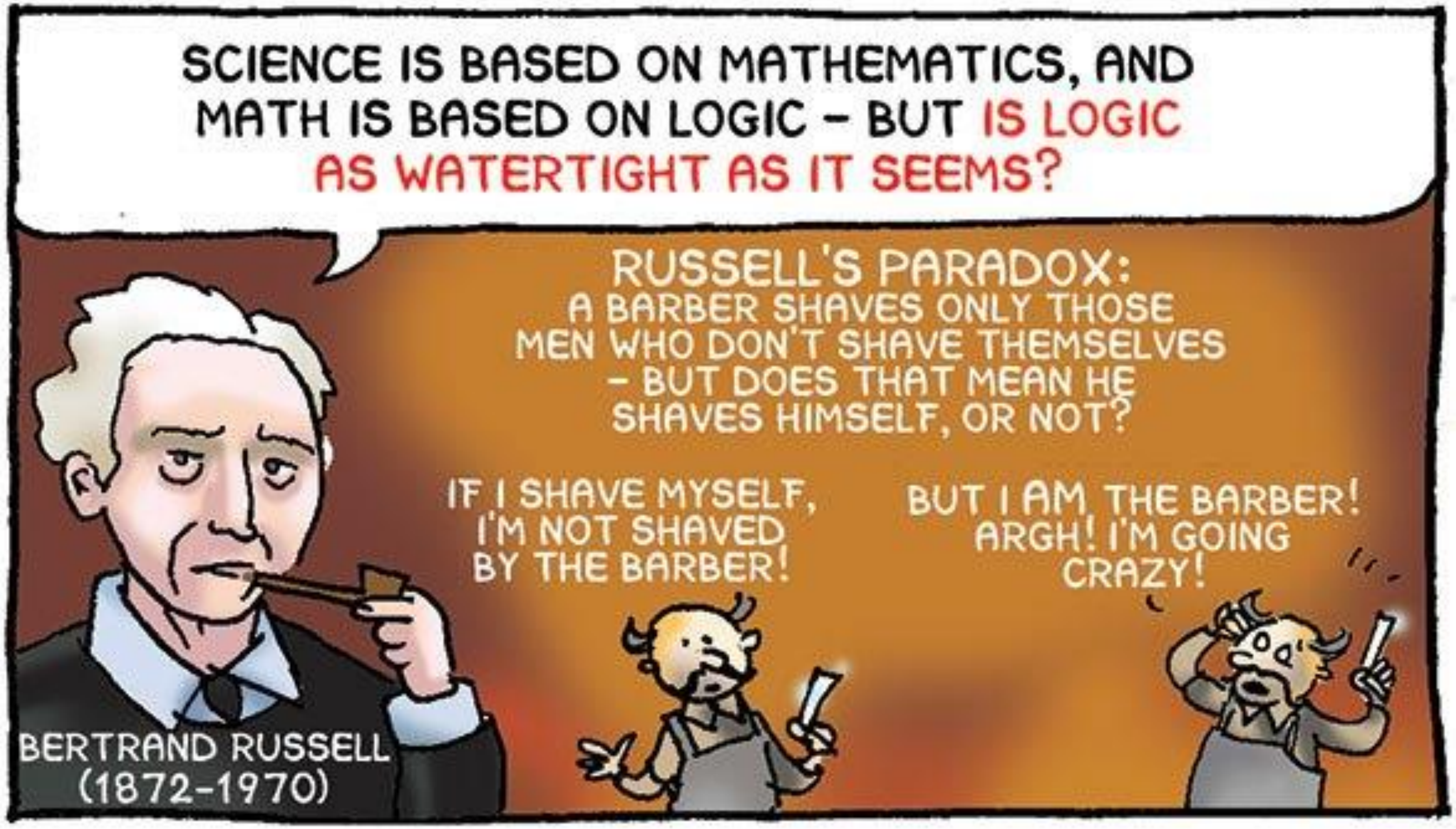
(i.e. if W is extraordinary, then it contains itself which contradicts the definition of W; thus it can't be extraordinary)

If W is not in W, then it satisfies the W property. So W is in W.

(i.e. if W is ordinary, then it contains itself which makes it extraordinary; thus it can't be ordinary)

What's wrong???

Barber's Paradox



Solution to Russell's Paradox

A man either shaves himself or not shaves himself.

A barber either shaves himself or not shaves himself.

Perhaps such a barber does not exist?

Actually this is the way out of the paradox.

Going back to the Russell's paradox,

we conclude that W cannot be a set,

because every set either contains itself or not,

but neither case can happen for W.

This paradox tells us that not everything we define is a set.

Later on mathematicians define sets more carefully,

e.g. using the sets that we already know.

Halting Problem

Now we mention one of the most famous problems in computer science.

The halting problem: Can we write a program which detects infinite loop?

We want a program H that given any program P and input I:

H(P,I) returns "halt" if P will terminate given input I;

H(P,I) returns "loop forever" if P will not terminate given input I.

And H itself must terminate in finite time.

The halting problem: Does such a program H exist?

NO!

The reasoning used in solving the halting problem is very similar to that of Russell's paradox.

Halting Problem

Program P consists of characters, and hence it is an input.

Construct a program Test(P) such that
- Test(P) loops forever if H(P,P) returns halt;
- Test(P) halts if H(P,P) returns loops forever.

- Let P=Test
  - Test(Test) means that Test is used as input to the Test program
  - H(Test, Test) means that Test is used as input to the Test program
  - If Test(Test) loops forever, then H(Test, Test) will return "loop forever" because P=Test in H(Test, Test) does not terminate; hence Test(Test) halts by construction. A contradiction!
  - If Test(Test) halts, then H(Test, Test) will return "halt" because P=Test halts; hence Test(Test) loops forever. A contradiction!

The halting problem: Does such a program H exist? NO!

Summary

Recall what we have covered so far.

- Basic Definitions (defining sets, membership, subsets, size)
- Operations on Sets (intersection, union, difference, complement,
  partition, power set, Cartesian product)
- Set Identities (Distributive law, DeMorgan's law,
  checking set identities – proof & disproof, algebraic)