

Denominator-layout Matrix (Vector) Derivative:

$$y, x \in \mathbb{R}^1, \quad \underline{y} \in \mathbb{R}^n, \quad \underline{x} \in \mathbb{R}^m, \quad Y \in \mathbb{R}^{m \times n}, \quad X \in \mathbb{R}^{p \times q}$$

$$\frac{\partial \text{scalar}}{\partial \text{scalar}} = \frac{\partial y}{\partial x}$$

$$\frac{\partial \text{vector}}{\partial \text{scalar}} = \frac{\partial \underline{y}}{\partial x} = \left[ \frac{\partial y_1}{\partial x} \quad \frac{\partial y_2}{\partial x} \quad \dots \quad \frac{\partial y_n}{\partial x} \right]$$

$$\frac{\partial \text{matrix}}{\partial \text{scalar}} = \frac{\partial Y}{\partial x} = \begin{bmatrix} \frac{\partial Y_{11}}{\partial x} & \frac{\partial Y_{21}}{\partial x} & \dots & \frac{\partial Y_{m1}}{\partial x} \\ \frac{\partial Y_{12}}{\partial x} & \frac{\partial Y_{22}}{\partial x} & \dots & \frac{\partial Y_{m2}}{\partial x} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial Y_{1n}}{\partial x} & \frac{\partial Y_{2n}}{\partial x} & \dots & \frac{\partial Y_{mn}}{\partial x} \end{bmatrix}$$

$$\frac{\partial \text{scalar}}{\partial \text{vector}} = \frac{\partial y}{\underline{x}} = \begin{bmatrix} \frac{\partial y}{\partial x_1} \\ \frac{\partial y}{\partial x_2} \\ \vdots \\ \frac{\partial y}{\partial x_m} \end{bmatrix}$$

$$\frac{\partial \text{vector}}{\partial \text{vector}} = \frac{\partial \underline{y}}{\underline{x}} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_2}{\partial x_1} & \dots & \frac{\partial y_n}{\partial x_1} \\ \frac{\partial y_1}{\partial x_2} & \frac{\partial y_2}{\partial x_2} & \dots & \frac{\partial y_n}{\partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_1}{\partial x_m} & \frac{\partial y_2}{\partial x_m} & \dots & \frac{\partial y_n}{\partial x_m} \end{bmatrix}$$

$$\frac{\partial \text{scalar}}{\partial \text{matrix}} = \frac{\partial y}{\underline{X}} = \begin{bmatrix} \frac{\partial y}{\partial X_{11}} & \frac{\partial y}{\partial X_{12}} & \dots & \frac{\partial y}{\partial X_{1q}} \\ \frac{\partial y}{\partial X_{21}} & \frac{\partial y}{\partial X_{22}} & \dots & \frac{\partial y}{\partial X_{2q}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y}{\partial X_{p1}} & \frac{\partial y}{\partial X_{p2}} & \dots & \frac{\partial y}{\partial X_{pq}} \end{bmatrix}$$

$X \in \mathbb{R}^{p \times q}$