Stochastic Processes

Lecture 10: Positive recurrence, Decomposition of state space, Limiting behavior, Period

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Two Examples

One dimensional symmetric random walk

Reflected random walks

P.r.? ✗

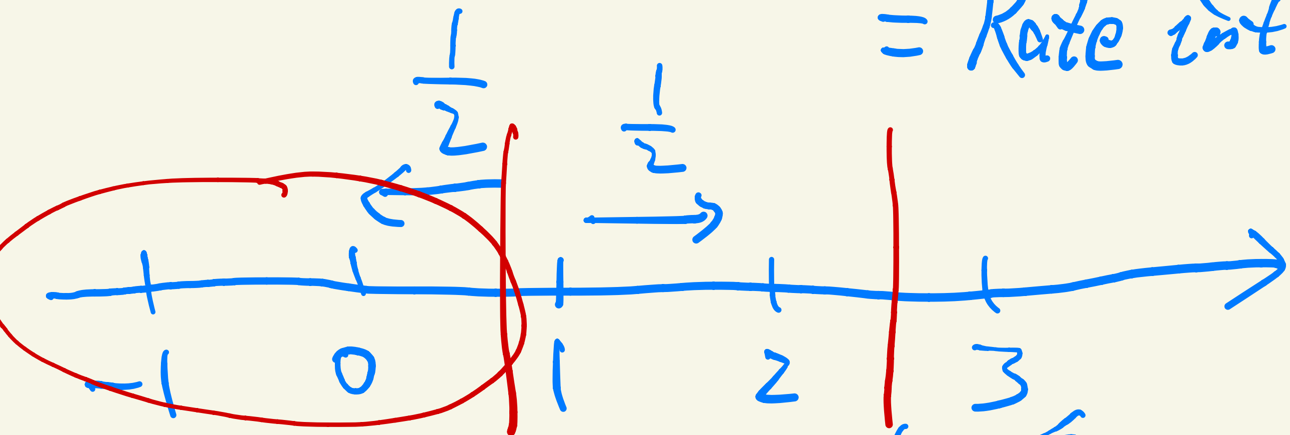
"Cut Method"

$S = A \cup A^c$

NULL Recurrent!

Rate out of A

$=$ Rate into A



$A = \{\cdots, -1, 0\}$

$A^c = \{1, 2, \cdots\}$

Rate out $\qquad$ Rate in

$$\pi(0)\cdot\frac{1}{2} = \pi(1)\cdot\frac{1}{2}$$

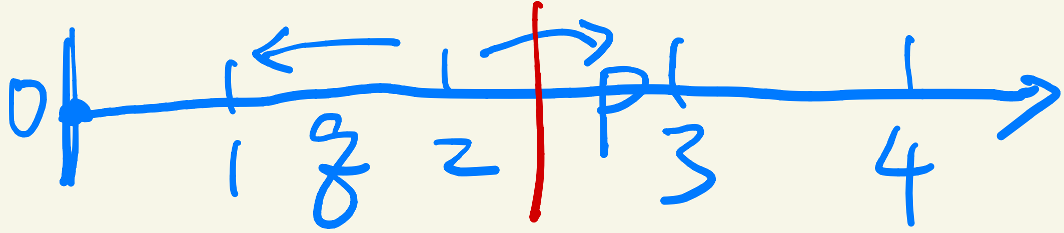
$\pi(0) = \pi(1)$

$$\pi(2)\cdot\frac{1}{2} = \pi(3)\cdot\frac{1}{2}$$

$\pi(2) = \pi(3)$

$\pi(i) = \pi(j) \quad \forall\, i, j$

Stationary distri? ✗



$p+q=1$

$P_{i,i+1}=p,\quad i\geq 0$

$P_{i,i-1}=q,\quad i\geq 1,\quad P_{0,0}=q$

$A=\{0,1,2\}\qquad A^c=\{3,4,\cdots\}$

$\pi(2)\cdot p=\pi(3)\cdot q$

$\pi(i)p=\pi(i+1)\cdot q$

$\pi(i+1)=\frac{p}{q}\pi(i)=\left(\frac{p}{q}\right)^{i+1}\pi(0)$

$\sum_{i=0}^{\infty}\pi(i)=1\Rightarrow\pi(0)=1-\frac{p}{q}$

$\pi(i)=\left(1-\frac{p}{q}\right)\cdot\left(\frac{p}{q}\right)^{i}.$

$\begin{cases}\text{P.r.} & p<q\\ \text{NULL recu.} & p=q\\ \text{transient.} & p>q\end{cases}$

$p<q$

Positive recurrence criterion

- Let $N_i(n) = \sum_{k=1}^{n} 1_{\{X_k=i\}}$ be the number of times visiting state i in $[1, n]$. Then

$$\mathbb{E}_i(N_i(n)) = \sum_{k=1}^{n} \mathbb{E}_i 1_{\{X_k=i\}} = \sum_{k=1}^{n} \mathbb{P}_i\{X_k = i\} = \sum_{k=1}^{n} P_{ii}^k.$$

THEOREM

State i is positive recurrent if and only if

$$\lim_{n\to\infty} \boxed{\frac{1}{n}\sum_{k=1}^{n} P_{ii}^k} > 0.$$

"$\Leftarrow$" ?

- Proof.

"$\Rightarrow$": $\frac{N_i(n)}{n} \to \pi(i) > 0$

$\lim_{n\to\infty} \frac{N_i(n)}{n} = \pi(i) > 0$

$\Rightarrow \lim_{n\to\infty} \boxed{\frac{\mathbb{E}_i N_i(n)}{n}} = \pi(i)$

Comparison with recurrence criterion

- Recall that state i is recurrent iff

$$\sum_{k=1}^{\infty} P_{ii}^{k} = \infty. \quad \lim_{n\to\infty} \boxed{\sum_{k=1}^{n} P_{ii}^{k}} = \infty$$

- State i is positive recurrent iff

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} P_{ii}^{k} > 0.$$

Lemma 1

Assume states i and j communicate. State i is p.r. iff state j is p.r.

- Proof: there exist k_1 and k_2 such that $P_{ij}^{k_1} > 0$ and $P_{ji}^{k_2} > 0$.
- Assume j is p.r. Then $\lim_{n\to\infty}(1/n)\sum_{k=1}^{n} P_{jj}^{k} > 0$. Lemma follows from

$$\Rightarrow \boxed{\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} P_{ii}^{k} > 0} \; ?$$

$$P_{ii}^{k_1+k+k_2} \geq P_{ij}^{k_1} P_{jj}^{k} P_{ji}^{k_2},$$

$$i \xrightarrow{k_1} j \xrightarrow{k} j \xrightarrow{k_2} i$$

$$\boxed{\frac{1}{n}\sum_{k=1}^{n+k_1+k_2} P_{ii}^{k}} = \boxed{\frac{1}{n}\sum_{k=1}^{n} P_{ii}^{k_1+k+k_2}} + \frac{1}{n}\sum_{k=1}^{k_1+k_2} P_{ii}^{k} > 0$$

when n is large enough. $\geq \frac{1}{n}\sum_{k=1}^{n} P_{jj}^{k} > 0 \ (n\to\infty)$

- The proof for solidarity of recurrence is left as exercise.

P^n

- Assume that the DTMC is irreducible.
- If it is positive recurrent, for every pair of states $i, j \in S$,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}(P^k)_{ji} = \frac{1}{\mathbb{E}_i(T_i)} > 0.$$

$$= \frac{1}{n}\sum_{k=1}^{n}\mathbb{E}_j[1\{X_k = i\}] = \frac{1}{n}\sum_{k=1}^{n} f(X_k) = \frac{1}{\mathbb{E}_i(T_i)}$$

Namely,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} P^k = P^{(\infty)},$$

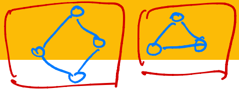
where $P_{ij}^{(\infty)} = \pi_j = 1/\mathbb{E}_j(T_j)$.

$$P^{(\infty)} = \begin{pmatrix} \pi_1, & \pi_2, & \cdots, & \pi_{|S|} \\ \pi_1, & \pi_2, & \cdots & \pi_{|S|} \\ \pi_1, & \pi_2, & \cdots, & \pi_{|S|} \end{pmatrix}.$$

- If it is not positive recurrent, for every pair of states

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}(P^k)_{ji} = 0. \qquad \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} P^k = 0.$$

Communicating classes



Definition

(a) A set $C \subset S$ is said to be a communicating class if i, j communicate for any $i, j \in C$ and i, j does not communicate if $i \in C$ and $j \notin C$.

(b) A communicating class is said to be *closed* if $i \in C$ and $i \to j$ imply $j \in C$.

Theorem

Let C be a communicating class. Then either all states in C are transient or all are recurrent.

Theorem

Every recurrent class is closed.

HW 5. Problem 3.

- The state space

$$S = T \cup C_1 \cup C_2 \cup \ldots,$$

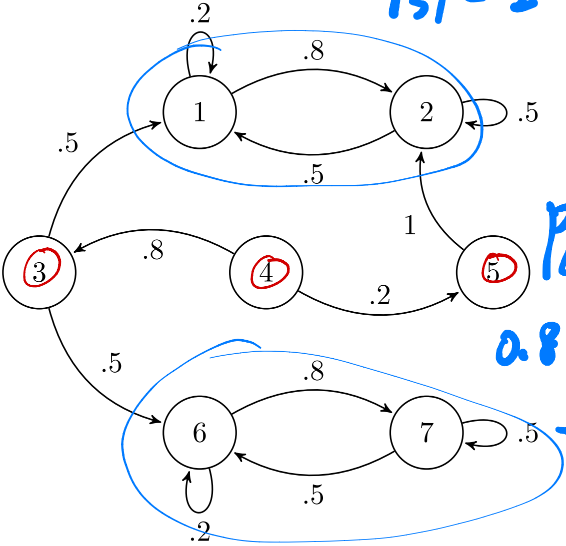
where C_i is a closed, communicating recurrent class, and T the set of transient states.

- For a finite state DTMC, there exists at least one (closed) recurrent class.
- Counter example when S is infinite.

One Dimension asymmetric R.W.
$p \neq \frac{1}{2}$.

A reducible DTMC

Consider the following DTMC.



$$P^{\infty}_{31} = \frac{1}{2} \cdot P^{\infty}_{11}$$

$$= \frac{1}{2} \cdot \frac{5}{13}$$

$$P^{\infty}_{41} = 0.8\, P^{\infty}_{31} + 0.2\, P^{\infty}_{51}$$

$$= P^{\infty}_{21}$$

Limiting behavior

- compute $\lim_{n\to\infty} P^n$.

$$\begin{pmatrix} 5/13 & 8/13 & 0 & 0 & 0 & 0 & 0 \\ 5/13 & 8/13 & 0 & 0 & 0 & 0 & 0 \\ (1/2)(5/13) & (1/2)(8/13) & 0 & 0 & 0 & (1/2)(5/13) & (1/2)(8/13) \\ (.6)(5/13) & (.6)(8/13) & 0 & 0 & 0 & (.4)(5/13) & (.4)(8/13) \\ 5/13 & 8/13 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5/13 & 8/13 \\ 0 & 0 & 0 & 0 & 0 & 5/13 & 8/13 \end{pmatrix}$$

- $S = T \cup C_1 \cup C_2 = \{3,4,5\} \cup \{1,2\} \cup \{6,7\}$
- When computing rows 1, 2, you can just forget about states except for 1 and 2 because there is no arrow going out. Same for rows 6, 7.

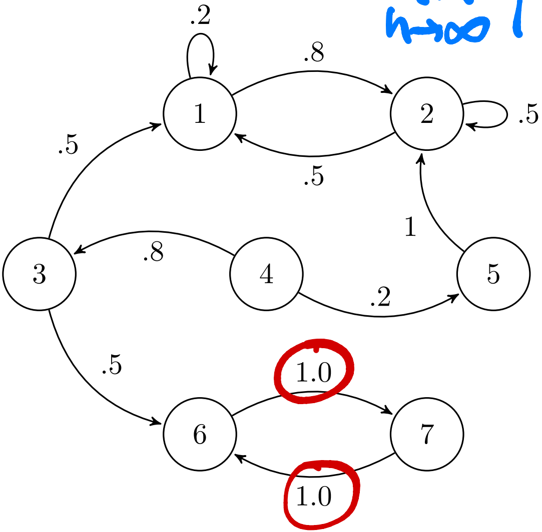
$\lim_{n\to\infty} P^n$

	1	2	3	4	5	6	7
1	$\frac{5}{13}$	$\frac{8}{13}$	0			0	
2	$\frac{5}{13}$	$\frac{8}{13}$					
3	$\frac{1}{2}\cdot\frac{5}{13}$	$\frac{1}{2}\cdot\frac{8}{13}$	0			?	?
4	$\square$	$\square$					
5	$\square$	$\square$					
6	0		0			$\square$	$\square$
7						$\square$	$\square$

Another reducible DTMC

Consider the following DTMC.

$\lim_{n \to \infty} P^n$ X



Limiting distribution?

- $\lim_{n\to\infty} P^n$ does not exist. $\lim_{n\to\infty}(P^n + P^{n+1})/2$ exists.

$$\begin{pmatrix}
5/13 & 8/13 & 0 & 0 & 0 & 0 & 0 \\
5/13 & 8/13 & 0 & 0 & 0 & 0 & 0 \\
(1/2)(5/13) & (1/2)(8/13) & 0 & 0 & 0 & (1/2)(.5) & (1/2)(.5) \\
(.6)(5/13) & (.6)(8/13) & 0 & 0 & 0 & (.4)(.5) & (.4)(.5) \\
5/13 & 8/13 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & .5 & .5 \\
0 & 0 & 0 & 0 & 0 & .5 & .5
\end{pmatrix}$$

Periodicity

Definition

The *period* of state i of a DTMC is $d(i) = \gcd\{n : P_{ii}^n > 0\}$.

Theorem (Solidarity property)

If state i and j communicate, then $d(i) = d(j)$.

- Assume $P_{ij}^{k_1} > 0$ and $P_{ji}^{k_2} > 0$. For $k \geq 0$,

$$P_{ii}^{k+k_1+k_2} \geq P_{ij}^{k_1} P_{jj}^{k} P_{ji}^{k_2}$$

- Take $k = 0$, $P_{ii}^{k_1+k_2} > 0$, which implies $d(i) \mid k_1 + k_2$.
- Whenever $P_{jj}^{k} > 0$, $P_{ii}^{k+k_1+k_2} > 0$, thus, $d(i) \mid k + k_1 + k_2$, which implies $d(i) \mid k$. Thus, $d(i) \leq d(j)$.

Periodicity and limit

Definition

An irreducible DTMC is *aperiodic* if $d = 1$. Otherwise, it's *periodic*.

Theorem

If an irreducible DTMC is aperiodic, then

$$\lim_{n\to\infty} P^n = P^{(\infty)}$$

exists, where $P_{ij}^{(\infty)} = 1/\mathbb{E}_j(T_j)$. *Therefore, when the DTMC is positive recurrent, every row of the limiting matrix* $P^{(\infty)}$ *is equal to the DTMC's stationary distribution* π.

The Theorem is false if the DTMC is periodic!

Random walks on circles

- R.w. on a circle of three points.
- R.w. on a circle of four points.