MAT3253 Tutorial 1

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I give proof synopsis in this documents, the detailed explanation can be referred in live tutorials. I understand THIS week's video quality is not HD, so feel free to ask questions.

1 Inequality

1. Triangle Inequality
   - Prove: for any $a, b \in C, |a+b| \leq |a| + |b|$
     - **Step 1**: Consider $|a+b|^2$ and write it out as by the definition of square of modulus.
     - **Step 2**: Use the property $Re(z) \leq |z|, \text{for} z = a\bar{b}$
     - **Step 3**: Obtain $|a+b|^2 \leq (|a|+|b|)^2$
   - Extend the result for any finite summation above by induction.
   - Give the condition for equality.
     - Equality occurs if and only if $\frac{a_i}{a_j} > 0, \forall i, j$
     - Intuition here is that triangle inequality is an equality when the terms are **positive** multiples of one another, i.e., lie on the same line in complex plane.
   - Prove: Given $\forall i = 1, \cdots, n, |a_i| < 1, \lambda_i \geq 0 and \sum \lambda_i = 1, then |\sum \lambda_i a_i| < 1$
     - Apply the result above for summing a finite number of terms directly.
2. Cauchy's Inequality
   - Prove (Lagrange's Identity): $|\sum_1^n a_i b_i|^2 = (\sum_1^n |a_i|^2)(\sum_1^n |b_i|^2) - \sum_{i<j} |a_i \bar{b_j} - a_j \bar{b_i}|^2$
     - Not required, but the proof is quite tedious and is straightforward expansion of the squares.
   - Prove: $|\sum_1^n a_i b_i|^2 \leq (\sum_1^n |a_i|^2)(\sum_1^n |b_i|^2)$
     - **Step 1**: Consider the fact $\sum_1^n |a_i - \lambda b_i|^2 \geq 0$, true for any complex number λ.
     - **Step 2**: As a result, $\lambda \sum \bar{a}_i \bar{b_i} + \bar{\lambda} \sum a_i b_i \leq \sum |a_i|^2 + |\lambda|^2 \sum |bi|^2$
     - **Step 3**: Set $\lambda = \frac{\sum_1^n a_i b_i}{\sum_1^n |b_i|^2}$, Cauchy's inequality follows.
     - **Step 4**: Obtain from step 1 that the equality criterion is that a_i is proportional to $\bar{b_i}, \forall i$.