24 Lecture 24 (Evaluation of real integrals)

Summary

- Trigonometric integral
- Integral of rational function
- Jordan lemma

The integral of a rational function $R(\cos\theta, \sin\theta)$ in $\cos\theta$ and $\sin\theta$, for θ from 0 to 2π, can be transformed to complex integral,

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta = \int_{|z|=1} \frac{1}{iz} R\Big(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\Big)\, dz.$$

The complex integral on the right is computed on the unit circle, with counter-clockwise orientation. Recall that $z = e^{i\theta}$ when z is on the unit circle, and $z'(\theta) = iz$. The integrand on the right is a rational function in z, and can be evaluated using residue theorem.

Example 24.1. Show that

$$\int_0^{2\pi} \frac{1}{1 + a\cos\theta}\, d\theta = \frac{2\pi}{\sqrt{1-a^2}} \quad \text{for } a \in \mathbb{R},\ |a| < 1$$

Let C be the unit circle with clock-wise orientation. Parameterize C by $z(\theta)$ for $0 \leq \theta \leq 2\pi$. The trigonometric integral can be written as

$$\begin{aligned}
\int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta} &= \int_C \frac{dz}{iz(1 + a(\frac{z+z^{-1}}{2}))} \\
&= \frac{2}{i} \int_C \frac{dz}{az^2 + 2z + a}.
\end{aligned}$$

The denominator is a quadratic function. Let α and β be the roots

$$\alpha = \frac{-1 + \sqrt{1-a^2}}{a}, \quad \beta = \frac{-1 - \sqrt{1-a^2}}{a}.$$

Both α and β are real roots. We can show that α is inside the unit circle, while β is outside the unit circle. For example, when $a > 0$, we have $\beta < -1$ and $0 > \alpha = 1/\beta > -1$. Likewise, we can show that when $a < 0$, we have $\beta > 1$ and $0 < \alpha < 1$. When $a = 0$, there is only one root at $z = 0$. We may assume $0 < |a| < 1$ in the following calculations.

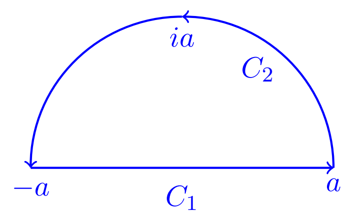


Figure 1: Boundary of a semi-circle

By the residue theorem (Theorem 21.2), we can compute

$$\begin{aligned}
\int_0^{2\pi} \frac{1}{1+a\cos\theta}\,d\theta &= 4\pi \operatorname{Res}\big(\frac{1}{a(z-\alpha)(z-\beta)};\alpha\big) \\
&= 4\pi\frac{1}{a}\Big(\frac{1}{z-\beta}\Big)\Big|_{z=\alpha} \\
&= \frac{4\pi}{a(\alpha-\beta)} \\
&= \frac{2\pi}{\sqrt{1-a^2}}.
\end{aligned}$$

Consider a real integral in the form

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx$$

where $P(x)$ and $Q(x)$ are polynomial function and $\deg Q(x) \geq 2 + \deg P(x)$. The function $Q(x)$ in the denominator is assumed to be analytic on the real axis.

We can use complex integral to compute the principal value

$$\lim_{a\to\infty}\int_{-a}^{a} \frac{P(x)}{Q(x)}\,dx$$

using contour shown in Figure 1.

The first part C_1 is a line segment from $-a$ to a. The real integral $\int_{-a}^{a} P(x)/Q(x)\,dx$ is the same as the complex integral $\int_{C_1} P(z)/Q(z)\,dz$.

The second part C_2 is a semi-circle in the upper half plane from a to $-a$. The assumption that $\deg Q \geq 2 + \deg P$ implies that the integral $\int_{C_2} P(z)/Q(z)\,dz$ approaches 0 as $a \to \infty$.

Example 24.2. Show that

$$\int_0^\infty \frac{1}{1+x^2}\,dx = \frac{\pi}{2}. \tag{24.1}$$

Since the integrand is an even function, it is equivalent to

$$\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx = \pi.$$

Referring to the contour in Fig. 1, when $a > 1$, the pole at $z = i$ is inside the contour, and the the residue is

$$\operatorname{Res}\big(\frac{1}{1+z^2}; i\big) = \Big(\frac{1}{z+i}\Big)\Big|_{z=i} = \frac{1}{2i}.$$

The integral along C_1 is

$$\int_{C_1} \frac{1}{1+z^2}\,dz = \int_{-a}^a \frac{1}{1+x^2}\,dx.$$

When z is a point on C_2, the function value is upper bounded by $\frac{1}{a^2-1}$. By ML inequality (Theorem 13.3),

$$\Big|\int_{C_2} \frac{1}{1+z^2}\,dz\Big| \leq \frac{1}{a^2-1}(\pi a) = O(1/a).$$

The modulus converges to zero as $a \to \infty$. Hence, the integral to be computed can be written as

$$\begin{aligned}\lim_{a\to\infty}\int_{-a}^a \frac{1}{1+x^2}\,dx &= \lim_{a\to\infty}\int_{C_1}\frac{1}{1+z^2}\,dz \\ &= 2\pi i \operatorname{Res}\big(\frac{1}{1+z^2}; i\big) - \lim_{a\to\infty}\int_{C_2}\frac{1}{1+z^2}\,dz \\ &= \pi - 0 = \pi.\end{aligned}$$

This proves (24.1).

Example 24.3. Derive

$$\int_{-\infty}^\infty \frac{1}{(1+x^2)^{n+1}}\,dx = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\pi$$

for integer $n \geq 1$.

We use the same contour as in Fig. 1. Let $f(z)$ denote the complex function $\frac{1}{(1+z^2)^{n+1}}$. By residue theorem (Theorem 21.2), we obtain

$$\int_{C_1} + \int_{C_2} = 2\pi i \operatorname{Res}(f; i).$$

We can use similar analysis as in the previous example to show that the integral over C_2 tends to 0 as the radius approach infinity. Hence, what we need to show is

$$2\pi i \operatorname{Res}(f; i) = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \pi. \tag{24.2}$$

In this example, the pole at $z = i$ has order $n+1$. We first calculate

$$\begin{aligned}\frac{d^n}{dz^n}(z-i)^{n+1} f(z) &= \frac{d^n}{dz^n} \frac{1}{(z+i)^{n+1}} \\ &= \frac{(-1)^n (n+1)(n+2)\cdots(2n)}{(z+i)^{2n+1}}\end{aligned}$$

Then we take limit as z tends to i and multiply by $2\pi i/n!$,

$$\begin{aligned}\frac{2\pi i}{n!} \lim_{z\to i} \frac{d^n}{dz^n}(z-i)^{n+1} f(z) &= \frac{2\pi i}{n!} \frac{(-1)^n (n+1)(n+2)\cdots(2n)}{(2i)^{2n+1}} \\ &= \frac{\pi}{n!} \frac{(n+1)(n+2)\cdots(2n)}{2^{2n}} \\ &= \pi \frac{(2n)!}{2^2 \cdot 4^2 \cdot 6^2 \cdots (2n)^2} \\ &= \pi \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}.\end{aligned}$$

This proves (24.2) and completes the derivation.

Lemma 24.1 (Jordan lemma)**.** *Consider the contour C_R shown in Fig. 2 Assume f is analytic on the contour C_R for all sufficiently large R. If $|f(z)| \leq M_R$ for z on the semi-circle C_R and $M_R \to 0$ as $R \to$, then*

$$\lim_{R\to\infty} \int_{C_R} f(z) e^{iaz}\, dz = 0 \quad \textit{for any real constant } a > 0.$$

Proof. We first prove the following Jordan inequality, which is an inequality for real integral,

$$\int_0^\pi e^{-R\sin\theta}\, d\theta < \frac{\pi}{R} \quad \text{for } R > 0. \tag{24.3}$$

For θ in the range $0 \leq \theta \leq \pi/2$, the value $\sin\theta$ is larger than or equal to $2\theta/\pi$. We can see this by comparing the curve of $\sin\theta$ for $0 \leq \theta \leq \pi/2$ and the line segment from the origin

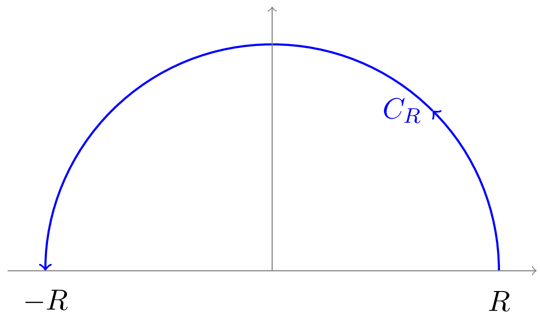


Figure 2: The semi-circular contour in Jordan lemma.

to the point $(\pi/2, 1)$. This yields

$$\begin{aligned}\int_0^{\pi/2} e^{-R\sin\theta}\,d\theta &\le \int_0^{\pi/2} e^{-R(2\theta/\pi)}\,d\theta \\ &= \Big[\frac{-\pi}{2R}e^{-R2\theta/\pi}\Big]_0^{\pi/2} \\ &= \frac{\pi}{2R}(1-e^{R}) \\ &< \frac{\pi}{2R}.\end{aligned}$$

By using the symmetry of the graph of sine function, we can see that the same argument apply to the second part of the interval from $\pi/2$ to π,

$$\int_{\pi/2}^{\pi} e^{-R\sin\theta}\,d\theta < \frac{\pi}{2R}.$$

This proves (24.3).

For real constant $a > 0$, we have

$$\int_{C_R} f(z)e^{iaz}\,dz = \int_0^{\pi} f(Re^{i\theta})e^{iaR(\cos\theta+i\sin\theta)}(Rie^{i\theta})\,d\theta.$$

By triangle inequality for complex integral (Theorem (13.1)),

$$\Big|\int_0^{\pi} f(Re^{i\theta})e^{iaR(\cos\theta+i\sin\theta)}(Rie^{i\theta})\,d\theta\Big| \le RM_R\int_0^{\pi} e^{-aR\sin\theta}\,d\theta.$$

By Jordan inequality, the integral on the right-hand side is less than or equal to π/R. As a result, the modulus of $\int_{C_R} f(z)e^{iaz}\,dz$ is upper bounded by πM_R, which approaches 0 as R approaches ∞. $\square$

Jordan lemma is useful in evaluating integral of type

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos(x)\, dx \quad \text{or} \quad \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \sin(x)\, dx$$

where $P(x)$ and $Q(x)$ are polynomials and $\deg Q - \deg P \geq 1$.

Example 24.4. Derive

$$\int_{-\infty}^{\infty} \frac{\cos(bx)}{1+x^2}\, dx = \frac{\pi}{e^b} \quad \text{for } b > 0.$$

Since $(\sin x)/(1+x^2)$ is an odd function, establishing the above equation is the same as showing

$$p.v. \int_{-\infty}^{\infty} \frac{e^{ibx}}{1+x^2}\, dx = \frac{\pi}{e^b} \quad \text{for } b > 0.$$

Let C_1 and C_2 be the contours in Figure 1. We compute the complex integral

$$\int_{C_1+C_2} \frac{e^{iz}}{1+z^2}\, dz$$

along the closed path formed by C_1 and C_2.

By Lemma 24.1,

$$\Big| \int_{C_2} \frac{e^{iz}}{1+z^2}\, dz \Big| \to 0, \qquad \text{as } a \to \infty.$$

Hence

$$\lim_{a\to\infty} \int_{C_1+C_2} \frac{e^{iz}}{1+z^2}\, dz = \int_{-\infty}^{\infty} \frac{e^{ibx}}{1+x^2}\, dx.$$

On the other hand, the residue of $\frac{e^{ibz}}{1+z^2}$ at $z = i$ is

$$\operatorname{Res}\big(\frac{e^{ibz}}{1+z^2}; i\big) = \frac{e^{ibz}}{z+i}\Big|_{z=i} = \frac{e^{-b}}{2i}.$$

By residue theorem (Theorem 21.2), we get

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}\, dx = \int_{-\infty}^{\infty} \frac{e^{ibx}}{1+x^2}\, dx = 2\pi i \frac{e^{-b}}{2i} = \frac{\pi}{e^b}.$$

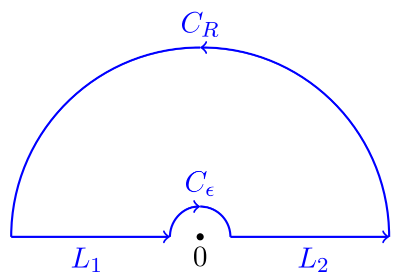
Example 24.5. Show that

$$\int_{0}^{\infty} \frac{\sin x}{x}\, dx = \frac{\pi}{2}.$$

Since $\frac{\sin x}{x}$ is an even function and $\frac{\cos x}{x}$ is odd, it is sufficient to prove

$$p.v. \quad \int_{-\infty}^{\infty} \frac{e^{ix}}{x}\, dx = \pi i.$$

The function $\frac{e^{iz}}{z}$ has a pole at the origin. In order to avoid the pole, we consider the following indented contour.



The outer semi-circle has radius R and the inner semi-circle has radius ϵ.

The function $\frac{e^{iz}}{z}$ is analytic inside the contour. By Cauchy theorem (Theorem 14.6), we have

$$\int_{L_1+L_2+C_R+C_\epsilon} \frac{e^{iz}}{z}\, dz = 0.$$

By Jordan lemma (Lemma 24.1), the integral of $\frac{e^{iz}}{z}$ along C_R approaches 0 as $R \to \infty$. The integrals on the real axis approaches $\int_{-\infty}^{\infty} e^{iz}/z\, dz$ as $R \to \infty$ and $\epsilon \to 0$. The problem reduces to proving

$$\int_{C_\epsilon} \frac{e^{iz}}{z}\, dz = -\pi i.$$

The integrand can be represented by Laurent series

$$\frac{e^{iz}}{z} = \frac{1}{z} + i - \frac{z}{2} - \frac{iz^2}{6} + \cdots$$

For small enough $\epsilon > 0$, the analytic part $+i - \frac{z}{2} - \frac{iz^2}{6} + \cdots$ is bounded (because it converges and is continuous at $z = 0$). By ML inequality (Theorem 13.3),

$$\Big| \int_{C_\epsilon} +i - \frac{z}{2} - \frac{iz^2}{6} + \cdots\, dz \Big| \to 0$$

as $\epsilon \to 0$. The integral of $1/z$ on C_ϵ is equal to

$$\int_{C_\epsilon} \frac{1}{z}\, dz = \int_{-\pi}^{0} \frac{i\epsilon e^{i\theta}}{\epsilon e^{i\theta}}\, d\theta = -\pi i.$$

This proves that

$$i\int_{-\infty}^{\infty}\frac{\sin x}{x}\,dx=\int_{-\infty}^{\infty}\frac{e^{iz}}{z}\,dz=\lim_{\substack{R\to 0\\ \epsilon\to 0}}\int_{L_1+L_2}\frac{e^{iz}}{z}\,dz=-\lim_{\epsilon\to 0}\int_{C_\epsilon}\frac{e^{iz}}{z}\,dz=i\pi.$$

25 Lecture 25 (Keyhole contour, analytic at infinity)

Summary

- Evaluating real integral in the form $\int_0^\infty P(x)/Q(x)\,dx$.
- Being analytic at the point at infinity
- Residue at the point at infinity.

We can use the keyhole contour in Fig. 3 to evaluate real integral

$$\int_0^\infty \frac{P(x)}{Q(x)}\,dx,$$

where $\deg Q \geq \deg P + 2$ and $Q(x) \neq 0$ for $x \geq 0$. We demonstrate the procedure using the following example:

$$\text{Evaluate } \int_0^\infty \frac{1}{x^3+1}\,dx.$$

The first step is to multiply the function to be integrated by a log function, and consider the complex integral

$$\int_C \frac{\log z}{z^3+1}\,dz,$$

over the keyhole contour C as shown in Fig. 3. For the complex log function we take the nonnegative real axis as the branch cut, i.e., for complex number in polar form $re^{i\theta}$, with $0<\theta<2\pi$, the log function is evaluated as

$$\log r + i\theta \quad \text{ for } 0<\theta<2\pi.$$

The contour C consists of four parts. The outer circle C_R has radius R and positive orientation. The inner circle C_ϵ has radius ϵ and negative orientation. The distance between L_1 and L_2 is 2δ. When $R\to\infty$, $\epsilon\to 0$ and $\epsilon\to 0$, the integrals along L_1 and L_2 have limits

$$\int_{L_1}\frac{\log z}{z^3+1}\,dz \to \int_0^\infty \frac{\log x}{x^3+1}\,dx$$
$$\int_{L_2}\frac{\log z}{z^3+1}\,dz \to -\int_0^\infty \frac{\log x + 2\pi i}{x^3+1}\,dx.$$

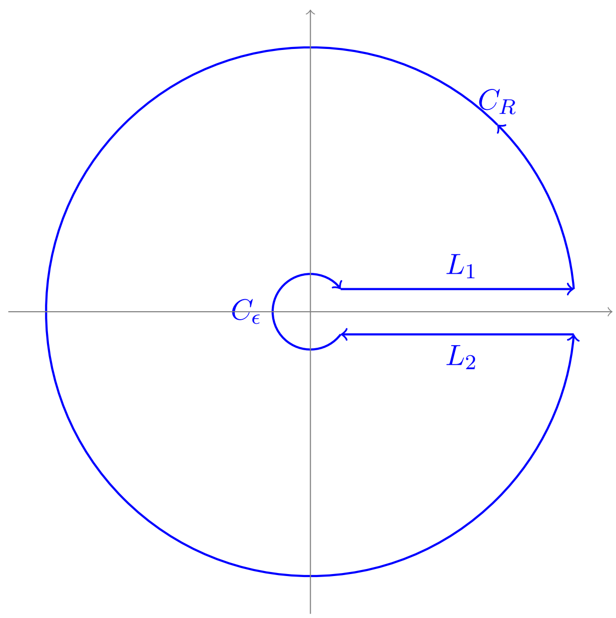


Figure 3: Keyhole contour

The integral over C_ϵ has modulus upper bounded by

$$\Big| \int_{C_\epsilon} \frac{\log z}{z^3+1}\, dz \Big| \leq 2\pi\epsilon M_\epsilon \max_{|z|=\epsilon} |\log z|$$

where M_ϵ denotes the maximum of $1/(z^3+1)$ on the circle $|z| = \epsilon$. Since it is assumed that $Q(z)$ is defined at $z = 0$, M_ϵ can be upper bounded by another constant independent of ϵ. In this example M_ϵ is approach 1 as ϵ approaches 0, and hence we can say that $M_\epsilon < 2$ for all sufficiently small ϵ. The modulus of $\log(z)$ is no more than the modulus of $\log(\epsilon) + i2\pi$. Hence, as $\epsilon \to 0$, the modulus of the integral of C_ϵ is upper bounded by a constant times $\epsilon |\log \epsilon|$, which decreases to zero as $\epsilon \to 0$.

For complex number z on C_R, the modulus $|\log(z)/(z^3+1)|$ is upper bounded by a constant times $\frac{\log R}{R^3-1}$. The integral over C_R approaches 0 in the order of $O(R\frac{\log R}{R^3})$.

Therefore,

$$\lim_{\substack{\epsilon\to 0\\ \delta\to 0\\ R\to\infty}} \int_{L_1+L_2+C_R+C_\epsilon} \frac{\log(z)}{z^3+1}\, dz = -2\pi i \int_0^\infty \frac{1}{x^3+1}\, dx.$$

We can re-write the above equation as

$$\int_0^\infty \frac{1}{x^3+1}\,dx = -\lim_{\substack{\epsilon\to 0\\ \delta\to 0\\ R\to\infty}} \frac{1}{2\pi i}\int_{L_1+L_2+C_R+C_\epsilon} \frac{\log(z)}{z^3+1}\,dz.$$

The polynomial z^3+1 has three roots, namely, -1, $e^{\pi i/3}$ and $e^{5\pi i/3}$. By residue theorem (Theorem 21.2), we can compute the integral by

$$\int_0^\infty \frac{1}{x^3+1}\,dx = -\Big[\operatorname{Res}\big(\frac{\log(z)}{z^3+1};-1\big) + \operatorname{Res}\big(\frac{\log(z)}{z^3+1};e^{\pi i/3}\big) + \operatorname{Res}\big(\frac{\log(z)}{z^3+1};e^{5\pi i/3}\big)\Big].$$

Since the pole of $\log(z)/(z^3+1)$ are all simple roots, we can evaluate the residues at -1, $e^{\pi i/3}$ and $e^{5\pi i/3}$ by

$$\begin{aligned}
\operatorname{Res}\big(\frac{\log(z)}{z^3+1};-1\big) &= \frac{\log(z)}{3z^2}\Big|_{-1} = \pi i\frac{1}{3}\\
\operatorname{Res}\big(\frac{\log(z)}{z^3+1};e^{\pi i/3}\big) &= \frac{\log(z)}{3z^2}\Big|_{e^{\pi i/3}} = \frac{\pi i}{3}\frac{e^{-2\pi i/3}}{3}\\
\operatorname{Res}\big(\frac{\log(z)}{z^3+1};e^{5\pi i/3}\big) &= \frac{\log(z)}{3z^2}\Big|_{e^{5\pi i/3}} = \frac{5\pi i}{3}\frac{e^{-10\pi/3}}{3}.
\end{aligned}$$

(See Question 5 in Homework 14.)

Adding the three residues, we get

$$\frac{i\pi}{3}\big[1+\frac{-1-\sqrt{3}i}{6}+5\frac{-1+\sqrt{3}i}{6}\big] = -\frac{2\sqrt{3}}{9}\pi.$$

Hence, the answer is

$$\int_0^\infty \frac{1}{x^3+1}\,dx = \frac{2\sqrt{3}}{9}\pi.$$

To understand the behavior of a function $f(z)$ at the point at infinity, we make a change of variable $w = 1/z$. The new variable $1/z$ is called the *local parameter* at ∞.

Definition 25.1. Given a complex function $f(z)$, make a change of variable and define a new function $g(w) = f(1/w)$. We say that the function $f(z)$ is analytic at $z = \infty$ if $g(w)$ is analytic at $w = 0$. The point at infinity is said to be a removable singularity (resp. pole, or essential singularity) if $g(w)$ has a removable regularity (resp. pole, or essential singularity) at $w = 0$.

Example 25.1. The function $f(z) = z$ has a simple pole at $z = \infty$, because $g(w) = f(1/w) = 1/w$ has a simple pole at $w = 0$.

Example 25.2. The function $f(z) = 1/z$ has a simple zero at $z = \infty$, because $g(w) = f(1/w) = w$ has a simple zero at $w = 0$.

Example 25.3. The function $f(z) = e^z$ has an essential singularity at $z = \infty$, because

$$g(w) = \exp(1/w) = 1 + \frac{1}{w} + \frac{1}{2w^2} + \cdots$$

has an essential singularity at $w = 0$.

Definition 25.2. Suppose $f(z)$ has finitely many singular points in the complex plane, so that f converges in the domain $R < |z|$ for some R. The *residue at ∞* of $f(z)$ is defined as

$$\mathrm{Res}(f;\infty) \triangleq \frac{1}{2\pi i}\int_{C_0} f(z)\, dz$$

where C_0 is a circle containing all singular points in the interior, with *clockwise orientation.*

The assumption that $f(z)$ has finitely many singular points is the same as assuming that the point at infinity is an isolated singular point.

By making a change of variable $w = 1/z$, $dw = -1/z^2\, dz$, we get

$$\frac{1}{2\pi i}\int_{C_0} f(z)\, dz = \frac{1}{2\pi i}\int_{C} \frac{-1}{w^2} f\big(\frac{1}{w}\big)\, dw = -\,\mathrm{Res}\,\Big(\frac{1}{w^2} f\big(\frac{1}{w}\big); 0\big).$$

where C is the image of C_0 under the transformation $w = 1/z$. In the w-plane, the function $\frac{-1}{w^2} f\big(\frac{1}{w}\big)$ has a isolated singularity at $w = 0$. This proves the following property.

Theorem 25.3. *Suppose f has finitely many singular points and γ is a contour with positive orientation, containing all singluar points in the interior. Then*

$$\frac{1}{2\pi i}\int_{\gamma} f(z)\, dz = \mathrm{Res}\,\Big(\frac{1}{w^2} f\big(\frac{1}{w}\big); 0\big).$$

Example 25.4. Evaluate

$$\int_{|z|=2} \frac{4z+1}{z(z-1)}\, dz.$$

There are two simple poles at $z = 0$ and $z = 1$. Both of them are inside the contour. Using the previous theorem, we can calculate the complex integral by

$$\begin{aligned}
\int_{|z|=2} \frac{4z+1}{z(z-1)}\,dz &= 2\pi i \operatorname{Res}\left(\frac{1}{w^2}\frac{4(1/w)+1}{(1/w)((1/w)-1)};0\right) \\
&= 2\pi i \operatorname{Res}\left(\frac{4+w}{w(1-w)};0\right) \\
&= 2\pi i \cdot 4 = 8\pi i.
\end{aligned}$$