MAT3253 Tutorial 3

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1 Sequences and Series

- Cauchy Criterion

  Theorem 1. $\{a_n\}$ is a sequence of complex numbers. It converges if and only if for each $\epsilon > 0$, $\exists$ positive integer N such that, $N \leq n < m \rightarrow |a_n - a_m| < \epsilon$

- Limit Supremum

  Definition 1. The limit supremum is a unique number associated with each sequence of real numbers $\{a_n\}$, provided $\{a_n\}$ is bounded above and does not converge to negative infinity. It is given by the following process:

  1. Let $b_n^{(k)} = \max(a_k, a_{k+1} \cdots, a_{k+n-1})$
  2. Monotone Convergence Theorem allows to define the numbers c_k, as the limit of the sequence $\{b_n^{(k)}\}$, as $n \rightarrow \infty$.
  3. limit supremum is the limit of the sequence $\{c_k\}$

  Theorem 2. let $\alpha = \limsup\limits_{n\rightarrow\infty} a_n$, then α satisfy

  1. $\forall \epsilon > 0, \exists\ k \in Z^+, s.t., n \geq k \rightarrow n_k < \alpha + \epsilon$
  2. $\forall \epsilon > 0$, there are infinitely many terms in $\{a_n\}$ that's larger than $\alpha - \epsilon$

  Conversely, if a number satisfy the above two conditions, it equals α.

- Uniform Convergence

  Definition 2. A sequence of complex functions $\{f_n\}$ that's defined on a common complex number set E converges uniformly to a function f if, $\forall \epsilon > 0, \exists N \in Z^+, s.t., n \geq N, z \in E \rightarrow |f_n(z) - f(z)| < \epsilon$

  Theorem 3. Under the same setting, $f_n \rightarrow f$ uniformly if and only if, $\forall \epsilon > 0, \exists N \in Z^+, s.t., m > n \geq N, z \in E \rightarrow |f_n(z) - f_m(z)| < \epsilon$

  Theorem 4. Under the same setting, given $f_n \rightarrow f$ uniformly. Then each f_n continuous on $E \rightarrow f$ continuous on E.

- Weierstrass M Test

  Theorem 5. For a series of functions $sum f_n(z)$ defined on a common complex set E, its majorant is a series of complex numbers $\sum a_n$, where $\forall n \in Z^+, \forall z \in E, |f_n(z)| \leq a_n$. Then the convergence of the majorant would imply the uniform convergence of $\sum f_n$

2 Power Series

- Circle of Convergence

  Theorem 6. let $\alpha = \limsup\limits_{n\rightarrow\infty} \sqrt[n]{a_n}, R = \frac{1}{\alpha}$. Then the power series $\sum_0^\infty a_n z^n$ converges in the interior of the circle of radius R centered at origin; diverges in the exterior.