12 Lecture 12 (Contour integral)

Summary

- Definition of integral of complex function of a real variable
- Definition of contour integral
- Independence of parameterization

Definition 12.1. Given a continuous function $f : \mathbb{R} \to \mathbb{C}$ mapping real numbers to complex numbers, with $u(t)$ and $v(t)$ as the real and imaginary parts,

$$f(t) = u(t) + iv(t)$$

for $t \in [a, b]$, we define the integral of f as

$$\int_a^b f(t)\,dt \triangleq \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt.$$

The real and imaginary parts are the limits of Riemann sums. Given a partition of $[a, b]$ into subintervals $[t_{k-1}, t_k]$, for $k = 1, 2, \dots, n$, with

$$a = t_0 < t_1 < t_2 < \cdots < t_n = b,$$

the corresponding Riemann sums are

$$\sum_{k=1}^n u(t_k^*)(t_k - t_{k-1}), \quad \text{and} \quad \sum_{k=1}^n v(t_k^*)(t_k - t_{k-1}),$$

where t_k^* is any point in the subinterval $[t_{k-1}, t_k]$. The limit is taken with $n \to \infty$ and $\max_k(t_k - t_{k-1}) \to 0$.

Example 12.1.

$$\int_0^1 e^{\pi i t}\,dt = \int_0^1 \cos(\pi t)\,dt + i\int_0^1 \sin(\pi t)\,dt = \frac{2i}{\pi}.$$

Review of work integral and flow integral

Given a vector field $\mathbf{F}(x, y) = (M(x, y), N(x, y))$, the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C M\, dx + N\, dy.$$

over a curve C has the physical interpretation of work done. It is the work done by the force $\mathbf{F}$ on a particle moving along the curve C.

On the other hand, if the vector field $\mathbf{F}(x, y) = (M(x, y), N(x, y))$ is regarded as the flow of some fluid, then the flux through a curve C is given by

$$\int_C \mathbf{F} \cdot \hat{\mathbf{n}}\, ds = \int_C M\, dy - N\, dx.$$

Here $\hat{\mathbf{n}}$ signifies a unit normal vector along the curve.

Definition 12.2. A parametric a curve C represented by

$$z : [a, b] \to \mathbb{C}$$
$$z(t) = x(t) + iy(t)$$

is said to be *smooth* if

(i) $x(t)$ and $y(t)$ are continuous on $[a, b]$,

(ii) $x(t)$ and $y(t)$ are differentiable on $[a, b]$, and

(iii) the vector $(x'(t), y'(t))$ is not equal to the zero vector for $t \in [a, b]$.

We note that the condition in (iii) means that the tangent vector is well-defined for all points on the curve. We write $z'(t) = x'(t) + iy'(t)$ as a tangent vector at t. To emphasize the direction/orientation of the curve from $t = a$ to $t = b$, we sometime use the notation "contour". If a curve can be divided into infinitely many parts and each part is smooth, then we call it a *piece-wise smooth* curve.

Remark. In the rest of the lecture notes, we shall not distinguish smooth curve and piece-wise smooth curve. All piece-wise smooth curve can be treated as a concatenation of smooth curves. All results for smooth curves also hold for piece-wise smooth curve.

Definition 12.3. Given a continuous complex-valued function $f : \mathbb{C} \to \mathbb{C}$ and a smooth curve C, the *complex integral* (or *contour integral*) of f over C is defined as

$$\int_C f(z)\, dz \triangleq \int_a^b f(z(t)) \cdot z'(t)\, dt \tag{12.1}$$

with the right-hand side defined as in Definition 12.1.

It is important to note that the integral depends on the direction of the curve C.

Physical interpretation Given a complex function $f(z) = u(x, y) + iv(x, y)$, define a vector field $\mathbf{F}(x, y) = (M(x, y), N(x, y))$ by

$$M(x, y) = u(x, y) \quad \text{and} \quad N(x, y) = v(x, y).$$

Then

$$\begin{aligned}\int_C f(z)\, dz &= \int_C (u + iv)(x' + iy')dt \\ &= \int_C (u + iv)(dx + idy)\end{aligned}$$

After formally expanding the product in the integrand, we see that

$$\int_C f(z)\, dz = \int_C M\, dx + N\, dy) + i \int_C M\, dy - N\, dx.$$

The real and imaginary parts can be interpreted as the work integral and the flux integral, respectively.

In order for Definition 12.3 to make sense, we need to first show that the right-hand side of (12.1) does not depend on the parameterization of curve.

Suppose C has two parameterizations

$$\begin{gathered} z(t) \text{ for } a \le t \le b \\ w(t) \text{ for } c \le t \le d. \end{gathered}$$

We can find a monotonically increasing $\lambda : [c, d] \to [a, b]$ such that

$$w(t) = z(\lambda(t)).$$

The main step is the chain rule $w'(t) = z'(\lambda(t))\lambda'(t)$.

$$\begin{aligned}\int_c^d f(w(t))w'(t)\, dt &= \int_c^d f(z(\lambda(t))z'(\lambda(t))\lambda'(t)\, dt \\ &= \int_c^d [u(z(\lambda(t))) + iv(z(\lambda(t)))][x'(\lambda(t)) + iy'(\lambda(t))]\lambda'(t)\, dt \\ &= \int_c^d u(z(\lambda(t)))x'(\lambda(t)) - v(z(\lambda(t))y'(\lambda(t))\lambda'(t)\, dt \\ &\quad + i \int_c^d u(z(\lambda(t)))y'(\lambda(t)) + v(z(\lambda(t))x'(\lambda(t))\lambda'(t)\, dt.\end{aligned}$$

Substitute $\tau = \lambda(t)$, $d\tau = \lambda'(t)\,dt$.

$$\begin{aligned}\int_c^d f(w(t))w'(t)\,dt &= \int_a^b u(z(\tau))x'(\tau) - v(z(\tau))y'(\tau)\,d\tau + \\ &\qquad + i\int_a^b u(z(\tau))y'(\tau) + v(z(\tau))x'(\tau)\,d\tau \\ &= \int_a^b f(z(\tau))z'(\tau)\,d\tau.\end{aligned}$$

This proves that the definition of complex integral is independent of path.

Example 12.2. Integrate $f(z) = \bar{z}$ over the curve $C: \ z(t) = 0 + it$ for $0 \le t \le 2$.

$$\begin{aligned}\int_C \bar{z}\,dz &= \int_0^2 (0+it)^*(i)\,dt \\ &= \int_0^2 (-it)i\,dt \\ &= \Big[\frac{t^2}{2}\Big]_0^2 = 2.\end{aligned}$$

Example 12.3. Integrate $f(z) = z^2$ over the curve $C: \ z(t) = \cos(t) + i\sin(t)$ for $0 \le t \le \pi$. The derivative of $z(t)$ is $z'(t) = -\sin(t) + i\cos(t)$.

$$\begin{aligned}\int_C z^2\,dz &= \int_0^\pi (\cos t + i\sin t)^2(-\sin t + i\cos t)\,dt \\ &= i\int_0^\pi (\cos 2t + i\sin 2t)(\cos t + i\sin t)\,dt \\ &= i\int_0^\pi (\cos 3t + i\sin 3t)\,dt \\ &= i\Big[\frac{\sin 3t}{3} - i\frac{\cos 3t}{3}\Big]_0^\pi \\ &= \frac{2}{3}.\end{aligned}$$

Definition 12.4. The *negation* of a curve C is the curve with the same locus with reverse direction. The negation of C is denoted by $-C$. Given two contours C_1 and C_2, with the end point of C_1 identical to the start point of C_2, then the *concatenated contour* (first traveling along C_1 and then along C_2) is denoted by $C_1 + C_2$.

Theorem 12.5. *For any continuous function $f(z)$ and $g(z)$, constant a, and smooth curves C, C_1 and C_2,*

$$\int_{-C} f(z)\,dz = -\int_C f(z)\,dz$$
$$\int_{C_1+C_2} f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$
$$\int_C f(z)+g(z)\,dz = \int_C f(z)\,dz + \int_C g(z)\,dz$$
$$\int_C af(z)\,dz = a\int_C f(z)\,dz.$$

The proof is omitted.

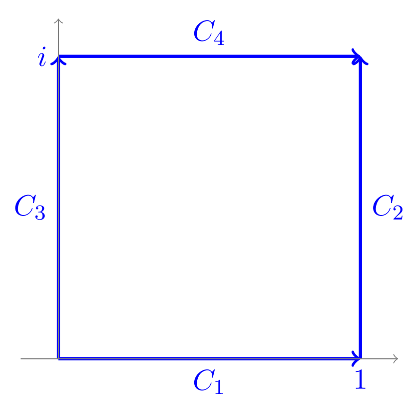
13 Lecture 13 (Independence of path)

Summary

- Triangle inequality for complex integral
- ML inequality
- Fundamental theorem of calculus for complex functions

A motivating example.

Example 13.1. Consider the function $f(z) = x^2 + iyx$. Compute the complex integral from 0 to $1+i$ (i) from 0 to 1 and then from 1 to $1+i$, (ii) from 0 to i and then from i to $1+i$.



Let C_1, C_2, C_3 and C_4 be directed path as indicated above. Parameterize C_1 and C_2,

$$\begin{aligned} C_1 : z(x) &= x + 0i, \quad 0 \le x \le 1 \\ z'(x) &= 1, \\ C_2 : z(y) &= 1 + iy, \quad 0 \le y \le 1 \\ z'(x) &= i. \end{aligned}$$

The complex integral along $C_1 + C_2$ is

$$\begin{aligned} \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz &= \int_0^1 x^2\,dx + \int_0^1 (1+iy)i\,dy \\ &= \frac{1}{3} + i - \frac{1}{2} \\ &= i - \frac{1}{6}. \end{aligned}$$

Next, parameterize C_3 and C_4 as

$$\begin{aligned} C_3 : z(x) &= iy, \quad 0 \le y \le 1 \\ z'(y) &= i, \\ C_4 : z(x) &= x + i, \quad 0 \le x \le 1 \\ z'(x) &= 1. \end{aligned}$$

The complex integral along $C_3 + C_4$ is

$$\begin{aligned} \int_{C_3} f(z)\,dz + \int_{C_4} f(z)\,dz &= \int_0^1 0\,dy + \int_0^1 (x^2 + ix)\,dx \\ &= \frac{1}{3} + \frac{i}{2}. \end{aligned}$$

It is obvious that the two values obtained from the two different paths are different.

Theorem 13.1 (Triangle inequality for complex integral)**.** *Suppose $g(t)$ is a continuous complex function from $[a,b]$ to $\mathbb{C}$. Then*

$$\Big| \int_a^b g(t)\,dt \Big| \le \int_a^b |g(t)|\,dt \tag{13.1}$$

Proof. Let α denote the complex integral $\alpha \triangleq \int_a^b g(t)\,dt$. If $\alpha = 0$, then (13.1) is obviously true, because the left-hand side is zero. Hence, we can suppose $\alpha \neq 0$. Write α in polar form $\alpha = re^{i\theta}$. (θ is defined because $\alpha \neq 0$.)

Then,

$$e^{-i\theta} \int_a^b g(t)\,dt = r$$

is a real number. We can re-write it as

$$\int_a^b e^{-i\theta} g(t)\,dt.$$

Let $u(t)$ and $v(t)$ be the real and imaginary parts of $e^{-i\theta}g(t)$, respectively. We get

$$\int_a^b u(t)\,dt = r, \quad \int_a^b v(t)\,dt = 0.$$

But

$$\begin{aligned} u(t) &\leq \sqrt{u^2(t)} \\ &\leq \sqrt{u^2(t) + v^2(t)} \\ &= |g(t)| \end{aligned}$$

By the monotonic property for real functions,

$$\Big| \int_a^b g(t)\,dt \Big| = r = \int_a^b u(t)\,dt \leq \int_a^b |g(t)|\,dt.$$

$\square$

Example 13.2.

$$\Big| \int_a^b e^{it}\,dt \Big| \leq \int_a^b |e^{it}|\,dt = \int_a^b dt = b - a.$$

Definition 13.2. The *length* of a smooth curve C, represented by $z(t)$, $a \leq t \leq b$, is defined as $\int_a^b |z'(t)|\,dt$.

Theorem 13.3 (ML inequality)**.** *If $|f(z)| \leq M$ for z on a smooth curve C and the length of C is equal to L, then*

$$\Big| \int_C f(z)\,dz \Big| \leq ML.$$

Proof. Apply the triangle inequality to

$$\Big|\int_C f(z)\,dz\Big| = \Big|\int_a^b f(z(t))z'(t)\,dt\Big|.$$

This yields

$$\Big|\int_C f(z)\,dz\Big| \le \int_a^b |f(z(t))|\cdot|z'(t)|\,dt \le M\int_a^b |z'(t)|\,dt = ML.$$

□

Theorem 13.4 (Fundamental theorem of calculus for complex function)**.** *Suppose $f(z)$ is a continuous complex function in a region R. Then the followings are equivalent:*

(i) $f(z)$ is the derivative of a function $F(z)$ in R; (The function $F(z)$ is necessarily analytic)

(ii) for any smooth contour C in R from z_1 to z_2,

$$\int_C f(z)\,dz = F(z_2) - F(z_1).$$

Proof. ((i) $\Rightarrow$ (ii)) Suppose C is a smooth curve in R parameterized by $z(t)$ for $a \le t \le b$. We have $z(a) = z_1$ and $z(b) = z_2$.

Let $g(t) = F(z(t))$. By chain rule for complex functions,

$$g'(t) = F'(z(t))z'(t) = f(z(t))z'(t).$$

The problem then reduces to fundamental theorem of calculus for real functions,

$$\begin{aligned}\int_C f(z)\,dz &= \int_a^b f(z(t))z'(t)\,dt \\ &= \int_a^b g'(t)\,dt \\ &= g(b) - g(a) \\ &= F(z_2) - F(z_1).\end{aligned}$$

((ii) $\Rightarrow$ (i)) Fix a base point z_0 in the R. Define $F(z) \triangleq \int_C f(z)\,dz$, where C is a smooth path from z_0 to z. Since it is assumed that the integral is independent of path, the integral only depends on the start point and end point. We can write

$$F(z) = \int_{z_0}^{z} f(w)\,dw.$$

We want to show that $F'(z) = f(z)$.

Consider

$$\begin{aligned} \frac{F(z+h)-F(z)}{h} - f(z) &= \frac{1}{h}\int_z^{z+h} f(w)\,dw - f(z) \\ &= \frac{1}{h}\int_z^{z+h} (f(w)-f(z))\,dw. \end{aligned}$$

Because f is continuous at z, given any $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(w) - f(z)| < \epsilon \quad \text{whenever} \quad |w-z| < \delta.$$

Since the integral is independent of path, we can take the line segement from z to $z+h$ as the path. The length is equal to $|h|$. When $|h|$ is smaller than δ, the modulus of $f(w)-f(z)$ is upper bounded by ϵ. We then apply ML inequality to get an upper bound,

$$\Big|\frac{F(z+h)-F(z)}{h} - f(z)\Big| \leq \frac{1}{|h|}\epsilon|h| \leq \epsilon$$

for all $|h| \leq \delta$. Hence,

$$\lim_{h\to 0}\frac{F(z+h)-F(z)}{h} = f(z).$$

$\square$

Suppose C is a piece-wise smooth curve $C = C_1 + C_2 + \cdots + C_n$, so that C_k is a differentiable curve from z_{k-1} to z_k, for $k = 1, 2, \ldots, n$. To apply Theorem 13.4 for piece-wise smooth curve, we can write

$$\begin{aligned} \int_C f(z)\,dz &= \sum_{k=1}^{\infty}\int_{C_k} f(z)\,dz \\ &= \sum_{k=1}^{n} F(z_k) - F(z_{k-1}) \\ &= F(z_n) - F(z_0). \end{aligned}$$

Example 13.3. Compute $\int_C z^3 - z\,dz$ for a curve C from z_0 to z_1. Since $z^4/4 - z^2/2$ is an anti-derivative of $z^3 - z$ exists, we can compute the integral by

$$\int_C z^3 - z\,dz = \left[\frac{z^4}{4} - \frac{z^2}{2}\right]_{z_0}^{z_1} = \frac{z_1^4}{4} - \frac{z_1^2}{2} - \frac{z_0^4}{4} + \frac{z_0^2}{2}.$$

14 Lecture 14 (Cauchy-Goursat theorem)

Summary

- Closed curve
- Cauchy-Goursat theorem for rectangle

Definition 14.1. A smooth (or piece-wise smooth) curve is said to be *closed* if the start point is the same as the end point.

As in multi-variable calculus, independence of path is equivalent to the condition that the integral over any closed curve is zero.

Theorem 14.2. *$\int_C f\,dz$ is path independent for any piece-wise smooth curve C in a domain, if and only if $\oint_C f\,dz = 0$ for any closed curve C in the domain.*

The proof is the same as in multi-variable calculus and is omitted.

Definition 14.3. A curve is called *simple* if there is no self-intersection.

The figure-8 curve, which looks like ∞, is a typical example of non-simple curve.

Remark. The famous Jordan curve theorem says that a simple closed curve C divides the plane into two components. One component is bounded and the other is unbounded, and the curve is the boundary of each component. Actually, we only need to assume that the curve is continuous (not necessarily smooth) in the Jordan curve theorem.

We need the following theorem called "Cantor's intersection theorem" from point-set topology.

Theorem 14.4. *Suppose K_1, K_2, $K_3, \ldots$ are nonempty compact sets (in some topological space) forming a decreasing sequence*

$$K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots$$

Then $\cap_{j=1}^{\infty} K_j$ is not empty.

The main theorem in this lecture is the following

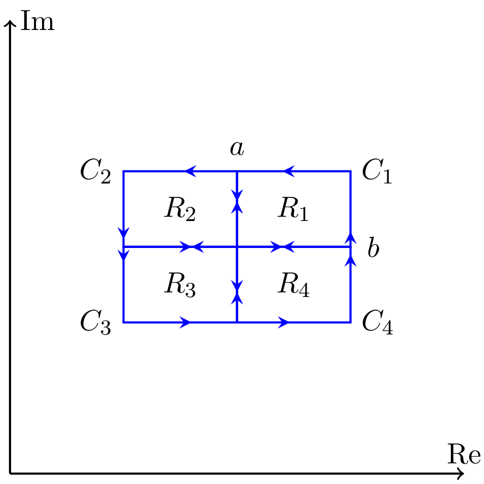
Theorem 14.5 (Cauchy-Goursat for rectangle). *Suppose $f(z)$ is analytic in a domain D and R is a rectangle contained inside D, with sides parallel to the real and imaginary axes. If C is the boundary of R, Then*

$$\oint_C f(z)\, dz = 0.$$

Remark. The function $f(z)$ in Theorem 14.5 is analytic at all points in R. Actually, $f(z)$ is analytic in an open neighborhood containing R. It is important to note that we only need to assume the first-order derivative of $f(z)$ exists. We do not need to assume that $f'(z)$ is continuous.

Proof. We can assume that the orientation of C is counter-clockwise. Suppose the width and height of R are a and b, respectively.

Suppose $\int_C f(z)\, dz = I$. We want to show that $|I| = 0$.



Divide rectangle R into four equal parts, R_1, R_2, R_3 and R_4. Let C_i be the boundary of R_i, for $i = 1, 2, 3, 4$, all in the counter-clockwise direction. Due to cancellations of internal lines, we have

$$I = \int_{C_1} f(z)\, dz + \int_{C_2} f(z)\, dz + \int_{C_3} f(z)\, dz + \int_{C_4} f(z)\, dz.$$

By triangle inequality,

$$|I| = \Big|\int_{C_1} f(z)\,dz\Big| + \Big|\int_{C_2} f(z)\,dz\Big| + \Big|\int_{C_3} f(z)\,dz\Big| + \Big|\int_{C_4} f(z)\,dz\Big|.$$

Suppose $R^{(1)}$ is the the rectangle among R_1 to R_4 that has the largest integral in magnitude,

$$\Big|\int_{\partial R^{(1)}} f(z)\,dz\Big| = \max_{j=1,2,3,4}\Big|\int_{C_j} f(z)\,dz\Big|$$

(Here $\partial R^{(1)}$ means the boundary of $R^{(1)}$.)

This implies

$$\frac{|I|}{4} \le \Big|\int_{\partial R^{(1)}} f(z)\,dz\Big|.$$

Recursively, for $k \ge 1$, divide $R^{(k)}$ into four equal parts, and let $R^{(k+1)}$ be the subrectangle that has the largest integral in absolute value. Thus,

$$R \supset R^{(1)} \supset R^{(2)} \supset R^{(3)} \supset R^{(4)} \supset \cdots$$

and

$$\frac{|I|}{4^k} \le \Big|\int_{\partial R^{(k)}} f(z)\,dz\Big|$$

for $k = 1, 2, 3, \ldots$ The perimeter of $R^{(k)}$ is $L/2^k$, for $k \ge 1$.

By the Theorem 14.4, there is a point z_0 that lies inside $R^{(k)}$ for all k. Since f is assumed to be analytic in R, it is complex differentiable at the point z_0. Suppose that the derivative is $f'(z_0)$. For any $h \in \mathbb{C}$ such that $z + h \in R$, by the definition of complex derivative, we have

$$f(z_0 + h) = f(z_0) + f'(z_0)h + \epsilon h \tag{14.1}$$

where $|\epsilon| \to 0$ as $|h| \to 0$.

Put $z = z_0 + h$ in (14.1). For any $k = 1, 2, \ldots$, the distance between a point z on the boundary of $R^{(k)}$ and z_0 can be bounded by

$$\begin{aligned} |z - z_0| &\le \text{ diagonal of } R^{(k)} \\ &\le \sqrt{\Big(\frac{a}{2^k}\Big)^2 + \Big(\frac{b}{2^k}\Big)^2} \\ &\le \frac{\max(a,b)\sqrt{2}}{2^k}. \end{aligned}$$

We now fix $\delta > 0$, and choose a sufficiently large integer k, such that $|\epsilon| < \delta$ for for all z on the boundary of $R^{(k)}$. This k certainly exists because $\frac{\max(a,b)\sqrt{2}}{2^k} \to 0$ as $k \to \infty$.

The integral of f over the boundary of $R^{(k)}$ is

$$\int_{\partial R^{(k)}} f(z)\,dz = \int_{\partial R^{(k)}} f(z_0) + f'(z_0)(z - z_0) + \epsilon(z - z_0)\,dz.$$

Because $f(z_0) + f'(z_0)(z - z_0)$ is a linear function in z, it has an anti-derivative. By Theorem 13.4, $\int_{\partial R^{(k)}} f(z_0) + f'(z_0)(z - z_0)\,dz$ is zero. So,

$$\int_{\partial R^{(k)}} f(z)\,dz = \int_{\partial R^{(k)}} \epsilon(z - z_0)\,dz.$$

To finish the proof, we upper bound $|I|$ using ML inequality

$$\begin{aligned} |I| &\le 4^k \Big| \int_{\partial R^{(k)}} \epsilon(z - z_0)\,dz \Big| \\ &\le 4^k \frac{\delta \max(a,b)\sqrt{2}}{2^k} \cdot \frac{2(a+b)}{2^k} = 2\sqrt{2}\max(a,b)(a+b)\delta. \end{aligned}$$

Here $2(a+b)$ is the perimeter of the original rectangle R. Since δ can be arbitrarily small, and a and b are constant, we conclude that $|I| = 0$. Hence $I = 0$. $\square$

There are several versions of Cauchy theorem in the literature. Another version is for simple closed curve C.

Theorem 14.6. *If $f(z)$ is analytic in a domain D and C is a simple closed curve in D, such that $f'(z)$ exists in the interior of C (the interior is well-defined by the Jordan curve theorem), then*

$$\oint_C f(z)\,dz = 0.$$

In Theorem 14.6, the condition that $f'(z)$ exists in the interior of C is essential (compare with the remark after Theorem 14.5). The following is a counter-example.

Example 14.1. The function $f(z) = 1/z$ is analytic in the punctured plane $\mathbb{C}\setminus\{0\}$. Consider the circle C_r with radius r and center equal to the origin. The integral $\int_{C_r} 1/z\,dz = 2\pi i$, which is not zero. We can compute the integral using the definition of integral.

$$\begin{aligned}\int_{C_r} 1/z\,dz &= \int_0^{2\pi} r^{-1}e^{-i\theta}(rie^{i\theta})\,d\theta \\ &= \int_0^{2\pi} i\,d\theta \\ &= 2\pi i.\end{aligned}$$

However, if C is a circle in the complex plane that does not contain the origin, then $\int_C 1/z\,dz = 0$ by Theorem 14.6.

15 Lecture 15 (Closed-curve theorem)

Summary

- Existence of local primitive in a disc
- Closed curve theorem

The closed-curve theorem is the analog of Green's theorem in multi-variable calculus. A difference between the closed-curve theorem and Green's theorem is that we only need to assume that the function is complex differentiable in the closed-curved theorem, no need to assume continuity of derivative.

Definition 15.1. Given a complex function $f(z)$ defined on a domain D, a function $F(z)$ such that $F'(z) = f(z)$ for all points $z \in D$ is called a *primitive* or a *primitive function*, or an *anti-derivative* of $f(z)$.

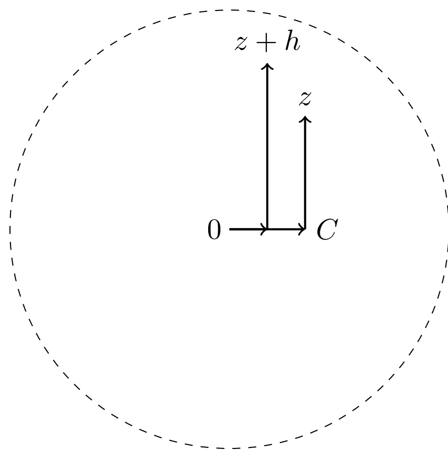
Example 14.1 indicates that in many cases it is impossible to have a primitive function on the largest domain on which the function $f(z)$ is defined. The anti-derivative of $1/z$ is the log function, which is multi-valued. We need to pick a branch cut to make the log function analytic. But once we make a branch cut, the domain is strictly smaller than the puncture plane $\mathbb{C} \setminus \{0\}$.

The best thing we can have is the existence of local primitive.

Theorem 15.2. *If $f(z)$ is analytic in an open disc, then f has a primitive in the open disc.*

Proof. Without loss of generality assume that open disc is centered at the origin. For each z in the disc, consider a polygonal path C that starts at the origin, goes to the right until it reaches $\text{Re}(z)$, and then moves up to the point z. Define a function $F(z)$ by

$$F(z) \triangleq \int_C f(w)\, dw.$$



We want to show that $F'(z) = f(z)$ for all z in the open disc.

Add a complex number h to z so that $z+h$ is inside the disc. The value $F(z+h)$ is computed by integrating on a piece-wise linear path from 0 to $\text{Re}(z+h)$ and then from $\text{Re}(z+h)$ to $z+h$. The difference $F(z+h) - F(z)$ is the integral with the path shown on the left below.

By Cauchy-Goursat theorem for rectangle, the integral of $f(z)$ along the rectangle is zero. Hence the path can be simplified to the path on the right, consisting of a horizontal path γ_1 and a upward vertical path γ_2. We can now write

$$F(z+h) - F(z) = \int_{\gamma_1+\gamma_2} f(z)\, dz.$$

Since $f(z)$ is analytic, it must be continuous at z,

$$\lim_{w \to z} f(w) = f(z).$$

Next we use the fact $\int_{\gamma_1+\gamma_2} f(z)\,dw = f(z)h$, to write

$$\begin{aligned}\Big|\frac{F(z+h)-F(z)}{h} - f(z)\Big| &= \Big|\frac{1}{h}\int_{\gamma_1+\gamma_2} f(w)\,dw - \frac{1}{h}\int_{\gamma_1+\gamma_2} f(z)\,dw\Big| \\ &= \frac{1}{|h|}\Big|\int_{\gamma_1+\gamma_2} f(w) - f(z)\,dw\Big|.\end{aligned}$$

Fix an arbitrarily small and positive ϵ. Find a sufficiently small $\delta > 0$ such that $|f(w) - f(z)| < \epsilon$ for all w with $|w - z| < \delta$. (This is possible by the continuity of f at z.) Then for $|h| < \delta$, by ML inequality,

$$\Big|\frac{F(z+h)-F(z)}{h} - f(z)\Big| \le \frac{1}{|h|}\epsilon \cdot (\text{ length of } \gamma_1 + \text{ length of } \gamma_2).$$

By the total length of γ_1 and γ_2 is less than $2|h|$ (because the length of each γ_1 and γ_2 is less than $|h|$). Therefore

$$\Big|\frac{F(z+h)-F(z)}{h} - f(z)\Big| \le 2\epsilon.$$

Since ϵ is arbitrarily small, this proves that $\frac{F(z+h)-F(z)}{h}$ tends to $f(z)$ as $h \to 0$. □

Theorem 15.3 (Closed-curve theorem for open disc)**.** *If f is analytic in an open disc, then the integral of f over any closed curve C in the disc is zero.*

Proof. By the previous theorem, $f(z)$ has a primitive $F(z)$ in the open disc. By the fundamental theorem of calculus for complex function Theorem 13.4, we have $\int_C f(z)\,dz$ for all C that lies inside the disc. □

Remark. If f is entire, then the radius of the open disc in Theorem 15.3 can be taken to be infinity.

Remark. The closed-curve theorem also holds for other simple shapes such as triangles, semi-circle, parallelogram, or a sector of a circle.

When a function failed to be complex differentiable at some point in an open disc, then the conclusion in Theorem 15.3 may or may not hold. The following is an example

Example 15.1. Let C_r be a circle with radius r centered at the origin.

$$\int_{C_r} z^n\,dz = \begin{cases} 0 & \text{if } n = 0, 1, 2, 3, \dots, \\ 2\pi i & \text{if } n = -1, \\ 0 & \text{if } n = -2, -3, -4, \dots. \end{cases}$$

When $n \geq 0$, z^n has a primitive throughout $\mathbb{C}$. By Theorem 13.4, $\int_{C_r} z^n\,dz = 0$. When $n = -1$, the integral was evaluated in Example 14.1. When $n \leq -2$, the function z^n is not defined at the origin. However, $z^{n+1}/(n+1)$ is a primitive of z^n for z in the punctured plane $\mathbb{C} \setminus \{0\}$. By Theorem 13.4, $\int_{C_r} z^n\,dz = 0$.

Example 15.2. We can apply Theorem 15.2 to define a branch of the log function as a primitive function of $1/z$. Let D be the set

$$D = \mathbb{C} \setminus \{x + iy \in \mathbb{C} :\, x \leq 0, y = 0\}.$$

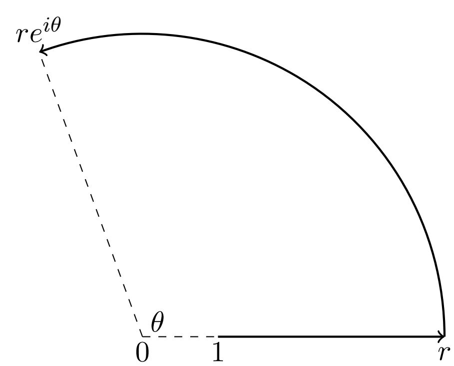
Take $z = 1$ to be a base point. Every point in D can be reached by a piece-wise linear path γ from 1 to $i\mathrm{Im}(z)$, and then from $i\mathrm{Im}(z)$ to z. The proof of Theorem 15.2 go through by defining the primitive of $1/z$ as

$$Log(z) \triangleq \int_\gamma \frac{1}{z}\,dz$$

for z in the domain defined above. Since a primitive exists, the integral is independent of path. We can write

$$Log(z) \triangleq \int_1^z \frac{1}{z}\,dz.$$

Suppose $z = re^{i\theta}$ with $-\pi < \theta < \pi$. The calculation of $Log(z)$ is easy when we take the path that first goes from 1 to r horizontally, and then travel along an arc from r to $re^{i\theta}$.



The integral from 1 to r is

$$\int_1^r \frac{1}{z}\,dz = \int_1^r \frac{1}{x}\,dx = \ln r.$$

The integral from r to $re^{i\theta}$ is

$$\begin{aligned}\int_r^{re^{i\theta}} \frac{1}{z}\,dz &= \int_0^\theta r^{-1}e^{-i\theta}(ire^{i\theta})\,d\theta \\ &= i\theta.\end{aligned}$$

Therefore the function

$$Log(z) = \ln(r) + i\theta$$

is a primitive of $1/z$ in the domain D. This the same as the function obtained from the inverse of the exponential function.

16 Lecture 16 (Cauchy integral formula)

<u>Summary</u>

- Cauchy theorem for multiply connected region
- Cauchy integral formula
- Taylor series expansion of analytic functions

Definition 16.1. A connected region is *simply connected* if every closed curve in the region can be continuously shrink to a single point, without leaving the region. A connected region that is not simply connected is said to be *multiply connected*.

A multiply connected region is a connected region with some "holes" inside. The "hole" is drawn in order to contain all points where the function is not defined.

A stronger version of Cauchy theorem is

Theorem 16.2. *Suppose D is a simply connected region and f is analytic in D. Then*

$$\oint_C f(z)\,dz = 0$$

for all closed and smooth curve C.

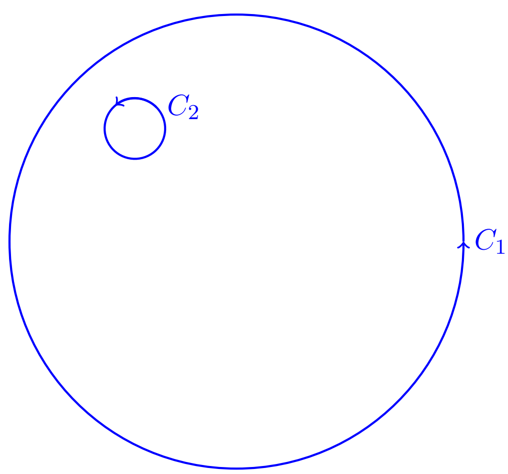
Proof sketch. We first pick a base point, say z_0, in D. For any other point z in D, we can connect z and z_0 by a piece-wise linear path C, with each piece parallel to either the real or imaginary axis. This can always be done because D is connected. Define a function $F(z)$ by $\int_C f(z)\,dz$. This is well-defined (does not depend on the choice of piece-wise linear path) because D is simply connected. Then we show as in Theorem 15.2 that $F(z)$ is a primitive of $f(z)$ in D. Therefore the integral of any closed curve in D is zero, by Theorem 14.2. $\square$

For multiply connected region, we can apply Cauchy theorem as follows.

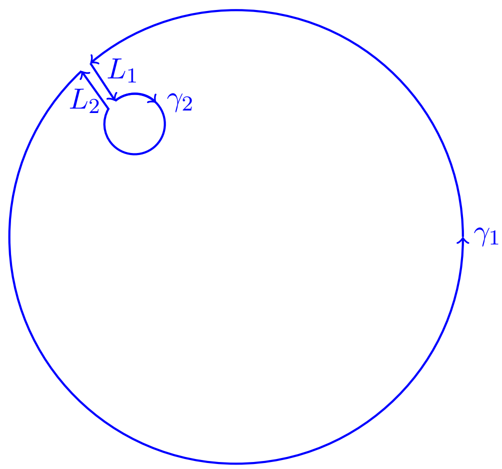
Theorem 16.3. *Consider a complex function f that is analytic inside a domain D. Circles C_1 and C_2 are in D so that C_2 is contained inside the outer circle C_1, both with positive orientation. If $f(z)$ is analytic in the region between C_1 and C_2, then*

$$\int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz = 0.$$

Remark. Actually, the function $f(z)$ in Theorem 16.3 is analytic in a region that contains C_1, C_2 and the area between C_1 and C_2.



Proof. Draw a narrow "river" that connects the exterior to the inner circle as follows.



Note that the orientation of γ_2 is opposite to that of C_2. The resulting contour encloses a simply connected region. By Theorem 16.2,

$$\int_{\gamma_1} + \int_{\gamma_2} + \int_{L_1} + \int_{L_2} = 0$$

as the gap between L_1 and L_2 approaches 0, the integrals over L_1 and L_2 cancel each other. The curve γ_1 becomes the circle C_1 and the curve γ_2 becomes the circle $-C_2$. Hence

$$\int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz = 0.$$

□

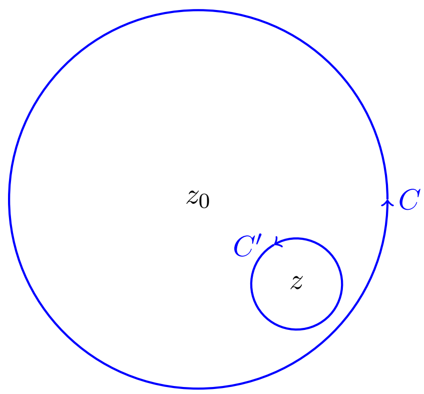
Theorem 16.3 can be extended to the case with two or more holes.

Theorem 16.4 (Cauchy integral formula)**.** *Consider a circle C with radius r and center at z_0. If $f(z)$ is analytic in a region that contain C and its interior, then*

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w-z}\,dw.$$

for all z in the interior of C.

Proof. In this prove we fix a complex number z that lies in the interior of the circle C, and draw a small circle C' with z as the center. The radius of C' is ρ, and ρ should be chosen so that C' lies completely inside C.



By Theorem 16.3, it suffices to prove

$$\int_{C'} \frac{f(w)}{w-z}\, dw = 2\pi i f(z).$$

The variable w represents a point on the circle C'.

Write

$$f(w) = f(z) + [f(w) - f(z)]$$

and decompose the complex integral into two parts

$$\int_{C'} \frac{f(w)}{w-z}\, dw = \int_{C'} \frac{f(z)}{w-z}\, dw + \int_{C'} \frac{f(w)-f(z)}{w-z}\, dw. \tag{16.1}$$

The first integral on the RHS of (16.1) equals

$$\int_{C'} \frac{f(z)}{w-z}\, dw = f(z) \int_{C'} \frac{1}{w-z}\, dw = 2\pi i f(z)$$

using a calculation similar to Example 14.1.

For the second integral on the RHS of (16.1), we let $g(w) = f(w) - f(z)$ and write the integral as

$$\int_{C'} \frac{g(w)}{w-z}\, dw.$$

Since C' is a compact set and $g(w)$ is a continuous function, the maximum value of $|g(w)|$ on C' is well-defined,

$$m_\rho \triangleq \sup_{z \in C'} |g(w)| = \max_{z \in C'} |g(w)|.$$

By ML inequality

$$\Big| \int_{C'} \frac{g(w)}{w-z}\, dw \Big| \le \frac{m_\rho}{\rho} 2\pi\rho = 2\pi m_\rho.$$

Since the function g is continuous, m_ρ approaches 0 as ρ approaches 0. Therefore

$$\int_{C'} \frac{f(w) - f(z)}{w - z}\, dw = \int_{C'} \frac{g(w)}{w - z}\, dw = 0.$$

The RHS of (16.1) is thus equal to $2\pi i f(z)$. This finished the derivation of the Cauchy's integral formula. $\square$

Example 16.1. Calculate $\int_C \frac{1}{z^2-1}\, dz$ for (i) C is the circle $C(1;1)$, (ii) C is the circle $C(-1;1)$, (iii) C is the circle $C(0;2)$. All contours have positive orientation.

The integrate can be factorized as

$$\frac{1}{z^2 - 1} = \frac{1}{(z-1)(z+1)}.$$

(i) The circle $C(1,1)$ with radius 1 and center $z = 1$ does not enclose the point $z = -1$. The fraction $\frac{1}{z+1}$ is analytic inside $C(1,1)$.

$$\int_{C(1,1)} \frac{\frac{1}{z+1}}{z-1}\, dz = 2\pi i \Big(\frac{1}{z+1}\Big)\Big|_{z=1} = \pi i.$$

(ii) The fraction $\frac{1}{z-1}$ is analytic inside $C(-1,1)$, because the point $z = 1$ is not inside the the circle $C(-1,1)$ with radius 1 and center $z = -1$.

$$\int_{C(-1,1)} \frac{\frac{1}{z-1}}{z+1}\, dz = 2\pi i \Big(\frac{1}{z-1}\Big)\Big|_{z=-1} = -\pi i.$$

(iii) Apply Theorem 16.3 on $C(0;2)$ as the outer circle and $C(1;1)$ and $C(-1;1)$ as the inner circle. The function is analytic in the area outside $C(1;1)$ and $C(-1;1)$. Therefore

$$\int_{C(0,2)} \frac{\frac{1}{z+1}}{z-1}\, dz = \pi i + (-\pi i) = 0.$$

Theorem 16.5 (Taylor expansion)**.** *Suppose $f(z)$ is analytic in a domain that contains the closure of the open disc $D(z_0; r)$. Then $f(z)$ has Taylor series expansion*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for z inside $D(z_0; r)$. Moreover, for $n = 0, 1, 2, 3, \ldots$, the coefficient a_n is given by

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(w)}{(w - z_0)^{n+1}}\, dw$$

with the integral taken over the circle $|z - z_0| = r$.

In Theorem 16.5, we require that $f(z)$ is defined on every point on the boundary of $D(z_0;r)$. Hence f should be analytic in a neighborhood that contain $D(z_0;r)$ in its interior.

Proof. By Theorem 16.3, we can take the circle C centered at z_0 as the curve in the Cauchy's integral formula,

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(w)}{w-z}\,dw.$$

The proof idea is to expand $1/(w-z)$ using geometric series,

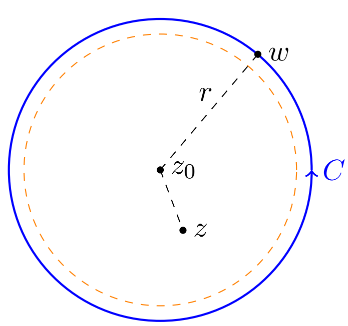
$$\begin{aligned}\frac{1}{w-z} &= \frac{1}{w-z_0-(z-z_0)}\\ &= \frac{1}{w-z_0}\Big(\frac{1}{1-\frac{z-z_0}{w-z_0}}\Big)\\ &= \frac{1}{w-z_0} + \frac{z-z_0}{(w-z_0)^2} + \frac{(z-z_0)^2}{(w-z_0)^3} + \frac{(z-z_0)^3}{(w-z_0)^4} + \cdots \end{aligned} \tag{16.2}$$

This geometric series converges because $|z-z_0| < |w-z_0|$ for all z in the open disc $D(z_0;r)$. If we can justify the exchange of infinite summation an contour integral, then we get

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\oint_C \sum_{n=0}^{\infty} f(w)\frac{(z-z_0)^n}{(w-z_0)^{n+1}}\,dw\\ &= \sum_{n=0}^{\infty}\Big(\frac{1}{2\pi i}\oint_C \frac{f(w)}{(w-z_0)^{n+1}}\Big)(z-z_0)^n. \end{aligned}$$

This is a Taylor series centered at z_0.

The remaining part of the proof is to make this argument rigorous. In fact, we will make use of uniform convergence implicitly. Consider a smaller circle $C(z_0;\rho)$ with radius $\rho < r$ (the dashed line in the following figure), and let z be a complex inside the smaller circle, i.e., $|z-z_0| < \rho$.



Let n be a positive integer. Consider the partial sum on the right-hand side of (16.2) up to degree n. The remainder is

$$\sum_{k=n+1}^{\infty} \frac{z-z_0)^k}{(w-z_0)^{k+1}} = \frac{(z-z_0)^{n+1}}{(w-z_0)^{n+2}} \frac{1}{1-\frac{z-z_0}{w-z_0}} = \frac{1}{(w-z)} \frac{(z-z_0)^{n+1}}{(w-z_0)^{n+1}}$$

We can thus write $1/(w-z)$ as

$$\frac{1}{w-z} = \frac{1}{w-z_0} + \frac{z-z_0}{(w-z_0)^2} + \cdots + \frac{(z-z_0)^n}{(w-z_0)^{n+1}} + \frac{1}{(w-z)} \frac{(z-z_0)^{n+1}}{(w-z_0)^{n+1}}.$$

Multiply each term by $f(w)$ and integrate over the curve C. (Since this is a finite sum, there is no problem in exchanging finite summation and integration.)

$$\frac{1}{2\pi i} \int_C \frac{f(w)}{w-z}\, dw = \sum_{k=0}^{n} \Big(\frac{1}{2\pi i} \int_C \frac{f(w)\, dw}{(w-z_0)^{k+1}} \Big) (z-z_0)^k + \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-z)} \frac{(z-z_0)^{n+1}}{(w-z_0)^{n+1}}\, dw.$$

The above equation holds for any $n \geq 0$. The last summation can be bounded by observing

$$\begin{aligned} |z-z_0| &< \rho \\ |w-z| &\geq \rho - |z-z_0| \\ |w-z_0| &= r, \end{aligned}$$

and $|f(z)|$ is bounded by some constant M for all z on the curve C. By ML inequality,

$$\Big| \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-z)} \frac{(z-z_0)^{n+1}}{(w-z_0)^{n+1}}\, dw \Big| \leq \frac{1}{2\pi} \frac{M}{\rho - |z-z_0|} \frac{\rho^{n+1}}{r^{n+1}} \cdot 2\pi r = \frac{M\rho}{\rho - |z-z_0|} \Big(\frac{\rho}{r}\Big)^n.$$

Take $n \to \infty$, the modulus of the remainder term approach 0. (This step depends on the assumption that $\rho < r$). Consequently, the Taylor series converges for all z in the disk $D(z_0; \rho)$. Since ρ can be any number less than r, we conclude that the Taylor series converges for all z in the disk $D(z_0; r)$. □

Since we can differentiate a Taylar series arbitrarily many times, Theorem 16.5 says that the function $f(z)$ can be differentiated arbitrarily many times at $z = z_0$. Apply Theorem 16.5 to each point z_0 in the domain of f, we obtain the following important theorem

Theorem 16.6. *If a function f is complex differentiable once for all points in a domain, then f is infinitely differentiable.*

Remark. The property in Theorem 16.6 is certainly false in real analysis. There are examples of real functions that can be differentiated once but not twice. There are also examples of real functions that can be differentiated twice but not differentiated three times, and so on.

Remark. Theorem 16.5 implies that we can always expand an analytic function f as a power series at any point z_0 in the domain of f. In some books, this is taken as the definition of analytic functions, i.e., some people define analytic function as a function that can be expanded as an a power series at any point z_0 in the domain. A function that can be differentiated once at any point in the domain is called *holomorphic* instead. Using these terminologies, holomorphic function is analytic by Theorem 16.5. Conversely, analytic function is holomorphic, because Taylor series can be differentiated term-wise. In the remainder of this lecture notes, holomorphic and analytic will be treated as synonyms.