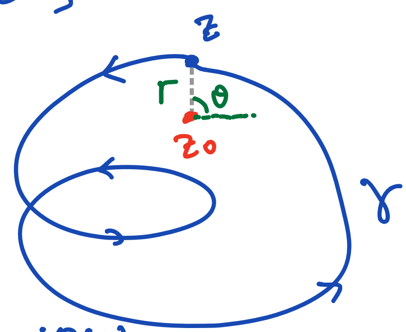
MAT 3253 Lecture 22

Curve $\gamma : \mathbb{R} \to \mathbb{C}$

Winding number

Consider a piecewise smooth curve

$$\gamma : [0,1] \to \mathbb{C} \setminus \{z_0\}$$



$$\gamma(t) = r(t)e^{i\theta(t)} + z_0$$

$$r(t) > 0 \quad \forall t$$

$$\gamma'(t) = r'(t)e^{i\theta(t)} + r(t)\, i\, \theta'(t) e^{i\theta(t)}$$

Theorem If γ is closed piecewise smooth curve

$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-z_0} \text{ is an integer.}$$

Proof

$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-z_0} = \frac{1}{2\pi i}\int_0^1 \frac{\gamma'(t)}{\gamma(t)}\,dt \qquad z - z_0$$

$$= \frac{1}{2\pi i}\int \frac{[r'(t) + r(t)\, i\, \theta'(t)]\cancel{e^{i\theta(t)}}}{r(t)\cancel{e^{i\theta(t)}}}\,dt$$

$$= \frac{1}{2\pi i}\int_0^1 \frac{r'(t)}{r(t)}\,dt + \frac{1}{2\pi}\int_0^1 \theta'(t)\,dt$$

$$= \frac{1}{2\pi i}\left[\cancel{\log(r(1)) - \log(r(0))}^{\,0}\right] + \frac{1}{2\pi}\int_0^1 \theta'(t)\,dt$$

$$= \frac{1}{2\pi}\left(\theta(1) - \theta(0)\right)$$

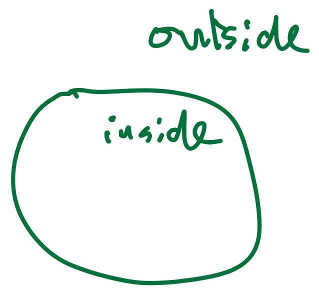
$= $ no. of time the curve γ going around the point z_0. $\blacksquare$

Def The winding number of a closed curve γ around a point z_0 is defined as

$$n(\gamma; z_0) \triangleq \frac{1}{2\pi i} \int_\gamma \frac{1}{z - z_0}\, dz$$

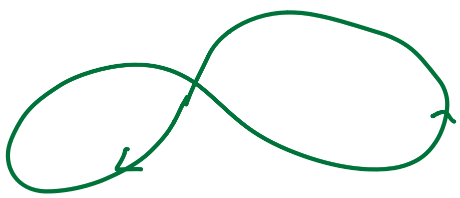
another notation $\quad \operatorname{ind}(\gamma; z_0)$

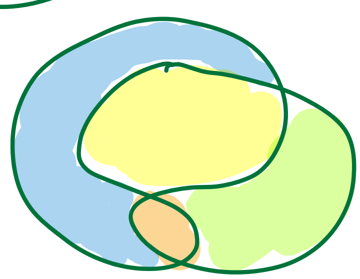
$\qquad\qquad\qquad\qquad w(\gamma; z_0)$

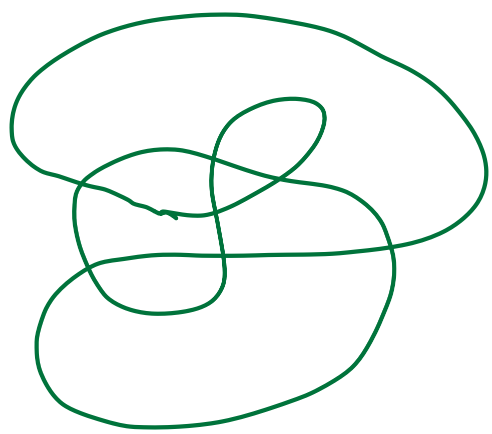


Jordan curve theorem

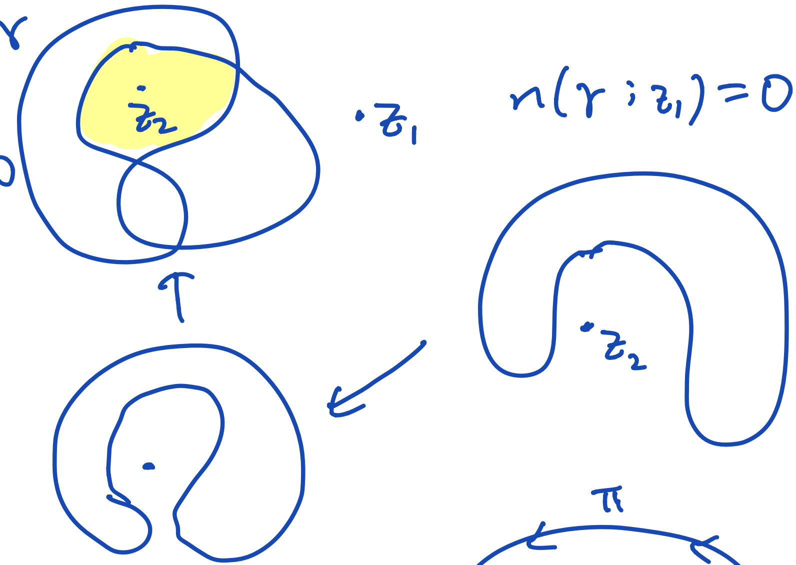
Any simple closed continuous curve divides the plane into two parts: interior and exterior.





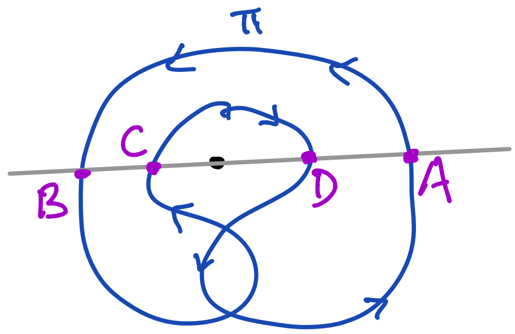


$n(\gamma; z_2) = 0$



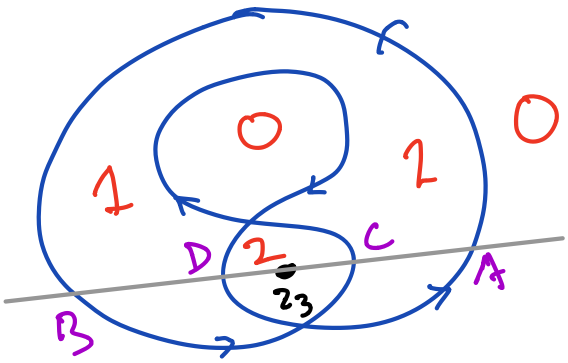
$n(\gamma; z_1) = 0$

$$\int_\gamma \frac{1}{z - z_0}\, dz$$



AB	π
BC	0
CD	$-\pi$
DA	0

total change $= 0$



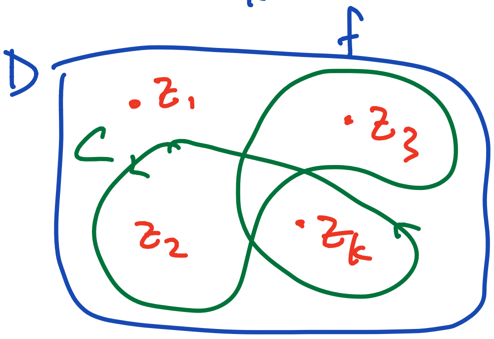
$n(\gamma; z_3) = 2$

AB	:	π
BC	:	π
CD	:	π
DA	:	π

4π in total

Generalized residue theorem

Suppose f is analytic is a domain except k isolated singular points $z_1, z_2 \dots z_k$. For any closed smooth C not intersecting any one of the k isolated points. Then



$$\int_C f(z)\,dz = 2\pi i \sum_{j=1}^{k} n(C; z_j)\,\mathrm{Res}(f; z_j)$$

Proof: Expand $f(z)$ at z_j using Laurent series

$$f(z) = \underbrace{p_j}_{p_j \text{ is a function}}((z-z_j)^{-1}) + \text{analytic part} \qquad j = 1, 2 \dots k$$

$$g(z) = f(z) - p_1((z-z_1)^{-1}) - p_2((z-z_2)^{-1}) - \dots - p_k((z-z_k)^{-1})$$

$g(z)$ is analytic throughout D.

$$\int_C g(z)\,dz = 0$$

$$\int_C f(z)\,dz = \sum_{j=1}^{k} \int_C p_j((z-z_j)^{-1})$$

$$= 2\pi i \sum_{j=1}^{k} \mathrm{Res}(f; z_j) \cdot \underbrace{\frac{1}{2\pi i}\int_C \frac{1}{z-z_j}\,dz}_{\triangleq\, n(C; z_j)}$$

$\boxed{/\!/}$

Argument principle

Suppose C is the boundary of a simply connected region and f is analytic inside a domain containing C.

(Count with multiplicity)

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = \text{no. of zeros inside } C.$$

Proof. Suppose $z_1, z_2 \dots z_k$ are zeros of f inside C.

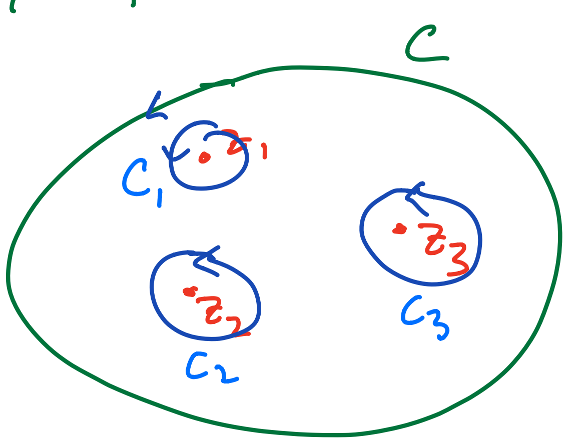
For $j = 1, 2, \dots, k$

$$f(z) = a_m (z-z_j)^m + a_{m+1}(z-z_j)^{m+1} + \dots$$

$$f'(z) = m_j a_m (z-z_j)^{m-1} + \text{analytic function}$$

$\uparrow \neq 0$

$$\frac{f'}{f} = \frac{m_j}{z-z_j} + \text{analytic function}$$



$$\int_C \frac{f'}{f} = \sum_{j=1}^{k} \int_{C_j} \frac{m_j}{z-z_j}\,dz$$

$$= 2\pi i\,(m_1 + m_2 + \dots + m_k)$$

$$\frac{1}{2\pi i}\int_C \frac{f'}{f} = m_1 + m_2 + \dots + m_k = \text{no. of zero counted with multiplicity}$$

$$\frac{1}{2\pi i}\int_{\gamma} \frac{f'(z)}{f(z)}\,dz$$ is the winding no. of $f(\gamma(t))$ around $z=0$.

$f\circ\gamma : \mathbb{R} \to \mathbb{C}$

is a closed curve