MAT2006 Tutorial #13

1. (Sequential Criterion for Integrability).

(a) Prove that a bounded function f is integrable on $[a, b]$ if and only if there exists a sequence of partitions $\{P_n\}_{n=1}^{\infty}$ satisfying

$$\lim_{n\to\infty}[U(f, P_n) - L(f, P_n)] = 0,$$

and in this case

$$\int_a^b f(x)dx = \lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} L(f, P_n).$$

(b) For each n, let P_n be the partition of $[0, 1]$ into n equal subintervals. Find formulas for $U(f, P_n)$ and $L(f, P_n)$ if $f(x) = x$.

(c) Use the sequential criterion for integrability from (a) to show directly that $f(x) = x$ is integrable on $[0, 1]$ and compute $\int_0^1 f(x)dx$.

2. Decide which of the following conjectures is true and supply a short proof. For those that are not true, give a counterexample.

(a) If $|f|$ is integrable on $[a, b]$, then f is also integrable on this set.

(b) Assume g is integrable and $g(x) \geq 0$ on $[a, b]$. If $g(x) > 0$ for an infinite number of points $x \in [a, b]$, then $\int_a^b g(x)dx > 0$.

(c) If g is continuous on $[a, b]$ and $g(x) \geq 0$ with $g(y_0) > 0$ for at least one point $y_0 \in [a, b]$, then $\int_a^b g(x) > 0$.

3. Decide whether each statement is true or false, providing a short justification for each conclusion.

(a) If $h' = g$ on $[a, b]$, then g is continuous on $[a, b]$.

(b) If g is continuous on $[a, b]$, then $g = h'$ for some h on $[a, b]$.

(c) If $H(x) = \int_a^x h(t)dt$ is differentiable at $c \in [a, b]$, then h is continuous at c.

If time allows, discuss the following question.

4. For each $n \in \mathbb{N}$, let

$$h_n(x) = \begin{cases} 1/2^n & \text{if } 0 \leq x \leq 1/2^n \\ 0 & \text{if } 1/2^n \leq x \leq 1, \end{cases}$$

and set $H(x) = \sum_{n=1}^{\infty} h_n(x)$. Show H is integrable and compute $\int_0^1 H(x)dx$.

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