The Chinese University of Hong Kong, Shenzhen
iDDA · Institute for Data and Decision Analytics
Andre Milzarek · Summer Semester 2020

MAT 3007 – Optimization

Exercise Sheet 1

Exercise E1.1 (Multiple Choice – Minimizer and LPs):
Answer the following multiple choice questions and decide whether the statements are true or false. Please explain your answer and give suitable (counter)examples.

a) Each global minimizer is also a local minimizer.

☐ True. ☐ False.

b) Each global minimizer is also a strict local minimizer.

☐ True. ☐ False.

c) There are functions $f : \mathbb{R} \to \mathbb{R}$ and intervals $I := [a, b] \subset \mathbb{R}$, $a < b$, $a, b \in \mathbb{R}$, such that every point $x \in I$ is a local minimum/minimizer of f.

☐ True. ☐ False.

d) There are functions $f : \mathbb{R} \to \mathbb{R}$ that have multiple global minima.

☐ True. ☐ False.

e) There are functions $f : \mathbb{R} \to \mathbb{R}$ that have multiple strict global minima.

☐ True. ☐ False.

f) There are functions $f : \mathbb{R} \to \mathbb{R}$ and intervals $I := [a, b] \subset \mathbb{R}$, $a < b$, $a, b \in \mathbb{R}$, such that every point $x \in I$ is a strict local minimum of f.

☐ True. ☐ False.

g) There are functions $f : \mathbb{R} \to \mathbb{R}$ that have local minimizer but no global minimizer.

☐ True. ☐ False.

We now consider an LP in its standard form:

$$\min_x \; c^\top x \qquad \text{s.t.} \qquad Ax = b, \quad x \geq 0,$$

where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ are given. Decide whether the following statements are true or false and explain your answer.

h) The set of all optimal solutions (assuming existence) must be bounded.

☐ True. ☐ False.

i) At every optimal solution, no more than m variables can be positive.

☐ True. ☐ False.

j) If there is more than one optimal solution, then there are infinitely many optimal solutions.

☐ True. ☐ False.

Exercise E1.2 (Point Clouds and Circles):
Let $y^1, y^2, ..., y^k$ be k different and given points in $\mathbb{R}^2$, i.e., it holds that $y^i \in \mathbb{R}^2$ for all $i = 1, ..., k$. We want to find a circle in $\mathbb{R}^2$ with *minimum* radius that contains all of these points.

Formulate this problem as a constrained nonlinear program with the center and radius of the circle as decision variables.

Assignment A1.1 (Optimal Design of a Building): (approx. *25* points)
CUHK-SZ plans to build a new research building with cuboid shape between upper and lower campus. In this exercise, we want to find an optimal design for the new building that fulfills the given specifications. Let ℓ and w denote the length and width of the building, respectively. Moreover, let h be the height, aboveground, and d be the height/depth, underground, of the building. The university has collected a list of requirements that need to be satisfied:

- The building should be at least as long as wide. Moreover, it should be at most twice as long as wide.
- The length ℓ of the building should not be larger than $40\,m$.
- The length ℓ should not exceed the height (aboveground) h of the building.
- All floors should have a uniform height of at least $3.50\,m$.
- At least 10% but at most 25% of the building should lie underground.
- The floor of the ground floor should be at ground level.
- The total amount of floor area in the building should be at least $10\,000\,m^2$.
- The average annual heating costs only depend on the outer surface of the building that lies aboveground. The costs are set to 1 000 RMB per m^2 of the outer surface. The annual heating costs should not exceed 5 000 000 RMB.

For simplification, we assume that the thickness of the walls, floors, and ceilings is negligible. The building should now be optimally designed such that the total amount of soil that needs to be excavated in order to build the building is minimized.

a) Formulate this problem as a constrained optimization problem.

   Hint: Besides the mentioned variables ℓ, w, h, and h, you can also introduce decision variables for the uniform height of the floors and for the number of floors aboveground and underground. Notice that the number of floors are integer variables.

b) Estimate or calculate a feasible point of this problem.

Assignment A1.2 (Production Planning): (approx. *25* points)
A company produces two kinds of products. The product of the first type requires 1/3 hours of assembly labor, 1/5 hours of testing, and \$2 worth of raw materials. The second product requires 1/4 hours of assembly, 1/4 hours of testing, and \$1 worth of raw materials. Given the current personnel of the company, there can be at most 100 hours of assembly labor and 70 hours of testing each day. Products of the first and second type have a market value of \$10 and \$8 respectively.

(a) Formulate a linear optimization problem that maximizes the daily profit of the company.

(b) Derive the standard form of the LP formulated in part (a).

(c) Consider the following modification to the original problem: Suppose that up to 60 hours of overtime assembly labor can be scheduled, at a cost of \$7 per hour. Can it be easily incorporated into the linear optimization formulation and how?

(d) Solve the original LP in part (a) using MATLAB.

Assignment A1.3 (Network Flow): (approx. *25* points)

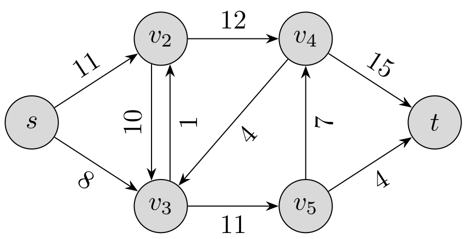
Let $G = (V, E)$ be a directed graph representing a given network. In particular, let $V = \{v_1, ..., v_n\}$ denote the set of nodes and let E be the set of edges in the network, i.e., we write $(i, j) \in E$ if and only if there exists a connection between the nodes v_i and v_j. With each edge (i, j), an associated capacity value $c_{ij} \geq 0$ is given that specifies the possible maximum flow of units along this edge.

In this exercise, we want to model the maximum flow problem from the source $s = v_1$ to the sink $t = v_n$ as a linear program. In particular, we want to determine the maximum number of units that can be transferred from the source s to the sink t via the network adhering to the capacity constraints. An optimal network flow should fulfill the conditions:

- The total flow into the sink t is maximized.
- For all edges $(i, j) \in E$, the capacity constraints are satisfied.
- Flow conservation: for each node v_i, $i \in \{2, ..., n-1\}$, the total incoming flow equals the total outgoing flow.

(a) Formulate the described maximum flow problem as a linear program.

(b) Consider the following network architecture and capacities:



Solve the flow network problem for this specific network using MATLAB.

Hint: You can set $c_{ij} = 0$ in case $(i, j) \notin E$.

Assignment A1.4 (A Robust LP Formulation): (approx. *25* points)

In this exercise, we consider the following optimization problem:

$$\min_{x \in \mathbb{R}^n} \; c^\top x \qquad \text{s.t.} \qquad \|Ax - b\|_\infty \leq \delta, \quad x \geq 0, \tag{1}$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, and $\delta \geq 0$ are given and $\|y\|_\infty = \max_{1 \leq i \leq p} |y_i|$ denotes the maximum norm of a vector $y \in \mathbb{R}^p$. In the case $\delta = 0$, problem (1) coincides with the standard form for linear programs. The choice $\delta > 0$ can be useful to model situations where A or b are not fully or exactly known, e.g., when A or b contaminated by noise or when they can only be estimated (from measurements, data, etc.).

(a) Rewrite the optimization problem (1) as a linear problem and verify that your reformulation is equivalent to problem (1).

We now consider a specific application of problem (1).

The fruit store in Pandora is producing two different fruit salads A and B. The smaller fruit salad A consists of "1/4 mango, 1/8 pineapple, 5 strawberries"; the larger fruit salad B consists of "1/2 mango, 1/4 pineapple, 1 strawberry". The profits per fruit salad and the total number of fruits in stock are summarized in the following table:

	Mango	Pineapple	Strawberry	Net profit
Fruit salad A	1/4	1/8	5	10 RMB
Fruit salad B	1/2	1/4	1	20 RMB
Stock / Resources	25	10	120	

Since the cooling unit in the warehouse is currently malfunctioning, all fruits need to be processed and completely used to make the fruit salads A and B.

(b) Given these constraints, formulate a linear program to maximize the total profits of the fruit store. Show that this program can be expressed in standard form

$$\min_{x \in \mathbb{R}^n} \; c^\top x \qquad \text{s.t.} \qquad Ax = b, \quad x \geq 0,$$

with $n = 2$ and $m = 3$.

(c) Analyze whether the linear program is solvable.

(d) One of the employee found some additional fruits in a storage crate and the manager of the fruit shop decides to determine the production plan by using the robust formulation (1). Consider the robust variant of the problem in part (b) with $\delta = 5$. Sketch the feasible set of this problem and solve the problem graphically – as discussed in lecture. Which constraints are active in the solution?

Calculate the optimal value, i.e., the total profit, of this problem. How many of the different fruits are used when following the optimal production plan?

Information: Exercises and Assignments

- There will be two kinds of exercises on the exercise sheet.
- Problems that are marked as "Assignment A?.?" (type A) need to be submitted and will be graded and taken into account for the final grade. (In total, the assignments will make up 40% of the final grade).
- Each student has to submit his or her own, individual solution sheet for grading. In order to avoid any form of plagiarism, please indicate on the cover sheet if you are working together with other students in smaller groups. Please put the names and student IDs of possible group members and collaborators on the cover sheet. Even if you are working in a group, please use your own words and ideas in the creation of your solution sheet to clarify and demonstrate your contribution.
- You can use the sample cover sheet provided on Blackboard.
- Problems that are labeled as "Exercise E?.?" (type E) are additional exercises that are intended to specifically practice different aspects of new topics or mathematical concepts and notations. Those problems will **not** be considered for grading and will generally **not** be corrected (even if you hand in your solutions)!
- Hand in your solutions **in time** before the specified deadline via Blackboard. Please hand in your (written, scanned, or copied) solutions as a pdf-document.

Information: MATLAB Code

- For those questions that ask you to write MATLAB code to solve the problem, please attach the code to the homework. Please also state the optimal solution and the optimal value that you have obtained in your solution sheet. You do not need to attach the outputs in the command window of MATLAB.

Sheet 1 is due on **Jun, 19th**. Submit your solutions before **Jun, 19th**, **11:00 am**.