1\. (a) Construct a CTMC with state space $\{0, 1, 2, \ldots\}$, the rate diagram where $\lambda = 95$ and $\mu = 1$ is given by

$$0 \underset{\mu}{\overset{\lambda}{\rightleftarrows}} 1 \underset{2\mu}{\overset{\lambda}{\rightleftarrows}} 2 \underset{3\mu}{\overset{\lambda}{\rightleftarrows}} 3 \underset{4\mu}{\overset{\lambda}{\rightleftarrows}} \cdots$$

(b) Yes, the CTMC is irreducible since all the states communicate.

(c) Yes, the CTMC is positive recurrent since it is irreducible and has a stationary distribution where $\pi_n = \dfrac{e^{\frac{\lambda}{\mu}} (\frac{\lambda}{\mu})^n}{n!}$.

(d) $E(\#$ in system$) = \frac{\lambda}{\mu} = 95$ calls.

2\. (a) Construct a CTMC with state space $\{0, 1\}$, where state 0 represents the machine is not working, state 1 represents the machine is working.

The change of state $0 \to 1$ follows exponential distribution with rate μ, the change of state $1 \to 0$ follows exponential distribution with rate λ.

Then the generator matrix is given by

$$G = \begin{bmatrix} -\mu & \mu \\ \lambda & -\lambda \end{bmatrix}$$

The rate diagram is given by $0 \underset{\lambda}{\overset{\mu}{\rightleftarrows}} 1$

(b) ① If $\lambda > \mu$, consider a CTMC $\tilde{X}$ with uniform holding time rate λ and "jump matrix"

$$J_\mu = \begin{bmatrix} \frac{\lambda - \mu}{\lambda} & \frac{\mu}{\lambda} \\ 1 & 0 \end{bmatrix}$$

② If $\mu > \lambda$, consider a CTMC $\tilde{X}$ with uniform holding time rate u and "jump matrix"

$$J_u = \begin{bmatrix} 0 & 1 \\ \frac{\lambda}{u} & \frac{u-\lambda}{u} \end{bmatrix}$$

(c) ① if $\lambda > u$, then the probability that it will be working at time t is $P_{11}(t) = \sum_{n=0}^{\infty} (J_u)_{11}^{n} \frac{(\lambda t)^n}{n!} e^{-\lambda t} = 0$

② if $u > \lambda$, then the probability that it will be working at time t is $P_{11}(t) = \sum_{n=0}^{\infty} (J_u)_{11}^{n} \frac{(\lambda t)^n}{n!} e^{-\lambda t} = \sum_{n=0}^{\infty} \left(\frac{u-\lambda}{u}\right)^n \frac{(\lambda t)^n}{n!} e^{-\lambda t}$

3. (a) $\hat{\lambda}_A = \hat{\lambda}_B = \hat{\lambda}_C = 12$ and

$$\hat{J} = \begin{bmatrix} 0 & 1/3 & 2/3 \\ 5/12 & 1/2 & 1/12 \\ 1/6 & 0 & 5/6 \end{bmatrix}$$

(b) $$P(10,2) = \begin{bmatrix} 3677/15995 & 704/3901 & 319/541 \\ 1226/5145 & 598/1499 & 653/1800 \\ 325/2304 & 557/10959 & 956/1183 \end{bmatrix}$$

$$\sum_{k=0}^{n} \frac{t^k G^k}{k!} = \begin{bmatrix} 0.2299 & 0.1805 & 0.5896 \\ 0.2383 & 0.3989 & 0.3628 \\ 0.1411 & 0.0508 & 0.8081 \end{bmatrix}$$

$$P(10,2) - \sum_{n=0}^{n} \frac{t^k G^k}{k!} = 10^{-9} \cdot \begin{bmatrix} 0.2054 & -0.0884 & -0.1170 \\ -0.1072 & 0.0461 & 0.0610 \\ -0.0309 & 0.0133 & 0.0176 \end{bmatrix}$$

Thus, the difference between two method is quite small.

(c)

$$P(10,2) - \sum_{k=0}^{20} \cdot e^{-(12\cdot 0.2)} \frac{(12\cdot 0.2)^k}{k!} \hat{J}^k \approx$$

$$\begin{bmatrix} 3.1752 \times 10^{-14} & 2.1261 \times 10^{-14} & 1.3856 \times 10^{-13} \end{bmatrix}$$

$$\begin{bmatrix} 3.1863\times10^{-14} & 2.1261\times10^{-14} & 1.3861\times10^{-13} \\ 3.1919\times10^{-14} & 2.1337\times10^{-14} & 1.3900\times10^{-13} \end{bmatrix}$$

Comparing this approximation in (b), (c) is more accurate.

(d) Since $G = VMV^{-1}$ and $M = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$, where

$\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of G. Then,

$$G^k = VM^kV^{-1} = V \begin{bmatrix} \lambda_1^k & 0 & 0 \\ 0 & \lambda_2^k & 0 \\ 0 & 0 & \lambda_3^k \end{bmatrix} V^{-1}$$

Then $$e^{1.5G} = \sum_{k=0}^{\infty} \frac{(1.5G)^k}{k!}$$

$$= V \left(\sum_{k=0}^{\infty} \begin{bmatrix} \frac{(1.5\lambda_1)^k}{k!} & 0 & 0 \\ 0 & \frac{(1.5\lambda_2)^k}{k!} & 0 \\ 0 & 0 & \frac{(1.5\lambda_3)^k}{k!} \end{bmatrix} \right) \cdot V^{-1}$$

$$= V \begin{bmatrix} e^{1.5\lambda_1} & 0 & 0 \\ 0 & e^{1.5\lambda_2} & 0 \\ 0 & 0 & e^{1.5\lambda_3} \end{bmatrix} V^{-1}$$

Thus,

$$P(1.5) = \begin{bmatrix} 0.16673 & 0.11130 & 0.72198 \\ 0.16687 & 0.11173 & 0.72141 \\ 0.16662 & 0.11097 & 0.72240 \end{bmatrix}$$

(e) Since $\pi G = 0$, where $\pi = [\pi_A, \pi_B, \pi_C]$, and $\pi_A + \pi_B + \pi_C = 1$.

Then solve for the system of equations, we get

$\pi = [1/6, 1/9, 13/18]$.

(f) Since $\pi^T P = \pi^T$, then after solving the system of equations,

we get $\pi^T = [1/6, 1/9, 13/18]$.

4. (a) Since $\pi G = 0$, where $\pi = [\pi_A, \pi_B, \pi_C]$, and $\pi_A + \pi_B + \pi_C = 1$.

Then solve for the system of equations, we get

$$\pi = [1/6,\ 1/9,\ 13/18].$$

(b) The jump matrix is

$$J = \begin{bmatrix} 0 & 1/3 & 2/3 \\ 5/6 & 0 & 1/6 \\ 1 & 0 & 0 \end{bmatrix}$$

with holding time rates $\lambda_A = 12$, $\lambda_B = 6$, $\lambda_C = 2$.

Since $\pi' J = \pi$, after solving the system of equations,

we get $\pi' = [18/37,\ 6/37,\ 13/37]$

(c) $\pi'_i = \frac{9}{37} \lambda_i \pi_i$, where $i \in \{A, B, C\}$.

and $\rho = \sum_{i \in \{A,B,C\}} \frac{\pi'_i}{\lambda_i} = \frac{9}{37}$

(d) In general, $\pi_i = \dfrac{\pi'_i / \lambda_i}{\sum_{j \in \{A,B,C\}} \frac{\pi'_j}{\lambda_j}}$, where $i \in \{A, B, C\}$.

and $\pi G = \pi(I - J) = 0$.