Solutions Manual for:
Understanding Analysis, Second Edition

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Author's note

What began as a desire to sketch out a simple answer key for the problems in *Understanding Analysis* inevitably evolved into something a bit more ambitious. As I was generating solutions for the nearly 200 odd-numbered exercises in the text, I found myself adding regular commentary on common pitfalls and strategies that frequently arise. My sense is that this manual should be a useful supplement to instructors teaching a course or to individuals engaged in an independent study. As with the textbook itself, I tried to write with the introductory student firmly in mind. In my teaching of analysis, I have come to understand the strong correlation between how students learn analysis and how they write it. A final goal I have for these notes is to illustrate by example how the form and grammar of a written argument are intimately connected to the clarity of a proof and, ultimately, to its validity.

The decision to include only the odd-numbered exercises was a compromise between those who view access to the solutions as integral to their educational needs, and those who strongly prefer that no solutions be available because of the potential for misuse. The total number of exercises was significantly increased for the second edition, and almost every even-numbered problem (in the regular sections of the text) is one that did not appear in the first edition. My hope is that this arrangement will provide ample resources to meet the distinct needs of these different audiences.

I would like to thank former students Carrick Detweiller, Katherine Ott, Yared Gurmu, and Yuqiu Jiang for their considerable help with a preliminary draft. I would also like to thank the readers of *Understanding Analysis* for the many comments I have received about the text. Especially appreciated are the constructive suggestions as well as the pointers to errors, and I welcome more of the same.

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Chapter 1

The Real Numbers

1.1 Discussion: The Irrationality of $\sqrt{2}$

1.2 Some Preliminaries

Exercise 1.2.1. (a) Assume, for contradiction, that there exist integers p and q satisfying

$$\left(\frac{p}{q}\right)^2 = 3. \tag{1}$$

Let us also assume that p and q have no common factor. Now, equation (1) implies

$$p^2 = 3q^2. \tag{2}$$

From this, we can see that p^2 is a multiple of 3 and hence p must also be a multiple of 3. This allows us to write $p = 3r$, where r is an integer. After substituting $3r$ for p in equation (2), we get $(3r)^2 = 3q^2$, which can be simplified to $3r^2 = q^2$. This implies q^2 is a multiple of 3 and hence q is also a multiple of 3. Thus we have shown p and q have a common factor, namely 3, when they were originally assumed to have no common factor.

A similar argument will work for $\sqrt{6}$ as well because we get $p^2 = 6q^2$ which implies p is a multiple of 2 and 3. After making the necessary substitutions, we can conclude q is a multiple of 6, and therefore $\sqrt{6}$ must be irrational.

(b) In this case, the fact that p^2 is a multiple of 4 does not imply p is also a multiple of 4. Thus, our proof breaks down at this point.

Exercise 1.2.2.

Exercise 1.2.3. (a) False, as seen in Example 1.2.2.

(b) True. This will follow from upcoming results about compactness in Chapter 3.

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(c) False. Consider sets $A = \{1, 2, 3\}, B = \{3, 6, 7\}$ and $C = \{5\}$. Note that $A \cap (B \cup C) = \{3\}$ is not equal to $(A \cap B) \cup C = \{3, 5\}$.

(d) True.

(e) True.

Exercise 1.2.4.

Exercise 1.2.5. (a) If $x \in (A \cap B)^c$ then $x \notin (A \cap B)$. But this implies $x \notin A$ or $x \notin B$. From this we know $x \in A^c$ or $x \in B^c$. Thus, $x \in A^c \cup B^c$ by the definition of union.

(b) To show $A^c \cup B^c \subseteq (A \cap B)^c$, let $x \in A^c \cup B^c$ and show $x \in (A \cap B)^c$. So, if $x \in A^c \cup B^c$ then $x \in A^c$ or $x \in B^c$. From this, we know that $x \notin A$ or $x \notin B$, which implies $x \notin (A \cap B)$. This means $x \in (A \cap B)^c$, which is precisely what we wanted to show.

(c) In order to prove $(A \cup B)^c = A^c \cap B^c$ we have to show,

$$(A \cup B)^c \subseteq A^c \cap B^c \text{ and,} \tag{1}$$

$$A^c \cap B^c \subseteq (A \cup B)^c. \tag{2}$$

To demonstrate part (1) take $x \in (A \cup B)^c$ and show that $x \in (A^c \cap B^c)$. So, if $x \in (A \cup B)^c$ then $x \notin (A \cup B)$. From this, we know that $x \notin$A and $x \notin B$ which implies $x \in A^c$ and $x \in B^c$. This means $x \in (A^c \cap B^c)$.

Similarly, part (2) can be shown by taking $x \in (A^c \cap B^c)$ and showing that $x \in (A \cup B)^c$. So, if $x \in (A^c \cap B^c)$ then $x \in A^c$ and $x \in B^c$. From this, we know that $x \notin A$ and $x \notin B$ which implies $x \notin (A \cup B)$. This means $x \in (A \cup B)^c$. Since we have shown inclusion both ways, we conclude that $(A \cup B)^c = A^c \cap B^c$.

Exercise 1.2.6.

Exercise 1.2.7. (a) $f(A) = [0, 4]$ and $f(B) = [1, 16]$. In this case, $f(A \cap B) = f(A) \cap f(B) = [1, 4]$ and $f(A \cup B) = f(A) \cup f(B) = [0, 16]$.

(b) Take $A = [0, 2]$ and $B = [-2, 0]$ and note that $f(A \cap B) = \{0\}$ but $f(A) \cap f(B) = [0, 4]$.

(c) We have to show $y \in g(A \cap B)$ implies $y \in g(A) \cap g(B)$. If $y \in g(A \cap B)$ then there exists an $x \in A \cap B$ with $g(x) = y$. But this means $x \in A$ and $x \in B$ and hence $g(x) \in g(A)$ and $g(x) \in g(B)$. Therefore, $g(x) = y \in g(A) \cap g(B)$.

(d) Our claim is $g(A \cup B) = g(A) \cup g(B)$. In order to prove it, we have to show,

$$g(A \cup B) \subseteq g(A) \cup g(B) \text{ and,} \tag{1}$$

$$g(A) \cup g(B) \subseteq g(A \cup B). \tag{2}$$

To demonstrate part (1), we let $y \in g(A \cup B)$ and show $y \in g(A) \cup g(B)$. If $y \in g(A \cup B)$ then there exists $x \in A \cup B$ with $g(x) = y$. But this means

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$x \in A$ or $x \in B$, and hence $g(x) \in g(A)$ or $g(x) \in g(B)$. Therefore, $g(x) = y \in g(A) \cup g(B)$.

To demonstrate the reverse inclusion, we let $y \in g(A) \cup g(B)$ and show $y \in g(A \cup B)$. If $y \in g(A) \cup g(B)$ then $y \in g(A)$ or $y \in g(B)$. This means we have an $x \in A$ or $x \in B$ such that $g(x) = y$. This implies, $x \in A \cup B$, and hence $g(x) \in g(A \cup B)$. Since we have shown parts (1) and (2), we can conclude $g(A \cup B) = g(A) \cup g(B)$.

Exercise 1.2.8.

Exercise 1.2.9. (a) $f^{-1}(A) = [-2, 2]$ and $f^{-1}(B) = [-1, 1]$. In this case, $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B) = [-1, 1]$ and $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B) = [-2, 2]$.

(b) In order to prove $g^{-1}(A \cap B) = g^{-1}(A) \cap g^{-1}(B)$, we have to show,

$$g^{-1}(A \cap B) \subseteq g^{-1}(A) \cap g^{-1}(B) \text{ and,} \tag{1}$$

$$g^{-1}(A) \cap g^{-1}(B) \subseteq g^{-1}(A \cap B). \tag{2}$$

To demonstrate part (1), we let $x \in g^{-1}(A \cap B)$ and show $x \in g^{-1}(A) \cap g^{-1}(B)$. So, if $x \in g^{-1}(A \cap B)$ then $g(x) \in (A \cap B)$. But this means $g(x) \in A$ and $g(x) \in B$, and hence $g(x) \in A \cap B$. This implies, $x \in g^{-1}(A) \cap g^{-1}(B)$.

To demonstrate the reverse inclusion, we let $x \in g^{-1}(A) \cap g^{-1}(B)$ and show $x \in g^{-1}(A \cap B)$. So, if $x \in g^{-1}(A) \cap g^{-1}(B)$ then $x \in g^{-1}(A)$ and $x \in g^{-1}(B)$. This implies $g(x) \in A$ and $g(x) \in B$, and hence $g(x) \in A \cap B$. This means, $x \in g^{-1}(A \cap B)$.

Similarly, in order to prove $g^{-1}(A \cup B) = g^{-1}(A) \cup g^{-1}(B)$, we have to show,

$$g^{-1}(A \cup B) \subseteq g^{-1}(A) \cup g^{-1}(B) \text{ and,} \tag{1}$$

$$g^{-1}(A) \cup g^{-1}(B) \subseteq g^{-1}(A \cup B). \tag{2}$$

To demonstrate part (1), we let $x \in g^{-1}(A \cup B)$ and show $x \in g^{-1}(A) \cup g^{-1}(B)$. So, if $x \in g^{-1}(A \cup B)$ then $g(x) \in (A \cup B)$. But this means $g(x) \in A$ or $g(x) \in B$, which implies $x \in g^{-1}(A)$ or $x \in g^{-1}(B)$. From this we know $x \in g^{-1}(A) \cup g^{-1}(B)$.

To demonstrate the reverse inclusion, we let $x \in g^{-1}(A) \cup g^{-1}(B)$ and show $x \in g^{-1}(A \cup B)$. So, if $x \in g^{-1}(A) \cap g^{-1}(B)$ then $x \in g^{-1}(A)$ or $x \in g^{-1}(B)$. This implies $g(x) \in A$ or $g(x) \in B$, and hence $g(x) \in A \cup B$. This means, $x \in g^{-1}(A \cup B)$.

Exercise 1.2.10.

Exercise 1.2.11. (a) There exist two real numbers a and b satisfying $a < b$ such that for all $n \in \mathbf{N}$ we have $a + 1/n \geq b$.

(b) For all real numbers $x > 0$, there exists $n \in \mathbf{N}$ satisfying $x \geq 1/n$.

(c) There exist two distinct rational numbers with the property that every number in between them is irrational.

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Exercise 1.2.12.

Exercise 1.2.13. (a) From Exercise 1.2.5 we know $(A_1 \cup A_2)^c = A_1^c \cap A_2^c$ which proves the base case. Now we want to show that

if we have $(A_1 \cup A_2 \cup \cdots \cup A_n)^c = A_1^c \cap A_2^c \cap \cdots \cap A_n^c$, then it follows that

$$(A_1 \cup A_2 \cup \cdots \cup A_{n+1})^c = A_1^c \cap A_2^c \cap \cdots \cap A_{n+1}^c.$$

Since the union of sets obey the associative law,

$$(A_1 \cup A_2 \cup \cdots \cup A_{n+1})^c = ((A_1 \cup A_2 \cup \cdots \cup A_n) \cup A_{n+1})^c$$

which is equal to

$$(A_1 \cup A_2 \cup \cdots \cup A_n)^c \cap A_{n+1}^c.$$

Now from our induction hypothesis we know that

$$(A_1 \cup A_2 \cup \cdots \cup A_n)^c = A_1^c \cap A_2^c \cap \cdots \cap A_n^c$$

which implies that

$$(A_1 \cup A_2 \cup \cdots \cup A_n)^c \cap A_{n+1}^c = A_1^c \cap A_2^c \cap \cdots \cap A_n^c \cap A_{n+1}^c.$$

By induction, the claim is proved for all $n \in \mathbf{N}$.

(b) Example 1.2.2 illustrates this phenomenon.

(c) In order to prove $\left(\bigcup_{n=1}^{\infty} A_n\right)^c = \bigcap_{n=1}^{\infty} A_n^c$ we have to show,

$$\left(\bigcup_{n=1}^{\infty} A_n\right)^c \subseteq \bigcap_{n=1}^{\infty} A_n^c \text{ and,} \tag{1}$$

$$\bigcap_{n=1}^{\infty} A_n^c \subseteq \left(\bigcup_{n=1}^{\infty} A_n\right)^c. \tag{2}$$

To demonstrate part (1), we let $x \in \left(\bigcup_{n=1}^{\infty} A_n\right)^c$ and show $x \in \bigcap_{n=1}^{\infty} A_n^c$. So, if $x \in \left(\bigcup_{n=1}^{\infty} A_n\right)^c$ then $x \notin A_n$ for all $n \in \mathbf{N}$. This implies x is in the complement of each A_n and by the definition of intersection $x \in \bigcap_{n=1}^{\infty} A_n^c$.

To demonstrate the reverse inclusion, we let $x \in \bigcap_{n=1}^{\infty} A_n^c$ and show $x \in \left(\bigcup_{n=1}^{\infty} A_n\right)^c$. So, if $x \in \bigcap_{n=1}^{\infty} A_n^c$ then $x \in A_n^c$ for all $n \in \mathbf{N}$ which means $x \notin A_n$ for all $n \in \mathbf{N}$. This implies $x \notin \left(\bigcup_{n=1}^{\infty} A_n\right)$ and we can now conclude $x \in \left(\bigcup_{n=1}^{\infty} A_n\right)^c$.

1.3 The Axiom of Completeness

Exercise 1.3.1. (a) A real number i is the greatest lower bound, or the infimum, for a set $A \subseteq \mathbf{R}$ if it meets the following two criteria:

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(i) i is a lower bound for A; i.e., $i \leq a$ for all $a \in A$, and

(ii) if l is any lower bound for A, then $l \leq i$.

(b) Lemma: Assume $i \in \mathbf{R}$ is a lower bound for a set $A \subseteq \mathbf{R}$. Then, $i = \inf A$ if and only if, for every choice of $\epsilon > 0$, there exists an element $a \in A$ satisfying $i + \epsilon > a$.

(i) To prove this in the forward direction, assume $i = \inf A$ and consider $i + \epsilon$, where $\epsilon > 0$ has been arbitrarily chosen. Because $i + \epsilon > i$, statement (ii) implies $i + \epsilon$ is not a lower bound for A. Since this is the case, there must be some element $a \in A$ for which $i + \epsilon > a$ because otherwise $i + \epsilon$ would be a lower bound.

(ii) For the backward direction, assume i is a lower bound with the property that no matter how $\epsilon > 0$ is chosen, $i + \epsilon$ is no longer a lower bound for A. This implies that if l is any number greater than i then l is no longer a lower bound for A. Because any number greater than i cannot be a lower bound, it follows that if l is some other lower bound for A, then $l \leq i$. This completes the proof of the lemma.

Exercise 1.3.2.

Exercise 1.3.3. (a) Because A is bounded below, B is not empty. Also, for all $a \in A$ and $b \in B$, we have $b \leq a$. The first thing this tells us is that B is bounded above and thus $\alpha = \sup B$ exists by the Axiom of Completeness. It remains to show that $\alpha = \inf A$. The second thing we see is that every element of A is an upper bound for B. By part (ii) of the definition of supremum, $\alpha \leq a$ for all $a \in A$ and we conclude that α is a lower bound for A.

Is it the greatest lower bound? Sure it is. If l is an arbitrary lower bound for A then $l \in B$, and part (i) of the definition of supremum implies $l \leq \alpha$. This completes the proof.

(b) We do not need to assume that greatest lower bounds exist as part of the Axiom of Completeness because we now have a proof that they exist. By demonstrating that the infimum of a set A is always equal to the supremum of a different set, we can use the existence of least upper bounds to assert the existence of greatest lower bounds.

Another way to achieve the same goal is to consider the set $-A = \{-a : a \in A\}$. If A is bounded below it follows that $-A$ is bounded above and it is not too hard to prove $\inf A = \sup(-A)$.

Exercise 1.3.4.

Exercise 1.3.5. (a) In the case $c = 0$, $cA = \{0\}$ and without too much difficulty we can argue that $\sup(cA) = 0 = c \sup A$. So let's focus on the case where $c > 0$. Observe that $c \sup A$ is an upper bound for cA. Now, we have to show if d is any upper bound for cA, then $c \sup A \leq d$. We know $ca \leq d$ for all $a \in A$, and thus $a \leq d/c$ for all $a \in A$. This means d/c is an upper bound for A, and by Definition 1.3.2 $\sup A \leq d/c$. But this implies $c \sup A \leq c(d/c) = d$, which is precisely what we wanted to show.

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(b) Assuming the set A is bounded below, we claim $\sup(cA) = c \inf A$ for the case $c < 0$. In order to prove our claim we first show $c \inf A$ is an upper bound for cA. Since $\inf A \leq a$ for all $a \in A$, we multiply both sides of the equation to get $c \inf A \geq ca$ for all $a \in A$. This shows that $c \inf A$ is an upper bound for cA. Now, we have to show if d is any upper bound for cA, then $c \inf A \leq d$. We know $ca \leq d$ for all $a \in A$, and thus $d/c \leq a$ for all $a \in A$. This means d/c is a lower bound for A and from Exercise 1.3.1, $d/c \leq \inf A$. But this implies $c \inf A \leq c(d/c) \leq d$, which is precisely what we wanted to show.

Exercise 1.3.6.

Exercise 1.3.7. Since a is an upper bound for A, we just need to verify the second part of the definition of supremum and show that if d is any upper bound then $a \leq d$. By the definition of upper bound $a \leq d$ because a is an element of A. Hence, by Definition 1.3.2, a is the supremum of A.

Exercise 1.3.8.

Exercise 1.3.9. (a) Set $\epsilon = \sup B - \sup A > 0$. By Lemma 1.3.8, there exists an element $b \in B$ satisfying $\sup B - \epsilon < b$, which implies $\sup A < b$. Because $\sup A$ is an upper bound for A, then b is as well.

(b) Take $A = [0, 1]$ and $B = (0, 1)$.

Exercise 1.3.10.

Exercise 1.3.11. (a) True. Observe that all elements of B are contained in A and hence $\sup A \geq b$ for all $b \in B$. By Definition 1.3.2 part (ii), $\sup B$ is less than or equal to any other upper bounds of B. Because $\sup A$ is an upper bound for B, it follows that $\sup B \leq \sup A$.

(b) True. Let $c = (\sup A + \inf B)/2$ from which it follows that

$$a \leq \sup A < c < \inf B \leq b.$$

(c) False. Consider, the open sets $A = (d, c)$ and $B = (c, f)$. Then $a < c < b$ for every $a \in A$ and $b \in B$, but $\sup A = c = \inf B$.

1.4 Consequences of Completeness

Exercise 1.4.1. (a) We have to show if $a, b \in \mathbf{Q}$, then ab and $a+b$ are elements of $\mathbf{Q}$. By definition, $\mathbf{Q} = \{p/q : p, q \in \mathbf{Z}, q \neq 0\}$. So take $a = p/q$ and $b = c/d$ where $p, q, c, d \in \mathbf{Z}$ and $q, d \neq 0$. Then, $ab = \frac{pc}{qd}$ where $pc, qd \in \mathbf{Z}$ because $\mathbf{Z}$ is closed under multiplication. This implies $ab \in \mathbf{Q}$. To see that $a + b$ is rational, write

$$\frac{p}{q} + \frac{c}{d} = \frac{pd + qc}{qd},$$

and observe that both $pd + qc$ and qd are integers with $qd \neq 0$.

(b) Assume, for contradiction, that $a + t \in \mathbf{Q}$. Then $t = (t + a) - a$ is the difference of two rational numbers, and by part (a) t must be rational as well. This contradiction implies $a + t \in \mathbf{I}$.

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Likewise, if we assume $at \in \mathbf{Q}$, then $t = (at)(1/a)$ would again be rational by the result in (a). This implies $at \in \mathbf{I}$.

(c) The set of irrationals is not closed under addition and multiplication. Given two irrationals s and t, $s + t$ can be either irrational or rational. For instance, if $s = \sqrt{2}$ and $t = -\sqrt{2}$, then $s + t = 0$ which is an element of $\mathbf{Q}$. However, if $s = \sqrt{2}$ and $t = 2\sqrt{2}$ then $s + t = \sqrt{2} + 2\sqrt{2} = 3\sqrt{3}$ which is an element of $\mathbf{I}$. Similarly, st can be either irrational or rational. If $s = \sqrt{2}$ and $t = -\sqrt{2}$, then $st = -1$ which is a rational number. However,if $s = \sqrt{2}$ and $t = \sqrt{3}$ then $st = \sqrt{2}\,\sqrt{3} = \sqrt{6}$ which is an irrational number.

Exercise 1.4.2.

Exercise 1.4.3. Let $x \in \mathbf{R}$ be arbitrary. To prove $\bigcap_{n=1}^{\infty}(0, 1/n) = \emptyset$ it is enough to show that $x \notin (0, 1/n)$ for some $n \in \mathbf{N}$. If $x \leq 0$ then we can take $n = 1$ and observe $x \notin (0, 1)$. If $x > 0$ then by Theorem 1.4.2 we know there exists an $n_0 \in \mathbf{N}$ such that $1/n_0 < x$. This implies $x \notin \bigcap_{n=1}^{\infty}(0, 1/n)$, and our proof is complete.

Exercise 1.4.4.

Exercise 1.4.5. We have to show the existence of an irrational number between any two real numbers a and b. By applying Theorem 1.4.3 on the real numbers $a-\sqrt{2}$ and $b-\sqrt{2}$ we can find a rational number r satisfying $a-\sqrt{2} < r < b-\sqrt{2}$. This implies $a < r + \sqrt{2} < b$. From Exercise 1.4.1(b) we know $r + \sqrt{2}$ is an irrational number between a and b.

Exercise 1.4.6.

Exercise 1.4.7. Now, we need to pick n_0 large enough so that

$$\frac{1}{n_0} < \frac{\alpha^2 - 2}{2\alpha} \quad \text{or} \quad \frac{2\alpha}{n_0} < \alpha^2 - 2.$$

With this choice of n_0, we have

$$(\alpha - 1/n_0)^2 > \alpha^2 - 2\alpha/n_0 = \alpha^2 - (\alpha^2 - 2) = 2.$$

This means $(\alpha - 1/n_0)$ is an upper bound for T. But $(\alpha - 1/n_0) < \alpha$ and $\alpha = \sup T$ is supposed to be the least upper bound. This contradiction means that the case $\alpha^2 > 2$ can be ruled out. Because we have already ruled out $\alpha^2 < 2$, we are left with $\alpha^2 = 2$ which implies $\alpha = \sqrt{2}$ exists in $\mathbf{R}$.

Exercise 1.4.8.

1.5 Cardinality

Exercise 1.5.1. Next let $n_2 = \min\{n \in \mathbf{N} : f(n) \in A\backslash\{f(n_1)\}\}$ and set $g(2) = f(n_2)$. In general, assume we have defined $g(k)$ for $k < m$, and let $g(m) = f(n_m)$ where $n_m = \min\{n \in \mathbf{N} : f(n) \in A\backslash\{f(n_1) \ldots f(n_{k-1})\}\}$.

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To show that $g : N \to A$ is 1–1, observe that $m \neq m'$ implies $n_m \neq n_{m'}$ and it follows that $f(n_m) = g(m) \neq g(m') = f(n_{m'})$ because f is assumed to be 1–1. To show that g is onto, let $a \in A$ be arbitrary. Because f is onto, $a = f(n')$ for some $n' \in \mathbf{N}$. This means $n' \in \{n : f(n) \in A\}$ and as we inductively remove the minimal element, n' must eventually be the minimum by at least the $n' - 1$st step.

Exercise 1.5.2.

Exercise 1.5.3. (a) Because A_1 is countable, there exists a 1–1 and onto function $f : \mathbf{N} \to A_1$.

If $B_2 = \emptyset$, then $A_1 \cup A_2 = A_1$ which we already know to be countable.

If $B_2 = \{b_1, b_2, \ldots, b_m\}$ has m elements then define $h : A_1 \cup B_2$ via

$$h(n) = \begin{cases} b_n & \text{if } n \leq m \\ f(n-m) & \text{if } n > m. \end{cases}$$

The fact that h is a 1–1 and onto follows immediately from the same properties of f.

If B_2 is infinite, then by Theorem 1.5.7 it is countable, and so there exists a 1–1 onto function $g : \mathbf{N} \to B_2$. In this case we define $h : A_1 \cup B_2$ by

$$h(n) = \begin{cases} f((n+1)/2) & \text{if } n \text{ is odd} \\ g(n/2) & \text{if } n \text{ is even.} \end{cases}$$

Again, the proof that h is 1–1 and onto is derived directly from the fact that f and g are both bijections. Graphically, the correspondence takes the form

$$\begin{array}{rcccccccc} \mathbf{N}: & & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\ & & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ A_1 \cup B_2: & & a_1 & b_1 & a_2 & b_2 & a_3 & b_3 & \cdots \end{array}$$

To prove the more general statement in Theorem 1.5.8, we may use induction. We have just seen that the result holds for two countable sets. Now let's assume that the union of m countable sets is countable, and show that the union of $m+1$ countable sets is countable.

Given $m + 1$ countable sets $A_1, A_2, \ldots, A_{m+1}$, we can write

$$A_1 \cup A_2 \cup \cdots \cup A_{m+1} = (A_1 \cup A_2 \cup \cdots \cup A_m) \cup A_{m+1}.$$

Then $C_m = A_1 \cup \cdots \cup A_m$ is countable by the induction hypothesis, and $C_m \cup A_{n+1}$ is just the union of two countable sets which we know to be countable. This completes the proof.

(b) Induction cannot be used when we have an infinite number of sets. It can only be used to prove facts that hold true for each value of $n \in \mathbf{N}$. See the discussion in Exercise 1.2.13 for more on this.

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(c) Let's first consider the case where the sets $\{A_n\}$ are disjoint. In order to achieve 1-1 correspondence between the set $\mathbf{N}$ and $\bigcup_{n=1}^{\infty} A_n$, we first label the elements in each countable set A_n as

$$A_n = \{a_{n1}, a_{n2}, a_{n3}, \ldots\}.$$

Now arrange the elements of $\bigcup_{n=1}^{\infty} A_n$ in an array similar to the one for $\mathbf{N}$ given in the exercise:

$$\begin{array}{llllll}
A_1 = a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & \cdots \\
A_2 = a_{21} & a_{22} & a_{23} & a_{24} & \cdots & \\
A_3 = a_{31} & a_{32} & a_{33} & \cdots & & \\
A_4 = a_{41} & a_{42} & \cdots & & & \\
A_5 = a_{51} & \cdots & & & & \\
\quad\vdots & & & & &
\end{array}$$

This establishes a 1–1 and onto mapping $g : \mathbf{N} \to \bigcup_{n=1}^{\infty} A_n$ where $g(n)$ corresponds to the element a_{jk} where (j,k) is the row and column location of n in the array for $\mathbf{N}$ given in the exercise.

If the sets $\{A_n\}$ are not disjoint then our mapping may not be 1–1. In this case we could again replace A_n with $B_n = A_n \backslash \{A_1 \cup \cdots \cup A_{n-1}\}$. Another approach is to use the previous argument to establish a 1–1 correspondence between $\bigcup_{n=1}^{\infty} A_n$ and an infinite *subset* of $\mathbf{N}$, and then appeal to Theorem 1.5.7.

Exercise 1.5.4.

Exercise 1.5.5. (a) The identity function $f(a) = a$ for all $a \in A$ shows that $A \sim A$.

(b) Since $A \sim B$ we know there is 1-1, onto function from A onto B. This means we can define another function $g : B \to A$ that is also 1-1 and onto. More specifically, if $f : A \to B$ is 1-1 and onto then $f^{-1} : B \to A$ exists and is also 1-1 and onto.

(c) We will show there exists a 1-1, onto function $h : A \to C$. Because $A \sim B$, there exists $g : A \to B$ that is 1–1 and onto. Likewise, $B \sim C$ implies that there exists $f : B \to C$ that is also 1-1 and onto. So let's define $h : A \to C$ by the composition $h = f \circ g$.

In order to show $f \circ g$ is 1-1, take $a_1, a_2 \in A$ where $a_1 \neq a_2$ and show $f(g(a_1)) \neq f(g(a_2))$. Well, $a_1 \neq a_2$ implies that $g(a_1) \neq g(a_2)$ because g is 1–1. And $g(a_1) \neq g(a_2)$ implies that $f(g(a_1)) \neq (f(g(a-2))$ because f is 1–1. This shows $f \circ g$ is 1–1.

In order to show $f \circ g$ is onto, we take $c \in C$ and show that there exists an $a \in A$ with $f(g(a)) = c$. If $c \in C$ then there exists $b \in B$ such that $f(b) = c$ because f is onto. But for this same $b \in B$ we have an $a \in A$ such that $g(a) = b$ since g is onto. This implies $f(b) = f(g(a)) = c$ and therefore $f \circ g$ is onto.

Exercise 1.5.6.

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Exercise 1.5.7. (a). The function $f(x) = (x, \frac{1}{3})$ is 1–1 from $(0,1)$ to S.

(b) Given $(x, y) \in S$, let's write x and y in their decimal expansions

$$x = .x_1x_2x_3\ldots \quad \text{and} \quad y = .y_1y_2y_3\ldots$$

where we make the convention that we always use the terminating form (or repeated 0s) over the repeating 9s form when the situation arises.

Now define $f : S \to (0,1)$ by

$$f(x, y) = .x_1y_1x_2y_2x_3y_3\ldots$$

In order to show f is 1–1, assume we have two distinct points $(x, y) \neq (w, z)$ from S. Then it must be that either $x \neq w$ or $y \neq z$, and this implies that in at least one decimal place we have $x_i \neq w_i$ or $y_i \neq z_i$. But this is enough to conclude $f(x, y) \neq f(w, z)$.

The function f is not onto. For instance the point $t = .555959595\ldots$ is not in the range of f because the ordered pair (x, y) with $x = .555\ldots$ and $y = .5999\ldots$ would not be allowed due to our convention of using terminating decimals instead of repeated 9s.

Exercise 1.5.8.

Exercise 1.5.9. (a) $\sqrt{2}$ is a root of the polynomial $x^2 - 2$, $\sqrt[3]{2}$ is a root of the polynomials $x^3 - 2$, and $\sqrt{3} + \sqrt{2}$ is a root of $x^4 - 10x^2 + 1$. Since all of these numbers are roots of polynomials with integer coefficients, they are all algebraic.

(b) Fix $n, m \in \mathbf{N}$. The set of polynomials of the form

$$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$$

satisfying $|a_n| + |a_{n-1}| + \cdots + |a_0| \leq m$ is *finite* because there are only a finite number of choices for each of the coefficients (given that they must be integers.) If we let A_{nm} be the set of all the roots of polynomials of this form, then because each one of these polynomials has at most n roots, the set A_{nm} is finite. Thus A_n, the set of algebraic numbers obtained as roots of any polynomial (with integer coefficients) of degree n, can be written as a countable union of finite sets

$$A_n = \bigcup_{m=1}^{\infty} A_{nm}.$$

It follows that A_n is countable.

(c) If A is the set of all algebraic numbers, then $A = \bigcup_{n=1}^{\infty} A_n$. Because each A_n is countable, we may use Theorem 1.5.8 to conclude that A is countable as well.

If T is the set transcendental numbers, then $A \cup T = \mathbf{R}$. Now if T were countable, then $\mathbf{R} = A \cup T$ would also be countable. But this is a contradiction because we know $\mathbf{R}$ is uncountable, and hence the collection of transcendental numbers must also be uncountable.

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Exercise 1.5.10.

Exercise 1.5.11. (a) For all $x \in A'$, there exists a unique $y \in B'$ satisfying $g(y) = x$. This means that there is a well-defined inverse function $g^{-1}(x) = y$ that maps A' onto B'. Setting

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g^{-1}(x) & \text{if } x \in A' \end{cases}$$

gives the desired 1–1, onto function from X to Y.

(b) To see that the sets $A_1, A_2, A_3, \ldots$ are pairwise disjoint, note that $A_1 \cap A_n = \emptyset$ for all $n \geq 2$ because $A_1 = X \backslash g(Y)$ and $A_n \subseteq g(Y)$ for all $n \geq 2$.

In the general case of $A_n \cap A_m$ where $1 < n < m$, note that if $x \in A_n \cap A_m$ then $f^{-1}(g^{-1}(x)) \in A_{n-1} \cap A_{m-1}$. Continuing in this way, we can show $A_1 \cap A_{m-n+1}$ is not empty, which is a contradiction. Thus $A_n \cap A_m = \emptyset$. Just to be clear, the disjointness of the A_n sets is not crucial to the overall proof, but it does help paint a clearer picture of how the sets A and A' are constructed.

(c) This is very straightforward. Each $x \in A$ comes from some A_n and so $f(x) \in f(A_n) \subseteq B$. Likewise, each $y \in B$ is an element of some $f(A_n)$ and thus $y = f(x)$ for some $x \in A_n \subseteq A$. Thus $f : A \to B$ is onto.

(d) Let $y \in B'$. Then $y \notin f(A_n))$ for all n, and because g is 1–1, $g(y) \notin A_{n+1}$ for all n. Clearly, $g(y) \notin A_1$ either and so g maps B' into A'.

To see that g maps B' onto A', let $x \in A'$ be arbitrary. Because $A' \subseteq g(Y)$, there exists $y \in Y$ with $g(y) = x$. Could y be an element of B? No, because $g(B) \subseteq A$. So $y \in B'$ and we have $g : B' \to A'$ is onto.

1.6 Cantor's Theorem

Exercise 1.6.1. The function $f(x) = (x - 1/2)/(x - x^2)$ is a 1–1, onto mapping from $(0, 1)$ to $\mathbf{R}$. This shows $(0, 1) \sim \mathbf{R}$, and the result follows using the ideas in Exercise 1.5.5.

Exercise 1.6.2.

Exercise 1.6.3. (a) If we imitate the proof to try and show that $\mathbf{Q}$ is uncountable, we can construct a real number x in the same way. This x will again fail to be in the range of our function f, *but there is no reason to expect x to be rational.* The decimal expansions for rational numbers either terminate or repeat, and this will not be true of the constructed x.

(b) By using the digits 2 and 3 in our definition of b_n we eliminate the possibility that the point $x = .b_1 b_2 b_3 \ldots$ has some other possible decimal representation (and thus it cannot exist somewhere in the range of f in a different form.)

Exercise 1.6.4.

Exercise 1.6.5. (a) $P(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$.

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(b) First, list the n elements of A. To construct a subset of A, we consider each element and associate either a 'Y' if we decide to include it in our subset or an 'N' if we decide not to include it. Thus, to each subset of A there is an associated sequence of length n of Ys and Ns. This correspondence is 1–1, and the proof is done by observing there are 2^n such sequences.

A second way to prove this result is with induction.

Exercise 1.6.6.

Exercise 1.6.7. This solution depends on the mappings chosen in the previous exercise. The key point is that no matter how this is done, the resulting set B should not be in the range of the particular 1–1 mapping used to create it.

Exercise 1.6.8.

Exercise 1.6.9. It is unlikely that there is a reasonably simple way to explicitly define a 1–1, onto mapping from $P(\mathbf{N})$ to $\mathbf{R}$. A more fruitful strategy is to make use of the ideas in Exercise 1.5.5 and 1.5.11. In particular, we have seen earlier in this section that $\mathbf{R} \sim (0,1)$. It is also true that $P(\mathbf{N}) \sim S$ where S is the set of all sequences consisting of 0s and 1s from Exercise 1.6.4. To see why, let $A \in P(\mathbf{N})$ be an arbitrary subset of $\mathbf{N}$. Corresponding to this set A is the sequence (a_n) where $a_n = 1$ if $n \in A$ and $a_n = 0$ otherwise. It is straightforward to show that this correspondence is both 1–1 and onto, and thus $P(\mathbf{N}) \sim S$.

With a nod to Exercise 1.5.5, we can conclude that $P(\mathbf{N}) \sim \mathbf{R}$ if we can demonstrate that $S \sim (0,1)$. Proving this latter fact is easier, but it is still not easy by any means. One way to avoid some technical details, is to use the Schröder–Bernstein Theorem (Exercise 1.5.11). Rather than finding a 1–1, onto function, the punchline of this result is that we will be done if we can find two 1–1 functions, one mapping $(0,1)$ into S, and the other mapping S into $(0,1)$. There are a number of creative ways to produce each of these functions.

Let's focus first on mapping $(0,1)$ into S. A fairly natural idea is to think in terms of binary representations. Given $x \in (0,1)$ let's inductively define a sequence (x_n) in the following way. First, bisect $(0,1)$ into the two parts $(0,1/2)$ and $[1/2,1)$. Then set $x_1 = 0$ if x is in the left half, and $x_1 = 1$ if x is in the right half. Now let I be whichever of these two intervals contains x, and bisect it using the same convention of including the midpoint in the right half. As before we set $x_2 = 0$ if x is in the left half of I and $x_2 = 1$ if x is in the right half. Continuing this process inductively, we get a sequence $(x_n) \in S$ that is uniquely determined by the given $x \in (0,1)$, and thus the mapping is 1–1.

It may seem like this mapping is onto S but it falls just short. Because of our convention about including the midpoint in the right half of each interval, we never get a sequence that is eventually all 1s, nor do we get the sequence of all 0s. This is fixable. The collection of all sequences in S that are NOT in the range of this mapping form a countable set, and it is not too hard to show that the cardinality of S with a countable set removed is the same as the cardinality of S. The other option is to use the Schröder–Bernstein Theorem mentioned previously. Having found a 1–1 function from $(0,1)$ into S, we just need to

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produce a 1–1 function that goes the other direction. An example of such a function would be the one that takes $(x_n) \in S$ and maps it to the real number with decimal expansion $.x_1x_2x_3x_4\ldots$. Because the only decimal expansions that aren't unique involve 9s, we can be confident that this mapping is 1–1.

The Schröder–Bernstein Theorem now implies $S \sim (0,1)$, and it follows that $P(\mathbf{N}) \sim \mathbf{R}$.

Exercise 1.6.10.

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Chapter 2

Sequences and Series

2.1 Discussion: Rearrangements of Infinite Series

2.2 The Limit of a Sequence

Exercise 2.2.1. Consider the sequence $(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \cdots)$ This sequence vercongеs to $x = 0$. To see this, note that we only have to produce a *single* $\epsilon > 0$ where the prescribed condition follows, and in this case we can take $\epsilon = 2$. This ϵ works because for all $N \in \mathbf{N}$, it is true that $n \geq N$ implies $|x_n - \frac{1}{2}| < 1$.

This is also an example of a vercongent sequence that is divergent. Notice that the "limit" $x = 0$ is not unique. We could also show this same sequence verconges to $x = 1$ by choosing $\epsilon = 3$.

In general, a vercongent sequence is a bounded sequence. By a bounded sequence, we mean that there exists an $M \geq 0$ satisfying $|x_n| \leq M$ for all $n \in \mathbf{N}$. In this case we can always take $x = 0$ and $\epsilon = M + 1$. Then $|x_n - x| = |x_n| < \epsilon$, and the sequence (x_n) verconges to 0.

Exercise 2.2.2.

Exercise 2.2.3. a) There exists at least one college in the United States where all students are less than seven feet tall.

b) There exists a college in the United States where all professors gave at least one student a grade of C or less.

c) At every college in the United States, there is a student less than six feet tall.

Exercise 2.2.4.

Exercise 2.2.5. a) The limit of (a_n) is zero. To show this let $\epsilon > 0$ be arbitrary. We must show that there exists an $N \in \mathbf{N}$ such that $n \geq N$ implies $|[[5/n]] - 0| <$

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ϵ. Well, pick $N = 6$. If $n \geq N$ we then have;

$$\left| \left[\left[\frac{5}{n} \right]\right] - 0 \right| = |0 - 0| < \epsilon,$$

because $[[5/n]] = 0$ for all $n > 5$.

b) Here the limit of a_n is 1. Let $\epsilon > 0$ be arbitrary. By picking $N = 7$ we have that for $n \geq N$,

$$\left| \left[\left[\frac{12 + 4n}{3n} \right]\right] - 1 \right| = |1 - 1| < \epsilon,$$

because $[[(12 + 4n)/3n]] = 1$ for all $n \geq 7$.

In these exercises, the choice of N does not depend on ϵ in the usual way. In exercise (b) for instance, setting $N = 7$ is a suitable response for every choice of $\epsilon > 0$. Thus, this is a rare example where a smaller $\epsilon > 0$ does not require a larger N in response.

Exercise 2.2.6.

Exercise 2.2.7. (a) The sequence $(-1)^n$ is *frequently* in the set 1.

(b) Definition (i) is stronger. "Frequently" does not imply "eventually", but "eventually" implies "frequently".

(c) A sequence (a_n) converges to a real number a if, given any ϵ-neighborhood $V_\epsilon(a)$ of a, (a_n) is *eventually* in $V_\epsilon(a)$.

(d) Suppose an infinite number of terms of a sequence (x_n) are equal to 2, then (x_n) is *frequently* in the interval (1.9, 2.1). However, (x_n) is not necessarily *eventually* in the interval (1.9, 2.1). Consider the sequence $(2, 0, 2, 0, 2, \cdots)$, for instance.

Exercise 2.2.8.

2.3 The Algebraic and Order Limit Theorems

Exercise 2.3.1. (a) Let $\epsilon > 0$ be arbitrary. We must find an N such that $n \geq N$ implies $|\sqrt{x_n} - 0| < \epsilon$. Because $(x_n) \to 0$, there exists $N \in \mathbf{N}$ such that $n \geq N$ implies $|x_n - 0| = x_n < \epsilon^2$. Using this N, we have $\sqrt{(x_n)^2} < \epsilon^2$, which gives $|\sqrt{x_n} - 0| < \epsilon$ for all $n \geq N$, as desired.

(b) Part (a) handles the case $x = 0$, so we may assume $x > 0$. Let $\epsilon > 0$. This time we must find an N such that $n \geq N$ implies $|\sqrt{x_n} - \sqrt{x}| < \epsilon$, for all $n \geq N$. Well,

$$\begin{aligned} |\sqrt{x_n} - \sqrt{x}| &= |\sqrt{x_n} - \sqrt{x}| \left(\frac{\sqrt{x_n} + \sqrt{x}}{\sqrt{x_n} + \sqrt{x}} \right) \\ &= \frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}} \\ &\leq \frac{|x_n - x|}{\sqrt{x}} \end{aligned}$$

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