MAT2002 ODEs
Nonlinear Differential Equations and Stability III

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Overview

1. Locally linear systems
   - Linear approximations to nonlinear systems
   - Trajectories of the locally linear system Vs. linear system
   - Competing Species
   - Damped pendulum
   - Undamped pendulum

Outline

1. Locally linear systems
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   - Undamped pendulum

Non-homogeneous linear systems

For linear homogeneous systems with constant coefficients:

$$\frac{d\mathbf{y}(t)}{dt} = \mathbf{A}\mathbf{y}(t), \quad \mathbf{A} \in \mathbb{R}^{2\times 2},$$

the behaviour of trajectories in the phase plane can be more or less determined by the eigenvalues of $\mathbf{A}$. Hence, the stability of critical points can also be deduced. However, for nonlinear autonomous systems, this is **not true** due to the following reasons:

- several or many critical points competing for influence of the trajectories;
- nonlinearity far away can affect stability of critical points.

To investigate nonlinear systems $\frac{d\mathbf{y}(t)}{dt} = \mathbf{f}(\mathbf{y})$, one idea to to approximate them with linear systems.

Linear approximations to nonlinear systems

In this subsection, our main objective is to investigate the behavior of trajectories of the nonlinear autonomous **two-dimensional** system of equations

$$\frac{d\mathbf{y}(t)}{dt} = \mathbf{f}(\mathbf{y}(t)), \quad \mathbf{y} \in \mathbb{R}^2,$$

with a critical point at $\mathbf{x}_*$, i.e., $\mathbf{f}(\mathbf{x}_*) = \mathbf{0}$.

Definition 17.1

A critical point $\mathbf{x}_*$ is **isolated** if there is a circle around $\mathbf{x}_*$ where no other critical points are in the circle.

Without loss of generality we can take $\mathbf{x}_*$ as the origin. Since, if $\mathbf{x}_* \neq \mathbf{0}$, then we can use the variable $\mathbf{z} = \mathbf{y} - \mathbf{x}_*$ and observe that

$$\mathbf{z}'(t) = \frac{d\mathbf{y}(t)}{dt} = \mathbf{f}(\mathbf{y}(t)) = \mathbf{f}(\mathbf{z}(t) + \mathbf{x}_*) =: \mathbf{h}(\mathbf{z}(t)),$$

with

$$\mathbf{h}(\mathbf{0}) = \mathbf{f}(\mathbf{x}_*) = \mathbf{0}.$$

That is, $\mathbf{0}$ is a critical point of $\mathbf{z}'(t) = \mathbf{h}(\mathbf{z}(t))$.

Linear approximations to nonlinear systems

The main idea of this section is to investigate the stability of the critical point $\mathbf{x}_*$ to the nonlinear system by studying an associated linear system. Using Taylor's theorem we can expand

$$\boxed{\mathbf{f}(\mathbf{y}) = \mathbf{f}(\mathbf{0}) + D\mathbf{f}(\mathbf{x})|_{\mathbf{x}=\mathbf{0}}\mathbf{y} + \mathbf{g}(\mathbf{y}) = D\mathbf{f}(\mathbf{x})|_{\mathbf{x}=\mathbf{0}}\mathbf{y} + \mathbf{g}(\mathbf{y})},$$

where $D\mathbf{f}$ is the **Jacobian matrix** of $\mathbf{f}$ defined as

$$D\mathbf{f}(\mathbf{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix},$$

and $\mathbf{g}(\mathbf{y})$ is a vector containing all higher order derivatives. Using this gives

$$\boxed{\frac{d\mathbf{y}(t)}{dt} = \mathbf{f}(\mathbf{y}(t)) = D\mathbf{f}(\mathbf{x})|_{\mathbf{x}=\mathbf{0}}\mathbf{y}(t) + \mathbf{g}(\mathbf{y}(t))}.$$

Linear approximations to nonlinear systems

The idea is that if $\mathbf{g}(\mathbf{y}(t))$ is "small" for trajectories $\{\mathbf{y}(t) : t \in I\}$ close to the critical point $\mathbf{0}$, then the nonlinear system $\frac{d\mathbf{y}(t)}{dt} = \mathbf{f}(\mathbf{y}(t)) = D\mathbf{f}(\mathbf{x})|_{\mathbf{x}=\mathbf{0}}\mathbf{y}(t) + \mathbf{g}(\mathbf{y}(t))$ should be well approximated by the linear system

$$\boxed{\frac{d\mathbf{y}(t)}{dt} = D\mathbf{f}(\mathbf{x})|_{\mathbf{x}=\mathbf{0}}\mathbf{y}(t)},$$

close to the critical point $\mathbf{0}$. This motivates the following definition.

Linear approximations to nonlinear systems

Definition 17.2

(Locally linear systems). We say that the nonlinear system

$$\frac{d\mathbf{y}(t)}{dt} = \mathbf{f}(\mathbf{y}(t)), \quad \mathbf{y} \in \mathbb{R}^n,$$

with an isolated critical point $\mathbf{0}$ is **locally linear** near $\mathbf{0}$ if there is a $n \times n$ matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ and a vector function $\mathbf{g}(\mathbf{y}(t))$ such that

$$\boxed{\frac{d\mathbf{y}(t)}{dt} = \mathbf{A}\mathbf{y}(t) + \mathbf{g}(\mathbf{y}(t)), \quad \lim_{\mathbf{y}\to\mathbf{0}} \frac{\|\mathbf{g}(\mathbf{y})\|}{\|\mathbf{y}\|} = 0},$$

where $\|\mathbf{y}\| = \sqrt{y_1^2 + \cdots + y_n^2}$ and $\|\mathbf{g}(\mathbf{y})\| = \sqrt{(g_1(y_1, \cdots, y_n))^2 + \cdots + (g_n(y_1, \cdots, y_n))^2}$.

Linear approximations to nonlinear systems

The condition

$$\lim_{\mathbf{y}\to\mathbf{0}} \frac{\|\mathbf{g}(\mathbf{y})\|}{\|\mathbf{y}\|} = 0$$

is how we quantify "smallness" of $\mathbf{g}(\mathbf{y})$, which means that the vector $\mathbf{g}(\mathbf{y})$ has smaller influence on the trajectories of the linear part $\mathbf{A}\mathbf{y}$. Furthermore, it is clear that the matrix $\mathbf{A}$ should be the Jacobian matrix $D\mathbf{f}(\mathbf{x})|_{\mathbf{x}=\mathbf{0}}$.

Linear approximations to nonlinear systems

Example 17.3

Consider the system

$$\frac{d\mathbf{y}(t)}{dt} = \begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix} \mathbf{y}(t) + \begin{pmatrix} -y_1^2 - y_1 y_2 \\ -0.75 y_1 y_2 - 0.25 y_2^2 \end{pmatrix}.$$

Then, setting

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix}, \quad \mathbf{g}(\mathbf{y}) = \begin{pmatrix} -y_1^2 - y_1 y_2 \\ -0.75 y_1 y_2 - 0.25 y_2^2 \end{pmatrix},$$

We can check that

(1) $\mathbf{0}$ is is indeed a critical point;

(2) The other critical points are $(0, 2), (1, 0)$ and $(1/2, 1/2)$. Thus, $\mathbf{0}$ is an isolated critical point;

Linear approximations to nonlinear systems

Example 17.5 continue

(3) To verify the condition

$$\lim_{\mathbf{y}\to\mathbf{0}} \frac{\|\mathbf{g}(\mathbf{y})\|}{\|\mathbf{y}\|} = 0,$$

it is useful to use polar coordinates: $y_1 = r\cos\theta$ and $y_2 = r\sin\theta$. Then $\|\mathbf{y}\| = r$ and

$$\begin{aligned}\|\mathbf{g}(\mathbf{y})\|^2 &= (r^2\cos^2\theta + r^2\cos\theta\sin\theta)^2 + (0.75r^2\cos\theta\sin\theta + 0.25r^2\sin^2\theta)^2 \\ &= r^4((\cos^2\theta + \cos\theta\sin\theta)^2 + (0.75\cos\theta\sin\theta + 0.25\sin^2\theta)^2).\end{aligned}$$

So that

$$\lim_{\mathbf{y}\to\mathbf{0}} \frac{\|\mathbf{g}(\mathbf{y})\|}{\|\mathbf{y}\|} = \lim_{r\to 0} r((\cos^2\theta + \cos\theta\sin\theta)^2 + (0.75\cos\theta\sin\theta + 0.25\sin^2\theta)^2)^{1/2} = 0.$$

Hence, the system is locally linear near $\mathbf{0}$.

Linear approximations to nonlinear systems

Remark

(a) Note that if the critical point is $\mathbf{x}_*$ instead of $\mathbf{0}$, then we want to derive a local linear system close to $\mathbf{x}_*$. It is convenient to set $\mathbf{z} = \mathbf{y} - \mathbf{x}_*$, so that $\mathbf{0}$ is the critical point for the system in the variable $\mathbf{z}$. Then, as before by Taylor expansion

$$\begin{aligned} \frac{d\mathbf{y}(t)}{dt} = \mathbf{z}'(t) = \mathbf{f}(\mathbf{z} + \mathbf{x}_*) &\approx \mathbf{f}(\mathbf{x}_*) + D\mathbf{f}(\mathbf{x}_*)\mathbf{z} + \mathbf{H}(\mathbf{z}) \\ &= D\mathbf{f}(\mathbf{x}_*)(\mathbf{y} - \mathbf{x}_*) + \mathbf{H}(\mathbf{y} - \mathbf{x}_*), \end{aligned}$$

where $\mathbf{H}$ contains terms of higher derivatives. The small condition now is given as

$$\lim_{\mathbf{y} \to \mathbf{x}_*} \frac{\|\mathbf{H}(\mathbf{y} - \mathbf{x}_*)\|}{\|\mathbf{y} - \mathbf{x}_*\|} = \lim_{\mathbf{z} \to \mathbf{0}} \frac{\|\mathbf{H}(\mathbf{z})\|}{\|\mathbf{z}\|} = 0.$$

(b) If $\mathbf{f}$ is a twice continuously differentiable vector function, then one can show that $\frac{d\mathbf{y}(t)}{dt} = \mathbf{f}(\mathbf{y}(t))$ is locally linear near the critical point.

(c) The matrix $D\mathbf{f}(\mathbf{x})|_{\mathbf{x}=\mathbf{0}}$ is a 2×2 matrix with constant coefficients.

Linear approximations to nonlinear systems

With the fact that $\mathbf{g}(\mathbf{y})$ is "small" compared to the linear part $D\mathbf{f}(\mathbf{x})|_{\mathbf{x}=\mathbf{0}}\mathbf{y}$, we hope that the trajectories near the critical point $\mathbf{0}$ for the locally linear system $\frac{d\mathbf{y}(t)}{dt} = D\mathbf{f}(\mathbf{x})|_{\mathbf{x}=\mathbf{0}}\mathbf{y} + \mathbf{g}(\mathbf{y})$ can be well approximated by studying the linear system $\frac{d\mathbf{y}(t)}{dt} = D\mathbf{f}(\mathbf{x})|_{\mathbf{x}=\mathbf{0}}\mathbf{y}(t)$, which we have studied previously. This turns out to be true in most cases, but not all. Indeed, the small perturbation $\mathbf{g}(\mathbf{y})$ sometimes could change the stability and type of the critical point.

Linear approximations to nonlinear systems

Theorem 17.4

(Stability for locally 2×2 linear system).

For the nonlinear system

$$\frac{d\mathbf{y}(t)}{dt} = \mathbf{f}(\mathbf{y}(t)), \quad \mathbf{y} \in \mathbb{R}^2.$$

Let r_1 and r_2 be the eigenvalues of $D\mathbf{f}(\mathbf{x})|_{\mathbf{x}=\mathbf{0}}$. Then, **except** *the cases*

(a) $r_1 = i\mu$ *for* $\mu \in \mathbb{R}$ *(and so* $r_2 = -i\mu$*)*

(b) $r_1 = r_2 \in \mathbb{R}$,

the type and stability of the critical point $\mathbf{0}$ *for the locally linear system* $\frac{d\mathbf{y}(t)}{dt} = D\mathbf{f}(\mathbf{x})|_{\mathbf{x}=\mathbf{0}}\mathbf{y} + \mathbf{g}(\mathbf{y})$ *and the linear system* $\frac{d\mathbf{y}(t)}{dt} = D\mathbf{f}(\mathbf{x})|_{\mathbf{x}=\mathbf{0}}\mathbf{y}(t)$ *are the* **same**.

More precisely, we have the following table for the critical point $\mathbf{0}$ *in the locally linear syetem* $\frac{d\mathbf{y}(t)}{dt} = \mathbf{f}(\mathbf{y}(t)) = D\mathbf{f}(\mathbf{x})|_{\mathbf{x}=\mathbf{0}}\mathbf{y} + \mathbf{g}(\mathbf{y})$:

Linear approximations to nonlinear systems

Theorem 17.6 continue

r_1, r_2	Type	Stability
$r_1 > r_2 > 0$	Node	Unstable
$r_1 < r_2 < 0$	Node	Asym.stable
$r_1 < 0 < r_2$	Saddle	Unstable
$r_1, r_2 = \lambda \pm i\mu$	Spiral	Unstable ($\lambda > 0$), Asym.stable ($\lambda < 0$)

For the other cases, we only have partial information as follows:

r_1, r_2	Type	Stability
$r_1 = r_2 > 0$	Node or Spiral	Unstable
$r_1 = r_2 < 0$	Node or Spiral	Asym.stable
$r_1, r_2 = \pm i\mu$	Center or Spiral	Undetermined

The proof of the theorem is beyond the scope of this course.

Linear approximations to nonlinear systems

Remark: For the nonlinear system

$$\frac{d\mathbf{y}(t)}{dt} = \mathbf{f}(\mathbf{y}(t)), \quad \mathbf{y} \in \mathbb{R}^2,$$

with an isolated critical point $\mathbf{0}$ is **locally linear** near $\mathbf{0}$.

(a) The critical point $\mathbf{x} = \mathbf{0}$ for the above nonlinear ODE system is unstable if $D\mathbf{f}(\mathbf{x})|_{\mathbf{x}=\mathbf{0}}$ has (at least) one eigenvalue with positive real part.

(b) The critical point $\mathbf{x} = \mathbf{0}$ for the above nonlinear ODE system is asymptotically stable stable if all eigenvalues of of $D\mathbf{f}(\mathbf{x})|_{\mathbf{x}=\mathbf{0}}$ have negative real parts.

(c) If $D\mathbf{f}(\mathbf{x})|_{\mathbf{x}=\mathbf{0}}$ has purely imaginary eigenvalues for $D\mathbf{f}(\mathbf{x})|_{\mathbf{x}=\mathbf{0}}$, then the stability of the critical point $\mathbf{x} = \mathbf{0}$ for the above nonlinear ODE system cannot be determined.

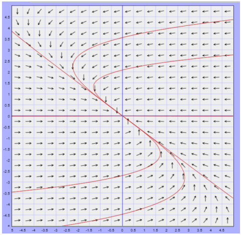
Linear approximations to nonlinear systems of competing Species

From above theorem, when the eigenvalues (r_1, r_2) of $D\mathbf{f}(\mathbf{x})|_{\mathbf{x}=\mathbf{0}}$ are not the same or not the case that they are purely imaginary numbers. The type and stability of the critical point $\mathbf{0}$ for the locally linear system $\frac{d\mathbf{y}(t)}{dt} = D\mathbf{f}(\mathbf{x})|_{\mathbf{x}=\mathbf{0}}\mathbf{y} + \mathbf{g}(\mathbf{y})$ and the linear system $\frac{d\mathbf{y}(t)}{dt} = D\mathbf{f}(\mathbf{x})|_{\mathbf{x}=\mathbf{0}}\mathbf{y}(t)$ are the **same**. Indeed, the nearby trajectories in the linear and locally linear system are topologically equivalent in these cases.

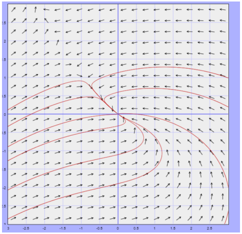
The feature of a critical point of the linear system $\frac{d\mathbf{y}(t)}{dt} = D\mathbf{f}(\mathbf{x})|_{\mathbf{x}=\mathbf{0}}\mathbf{y}$ would be carried out into the locally nonlinear system $\frac{d\mathbf{y}(t)}{dt} = D\mathbf{f}(\mathbf{x})|_{\mathbf{x}=\mathbf{0}}\mathbf{y} + \mathbf{g}(\mathbf{y})$.

The the pattern of trajectories near critical point $\mathbf{0}$ for $\frac{d\mathbf{y}(t)}{dt} = D\mathbf{f}(\mathbf{x})|_{\mathbf{x}=\mathbf{0}}\mathbf{y} + \mathbf{g}(\mathbf{y})$ are similar as the trajectories near critical point $\mathbf{0}$ for the linear system $\frac{d\mathbf{y}(t)}{dt} = D\mathbf{f}(\mathbf{x})|_{\mathbf{x}=\mathbf{0}}\mathbf{y}(t)$.

Phase portrait for node of the linear and locally nonlinear system



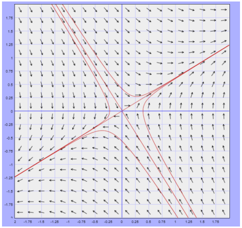
(a) $\frac{d\mathbf{y}(t)}{dt} = \begin{pmatrix} -1 & -1 \\ 0 & -0.25 \end{pmatrix} \mathbf{y}(t)$



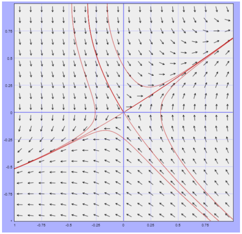
(b) $\frac{d\mathbf{y}(t)}{dt} = \begin{pmatrix} -1 & -1 \\ 0 & -0.25 \end{pmatrix} \mathbf{y}(t) + \begin{pmatrix} -0.1y_1y_2 \\ 0.2y_1^2 \end{pmatrix}$

There are two straight line trajectories in the linear system near the Nodal point, however, such two straight lines may be not exist in the locally nonlinear system, there is only a pair of trajectories that are tangent with other trajectories at the origin in the locally nonlinear system. The feature of the trajectories are qualitatively similar.

Phase portrait for saddle of the linear and locally nonlinear system



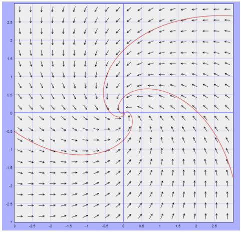
(c) $\frac{d\mathbf{y}(t)}{dt} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \mathbf{y}(t)$



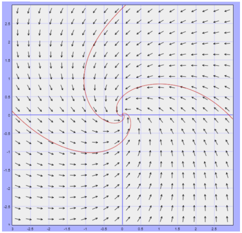
(d) $\frac{d\mathbf{y}(t)}{dt} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \mathbf{y}(t) + \begin{pmatrix} y_2^2 \\ sin(y_1) - y_1 \end{pmatrix}$

There are two straight lines in the linear system near the saddle point, where all other trajectories tend to tangent with one of the straight line, however, such two straight lines may be changed in to two curves in the locally nonlinear system.

Phase portrait for the spiral of the linear and locally nonlinear system



(e) $\frac{d\mathbf{y}(t)}{dt} = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \mathbf{y}(t)$



(f) $\frac{d\mathbf{y}(t)}{dt} = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \mathbf{y}(t) + \begin{pmatrix} 0.1y_1y_2 \\ -0.2y_1y_2 \end{pmatrix}$

Orientation and stability of the spiral in the linear system is carried into the nonlinear system.

Nonlinear systems of competing Species

Now, we will explore the application of phase plane analysis to some problems in population dynamics. These problems involve two interacting populations and are extensions of the discussion for a single population.

Mathematical modeling for competing Species

Suppose that in some closed environment there are two similar species competing for a limited food supply—for example, two species of fish in a pool that do not prey on each other but do compete for the available food. Let x and y be the populations of the two species at time t. As discussed in previous lecture, we assume that the population of each of the species, in the absence of the other, is governed by a logistic equation.
Thus

$$dx/dt = x(\epsilon_1 - \sigma_1 x), \tag{1a}$$

$$dy/dt = y(\epsilon_2 - \sigma_2 y), \tag{1b}$$

respectively, where ϵ_1 and ϵ_2 are the growth rates of the two populations, and ϵ_1/σ_1 and ϵ_2/σ_2 are their saturation levels.

Nonlinear systems of competing Species

Mathematical modeling for competing Species

However, when both species are present, each will tend to diminish the available food supply for the other. In effect, they reduce each others growth rates and saturation populations. The simplest expression for reducing the growth rate of species x due to the presence of species y is to replace the growth rate factor $\epsilon_1 - \sigma_1 x$ in Eq. (1a) by $\epsilon_1 - \sigma_1 x - \alpha_1 y$, where α_1 is a measure of the degree to which species y interferes with species x. Similarly,in Eq. (1b) we replace $\epsilon_2 - \sigma_2 y$ by $\epsilon_2 - \sigma_2 y - \alpha_2 x$. Thus we have the system of equations

$$\begin{aligned} dx/dt &= x(\epsilon_1 - \sigma_1 x - \alpha_1 y), \\ dy/dt &= y(\epsilon_2 - \sigma_2 y - \alpha_2 x). \end{aligned} \tag{2}$$

The values of the positive constants $\epsilon_1, \sigma_1, \alpha_1, \epsilon_2, \sigma_2$, and α_2 depend on the particular species under consideration and in general must be determined from observations. We are interested in solutions of Eqs. (2) for which x and y are nonnegative.

Linear approximations to nonlinear systems of competing Species

In the following two examples we discuss two typical problems in some detail.

Example 17.8

Discuss the qualitative behavior of solutions of the system

$$
\begin{aligned}
dx/dt &= x(1-x-y), \\
dy/dt &= y(0.75-y-0.5x).
\end{aligned} \tag{3}
$$

We find the critical points by solving the system of algebraic equations

$$
x(1-x-y)=0, \qquad y(0.75-y-0.5x)=0. \tag{4}
$$

We can find four critical points,namely, $(0,0)$, $(0,0.75)$, $(1,0)$, and $(0.5,0.5)$.

Competing Species

Example 17.8

The first three of critical points involve the absence of one or both species; only the last critical point corresponds to the presence of both species.

Other solutions are represented as trajectories in the xy-plane that describe the evolution of the populations in time. To begin to discover their qualitative behavior, we can proceed in the following way.

First, observe that the coordinate axes are themselves trajectories. This follows directly from Eqs. (3) since $dx/dt = 0$ on the y-axis (where $x = 0$) and, similarly, $dy/dt = 0$ on the x-axis (where $y = 0$). Thus no other trajectories can cross the coordinate axes. For a population problem only nonnegative values of x and y are significant, and we conclude that any trajectory that starts in the first quadrant remains there for all time.

Competing Species

Example 17.8

A direction field for the system (3) in the positive quadrant is shown in Figure 1; the black dots in this figure are the critical points or equilibrium solutions. Based on the direction field, it appears that the point $(0.5, 0.5)$ attracts other solutions and is therefore asymptotically stable, while the other three critical points are unstable. To confirm these conclusions, we can look at the linear approximations near each critical point.

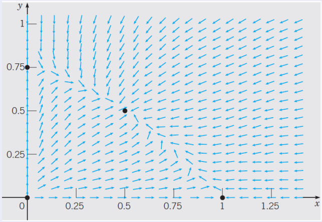


Figure 3. Critical points and direction field for the system (3).

Competing Species

Example 17.8

The system (3) is locally linear in the neighborhood of each critical point. Alternatively, we can evaluate the Jacobian matrix **J** at each critical point to obtain the coefficient matrix in the approximating linear system; see Eq. (13) in Section 9.3 in the textbook. When several critical points are to be investigated, it is usually better to use the Jacobian matrix. For the system (3), we have

$$F(x,y) = x(1-x-y), \qquad G(x,y) = y(0.75-y-0.5x), \tag{5}$$

so

$$\mathbf{J} = \begin{pmatrix} 1-2x-y & -x \\ -0.y & 0.75-2y-0.5x \end{pmatrix}. \tag{6}$$

We will now examine each critical point in turn.

Competing Species

Example 17.8

$\mathbf{x} = \mathbf{0}, \mathbf{y} = \mathbf{0}$. This critical point corresponds to the state in which neither species is present. To determine what happens near the origin we can set $x = y = 0$ in Eq. (6), which leads to the corresponding linear system

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0.75 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}. \tag{7}$$

The eigenvalues and eigenvectors of the system (7) are

$$r_1 = 1, \quad \xi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad r_2 = 0.75, \quad \xi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{8}$$

so the general solution of the system is

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 0 \end{pmatrix}e^{t} + c_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}e^{0.75t}. \tag{9}$$

Thus the origin is an unstable node of both the linear system (7) and the nonlinear system (3). In the neighborhood of the origin, all trajectories are tangent to the y-axis except for one trajectory that lies along the x-axis. If either or both of the species are present in small numbers, the population(s) will grow.

Competing Species

Example 17.8

$\mathbf{x} = \mathbf{1}, \mathbf{y} = \mathbf{0}$. This corresponds to a state in which species x is present but species y is not.

Using the substitution $x = 1 + u$, $y = 0 + v$ in Eqs. (3), retaining only the terms that are linear in u and v.

By evaluating $\mathbf{J}$ from Eq. (6) at $(x, y) = (1, 0)$, we find that the corresponding linear system for (u, v) is

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 0 & 0.25 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}. \tag{10}$$

It eigenvalues and eigenvectors are

$$r_1 = -1, \quad \xi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad r_2 = 0.25, \quad \xi^{(2)} = \begin{pmatrix} 4 \\ -5 \end{pmatrix}, \tag{11}$$

and its general solution is

$$\begin{pmatrix} u \\ v \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 0 \end{pmatrix}e^{-t} + c_2\begin{pmatrix} 4 \\ -5 \end{pmatrix}e^{0.25t}. \tag{12}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 0 \end{pmatrix}e^{-t} + c_2\begin{pmatrix} 4 \\ -5 \end{pmatrix}e^{0.25t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{13}$$

Competing Species

Example 17.8

Since the eigenvalues have opposite signs, the point $(1, 0)$ is a saddle point,and so it is an unstable equilibrium point of the linear system (10) and of the nonlinear system (3). The behavior of the trajectories near $(1, 0)$ can be seen from Eq. (13). If $c_2 = 0$, then there is one pair of trajectories that approaches the critical point along the x-axis. In other words, if the y population is initially zero, then it remains zero forever. All other trajectories depart from the neighborhood of $(1, 0)$; if y is initially small and positive, then the y population grows with time.

Competing Species

Example 17.8

$\mathbf{x} = \mathbf{0}, \mathbf{y} = \mathbf{0.75}$. This critical point is a state where species y is present but x is not.

Using the substitution $x = u$, $y = 0.75 + v$ in Eqs. (3), retaining only the terms that are linear in u and v.

The analysis is similar to that for the point $(1, 0)$. The corresponding linear system is

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0.25 & 0 \\ -0.375 & -0.75 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}. \tag{14}$$

The eigenvalues and eigenvectors are

$$r_1 = 0.25, \quad \xi^{(1)} = \begin{pmatrix} 8 \\ -3 \end{pmatrix}; \quad r_2 = -0.75, \quad \xi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{15}$$

Competing Species

Example 17.8

so the general solution of Eq. (14) is

$$\begin{pmatrix} u \\ v \end{pmatrix} = c_1 \begin{pmatrix} 8 \\ -3 \end{pmatrix} e^{0.25t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-0.75t}. \tag{16}$$

or

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 8 \\ -3 \end{pmatrix} e^{0.25t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-0.75t} + \begin{pmatrix} 0 \\ 0.75 \end{pmatrix}. \tag{17}$$

Thus the point $(0, 0.75)$ is also a saddle point. All trajectories leave the neighborhood of this point except one pair that approaches along the y-axis. If the x population is initially zero, it will remain zero, but a small positive x population will grow.

Competing Species

Example 17.8

$\mathbf{x} = \mathbf{0.5}, \mathbf{y} = \mathbf{0.5}$. This critical point corresponds to a mixed equilibrium state, or coexistence, in the competition between the two species.
Using the substitution $x = 0.5 + u$, $y = 0.5 + v$ in Eqs. (3), retaining only the terms that are linear in u and v, one has the linear system

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -0.5 & -0.5 \\ -0.25 & -0.5 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}. \tag{18}$$

The eigenvalues and eigenvectors of the corresponding linear system are

$$\begin{aligned} r_1 &= (-2+\sqrt{2})/4 \cong -0.146, \qquad \xi^{(1)} = \begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix}; \\ r_1 &= (-2-\sqrt{2})/4 \cong -0.854, \qquad \xi^{(2)} = \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}. \end{aligned} \tag{19}$$

Competing Species

Example 17.8

Therefore, the general solution of Eq. (18) is

$$\begin{pmatrix} u \\ v \end{pmatrix} = c_1 \begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix} e^{-0.146t} + c_2 \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} e^{-0.854t}. \tag{20}$$

or

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix} e^{-0.146t} + c_2 \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} e^{-0.854t} + \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}. \tag{21}$$

Since both eigenvalues are negative, the critical point $(0.5, 0.5)$ is an asymptotically stable node of the linear system (18) and of the nonlinear system (3). All nearby trajectories approach the critical point as $t \to \infty$. One pair of trajectories approaches the critical point along the line with slope $\sqrt{2}/2$ determined from the eigenvector $\xi^{(2)}$. All other trajectories approach the critical point tangent to the line with slope $-\sqrt{2}/2$ determined from the eigenvector $\xi^{(1)}$.

Competing Species

Example 17.8

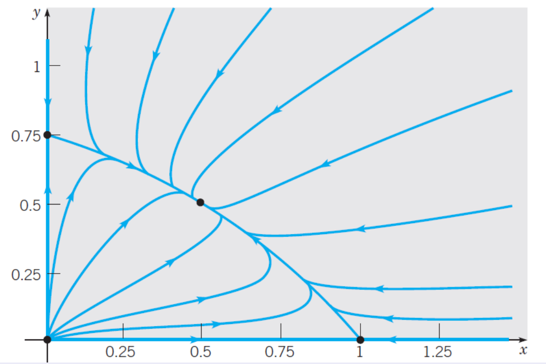


Figure 4. A phase portrait of the system (3).

Competing Species

Example 17.8

A phase portrait for the system (3) is shown in Figure 2. By looking closely at the trajectories near each critical point, you can see that they behave in the manner predicted by the linear system near that point. In addition, note that the quadratic terms on the right side of Eqs. (3) are all negative. Since for x and y large and positive these terms are the dominant ones, it follows that far from the origin in the first quadrant both x' and y' are negative; that is, the trajectories are directed inward. Thus all trajectories that start at a point (x_0, y_0) with $x_0 > 0$ and $y_0 > 0$ eventually approach the point $(0.5, 0.5)$. In other words, $(0.5, 0.5)$ is the globally attractor in the entire open first quadrant. The entire open first quadrant is the basin of attraction of $(0.5, 0.5)$.

This example shows that the competition between two species leads to an equilibrium state of **coexistence**.

Linear approximations to Damped pendulum system

Now, we will explore the application of phase plane analysis to the pendulum system.
Recall from Chapter 1, the equation for the motion of a damped pendulum is

$$\theta'' + \gamma\theta' + w^2 \sin\theta = 0,$$

where θ is the angle the pendulum makes with the vertical line, and the parameter $\gamma > 0$ is a damping factor taking into account friction forces. As this is a second order nonlinear equation, we can express this into a first order system:
Introducing the notation

$$y_1 = \theta, \quad y_2 = \theta',$$

then

$$\frac{d\mathbf{y}(t)}{dt} = \begin{pmatrix} y_2 \\ -w^2 \sin y_1 - \gamma y_2 \end{pmatrix} =: \mathbf{f}(\mathbf{y}). \tag{22}$$

Linear approximations to Damped pendulum system

Step.1 The critical points of the above systems satisfy

$$y_2 = 0, \quad \sin y_1 = 0,$$

and so the critical points are $(\pm n\pi, 0)$ for $n \in \mathbb{Z}$.

Step.2 Check to see if (22) is locally linear near the critical points. First for (0,0) we write (22) as

$$\frac{d\mathbf{y}(t)}{dt} = \begin{pmatrix} 0 & 1 \\ -w^2 & \gamma \end{pmatrix} \mathbf{y} - w^2 \begin{pmatrix} 0 \\ \sin y_1 - y_1 \end{pmatrix} =: \mathbf{A}\mathbf{y}(t) + \mathbf{g}(\mathbf{y}(t)).$$

Then, for $\mathbf{y}$ close to (0,0) we check

$$\lim_{\mathbf{y}\to\mathbf{0}} \frac{\|\mathbf{g}(\mathbf{y})\|}{\|\mathbf{y}\|} = 0.$$

For small y_1, by Taylor's expansion we have

$$\sin y_1 = y_1 - \frac{y_1^3}{3!} + \ldots.$$

Linear approximations to Damped pendulum system

And by polar coordinates $y_1 = r\cos\phi$, $y_2 = r\sin\phi$, we find that

$$\|\mathbf{g}(\mathbf{y})\| = w^2|\sin y_1 - y_1| = w^2\left|r^3\frac{\cos^3\phi}{3!} - r^5\frac{\cos^5\phi}{5!} + \dots\right|.$$

Thus,

$$\lim_{\mathbf{y}\to 0}\frac{\|\mathbf{g}(\mathbf{y})\|}{\|\mathbf{y}\|} = \lim_{r\to 0} w^2 r^2\left|\frac{\cos^3\phi}{3!} - r^2\frac{\cos^5\phi}{5!} + \dots\right| = 0.$$

That is, (22) is locally linear near (0,0). What about near $(\pi, 0)$? For this we employ a transformation

$$\mathbf{z} = \mathbf{y} - \begin{pmatrix} \pi \\ 0 \end{pmatrix},$$

and so if $\mathbf{z}$ is small, then $\mathbf{y}$ is close to the critical point $(\pi, 0)$. Then, it is clear that

$$\mathbf{z}'(t) = \frac{d\mathbf{y}(t)}{dt} = \mathbf{f}(\mathbf{y}(t)) = \mathbf{f}(\mathbf{z}(t) + (\pi, 0)) = \begin{pmatrix} z_2 \\ -w^2\sin(z_1+\pi) - \gamma z_2 \end{pmatrix} = \begin{pmatrix} z_2 \\ w^2\sin z_1 - \gamma z_2 \end{pmatrix},$$

upon using the addition formula for $\sin(\cdot)$:

$$\sin(z_1+\pi) = \sin z_1\cos\pi + \cos z_1\sin\pi = -\sin z_1.$$

Linear approximations to Damped pendulum system

Thus,

$$\mathbf{z}'(t) = \begin{pmatrix} 0 & 1 \\ w^2 & -\gamma \end{pmatrix} \mathbf{z}(t) + \begin{pmatrix} 0 \\ w^2(\sin z_1 - z_1) \end{pmatrix} =: \mathbf{B}\mathbf{z}(t) + \mathbf{h}(\mathbf{z}(t)). \tag{23}$$

Similar arguments as before show that

$$\lim_{\mathbf{z}\to\mathbf{0}} \frac{\|\mathbf{h}(\mathbf{z})\|}{\|\mathbf{z}\|} = 0$$

if we use polar coordinates and Taylor expansion for $\sin(\cdot)$ for small values of z_1. Hence, (22) is also locally linear near $(\pi, 0)$.

The same arguments can be used to show that (22) is locally linear near all the critical points $(\pm n\pi, 0)$ for $n \in \mathbb{Z}$. We now investigate the stability of the associated linear system and infer results for the locally linear system.

Linear approximations to Damped pendulum system

Step.3 Near the critical point (0,0), (22) can be expressed as the locally linear system

$$\frac{d\mathbf{y}(t)}{dt} = \begin{pmatrix} 0 & 1 \\ -w^2 & -\gamma \end{pmatrix} \mathbf{y} - w^2 \begin{pmatrix} 0 \\ \sin y_1 - y_1 \end{pmatrix} =: \mathbf{A}\mathbf{y}(t) + \mathbf{g}(\mathbf{y}(t)).$$

Note that one can also obtain the matrix $\mathbf{A}$ by computing the Jacobian matrix for $\mathbf{f}$, which we will do below:

$$D\mathbf{f}(\mathbf{y}) = \begin{pmatrix} 0 & 1 \\ -w^2 \cos y_1 & -\gamma \end{pmatrix}.$$

At the critical point (0,0) we see that the Jacobian matrix $D\mathbf{f}(\mathbf{x})|_{\mathbf{x}=\mathbf{0}}$ coincides with $\mathbf{A}$. Now by computing the eigenvalues of $\mathbf{A}$, we first determine the type and stability of the critical point $\mathbf{0}$ to the linear system $\frac{d\mathbf{y}(t)}{dt} = \mathbf{A}\mathbf{y}(t)$. We have

$$\det(\mathbf{A} - r\mathbf{I}) = r^2 + \gamma r + w^2 = 0,$$

and so

$$r_1 = -\frac{\gamma}{2} + \frac{1}{2}\sqrt{\gamma^2 - 4w^2}, \quad r_2 = -\frac{\gamma}{2} - \frac{1}{2}\sqrt{\gamma^2 - 4w^2}.$$

Linear approximations to Damped pendulum system

The classification for the type and stability of the critical point (0,0) is as follows:

(1) if $\gamma^2 > 4w^2$, then r_1, r_2 are distinct negative eigenvalues and $\mathbf{0}$ is an asym. stable node.

(2) if $\gamma^2 = 4w^2$, then r_1, r_2 are equal but negative eigenvalues and $\mathbf{0}$ is an asym. stable node or an asym. stable spiral.

(3) if $\gamma^2 < 4w^2$, then r_1, r_2 are complex conjugate pairs of eigenvalues with negative real part, and $\mathbf{0}$ is an asym. stable spiral.

In fact the same classification holds for all critical points of the form $(\pm 2m\pi, 0)$ for $m \in \mathbb{Z}$, since $D\mathbf{f}((\pm 2m\pi, 0)) = \mathbf{A}$. Then, by Thm.17.4, the critical points $(\pm 2m\pi, 0)$ for $m \in \mathbb{Z}$ to the system (22) has the same type and stability as stated above.

Now we look at the critical point $(\pi, 0)$, which we transform to (0,0) when studying the locally linear system (23).

Linear approximations to Damped pendulum system

Observe that

$$D\mathbf{f}((\pi,0)) = \begin{pmatrix} 0 & 1 \\ w^2 & -\gamma \end{pmatrix} = \mathbf{B},$$

and the eigenvalues for $\mathbb{B}$ are

$$r_1 = -\frac{\gamma}{2} + \frac{1}{2}\sqrt{\gamma^2 + 4w^2}, \quad r_2 = -\frac{\gamma}{2} - \frac{1}{2}\sqrt{\gamma^2 + 4w^2}.$$

Note that

$$\sqrt{\gamma^2 + 4w^2} > \sqrt{\gamma^2} = \gamma,$$

and so r_1 is positive and r_2 is negative. This implies that (0,0) as a critical point to the transformed system (23) is an unstable saddle point. This means that after transforming back and also using Thm. 17.4, $(\pi, 0)$ is an unstable saddle point for the pendulum system (22). A similar analysis then shows that $(\pm(2m+1)\pi, 0)$ for $m \in \mathbb{Z}$ are all unstable saddle points.

Linear approximations to Damped pendulum system

Step.4 We summarize the above with a phase portrait. Thanks to (22), we now know that the "odd" critical points $(\pm(2m+1)\pi, 0)$ for $m \in \mathbb{Z}$ are all unstable saddle points, and depending on the values of γ and w^2, the "even" critical points $(\pm 2m\pi, 0)$ can be nodes or spirals, but they are always asym. stable if $\gamma > 0$. In Fig. 3, we show the phase portrait highlighting the critical points $(0, 0)$, $(\pi, 0)$ and $(2\pi, 0)$ for the parameters $\gamma = 1, w = 1$ (so that we have spirals). In Fig. 4 we show the phase portrait for $\gamma = 5, w = 1$ (so that we have nodes).

Linear approximations to Damped pendulum system

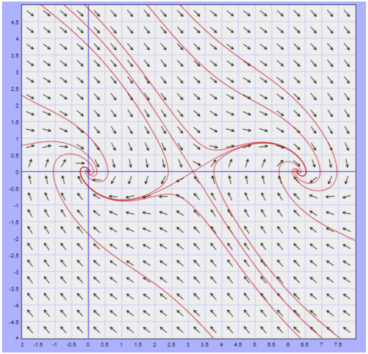


Fig. 3. Phase portrait for the damped pendulum with $\gamma = w = 1$, (0,0) is an asym. stable spiral sink, (π, 0) is a saddle point, (2π,0) is an asym. stable spiral sink.

Linear approximations to Damped pendulum system

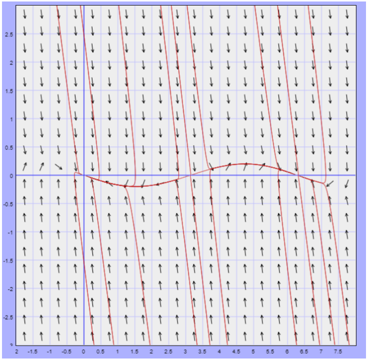


Fig. 4. Phase portrait for the damped pendulum with $\gamma = 5, w = 1$, (0,0) is an asym. stable node, (π, 0) is a saddle point, (2π,0) is an asym. stable node.

Linear approximations to Undamped pendulum system

Suppose we have no damping in the pendulum, which is equivalent to setting γ to zero in (22). Then, still we have critical points $(\pm n\pi, 0)$ for $n \in \mathbb{Z}$. Computing the Jacobian matrix near the "odd" critical points $(\pm(2m+1)\pi, 0)$ for $m \in \mathbb{Z}$ yields

$$D\mathbf{f}((\pm(2m+1)\pi, 0)) = \begin{pmatrix} 0 & 1 \\ w^2 & 0 \end{pmatrix},$$

with eigenvalues $r_1 = w, r_2 = -w$. This gives that $(\pm(2m+1)\pi, 0)$ are all unstable saddle points.

For the "even" critical points $(\pm 2m\pi, 0)$ for $m \in \mathbb{Z}$, we have

$$D\mathbf{f}((\pm 2m\pi, 0)) = \begin{pmatrix} 0 & 1 \\ -w^2 & 0 \end{pmatrix},$$

with eigenvalues $r_1 = iw, r_2 = -iw$. Note that we have purely imaginary eigenvalues and thus Thm. 17.4 cannot be used to deduce the stability of the critical points $(\pm 2m\pi, 0)$.

Linear approximations to Undamped pendulum system

In next slide, we will present a method to deduce the stability of the "even" critical point in the undamped pendulum system.

Appendix: Stability discussion for general $n \times n$ linear system

All the above discussion can be extended into the general autonomous ODE system $\mathbf{y}' = \mathbf{f}(\mathbf{y})$ with n unknown functions $\mathbf{y} = (y_1(t), \cdots, y_n(t))^T$.

The stability question can be completely resolved for each solution of the linear differential equation

$$\mathbf{y}'(t) = \mathbf{A}\mathbf{y}(t),$$

where $\mathbf{A}$ is a $n \times n$ matrix.

Appendix: Stability discussion for general $n \times n$ linear system

Theorem 17.5

For the above linear ODE system: $\mathbf{y}'(t) = \mathbf{A}\mathbf{y}(t), \quad \mathbf{A} \in \mathbb{R}^{n\times n}$.
(1) zero solution is stable if all the eigenvalues of $\mathbf{A}$ *have negative real part.*
(2) zero solution is unstable if at least one of the eigenvalues of $\mathbf{A}$ *has positive real part.*
(3) Suppose all eigenvalues of A have real parts ≤ 0 and $\lambda_1 = i\sigma_1, \cdots, \lambda_l = i\sigma_l$ have zero real part, let $\lambda_j = i\sigma_j$ have multiplicity k_j, this means that the characteristic polynomial of $\mathbf{A}$ *can be factorized as*

$$p_A(\lambda) = \det(\lambda I - A) = (\lambda - i\sigma_1)^{k_1} \cdots (\lambda - i\sigma_l)^{k_l} q(\lambda)$$

where all roots of $q(\lambda)$ have negative real parts. Then the zero solution of the above ODE system is stable if $\mathbf{A}$ *has k_j linearly independently eigenvectors for each eigenvalue $\lambda_j = i\sigma_j$, $j = 1, \cdots, l$. Otherwise, zero solution is unstable.*

This theorem can be found in the book "Differential Equations and Their Applications, Matrin Braun, Fourth Edition."

Appendix: Stability discussion for general $n \times n$ linear system

Example: Consider the following ODE system

$$\frac{d\mathbf{y}(t)}{dt} = \begin{pmatrix} 2 & -3 & 0 \\ 0 & -6 & -2 \\ -6 & 0 & -3 \end{pmatrix} \mathbf{y} \tag{24}$$

The characteristic polynomial is given by

$$p_A(\lambda) = \det(\lambda I - A) = -\lambda^2(\lambda + 7)$$

Eigenvalues are $\lambda = 0$ (algebraic multiplicity is 2) and $\lambda = -7$.
The eigenspace corresponding to $\lambda = 0$ is

$$\mathbf{Span}(3, 2, -6)^T.$$

since $\lambda = 0$ is an eigenvalue of multiplicity two and $\mathbf{A}$ has only one linearly independent eigenvector corresponding to eigenvalue 0. Thus, the zero solution is unstable.

Appendix: Stability results for general autonomous ODE system

Theorem 17.6

Consider the ODE system

$$\frac{d\mathbf{y}(t)}{dt} = \mathbf{A}\mathbf{y}(t) + \mathbf{g}(\mathbf{y}(t)).$$

If $\lim_{\mathbf{y}\to 0} \frac{\|\mathbf{g}(\mathbf{y})\|}{\|\mathbf{y}\|} = 0$ *and* $\mathbf{g}(\mathbf{0}) = \mathbf{0}$*, then*
(1) the zero solution $\mathbf{y}(t) \equiv 0$ *is asymptotically stable if the equilibrium solution* $\mathbf{y}(t) \equiv 0$ *of the linearized system* $\frac{d\mathbf{y}(t)}{dt} = \mathbf{A}\mathbf{y}(t)$ *is asymptotically stable. Equivalently, the zero solution of the above nonlinear system is asymptotically stable if all the eigenvalues of* $\mathbf{A}$ *have negative real parts.*
(2) the zero solution $\mathbf{y}(t) \equiv 0$ *of the above nonlinear system is unstable if at least one of the eigenvalues of* $\mathbf{A}$ *has positive real part.*
(3) the stability of the equilibrium solution $\mathbf{y}(t) \equiv 0$ *of nonlinear system cannot be determined from the stability of the equilibrium solution* $\mathbf{y}(t) \equiv 0$ *of* $\frac{d\mathbf{y}(t)}{dt} = \mathbf{A}\mathbf{y}(t)$ *if all the eigenvalues of* $\mathbf{A}$ *have real part* ≤ 0 *but at least one eigenvalue of* $\mathbf{A}$ *has zero real part.*

This theorem can be found in the book "Differential Equations and Their Applications, Matrin Braun, Fourth Edition."

Appendix: Stability discussion for general $n \times n$ linear system

Example: Consider the system of differential equations

$$\frac{dx_1}{dt} = -2x_1 + x_2 + 3x_3 + 9x_2^3$$
$$\frac{dx_2}{dt} = -6x_2 - 5x_3 + 7x_3^5$$
$$\frac{dx_3}{dt} = -x_3 + x_1^2 + x_2^2$$

We can write it as

$$\frac{d\mathbf{y}}{dt} = A\mathbf{y}(t) + \mathbf{g}(t)$$

where

$$\mathbf{A} = \begin{pmatrix} -2 & 1 & 3 \\ 0 & -6 & -5 \\ 0 & 0 & -1 \end{pmatrix}, \quad \mathbf{g}(t) = \begin{pmatrix} 9x_2^3 \\ 7x_3^5 \\ x_1^2 + x_2^2 \end{pmatrix}$$

Appendix: Stability discussion for general $n \times n$ linear system

The zero solution of

$$\frac{d\mathbf{y}}{dt} = A\mathbf{y}(t)$$

is asymptotically stable since the eigenvalues are $-2, -6, -1$. Thus, the zero solution of the nonlinear system

$$\frac{d\mathbf{y}}{dt} = A\mathbf{y}(t) + \mathbf{g}(t)$$

is also asymptotically stable.