Sheet 3 118010350

A3.1 The linear program:

$$\text{maximize } 5x_1 + 2x_2 + 5x_3$$

$$\text{subject to } 2x_1 + 3x_2 + x_3 \leq 4$$

$$x_1 + 2x_2 + 3x_3 \leq 7$$

$$x_1, x_2, x_3 \geq 0$$

$$b = \begin{bmatrix} 5 \\ 2 \\ 5 \end{bmatrix}, \quad A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \\ 1 & 3 \end{bmatrix}, \quad c = \begin{bmatrix} 4 \\ 7 \end{bmatrix}.$$

(a) The dual problem of the LP:

$$\text{minimize } 4y_1 + 7y_2$$

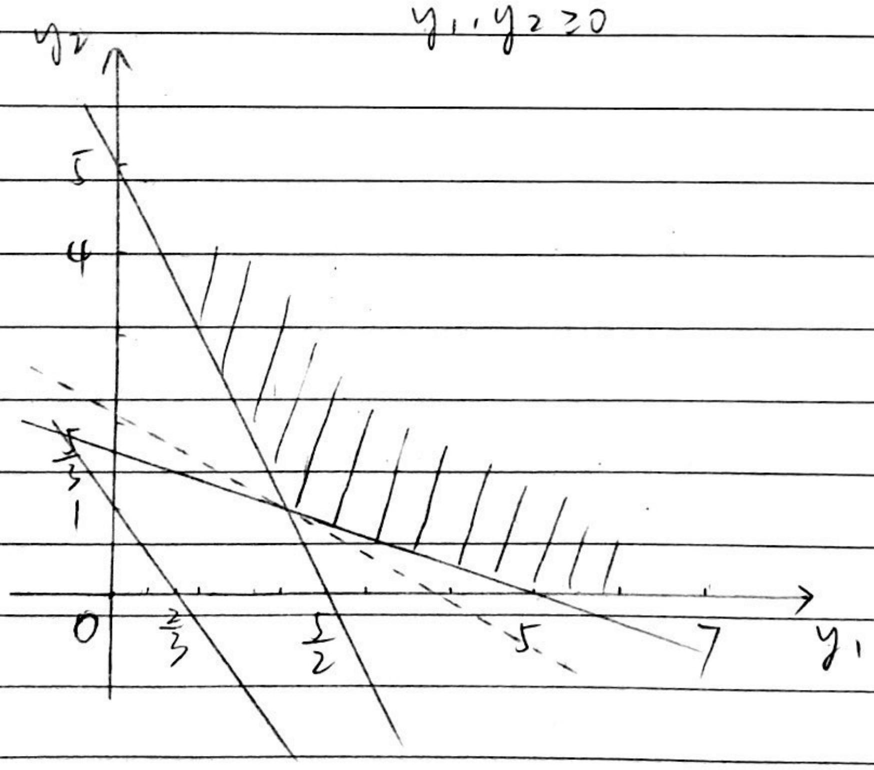
$$\text{subject to } 2y_1 + y_2 \geq 5$$

$$3y_1 + 2y_2 \geq 2$$

$$y_1 + 3y_2 \geq 5$$

$$y_1, y_2 \geq 0$$

(b)



we can get $y_1 = 2$, $y_2 = 1$,

this $\min(4y_1 + y_2) = 15$.

(c) Use the complementarity conditions:

① we find that $x_1=1, x_2=0, x_3=2$ is primal feasible

② we find that $y_1=2, y_2=1$ is dual feasible.

③ want to find $x=[x_1,x_2,x_3]^T$, $y=[y_1,y_2]^T$

Such that,

$$x_1(2y_1+y_2-5)=0$$

$$x_2(3y_1+2y_2-2)=0$$

$$x_3(y_1+3y_2-5)=0$$

and

$$y_1(2x_1+3x_2+x_3-4)=0$$

$$y_2(x_1+2x_2+3x_3-7)=0.$$

Since $x=[1,0,2]^T$ and $y=[2,1]^T$ is satisfied ①,②,③.

then we get that x,y are optimal solutions.

Thus, $\max(5x_1+2x_2+5x_3)=15$.

A 2.2. The general linear program:

$$\min_x c^T x.$$

$$\text{s.t. } Ax\le b,\ Cx=d.$$

where $A\in\mathbb{R}^{m\times n}$, $C\in\mathbb{R}^{p\times n}$, $c\in\mathbb{R}^n$, $b\in\mathbb{R}^m$, $d\in\mathbb{R}^p$.

(a) The dual problem of the LP:

$$\max_y \begin{bmatrix} b \\ d \end{bmatrix}^T y$$

$$\text{s.t. } \begin{bmatrix} A \\ C \end{bmatrix}^T y = c$$

$$y_i\le 0,\ i=1,\dots m$$

$$y_j\in\mathbb{R},\ i=m+1,\dots m+p.$$

where $y\in\mathbb{R}^{(m+p)}$

(b) we trasform the dual problem in (a) again

$$(a) \Leftrightarrow \min_y \begin{bmatrix} -b \\ -d \end{bmatrix}^T \cdot y$$

$$\text{s.t.} \begin{bmatrix} A \\ c \end{bmatrix}^T \cdot y = c$$

$$y_i \le 0, \quad i = 1, \dots, m.$$

$$y_j \in \mathbb{R}, \quad j = m+1, \dots m+p.$$

The dual problem of the dual problem:

$$\max_x \ c^T x$$

$$\text{s.t. } Ax \ge -b, \ cx = -d$$

$$x_i \in \mathbb{R}, \ i = 1, \dots n.$$

we let $z = -x$, then the dual problem:

$$\min_z \ c^T z$$

$$\text{s.t } Az \le b, \ cz = d.$$

$$z_i \in \mathbb{R}, \ i = 1, \dots n.$$

Thus, the dual of dual is equivalent to problem (2)

A 3.3. (a) Primal: maximize $2x_1 - x_2$

subject to $x_1 - x_2 \le 1$

$-x_1 + x_2 \le -2$

$x_1, x_2 \ge 0.$

Dual: minimize $y_1 - 2y_2$

subject to $y_1 - y_2 \ge 2$

$-y_1 + y_2 \ge -1$

$y_1, y_2 \ge 0$

(b) Primal: minimize $x_1+2x_2+x_3$

subject to $x_1+x_2=1$

$x_2+x_3=1$

$x_1, x_2 \geq 0,\ x_3 \leq 0$

Dual: maximize y_1+y_2

subject to $y_1 \leq 1$

$y_1+y_2 \leq 2$

$y_2 \geq 1$

(c) Primal: maximize $5x_1+2x_2+5x_3$

subject to $2x_1+3x_2+x_3 \leq 4$

$x_1+2x_2+3x_3 \leq 7$

$x_1, x_2, x_3 \geq 0$

Dual: minimize $4y_1+7y_2$

subject to $2y_1+y_2 \geq 5$

$3y_1+2y_2 \geq 2$

$y_1+3y_2 \geq 5$

$y_1, y_2 \geq 0$

(d) Primal: $\underset{x_1, x_2 \in \mathbb{R}}{\text{minimize}}$ $2x_1+x_2$

subject to $x_1+x_2=1$

$-x_1+x_2=1$

Dual: $\underset{y_1, y_2 \in \mathbb{R}}{\text{maximize}}$ y_1+y_2

subject to $y_1-y_2=2$

$y_1+y_2=1$

A3.4 (a) The linear program:

$$\max_{x,t} \quad t$$

$$\text{s.t.} \quad Ax \geq t\mathbb{1} \quad \text{①}$$

$$\mathbb{1}^T x = 1 \quad \text{②}$$

$$x \geq 0 \quad \text{③}$$

① Here $x \in \mathbb{R}^4$, $x = [x_1\ x_2\ x_3\ x_4]^T$, x_i denotes the probability of player I calling out number i; $t \in \mathbb{R}$, t denote the lower bound of player I's winning when player II calls out number i, $i = 1,2,3,4$.

The constraint ① ensures that when player II calls out number i, the player I's winning is no smaller than t.

The constraint ② ensures that the sum of the probability of calling out number i is 1.

The constraint ② ensures that the probability of calling out number i is nonnegative.

The objective function maximize t, get the max t so that the strategy makes player I's winning reach maximum whenever which number is called by player II.

② Using MATLAB

we get $p^* = -1.7266 \times 10^{-9} \approx 0$, $t = -1.7266 \times 10^{-9}$ and $x = [x_1\ x_2\ x_3\ x_4]^T$

$x_1 = 0.1297$, $x_2 = 0.3797$, $x_3 = 0.3703$, $x_4 = 0.1203$.

(b) The primal problem written in a compact way:

$$\max_{x,t} \quad 0^T \cdot x - t$$

$$\text{s.t.} \quad [A\ \ {-\mathbb{1}}] \cdot \begin{bmatrix} x \\ t \end{bmatrix} \geq 0$$

$$[\mathbb{1}^T\ \ 0] \cdot \begin{bmatrix} x \\ t \end{bmatrix} = 1$$

$$x \geq 0, \quad t \text{ free}$$

The dual problem of (a):

$$\min_{y,m} \quad m$$

$$\text{s.t.} \quad [A^T \ \mathbb{1}] \cdot \begin{bmatrix} y \\ m \end{bmatrix} \geq 0 \quad \text{---} \quad ④$$

$$[-\mathbb{1}^T \ 0] \cdot \begin{bmatrix} y \\ m \end{bmatrix} = 1 \quad \text{---} \quad ⑤$$

$$y \leq 0, \ m \text{ free} \quad \text{---} \quad ⑥$$

① Using MATLAB,
we get $d^* = 7.5470 \times 10^{-10} \approx 0$.

② Here $y \in \mathbb{R}^4$, $y = [y_1, y_2, y_3, y_4]^T$, y_i denotes the oppisite value of the probability of player II calling out number i; $m \in \mathbb{R}$, m denote the upper bound of $(-1) \cdot$ player II's winning (i.e. player I's loss) when player I calls out number i, $i = 1, 2, 3, 4$

The constraint ④ ensures that when player I calls out nuber i, the player II's loss is no larger than m

The constraint ⑤ ensures that the sum of probability of calling out number i is 1

The constraint ⑥ ensures that the probability of calling out number i is nonnegative

The objective function minimize m, get mint so that the strategy makes player I's loss reach minimum

(c) Define $P = \{x \in \mathbb{R}^4 : x \geq 0,\ \mathbb{1}^T x = 1\}$.

Consider the primal problem and the dual problem,

the primal: $\max t$	The dual: $\min m$
s.t. $x \in P$, $Ax \geq t\mathbb{1}$	s.t. $y \in P$, $A^T y \leq m\mathbb{1}$

Then consider the expression $\max\limits_{x \in P} \min\limits_{y \in P} y^T A x$.

Since $\max\limits_{x \in P} \min\limits_{y \in P} y^T A x = \min\limits_{y \in P} \max\limits_{x \in P} y^T A x$.

Then, $\min\limits_{y \in P} y^T A x \leq \max\limits_{x \in P} \min\limits_{y \in P} y^T A x = \min\limits_{y \in P} \max\limits_{x \in P} y^T A x \leq \max\limits_{x \in P} y^T A x$
$(\forall x \in P)$ $(\forall y \in P)$

Suppose the primal problem: optimal value is p^*, optimal solution is x^*;

the dual problem: optimal value is d^*, optimal solution is y^*

Since $Ax^* \geq p^* \mathbb{1}$, then $\min\limits_{y \in P} y^T p^* \mathbb{1} \leq \min\limits_{y \in P} y^T A x^*$

Since $A^T y^* \leq d^* \mathbb{1}$, then $y^{*T} A \leq d^* \mathbb{1}^T$

then $\max\limits_{x \in P} y^{*T} A x \leq \max\limits_{y \in P} d^* \mathbb{1}^T x$

By Strong Duality Theorem, we have $p^* = d^* = 0$

and $x \geq 0$, $y \geq 0$, and through the inequality above,

we have $\min\limits_{y \in P} p^* y^T \mathbb{1} \leq \max\limits_{x \in P} \min\limits_{y \in P} y^T A x \leq \max\limits_{y \in P} d^* \mathbb{1}^T x$

Since $\min\limits_{y \in P} p^* y^T \mathbb{1} = 0$, and $\max\limits_{y \in P} d^* \mathbb{1}^T x = 0$,

then we can get $\max\limits_{x \in P} \min\limits_{y \in P} y^T A x = 0$

Therefore, $p^* = \max\limits_{x \in P} \min\limits_{y \in P} y^T A x = d^*$.

By Strong Duality Theorem, we can get that $p^* = d^*$.
and here we denote x^*, y^* as optimal solution.
Since $p^* = d^*$, and $Ax^* \geq p^* \mathbb{1}$, $A^T y^* \leq d^* \mathbb{1}$
thus, we get that $Ax^* = A^T y^* = p^* \mathbb{1} = d^* \mathbb{1}$.
Consider $\max_{x \in P} (\min_{y \in P} y^T A x)$.

(d). ① we consider player II's winnings.

$$LP: \max_{y,t} t. \quad s.t \ (-A)y \geq t\mathbb{1}$$
$$\mathbb{1}^T y = 1$$
$$y \geq 0$$

using MATLAB, find that the optimal value is also 0.
Thus, the game is fair.

② When using number 1 and 2, now matrix A changes.

$$A = \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix} \quad -A = \begin{bmatrix} 2 & -3 \\ -3 & 4 \end{bmatrix}.$$

Consider player I's winnings.

$$LP: \max_{x,t} t \quad s.t \ Ax \geq t\mathbb{1}$$
$$\mathbb{1}^T x = 1 \qquad p^* = 0.0833$$
$$x \geq 0$$

Consider player II's winnings.

$$LP: \max_{y,t} t \quad s.t. \ A^T y \geq t\mathbb{1}$$
$$\mathbb{1}y = 1 \qquad d^* = -0.0833$$
$$y \geq 0$$

we find that in this case, the optimal values are different.
thus the game is not fair, and player I is better in the game

A3.5 $\underset{x \in \mathbb{R}^n}{\text{minimize}} \|Ax-b\|_\infty$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$

(a) The linear program of (4):

$$\underset{x,t}{\text{minimize}} \quad t$$

$$\text{subject to} \quad Ax-b \le t\mathbb{1}$$
$$-Ax+b \le t\mathbb{1}$$
$$(t \ge 0)$$

(b) The primal problem can be rewritten as:

$$\underset{x,t}{\text{minimize}} \quad 0^T \cdot x + t$$

$$\text{subject to} \quad \begin{bmatrix} A & -\mathbb{1} \\ -A & -\mathbb{1} \end{bmatrix} \cdot \begin{bmatrix} x \\ t \end{bmatrix} \le \begin{bmatrix} b \\ -b \end{bmatrix}$$
$$(t \ge 0)$$

The dual problem of the primal problem:

$$\underset{y,z}{\text{maximize}} \quad b^T \cdot y - b^T \cdot z.$$

$$\text{subject to} \quad \begin{bmatrix} A^T & -A^T \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = 0$$
$$\begin{bmatrix} -\mathbb{1}^T & -\mathbb{1}^T \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} \le 1$$
$$y \le 0, \ z \le 0$$

(c) Consider the dual problem in (b),

Let $K \in \mathbb{R}^m$, and $k = y - z$.

Then the dual problem can be rewritten as:

$$\underset{K}{\text{maximize}} \quad b^T K.$$

$$\text{subject to} \quad A^T K = 0$$

since $\|y\|_1 + \|z\|_1 \le 1$, and $y \le 0, z \le 0$

then $\|K\|_1 = \|y - z\|_1 \le \|y\|_1 + \|z\|_1 \le 1$.

thus, $\|K\|_1 \le 1$.

Thus, the dual problem is equivalent to:

$$\max_{y \in \mathbb{R}^m} b^T y$$

$$\text{s.t. } A^T y = 0, \ \|y\|_1 \le 1$$

(d). we know $\min_x \|Ax-b\|_\infty$ is equivalent to the primal problem in (a), and $\max_{y \in Y} b^T y$ is equivalent to the dual problem in (b). here $Y = \{y \in \mathbb{R}^m : A^T y = 0, \|y\|_1 \le 1\}$.

① Since we can always find $y = 0$, such that $y \in Y$, thus the feasible set Y is nonempty, there exists a basic feasible solution of dual problem.

② Also, we can easily know that the dual problem is bounded, due to the constriants.

Thus, the dual problem has an optimal solution.

By strong duality theorem, the primal problem also has an optimal solution, and the optimal values of them are the same, that is

$$\min_x \|Ax-b\|_\infty = \max_{y \in Y} b^T y$$

(e) using MATLAB.

Run the codes of the primal and the dual, we can find the runtime is simliar to each other

Maxleaf