Stochastic Processes

Lecture 13: Exponential distribution and Poisson processes

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Exponential r.v.

Definition

A random variable X is said to have exponential distribution with rate λ (with mean $1/\lambda$) if it has c.d.f. of

$$F(x) = 1 - e^{-\lambda x}$$

and p.d.f. of

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}.$$

Exponential r.v.

- $\mathbb{E}(X) = 1/\lambda$
- $\mathrm{Var}(X) = 1/\lambda^2$
- Memoryless property

$$\mathbb{P}(X > t + s | X > s) = \mathbb{P}(X > t) \quad \text{for any } t, s \in \mathbb{R}_+.$$

- Strong memoryless property

$$\mathbb{P}(X > t + S | X > S) = \mathbb{P}(X > t) \quad \text{for any } t \in \mathbb{R}_+,$$

where $S \geq 0$ is a r.v. that is independent of X.

Competing exponential clocks

- Suppose that X_1 and X_2 denote the lifetime of two light bulbs. Suppose that $X_1 \sim \mathrm{Exp}(\lambda_1), X_2 \sim \mathrm{Exp}(\lambda_2)$, and they are independent.
- Let $X = \min(X_1, X_2)$. Then X denote the time when any one of two bulbs fails.

$$X \sim \mathrm{Exp}(\lambda_1 + \lambda_2).$$

- $\mathbb{E}(X_1) = 2$ hours and $\mathbb{E}(X_2) = 6$ hours. Then,

$$\mathbb{E}(X) = \frac{1}{\frac{1}{2} + \frac{1}{6}} = \frac{1}{4/6} = 1.5 \text{ hours.}$$

- How about $X = \min(X_1, X_2, X_3)$?

Maximum

- How about the expectation of $\max(X_1, X_2)$?
- We can use the fact that $X_1 + X_2 = \min(X_1, X_2) + \max(X_1, X_2)$.

$$\begin{aligned}\mathbb{E}(\max(X_1, X_2)) =& \mathbb{E}(X_1 + X_2) - \mathbb{E}(\min(X_1, X_2)) = 8 - 1.5 \\ =& 6.5 \text{ hours.}\end{aligned}$$

- How about the expectation of $\max(X_1, X_2, X_3)$?

Corollary

$$
\begin{aligned}
\mathbb{P}(X_1 < X_2) &= \frac{\lambda_1}{\lambda_1 + \lambda_2}, \\
\mathbb{P}(X_1 > X_2) &= \frac{\lambda_2}{\lambda_1 + \lambda_2}, \\
\mathbb{P}(X_1 = X_2) &= 0.
\end{aligned}
$$

Proof.

$$
\begin{aligned}
\mathbb{P}(X_1 > X_2) &= \int_0^\infty \mathbb{P}(X_1 > X_2 | X_2 = t) f_{X_2}(t) dt \\
&= \int_0^\infty \mathbb{P}(X_1 > t | X_2 = t) \lambda_2 e^{-\lambda_2 t} dt \\
&= \int_0^\infty \lambda_1 e^{-\lambda_1 t} \lambda_2 e^{-\lambda_2 t} dt \\
&= \frac{\lambda_2}{\lambda_1 + \lambda_2}.
\end{aligned}
$$

A homework problem

- A bank has two tellers, John and Mary. John's processing times are iid exponential distributions X_1 with mean 6 minutes. Mary's processing times are iid exponential distributions X_2 with mean 4 minutes. A car with three customers A, B, C shows up at 12:00 noon and two tellers are both free.
- Suppose that Mary serves A and John serves B. What is the probability that C finishes service before A?

$$\mathbb{P}(B \text{ before } A) = \frac{1/6}{1/4 + 1/6} = \frac{4}{4+6} = \frac{4}{10}$$

$$\mathbb{P}(C \text{ before } A) =$$

- What is the probability that C finishes last?

$$\mathbb{P}(C \text{ finishes last}) =$$

Poisson distribution

Definition

A r.v. X is said to have a Poisson distribution with parameter μ if

$$\mathbb{P}\{X = k\} = \frac{\mu^k}{k!} e^{-\mu}, \quad k \in \mathbb{Z}_+.$$

- $\mathbb{E}(X) = \mu$, $\mathrm{Var}(X) = \mu$.
- Moment generating function (mgf)

$$m(s) = \mathbb{E}\left[e^{sX}\right] = e^{\mu(e^s - 1)} \text{ for } s \leq 0.$$

- Sum of independent Poisson r.v.'s; μ is "large", $X \sim N(\mu, \mu)$.
- $\mathrm{Bino}(n, p) \approx \mathrm{Poisson}(np)$ when n is large, p is small, and np is "moderate".

Poisson processes

Definition

A stochastic process $N = \{N(t), t \geq 0\}$ is said to be have *independent increments* if $N(s_1, t_1]$, …, $N(s_K, t_K]$ are independent for any $K \geq 1$ and any non-overlapping intervals $(s_1, t_1]$, …, $(s_K, t_K]$, where $N(s, t] \equiv N(t) - N(s)$.

Definition

A stochastic process $N = \{N(t), t \geq 0\}$ is said to be a (homogeneous) Poisson process with constant rate $\lambda > 0$ if (a) it has independent increments, (b) $N(s, t] \sim \text{Poisson}(\lambda(t - s))$ for any $0 \leq s < t$, (c) $N(0) = 0$.

- Customer/order/packet arrivals are modeled by Poisson processes, where $N(t)$ is the number of arrivals in time interval $(0, t]$.
- $\mathbb{E}(N(t)) = \lambda t$; therefore λ is the arrival rate.
- Why Poisson arrival process?

An example

- Assume N is a Poisson process with rate $\lambda = 2/\text{minutes}$
- Find the probability that there are exactly 4 arrivals in first 3 minutes.

$$\mathbb{P}(N(3) - N(0) = 4) = \frac{(2(3-0))^4}{4!} e^{-2(3-0)} = \frac{6^4}{4!} e^{-6} = 0.1339$$

- What is the probability that exactly two arrivals in $[0, 2]$ and at least 3 arrivals in $[1, 3]$?

$$\begin{aligned}\mathbb{P}(\{N(2) =& 2\} \cap \{N(3) - N(1) \geq 3\}) \\ =& \mathbb{P}(N(1) = 0)\mathbb{P}(N(2) - N(1) = 2)\mathbb{P}(N(3) - N(2) \geq 1) \\ &+ \mathbb{P}(N(1) = 1)\mathbb{P}(N(2) - N(1) = 1)\mathbb{P}(N(3) - N(2) \geq 2) \\ &+ \mathbb{P}(N(1) = 2)\mathbb{P}(N(2) - N(1) = 0)\mathbb{P}(N(3) - N(2) \geq 3)\end{aligned}$$

An example

- Computing

$$\begin{aligned}\mathbb{P}(N(3)-N(2)\geq 1) &= 1-\mathbb{P}(N(3)-N(2)<1)\\ &= 1-\mathbb{P}(N(3)-N(2)=0) = 1-\frac{2^0}{0!}e^{-2} = 1-e^{-2}\end{aligned}$$

- What is the probability that there is no arrival in $[0,4]$?

$$\mathbb{P}(N(4)-N(0)=0)=e^{-8}$$

- Let T_1 be the arrival time of the first customer. Is T_1 a continuous or discrete random variable?
- What is the probability that the first arrival will take at least 4 minutes?

$$\mathbb{P}(T_1>4)=\mathbb{P}(N(4)=0)=e^{-8} \tag{1}$$

In plain English, "the first arrival takes at least 4 minutes" is equivalent to "there is no arrival for the first 4 minutes."