Stochastic Processes

Lecture 5: Discrete Time Markov Chains

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Constructions of a DTMC

Theorem

(a) Suppose there is a function

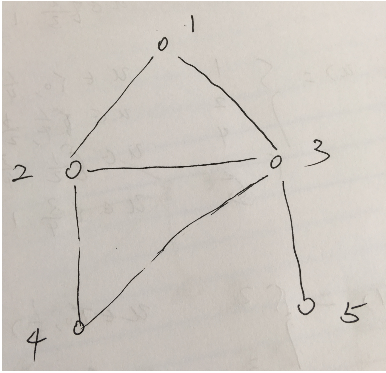
$$f : (i, u) \in S \times \mathcal{U} \mapsto f(i, u) \in S.$$

(b) $\{U_n : n = 1, 2, 3, \cdots\}$ *is an i.i.d. sequence of random variables taking values in* $\mathcal{U}$.

(c) $X_{n+1} = f(X_n, U_{n+1})$.

Then, $\{X_n : n = 1, 2, \cdots\}$ *is a DTMC.*

- Let (V, E) be a graph



- $V = \mathbb{Z}^2$
- $V = \mathbb{Z}_+^2$

f for the random walk

Random walk on $\mathbb{Z}_+^2$

- $f((2,3),d)$

Random walk on $\mathbb{Z}_+^2$

- $f((2,3),d)$
- $f((2,0),d)$
- $f((0,0),d)$

A single-server queue

- In time slot n, there are A_n number of customer (packet) arrivals; $\{A_n : n = 1, 2, \dots\}$ is an i.i.d. sequence

$$A_n = \begin{cases} 0 & \text{with probability } .4 \\ 1 & \text{with probability } .5 \\ 2 & \text{with probability } .1 \end{cases}$$

- there is a single server. It processes one customer a time. It takes exactly one time slot to complete a service.
- X_n is the number packets at the end of slot n, after the processing is completed.

Dynamics of X_n

Service time has two timeslots

When service time is geometrically distributed

A hospital model with a single ward

- A hospital with N beds
- A_n is the number of new patients arriving at the hospital on day n.
- Each admitted patient occupies a bed a random number of days, following a geometric distribution with probability p going home each day.
- X_n is the number of patients either waiting for a bed or occupying a bed at the beginning of day n.

Transient analysis: an Example

- Let $X = \{X_n : n = 0, 1, 2, \cdots\}$ be a DTMC on $S = \{1, 2, 3\}$ with transition matrix

$$P = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{pmatrix} 0 & .8 & .2 \\ .5 & 0 & .5 \\ 1 & 0 & 0 \end{pmatrix}.$$

- You need to be able to interpret transition probabilities $P_{i,j}$ by looking at this matrix. For example,

$$P_{2,3} = .5 = \mathbb{P}(X_{n+1} = 3 | X_n = 2), \quad P_{3,2} = 0.$$

- Compute

$$\mathbb{P}(X_{n+2} = 3, X_{n+1} = 1 | X_n = 2)$$

Matrix power

- Compute P^2 or P^3.

$$P^2 = \begin{pmatrix} 0.6 & 0.0 & 0.4 \\ 0.5 & 0.4 & 0.1 \\ 0.0 & 0.8 & 0.2 \end{pmatrix}, \qquad P^3 = \begin{pmatrix} 0.4 & 0.48 & 0.12 \\ 0.3 & 0.4 & 0.3 \\ 0.6 & 0.0 & 0.4 \end{pmatrix}$$

Transient probabilities

- Example 1:

$$\begin{aligned}\mathbb{P}(X_4 = 3 | X_2 = 1) &= \sum_{i=1}^{3} \mathbb{P}(X_4 = 3, X_3 = i | X_2 = 1) \\ &= \sum_{i=1}^{3} P_{1,i} P_{i,3} = P^2_{1,3}.\end{aligned}$$

- Example 2:

$$\mathbb{P}(X_4 = 3, X_2 = 1 | X_1 = 2) = P_{2,1} P^2_{1,3}$$

- Example 3:

$$\mathbb{P}(X_{10} = 1, X_4 = 3, X_2 = 1 | X_1 = 2) = P_{2,1} P^2_{1,3} P^6_{3,1}$$

Transient probabilities

Expected value

- $\mathbb{E}(X_3|X_0 = 1)$
- $\text{Var}(X_3|X_0 = 1)$

Expected value

- Expected profit on "day 3": $g(1) = -\$5$, $g(2) = \$1$, $g(3) = \$10$.

$$\mathbb{E}[g(X_3)|X_0 = 1)]$$

Expected value

- Expected profit on "day 3": $g(1) = -\$5$, $g(2) = \$1$, $g(3) = \$10$.

$$\begin{pmatrix} 0.4 & 0.48 & 0.12 \\ 0.3 & 0.4 & 0.3 \\ 0.6 & 0.0 & 0.4 \end{pmatrix} \begin{pmatrix} -5 \\ 1 \\ 10 \end{pmatrix} = \begin{pmatrix} -0.32 \\ 1.90 \\ 1.00 \end{pmatrix}$$

Expected total profit

- Expected total profit in three days

$$\mathbb{E}[g(X_1) + g(X_2) + g(X_3)|X_0 = 1)]$$

- Expected total discounted profit in three days

$$\mathbb{E}[\beta g(X_1) + \beta^2 g(X_2) + \beta^3 g(X_3)|X_0 = 1)]$$

100 Yuan on day 1 is worthy 95 Yuan today if $\beta = .95$.