# MAT2002 Ordinary Differential Equations Review for the first 3 weeks

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### Definition of ODE and IVP

#### Definition R.1

An ordinary differential equation is an equation involving  $\underline{ONE}$  independent variable  $t \in I(I \text{ is an interval})$  and  $\underline{ONE}$  dependent variable y of the form

$$F(t, y, y', y'', \dots, y^{(n)}) = 0.$$

Given constants  $t_0, t_1, \ldots, t_{n-1} \in I$  and  $y_0, y_1, \ldots, y_{n-1} \in \mathbb{R}$ ,we call

$$\begin{cases} F(t, y, y', y'', \dots, y^{(n)}) = 0, \\ y(t_0) = y_0, \frac{dy}{dt}(t_1) = y_1, \dots, \frac{d^{(n-1)}y}{dt^{n-1}}(t_{n-1}) = y_{n-1}, \end{cases}$$

an initial value problem (IVP).

The <u>order</u> of an ODE is the **highest order** of derivative in the ODE.

## Definition related to ODE

### Definition R.2

(a) An ODE  $F(t, y, y', y'', \dots, y^{(n)}) = 0$  is <u>linear</u> is if F is a <u>linear function</u> of  $y, \frac{dy}{dt}, \dots, \frac{d^ny}{dt^n}$ . Otherwise, it is a <u>non-linear</u> ODE. The general <u>linear</u> ODE of order n is

$$a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \cdots + a_0(t)y(t) = f(t),$$

for some given functions  $a_0, a_1, \ldots, a_n$  and f.

(b) An ODE is called <u>autonomous</u> if the independent variable does not appear explicitly (only in the derivatives), the <u>autonomous</u> ODE has the form:  $F(y, y', y'', \dots, y^{(n)}) = 0$ . Otherwise it is a <u>non-autonomous</u> ODE.

# Real-world mathematical modelling

- (a) Motion of a falling object. Two force acting on the object: gravitational force and air resistance force (assume to be proportional to the velocity). Need to use Newton's second law.
- (b) Motion of a pendulum. Need to use Newton's second law.
- (c) Modeling the growth of cows.

### 1st-order linear ODE

$$\begin{cases}
\frac{dy}{dt} = p(t)y + q(t), \\
y(t_0) = y_0,
\end{cases}$$
(1)

General solution:

$$y(t) = \frac{1}{\mu(t)} \left[ \int \mu(t) q(t) dt + c \right], \text{ where } \mu(t) = \exp\left( -\int p(t) dt \right).$$
 (2)

In order to use the initial condition more easily, the general solution can be rewritten as

$$y(t) = \frac{1}{\mu(t)} \left[ \int_{t_0}^t \mu(t) q(t) dt + c \right], \quad \text{where} \quad \mu(t) = \exp\left( - \int_{t_0}^t p(t) dt \right). \quad (3)$$

Using the initial condition, one can get  $c = y_0$ . The solution to IVP is

$$y(t) = \frac{1}{\mu(t)} \left[ \int_{t_0}^t \mu(t) q(t) dt + y_0 \right], \quad \text{where} \quad \mu(t) = \exp\left( - \int_{t_0}^t \rho(t) dt \right). \tag{4}$$

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# 1st-order non-linear ODE-separable equation

For 1st-order non-linear ODE, a general method is still missing. We can only solve some special type of 1st-order non-linear ODEs.

### Definition R.3

(Separable equation). A first order ODE y' = f(t, y) is **separable** if it can be written in the form

$$M(t) + N(y)\frac{dy}{dt} = 0 (5)$$

for some functions M and N.

Suppose there exist functions m and n such that

$$m'=M$$
,  $n'=N$ .

Then (5) can be written as

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$$\frac{d}{dt}m(t)+\frac{d}{dt}n(y(t))=0.$$

Integrating yields the general (implicit) solution

$$m(t) + n(y(t)) = c$$
,  $c \in \mathbb{R}$  (6)

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## 1st-order non-linear ODE-Exact equation

### Definition R.4

(Exact equation). A first order ODE  $M(t,y) + N(t,y) \frac{dy}{dt} = 0$  is an **exact equation** if there exists a function  $\Psi(t,y)$  such that

$$\frac{\partial \Psi}{\partial t}(t,y) = M(t,y), \quad \frac{\partial \Psi}{\partial y}(t,y) = N(t,y). \tag{7}$$

The general solution y(t) to the ODE is given implicitly as  $\Psi(t,y(t))=c,\ c\in\mathbb{R}.$ 

#### Remark:

$$M(t,y) + N(t,y) \frac{dy}{dt} = 0$$
 ( $M_y$  and  $N_t$  are continuous) is exact  $\Leftrightarrow M_y = N_t$ .

# 1st-order non-linear ODE-(non-Exact equation)

If  $M(t,y) + N(t,y) \frac{dy}{dt} = 0$  is non-exact equation. Try to look for factor  $\mu(t,y)$  s.t.  $\mu M(t,y) + \mu N(t,y) \frac{dy}{dt} = 0$  is exact  $\Leftrightarrow (\mu M)_y = (\mu N)_t$ .

- (1)  $\mu(t,y) = \mu(t)$ . Compute  $K(t,y) = \frac{M_y N_t}{N}(t,y)$ ; If K(t,y) = K(t), take  $\mu(t) = e^{\int K(t)dt}$ .
- (2)  $\mu(t,y) = \mu(y)$ . Compute  $H(t,y) = \frac{N_t M_y}{M}(t,y)$ ; If H(t,y) = H(y), take  $\mu(y) = e^{\int H(y) dy}$ .
- (3)  $\mu(t,y) = \mu(ty) = \mu(z), z = ty$ . Compute  $L(t,y) = \frac{N_t M_y}{tM yN}$ ; If L(t,y) = L(ty) = L(z) (z = ty), take  $\mu(z) = e^{\int L(z)dz}$ .

For 1st order nonlinear non-exact ODE, there is no general method.

## Existence and Uniqueness

Next, we will address the theoretical fundamental question, is the 1st order nonlinear ODE  $y' = f(t, y), y(t_0) = y_0$  has a solution? If it is, is the solution the only solution?

# Existence and Uniqueness for first order linear ODE

#### Theorem R.5

(Existence and Uniqueness for first order linear ODEs). Suppose functions p and q are continuous on  $(\alpha, \beta) \subset \mathbb{R}$   $(\alpha, \beta)$  are some real

numbers). Then, for any  $t_0 \in (\alpha, \beta)$ ,  $y_0 \in \mathbb{R}$ , there exists a unique function y(t) satisfying

$$\frac{dy}{dt} = p(t)y + q(t), \quad \forall t \in (\alpha, \beta),$$
$$y(t_0) = y_0.$$

And the solution is defined throughout the interval  $(\alpha, \beta)$ .

The solution **globally** exists in the interval  $(\alpha, \beta)$  in which p and q are continuous.

# Existence and Uniqueness for first order general ODE

#### Theorem R.6

Consider the initial value problem (IVP)

$$\frac{dy}{dt}=f(t,y), \quad y(t_0)=y_0.$$

Let R be a closed rectangle

$$R = \{(t,y)||t-t_0| \le a, \quad |y-y_0| \le b\}(a>0,b>0).$$

Assume that both f(t,y) and  $\frac{\partial f}{\partial y}$  are continuous on R. Then the IVP has a unique solution y=y(t) defined on the interval  $(t_0-h,t_0+h)$ , where  $h=\min\left(\frac{b}{M},a\right)$  and  $M=\max_{(t,y)\in R}|f(t,y)|$ .

The solution only **locally** exists in the interval  $[t_0 - a, t_0 + a]$ .

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# Existence and uniqueness of the solution

### Proof.

$$\phi_0(t) = y_0.$$

$$\phi_1(t) = y_0 + \int_{t_0}^t f(s, \phi_0(s)) ds.$$

$$\vdots$$

$$\phi_{n+1}(t) = y_0 + \int_{t_0}^t f(s, \phi_n(s)) ds$$

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# Existence and uniqueness of the solution

### Proof.

We have used the following four steps to prove the theorem

- Show all  $\{\phi_n(t)\}_{n=0}^{\infty}$  satisfy  $|\phi_n(t)-y_0| \leq b$ ,  $\forall t \in (t_0-h,t_0+h)$  (We need  $(t,\phi_n(t)) \in R$  for  $t \in (t_0-h,t_0+h)$  in order to show  $\{\phi_n(t)\}_{n=0}^{\infty}$  is uniformly convergent)
- Show the sequence  $\{\phi_n(t)\}_{n=0}^{\infty}$  is uniformly convergent.
- Show the limit function of  $\{\phi_n(t)\}_{n=0}^{\infty}$  is the solution of the integration equation (\*\*).
- Show the uniqueness of the solution.

