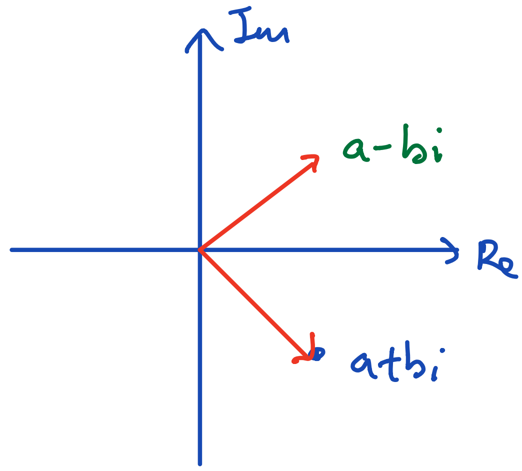
MAT 3253 Lecture 2

$z = a+bi$

$\bar{z} \triangleq a-bi$

$z^* \triangleq a-bi$



$|z| = |a+bi| = \sqrt{a^2+b^2}$ ← modulus, absolute value
radius

* $z \cdot \bar{z} = |z|^2$

$$(a+bi)(a-bi) = a^2+b^2 - abi + abi = a^2+b^2$$

* $\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$ ← $\overline{(a+bi)(c+di)} = [(ac-bd+(ad+bc)i]^*$

$\overline{z_1+z_2} = \bar{z}_1 + \bar{z}_2$

$$= ac-bd-(ad+bc)i$$

$$(a-bi)(c-di) = ac-bd+(-ad-bc)i$$

* $|z_1 z_2| = |z_1| \cdot |z_2|$

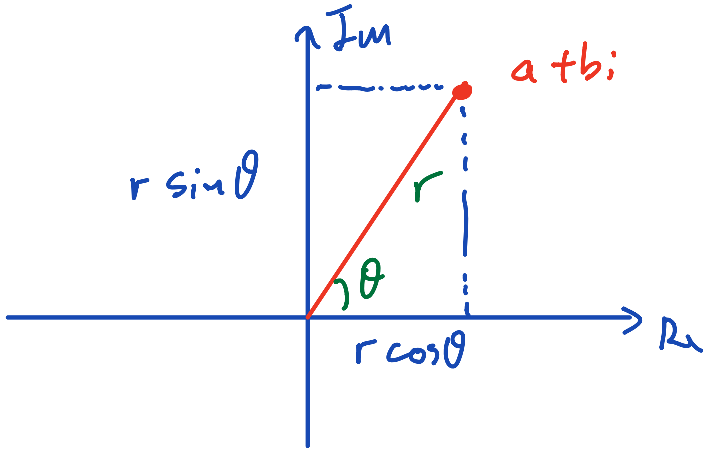
Proof. $|z_1 z_2|^2 = (z_1 z_2)(z_1 z_2)^*$

$$= z_1 z_2 \, z_1^* z_2^*$$

$$= z_1 z_1^* \cdot z_2 z_2^*$$

$$= |z_1|^2 \cdot |z_2|^2$$

* $((z)^*)^* = z$

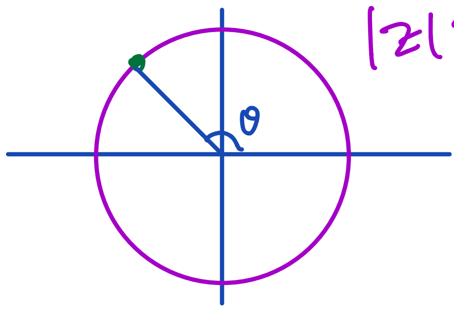


$$a+bi = r\cos\theta + i\,r\sin\theta$$

$$= \underbrace{r(\cos\theta + i\sin\theta)}$$

Note: textbooks use the notation $r\,\mathrm{cis}\,\theta$

$$|\cos\theta + i\sin\theta| = \cos^2\theta + \sin^2\theta = 1$$



$|z|=1$

unit circle

$$z_1 = r_1\cos\theta_1 + i\,r_1\sin\theta_1$$

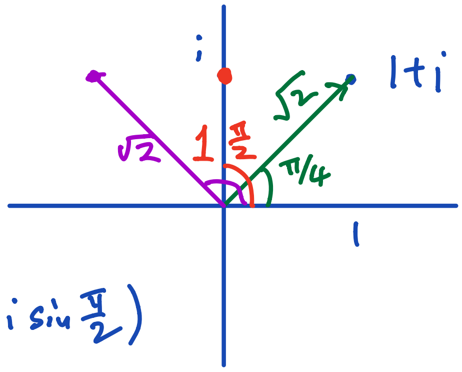
$$z_2 = r_2\cos\theta_2 + i\,r_2\sin\theta_2$$

$$z_1 z_2 = r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$

$$= r_1 r_2\Big[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + (\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)\,i\Big]$$

$$= \underbrace{r_1 r_2}\Big[\cos\underbrace{(\theta_1+\theta_2)} + i\sin(\underbrace{\theta_1+\theta_2})\Big]$$

$$(1+i)\cdot i = i-1$$



$$\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\cdot\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$$

$$=\sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right)+i\sin\left(\frac{3\pi}{4}\right)\right)$$

Fix $a+bi \longleftrightarrow \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

What is the action of multiplying $(a+bi)(x+yi)$?

$$\underbrace{\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}}\left|\begin{matrix} -y \\ x \end{matrix}\right]$$

forget this temporarily

$\boxed{a+bi = r\cos\theta + i\, r\sin\theta}$

$$\underbrace{r}_{\text{expasion}}\underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{\text{rotation matrix}}\begin{bmatrix} x \\ y \end{bmatrix} = r\begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}$$

$\theta = \frac{\pi}{2}$

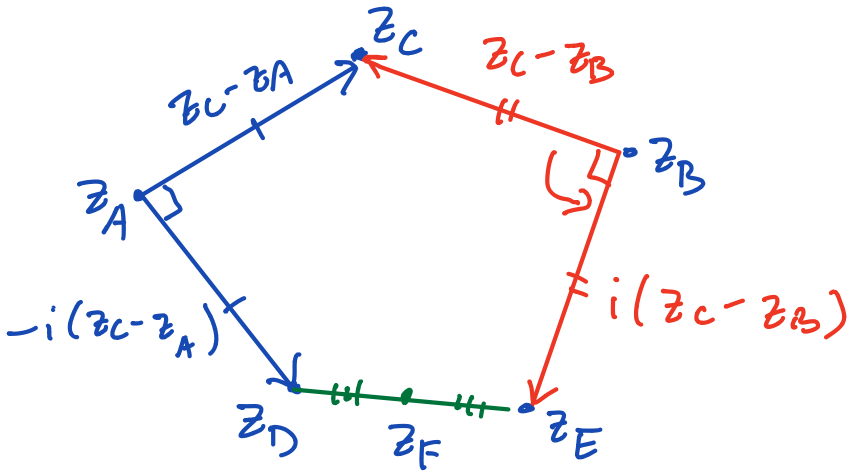
$r=1$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \longleftrightarrow i$$

$\theta = -\frac{\pi}{2}$

$r=1$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \longleftrightarrow -i$$



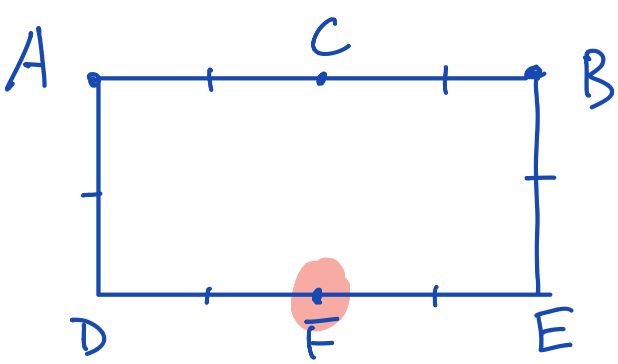
$$z_D = -i(z_C - z_A) + z_A$$

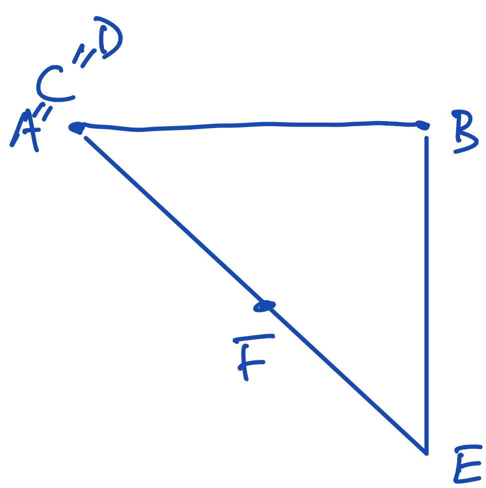
$$z_E = i(z_C - z_B) + z_B$$

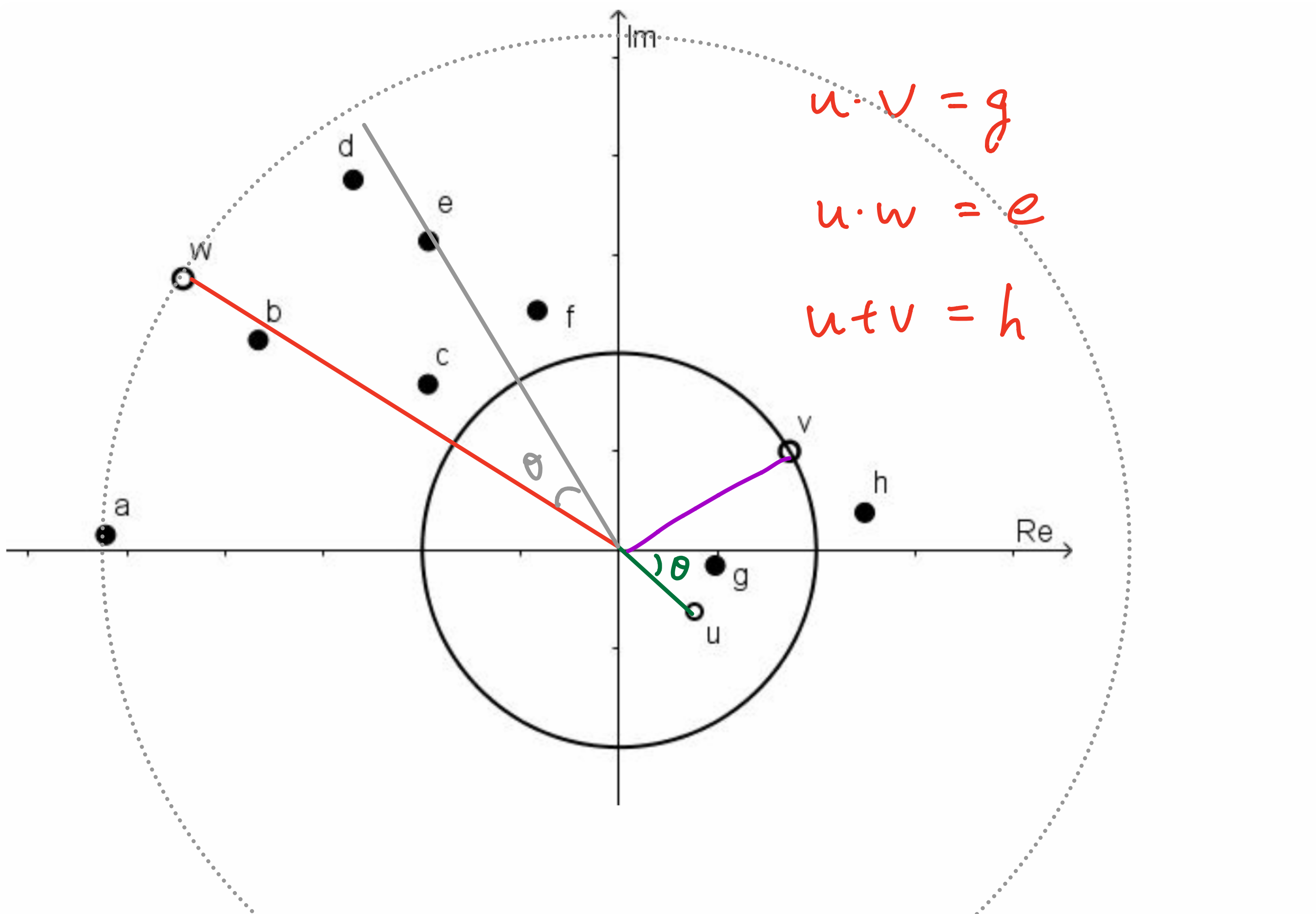
$$z_F = \frac{z_D + z_E}{2} = \frac{1}{2}\left[-i(z_C - z_A) + z_A + i(z_C - z_B) + z_B\right]$$

$$= \frac{1}{2}\left[-iz_C + iz_A + z_A + iz_C - iz_B + z_B\right]$$

$$= \frac{1}{2}\left[(i+1)z_A + (1-i)z_B\right]$$







In this diagram, there are several points and the unit circle (the circle with centre at the origin and radius 1).

The points a to h are the sum or product of combinations of u, v and w.
Can you work out which of the expression below describes each point?

$u+v$
$u+w$
$v+w$
$u+v+w$
uv
uw
vw
uvw

$$(r_1\cos\theta_1 + i r_1 \sin\theta_1)(r_2\cos\theta_2 + i r_2\sin\theta_2)$$

suppose $r_1 = r_2 = 1$

$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$

$$= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)$$

If $\theta_1 = \theta_2$

$$(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta$$

By induction, for positive integer n

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

$$(\cos\theta + i\sin\theta)^{-1} = \frac{1}{\cos\theta + i\sin\theta}$$

$$= \cos\theta - i\sin\theta$$

$$= \cos(-\theta) + i\sin(-\theta)$$

For $n \in \mathbb{Z}$ De Moivre formula

$$\boxed{(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta}$$

Application

$$\cos 5\theta = \mathrm{Re}\left[(\cos\theta + i\sin\theta)^5\right]$$

expand to get an expression in $\cos\theta$ and $\sin\theta$.