MAT2002 Ordinary Differential Equations
System of first order linear equations II

Dongdong He

The Chinese University of Hong Kong (Shenzhen)

March 25, 2021

Overview

1. Basic theory of system of first order linear equations

2. Homogeneous system with constant coefficients
   - Two-by-two matrices

Outline

1. Basic theory of system of first order linear equations

2. Homogeneous system with constant coefficients
   - Two-by-two matrices

System of first order equations

The general first order linear system is

$$\mathbf{y}' = \mathbf{P}(t)\mathbf{y}(t) + \mathbf{g}(t),$$

for given $\mathbf{g}(t) = (g_1(t), \ldots, g_n(t))^T$ and $\mathbf{P}(t)$ is a square matrix of functions

$$\mathbf{P}(t) = \begin{pmatrix} p_{11}(t) & \ldots & p_{1n}(t) \\ \vdots & \ddots & \vdots \\ p_{n1}(t) & \ldots & p_{nn}(t) \end{pmatrix}.$$

In the following, we assume that all $p_{ij}(t)$ and $g(t)$ are continuous in some interval I.
Again, we first look at the corresponding homogeneous first order linear system

$$\mathbf{y}' = \mathbf{P}(t)\mathbf{y}(t).$$

Notations

Recall that

(a) Second order equations $\rightarrow$ 2 L.I. (linearly independent) solutions to the homogeneous equation;

(b) nth order equations $\rightarrow$ n L.I. solutions to the homogeneous equation;

and so for a system of n first order equations, we expect n L.I. solutions to the homogeneous system.

Let us use the following notation:

$$\mathbf{y}_j(t) = j - \text{th solution}, \qquad y_{ij}(t) = i\text{−th component of the } j\text{−th solution}$$

This means that

$$\mathbf{y}_j(t) = \begin{pmatrix} y_{1j}(t) \\ y_{2j}(t) \\ \vdots \\ y_{nj}(t) \end{pmatrix}.$$

Principle of superposition

Theorem 10.1

(Principle of superposition). *Let* $\mathbf{y}_1, \cdots, \mathbf{y}_n$ *be* n *solutions to the homogeneous system*

$$\mathbf{y}'(t) = \mathbf{P}(t)\mathbf{y}(t) \tag{1}$$

then any linear combination

$$\phi(t) = c_1\mathbf{y}_1(t) + \cdots + c_n\mathbf{y}_n(t)$$

is also a solution for any $c_1, \cdots, c_n \in \mathbb{R}$.

The natural question is: Can every solution to the homogeneous system (1) be written as a linear combination of n solutions $\mathbf{y}_1, \cdots, \mathbf{y}_n$? The answer is **YES**, with some analogue of Wronskian for system of equations.

Wronskian

Definition 10.2

(Wronskian). Let $\mathbf{y}_1(t), \ldots, \mathbf{y}_n(t)$ be n solutions to the homogeneous system (1). We define the matrix

$$\mathbf{X}(t) := \begin{pmatrix} y_{11}(t) & y_{12}(t) & \ldots & y_{1n}(t) \\ y_{21}(t) & y_{22}(t) & \ldots & y_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1}(t) & y_{n2}(t) & \ldots & y_{nn}(t) \end{pmatrix} = \begin{pmatrix} | & | & \ldots & | \\ \mathbf{y}_1(t) & \mathbf{y}_2(t) & \ldots & \mathbf{y}_n(t) \\ | & | & \ldots & | \end{pmatrix},$$

where the i-th column of $\mathbf{X}$ is the vector $\mathbf{y}_i(t)$. Then we set the Wronskian $W(\mathbf{y}_1, \ldots, \mathbf{y}_n)[t]$ to be

$$W(\mathbf{y}_1, \ldots, \mathbf{y}_n)[t] := \det \mathbf{X}(t).$$

Remark 1

Note that this definition of Wronskian **does not involve derivatives***!*

Fundamental solution set

For any point $t \in I$

$$\begin{aligned} &\alpha_1 \mathbf{y}_1(t) + \cdots + \alpha_n \mathbf{y}_n(t) = \mathbf{0}, \\ \Leftrightarrow &[\mathbf{y}_1(t), \cdots, \mathbf{y}_n(t)]\mathbf{c} = \mathbf{0}, \end{aligned}$$

where $\mathbf{c} = (c_1, \cdots, c_n)^T$.

It is easy to show that

Fact

$$\begin{aligned} W(\mathbf{y}_1, \ldots, \mathbf{y}_n)[t] \neq 0, \forall\ t \in I &\Leftrightarrow \det(\mathbf{X}(\mathbf{t})) \neq 0, \forall\ t \in I \\ &\Leftrightarrow \{\mathbf{y}_1, \ldots, \mathbf{y}_n\} \textit{ are L.I. at each point in } I. \end{aligned}$$

Fundamental solution set

Theorem 10.3

Let $\mathbf{y}_1(t), \ldots, \mathbf{y}_n(t)$ *be* n *solutions to the homogenous system (1) defined on an open interval* I*. Then,* $\mathbf{y}_1(t), \ldots, \mathbf{y}_n(t)$ *are linearly independent for each point in the interval* I **if and only if** *the Wronskian* $W(\mathbf{y}_1, \ldots, \mathbf{y}_n)[t]$ *is non-zero for* $t \in I$*. In this case, we say that* $\{\mathbf{y}_1(t), \ldots, \mathbf{y}_n(t)\}$ *forms a fundamental set of solutions (FSS), and any solution* $\phi(t)$ *to the homogeneous system (1) can be expressed as a linear combination:*

$$\phi(t) = c_1\mathbf{y}_1(t) + \cdots + c_n\mathbf{y}_n(t),$$

for constants $c_1, \ldots, c_n \in \mathbb{R}$ *in* **exactly one way***. That is, the constants* $c_1, \ldots, c_n$ *are* **uniquely** *determined.*

Fundamental solution set

Proof.

The aim is to show if $\mathbf{y}_1, \ldots, \mathbf{y}_n$ are linearly independent at each point in I (or equivalently $W(\mathbf{y}_1, \ldots, \mathbf{y}_n) \neq 0$), then any solution can be written as a linear combination of $\mathbf{y}_1, \ldots, \mathbf{y}_n$. Let ϕ be any solution to be homogeneous system (1) for $t \in I$, where I is an open interval. Let $t_0 \in I$ and denote the vector

$$\boldsymbol{\xi} := \phi(t_0) = (\xi_1, \ldots, \xi_n)^T.$$

Then, we find values $c_1, \ldots, c_n \in \mathbb{R}$ that satisfies

$$c_1\mathbf{y}_1(t_0) + \cdots + c_n\mathbf{y}_n(t_0) = \xi,$$

or equivalently

$$\begin{aligned} c_1 y_{11}(t_0) + \cdots + c_n y_{1n}(t_0) &= \xi_1, \\ &\vdots \\ c_1 y_{n1}(t_0) + \cdots + c_n y_{nn}(t_0) &= \xi_n \end{aligned}$$

□

Fundamental solution set

Proof.

or also equivalently

$$\begin{pmatrix} y_{11}(t_0) & \cdots & y_{1n}(t_0) \\ \vdots & \ddots & \vdots \\ y_{1n}(t_0) & \cdots & y_{nn}(t_0) \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}.$$

As the Wronskian is not zero at t_0, the matrix is invertible and hence there is a unique solution $(c_1^*, \ldots, c_n^*)^T$ to the above problem. Now we define a new function $\boldsymbol{\eta}$ by

$$\boldsymbol{\eta}(t) = c_1^* \mathbf{y}_1(t) + \cdots + c_n^* \mathbf{y}_n(t), \qquad \forall t \in I.$$

It is clear that $\boldsymbol{\eta}(t_0) = \boldsymbol{\xi} = \phi(t_0)$. Hence, both η and ϕ are solutions to the IVP

$$\mathbf{y}'(t) = \mathbf{P}(t)\mathbf{y}(t), \qquad \mathbf{y}(t_0) = \xi.$$

By uniqueness we must have $\boldsymbol{\eta} = \phi$ and thus

$$\phi(t) = c_1^* \mathbf{y}_1(t) + \cdots + c_n^* \mathbf{y}_n(t), \qquad \forall t \in I.$$

□

Fundamental matrix

$$\mathbf{y}'(t) = \mathbf{P}(t)\mathbf{y}(t) \tag{2}$$

Definition 10.4

Suppose that $\mathbf{y}_1(t), \ldots, \mathbf{y}_n(t)$ form a fundamental set of solutions for the homogeneous linear system (1). Then the matrix

$$\Psi(t) = \begin{pmatrix} y_{11}(t) & y_{12}(t) & \ldots & y_{1n}(t) \\ y_{21}(t) & y_{22}(t) & \ldots & y_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1}(t) & y_{n2}(t) & \ldots & y_{nn}(t) \end{pmatrix} = \begin{pmatrix} | & | & \ldots & | \\ \mathbf{y}_1(t) & \mathbf{y}_2(t) & \ldots & \mathbf{y}_n(t) \\ | & | & \ldots & | \end{pmatrix} \tag{3}$$

whose columns are the vectors $\mathbf{y}_1(t), \ldots, \mathbf{y}_n(t)$ is called a **fundamental matrix** of the system (1).

Liouville's formula

Since we have an analogue of the Wronskian for system of equations, we should expect an analogue of Abel's theorem as well. For systems of equations, this is called **Liouville's formula**.

Theorem 10.5

(Liouville's formula). *Let $\mathbf{y}_1, \ldots, \mathbf{y}_n$ be n solutions to the homogeneous equation (1) in the open interval I. Then, the Wronskian is given by*

$$W(\mathbf{y}_1, \ldots, \mathbf{y}_n)[t] = c \exp\left(\int tr(\mathbf{P}(t)) dt \right),$$

where the trace of a matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ is defined as

$$tr(\mathbf{A}) := \sum_{i=1}^{n} a_{ii} \qquad \text{(sum of the diagonal entries)},$$

and c is a constant not depending on $t \in I$. Consequently, the Wronskian is either always zero for $t \in I$ or never zero for $t \in I$.

Proof for Liouville's formula

Proof.

We will only prove this for the case $n = 2$: Let $\mathbf{y}_1, \mathbf{y}_2$ be two solutions to the homogeneous system (1), i.e.,

$$\mathbf{y}'(t) = \mathbf{P}(t)\mathbf{y}(t), \quad \mathbf{P}(t) \in \mathbb{R}^{2\times 2} \text{ for } t \in I.$$

Then, the Wronskian is

$$W(\mathbf{y}_1, \mathbf{y}_2)[t] = \begin{vmatrix} y_{11}(t) & y_{12}(t) \\ y_{21}(t) & y_{22}(t) \end{vmatrix} = y_{11}(t)y_{22}(t) - y_{12}(t)y_{21}(t).$$

Taking the derivative leads to

$$\begin{aligned}
\frac{d}{dt}W[t] &= y_{11}'(t)y_{22}(t) - y_{12}'(t)y_{21}(t) + y_{11}(t)y_{22}'(t) - y_{12}(t)y_{21}'(t) \\
&= \begin{vmatrix} y_{11}'(t) & y_{12}'(t) \\ y_{21}(t) & y_{22}(t) \end{vmatrix} + \begin{vmatrix} y_{11}(t) & y_{12}(t) \\ y_{21}'(t) & y_{22}'(t) \end{vmatrix} \\
&= \begin{vmatrix} p_{11}y_{11} + p_{12}y_{21} & p_{11}y_{12} + p_{12}y_{22} \\ y_{21}(t) & y_{22}(t) \end{vmatrix} + \begin{vmatrix} y_{11}(t) & y_{12}(t) \\ p_{21}y_{11} + p_{22}y_{21} & p_{21}y_{12} + p_{22}y_{22} \end{vmatrix} \\
&= (p_{11} + p_{22})(y_{11}y_{22} - y_{12}y_{21}) = (p_{11} + p_{22})W[t],
\end{aligned}$$

Proof for Liouville's formula

Proof.

where we have used from the fact that $\mathbf{y}_1, \mathbf{y}_2$ solve $\mathbf{y}'(t) = \mathbf{P}(t)\mathbf{y}(t)$ to deduce

$$\begin{pmatrix} y_{11}' \\ y_{21}' \end{pmatrix} = \begin{pmatrix} p_{11}y_{11} + p_{12}y_{21} \\ p_{21}y_{11} + p_{22}y_{21} \end{pmatrix}, \quad \begin{pmatrix} y_{12}' \\ y_{22}' \end{pmatrix} = \begin{pmatrix} p_{11}y_{12} + p_{12}y_{22} \\ p_{21}y_{12} + p_{22}y_{22} \end{pmatrix}.$$

This implies we have

$$\frac{d}{dt}W[t] = (p_{11} + p_{22})W[t] = tr(\mathbf{P}(t))W[t].$$

□

Next question: "does the fundamental set of solutions always exists?". Answer is Yes.

Existence of fundamental set of solutions

Theorem 10.6

(Existence of at least one fundamental set of solutions). *Let*

$$\mathbf{e}_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

where the entry 1 appears in the $i-$th row, and let $\mathbf{y}_i$ be the unique solution to the IVP

$$\mathbf{y}'(t) = \mathbf{P}(t)\mathbf{y}(t) \; for \; t \in I,$$
$$\mathbf{y}(t_0) = \mathbf{e}_i,$$

for $t_0 \in I$. Then, the functions $\mathbf{y}_1(t), \ldots, \mathbf{y}_n(t)$ form a fundamental set of solutions to the homogeneous system $\mathbf{y}'(t) = \mathbf{P}(t)\mathbf{y}(t)$.

Existence of fundamental set of solutions

Proof.

Simply compute the Wronskian at t_0:

$$W(\mathbf{y}_1, \ldots, \mathbf{y}_n)[t_0] = \det \mathbf{I} = 1 \neq 0.$$

□

Note that the fundamental set of solutions is not unique.
Let $\mathbf{y}_i$ $(i = 1, \cdots, n)$be the unique solution to the IVP

$$\mathbf{y}'(t) = \mathbf{P}(t)\mathbf{y}(t) \text{ for } t \in I,$$
$$\mathbf{y}(t_0) = \mathbf{s}_i,$$

for $t_0 \in I$. Then, the functions $\mathbf{y}_1(t), \ldots, \mathbf{y}_n(t)$ form a fundamental set of solutions as long as $W(\mathbf{y}_1, \ldots, \mathbf{y}_n)[t_0] = \det[\mathbf{s}_1|\mathbf{s}_2| \cdots |\mathbf{s}_n] \neq 0$.

Complex-valued solution

Just as for second order equations, you will see that a linear ODE system with real-valued coefficients may give rise to complex-valued solutions. But again we also have the following theorem.

Theorem 10.7

If $\mathbf{y}(t) = \mathbf{u}(t) + i\mathbf{v}(t)$ is a complex-valued solution to the homogeneous system (1), where the entries of $\mathbf{P}(t)$ are **real-valued** *functions, and the vectors $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are also real-valued, then $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are both solutions to the homogeneous system (1).*

Summary

Summary: The fundamental set of solutions $\mathbf{y}_1, \ldots, \mathbf{y}_n$ to $\mathbf{y}'(t) = \mathbf{P}(t)\mathbf{y}(t)$ always exists and any solution ϕ to the homogeneous system can be written **uniquely** as a linear combination of $\mathbf{y}_1, \ldots, \mathbf{y}_n$.

Next question: How to find the FSS? Indeed, again, no method can be used for general matrix $\mathbf{P}(t)$. We can only deal with the case when $\mathbf{P}(t)$ is a matrix with constant entries.

Outline

1 Basic theory of system of first order linear equations

2 Homogeneous system with constant coefficients

- Two-by-two matrices

Homogeneous system with constant coefficients

In the following, we focus on $\mathbf{P}(t) = \mathbf{A}$, where $\mathbf{A}$ is a square matrix with real, constant coefficients (not functions of t), and our goal is to derive explicit formula for the FSS $(\mathbf{y}_1, \ldots, \mathbf{y}_n)$.

Homogeneous system with constant coefficients

We now focus on systems of the form

$$\mathbf{y}'(t) = \mathbf{A}\mathbf{y}(t), \qquad t \in I, \tag{4}$$

where $\mathbf{A} \in \mathbb{R}^{n\times n}$. There are one special case which we can already deal with.

In the case $n = 1$, then $\mathbf{A}$ is just a scalar, i.e., $\mathbf{A} = a \in \mathbb{R}$, then (4) becomes

$$y'(t) = ay(t) \Rightarrow y(t) = ce^{at}, c \in \mathbb{R}.$$

Linear ODEs with general matrix

What about a general matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$? The idea is to try

$$\mathbf{y}(t) = \boldsymbol{\xi} e^{rt},$$

where ξ is a **constant vector** (not depending on t) and $r \in \mathbb{C}$. We have to determine the constant r and the constant vector ξ to obtain a solution. Substituting this function into the equation yields

$$\mathbf{0} = \mathbf{y}'(t) - \mathbf{A}\mathbf{y}(t) = e^{rt}(r\boldsymbol{\xi} - \mathbf{A}\boldsymbol{\xi}) = e^{rt}(\mathbf{A} - r\mathbf{I})\boldsymbol{\xi}.$$

Since the exponential term is never zero, we see that for $\boldsymbol{\xi} e^{rt}$ to be a solution to the homogeneous system, we require

$$\boxed{(\mathbf{A} - r\mathbf{I})\boldsymbol{\xi} = \mathbf{0}},$$

i.e., the constant r should be an **eigenvalue** of the matrix $\mathbf{A}$ with corresponding **eigenvector** ξ.

Let's first discuss the simple cases $\mathbf{A} \in \mathbb{R}^{2 \times 2}$, and late we will discuss the general case $\mathbf{A} \in \mathbb{R}^{n \times n}$.

Two-by-two matrices

Let $\mathbf{A} \in \mathbb{R}^{2\times 2}$ be a two-by-two matrix with real entries. Then, $\mathbf{A}$ has two eigenvalues. What are the possibilities for the eigenvalues r_1 and r_2?

(1) $r_1, r_2 \in \mathbb{R}$, $r_1 \neq r_2$ - real and distinct;

(2) $r_1, r_2 \in \mathbb{C}$, $r_1 = \delta + i\mu$, $\delta, \mu \in \mathbb{R}$ with $r_2 = \delta - i\mu$ - complex conjugate pair;

(3) $r_1 = r_2 \in \mathbb{R}$ - repeated and real.

3(a) $r_1 = r_2 \in \mathbb{R}$, there are two linearly independent eigenvectors.

3(b) $r_1 = r_2 \in \mathbb{R}$, there is only one linearly independent eigenvector.

Case 1: Real distinct eigenvalues

2×2 **matrix: Case 1 - Real distinct eigenvalues**. Let $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_1$ be the corresponding eigenvectors to r_1 and r_2. Note that $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_1$ are linearly independent. Then, we can compute the Wronskian to see that for the functions $\mathbf{y}_1(t) = \boldsymbol{\xi}_1 e^{r_1 t}$ and $\mathbf{y}_2(t) = \boldsymbol{\xi}_2 e^{r_2 t}$,
$(\boldsymbol{\xi}_1 = [\xi_{11}, \xi_{21}]^T, \boldsymbol{\xi}_2 = [\xi_{21}, \xi_{22}]^T)$

$$
\begin{aligned}
W(\mathbf{y}_1, \mathbf{y}_2)[t] = \begin{vmatrix} \xi_{11} e^{r_1 t} & \xi_{12} e^{r_2 t} \\ \xi_{21} e^{r_1 t} & \xi_{22} e^{r_2 t} \end{vmatrix} = e^{r_1 t} \begin{vmatrix} \xi_{11} & \xi_{12} e^{r_2 t} \\ \xi_{21} & \xi_{22} e^{r_2 t} \end{vmatrix} \\
= e^{(r_1 + r_2)t} \begin{vmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{vmatrix} \neq 0.
\end{aligned}
$$

for any $t \in I$. Hence, by Theorem 10.3, the general solution to the homogeneous system (4) is

$$
\boxed{\mathbf{y}(t) = c_1 e^{r_1 t} \boldsymbol{\xi}_1 + c_2 e^{r_2 t} \boldsymbol{\xi}_2}.
$$

2×2 matrix: Case 1: Real distinct eigenvalues

Example 10.8

For

$$\mathbf{y}' = \mathbf{A}\mathbf{y}, \quad \mathbf{A} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$$

with eigenvalues and corresponding eigenvectors

$$r_1 = 3, \qquad \boldsymbol{\xi}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \qquad r_2 = -1, \qquad \boldsymbol{\xi}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix},$$

the general solution is

$$\mathbf{y}(t) = c_1 e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

2×2 matrix: Case 2-Comples conjugate eigenvalues

Case 2 - Comples conjugate eigenvalues. Let $r_1 = \delta + i\mu$, with $\delta, \mu \in \mathbb{R}$ and corresponding eigenvector $\boldsymbol{\xi}_1 = \mathbf{u} + i\mathbf{v}$. Then $r_2 = \delta - i\mu$, with $\delta, \mu \in \mathbb{R}$ and corresponding eigenvector $\boldsymbol{\xi}_2 = \mathbf{u} - i\mathbf{v}$. ($\mathbf{u} + i\mathbf{v}$ and $\mathbf{u} - i\mathbf{v}$ are linearly independent)
Moreover, $\mathbf{x}_1(t) = (\mathbf{u} + i\mathbf{v})e^{(\delta + i\mu)t}, \quad \mathbf{x}_2(t) = (\mathbf{u} - i\mathbf{v})e^{(\delta - i\mu)t}$ are linearly independent solutions.
The general solution of the homogeneous system (4) is

$$\mathbf{y}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t).$$

But the disadvantage of using $\mathbf{x}_1$ and $\mathbf{x}_2$ is that they are complex-valued.

Complex-valued solution

Now we can rewrite the above solutions $\mathbf{x}_1$ and $\mathbf{x}_2$ as

$$\begin{aligned}
\mathbf{x}_1 &= (\mathbf{u} + i\mathbf{v})e^{\delta t}(\cos(\mu t) + i\sin(\mu t)) \\
&= e^{\delta t}[\mathbf{u}\cos(\mu t) - \mathbf{v}\sin(\mu t)] + ie^{\delta t}[\mathbf{u}\sin(\mu t) + \mathbf{v}\cos(\mu t)]. \\
\mathbf{x}_2 &= (\mathbf{u} - i\mathbf{v})e^{\delta t}(\cos(\mu t) - i\sin(\mu t)) \\
&= e^{\delta t}[\mathbf{u}\cos(\mu t) - \mathbf{v}\sin(\mu t)] - ie^{\delta t}[\mathbf{u}\sin(\mu t) + \mathbf{v}\cos(\mu t)].
\end{aligned}$$

Using the above Theorem 10.7 we can see that the real and imaginary parts of $\mathbf{x}_1$ are also solutions. Hence, we define

$$\mathbf{y}_1(t) = e^{\delta t}(\mathbf{u}\cos(\mu t) - \mathbf{v}\sin(\mu t)), \quad \mathbf{y}_2(t) = e^{\delta t}(\mathbf{u}\sin(\mu t) + \mathbf{v}\cos(\mu t)).$$

Since $\mathbf{u} + i\mathbf{v}$ and $\mathbf{u} - i\mathbf{v}$ are linearly independent, $0 \neq \det([\mathbf{u} + i\mathbf{v}, \mathbf{u} - i\mathbf{v}]) = \det(2\mathbf{u}, \mathbf{u} - i\mathbf{v}) = 2\det([\mathbf{u}, \mathbf{u} - i\mathbf{v}]) = 2\det([\mathbf{u}, -i\mathbf{v}]) = -2i\det([\mathbf{u}, \mathbf{v}])$. Thus, $\det(\mathbf{u}, \mathbf{v}) \neq 0$.

Therefore, the real and imaginary parts of a complex eigenvector are linearly independent.

2×2 matrix: Case 2-Comples conjugate eigenvalues

We can check that the Wronskian for $\mathbf{y}_1$ and $\mathbf{y}_2$ is non-zero.

$$
\begin{aligned}
& \det([\mathbf{y}_1, \mathbf{y}_2]) \\
=& e^{2\delta t} \det([\mathbf{u}\cos(\mu t) - \mathbf{v}\sin(\mu t), \mathbf{u}\sin(\mu t) + \mathbf{v}\cos(\mu t)]) \\
=& e^{2\delta t} \det([\mathbf{u}, \mathbf{v}]) \det\left(\begin{bmatrix} \cos(\mu t) & \sin(\mu t) \\ -\sin(\mu t) & \cos(\mu t) \end{bmatrix} \right) \\
=& e^{2\delta t} \det([\mathbf{u}, \mathbf{v}]) \neq 0.
\end{aligned}
$$

Then, by Theorem 10.3 we see that the general solution to the homogeneous system (4) is

$$
\begin{aligned}
\mathbf{y}(t) =& c_1\mathbf{y}_1(t) + c_2\mathbf{y}_2(t) \\
=& e^{\delta t}\left(c_1\left(\cos(\mu t)\mathbf{u} - \sin(\mu t)\mathbf{v}\right) + c_2\left(\cos(\mu t)\mathbf{v} + \sin(\mu t)\mathbf{u}\right)\right)
\end{aligned}
$$

2×2 matrix: Case 2-Comples conjugate eigenvalues

Example 10.9

For

$$\mathbf{y}' = \mathbf{A}\mathbf{y}, \quad \mathbf{A} = \begin{pmatrix} -3 & -2 \\ 4 & 1 \end{pmatrix}$$

with eigenvalues $r_1 = \overline{r_2}$ and corresponding eigenvectors $\mathbf{x}_1 = \overline{\mathbf{x}_2}$:

$$r_{1,2} = -1 \pm 2i, \qquad \xi_{1,2} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \pm i \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

For the general complex solution is

$$\begin{aligned} \mathbf{y}(t) =& c_1 e^{(-1+2i)t} \left(\begin{pmatrix} -1 \\ 1 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right), \\ &+ c_2 e^{(-1-2i)t} \left(\begin{pmatrix} -1 \\ 1 \end{pmatrix} - i \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right). \end{aligned}$$

2×2 matrix: Case 2-Comples conjugate eigenvalues

Example 10.9

For the general real solution is

$$\begin{aligned}\mathbf{y}(t) =& c_1 e^{-t} \left(\cos(2t) \begin{pmatrix} -1 \\ 1 \end{pmatrix} - \sin(2t) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \\ &+ c_2 e^{-t} \left(\sin(2t) \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \cos(2t) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right).\end{aligned}$$

2×2 matrix: Case 3(a) - Repeated real eigenvalues. geo. mult.=alg. mult.

Case 3 - Repeated real eigenvalues. If $r_1 = r_2$, then we have an eigenvalue with algebraic multiplicity of two. We need to divide our analysis into two subcases:

(a) The geometric multiplicity is also two, which implies there are two linearly independent eigenvectors $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2$ corresponding to $r_1 = r_2 =: \lambda$. Then, going back to Case 1, the general solution to the homogeneous system (4) is

$$\mathbf{y}(t) = c_1 e^{\lambda t} \boldsymbol{\xi}_1 + c_2 e^{\lambda t} \boldsymbol{\xi}_2.$$

2×2 matrix: Case 3(a) - Repeated real eigenvalues. geo. mult.=alg. mult.

Example 10.10

For

$$\mathbf{y}' = \mathbf{A}\mathbf{y}, \quad \mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

with eigenvalues and corresponding eigenvectors

$$r_1 = 2, \qquad \boldsymbol{\xi}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad r_2 = 2, \qquad \boldsymbol{\xi}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

the general solution is

$$\mathbf{y}(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

2×2 matrix: Case 3(b) - Repeated real eigenvalues. geo. mult. $<$ alg. mult.

Remark: In all above cases for the 2×2 matrix $\mathbf{A}$, the number of linearly independent eigenvectors equals to $n = 2$. Thus, $\mathbf{A}$ is diagonalizable, the solutions can be found easily. However, if there is a repeated eigenvalue $r_1 = r_2 =: \lambda$ with geometric multiplicity **strictly less** than its algebraic multiplicity, $\mathbf{A}$ is not diagonalizable. This case is more complicated.

2×2 matrix: Case 3(b) - Repeated real eigenvalues. geo. mult. $<$ alg. mult.

(b) If the geometric multiplicity of the eigenvalue λ is one, then there is only one eigenvector $\boldsymbol{\xi}$ corresponding to the eigenvalue λ. We know one solution is

$$\mathbf{y}_1 = \boldsymbol{\xi} e^{\lambda t},$$

what about a second solution that is linearly independent at each point t? As with second order equations, let's first try

$$\mathbf{z}(t) = t\boldsymbol{\xi} e^{\lambda t}.$$

Differentiating and plugging this into the homogeneous system (4) leads to

$$\mathbf{z}'(t) - \mathbf{A}\mathbf{z}(t) = \boldsymbol{\xi}(\lambda t e^{\lambda t} + e^{\lambda t}) - \mathbf{A}\boldsymbol{\xi} t e^{\lambda t} = (\boldsymbol{\xi}\lambda - \mathbf{A}\boldsymbol{\xi}) t e^{\lambda t} + \boldsymbol{\xi} e^{\lambda t}.$$

2×2 matrix: Case 3(b) - Repeated real eigenvalues. geo. mult. $<$ alg. mult.

We observe there are two terms: one involving the coefficient $te^{\lambda t}$ and the other involving just the coefficient $e^{\lambda t}$. Since we want $\mathbf{z}$ to be a solution, both terms must vanish. Hence, we require

$$\mathbf{A}\boldsymbol{\xi} = \lambda\boldsymbol{\xi}, \qquad \boldsymbol{\xi} = \mathbf{0}.$$

The first condition amounts to saying $\boldsymbol{\xi}$ is an eigenvector for λ, which is true by definition, but the second condition leads to a contradiction. Therefore, we deduce that the solution to the homogeneous system (4) cannot be of the form $t\boldsymbol{\xi}e^{\lambda t}$.

To modify the second solution form, we try

$$\mathbf{w}(t) = (\boldsymbol{\xi} t + \boldsymbol{\eta})\, e^{\lambda t},$$

for some constant vector $\boldsymbol{\eta}$ to be determined. Then, computing $\mathbf{w}'(t) - \mathbf{A}\mathbf{w}(t)$ gives

$$\mathbf{w}'(t) - \mathbf{A}\mathbf{w}(t) = te^{\lambda t}(\lambda\boldsymbol{\xi} - \mathbf{A}\boldsymbol{\xi}) + e^{\lambda t}(\lambda\boldsymbol{\eta} - \mathbf{A}\boldsymbol{\eta} + \boldsymbol{\xi}).$$

2×2 matrix: Case 3(b) - Repeated real eigenvalues. geo. mult. $<$ alg. mult.

Hence, for $\mathbf{w}$ to be a solution we need

$$\mathbf{A}\boldsymbol{\xi} = \lambda\boldsymbol{\xi}, \qquad (\mathbf{A} - \lambda\mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}.$$

We need to ask two questions:

- Does $\boldsymbol{\eta}$ such that $(\mathbf{A} - \lambda\mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}$ exists?
- If the $\boldsymbol{\eta}$ exists, are two solutions $\boldsymbol{\xi}e^{\lambda t}$ and $(\boldsymbol{\xi}t + \boldsymbol{\eta})\, e^{\lambda t}$ linearly independent?

2×2 matrix: Case 3(b) - Repeated real eigenvalues. geo. mult. $<$ alg. mult.

For the first question, take another vector $\mathbf{v}$ that is not a constant multiple of the eigenvector $\boldsymbol{\xi}$ ($\mathbf{v} \notin \mathbf{Span}\ \{\boldsymbol{\xi}\}$ and geometrical multiplicity of λ=1 implies $\mathbf{v}$ is not the eigenvector). Then, since $\boldsymbol{\xi}$ is a vector in $\mathbb{R}^2$, we see that $\mathbf{v}$ and $\boldsymbol{\xi}$ must be linearly independent (if $\mathbf{v}$ is not a constant multiple of $\boldsymbol{\xi}$), and hence they also form a basis of $\mathbb{R}^2$. So every vector $\mathbf{x} \in \mathbb{R}^2$ can be written as a linear combination of $\mathbf{v}$ and $\boldsymbol{\xi}$.

Define the vector $\mathbf{u} = (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}$ ($\mathbf{u} \neq \mathbf{0}$). Then, we can find constants $\alpha, \beta \in \mathbb{R}$ such that

$$\mathbf{u} = \alpha\mathbf{v} + \beta\boldsymbol{\xi}.$$

Now apply $\mathbf{A} - \lambda\mathbf{I}$ to both sides gives

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{u} = \alpha(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} + \beta(\mathbf{A} - \lambda\mathbf{I})\boldsymbol{\xi} = \alpha(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \alpha\mathbf{u},$$

since $\boldsymbol{\xi}$ is an eigenvector of $\mathbf{A}$.

2×2 matrix: Case 3(b) - Repeated real eigenvalues. geo. mult. $<$ alg. mult.

Rearranging gives

$$\mathbf{A}\mathbf{u} = (\lambda + \alpha)\mathbf{u},$$

and so $\mathbf{u}$ is an eigenvector corresponding to eigenvalue $\alpha + \lambda$. But, since $\mathbf{A}$ has only one repeated eigenvalue λ, there is no other possible eigenvalues and hence α **must be zero**. From this, we see that

$$\mathbf{u} = \beta\boldsymbol{\xi}, \qquad \beta \neq 0.$$

that is $\mathbf{u}$ is parallel to $\boldsymbol{\xi}$. Recalling the definition of $\mathbf{u}$, we see that

$$\mathbf{u} = (\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \beta\boldsymbol{\xi},$$

and if we set $\boldsymbol{\eta} = \frac{1}{\beta}\mathbf{v}$, we see that

$$(\mathbf{A} - \lambda\mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}.$$

2×2 matrix: Case 3(b) - Repeated real eigenvalues. geo. mult. $<$ alg. mult.

For the second question, since $\boldsymbol{\xi}$ is an eigenvector corresponding to λ, the vector $\boldsymbol{\eta}$ exists such that

$$\mathbf{A}\boldsymbol{\xi} = \lambda\boldsymbol{\xi}, \qquad (\mathbf{A} - \lambda\mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}.$$

then we have two solutions

$$\mathbf{y}_1(t) = \boldsymbol{\xi} e^{\lambda t}, \qquad \mathbf{y}_2(t) = (t\boldsymbol{\xi} + \boldsymbol{\eta}) e^{\lambda t},$$

where $\boldsymbol{\xi} = [\xi_1, \xi_2]^T, \boldsymbol{\eta} = [\eta_1, \eta_2]^T$.
Claim: $\mathbf{y}_1(t) = \boldsymbol{\xi} e^{\lambda t}, \qquad \mathbf{y}_2(t) = (t\boldsymbol{\xi} + \boldsymbol{\eta}) e^{\lambda t}$ forms a FSS.
Computing the Wronskian gives

$$W(\mathbf{y}_1, \mathbf{y}_2)[t] = e^{2\lambda t} \begin{vmatrix} \xi_1 & t\xi_1 + \eta_1 \\ \xi_2 & t\xi_2 + \eta_2 \end{vmatrix} = e^{2\lambda t} \begin{vmatrix} \xi_1 & \eta_1 \\ \xi_2 & \eta_2 \end{vmatrix},$$

and so the Wronskian is non-zero if and only if $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are linearly independent.

2×2 matrix: Case 3(b) - Repeated real eigenvalues. geo. mult. $<$ alg. mult.

Now suppose there are constants α_1, α_2 such that $\alpha_1 \boldsymbol{\xi} + \alpha_2 \boldsymbol{\eta} = \mathbf{0}$. Since $\mathbf{A} \neq \lambda \mathbf{I}$ (otherwise $\boldsymbol{\eta}$ would not exist), applying $\mathbf{A} - \lambda \mathbf{I}$ leads to

$$\mathbf{0} = \alpha_1 (\mathbf{A} - \lambda \mathbf{I}) \boldsymbol{\xi} + \alpha_2 (\mathbf{A} - \lambda \mathbf{I}) \boldsymbol{\eta} = \alpha_2 \boldsymbol{\xi},$$

since $\boldsymbol{\xi}$ is an eigenvector corresponding to λ. This implies that $\alpha_2 = 0$, since $\boldsymbol{\xi}$ is non-zero. Then, going back we see that

$$\alpha_1 \boldsymbol{\xi} = \mathbf{0} \Rightarrow \alpha_1 = 0.$$

Hence, we see that $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are linearly independent.

$$\mathbf{y}_1(t) = \boldsymbol{\xi} e^{\lambda t}, \qquad \mathbf{y}_2(t) = (t\boldsymbol{\xi} + \boldsymbol{\eta}) e^{\lambda t}$$

form a fundamental solution set.
Thus, by Theorem 10.3, the general solution is

$$\mathbf{y}(t) = c_1 \boldsymbol{\xi} e^{\lambda t} + c_2 (t\boldsymbol{\xi} + \boldsymbol{\eta}) e^{\lambda t}.$$

Remark: $\mathbf{A}\xi = \lambda\xi$, $(\mathbf{A} - \lambda\mathbf{I})\eta = \boldsymbol{\xi}$ gives $(\mathbf{A} - \lambda\mathbf{I})^2\eta = \mathbf{0}$. $\boldsymbol{\eta}$ is called the **generalized eigenvector**.

Definition

Let λ be an eigenvalue of matrix $\mathbf{A}$, a nonzero vector $\boldsymbol{\eta}$ is called a **generalized eigenvector** if there is a positive integer p such that

$$(\mathbf{A} - \lambda\mathbf{I})^p\boldsymbol{\eta} = \mathbf{0}.$$

And the **generalized eigenvector** η is called the **generalized eigenvector** of rank p (p is some positive integer) the matrix $\mathbf{A}$ and corresponding to the eigenvalue λ if

- $(\mathbf{A} - \lambda\mathbf{I})^p\boldsymbol{\eta} = \mathbf{0}$
- $(\mathbf{A} - \lambda\mathbf{I})^{p-1}\boldsymbol{\eta} \neq \mathbf{0}$

Here, since $(\mathbf{A} - \lambda\mathbf{I})\eta = \boldsymbol{\xi} \neq \mathbf{0}$ and $(\mathbf{A} - \lambda\mathbf{I})^2\eta = \mathbf{0}$, η is the **generalized eigenvector** of rank 2. Since $(\mathbf{A} - \lambda\mathbf{I})\boldsymbol{\xi} = \mathbf{0}$ and $(\mathbf{A} - \lambda\mathbf{I})^0\boldsymbol{\xi} \neq \mathbf{0}$, $\boldsymbol{\xi}$ is the **generalized eigenvector** of rank 1 (just usual eigenvector).

2×2 matrix: Case 3(b) - Repeated real eigenvalues. geo. mult. $<$ alg. mult.

Example 10.11

For

$$\mathbf{y}' = \mathbf{A}\mathbf{y}, \quad \mathbf{A} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix},$$

the eigenvalues are $r_1 = r_2 = \lambda = 2$, i.e., algebraic multiplicity is two, while the eigenvector corresponding to λ is

$$\boldsymbol{\xi} = \begin{pmatrix} -1 \\ 1 \end{pmatrix},$$

and so the geometric multiplicity is one. We now need to find a vector $\boldsymbol{\eta}$ such that

$$(\mathbf{A} - 2\mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}.$$

2×2 matrix: Case 3(b) - Repeated real eigenvalues. geo. mult. $<$ alg. mult.

Example continue

Computing $\mathbf{A} - 2\mathbf{I}$ gives

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow -\eta_1 - \eta_2 = 1.$$

We can take $\eta_1 = 0$ and $\eta_2 = -1$, leading to the general solution

$$\mathbf{y}(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \left(te^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{2t} \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right).$$

The general case: $n \times n$ matrix

Question: Can we have a systematic way to solve for $\mathbf{y}' = \mathbf{A}\mathbf{y}$ with a general matrix $\mathbf{A}_{\mathbf{n}\times\mathbf{n}}$ with real constant entries?

Answer: Yes.
There are two methods that can be used, one is using the matrix exponential (need to compute $e^{\mathbf{A}t}$), the other is using the eigenvalue and eigenvectors.