CSC 4020 Fundamentals of Machine Learning:
Bayesian Networks

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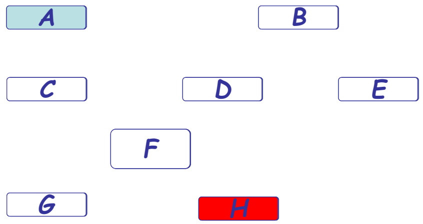
[Slide credit: Eric Xing]

Representing Multivariate Distribution

- **Representation: what is the joint probability dist. on multiple variables?**

$$P(X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8,)$$

  - **How many state configurations in total? --- 2^8**
  - **Are they all needed to be represented?**
  - **Do we get any scientific/medical insight?**



- **Factored representation: the chain-rule**

$$\begin{aligned} & P(X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8) \\ = & P(X_1)P(X_2 \mid X_1)P(X_3 \mid X_1, X_2)P(X_4 \mid X_1, X_2, X_3)P(X_5 \mid X_1, X_2, X_3, X_4)P(X_6 \mid X_1, X_2, X_3, X_4, X_5) \\ & P(X_7 \mid X_1, X_2, X_3, X_4, X_5, X_6)P(X_8 \mid X_1, X_2, X_3, X_4, X_5, X_6, X_7) \end{aligned}$$

  - **This factorization is true for any distribution and any variable ordering**
  - **Do we save any parameterization cost?**

- **If X_i's are independent: ($P(X_i|\cdot)= P(X_i)$)**

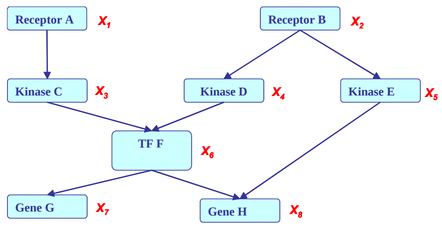
$$\begin{aligned} & P(X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8) \\ = & P(X_1)P(X_2)P(X_3)P(X_4)P(X_5)P(X_6)P(X_7)P(X_8) = \prod P(X_i) \end{aligned}$$

  - **What do we gain?**
  - **What do we lose?**

Two types of GMs

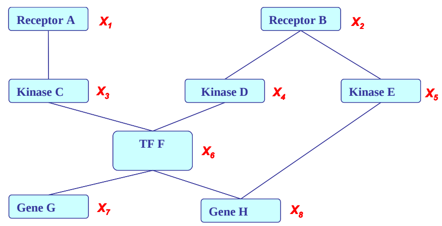
- **Directed edges give causality relationships (Bayesian Network or Directed Graphical Model):**

$$P(X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8)$$
$$= P(X_1)\, P(X_2)\, P(X_3|\, X_1)\, P(X_4|\, X_2)\, P(X_5|\, X_2)\, P(X_6|\, X_3, X_4)\, P(X_7|\, X_6)\, P(X_8|\, X_5, X_6)$$

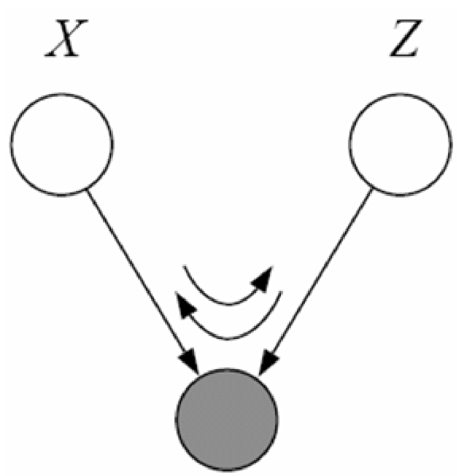


- **Undirected edges simply give correlations between variables (Markov Random Field or Undirected Graphical model):**

$$P(X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8)$$
$$= 1/Z \exp\{E(X_1)+E(X_2)+E(X_3, X_1)+E(X_4, X_2)+E(X_5, X_2) + E(X_6, X_3, X_4)+E(X_7, X_6)+E(X_8, X_5, X_6)\}$$



- **Representation of directed GM**



Example: The Dishonest Casino

A casino has two dice:

- Fair dice

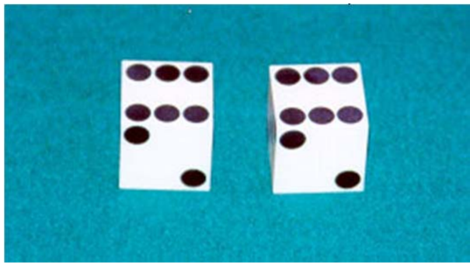
  P(1) = P(2) = P(3) = P(5) = P(6) = 1/6

- Loaded dice

  P(1) = P(2) = P(3) = P(5) = 1/10

  P(6) = 1/2

Casino player switches back-&-forth between fair and loaded dice once every 20 turns



Game:

1. You bet $1
2. You roll (always with a fair die)
3. Casino player rolls (maybe with fair die, maybe with loaded die)
4. Highest number wins $2

Puzzles regarding the dishonest casino

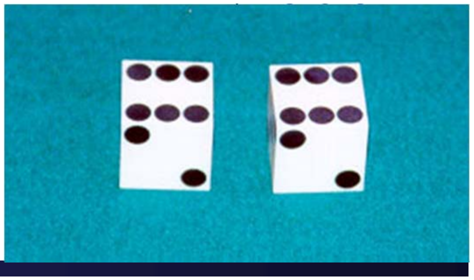
GIVEN: A sequence of rolls by the casino player

1245526462146146136136661664661636616366163616515615115146123562344

QUESTION

- How likely is this sequence, given our model of how the casino works?
  - This is the **EVALUATION** problem
- What portion of the sequence was generated with the fair die, and what portion with the loaded die?
  - This is the **DECODING** question
- How “loaded” is the loaded die? How “fair” is the fair die? How often does the casino player change from fair to loaded, and back?
  - This is the **LEARNING** question

Knowledge Engineering



- Picking variables
  - Observed
  - Hidden
- Picking structure
  - CAUSAL
  - Generative
  - Coupling
- Picking Probabilities
  - Zero probabilities
  - Orders of magnitudes
  - Relative values

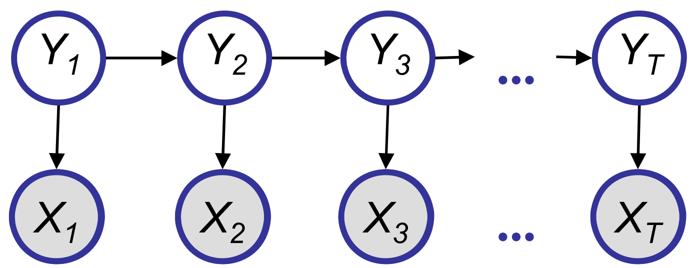
Hidden Markov Model

The underlying source:

Speech signal
genome function
dice

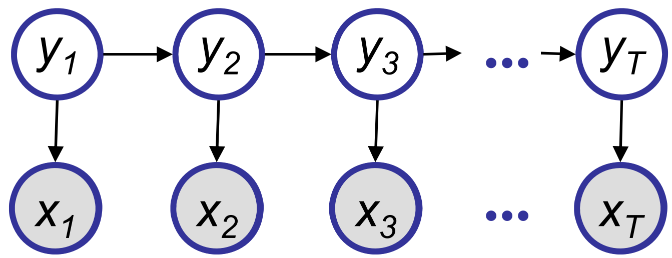
The sequence:

Phonemes
DNA sequence
sequence of rolls



Probability of a parse

- Given a sequence $\mathbf{x} = x_1 \ldots\ldots x_T$ and a parse $\mathbf{y} = y_1, \ldots\ldots, y_T$,
- To find how likely is the parse: (given our HMM and the sequence)



$$
\begin{aligned}
p(\mathbf{x}, \mathbf{y}) &= p(x_1 \ldots\ldots x_T, y_1, \ldots\ldots, y_T) \qquad \text{(Joint probability)} \\
&= p(y_1)\, p(x_1 \mid y_1)\, p(y_2 \mid y_1)\, p(x_2 \mid y_2) \ldots p(y_T \mid y_{T-1})\, p(x_T \mid y_T) \\
&= p(y_1)\, P(y_2 \mid y_1) \ldots p(y_T \mid y_{T-1}) \times p(x_1 \mid y_1)\, p(x_2 \mid y_2) \ldots p(x_T \mid y_T) \\
&= p(y_1, \ldots\ldots, y_T)\, p(x_1 \ldots\ldots x_T \mid y_1, \ldots\ldots, y_T)
\end{aligned}
$$

- Marginal probability: $p(\mathbf{x}) = \sum_{\mathbf{y}} p(\mathbf{x}, \mathbf{y}) = \sum_{y_1} \sum_{y_2} \cdots \sum_{y_N} \pi_{y_1} \prod_{t=2}^{T} a_{y_{t-1}, y_t} \prod_{t=1}^{T} p(x_t \mid y_t)$
- Posterior probability: $p(\mathbf{y} \mid \mathbf{x}) = p(\mathbf{x}, \mathbf{y}) / p(\mathbf{x})$

- We will learn how to do this explicitly (polynomial time)

Bayesian Network:

- A BN is a directed graph whose nodes represent the random variables and whose edges represent direct influence of one variable on another.

- It is a data structure that provides the skeleton for representing **a joint distribution** compactly in a **factorized** way;

- It offers a compact representation for **a set of conditional independence assumptions** about a distribution;

- We can view the graph as encoding a generative sampling process executed by nature, where the value for each variable is selected by nature using a distribution that depends only on its parents. In other words, each variable is a stochastic function of its parents.

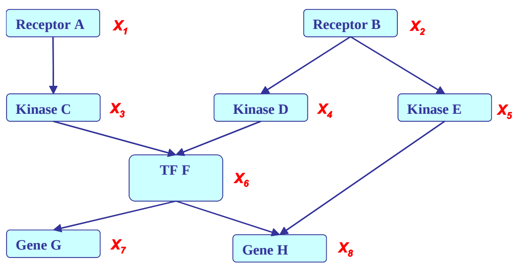
Bayesian Network: Factorization Theorem

- **Theorem:**

  Given a DAG, The most general form of the probability distribution that is consistent with the graph factors according to "node given its parents":

$$P(\mathbf{X}) = \prod_{i=1:d} P(X_i \mid \mathbf{X}_{\pi_i})$$

where $\mathbf{X}_{\pi_i}$ is the set of parents of X_i, d is the number of nodes (variables) in the graph.

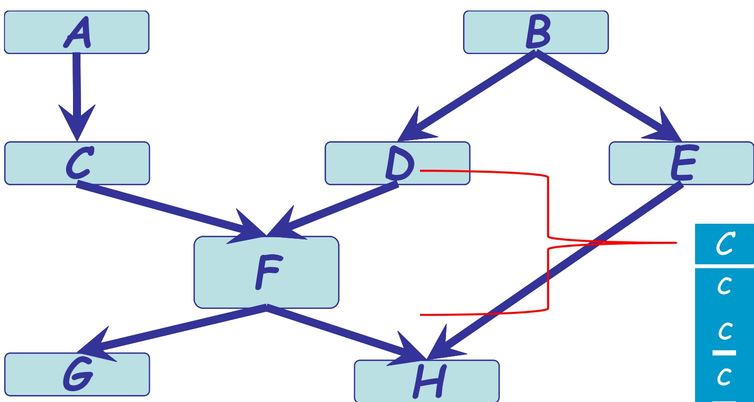


$$P(X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8)$$

$$= P(X_1)\, P(X_2)\, P(X_3 | X_1)\, P(X_4 | X_2)\, P(X_5 | X_2)\, P(X_6 | X_3, X_4)\, P(X_7 | X_6)\, P(X_8 | X_5, X_6)$$

Specification of a directed GM

- There are two components to any GM:
  - the *qualitative* specification
  - the *quantitative* specification

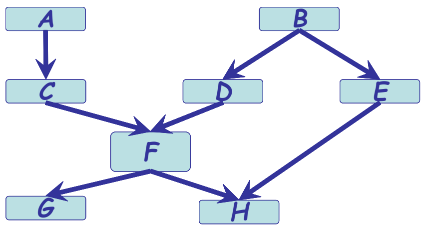


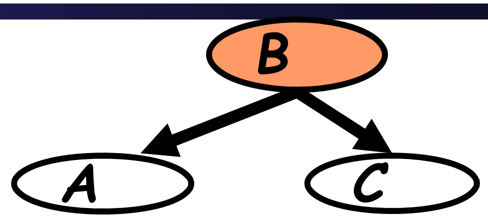
C	D	$P(F \mid C,D)$	
c	d	0.9	0.1
c	$\bar{d}$	0.2	0.8
$\bar{c}$	d	0.9	0.1
$\bar{c}$	$\bar{d}$	0.01	0.99

Qualitative Specification

- Where does the qualitative specification come from?
  - Prior knowledge of causal relationships
  - Prior knowledge of modular relationships
  - Assessment from experts
  - Learning from data
  - We simply link a certain architecture (e.g. a layered graph)
  - …

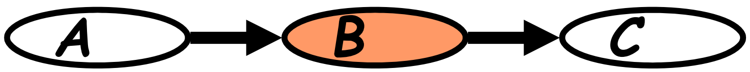
Local Structures & Independencies





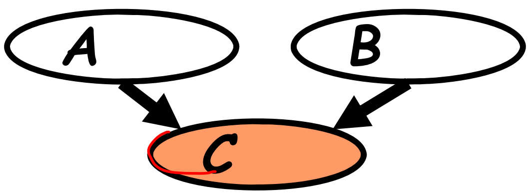
- Common parent
  - Fixing B decouples A and C

    "given the level of gene B, the levels of A and C are independent"



- Cascade
  - Knowing B decouples A and C

    "given the level of gene B, the level gene A provides no extra prediction value for the level of gene C"



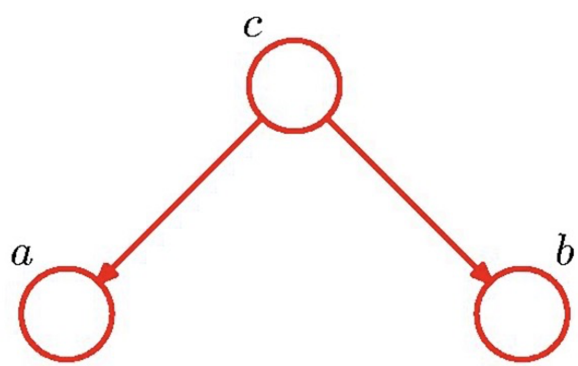
- V-structure
  - Knowing C couples A and B

    because A can "explain away" B w.r.t. C

    "If A correlates to C, then chance for B to also correlate to B will decrease"

- The language is compact, the concepts are rich!

Common parent

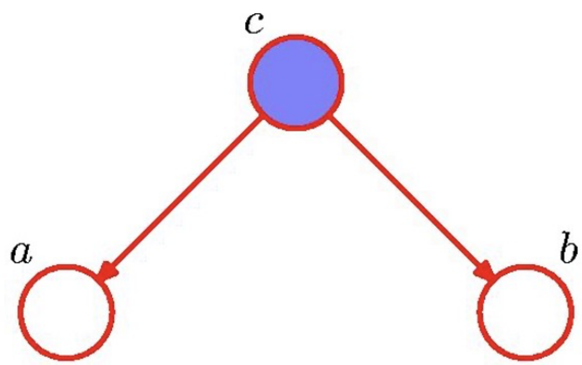


$$p(a, b, c) = p(a|c)p(b|c)p(c)$$

$$p(a, b) = \sum_c p(a|c)p(b|c)p(c)$$

$$a \not\perp\!\!\!\perp b \mid \emptyset$$

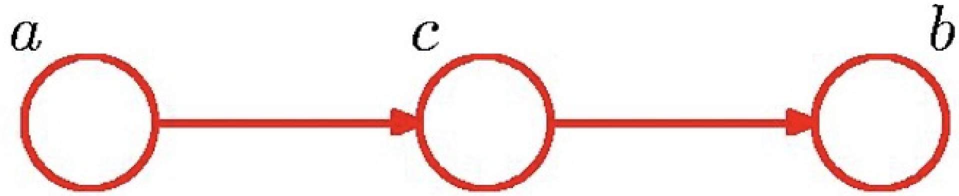
Common parent



$$\begin{aligned} p(a,b|c) &= \frac{p(a,b,c)}{p(c)} \\ &= p(a|c)p(b|c) \end{aligned}$$

$$a \perp\!\!\!\perp b \mid c$$

Chain

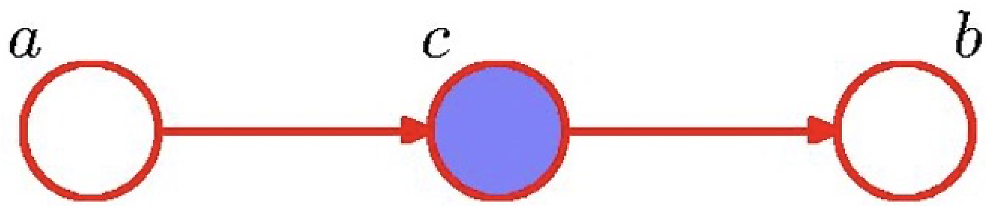


$$p(a, b, c) = p(a)p(c|a)p(b|c)$$

$$p(a, b) = p(a) \sum_{c} p(c|a)p(b|c) = p(a)p(b|a)$$

$$a \not\perp\!\!\!\perp b \mid \emptyset$$

Chain



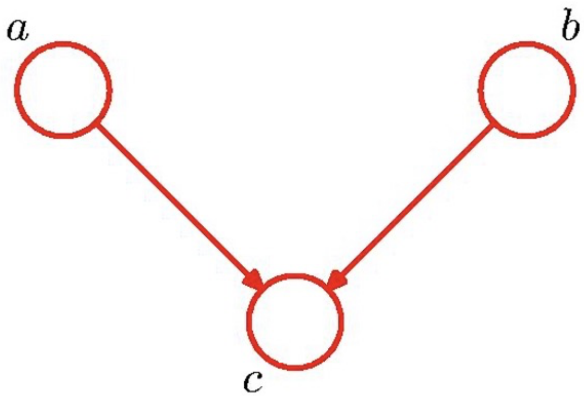
$$p(a,b|c) = \frac{p(a,b,c)}{p(c)}$$

$$= \frac{p(a)p(c|a)p(b|c)}{p(c)}$$

$$= p(a|c)p(b|c)$$

$$a \perp\!\!\!\perp b \mid c$$

V-structure

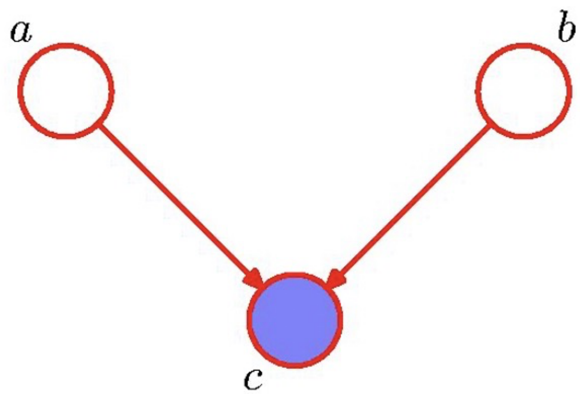


$$p(a, b, c) = p(a)p(b)p(c|a, b)$$

$$p(a, b) = p(a)p(b)$$

$$a \perp\!\!\!\perp b \mid \emptyset$$

V-structure

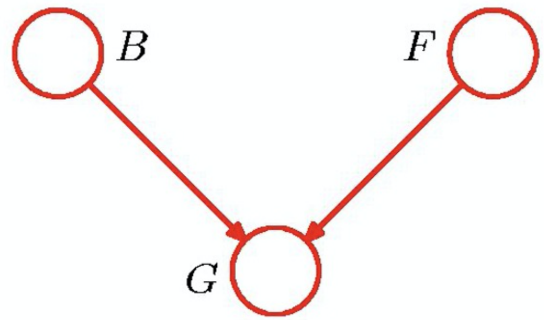


$$
\begin{aligned}
p(a,b|c) &= \frac{p(a,b,c)}{p(c)} \\
&= \frac{p(a)p(b)p(c|a,b)}{p(c)}
\end{aligned}
$$

$$a \not\perp\!\!\!\perp b \mid c$$

One example: “Am I out of fuel?”

$$
\begin{aligned}
p(G=1|B=1,F=1) &= 0.8 \\
p(G=1|B=1,F=0) &= 0.2 \\
p(G=1|B=0,F=1) &= 0.2 \\
p(G=1|B=0,F=0) &= 0.1
\end{aligned}
$$



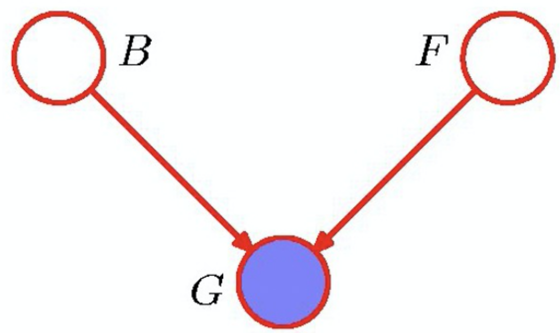
$$
\begin{aligned}
p(B=1) &= 0.9 \\
p(F=1) &= 0.9
\end{aligned}
$$

and hence

$$p(F=0) = 0.1$$

B = Battery (0=flat, 1=fully charged)
F = Fuel Tank (0=empty, 1=full)
G = Fuel Gauge Reading (0=empty, 1=full)

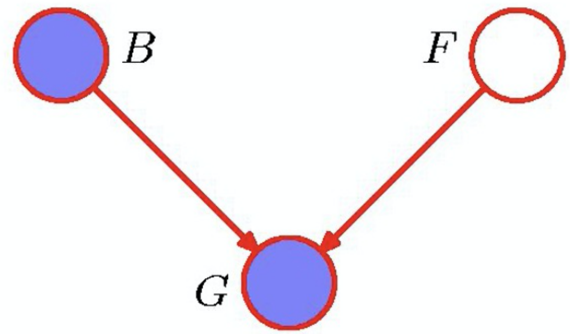
One example: “Am I out of fuel?”



$$p(F=0|G=0) = \frac{p(G=0|F=0)p(F=0)}{p(G=0)} \simeq 0.257$$

Probability of an empty tank increased by observing $G = 0$.

One example: "Am I out of fuel?"



$$p(F=0|G=0,B=0) = \frac{p(G=0|B=0,F=0)p(F=0)}{\sum_{F\in\{0,1\}} p(G=0|B=0,F)p(F)} \simeq 0.111$$

Probability of an empty tank reduced by observing $B = 0$.
This referred to as "explaining away".