CSC3001: Discrete Mathematics

Assignment 3

Instructions:

1. Print out this question paper (**two-sided**) and write down your full working on the blank area.
2. You can have discussions with your classmates. However, make sure all the solutions you submit are your own work. Any plagirism will be given **ZERO** mark.
3. Submission of this assignment should **NOT** be later than **5pm on 29th of November**.
4. Before your submission, please **make a softcopy** of your work for further discussion in a tutorial.
5. After making your softcopy, submit your assignment to the dropbox located on the 4th floor in Chengdao Building.

Student Number: ____________________ **Name:** _______________

1. (*20 points*) Determine whether a graph exists for the following degree sequences:

$$1)\ 4,4,5,5,5,5 \qquad 2)\ 1,2,3,4,4 \qquad 3)\ 1,1,1,2,2,3$$

If it exists, draw all the possibilities up to isomorphism and determine whether these graphs are planar; otherwise, explain why it does not exist.

Solution.

1) Yes. The unique graph is obtained by removing an edge from K_6. Since the graph is connected and does not satisfy the equation $m \leq 3n - 6$, the graph is not planar.

2) No. If we start drawing the two vertices of degree 4, then the remaining vertices will be of degree at least 2, which is a contradiction.

3) Yes, there are three non-isomorphic such graphs, which are all planar.

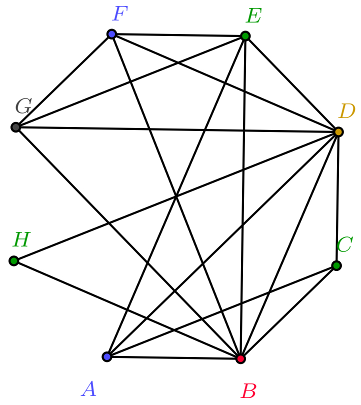
2. (*20 points*) There are a few lockers in a building for the delivery men to deposite parcels. Each locker stores only one parcel at a time. There were 7 residents who picked up the parcels on a day. The system records the deposite time and pick-up time for each parcel as follows. (**Note:** the letters in the table entries indicate the different parcels.)

Delivery man / Deposite time	Rabit	Turtle	Dinosaur	Wolf
7:05	A			
8:13		B		
8:20		C		
8:57			D	
10:04				E
11:51	F			
11:53	G			
14:11				H

Residents / Pick-up time	Kiwi	Moa	Morepork	Tuatara	Kakapo	Penguin	Emu
9:31					C		
10:50							A
11:58			E				
12:01					F		
12:42	G						
15:23				H			
16:15						B	
17:35		D					

Model this problem as a graph problem and determine the least number of lockers in the building.

Solution. Represent the parcels by vertices, and join an edge between two vertices if their storing periods have an overlap. Then we obtain a graph as follows, which is easily found to have the chromatic number 5. So there are at least 5 lockers in the building.



3. (*20 points*) In a standard stable matching problem, boys' and girls' preference are provided as follows. In order to obtain a stable matching in this problem, the boys are

Boys' preference

boy	preference order
A	b, a, d, e, c
B	d, b, a, c, e
C	b, e, c, d, a
D	a, d, c, b, e
E	b, d, a, e, c

Girls' preference

girl	preference order
a	E, B, A, D, C
b	D, C, B, A, E
c	B, C, D, E, A
d	A, E, D, C, B
e	D, B, E, C, A

making proposals. But these girls have learned "*Discrete Mathematics*". They complained that this is unfair, and two of them lied in their preference. By the end of the marrying procedure, one obtains the following stable matching:

$$(A, a), \quad (B, c), \quad (C, b), \quad (D, e), \quad (E, d)$$

Can you figure out which girls lied?

Solution. Following the instructions of marrying procedure, it is easy to see that the preference order of A, B should be swapped for girl "a"; and the preference order of C, D should be swapped for girl "b".

4. (*20 points*) Recall that an *n-stair* is the collection of the unit squares bounded by x-axis, $y = x$ and $x = n + 1$. An *n-stair graph* is obtained by replacing each of the unit squares by a vertex where two vertices are adjacent if two unit squares share a common side. A *Hamiltonian path* of a graph is a simple path that visits all the vertices of the graph. Proof that no n-stair graph has a Hamiltionian path for any $n \geq 3$.

Proof. By exhausting the cases, it is readily seen that a 3-stair graph has no Hamiltionian path.
Given an $n > 3$, suppose there is a Hamiltonian path for the n-stair graph. Consider the top three stairs, which forms a 3-stair subgraph. Then the Hamiltionian path would pass through this 3-stair subgraph, and so the Hamiltionian path could cross the buttom border once, twice or thrice. But if the Hamiltionian path crosses the buttom border once or twice, then it would mean a 3-stair graph has a Hamiltionian path, which is a contradiction. So the Hamiltionian path must cross the border thrice. Orient the Hamiltonian path so that it enters and exits the subgraph exactly twice and once respectively. Then there are only two possibilities, which both turn out to fail by simple attempts. □

5. (*20 points*) Let G be a bipartite graph with bipartition (A, B) where $|A| = |B| = 2n$. Suppose that $|N(X)| \geq |X|$ for every $X \subseteq A$ with $|X| \leq n$, and $|N(Y)| \geq |Y|$ for every $Y \subseteq B$ with $|Y| \leq n$. Prove that G has a perfect matching.

Proof. We shall prove that Hall's condition holds for all subsets of A. To the contrary, assume that there exists $X \subseteq A$ with $|X| > n$ and $|N(X)| < |X|$.
Let $X' \subset X$ be such that $|X'| = n$. Then

$$|N(X)| \geq |N(X')| \geq |X'| = n.$$

Let $Y = B - N(X)$. Since $N(X) \subseteq B$, we have

$$|Y| = |B - N(X)| = |B| - |N(X)| \leq 2n - n = n \qquad \Rightarrow \qquad |N(Y)| \geq |Y|$$

Also note that there is no edges between X and Y, so

$$N(Y) \subseteq A - X$$

This yields

$$|A| \geq |X| + |N(Y)| > |N(X)| + |Y| = |B|, \text{ a contradiction.}$$

□

6. (*10 points*) **[Bonus Question]** A kindergarten is organizing a game with a group of kids, where each kid may or may not know each other. Prove that there is always a way to divide the group into two rooms so that each kid knows an even number of kids in his/her own room. (**Note:** We allow an empty room.)

Proof. Let $n \in \mathbb{Z}^+$ and set

$$P(n): \text{ There is a way to divide a group of } n \text{ kids to meet the condition}$$

Apparently $P(1)$ holds. Assume $P(k)$ also holds. Consider $n = k + 1$. W.l.o.g, assume the group V has a kid u knowing friends $\mathsf{N}(u) = \{v_1, \ldots, v_k\}$ with k odd. Set

$$\begin{aligned}
V' =& V \setminus \{u\} \\
E =& \{xy \mid x, y \in V' \text{ and they know each other}\} \\
E' =& \{xy \in E \mid \{x, y\} \not\subseteq \mathsf{N}(u)\} \cup \{v_i v_j \mid v_i v_j \notin E\} \\
G =& (V, E \cup \{uv_1, \ldots, uv_k\}) \\
G' =& (V', E')
\end{aligned}$$

By $P(k)$ there is a partition $V' = V_1 \cup V_2$ such that each $v \in V_i$ is of even degree in $G'|_{V_i}$. W.l.o.g, assume u knows even number of kids in V_1. We claim that

$$W_1 = V_1 \cup \{u\}, \quad W_2 = V_2$$

is the desired partition for G. It suffices to consider a vertex $v \in V_1$.
If $v \notin \mathsf{N}(u)$, then v does not know u, so $\mathsf{deg}(v)$ is the same in both $G|_{W_1}$ and $G'|_{V_1}$.
If $v \in \mathsf{N}(u)$, then $\mathsf{deg}(v)$ is odd in $G|_{V_1}$, and so $\mathsf{deg}(v)$ is even in $G|_{W_1}$. $\square$