9 Lecture 9 (Euler's formula)

Summary

- Complex exponential function (second definition)
- Complex sine and cosine function
- Euler's formula

For real number x, the meaning of e^x can be defined in several ways.

- First define the constant e by $e = \lim_{n\to\infty}(1+1/n)^n$. For integer a, define e^a by $\underbrace{e \cdot e \cdots e}_{a \text{ factors}}$. For integer b, let $e^{1/b}$ be the number y such that $y^b = e$. For rational number a/b, define $e^{a/b}$ by $(e^a)^{(1/b)}$. For irrational number x, we approximate x by taking a sequence of rational numbers $(a_k/b_k)_{k=1}^{\infty}$ that converges to x, and defined e^x by
$$e^x \triangleq \lim_{k\to\infty} e^{a_k/b_k}.$$
- First define the log function by
$$\log(y) \triangleq \int_1^y \frac{1}{y}\,dy$$
for $y > 0$, and define e^x be the inverse function of $\log(y)$. That is, given x_0, let e^{x_0} be the number y_0 such that
$$x_0 = \int_1^{y_0} \frac{1}{y}\,dy$$
- Given a real number x, define e^x by power series
$$e^x \triangleq \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

All three definitions are equivalent. We will use the third one to extend e^x to complex numbers.

Definition 9.1. Given a complex number z, define e^z by power series
$$e^z \triangleq \exp(z) \triangleq \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots .$$

It is obvious that when z is a real number the above definition reduces to the real exponential function. It is also easy to see $\exp(0) = 1$. However it is not obvious from this definition that e^z is never equal to zero.

Remark. In this lecture we first neglect all the convergence issue. Assume that the series are convergent for all z, and furthermore, assume that the convergence is absolute. This is analogous to the real power series. Moreover, we will assume in this lecture that the convergence is uniform, so that we can re-arrange the order of terms and compute derivative term-wise. We shall return to these questions in the next lecture.

We establish below a fundamental property of the function $\exp(z)$.

Theorem 9.2. *For any complex numbers z_1 and z_2,*

$$\exp(z_1 + z_2) = \exp(z_1)\exp(z_2).$$

Proof. We use a fact about product of infinite series: Given two absolutely convergent series $\sum_k a_k$ and $\sum_k b_k$, the product $(\sum_k a_k)(\sum_k b_k)$ is equal to $\sum_k c_k$, where c_k is defined by the convolution

$$c_k \triangleq a_0 b_k + a_1 b_{k-1} + a_2 b_{k-2} + \cdots + a_{k-1} b_1 + a_k b_0.$$

Apply this fact to $a_k = z_1^k/k!$ and $b_k = z_2^k/k!$. For $k \geq 0$, the coefficient c_k is

$$\begin{aligned} c_k &= \sum_{\ell=0}^{k} \frac{z_1^\ell}{\ell!} \frac{z_2^{k-\ell}}{(k-\ell)!} \\ &= \frac{1}{k!} \sum_{\ell=0}^{k} \binom{k}{\ell} z_1^k z_2^{k-\ell} \\ &= \frac{1}{k!}(z_1 + z_2)^k. \end{aligned}$$

Hence

$$\exp(z_1)\exp(z_2) = \sum_{n=1}^{\infty} \frac{(z_1 + z_2)^n}{n!} \triangleq \exp(z_1 + z_2).$$

□

Using this theorem we can derive some immediate corollaries.

Theorem 9.3. *For any $z \in \mathbb{C}$,*

- $e^{-z} = (e^z)^{-1}$*;*
- $e^z \neq 0$*;*
- $e^{nz} = (e^z)^n$ *for integer n.*

Proof. In the previous theorem, let $z_1 = z$ and $z_2 = -z$,

$$e^z e^{-z} = e^{z-z} = e^0 = 1.$$

Therefore $(e^z)^{-1} = e^{-z}$. We see that the value of e^z cannot equal to 0 because it is the reciprocal of some complex number.

For positive integer n, we can prove $e^{nz} = (e^z)^n$ by repeatedly applying the previous theorem. For negative integer n, we apply the first part in this theorem. $\square$

Definition 9.4. Let the *complex sine* and *complex cosine function* be define by

$$\sin(z) \triangleq \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$
$$\cos(z) \triangleq \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}.$$

Since the coefficients of the sine and cosine power series are the same as in the real case, $\sin(z)$ and $\cos(z)$ reduce to the real sine and cosine function when z is a real number. The main relationship between the exponential and the sinusoidal functions is recorded in the next theorem.

Theorem 9.5 (Euler's formula)**.**

$$e^{iz} = \cos(z) + i\sin(z). \tag{9.1}$$

Proof. The power series expansion of e^{iz} is

$$e^{iz} = 1 + iz - \frac{z^2}{2!} - \frac{iz^3}{3!} + \frac{z^4}{4!} + \frac{iz^5}{5!} - \frac{z^6}{6!} + \cdots$$

Assuming that we can exchange the order of adding the terms in the powers, we can separate the terms without i and the terms with i. By direct comparison, we can re-arrange the sum to $\cos(z) + i\cos(z)$. □

In particular when z is a real number ϕ, we have

$$e^{i\phi} = \cos\phi + i\sin\phi. \tag{9.2}$$

Using Theorem 9.5, we can express sin and cos in terms of exp.

Theorem 9.6. *For any $z \in \mathbb{C}$,*

$$\begin{aligned}\cos(z) &= \frac{e^{iz} + e^{iz}}{2}\\ \sin(z) &= \frac{e^{iz} - e^{iz}}{2i}.\end{aligned}$$

Proof. We use the properties that $\cos(z)$ is an even function and $\sin(z)$ is an odd function. This can be easily seen because the complex cosine function is even because all the terms in the power series that defines $\cos(z)$ in Def. 9.4 have even power. The terms in the power series that defines $\sin(z)$ all have odd powers. Hence

$$e^{-iz} = \cos(z) - i\sin(z) \tag{9.3}$$

By adding (9.3) to (9.1), we get

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

after some re-arrangement of terms. Similarly, by subtracting (9.3) from (9.1), we get

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}.$$

□

The result in Theorem 9.6 holds for any complex numbers. In particular, when we restrict z to a real number ϕ, we can get

$$\begin{aligned}\cos(\phi) &= \frac{e^{i\phi} + e^{i\phi}}{2}\\ \sin(\phi) &= \frac{e^{i\phi} - e^{i\phi}}{2i}.\end{aligned}$$

We end this lecture by show that the definition of exp by power series is the same as that given in previous lecture.

Theorem 9.7. *If $z = x + iy$, then*

$$e^{x+iy} = e^x(\cos y + i \sin y).$$

Proof. Apply Theorem 9.2 to $z_1 = x$ and $z_2 = iy$,

$$e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y).$$

The last equality follows from (9.2). $\square$