Lecture 2: Newsvendor Model

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Expected Outcome

- Learn how to compute expected profit function
- Learn how to derive the optimal ordering quantity

News Vendor Problems

- A store sells perishable product, say, paper version of New York Times.
- Selling price $c_p = \$1.00$
- Variable cost $c_v = \$0.25$
- Salvage value $c_s = \$0.00$
- How many copies should the store order from the publisher the previous night?

Demand distribution

- Suppose the demand D has the following distribution

d	10	15	20	25	30
$\mathbb{P}(D = d)$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{4}$

- Profit

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- Profit

$$\begin{aligned}\text{Profit}(q, D) &= \min(q, D)c_p - qc_v + \max(q - D, 0)c_s \\ &= (q \wedge D)c_p - qc_v + (q - D)^+ c_s\end{aligned}$$

Notation

$$x^+ = \max\{x, 0\} = \begin{cases} x, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases},$$

$$x^- = \max\{-x, 0\} = \begin{cases} -x, & \text{if } x \leq 0 \\ 0, & \text{if } x > 0 \end{cases}$$

For example, $7^+ = 7, (-7)^+ = 0, 7^- = 0, (-7)^- = 7$. Therefore, for every real number x, $x = x^+ - x^-$.

How many to order?

- Order $q = 20$ copies every day for $n = 100000$ days

$$\text{total } 1188870, \qquad \text{average } 11.8887.$$

- Order $q = 22$ copies every day for $n = 100000$ days

$$\text{total } 1239199, \qquad \text{average } 12.3920.$$

Predict average profit per day

- Order $q = 20$ copies:

Demand	Profit	Probability
10	5	1/4
15	10	1/8
>=20	15	5/8

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Demand	Profit	Probability
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15	10	1/8
>=20	15	5/8

- Expected profit

$$5(1/4) + 10(1/8) + 15(5/8) = 11.875.$$

Objective

- Maximize the expected profit for a day

$$
\begin{aligned}
h(q) = \mathbb{E}[\text{Profit}(q, D)] = c_p \mathbb{E}(q \wedge D) - q c_v \\
+ c_s \mathbb{E}(q - D)^+.
\end{aligned} \tag{1}
$$

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- $h(20) = 11.875.$

Optimal order quantity

- Order $q = 25, 26$ copies:

Demand	Profit(25)	Probability	Profit(26)
10	3.75	1/4	3.5
15	8.75	1/8	8.5
20	13.75	1/8	13.5
25	18.75	1/4	18.5
30	18.75	1/4	19.5

- Expected profit

$$\begin{aligned} h(25) = & 3.75(1/4) + 18.75(1/8) + 13.75(1/8) + 18.75(1/4) \\ & + 18.75(1/4) = 13.125. \end{aligned}$$

- $h(26) = 13.125$
- $h(27) = 13.125$
- $h(30) = 13.125$

Example: $\mathbb{E}(q \wedge D)$ and $\mathbb{E}(q - D)^+$

Assume that D follows the following distribution.

d	20	25	30	35
$\mathbb{P}[D = d]$	0.1	0.2	0.4	0.3
$30 \wedge d$	20	25	30	30
$(30 - d)^+$	10	5	0	0
$24 \wedge d$	20	24	24	24
$(24 - d)^+$	4	0	0	0

Then,

$$\begin{aligned}
\mathbb{E}[(30 \wedge D)] &= 20(0.1) + 25(0.2) + 30(0.4) + 30(0.3) \\
&= 2 + 5 + 12 + 9 = 28 \\
\mathbb{E}[(30 - D)^+] &=
\end{aligned}$$

Example

Let $D \sim \text{Uniform}(20, 40)$. What would $\mathbb{E}[25 \wedge D]$ and $\mathbb{E}[(25 - D)^+]$ be? From uniform distribution, we have

$$f(x) = \begin{cases} 1/20, & \text{if } 20 \leq x \leq 40 \\ 0, & \text{otherwise.} \end{cases}$$

Then,

Example

Let $D \sim \text{Uniform}(20, 40)$. What would $\mathbb{E}[25 \wedge D]$ and $\mathbb{E}[(25 - D)^+]$ be? From uniform distribution, we have

$$f(x) = \begin{cases} 1/20, & \text{if } 20 \leq x \leq 40 \\ 0, & \text{otherwise.} \end{cases}$$

Then,

$$\begin{aligned}
\mathbb{E}[25 \wedge D] &= \int_{20}^{40} (25 \wedge x) f(x) dx = \int_{20}^{40} \frac{1}{20} (25 \wedge x) dx \\
&= \int_{20}^{25} \frac{1}{20} x dx + \int_{25}^{40} \frac{1}{20} 25 dx \\
&= \frac{1}{20} \frac{1}{2} \left(25^2 - 20^2 \right) + \frac{25}{20} 15
\end{aligned}$$

$$
\begin{aligned}
\mathbb{E}[(25-D)^+] =& \int_{20}^{40}(25-x)^+ f(x)dx = \int_{20}^{25}(25-x)\frac{1}{20}dx \\
=& \frac{1}{20}\left(\int_{20}^{25} 25dx - \int_{20}^{25} xdx\right) \\
=& \frac{1}{20}\left(25\cdot 5 - \frac{1}{2}(25^2-20^2)\right).
\end{aligned}
$$

$\mathbb{E}(25-D)^+$

$$
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=& \frac{1}{20}\left(25\cdot 5 - \frac{1}{2}(25^2-20^2)\right).
\end{aligned}
$$

Lemma

The following identity holds

$$
q = D \wedge q + (q-D)^+. \tag{2}
$$

Thus $\mathbb{E}(25-D)^+ = 25 - \mathbb{E}(D\wedge 25)$.

General continuous r.v.

$$
\begin{aligned}
&F(x) = \mathbb{P}[D \le x] = \int_0^x f(t)dt, \\
&\mathbb{E}[y \wedge D] = \int_0^\infty (y \wedge x) f(x)dx = \int_0^y f(x)xdx + \int_y^\infty f(x)ydx \\
&\qquad\qquad = \int_0^y f(x)xdx + y\int_y^\infty f(x)dx \\
&\qquad\qquad = \int_0^y f(x)xdx + y(1 - F(y)) \\
&\mathbb{E}(y - D)^+ = y - \mathbb{E}(y \wedge D) = yF(y) - \int_0^y f(x)xdx.
\end{aligned}
$$

$h(y)$ and optimal order quantity

$$
\begin{aligned}
h(y) &= c_p\mathbb{E}(y \wedge D) - c_v y + \mathbb{E}(y - D)^+ c_s \\
&= c_p\Big(\int_0^y f(x)xdx + y(1 - F(y))\Big) - c_c y \\
&\quad + c_s\Big(yF(y) - \int_0^y f(x)xdx\Big) \\
&= (c_p - c_s)\Big(\int_0^y f(x)xdx - yF(y)\Big) + (c_p - c_v)y
\end{aligned}
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h'(y) = -(c_p - c_s)F(y) + (c_p - c_v).
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$$

The optimal order quantity y^* satisfies

$$
F(y^*) = \frac{c_p - c_v}{c_p - c_s}.
$$

Optimal order quantity (discrete case)

- $y*$ is the smallest y such that

$$F(y) \geq \frac{c_p - c_v}{c_p - c_s}.$$

- Example,

$$\frac{c_p - c_v}{c_p - c_s} = \frac{1 - .25}{1 - 0} = .75$$

d	10	15	20	25	30
$\mathbb{P}(D = d)$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{4}$
$F(d)$	$\frac{1}{4}$				

- $y* =$