MAT3253 Homework 9

Due date: 2 Apr.

Question 1. (Brown&Churchill Ex.45.2) By finding an antiderivative, evaluate each of these integrals, where the path is any contour between the indicated limits of integration:

(a) $\int_i^{i/2} e^{\pi z}\, dz$;
(b) $\int_0^{\pi+2i} \cos(z/2)\, dz$;
(c) $\int_1^3 (z-2)^3\, dz$.

Question 2. (Bak&Newman Chapter 4.9) Evaluate $\int_C (z-i)\, dz$ where C is the parabolic segment:

$$z(t) = t + it^2, \quad -1 \le t \le 1.$$

(a) by applying the fundamental theorem of calculus for complex functions
(b) by integrating along the straight line from $-1+i$ to $1+i$ and apply the closed-curve theorem.

Question 3. (Modified from Brown&Churchill Ex.49.1) Apply the closed-curve theorem to show that

$$\int_C f(z)\, dz = 0$$

when the contour C is the unit circle $|z| = 1$, in either direction, and when

(a) $f(z) = \frac{z^2}{z-3}$ (b) $f(z) = ze^{-z}$ (c) $f(z) = \frac{1}{z^2+2z+2}$
(d) $f(z) = sech(z)$ (e) $f(z) = \tan(z)$ (f) $f(z) = Log(z+2)$.

In part (d), the function *sech* is hyperbolic secant $2/(e^{iz} + e^{-iz})$. In part (f), *Log* is the principal value of the complex log function defined on the domain $\mathbb{C} \setminus \{x + iy : x \le 0, y = 0\}$.

Question 4. (Brown&Churchill Ex.49.4) Use the following method to derive the integration formula

$$\int_0^\infty e^{-x^2} \cos(2bx)\, dx = \frac{\sqrt{\pi}}{2} e^{-b^2} \quad (b > 0).$$

(a) Show that the sum of the integrals of e^{-z^2} along the lower and upper horizontal legs of the rectangular path in Fig. 1 can be written

$$2\int_0^a e^{-x^2}\, dx - 2e^{b^2} \int_0^a e^{-x^2} \cos(2bx)\, dx$$

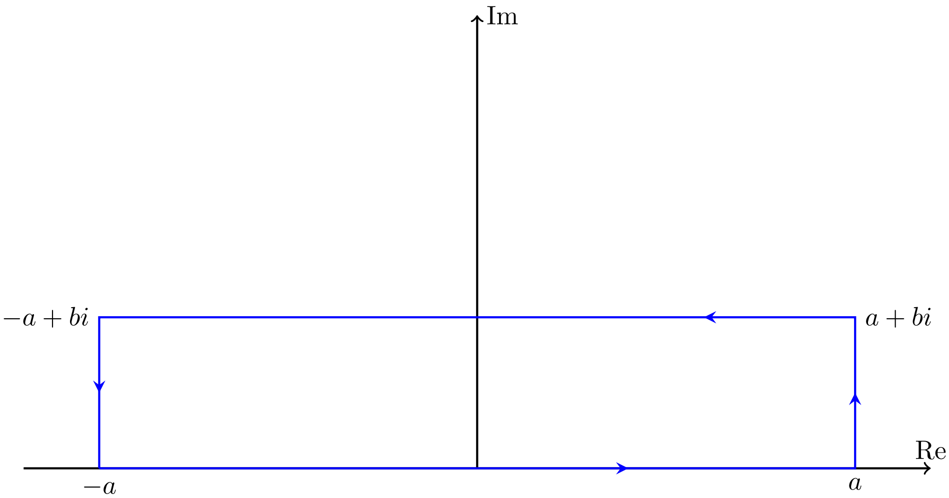

Figure 1: The contour in question 4.

and that the sum of the integrals along the vertical legs on the right and left can be written

$$ie^{-a^2}\int_0^b e^{y^2}e^{-i2ay}\,dy - ie^{-a^2}\int_0^b e^{y^2}e^{i2ay}, dy.$$

Thus, with the aide of the Cauchy-Goursat theorem for rectangle, show that

$$\int_0^a e^{-x^2}\cos(2bx)\,dx = e^{-b^2}\int_0^a e^{-x^2}\,dx + e^{-(a^2+b^2)}\int_0^b e^{y^2}\sin(2ay)\,dy.$$

(b) By accepting the fact that

$$\int e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$$

and observing that

$$\Big|\int_0^b e^{y^2}\sin(2ay)\,dy\Big| \le \int_0^b e^{y^2}dy,$$

obtain the desired integration formula by letting a tend to infinity in the equation at the end of part (a).

Question 5. (Brown&Churchill Ex.49.7) Show that if C is a positively oriented simple closed contour, then the area of the region enclosed by C can be written

$$\frac{1}{2i}\int_C \bar{z}\,dz.$$

(Note that the integral is well-defined even though the function $f(z) = \bar{z}$ is not analytic anywhere.)