Alternative Intuitive Proof of Hall’s Theorem

Let us make things intuitive by using the marriage metaphor, where the vertex sets correspond to boys and girls respectively, and the perfect matching problem becomes the marriage problem. Then Hall’s theorem can be stated as follows.

Hall’s Theorem
For a set of m boys and a set of m girls, a necessary and sufficient condition for a solution of the marriage problem is that each set of k boys collectively knows at least k girls, for $1 \leq k \leq m$.

Proof. (Necessity) If there is marriage (i.e. perfect matching), then with every set of k ($1 \leq k \leq m$) boys, the number of girls they collectively know cannot be less than k.

(Sufficiency) We use induction on the number of boys m, and assume that the theorem is true if the number of boys is less than m. Note that the theorem is true if $m = 1$. Suppose now that there are m boys. There are two cases to consider.

(i) If every k boys (where $k < m$) collectively know at least $k+1$ girls (cf. $N(S) > |S|$ or $N(S) \geq |S|+1$), so that the condition is always true “with one girl to spare”, then we take any edge and marry the boy and the girl at the ends of that edge. The original condition then remains true for the other $m-1$ boys, who can be married by induction, completing the proof for this case.

(ii) If now there is a set of k boys ($k < m$) who collectively know *exactly* k girls, then these k boys can be married by induction to the k girls, leaving $m-k$ boys still to be married. But any collection of h of these $m-k$ boys, for $h \leq m-k$, must know at least h of the remaining girls, since otherwise these h boys, together with the above collection of k boys, would collectively know fewer than $h+k$ girls, contrary to our assumption. It follows that the original condition applies to the $m-k$ boys. They can therefore be married by induction in such a way that everyone is happy. ❑