Chapter 8. The Riemann Integral *

1 The Definition of the Riemann Integral

Throughout this section, it is assumed that we are working with a bounded function f on a closed interval $[a, b]$, meaning that there exists an $M > 0$ such that $|f(x)| \leq M$ for all $x \in [a, b]$.

1.1 Partitions, Upper Sums, and Lower Sums

Definition 1. A *partition* P of $[a, b]$ is a finite set of points from $[a, b]$ that includes both a and b. The notational convention is to always list the points of a partition $P = \{x_0, x_1, x_2, \ldots, x_n\}$ in increasing order; thus,

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

For each subinterval $[x_{k-1}, x_k]$ of P, let

$$m_k = \inf\{f(x) \mid x \in [x_{k-1}, x_k]\} \qquad \text{and} \qquad M_k = \sup\{f(x) \mid x \in [x_{k-1}, x_k]\}.$$

The *lower sum* of f with respect to P is given by

$$L(f, P) = \sum_{k=1}^{n} m_k(x_k - x_{k-1}).$$

Likewise, we define the *upper sum* of f with respect to P by

$$U(f, P) = \sum_{k=1}^{n} M_k(x_k - x_{k-1}).$$

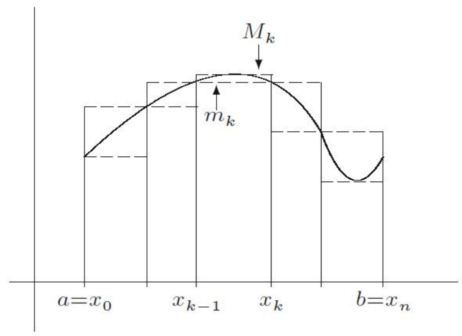


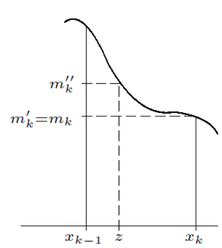
Figure 1: Upper and lower sums.

For a particular partition P, it is clear that $U(f,P) \geq L(f,P)$. The fact that this same inequality holds if the upper and lower sums are computed with respect to different partitions is the content of the next two lemmas.

Definition 2. A partition Q is a refinement of a partition P if Q contains all of the points of P; that is, if $P \subset Q$.

Lemma 1. *If $P \subset Q$, then $L(f,P) \leq L(f,Q)$, and $U(f,P) \geq U(f,Q)$.*

Proof. Consider what happens when we refine P by adding a single point z to some subinterval $[x_{k-1}, x_k]$ of P.



Focusing on the lower sum for a moment, we have

$$\begin{aligned} m_k(x_k - x_{k-1}) =& m_k(x_k - z) + m_k(z - x_{k-1}) \\ \leq& m_k'(x_k - z) + m_k''(z - x_{k-1}), \end{aligned}$$

where

$$m_k' = \inf\{f(x) \mid x \in [z, x_k]\}, \qquad m_k'' = \inf\{f(x) \mid x \in [x_{k-1}, z]\}.$$

*Lecture notes for CUHKSZ course MAT2006: Elementary Real Analysis.

are each necessarily larger than or equal to m_k.

By induction, we have $L(f, P) \leq L(f, Q)$, and an analogous argument holds for the upper sums. $\square$

Lemma 2. *If P_1 and P_2 are any two partitions of $[a, b]$, then $L(f, P_1) \leq U(f, P_2)$.*

Proof. Let $Q = P_1 \cup P_2$ be the so-called common refinement of P_1 and P_2. Because $P_1 \subset Q$ and $P_2 \subset Q$, it follows that

$$L(f, P_1) \leq L(f, Q) \leq U(f, Q) \leq U(f, P_2).$$

$\square$

1.2 Integrability

Intuitively, it helps to visualize a particular upper sum as an overestimate for the value of the integral and a lower sum as an underestimate. As the partitions get more refined, the upper sums get potentially smaller while the lower sums get potentially larger. A function is integrable if the upper and lower sums "meet" at some common value in the middle.

Rather than taking a limit of these sums, we will instead make use of the Axiom of Completeness and consider the *infimum* of the upper sums and the *supremum* of the lower sums.

Definition 3. Let $\mathcal{P}$ be the collection of all possible partitions of the interval $[a, b]$. The *upper integral* of f is defined to be

$$U(f) = \inf\{U(f, P) \mid P \in \mathcal{P}\}.$$

In a similar way, define the *lower integral* of f by

$$L(f) = \sup\{L(f, P) \mid P \in \mathcal{P}\}.$$

Lemma 3. *For any bounded function f on $[a, b]$, it is always the case that $U(f) \geq L(f)$.*

Proof. For any partitions $P, Q \in \mathcal{P}$ of the interval $[a, b]$, we have

$$L(f, P) \leq U(f, Q),$$

which means that $U(f, Q)$ is an upper bound of $\{L(f, P) \mid P \in \mathcal{P}\}$, and hence

$$L(f) = \sup\{L(f, P) \mid P \in \mathcal{P}\} \leq U(f, Q).$$

Note that $Q \in \mathcal{P}$ is arbitrary, thus $L(f)$ is a lower bound of $\{U(f, P) \mid P \in \mathcal{P}\}$, which further implies that

$$L(f) \leq \inf\{U(f, P) \mid P \in \mathcal{P}\} = U(f).$$

$\square$

Definition 4 (Riemann Integrability)**.** A bounded function f defined on the interval $[a, b]$ is *Riemann-integrable* if $U(f) = L(f)$. In this case, we define $\int_a^b f(x)dx$ to be this common value; namely,

$$\int_a^b f(x)dx = U(f) = L(f).$$

The modifier "Riemann" in front of "integrable" accurately suggests that there are other ways to define the integral. In fact, our work in this chapter will expose the need for a different approach. In this chapter, the Riemann integral is the only method under consideration, so it will usually be convenient to drop the modifier "Riemann" and simply refer to a function as being "integrable."

2 Criteria for Integrability

To summarize the situation thus far, it is always the case for a bounded function f on $[a, b]$ that

$$\sup\{L(f, P) \mid P \in \mathcal{P}\} = L(f) \leq U(f) = \inf\{U(f, P) \mid P \in \mathcal{P}\}.$$

The function f is integrable if the inequality is an equality. The major thrust of our investigation of the integral is to describe, as best we can, the class of integrable functions. The preceding inequality reveals that integrability is really equivalent to the existence of partitions whose upper and lower sums are arbitrarily close together.

Theorem 4 (Integrability Criterion)**.** *A bounded function f is integrable on $[a, b]$ if and only if, for every $\epsilon > 0$, there exists a partition P_ϵ of $[a, b]$ such that*

$$U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon.$$

Proof. Let $\epsilon > 0$. If such a partition P_ϵ exists, then

$$0 \leq U(f) - L(f) \leq U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon.$$

Because ϵ is arbitrary, it must be that $U(f) = L(f)$, so f is integrable.

The proof of the converse statement is a familiar triangle inequality argument. Because $U(f)$ is the greatest lower bound of the upper sums, we know that, given some $\epsilon > 0$, there must exist a partition P_1 such that

$$U(f, P_1) \leq U(f) + \frac{\epsilon}{2}.$$

Likewise, there exists a partition P_2 satisfying

$$L(f, P_2) \geq L(f) - \frac{\epsilon}{2}.$$

Now, let $P_\epsilon = P_1 \cup P_2$ be the common refinement. Keeping in mind that the integrability of f means $U(f) = L(f)$, we can write

$$\begin{aligned} U(f, P_\epsilon) - L(f, P_\epsilon) \leq & U(f, P_1) - L(f, P_2) \\ < & \left(U(f) + \frac{\epsilon}{2}\right) - \left(L(f) - \frac{\epsilon}{2}\right) \\ = & \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \quad \square \end{aligned}$$

It is noticed that integrability is closely tied to the concept of continuity. To make this observation more precise, let $P = \{x_0, x_1, x_2, \ldots, x_n\}$ be an arbitrary partition of $[a, b]$, and define $\Delta x_k = x_k - x_{k-1}$. Then,

$$U(f, P) - L(f, P) = \sum_{k=1}^{n} (M_k - m_k)\Delta x_k,$$

where M_k and m_k are the supremum and infimum of the function on the interval $[x_{k-1}, x_k]$, respectively. Our ability to control the size of $U(f, P) - L(f, P)$ hinges on the differences $M_k - m_k$, which we can interpret as the variation in the range of the function over the interval $[x_{k-1}, x_k]$. Restricting the variation of f over arbitrarily small intervals in $[a, b]$ is precisely what it means to say that f is uniformly continuous on this set.

Theorem 5. *If f is continuous on $[a, b]$, then it is integrable.*

Proof. Because f is continuous on a compact set, it must be bounded. It is also uniformly continuous for the same reason. This means that, given $\epsilon > 0$, there exists a $\delta > 0$ so that $|x - y| < \delta$ guarantees

$$|f(x) - f(y)| < \frac{\epsilon}{b - a}.$$

Now, let P be a partition of $[a, b]$ where $\Delta x_k = x_k - x_{k-1}$ is less than δ for every subinterval of P. Given a particular subinterval $[x_{k-1}, x_k]$ of P, we know from the Extreme Value Theorem that the supremum $M_k = f(z_k)$ for some $z_k \in [x_{k-1}, x_k]$. In addition, the infimum m_k is attained at some point y_k also in the interval $[x_{k-1}, x_k]$. But this means $|z_k - y_k| < \delta$, so

$$M_k - m_k = f(z_k) - f(y_k) < \frac{\epsilon}{b - a}.$$

Finally,

$$U(f, P) - L(f, P) = \sum_{k=1}^{n} (M_k - m_k)\Delta x_k < \frac{\epsilon}{b - a} \sum_{k=1}^{n} \Delta x_k = \epsilon,$$

and f is integrable by the criterion given in Theorem 4. $\square$

3 Integrating Functions with Discontinuities

The fact that continuous functions are integrable is not so much a fortunate discovery as it is evidence for a well-designed integral. Riemann's integral is a modification of Cauchy's definition of the integral, and Cauchy's definition was crafted specifically to work on continuous functions. The interesting issue is discovering just how dependent the Riemann integral is on the continuity of the integrand.

Example 3.1. Consider the function

$$f(x) = \begin{cases} 1 & \text{for } x \neq 1 \\ 0 & \text{for } x = 1, \end{cases}$$

it is integrable on $[0, 2]$.

The notation in the following proof is more cumbersome, but the essence of the argument is that the misbehavior of the function at its discontinuity is isolated inside a particularly small subinterval of the partition.

Theorem 6. *If $f : [a, b] \to \mathbb{R}$ is bounded, and f is integrable on $[c, b]$ for all $c \in (a, b)$, then f is integrable on $[a, b]$. An analogous result holds at the other endpoint.*

Proof. Let $\epsilon > 0$. As usual, our task is to produce a partition P such that $U(f, P) - L(f, P) < \epsilon$. For any partition, we can always write

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{k=1}^{n} (M_k - m_k)\Delta x_k \\ &= (M_1 - m_1)\Delta x_1 + \sum_{k=2}^{n} (M_k - m_k)\Delta x_k, \end{aligned}$$

so the first step is to choose x_1 close enough to a so that

$$(M_1 - m_1)(x_1 - a) < \frac{\epsilon}{2}.$$

This is not too difficult. Because f is bounded, we know there exists $M > 0$ satisfying $|f(x)| \leq M$ for all $x \in [a, b]$. Noting that $M_1 - m_1 \leq 2M$, let's pick x_1 so that

$$x_1 - a < \frac{\epsilon}{4M}.$$

Now, by hypothesis, f is integrable on $[x_1, b]$, so there exists a partition P_1 of $[x_1, b]$ for which

$$U(f, P_1) - L(f, P_1) < \frac{\epsilon}{2}.$$

Finally, we let $P = \{a\} \cup P_1$ be a partition of $[a, b]$, from which it follows that

$$U(f, P) - L(f, P) \leq (2M)(x_1 - a) + (U(f, P_1) - L(f, P_1)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \qquad \square$$

Theorem 6 enables us to prove that a bounded function on a closed interval with a single discontinuity at an endpoint is still integrable. In the next section, we will prove that integrability on the intervals $[a, b]$ and $[b, d]$ is equivalent to integrability on $[a, d]$. This property, together with an induction argument, leads to the conclusion that any function with a finite number of discontinuities is still integrable. What if the number of discontinuities is infinite?

Example 3.2. Recall Dirichlet's function

$$D(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

If P is some partition of $[0, 1]$, then the density of the rationals in $\mathbb{R}$ implies that every subinterval of P will contain a point where $D(x) = 1$. It follows that $U(D, P) = 1$. On the other hand, $L(D, P) = 0$ because the irrationals are also dense in $\mathbb{R}$. Because this is the case for every partition P, we see that the upper integral $U(D) = 1$ and the lower integral $L(D) = 0$. The two are not equal, so we conclude that Dirichlet's function is *not* integrable.

How discontinuous can a function be before it fails to be integrable? Before jumping to the hasty (and incorrect) conclusion that the Riemann integral fails for functions with more than a finite number of discontinuities, we should realize that Dirichlet's function is discontinuous at every point in $[0, 1]$. It would be useful to investigate a function where the discontinuities are infinite in number but do not necessarily make up all of $[0, 1]$. Thomae's function, also defined in Section 5.1, is one such example. The discontinuous points of this function are precisely the rational numbers in $[0, 1]$. In the exercises to follow we will see that Thomae's function is Riemann-integrable, raising the bar for allowable discontinuous points to include potentially infinite sets.

The conclusion of this story is contained in the doctoral dissertation of Henri Lebesgue, who presented his work in 1901. Lebesgue's elegant criterion for Riemann integrability is explored in great detail at the end of this Chapter. For the moment, though, we will take a short detour from questions of integrability and construct a proof of the celebrated Fundamental Theorem of Calculus.

4 Properties of the Integral

Theorem 7. *Assume $f : [a, b] \to \mathbb{R}$ is bounded, and let $c \in (a, b)$. Then, f is integrable on $[a, b]$ if and only if f is integrable on $[a, c]$ and $[c, b]$. In this case, we have*

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Proof. If f is integrable on $[a, b]$, then for $\epsilon > 0$ there exists a partition P such that $U(f, P) - L(f, P) < \epsilon$. Because refining a partition can only potentially bring the upper and lower sums closer together, we can simply add c to P if it is not already there. Then, let $P_1 = P \cap [a, c]$ be a partition of $[a, c]$, and $P_2 = P \cap [c, b]$ be a partition of $[c, b]$. It follows that

$$U(f, P_1) - L(f, P_1) < \epsilon, \qquad U(f, P_2) - L(f, P_2) < \epsilon,$$

implying that f is integrable on $[a, c]$ and $[c, b]$.

Conversely, if we are given that f is integrable on the two smaller intervals $[a, c]$ and $[c, b]$, then given an $\epsilon > 0$ we can produce partitions P_1 and P_2 of $[a, c]$ and $[c, b]$, respectively, such that

$$U(f, P_1) - L(f, P_1) < \frac{\epsilon}{2}, \qquad U(f, P_2) - L(f, P_2) < \frac{\epsilon}{2},$$

Letting $P = P_1 \cup P_2$ produces a partition of $[a, b]$ for which

$$U(f, P) - L(f, P) < \epsilon.$$

Thus, f is integrable on $[a, b]$.

Continuing to let $P = P_1 \cup P_2$ as earlier, we have

$$\begin{aligned}\int_a^b f(x)dx \le U(f, P) &< L(f, P) + \epsilon \\ &= L(f, P_1) + L(f, P_2) + \epsilon \\ &\le \int_a^c f(x)dx + \int_c^b f(x)dx + \epsilon,\end{aligned}$$

which implies $\int_a^b f(x)dx \le \int_a^c f(x)dx + \int_c^b f(x)dx$. To get the other inequality, observe that

$$\begin{aligned}\int_a^c f(x)dx + \int_c^b f(x)dx &\le U(f, P_1) + U(f, P_2) \\ &< L(f, P_1) + L(f, P_2) + \epsilon \\ &= L(f, P) + \epsilon \\ &\le \int_a^b f(x)dx + \epsilon.\end{aligned}$$

Because $\epsilon > 0$ is arbitrary, we must have $\int_a^c f(x)dx + \int_c^b f(x)dx \leq \int_a^b f(x)dx$, so

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx. \qquad \square$$

Theorem 8. *Assume f and g are integrable functions on the interval $[a, b]$.*

(i) *The function $f + g$ is integrable on $[a, b]$ with $\int_a^b (f + g)dx = \int_a^b f dx + \int_a^b g dx$.*

(ii) *For $k \in \mathbb{R}$, the function kf is integrable with $\int_a^b (kf)dx = k \int_a^b f dx$.*

(iii) *If $m \leq f(x) \leq M$ on $[a, b]$, then $m(b - a) \leq \int_a^b f(x)dx \leq M(b - a)$.*

(iv) *If $f(x) \leq g(x)$ on $[a, b]$, then $\int_a^b f(x) \leq \int_a^b g(x)dx$.*

(v) *The function $|f|$ is integrable and $|\int_a^b f(x)dx| \leq \int_a^b |f(x)|dx$.*

To this point, the quantity $\int_a^b f(x)dx$ is only defined in the case where $a < b$.

Definition 5. If f is integrable on the interval $[a, b]$, define

$$\int_b^a f(x)dx = -\int_a^b f(x)dx.$$

Also, for $c \in [a, b]$ define

$$\int_c^c f(x)dx = 0.$$

Uniform Convergence and Integration

If $\{f_n\}$ is a sequence of integrable functions on $[a, b]$, and if $f_n \to f$, then we are inevitably going to want to know whether

$$\int_a^b f_n(x)dx \to \int_a^b f(x)dx. \tag{4.1}$$

This is an archetypical instance of one of the major themes of analysis: When does a mathematical manipulation such as integration respect the limiting process?

If the convergence is pointwise, then any number of things can go wrong. It is possible for each f_n to be integrable but for the limit f not to be integrable. Even if the limit function f is integrable, equation (4.1) may fail to hold. As an example of this, let

$$f_n = \begin{cases} n & \text{if } 0 \leq x \leq \frac{1}{n} \\ 0 & \text{if } x = 0 \text{ or } x \geq \frac{1}{n}. \end{cases}$$

Each f_n has two discontinuities on $[0,1]$ and so is integrable with $\int_0^1 f(x)dx = 1$. For each $x \in [0,1]$, we have $\lim_{n \to]\infty} f_n(x) = 0$ so that $f_n \to 0$ pointwise on $[0,1]$. But now observe that the limit function $f = 0$ certainly integrates to 0, and

$$0 \neq \lim_{n \to \infty} \int_0^1 f_n(x)dx.$$

One way to resolve all of these problems is to add the assumption of uniform convergence.

Theorem 9 (Integrable Limit Theorem)**.** *Assume that $f_n \to f$ uniformly on $[a,b]$ and that each f_n is integrable. Then, f is integrable and*

$$\lim_{n \to \infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx.$$

Proof. The proof of f is integrable on $[a,b]$ is left as an exercise.

The property of Theorem 8 yields that

$$\left| \int_a^b f_n(x)dx - \int_a^b f(x)dx \right| = \left| \int_a^b [f_n(x)dx - f(x)]dx \right| \leq \int_a^b |f_n(x)dx - f(x)|dx.$$

Let $\epsilon > 0$ be arbitrary. Because $f_n \to f$ uniformly, there exists an N such that

$$|f_n(x) - f(x)| \leq \frac{\epsilon}{b-a} \qquad \forall n \geq N \quad \forall x \in [a,b].$$

Thus, for $n \geq N$ we see that

$$\left| \int_a^b f_n(x)dx - \int_a^b f(x)dx \right| \leq \int_a^b |f_n(x)dx - f(x)|dx \leq \int_a^b \frac{\epsilon}{b-a} dx = \epsilon,$$

and the result follows. $\square$

5 The Fundamental Theorem of Calculus

The derivative and the integral have been independently defined, each in its own rigorous mathematical terms. The definition of the derivative is motivated by the problem of finding slopes of tangent lines and is given in terms of functional limits of difference quotients. The definition of the integral grows out of the desire to calculate areas under nonconstant functions and is given in terms of supremums and infimums of finite sums. The Fundamental Theorem of Calculus reveals the remarkable inverse relationship between the two processes.

The result is stated in two parts. The first is a computational statement that describes how an antiderivative can be used to evaluate an integral over a particular interval. The second statement is more theoretical in nature, expressing the fact that every continuous function is the derivative of its indefinite integral.

Theorem 10 (Fundamental Theorem of Calculus)**.** *(i) If $f : [a, b] \to \mathbb{R}$ is integrable, and $F : [a, b] \to \mathbb{R}$ satisfies $F'(x) = f(x)$ for all $x \in [a, b]$, then*

$$\int_a^b f(x) = F(b) - F(a).$$

(ii) Let $g : [a, b] \to \mathbb{R}$ be integrable, and for $x \in [a, b]$, define

$$G(x) = \int_a^x g(t)dt.$$

Then G is continuous on $[a, b]$. If g is continuous at some point $c \in [a, b]$, then G is differentiable at c and $G'(c) = g(c)$.

Proof. (i) Let P be a partition of $[a, b]$ and apply the Mean Value Theorem to F on a typical subinterval $[x_{k-1}, x_k]$ of P. This yields a point $t_k \in (x_{k-1}, x_k)$ where

$$F(x_k) - F(x_{k-1}) = f(t_k)(x_k - x_{k-1}).$$

Now, consider the upper and lower sums $U(f, P)$ and $L(f, P)$. Because $m_k \leq f(t_k) \leq M_k$, (where m_k and M_k are the infimum and supremum of f on $[x_{k-1}, x_k]$ respectively) it follows that

$$L(f, P) \leq \sum_{k=1}^{n} [F(x_k) - F(x_{k-1})] \leq U(f, P).$$

But notice that the sum in the middle telescopes so that

$$\sum_{k=1}^{n} [F(x_k) - F(x_{k-1})] = F(b) - F(a),$$

which is independent of the partition P. Thus we have

$$L(f) \le F(b) - F(a) \le U(f).$$

Because $L(f) = U(f) = \int_a^b f(x)dx$, we conclude that $\int_a^b f(x)dx = F(b) - F(a)$.

(ii) To prove the second statement, take $x > y$ in $[a, b]$ and observe that

$$|G(x) - G(y)| = \left|\int_a^x g(t)dt - \int_a^y g(t)dt\right| = \left|\int_y^x g(t)dt\right| \le \int_y^x |g(t)|dt \le M(x - y).$$

where $M > 0$ is a bound on $|g|$. This shows that G is Lipschitz and so is uniformly continuous on $[a, b]$.

Now, let's assume that g is continuous at $c \in [a, b]$. In order to show that $G'(c) = g(c)$, we rewrite the limit for $G'(c)$ as

$$\lim_{x \to c} \frac{G(x) - G(c)}{x - c} = \lim_{x \to c} \frac{1}{x - c} \int_c^x g(t)dt$$

We would like to show that this limit equals $g(c)$. Thus, given an $\epsilon > 0$, we must produce a $\delta > 0$ such that if $|x - c| < \delta$, then

$$\left|\frac{1}{x - c} \int_c^x g(t)dt - f(c)\right| < \epsilon.$$

The assumption of continuity of g implies there exists $\delta > 0$ such that

$$|g(x) - g(c)| < \epsilon, \qquad \forall |x - c| < \delta.$$

It then follows that

$$\begin{aligned}\left|\frac{1}{x - c} \int_c^x g(t)dt - f(c)\right| &= \left|\frac{1}{x - c} \int_c^x [g(t) - g(c)]dt\right| \\ &\le \frac{1}{x - c} \int_c^x |g(t) - g(c)|\, dt \\ &< \frac{\epsilon}{x - c} \int_c^x dt = \epsilon. \quad \square\end{aligned}$$

6 Lebesgue's Criterion for Riemann Integrability

We now return to our investigation of the relationship between continuity and the Riemann integral. We have proved that continuous functions are integrable and that the integral also exists for functions with only a finite number of discontinuities. At the opposite end of the spectrum, we saw that Dirichlet's function which is discontinuous at every point on $[0, 1]$, fails to be Riemann-integrable. The next examples show that the set of discontinuities of an integrable function can be infinite and even uncountable.

Riemann-integrable Functions with Infinite Discontinuities

Recall that Thomae's function (also known as Riemann's function)

$$t(x) = \begin{cases} 1 & \text{if } x = 0 \\ 1/n & \text{if } x = m/n \in \mathbb{Q} \setminus \{0\} \text{ is in lowest term with } n > 0 \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

is continuous on the set of irrationals and has discontinuities at every rational point. Let's prove that Thomae's function is integrable on $[0, 1]$ with $\int_0^1 t(x)dx = 0$.

Let $\epsilon > 0$. The strategy, as usual, is to construct a partition P_ϵ of $[0, 1]$ for which $U(t, P_\epsilon) - L(t, P_\epsilon) < \epsilon$.

Exercise 1. (a) First, argue that $L(t, P) = 0$ for any partition P of $[0, 1]$.

(b) Consider the set of points $D_{\epsilon/2} = \{x \mid t(x) \geq \epsilon/2\}$. How big is $D_{\epsilon/2}$?

(c) To complete the argument, explain how to construct a partition P_ϵ of $[0, 1]$ so that $U(t, P_\epsilon) < \epsilon$.

We have since learned that the Cantor set C is a compact, uncountable subset of the interval $[0, 1]$.

Exercise 2. Define

$$h(x) = \begin{cases} 1 & \text{if } x \in C \\ 0 & \text{if } x \notin C. \end{cases}$$

(a) Show h has discontinuities at each point of C and is continuous at every point of the complement of C. Thus, h is not continuous on an uncountably infinite set.

(b) Now prove that h is integrable on $[0, 1]$.

Sets of Measure Zero

Thomae's function fails to be continuous at each rational number in [0, 1]. Although this set is infinite, we have seen that any infinite subset of $\mathbb{Q}$ is countable. Countably infinite sets are the smallest type of infinite set. The Cantor set is uncountable, but it is also small in a sense that we are now ready to make precise. Previously we presented an argument that the Cantor set has zero "length." The term "length" is awkward here because it really should only be applied to intervals or finite unions of intervals, which the Cantor set is not. There is a generalization of the concept of length to more general sets called the *measure* of a set. Of interest to our discussion are subsets that have *measure zero*.

Definition 6. A set $A \subset \mathbb{R}$ has measure zero if, for all $\epsilon > 0$, there exists a countable collection of open intervals O_n with the property that A is contained in the union of all of the intervals O_n and the sum of the lengths of all of the intervals is less than or equal to ϵ. More precisely, if $|O_n|$ refers to the length of the interval O_n, then we have

$$A \subset \bigcup_{n=1}^{\infty} O_n \qquad \text{and} \qquad \sum_{n=1}^{\infty} |O_n| \leq \epsilon.$$

Example 6.1. A finite set $A = \{a_1, a_2, \ldots, a_N\}$ has measure zero.

Exercise 3. (i) Show that any countable set has measure zero.
(ii) Prove that the Cantor set has measure zero.

Exercise 4. Show that if two sets A and B each have measure zero, then $A \cup B$ has measure zero as well. In addition, discuss the proof of the stronger statement that the countable union of sets of measure zero also has measure zero. (This second statement is true, but a completely rigorous proof requires a result about double summations discussed previously.)

α-Continuity, review

Definition 7. Let f be defined on $[a, b]$, and let $\alpha > 0$. The function f is *α-continuous* at $x \in [a, b]$ if there exists $\delta > 0$ such that for all $y, z \in (x - \delta, x + \delta)$ it follows that $|f(y) - f(z)| < \alpha$.

Let f be a bounded function on $[a, b]$. For each $\alpha > 0$, define D^α to be the set of points in $[a, b]$ where the function f fails to be α-continuous; that is,

$$D^\alpha = \{x \in [a, b] \mid D^\alpha \text{ is not } \alpha\text{-continuous at } x\}.$$

The concept of α-continuity was previously introduced in the proof of "the set of discontinuity D_f must be F_σ". Several of the ensuing exercises appeared as exercises in this section as well.

Exercise 5. If $\alpha < \alpha'$, then $D^{\alpha'} \subset D^\alpha$.

Now, let

$$D_f = \{x \in [a, b] \mid D^\alpha \text{ is not continuous at } x\}.$$

Exercise 6. (a) Let $\alpha > 0$ be given. Show that if f is continuous at $x \in [a, b]$, then it is α-continuous at x as well. Explain how it follows that $D^\alpha \subset D_f$.

(b) Show that if f is not continuous at x, then f is not α-continuous for some $\alpha > 0$. Now, explain why this guarantees that

$$D_f = \bigcup_{n=1}^{\infty} D^{\alpha_n}, \qquad \text{where } \alpha_n = \frac{1}{n}.$$

Exercise 7. Prove that for a fixed $\alpha > 0$, the set D^α is closed.

Just as with continuity, α-continuity is defined pointwise, and just as with continuity, uniformity is going to play an important role.

For a fixed $\alpha > 0$, a function $f : A \to \mathbb{R}$ is *uniformly α-continuous* on A if there exists a $\delta > 0$ such that whenever x and y are points in A satisfying $|x - y| < \delta$, it follows that $|f(x) - f(y)| < \alpha$. By a similar manner in showing "A function that is continuous on a compact set K is uniformly continuous on K", it is completely straightforward to show that if f is α-continuous at every point on some compact set K, then f is uniformly α-continuous on K.

6.1 Lebesgue's Theorem

We are now prepared to completely categorize the collection of Riemann-integrable functions in terms of continuity.

Theorem 11 (Lebesgue's Theorem)**.** *Let f be a bounded function defined on the interval $[a, b]$. Then, f is Riemann-integrable if and only if the set of points where f is not continuous has measure zero.*

Proof. Let $M > 0$ satisfy $|f(x)| \leq M$ for all $x \in [a, b]$, and let D_f and D^α be defined as previously. Let's first assume that D has measure zero and prove that our function is integrable.

($\Leftarrow$) Let $\epsilon > 0$ and

$$\alpha = \frac{\epsilon}{2(b-a)}.$$

Exercise 8. Show that there exists a finite collection of disjoint open intervals $\{G_1, G_2, \ldots, G_N\}$ whose union contains D^α and that satisfies

$$\sum_{n=1}^{N} |G_n| \leq \frac{\epsilon}{4M}.$$

Let K be what remains of the interval $[a, b]$ after the open intervals G_n are all removed; that is, $K = [a, b] \setminus \left(\bigcup_{n=1}^{N} G_n \right)$. Then K is closed and hence compact. Therefore the fact that f is α-continuous on K implies it is uniformly α-continuous on K.

Exercise 9. Finish the proof in this direction by explaining how to construct a partition P_ϵ of $[a, b]$ such that $U(f, P_\epsilon) - L(f, P_\epsilon) \leq \epsilon$. It will be helpful to break the sum

$$U(f, P_\epsilon) - L(f, P_\epsilon) = \sum_{k=1}^{n} (M_k - m_k)\Delta x_k$$

into two parts, one over those subintervals that contain points of D^α and the other over subintervals that do not.

($\Rightarrow$) For the other direction, assume f is Riemann-integrable. We must argue that the set D_f of discontinuities of f has measure zero.

Let $\epsilon > 0$ be arbitrary, and fix $\alpha > 0$. Because f is Riemann-integrable, there exists a partition P_ϵ of $[a, b]$ such that $U(f, P_\epsilon) - L(f, P_\epsilon) < \alpha\epsilon$.

Exercise 10. (a) Prove that D^α has measure zero. Point out that it is possible to choose a cover for D^α that consists of a finite number of open intervals.

(b) Show how this implies that D_f has measure zero.

□

Our main agenda in the remainder of this section is to employ Lebesgue's Theorem in our pursuit of a non-integrable derivative, but this elegant result has a number of other applications.

Exercise 11. (a) Show that if f and g are integrable on $[a, b]$, then so is the product fg.

(b) Show that if g is integrable on $[a, b]$ and f is continuous on the range of g, then the composition $f \circ g$ is integrable on $[a, b]$.

If we instead assume that f is integrable and g is continuous, it actually doesn't follow that the composition $f \circ g$ is an integrable function.

A Nonintegrable Derivative

To this point, our one example of a nonintegrable function is Dirichlet's nowherecontinuous function. We close this section with another example that has special significance. The content of the Fundamental Theorem of Calculus is that integration and differentiation are inverse processes of each other. If a function f is differentiable on $[a, b]$, then part (i) of the Fundamental Theorem tells us that

$$\int_a^b f'(x)dx = f(b) - f(a), \tag{6.1}$$

provided f' is integrable. But shouldn't f' be integrable just by virtue of being a derivative? A curious side-effect of staring at equation (6.1) for any length of time is that it starts to feel as though every derivative should be integrable because we have an obvious candidate for what the value of the integral ought to be. Alas, for the Riemann integral at least, reality comes up short of our expectations. What follows is the construction of a differentiable function f for which equation (6.1) fails because $\int_a^b f'$ does not exist.

We will once again be interested in the Cantor set

$$C = \bigcap_{n=1}^{\infty} C_n.$$

As an initial step, let's create a function $f(x)$ that is differentiable on $[0,1]$ and whose derivative $f'(x)$ has discontinuities at every point of C. The key ingredient for this construction is the function

$$g(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \geq 0 \\ 0 & \text{for } x < 0. \end{cases}$$

Exercise 12. (a) Find $g'(0)$.

(b) Use the standard rules of differentiation to compute $g'(x)$ for $x \neq 0$.

(c) Explain why, for every $\delta > 0$, $g'(x)$ attains every value between 1 and -1 as x ranges over the set $(-\delta, \delta)$. Conclude that g' is not continuous at $x = 0$.

Now, we want to transport the behavior of g around zero to each of the endpoints of the closed intervals that make up the sets C_n used in the definition of the Cantor set. The formulas are awkward but the basic idea is straightforward. Start by setting

$$f_0(x) = 0 \quad \text{on} \quad C_0 = [0,1].$$

To define f_1 on $[0,1]$, first assign

$$f_1(x) = 0 \quad \text{for all} \quad x \in C_1 = [0,1/3] \cup [2/3,1].$$

In the remaining open middle third, put translated "copies" of g oscillating toward the two endpoints. In terms of a formula, we have

$$f_1(x) = \begin{cases} 0 & \text{if } x \in [0,1/3] \\ g(x-1/3) & \text{if } x \text{ just to the right of } 1/3 \\ g(2/3-x) & \text{if } x \text{ just to the left of } 2/3 \\ 0 & \text{if } x \in [2/3,0]. \end{cases}$$

Finally, we splice the two oscillating pieces of f_1 together in a way that makes f_1 differentiable and such that

$$|f_1(x)| \leq (x - 1/3)^2 \qquad \text{and} \qquad |f_1(x)| \leq (x - 2/3)^2.$$

This splicing is no great feat, and we will skip the details so as to keep our attention focused on the two endpoints 1/3 and 2/3. These are the points where $f_1'(x)$ fails to be continuous.

To define $f_2(x)$, we start with $f_1(x)$ and do the same trick as before, this time in the two open intervals $(1/9, 2/9)$ and $(7/9, 8/9)$. The result is a differentiable function that is zero on C_2 and has a derivative that is not continuous on the set

$$\left\{\frac{1}{9}, \frac{2}{9}, \frac{1}{3}, \frac{2}{3}, \frac{7}{9}, \frac{8}{9}\right\}.$$

Continuing in this fashion yields a sequence of functions $f_0, f_1, f_2, \ldots$ defined on $[0, 1]$.

Exercise 13. (a) If $c \in C$, what is $\lim_{n\to\infty} f_n(c)$?

(b) Why does $\lim_{n\to\infty} f_n(x)$ exist for $x \notin C$?

Now, set

$$f(x) = \lim_{n\to\infty} f_n(x).$$

Exercise 14. (a) Explain why $f'(x)$ exists for all $x \notin C$.

(b) If $c \in C$, argue that $|f(x)| \leq (x - c)^2$ for all $x \in [0, 1]$. Show how this implies $f'(c) = 0$.

(c) Give a careful argument for why $f'(x)$ fails to be continuous on C. Remember that C contains many points besides the endpoints of the intervals that make up $C_1, C_2, C_3, \ldots$.

Let's take inventory of the situation. Our goal is to create a nonintegrable derivative. Our function $f(x)$ is differentiable, and f' fails to be continuous on C. We are not quite done.

Exercise 15. Why is $f'(x)$ Riemann-integrable on $[0, 1]$?

The reason the Cantor set has measure zero is that, at each stage, 2^{n-1} open intervals of length $1/3^n$ are removed from C_{n-1}. The resulting sum

$$\sum_{n=1}^{\infty} 2^{n-1}\left(\frac{1}{3^n}\right)$$

converges to one, which means that the approximating sets C_1, C_2, C_3, ... have total lengths tending to zero. Instead of removing open intervals of length $1/3^n$ at each stage, let's see what happens when we remove intervals of length $1/3^{n+1}$.

Exercise 16. Show that, under these circumstances, the sum of the lengths of the intervals making up each C_n no longer tends to zero as $n \to \infty$. What is this limit?

If we again take the intersection $\bigcap_{n=1}^{\infty} C_n$, the result is a Cantor-type set with the same topological properties – it is closed, compact, perfect, and contains no intervals. But a consequence of the previous exercise is that it no longer has measure zero. This is just what we need to define our desired function. By repeating the preceding construction of $f(x)$ on this new Cantor-type set of strictly positive measure, we get a differentiable function whose derivative has too many points of discontinuity. By Lebesgue's Theorem, this derivative cannot be integrated using the Riemann integral.

Exercise 17. As a final gesture, provide the example of an integrable function f and a continuous function g where the composition $f \circ g$ is properly defined but not integrable.