Stochastic Processes

Lecture 11: Dominated convergence theorem, Some proofs, Convergence rate

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Some unresolved issues

- The recurrence criterion

$$\mathbb{E}[N_i|X_0 = i] = \mathbb{E}[\sum_{n=1}^{\infty} 1_{\{X_n=i\}}|X_0 = i] \stackrel{?}{=} \sum_{n=1}^{\infty} P_{ii}^n$$

- Positive recurrence criterion

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} P_{ii}^k \stackrel{?}{=} \mathbb{E}_i[\lim_{n\to\infty} N_i(n)/n] = \frac{1}{\mathbb{E}_i[T_i]}$$

- Assume the DTMC is irreducible and i is positive recurrent.

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} P_{ji}^k \stackrel{?}{=} \frac{1}{\mathbb{E}_i(T_i)}$$

Two useful results

Theorem (Dominated convergence theorem)

Suppose that $S = \{1, 2, \dots\}$ *and that* $a_i \geq 0$ *and* $\sum_{i \in S} a_i < \infty$. *Assume that (a)* $\left| b_i^{(n)} \right| \leq a_i$ *for each* n *and each* i, *and (b)* $\lim_{n \to \infty} b_i^{(n)}$ *exists for each* $i \in S$. *Then*

$$\lim_{n \to \infty} \sum_{i \in S} b_i^{(n)} = \sum_{i \in S} \lim_{n \to \infty} b_i^{(n)}. \tag{1}$$

Theorem (Fubini-Tonelli)

Suppose that $a_{n,k} \geq 0$ *for positive integers* n *and* k. *Then*

$$\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} a_{n,k} = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} a_{n,k}.$$

Stationary distribution implies positive recurrence

Theorem

Assume that DTMC X is irreducible and has a stationary distribution π. Then, the stationary distribution is unique and is given by $\pi(i) = 1/\mathbb{E}_i(T_i)$ for $i \in S$. As a consequence, X is positive recurrent.

- Proof: Since $\pi = \pi P$, we have $\pi = \pi P^k$ for each $k \geq 1$. In particular, we have

$$\pi = \pi \frac{1}{n} \sum_{k=1}^{n} P^k, \quad \pi_i = \sum_{j \in S} \pi_j \frac{1}{n} \sum_{k=1}^{n} P_{ji}^k.$$

- Therefore

$$\pi_i = \lim_{n \to \infty} \sum_{j \in S} \pi_j \frac{1}{n} \sum_{k=1}^{n} P_{ji}^k \overset{\text{d.c.t.}}{=} \sum_{j \in S} \lim_{n \to \infty} \pi_j \frac{1}{n} \sum_{k=1}^{n} P_{ji}^k$$
$$= \sum_{j \in S} \pi_j \frac{1}{\mathbb{E}_i(T_i)} = \frac{1}{\mathbb{E}_i(T_i)} \sum_{j \in S} \pi_j = \frac{1}{\mathbb{E}_i(T_i)}.$$

SLLN for DTMC

Theorem (SLLN)

Assume that state i is positive recurrent and $f : S \to \mathbb{R}$ is bounded.

$$\mathbb{P}_i \left\{ \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(X_k) = c \right\} = 1,$$

where

$$c = \frac{\mathbb{E}_i\Big(\sum_{k=0}^{T_i-1} f(X_k) \Big)}{\mathbb{E}_i(T_i)}.$$

Try to derive Ergodic Theorem from it. ☺

Proof of SLLN for DTMC

- Define $T_i^{(\ell)}$ to be the first n that X_n is the ℓth visit to state i.
- Then SLLN implies that

$$\mathbb{P}_i\left\{\lim_{\ell\to\infty}\frac{1}{\ell}\sum_{k=0}^{T_i^{(\ell)}-1} f(X_k) = c_1\right\} = 1,$$

where

$$c_1 = \mathbb{E}_i\Big(\sum_{k=0}^{T_i-1} f(X_k)\Big).$$

- Then SLLN implies that

$$\mathbb{P}_i\left\{\lim_{\ell\to\infty}\frac{1}{\ell}T_i^{(\ell)} = c_2\right\} = 1,$$

where

$$c_2 = \mathbb{E}_i T_i < \infty.$$

- For each n, there some $\ell = \ell(n)$ such that $T_i^{(\ell)} \le n < T_i^{(\ell+1)}$.
- $\ell(n) \to \infty$ as $n \to \infty$,
- Assume $f \ge 0$,

$$\frac{1}{n}\sum_{\ell=1}^{\ell(n)} \text{cycle } \ell \text{ cost} \le \frac{1}{n}\sum_{k=1}^{n} f(X_k) \le \frac{1}{n}\sum_{\ell=1}^{\ell(n)+1} \text{cycle } \ell \text{ cost},$$

- cycle ℓ cost

$$c(\ell) = \sum_{k=T_i^{(\ell-1)}}^{T_i^{(\ell)}-1} f(X_k).$$

- Thus,

$$\frac{1}{n}\sum_{k=1}^{n} f(X_k) \le \frac{1}{T_i^{(\ell(n))}}\sum_{\ell=1}^{\ell(n)+1} c(\ell) = \frac{\ell(n)}{T_i^{(\ell(n))}}\frac{1}{\ell(n)}\sum_{\ell=1}^{\ell(n)+1} c(\ell)$$

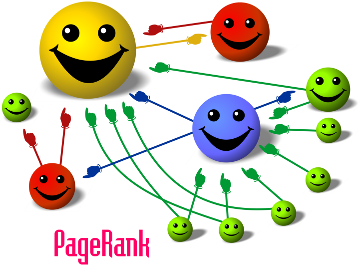
- which converges to c_1/c_2 with probability one.

Positive recurrence criterion

- Suppose that you type "healthy food store" into a search engine. Each of the five web pages, A, B, C, D, E, contains the relevant information on the subject. Suppose that

  A has links to B and C,
  B has links to A and D,
  C has link to D and E,
  D has link to A, B, and C,
  E has link to B.

  Compute the "PageRank" of these five web pages.



PageRank II

- $$P = \begin{pmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

  Using **Python** to compute P^{100}, one obtains the stationary distribution

  $$0.215385 \quad 0.276923 \quad 0.184615 \quad 0.230769 \quad 0.092308.$$

  Thus the PageRank of these five pages is

  B, D, A, C, and E

  with webpage B listed at the top.

Rate of convergence

- Consider a 2-state DTMC.

$$P = \begin{pmatrix} .99 & .01 \\ .01 & .99 \end{pmatrix}, P^{100} = \begin{pmatrix} 0.5663 & 0.4337 \\ 0.4337 & 0.5663 \end{pmatrix}, P^{200} = \begin{pmatrix} 0.5088 & 0.4912 \\ 0.4912 & 0.5088 \end{pmatrix}$$

$$P^{500} = \begin{pmatrix} .5000 & .5000 \\ .5000 & .5000 \end{pmatrix},$$

$$P^{500} = \begin{pmatrix} 0.500020511992570 & 0.499979488007424 \\ 0.499979488007424 & 0.500020511992569 \end{pmatrix}$$

$$P^{5000} = \begin{pmatrix} 0.499999999999966 & 0.499999999999965 \\ 0.499999999999965 & 0.499999999999965 \end{pmatrix}$$

$\pi = (.5, .5)$.

Rate of convergence: II

- Suppose

$$P = \begin{pmatrix} .6 & .4 \\ .4 & .6 \end{pmatrix}, \quad P^{30} = \begin{pmatrix} 0.500000000000000 & 0.500000000000000 \\ 0.500000000000000 & 0.500000000000000 \end{pmatrix}$$

$$\begin{pmatrix} .6 & .4 \\ .4 & .6 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & .2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1}$$

- $$\begin{pmatrix} .6 & .4 \\ .4 & .6 \end{pmatrix}^n = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & (.2)^n \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1}$$

Card shuffling