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Midterm

Oct 22nd

If no derivation only results, at most half score; Independent grading, e.g., if step 2 depends on result of step 1, gradings of step 1 and 2 should be independent.

1. (20 points) At the beginning of each day, a machine that was working the previous day is tested. With probability p, the machine is found to have failed. If it is failed, it takes one day to repair it so that it is working again the next day. The machine can produce products worth value of \$100 each day if it works properly. The factory can add oil to the machine to help lubricate the machine. There are two strategies:

   (i) No oil is added to the machine, and p is 0.6 in this case. (no lubrication)
   (ii) Oil is added to the machine each day so that p will be decreased to 0.25. (lubrication)

   (a) (6 points) Suppose each day is in state 0 if the machine fails and state 1 if it is working. What is the one-step transition probability matrix and two-step transition probability matrix for any given p?

   (b) (6 points) What is the long run fraction of time that the machine is in failure in (i) and (ii)?

   (c) (8 points) Suppose that adding oil costs 10 dollars each day. Calculate the long run average revenue each day under (i) and (ii). Which strategy is better?

   Solution.

   (a)
$$P = \begin{pmatrix} 0 & 1 \\ p & 1-p \end{pmatrix}$$
$$P^2 = \begin{pmatrix} p & 1-p \\ p(1-p) & p+(1-p)^2 \end{pmatrix}$$

   One step transition: 3 points; Two step transition: 3 points

   (b)
$$\pi P = \pi \implies \pi = (\frac{p}{1+p}, \frac{1}{1+p})$$

   The fail probability is $\pi_0 = \frac{p}{1+p}$. Case 1: 3 points; Case 2: 3 points

   (c) No lubrication, $100 \times \frac{1}{1+0.6} = 62.5$.
   Lubrication, $100 \times \frac{1}{1+0.5} - 10 = 70$.
   Lubrication is better. Case 1 Profit: 4 points; Case 2 Profit: 4 points

2. (25 points) A store sells a perishable product. Suppose that the weekly demand D for the product is i.i.d with the following distribution.

d	5	10	20	30
$\mathbb{P}(D=d)$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

   Sales happen from Monday-Friday. Unused items from one week *cannot* be used in the following week. Suppose that each item sells at \$200. Each item costs \$100 to order, and each leftover item by Friday evening has a salvage value of \$50. The store makes a weekly order from a supplier each Friday evening for the product to arrive early Monday morning (before sales commence).

   (a) (8 points) Find the optimal order quantity to maximize the expected profit per week.

(b) (6 points) Compute the expected profit per week, given that the optimal order quantity from part (a) is ordered.

(c) (6 points) Compute the 95% confidence interval of the cumulative profit in 100 weeks, given that the optimal order quantity from part (a) is ordered. (leaving your answer as an expression is fine.)

(d) (5 points) Suppose that on the evening of a fixed Friday, the store receives as donation of 6 fresh items of product A, free of charge. Furthermore, there is now a $500 fixed cost to make an order from the supplier. Should the store place an order that Friday evening? If so, how many items should it order, in order to maximize the expected profit for the following week? Show your work to support your claim.

Solution.

(a) The cdf is

d	5	10	20	30
$\mathbb{P}\{D = d\}$	1/2	1/6	1/6	1/6
$\mathbb{P}\{D \leq d\}$	1/2	4/6	5/6	1.

$$\frac{c_p - c_v}{c_p - c_s} = \frac{200 - 100}{200 - 50} = \frac{100}{150} = \frac{2}{3}.$$

So the optimal order quantity is 10. Derivation: 6 points; Result: 2 points

(b) The optimal expected profit per week is

$$\begin{aligned}
&200\mathbb{E}[\min(10, D)] + 50\mathbb{E}(10 - D)^+ - 10(100) \\
&= 200(15/2) + 50(5/2) - 10(100) \\
&= 625.
\end{aligned}$$

Derivation: 4 points; Result: 2 points

(c) Let $P(10)$ be the single period profit when the order quantity is 10. The desired C.I. is $[62500 - 1.96\sqrt{100}\sigma, 62500 + 1.96\sqrt{100}\sigma]$, where $\sigma^2 = \mathbb{E}P^2(10) - (\mathbb{E}P(10))^2 = \frac{1}{2}(250^2 + 1000^2) - 625^2$. Derivation: 4 points; Result: 2 points

(d) **If we do not order,** the expected profit is

$$\begin{aligned}
&200\mathbb{E}[\min(6, D)] + 50\mathbb{E}(6 - D)^+ \\
&= 200(5/2 + 6/2) + 50(1/2) \\
&= 1100 + 25 = 1125.
\end{aligned}$$

If order, then order 4 items. The expected profit is

$$\begin{aligned}
&200\mathbb{E}[\min(10, D)] + 50\mathbb{E}(10 - D)^+ - 4(100) - 500 \\
&= 200(15/2) + 50(5/2) - 4(100) - 500 \\
&= 1625 - 900 = 725
\end{aligned}$$

The solution is do not order. Profit when we order: 2 points; Profit when we do not order: 2 points; Result: 1 points

3. (20 points) I have N umbrellas, some at home, some in the office. I keep moving between home and office. I take an umbrella with me only if it rains. If it does not rain I leave the umbrella behind (at home or in the office). It may happen that all umbrellas are in one place, I am at the other, it starts raining and must leave, so I get wet. The probability of rain is $p = 0.6$, independent of other events.

(a) (8 points) Let X_n be the number of umbrellas in the place where I am currently at (home or office) in step n. Prove that $\{X_n\}$ is a DTMC. Write down the state space and transition probability matrix when $N = 4$.

(b) (8 points) What is the probability that I get wet (in the long run) when $N = 4$?

(c) (4 points) How many umbrellas should I have at least, so that the probability I get wet (in the long run) is less than 1%?

Solution.

(a) $X_{n+1} = N - X_n$ if no rain and $X_{n+1} = \min\{N, N - X_n + 1\}$ if it rains. Since the event of rain is independent of other events, it is a DTMC. State space $S = \{0, 1, 2, 3, 4\}$, and

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1-p & p \\ 0 & 0 & 1-p & p & 0 \\ 0 & 1-p & p & 0 & 0 \\ 1-p & p & 0 & 0 & 0 \end{pmatrix}$$

Proof of Markovian: 3 points; State space: 2 points; Transition matrix: 3 points

(b) Solving $\pi P = \pi$ and $\sum \pi_i = 1$ gives $\pi_0 = \frac{1-p}{1-p+N}$ and $\pi_1 = \pi_2 = ...\pi_N = \frac{1}{1-p+N}$. The prob. of getting wet is $\pi_0 p = 3/55$. Stationary distri.: 5 points; Expression of wet prob.: 3 points

(c) In order that the prob. of getting wet is less than 1%, solving $\frac{(1-p)p}{1-p+N} \leq 0.01$ gives $N \geq 24$. Derivation: 3 points; Result: 1 point

4. (30 points) Let $X = \{X_n : n = 0, 1, 2, \ldots\}$ be a discrete time Markov chain on state space $S = \{1, 2, 3, 4\}$ with transition matrix

$$P = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & 0 & \frac{3}{4} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & 0 & \frac{3}{4} & 0 \end{pmatrix}.$$

(a) (2 points) Draw a transition diagram.

(b) (4 points) Find $\mathbb{P}\{X_2 = 4 | X_0 = 2\}$. Derivation: 3 points; Result: 1 point

(c) (5 points) Find $\mathbb{P}\{X_2 = 2, X_4 = 4, X_5 = 1 | X_0 = 2\}$. Derivation: 4 points; Result: 1 point

(d) (5 points) What is the period of each state? Derivation: 4 points; Result: 1 point

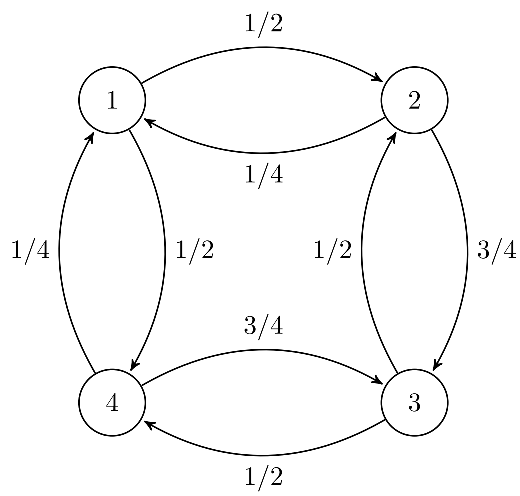
(e) (5 points) Let $\pi = (\frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{4})$. Is π the unique stationary distribution of X? Explain your answer. stationary: 3 points; unique: 2 points

(f) (5 points) Let P^n be the nth power of P. Does $\lim_{n\to\infty} P^n_{1,4} = \frac{1}{4}$ hold? Explain your answer. Derivation: 4 points; Result: 1 point

(g) (4 points) Let T_1 be the first $n \geq 1$ such that $X_n = 1$. Compute $\mathbb{E}(T_1 | X_0 = 1)$. (If it takes you a long time to compute it, you are likely on a wrong track.) Other methods are acceptable, if not using stationary distribution.

Solution.

(a)



(b) $\mathbb{P}\{X_2 = 4|X_0 = 2\} = P_{21}P_{14} + P_{23}P_{34} = \frac{1}{4} \times \frac{1}{2} + \frac{3}{4} \times \frac{1}{2} = \frac{1}{2}$.

(c)

$$\begin{aligned}
&\mathbb{P}\{X_2 = 2, X_4 = 4, X_5 = 1|X_0 = 2\} \\
&= \mathbb{P}\{X_5 = 1|X_4 = 4\}\mathbb{P}\{X_4 = 4|X_2 = 2\}\mathbb{P}\{X_2 = 2|X_0 = 2\} \\
&= P_{41} \times \frac{1}{2} \times (\frac{1}{4} \times \frac{1}{2} + \frac{3}{4} \times \frac{1}{2}) \\
&= \frac{1}{4} \times \frac{1}{2} \times \frac{1}{2} \\
&= \frac{1}{16}.
\end{aligned}$$

(d) Each state has period 2.

(e) Yes. Since $\pi = (\frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{4})$ satisfy $\pi = \pi P$ and sum of the elements are 1, then π must be the stationary distribution. It is unique since this DTMC is irreducible.

(f) No. The limit does not exist since the DTMC has period 2.

(g) $\mathbb{E}(\tau_1|X_0 = 1) = \frac{1}{\pi_1} = 8$.

5. (5 points) A reflected one dimensional random walk is defined as

$$P_{i,i+1} = 1 - P_{i,i-1} = p, \text{ for } i \geq 1; P_{00} = 1 - p.$$

Is the random walk transient or recurrent when $p = 1/2$? Prove your claim.

Solution. Firstly, note that the random walk in the text denoted as $\{X_n\}$ is **irreducible**, we only need to show state 0 is recurrent (1 points).

Let T_0 be the first return time to 0. We need to show $\mathbb{P}_0(T_0 < \infty) = 1$. By first step analysis, $\mathbb{P}_0(T_0 < \infty) = 1 - p + p\mathbb{P}_1(T_0 < \infty)$, it reduces to show $\mathbb{P}_1(T_0 < \infty) = 1$.

Method 1: (4 points) Again by first step analysis,

$$\mathbb{P}_1(T_0 < \infty) = 1 - p + p\mathbb{P}_2(T_0 < \infty), \quad \mathbb{P}_i(T_0 < \infty) = (1-p)\mathbb{P}_{i-1}(T_0 < \infty) + p\mathbb{P}_{i+1}(T_0 < \infty), \ i \geq 2$$

We see that for $p = 1/2$, $(1, \mathbb{P}_1(T_0 < \infty), \mathbb{P}_2(T_0 < \infty), ...)$ forms an arithmetic sequence, and it is easy to show that $\mathbb{P}_1(T_0 < \infty) = 1$ by contradiction (otherwise, $\mathbb{P}_i(T_0 < \infty)$ would be negative for large i).

Method 2: (4 points) Consider a random walk $\{Y_n\}$ defined as $P^Y_{i,i+1} = 1 - P^Y_{i,i-1} = p$ for all i. $\{X_n\}$ and $\{Y_n\}$ can be coupled in such a way that $X_0 = Y_0 = 1$ and $X_n = Y_n$ before $\{X_n\}$ hits 0. Therefore, $\mathbb{P}_1(T_0 < \infty) = \mathbb{P}^Y_1(T_0 < \infty) = 1$, where $\mathbb{P}^Y_1(T_0 < \infty) = 1$ follows from that the recurrent property of $\{Y_n\}$ when $p = 1/2$ and the claim in Problem 4 of Homework 5 or the first lemma of Lecture 11.

Other methods are acceptable. Merely showing that $\{Y_n\}$ is recurrent can give 1 points.