MAT3253 Homework 4

Due date: 26 Feb.

Question 1. For $k = 1, 2, 3, \ldots$, let a_k and b_k be the real and imaginary parts of $(5+i)^k$, respectively. Evaluate the infinite series

$$\sum_{k=1}^{\infty} \frac{a_k b_k}{28^k}.$$

(Hint: Use the properties that $\mathrm{Re}(w) = \frac{w+\bar{w}}{2}$ and $\mathrm{Im}(w) = \frac{w-\bar{w}}{2i}$.)

Question 2. Write a complex-valued function f with the input z in polar form $r\cos\theta + ir\sin\theta$ and output in cartesian form

$$f(z) = u(r,\theta) + iv(r,\theta),$$

where u and v are the real and imaginary parts of f, respectively. Derive the Cauchy-Riemann equations when the input variable is in polar coordinates,

$$\begin{cases} u_r = \frac{1}{r} v_\theta, \\ v_r = -\frac{1}{r} u_\theta, \end{cases}$$

for nonzero r. (The subscripts $_r$ and $_\theta$ means partial derivatives with respect to variables r and θ respectively.) (Hint: Compute the limits

$$\lim_{\Delta r \to 0} \frac{u(r+\Delta r,\theta) + iv(r+\Delta r,\theta) - u(r,\theta) - iv(r,\theta)}{(\Delta r)(\cos\theta + i\sin\theta)}$$
$$\lim_{\Delta\theta \to 0} \frac{u(r,\theta+\Delta\theta) + iv(r,\theta+\Delta\theta) - u(r,\theta) - iv(r,\theta)}{r(\cos(\theta+\Delta\theta) + i\sin(\theta+\Delta\theta)) - r(\cos\theta + i\sin\theta)}$$

and equate the real and imaginary parts.)

Question 3. Similar to the cartesian case, if a complex function f satisfies the Cauchy-Riemann equations in Question 2 at a point $r_0(\cos\theta_0 + i\sin\theta_0)$, and has continuous partial derivatives u_r, v_r, u_θ, v_θ at (r_0, θ_0), then f is complex differentiable at $r_0(\cos\theta_0 + i\sin\theta_0)$. (Assume $r_0 \neq 0$.)

Use this sufficient condition to show that the complex function $f(z) = \sqrt{r}(\cos(\theta/2) + i\sin(\theta/2))$ is analytic in the domain $r > 0$ and $-\pi < \theta < \pi$, and compute its complex derivative.