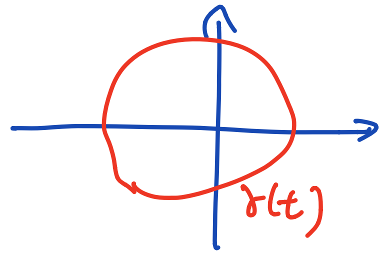
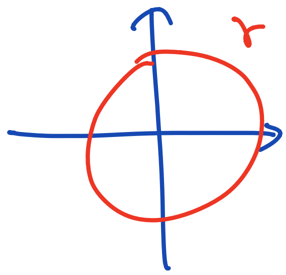
MAT 3253 Lecture 23

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz \qquad [a,b] \xrightarrow{\gamma} \mathbb{C}$$

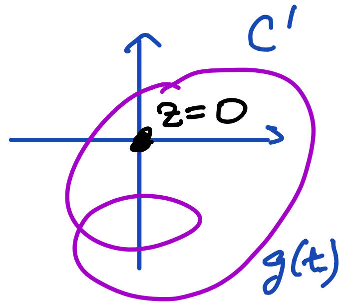
$[a,b] \xrightarrow{\gamma}$



$[a,b] \xrightarrow{\gamma}$



$\xrightarrow{f}$



$g(t) = f(\gamma(t))$

$n(f\circ\gamma ; 0)$

$$2\pi i\, n(C' ; 0) = \int_{C'} \frac{1}{z}\,dz$$

$$= \int_a^b \frac{g'(t)}{g(t)}\,dt$$

$$= \int_a^b \frac{f'(\gamma(t))\cdot \gamma'(t)}{f(\gamma(t))}\,dt$$

$$\int_\gamma \frac{f'}{f}\,dz = \int_a^b \frac{f'(\gamma(t))}{f(\gamma(t))}\cdot \gamma'(t)\,dt$$

They are equal.

Rouché theorem (Rouche)

γ : Simple closed curve, positively oriented.

If f and g are functions analytic on γ and inside γ.

(*) $|f(z)| > |g(z)|$ for all z on γ, then no. of zero of $f+g$ inside γ is the same as no. of zero of f inside γ.

* $f \neq 0$ on γ by assumption

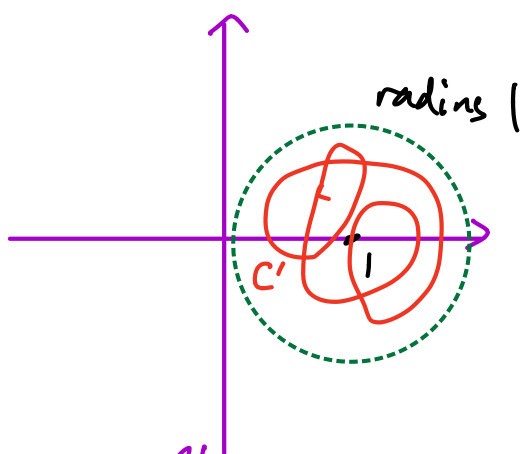
* $f+g \neq 0$ on γ by assumption

Proof

$$\left| \frac{g(\gamma(t))}{f(\gamma(t))} \right| < 1$$

Let C' be the curve parametrized by

$$1 + \frac{g(\gamma(t))}{f(\gamma(t))}$$



$$n(C'; 0) = 0$$

$$n((1+\tfrac{g}{f})\circ\gamma \,;\, 0) = 0$$

$$\frac{1}{2\pi i}\int_\gamma \frac{f'+g'}{f+g}\,dz \overset{?}{=} \frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,dz$$

$$f+g = f\left(1+\frac{g}{f}\right)$$

$$f'+g' = f'\left(1+\frac{g}{f}\right) + f\left(1+\frac{g}{f}\right)'$$

$$\frac{f'+g'}{f+g} = \frac{f'(1+g/f)}{f+g} + \frac{f}{f+g}\left(1+\frac{g}{f}\right)'$$

$$\int_\gamma \frac{(f+g)'}{f+g} = \int_\gamma \frac{f'}{f} + \int_\gamma \frac{(1+g/f)'}{\left(1+\frac{g}{f}\right)}$$

no. of zeros $f+g$ inside γ = no. of zeros of f inside γ

$$+\underbrace{n\left(\left(1+\frac{g}{f}\right)\circ\gamma\,;0\right)}_{=0} \qquad \blacksquare$$

<u>Example</u> $z^{100}+3z^3-1$

find the no. of zeros inside $|z|\le 1$

$$\underbrace{3z^3}_{f(z)} + \underbrace{z^{100}-1}_{g(z)}$$

$$|3z^3| = 3 \qquad \text{for } |z|=1$$

$$|z^{100}-1| \le |z^{100}|+1 \le 2 \qquad \text{for } |z|=1$$

By Rouche theorem

no. of zeros of $z^{100}+3z^3-1$ is the same

as no. of zeros of $3z^3$ inside $|z|\le 1$.

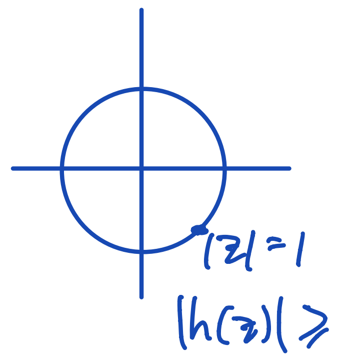
and this no. is equal to 3.

$z^3=0$ has a triple root at 0

Example Consider a polynomial of degree n with leading coefficient 1,

$$h(z) = z^n + c_1 z^{n-1} + c_2 z^{n-2} + \dots + c_n.$$

Show that $|h(z)|$ is larger than or equal to 1 for some point $|z|=1$.



Suppose $\forall z$ with $|z|=1$, $|h(z)|<1$

Apply Rouche theorem with

$$f(z) = z^n \quad \text{and} \quad g(z) = -h(z).$$

We have $|f(z)| = 1 > |-h(z)| = |g(z)|$ for all $|z|=1$.

By Rouche theorem f and $f+g$ has the same no. of zeros inside the unit circle. However $f(z) = z^n$ has exactly n zeros (counted with multiplicity) inside the unit circle, but $f(z)+g(z) = -c_1 z^{n-1} - c_2 z^{n-2} - \dots c_n$ has at most $n-1$ zeros.

Evaluation of real integral

Example
$$\int_0^{2\pi} \frac{d\theta}{1+a\cos\theta} = \frac{2\pi}{\sqrt{1-a^2}} \qquad |a|<1 \quad a \in \mathbb{R}$$

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+z^{-1}}{2} \qquad |z|=1$$

$$z(\theta) = e^{i\theta} \qquad z'(\theta) = ie^{i\theta} = iz$$

$$\int_0^{2\pi} \frac{d\theta}{1+a\cos\theta} = \int_{|z|=1} \frac{dz}{\underbrace{iz}_{\uparrow \frac{1}{z'(t)}}\left(1+a\left(\frac{z+1/z}{2}\right)\right)}$$

$$= \frac{2}{i}\int_{|z|=1} \frac{dz}{2z+az^2+a}$$

$$= 4\pi \operatorname{Res}\left(\frac{1}{az^2+2z+a} ; \frac{-1+\sqrt{1-a^2}}{a}\right)$$

$$az^2+2z+a$$

$$\frac{-2\pm\sqrt{4-4a^2}}{2a}$$

$$= \frac{1}{a}\left(\pm\sqrt{1-a^2}-1\right)$$

$$\alpha = \frac{-1+\sqrt{1-a^2}}{a}$$

$$\beta = \frac{-1-\sqrt{1-a^2}}{a}$$

$$4\pi \operatorname{Res}\left(\frac{1}{a(z-\alpha)(z-\beta)} ; \alpha\right)$$

$$= 4\pi \frac{1}{a}\left(\frac{1}{z-\beta}\right)\Big|_{z=\alpha}$$

$$= \frac{4\pi}{a(\alpha-\beta)}$$

$$= \frac{2\pi}{\sqrt{1-a^2}}$$