MAT 2006 Tutorial 5.

1. (a) $x_n = \dfrac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}$

$x_{n+1} = \dfrac{2n+1}{2n+2} x_n \le x_n$, $0 < x_n \le 1$.

MCT $\Rightarrow \{x_n\}$ cvgs. $x = \lim_{n\to\infty} x_n$.

✓ Method 1.

$x_n = \prod_{k=1}^{n} \dfrac{2k-1}{2k} = \prod_{k=1}^{n} \left(1 - \dfrac{1}{2k}\right)$

$\le \prod_{k=1}^{n} e^{-\frac{1}{2k}} = e^{\sum_{k=1}^{n} -\frac{1}{2k}}$ $(1 - x \le e^{-x})$.

$\sum_{k=1}^{n} -\dfrac{1}{2k} = -\dfrac{1}{2}\sum_{k=1}^{n} \dfrac{1}{k} \to -\infty$, as $n\to\infty$

$\lim_{x\to-\infty} e^{x} = 0 \Rightarrow \lim_{n\to\infty} e^{\sum_{k=1}^{n} -\frac{1}{2k}} = 0$

Squeeze $\Rightarrow \lim_{n\to\infty} x_n = 0$.

✓ Method 2. Telescope everything.

$\left(\dfrac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\right)^2 = \dfrac{1}{2}\cdot\dfrac{1}{2}\cdot\dfrac{3}{4}\cdot\dfrac{3}{4}\cdot\dfrac{5}{6}\cdot\dfrac{5}{6}\cdots\dfrac{2n-1}{2n}\cdot\dfrac{2n-1}{2n}$

$\left(\dfrac{n}{n+1} \le \dfrac{n+1}{n+2}\right)$ $\le \dfrac{1}{2}\cdot\dfrac{2}{3}\cdot\dfrac{3}{4}\cdot\dfrac{4}{5}\cdot\dfrac{5}{6}\cdot\dfrac{6}{7}\cdots\dfrac{2n-1}{2n}\cdot\dfrac{2n}{2n+1}$.

$= \dfrac{1}{2n+1}$

$\Rightarrow x_n \le \sqrt{\dfrac{1}{2n+1}} \to 0$ as $n\to\infty$.

Squeeze $\Rightarrow \lim_{n\to\infty} x_n = 0$.

✓ Method 3 $\dfrac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)} = \dfrac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdots(2n-1)(2n)}{(2\cdot 4\cdot 6\cdots(2n))^2} = \dfrac{(2n)!}{(2^n(n!))^2}$

Stirling. $\lim_{n\to\infty} \dfrac{n!}{\sqrt{2\pi n}\cdot\left(\frac{n}{e}\right)^n} = 1$ $= \dfrac{(2n)!}{2^{2n}(n!)^2}$

$\lim_{n\to\infty} \dfrac{(2n)!}{2^{2n}(n!)^2} = \lim_{n\to\infty} \dfrac{\sqrt{2\pi(2n)}\left(\frac{2n}{e}\right)^{2n}}{2^{2n}\sqrt{2\pi n}\cdot\left(\frac{n}{e}\right)^{n}} = \lim_{n\to\infty} \dfrac{\sqrt{4\pi}}{2\pi}\cdot\dfrac{1}{\sqrt{n}} = 0$.

(b) $x_n \le \sum_{k=n^2}^{(n+1)^2} \frac{1}{\sqrt{n^2}} = \frac{1}{n} \sum_{k=n^2}^{(n+1)^2} 1 = \frac{(n+1)^2 - n^2+1}{n} = \frac{2n+1}{n} \to 2$

as $n \to \infty$.

$x_n \ge \sum_{k=n^2}^{(n+1)^2} \frac{1}{\sqrt{(n+1)^2}} = \sum_{k=n^2}^{(n+1)^2} \frac{1}{n+1} = \frac{2n+2}{n+1} = 2$

$\Rightarrow \lim_{n\to\infty} x_n = 2$.

2. proof. ① If $x = k\pi$ $(k \in \mathbb{Z})$, then $\sin(nx) = 0$.

so $\sum_{n=1}^{\infty} \sin(nx) = 0$, cvgs.

② if $x \ne k\pi$. if $\sum_{n=1}^{\infty} \sin(nx)$ cvgs, then $\lim_{n\to\infty} \sin(nx) = 0$.

$\sin^2(nx) + \cos^2(nx) = 1$, $\forall n$. $\Rightarrow \lim_{n\to\infty} \cos^2(nx) = 1$.

$\underbrace{\sin((n+1)x)}_{\to 0} = \underbrace{\sin(nx)}_{\to 0} \cos x + \cos(nx) \sin x$

$\Rightarrow \lim_{n\to\infty} \cos(nx) \underbrace{\sin x}_{\ne 0} \to 0 \Rightarrow \lim_{n\to\infty} \cos(nx) = 0$ Contradiction!

3. $a_n = \frac{1}{n}$. $\sum_{n=1}^{\infty} a_n$ dvgs

depends on ε & p.

$(*) \Leftrightarrow$. $\forall \varepsilon > 0$. $\forall p \in \mathbb{N}$, $\exists N \in \mathbb{N}$. s.t. $\forall n \ge N$.

$|a_{n+1} + \cdots + a_{n+p}| < \varepsilon$.

Cauchy Criterion

$\Leftrightarrow$ $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$, s.t. $\forall m > n \ge N$.

$|a_{n+1} + \cdots + a_m| < \varepsilon$.

$\Leftrightarrow$ $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$, s.t. $\forall n \ge N$ $\forall p \in \mathbb{N}$.

$m = n+p$, $p \in \mathbb{N}$. depends only on ε.

$|a_{n+1} + \cdots + a_{n+p}| < \varepsilon$.

4. Proof: $\sum_{n=1}^{\infty} a_n$ cvgs $\Rightarrow \lim_{n\to\infty} a_n = 0.$

$\Rightarrow \exists N\in\mathbb{N}$. s.t $\forall n\geq N$. $|a_n - 0| < 1 \Rightarrow 0\leq a_n < 1$

$\Rightarrow \forall n\geq N$ $0\leq a_n^2 < a_n \Rightarrow \sum_{n=1}^{\infty} a_n^2$ cvgs.

5. (a) $\lim_{n\to\infty} (-1)^n \frac{1}{\sqrt[n]{n}}$ does not exist $\Rightarrow$ dvgs.

$\left(\frac{1}{\sqrt[n]{n}} \to 1\right).$

(b) $\left|(-1)^n \frac{\sqrt{n}}{n+2020}\right| = \frac{\sqrt{n}}{n+2020} \geq \frac{1}{n+2020} > 0.$

$\sum_{n=1}^{\infty} \frac{1}{n+2020}$ dvgs $\Rightarrow \sum_{n=1}^{\infty} \left|(-1)^n \frac{\sqrt{n}}{n+2020}\right|$ dvgs

cvgs: W.T.S. $\left\{\frac{\sqrt{n}}{n+2020}\right\} \downarrow 0.$

$\frac{\sqrt{n}}{n+2020} = \frac{1}{\sqrt{n} + \frac{2020}{\sqrt{n}}}$

If $n\geq 2020$, $\frac{1}{\sqrt{n} + \frac{2020}{\sqrt{n}}}$

$\Rightarrow \sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+2020}$ cvgs. By alternating series test

6. Proof. (a) Ratio test: If $|x|<1$.

$\lim_{n\to\infty} \left|\frac{\frac{x^{n+1}}{n+1}}{\frac{x^n}{n}}\right| = \lim_{n\to\infty} \frac{n}{n+1}|x| = |x| < 1 \Rightarrow$ cvgs.

(b) Ratio test:

$\lim_{n\to\infty} \left|\frac{x^{n+1}/(n+1)!}{x^n/n!}\right| = \lim_{n\to\infty} \frac{x}{n+1} = 0 < 1 \Rightarrow$ cvgs