Chapter 3. Sequences and Series *

1 Introduction and Discussion: Rearrangement of Infinite Series

Consider the series

$$S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \cdots = \ln 2. \tag{1.1}$$

The symbols $+$, $-$, and $=$ in the above equation are deceptively familiar notions being used in a very unfamiliar way. The crucial question is whether or not properties of addition and equality that are well understood for finite sums remain valid when applied to infinite objects such as equation (1.1). The answer, as we are about to witness, is somewhat ambiguous.

If we rearrangement the series in (1.1) in the following way

$$\left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{12} + \cdots$$

we obtain a series which is equal to $S/2$. Indeed, addition, in this infinite setting, is NOT commutative!

Let's look at a similar rearrangement of the series

$$\sum_{n=0}^{\infty} (-1/2)^n = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \cdots = \frac{2}{3}. \tag{1.2}$$

This time, some computational experimentation with the "one positive, two negatives" rearrangement

$$\left(1 - \frac{1}{2}\right) - \frac{1}{8} + \left(\frac{1}{4} - \frac{1}{32}\right) - \frac{1}{128} + \cdots = \frac{2}{3}$$

Infinite addition is commutative in some instances but not in others.

*Lecture notes for CUHKSZ course MAT2006: Elementary Real Analysis.

Associativity of addition might also have problems in a infinite series. Consider

$$\sum_{n=0}^{\infty}(-1)^n = 1 - 1 + 1 - 1 + 1 - 1 + \cdots .$$

Group the term one way yields

$$(1-1) + (1-1) + (1-1) + \cdots = 0 + 0 + 0 + \cdots = 0,$$

whereas grouping in another yields

$$1 + (-1+1) + (-1+1) + \cdots = 1 + 0 + 0 + 0 + \cdots = 1.$$

Manipulations that are legitimate in finite settings do not always extend to infinite settings. Deciding when they do and why they do not is one of the central themes of analysis.

It is the pathologies that give rise to the need for rigor. A satisfying resolution to the questions raised will require that we be absolutely precise about what we mean as we manipulate these infinite objects. It may seem that progress is slow at first, but that is because we do not want to fall into the trap of letting the biases of our intuition corrupt our arguments. Rigorous proofs are meant to be a check on intuition, and in the end we will see that they vastly improve our mental picture of the mathematical infinite.

2 The Limit of a Sequence

An understanding of infinite series depends heavily on a clear understanding of the theory of sequences. In fact, most of the concepts in analysis can be reduced to statements about the behavior of sequences. Thus, we will spend a significant amount of time investigating sequences before taking on infinite series.

Definition 1. A *sequence* is a function whose domain is $\mathbb{N}$.

Given a function $f : \mathbb{N} \to \mathbb{R}$, $f(n)$ is just the nth term on the list.

Example 2.1. Each of the following are common ways to describe a sequence.

(i) $\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots\}$; [Attention: don't confuse it with the notation of sets – the sequence specifies a order, while a set does not.]

(ii) $\{\frac{1}{2^n}\}_{n=1}^{\infty}$;

(iii) $\{a_n\}$ where $a_n = 1/2^n$ for each $n \in \mathbb{N}$;

(iii) $\{a_n\}$ where $a_1 = 1/2$ and $a_{n+1} = a_n/2$.

On occasion, it will be more convenient to index a sequence beginning with $n = 0$ or $n = n_0$ for some natural number n_0 different from 1. These minor variations should cause no confusion. What is essential is that a sequence be an *infinite* list of real numbers. What happens at the beginning of such a list is of little importance in most cases. The business of analysis is concerned with the behavior of the infinite "tail" of a given sequence.

Definition 2 (Convergence of a Sequence)**.** A sequence $\{a_n\}$ *converges* to a real number a if, for every positive number ϵ, there exists an $N \in \mathbb{N}$ such that whenever $n \geq N$ it follows that $|a_n - a| < \epsilon$.

By commonly used mathematical notation, we write

$$\begin{aligned} &\forall \epsilon > 0, \quad \exists N \in \mathbb{N}, \quad \text{s.t.} \\ &|a_n - a| < \epsilon, \qquad \forall n \geq N. \end{aligned}$$

To indicate that $\{a_n\}$ converges to a, we usually write either $\lim a_n = a$ or $\{a_n\} \to a$. The notation $\lim_{n\to\infty} a_n = a$ is also standard.

Definition 3 (Neighborhood)**.** Given a real number $a \in \mathbb{R}$ and a positive number $\epsilon > 0$, the set

$$V_\epsilon(a) = \{x \in \mathbb{R} \mid |x - a| < \epsilon\}$$

is called the ϵ-*neighborhood* of a.

Notice that $V_\epsilon(a)$ consists of all of those points whose distance from a is less than ϵ. Said another way, $V_\epsilon(a)$ is an open interval, centered at a, with radius ϵ.

Recasting the definition of convergence in terms of ϵ-neighborhoods gives a more geometric impression of what is being described.

Definition 4 (Convergence of a Sequence: Topological Version)**.** A sequence $\{a_n\}$ converges to a if, given any ϵ-neighborhood V_ϵ of a, there exists a point in the sequence after which all of the terms are in $V_\epsilon(a)$. In other words, every ϵ-neighborhood contains all but a finite number of the terms of $\{a_n\}$.

Definition 2 and Definition 4 say precisely the same thing; the natural number N in the original version of the definition is the point where the sequence $\{a_n\}$ enters $V_\epsilon(a)$, never to leave. It should be apparent that the value of N depends on the choice of ϵ. The smaller the ϵ-neighborhood, the larger N may have to be.

Example 2.2. Consider the sequence $\{a_n\}$, where $a_n = \frac{1}{\sqrt{n}}$.

Intuitively, we have

$$\lim_{n\to\infty} a_n = 0.$$

Proof. Let $\epsilon > 0$ be an arbitrary positive number. Choose a natural number N satisfying

$$N > \frac{1}{\epsilon^2}.$$

We now verify that this choice of N has the desired property. Let $n \geq N$. Then,

$$n > \frac{1}{\epsilon^2} \quad \text{implies} \quad \frac{1}{\sqrt{n}} < \epsilon \qquad \text{and hence} \quad |a_n - 0| < \epsilon.$$

$\square$

Quantifiers

The definition of convergence given earlier is the result of hundreds of years of refining the intuitive notion of limit into a mathematically rigorous statement. The logic involved is complicated and is intimately tied to the use of the quantifiers "for all" and "there exists." Learning to write a grammatically correct convergence proof goes hand in hand with a deep understanding of why the quantifiers appear in the order that they do.

The definition begins with the phrase,

"*For all* $\epsilon > 0$, *there exists* $N \in \mathbb{N}$ *such that* ..."

Looking back at our first example, we see that our formal proof begins with, "Let $\epsilon > 0$ be an arbitrary positive number." This is followed by a construction of N and then a demonstration that this choice of N has the desired property. This, in fact, is a basic outline for how every convergence proof should be presented.

Template for a proof that $\{x_n\} \to x$:

- "Let $\epsilon > 0$ be arbitrary."
- Demonstrate a choice for $N \in \mathbb{N}$. This step usually requires the most work, almost all of which is done prior to actually writing the formal proof.
- Now, show that N actually works.
- "Assume $n \geq N$"
- With N well chosen, it should be possible to derive the inequality $|x_n - x| < \epsilon$.

Example 2.3. Show that
$$\lim_{n\to\infty}\left(\frac{n+1}{n}\right)=1.$$

Theorem 1 (Uniqueness of Limits)**.** *The limit of a sequence, when it exists, must be unique.*

Definition 5 (Divergence)**.** A sequence that does not converge is said to *diverge*.

3 Algebraic and Order Limit Theorem

The real purpose of creating a rigorous definition for convergence of a sequence is so that we can confidently *prove statements about convergent sequences in general.* We are ultimately trying to resolve arguments about what is and is not true regarding the behavior of limits with respect to the mathematical manipulations we intend to inflict on them.

As a first example, let us prove that convergent sequences are bounded. Recall that a set A is bounded if it is both bounded above and bounded below. Or equivalently, there exists $M>0$ such that $|a|<M$ for all $a\in A$.

Theorem 2. *Every convergent sequence is bounded.*

Proof. Assume $\{x_n\}$ converges to a limit ℓ. This means that given a particular value of ϵ, say $\epsilon=1$, we know there must exist an $N\in\mathbb{N}$ such that if $n\geq N$, then x_n is in the interval $(\ell-1,\ell+1)$. Not knowing whether ℓ is positive or negative, we can certainly conclude that
$$|x_n|\leq|\ell|+1,\qquad\forall n\geq N.$$
Put
$$M=\max\{|x_1|,|x_2|,\cdots,|x_{N-1}|,|\ell|+1\}.$$
It follows that $|x_n|\leq M$ for all $n\in\mathbb{N}$, as desired. $\square$

Theorem 3 (Algebraic Limit Theorem)**.** *Let* $\lim a_n=a$*, and* $\lim b_n=b$*. Then,*

(i) $\lim(ca_n)=ca$*, for all* $c\in\mathbb{R}$*;*
(ii) $\lim(a_n+b_n)=a+b$*;*
(iii) $\lim(a_nb_n)=ab$*;*
(iv) $\lim(a_n/b_n)=a/b$ *provided that* $b\neq0$*.*

Theorem 4 (Order Limit Theorem)**.** *Assume* $\lim a_n=a$ *and* $\lim b_n=b$*. Then,*

(i) *If* $a_n\geq0$ *for all* $n\in\mathbb{N}$*, then* $a\geq0$*.*
(ii) *If* $a_n\leq b_n$ *for all* $n\in\mathbb{N}$*, then* $a\leq b$*.*
(iii) *If there exists* $c\in\mathbb{R}$ *for which* $c\leq b_n$ *for all* $n\in\mathbb{N}$*, then* $c\leq b$*. Similarly, if* $a_n\leq c$ *for all* $n\in\mathbb{N}$*, then* $a\leq c$*.*

Theorem 5 (Squeeze Theorem)**.** *Show that if* $x_n \leq y_n \leq z_n$ *for all* $n \in \mathbb{N}$*, and if* $\lim x_n = \lim z_n = \ell$*, then* $\lim y_n = \ell$ *as well.*

Exercise 1 (Some special sequences)**.** (i) If $p > 0$, then $\lim \frac{1}{n^p} = 0$;
(ii) If $p > 0$, then $\lim \sqrt[n]{p} = 1$;
(iii) $\lim \sqrt[n]{n} = 1$;
(iv) If $0 < |x| < 1$, then $\lim |x|^n = 0$.

Exercise 2 (Cesaro Means)**.** (i) Show that if $\{x_n\}$ is a convergent sequence, then the sequence given by the averages

$$y_n = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

also converges to the same limit.

(ii) Give an example to show that it is possible for the sequence $\{y_n\}$ of averages to converge even if $\{x_n\}$ does not.

Exercise 3. Show that

$$\text{(i)} \qquad \lim_{n\to\infty} \sqrt[n]{1 + \frac{a}{n}} = 1, \quad \text{where} \quad a > 0.$$

$$\text{(ii)} \qquad \lim_{n\to\infty} \frac{n^k}{n!} = 0, \quad \text{where} \quad k \in \mathbb{N}.$$

$$\text{(iii)} \qquad \lim_{n\to\infty} \frac{n^k}{a^n} = 0, \quad \text{where} \quad a > 1, \quad k \in \mathbb{N}.$$

$$\text{(iv)} \qquad \lim_{n\to\infty} \frac{a^n}{n!} = 0, \quad \text{where} \quad a \in \mathbb{R}.$$

$$\text{(v)} \qquad \lim_{n\to\infty} \sqrt[2n+1]{n^2 + n} = 1.$$

$$\text{(vi)} \qquad \lim_{n\to\infty} \sqrt[n]{\frac{a^n}{n} + \frac{b^n}{n^2}} = b, \quad \text{where} \quad b \geq a > 0.$$

4 The Monotone Convergence Theorem

We showed in Theorem 2 that convergent sequences are bounded. The converse statement is certainly not true. It is not too difficult to produce an example of a bounded sequence that does not converge. On the other hand, if a bounded sequence is *monotone*, then in fact it does converge.

Definition 6. A sequence $\{a_n\}$ is *increasing* if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$ and *decreasing* if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$. A sequence is *monotone* if it is either increasing or decreasing.

Theorem 6 (Monotone Convergence Theorem)**.** *If a sequence is monotone and bounded, then it converges.*

Proof. Let $\{a_n\}$ be monotone and bounded. Assume the sequence is increasing (the decreasing case is handled similarly). Consider the set of points $\{a_n \,|\, n \in \mathbb{N}\}$. By assumption, this set is bounded, so we can let

$$s = \sup\{a_n \,|\, n \in \mathbb{N}\}.$$

We claim that $\lim_{n\to\infty} a_n = s$. To prove this, let $\epsilon > 0$. Because s is the least upper bound for $\{a_n \,|\, n \in \mathbb{N}\}$, $s - \epsilon$ is not an upper bound, so there exists a point in the sequence a_N such that $s - \epsilon < a_N$. Now, the fact that $\{a_n\}$ is increasing implies that if $n \geq N$, then $a_n \geq a_N$. Hence,

$$s - \epsilon \leq a_N \leq a_n \leq s < s + \epsilon$$

which implies $|a_n - s| \leq \epsilon$, as desired. □

The Monotone Convergence Theorem is extremely useful for the study of infinite series, largely because it asserts the convergence of a sequence without explicit mention of the actual limit.

Definition 7. Let $\{b_n\}$ be a sequence. An *infinite series* is a formal expression of the form

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + b_3 + \cdots .$$

We define the corresponding sequence of *partial sums* $\{s_m\}$ by

$$s_m = \sum_{n=1}^{m} b_n = b_1 + b_2 + \cdots + b_m,$$

and say that the series $\sum_{n=1}^{\infty} b_n$ *converges* to B if the sequence $\{s_m\}$ converges to B. In this case, we write

$$\sum_{n=1}^{\infty} b_n = B.$$

Example 4.1. Consider

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

Because the terms in the sum are all positive, the sequence of partial sums given by

$$s_m = \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{m^2}$$

is increasing. The question is whether or not we can find some upper bound on $\{s_m\}$. To this end, observe

$$\begin{aligned} s_m =& 1 + \frac{1}{2\cdot 2} + \frac{1}{3\cdot 3} + \frac{1}{m\cdot m} \\ <& 1 + \frac{1}{2\cdot 1} + \frac{1}{3\cdot 2} + \frac{1}{m\cdot (m-1)} \\ =& 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{m-1} - \frac{1}{m}\right) \\ =& 2 - \frac{1}{m} \\ <& 2. \end{aligned}$$

Thus, 2 is an upper bound for the sequence of partial sums, so by the Monotone Convergence Theorem, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges to some (for the moment) unknown limit less than 2. (Finding the value of this limit is the subject of a later chapter.)

Example 4.2 (Harmonic series)**.** Consider

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

Again, we have an increasing sequence of partial sums,

$$s_m = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{m}.$$

However, the sequences $\{s_m\}$ is unbounded.

$$\begin{aligned} s_{2^k} =& 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^{k-1}+1} + \cdots + \frac{1}{2^k}\right) \\ >& 1 + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2} \\ =& 1 + \frac{k}{2}, \end{aligned}$$

which is unbounded. Thus, despite the incredibly slow pace, the sequence of partial sums of $\sum_{n=1}^{\infty} \frac{1}{n}$ eventually surpasses every number on the positive real line. Because convergent sequences are bounded, the harmonic series diverges.

5 Subsequences and the Bolzano–Weierstrass Theorem

5.1 Subsequences

Definition 8. Let $\{a_n\}$ be a sequence of real numbers, and let $n_1 < n_2 < n_3 < \cdots$ be an increasing sequence of natural numbers. Then the sequence

$$a_{n_1},\ a_{n_2},\ a_{n_3},\ a_{n_4},\ \ldots$$

is called a subsequence of $\{a_n\}$ and is denoted by $\{a_{n_k}\}$, where $k \in \mathbb{N}$ indexes the subsequence.

Theorem 7. *Subsequences of a convergent sequence converge to the same limit as the original sequence.*

Proof. Assume $\{a_n\} \to a$, and let $\{a_{n_k}\}$ be a subsequence. Given $\epsilon > 0$, there exists N such that $|a_n - a| < \epsilon$ whenever $n \geq N$. Because $n_k \geq k$ for all k, the same N will suffice for the subsequence; that is, $|a_{n_k} - a| < \epsilon$ whenever $k \geq N$. □

Example 5.1. Let $0 < b < 1$. Because

$$b > b^2 > b^3 > b^4 > \cdots > 0,$$

the sequence $\{b_n\}$ is decreasing and bounded below. The Monotone Convergence Theorem allows us to conclude that b_n converges to some ℓ satisfying $b > \ell \geq 0$. To compute ℓ, notice that $\{b^{2n}\}$ is a subsequence, so $\{b^{2n}\} \to \ell$ by Theorem 7. But $b^{2n} = b^n \cdot b^n$, so by the Algebraic Limit Theorem, $\{b^{2n}\} \to \ell \cdot \ell = \ell^2$. Because limits are unique (Theorem 1), $\ell^2 = \ell$, and thus $\ell = 0$.

Example 5.2 (Divergence Criterion)**.** Consider the sequence

$$\{(-1)^{n+1}\}_{n=1}^{\infty} = \{1, -1, 1, -1, 1, -1, \ldots\},$$

which does not converge to any proposed limit. Notice that

$$\{1, 1, 1, 1, 1, \cdots\}$$

is a subsequence that is converge to 1. Also

$$\{-1, -1, -1, -1, -1, \cdots\}$$

is a different subsequence of the original sequence that converges to -1. Because we have two subsequences converging to two different limits, we can rigorously conclude that the original sequence diverges.

5.2 The Bolzano–Weierstrass Theorem

In the previous example, it was rather easy to spot a convergent subsequence (or two) hiding in the original sequence. For bounded sequences, it turns out that it is always possible to find at least one such convergent subsequence.

Theorem 8 (Bolzano–Weierstrass Theorem)**.** *Every bounded sequence contains a convergent subsequence.*

Proof. Let $\{a_n\}$ be a bounded sequence so that there exists $M > 0$ satisfying $|a_n| \leq M$ for all $n \in \mathbb{N}$. Bisect the closed interval $[-M, M]$ into the two closed intervals $[-M, 0]$ and $[0, M]$. (The midpoint is included in both halves.) Now, it must be that at least one of these closed intervals contains an infinite number of the terms in the sequence $\{a_n\}$. Select a half for which this is the case and label that interval as I_1. Then, let a_{n_1} be some term in the sequence $\{a_n\}$ satisfying $a_{n_1} \in I_1$.

Next, we bisect I_1 into closed intervals of equal length, and let I_2 be a half that again contains an infinite number of terms of the original sequence. Because there are an infinite number of terms from $\{a_n\}$ to choose from, we can select an a_{n_2} from the original sequence with $n_2 > n_1$ and $a_{n_2} \in I_2$. In general, we construct the closed interval I_k by taking a half of I_{k-1} containing an infinite number of terms of $\{a_n\}$ and then select $n_k > n_{k-1} > \cdots > n_2 > n_1$ so that $a_{n_k} \in I_k$.

We claim that $\{a_{n_k}\}$ is a convergent subsequence, and we need find its limit first. The sets

$$I_1 \supset I_2 \supset I_3 \supset I_4 \supset \cdots$$

form a nested sequence of closed intervals, and by the Nested Interval Property there exists at least one point $x \in \mathbb{R}$ contained in every I_k. This provides us with the candidate we were looking for. It just remains to show that $a_{n_k} \to x$.

Let $\epsilon > 0$. By construction, the length of I_k is $M/2^{k-1}$ which converges to zero. Choose N so that $k \geq N$ implies that the length of I_k is less than ϵ. Because x and a_{n_k} are both in I_k, it follows that $|a_{n_k} - x| < \epsilon$. $\square$

Exercise 4. Assume $\{a_n\}$ is a bounded sequence with the property that every convergent subsequence of $\{a_n\}$ converges to the same limit $a \in \mathbb{R}$. Show that $\{a_n\}$ must converge to a.

5.3 Upper and Lower Limits

Given a bounded sequence $\{a_n\}$, not necessarily convergent, the sequence

$$\big\{ \sup\{a_n \,|\, n \geq m\} \big\}_{m=1}^{\infty}$$

is clearly decreasing. Then the Monotone Convergence Theorem says that the above sequence is convergent, and we define the *upper limit* of $\{a_n\}$ by

$$\limsup_{n\to\infty} a_n = \lim_{m\to\infty} \sup\{a_n \,|\, n \geq m\}.$$

Similarly, we can define the *lower limit* of $\{a_n\}$ by

$$\liminf_{n\to\infty} a_n = \lim_{m\to\infty} \inf\{a_n \,|\, n \geq m\}.$$

Remark. The notation $\overline{\lim}$ and $\underline{\lim}$ is also sometimes used to stand the $\limsup$ and $\liminf$ respectively. In certain circumstances, we may also allowed the upper and lower limits assume the value $+\infty$ or $-\infty$ for the sake of convenience.

Exercise 5. Show that for a bounded sequence $\{x_n\}$

$$\limsup_{n\to\infty} x_n = \inf\Big\{\sup\{x_n\}_{n=m}^{\infty} \,|\, m \in \mathbb{N}\Big\},$$

and that

$$\liminf_{n\to\infty} x_n = \sup\Big\{\inf\{x_n\}_{n=m}^{\infty} \,|\, m \in \mathbb{N}\Big\}.$$

This is the definition of upper and lower limits used in [Tao].

Exercise 6. For a bounded sequence $\{x_n\}$, the Bolzano–Weierstrass Theorem says that there exists a convergent subsequence. Let E be the set of real numbers s such that $x_{n_k} \to s$ for some subsequence $\{x_{n_k}\}$. Show that

$$\limsup_{n\to\infty} x_n = \sup E \qquad \text{and} \qquad \liminf_{n\to\infty} x_n = \inf E.$$

This is the definition of upper and lower limits used in [Rudin].

Exercise 7. For the following sequences, find their upper and lower limits.

$$\text{(i)} \quad \{(-1)^n\}_{n=1}^{\infty}, \qquad \text{(ii)} \quad \{(-1)^n n\}_{n=1}^{\infty}, \qquad \text{(iii)} \quad \left\{(-1)^n \frac{1}{n}\right\}_{n=1}^{\infty}$$

Theorem 9. *A sequence $\{x_n\}$ is convergent if and only if $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n$. In this case, all three share the same value.*

Corollary 10. *If every subsequence of $\{x_n\}$ converges to the same limit x, then $\{x_n\}$ converges to x.*

6 The Cauchy Criterion

Definition 9. A sequence $\{a_n\}$ is called a *Cauchy sequence* if, for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $m, n \geq N$ it follows that $|a_n - a_m| < \epsilon$.

As we have discussed, the definition of convergence asserts that, given an arbitrary positive ϵ, it is possible to find a point in the sequence after which the terms of the sequence are all closer to the limit a than the given ϵ. On the other hand, a sequence is a Cauchy sequence if, for every $\epsilon > 0$, there is a point in the sequence after which the terms are all closer to each other than the given ϵ. To spoil the surprise, we will argue in this section that in fact these two definitions are equivalent: Convergent sequences are Cauchy sequences, and Cauchy sequences converge. The significance of the definition of a Cauchy sequence is that there is no mention of a limit. This is somewhat like the situation with the Monotone Convergence Theorem in that we will have another way of proving that sequences converge without having any explicit knowledge of what the limit might be.

Theorem 11. *Every convergent sequence is a Cauchy sequence.*

Proof. Assume $\{x_n\}$ converges to x. Given any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $|x_n - x| < \epsilon/2$ whenever $n \geq N$. Hence, for every $m, n \geq N$, we have

$$|x_m - x_n| = |(x_m - x) - (x_n - x)| \leq |x_m - x| + |x_n - x| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus $\{x_n\}$ is a Cauchy sequence. $\square$

The converse is a bit more difficult to prove, mainly because, in order to prove that a sequence converges, we must have a proposed limit for the sequence to approach. We have been in this situation before in the proofs of the Monotone Convergence Theorem and the Bolzano–Weierstrass Theorem. Our strategy here will be to use the Bolzano–Weierstrass Theorem. We shall need the following lemma.

Lemma 12. *Cauchy sequences are bounded.*

Proof. Given $\epsilon = 1$, there exists an N such that $|x_m - x_n| \leq 1$ for all $m, n \geq N$. Thus, we must have $|x_n| \leq |x_N| + 1$ for all $n \geq N$. It follows that

$$M = \max\{|x_1|, |x_2|, \ldots, |x_{N-1}|, |x_N| + 1\}$$

is a bound for the sequence $\{x_n\}$. $\square$

Theorem 13 (Cauchy Criterion). *A sequence converges if and only if it is a Cauchy sequence.*

Proof. (⇒) This direction is Theorem 11.

(⇐) For this direction, we start with a Cauchy sequence $\{x_n\}$. Lemma 12 guarantees that $\{x_n\}$ is bounded, so we may use the Bolzano–Weierstrass Theorem to produce a convergent subsequence $\{x_{n_k}\}$. Set

$$x = \lim_{k\to\infty} x_{n_k}$$

We are going to show that the original sequence $\{x_n\}$ converge to x.

Let $\epsilon > 0$. Because $\{x_n\}$ is Cauchy, there exists N such that

$$|x_n - x_m| \leq \frac{\epsilon}{2}$$

whenever $m, n \geq N$. Now, we also know that $\{x_{n_k}\} \to x$, so choose a term in this subsequence, call it x_{n_K}, with $n_K \geq N$ and

$$|x_{n_K} - x| \leq \frac{\epsilon}{2}.$$

Then, whenever $n \geq N$

$$\begin{aligned}
|x_n - x| =& |(x_n - x_{n_K}) + (x_{n_K} - x)| \\
\leq& |x_n - x_{n_K}| + |x_{n_K} - x| \\
<& \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
=& \epsilon.
\end{aligned}$$

Hence the sequence $\{x_n\}$ converges to x. □

The Cauchy Criterion is named after the French mathematician Augustin Louis Cauchy. Cauchy is a major figure in the history of many branches of mathematics – number theory and the theory of finite groups, to name a few – but he is most widely recognized for his enormous contributions in analysis, especially complex analysis. He is deservedly credited with inventing the ϵ-based definition of limits we use today, although it is probably better to view him as a pioneer of analysis in the sense that his work did not attain the level of refinement that modern mathematicians have come to expect. The Cauchy Criterion, for instance, was devised and used by Cauchy to study infinite series, but he never actually proved it in both directions. The fact that there were gaps in Cauchy's work should not diminish his brilliance in any way. The issues of the day were both difficult and subtle, and Cauchy was far and away the most influential in laying the groundwork for modern standards of rigor. Karl Weierstrass played a major role in sharpening Cauchy's arguments. We will hear a good deal more from Weierstrass, most notably in a later chapter when we take up uniform convergence. Bernhard Bolzano was working in Prague and was writing and thinking about many of these same issues surrounding limits and continuity. Because his work was not widely available to the rest of the mathematical community, his historical reputation never achieved the distinction that his impressive accomplishments would seem to merit.

Exercise 8. Give an example of each of the following, or argue that such a request is impossible.

(i) A Cauchy sequence that is not monotone.
(ii) A Cauchy sequence with an unbounded subsequence.
(iii) A divergent monotone sequence with a Cauchy subsequence.
(iv) An unbounded sequence containing a subsequence that is Cauchy.

Completeness Revisited

In the previous chapter, we established the Axiom of Completeness (AoC) to be the assertion that nonempty sets bounded above have least upper bounds (LUBP). We then used this axiom as the crucial step in the proof of the Nested Interval Property (NIP). We have also proved that the LUBP and the Dedekind's Cut Property (CP) is equivalent. In this chapter, LUBP was the central step in the Monotone Convergence Theorem (MCT), and NIP was the key to proving the Bolzano–Weierstrass Theorem (BW). Finally, we needed BW in our proof of the Cauchy Criterion (CC) for convergent sequences. The list of implications then looks like

$$\text{CP} \iff \text{LUBP} \implies \begin{cases} \left.\begin{array}{l}\text{(AP)}\\ \text{NIP}\end{array}\right\} \implies \text{BW} \implies \text{CC} \\ \text{MCT} \end{cases}$$

But this one-directional list is not the whole story. Recall that in our original discussions about completeness, the fundamental problem was that the rational numbers contained "gaps." The reason for moving from the rational numbers to the real numbers to do analysis is so that when we encounter a sequence that looks as if it is converging to some number – say $\sqrt{2}$ – then we can be assured that there is indeed a number there that we can call the limit. The assertion that "nonempty sets bounded above have least upper bounds" is simply one way to mathematically articulate our insistence that there be no "holes" in our ordered field, but it is not the only way. Instead, we could have taken MCT to be our defining axiom and used it to prove NIP and LUBP.

How about NIP? Could this property serve as a starting point for a proper axiomatic treatment of the real numbers? Almost. We can show that NIP implies LUBP (exercise), but to prevent the argument from making implicit use of LUBP we needed an extra assumption that is equivalent to the Archimedean Property. This extra hypothesis is unavoidable.

Whereas LUBP and MCT can both be used to prove that $\mathbb{N}$ is not a bounded subset of $\mathbb{R}$, there is no way to prove this same fact starting from NIP. The upshot is that NIP is a perfectly reasonable candidate to use as the fundamental axiom of the real numbers provided that we also include the Archimedean Property as a second unproven assumption.

In fact, if we assume the Archimedean Property holds, then CP, LUBP, NIP, MCT, BW, and CC are equivalent in the sense that once we take any one of them to be true, it is possible to derive the other five. However, because we have an example of an ordered field that is not complete – namely, the set of rational numbers – we know it is impossible to prove any of them using only the field and order properties. Just how we decide which should be the axiom and which then become theorems depends largely on preference and context, and in the end is not especially significant. What is important is that we understand all of these results as belonging to the same family, each asserting the completeness of $\mathbb{R}$ in its own particular language.

One loose end in this conversation is the curious and somewhat unpredictable relationship of the Archimedean Property to these other results. As we have mentioned, the Archimedean Property follows as a consequence of LUBP as well as MCT, but not from NIP. Starting from BW, it is possible to prove MCT and thus also the Archimedean Property. On the other hand, the Cauchy Criterion is like NIP in that it cannot be used on its own to prove the Archimedean Property.

Exercise 9 (MCT implies AP)**.** We used the LUBP to prove the Archimedean Property (AP) of $\mathbb{R}$ (Theorem 4 of Chapter 2). Show that the Monotone Convergence Theorem can also be used to prove the Archimedean Property without making any use of LUBP.

Exercise 10 (MCT implies NIP)**.** Use the Monotone Convergence Theorem to supply a proof for the Nested Interval Property that doesn't make use of LUBP.

Exercise 11 (NIP+AP implies LUBP)**.** Assume the Nested Interval Property is true. Use the technique in proving the Bolzano–Weierstrass Theorem to provide a proof of the LUBP. To prevent the argument from being circular, assume also that $1/2^n \to 0$. (Why precisely is this last assumption needed to avoid circularity?)

The above three exercises means that MCT implies LUBP, and thence they are equivalent.

$$\text{MCT} \quad \Longrightarrow \left\{ \begin{matrix} \text{(AP)} \\ \text{NIP} \end{matrix} \right\} \Longrightarrow \quad \text{LUBP}.$$

Exercise 12 (BW implies MCT)**.** Assume the Bolzano–Weierstrass Theorem is true and use it to construct a proof of the Monotone Convergence Theorem without making any appeal to the Archimedean Property. This shows that BW, LUBP, and MCT are all equivalent.

Exercise 13 (CC+AP implies BW)**.** Use the Cauchy Criterion to prove the Bolzano–Weierstrass Theorem, and find the point in the argument where the Archimedean Property is implicitly required.

7 Infinite Series

7.1 Properties of Infinite Series

Recall that the convergence of the series $\sum_{k=1}^{\infty} a_k$ is defined in terms of the sequence of partial sums $\{s_n\}$, where

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n.$$

Specifically, the statement

$$\sum_{k=1}^{\infty} a_k = A \qquad \text{means that} \qquad \lim_{n\to\infty} s_n = A.$$

It is for this reason that we can immediately translate many of our results from the study of sequences into statements about the behavior of infinite series.

Theorem 14 (Algebraic Limit Theorem for Series)**.** *If $\sum_{k=1}^{\infty} a_k = A$ and $\sum_{k=1}^{\infty} b_k = B$, then*
(i) $\sum_{k=1}^{\infty} ca_k = cA$ *for all* $c \in \mathbb{R}$ *and*
(ii) $\sum_{k=1}^{\infty}(a_k + b_k) = A + B$.

Proof. (i) In order to show that $\sum_{k=1}^{\infty} ca_k = cA$, we must argue that the sequence of partial sums

$$t_m = ca_1 + ca_2 + \cdots + ca_m$$

converges to cA. But we are given that $\sum_{k=1}^{\infty} a_k$ converges to A, meaning that the partial sums

$$s_m = a_1 + a_2 + \cdots + a_m$$

converge to A. Because $t_m = cs_m$, applying the Algebraic Limit Theorem for sequences (Theorem 3) yields $\{t_m\} \to cA$, as desired.

The proof of part (ii) is analogous and is left as an unofficial exercise.

□

One way to summarize part (i) of the above theorem is to say that infinite addition still satisfies the distributive property. Part (ii) verifies that series can be added in the usual way. Missing from this theorem is any statement about the product of two infinite series. At the heart of this question is the issue of commutativity, which requires a more delicate analysis and so is postponed until the next section.

Theorem 15 (Cauchy Criterion for Series)**.** *The series $\sum_{k=1}^{\infty} a_k$ converges if and only if, given $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $n > m \geq N$ it follows that*

$$|a_{m+1} + a_{m+2} + \cdots + a_n| < \epsilon.$$

Proof. Observe that

$$|s_n - s_m| = |a_{m+1} + a_{m+2} + \cdots + a_n|$$

and apply the Cauchy Criterion for sequences. □

The Cauchy Criterion leads to economical proofs of several basic facts about series.

Theorem 16. *If the series $\sum_{k=1}^{\infty} a_k$ converges, then $\{a_k\} \to 0$.*

Proof. Consider the special case $n = m + 1$ in the Cauchy Criterion for Series. □

Every statement of this result should be accompanied with a reminder to look at the harmonic series (Example 4.2) to erase any misconception that the converse statement is true. Knowing $\{a_k\}$ tends to 0 does not imply that the series converges – it is only a necessary condition for a series converges.

Exercise 14. (i) Assume the series $\sum_{n=0}^{\infty} a_n$ converges, with a_n decreasing and $a_n > 0$, show that $\lim_{n\to\infty} na_n = 0$.

(ii) Find a convergent series $\sum_{n=0}^{\infty} a_n$ satisfies

$$(a) \quad a_n > 0 \qquad \forall n \in \mathbb{N}, \qquad \text{and} \qquad (b) \quad a_n \neq o\left(\frac{1}{n}\right).$$

7.2 Convergence Tests

Comparison Test

Theorem 17 (Comparison Test)**.** *Assume a_k and b_k are sequences satisfying $0 \leq a_k \leq b_k$ for all $k \in \mathbb{N}$.*

(i) *If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.*

(ii) *If $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ diverges.*

sketch of proof. Both statements follow immediately from the Cauchy Criterion for Series and the observation that

$$|a_{m+1} + a_{m+2} + \cdots + a_n| \leq |b_{m+1} + b_{m+2} + \cdots + b_n|.$$

□

Exercise 15. (a) Provide the details for the proof of the Comparison Test using the Cauchy Criterion for Series.

(b) Give another proof for the Comparison Test, this time using the Monotone Convergence Theorem.

This is a good point to remind ourselves again that statements about convergence of sequences and series are immune to changes in some finite number of initial terms. In the Comparison Test, the requirement that $0 \leq a_k \leq b_k$ does not really need to hold for all $k \in \mathbb{N}$ but just needs to be eventually true. A weaker, but sufficient, hypothesis would be to assume that there exists some point $M \in \mathbb{N}$ such that the inequality $a_k \leq b_k$ is true for all $k \geq M$.

Exercise 16. Show that the series $\sum_{n=1}^{\infty} 1/n^p$ diverges if $0 < p < 1$.

Exercise 17. Show that if the series $\sum_{n=1}^{\infty} a_n$ converges and $a_n \geq 0$ for all $n \in \mathbb{N}$, then $\sum_{n=1}^{\infty} a_n^2$ also converges.

The Comparison Test is used to deduce the convergence or divergence of one series based on the behavior of another. Thus, for this test to be of any great use, we need a catalog of series we can use as measuring sticks. The Cauchy Condensation Test can be used to led to the general statement that the series $\sum_{n=1}^{\infty} 1/n^p$ converges if and only if $p > 1$.

The next example summarizes the situation for another important class of series.

Example 7.1 (Geometric Series)**.** A series is called *geometric* if it is of the form

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \cdots .$$

If $r = 1$ and $a \neq 0$, the series evidently diverges. For $r \neq 1$, the algebraic identity

$$(1 - r)(1 + r + r^2 + \cdots + r^{m-1}) = 1 - r^m$$

enables us to rewrite the partial sum

$$s_m = a + ar = ar^2 + \cdots + ar^{m-1} = \frac{a(1 - r^m)}{1 - r}.$$

Now the Algebraic Limit Theorem for sequences justify the conclusion

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1 - r}$$

if and only if $|r| < 1$.

Absolute Convergence Test

Although the Comparison Test requires that the terms of the series be positive, it is often used in conjunction with the next theorem to handle series that contain some negative terms.

Theorem 18 (Absolute Convergence Test)**.** *If the series $\sum_{n=1}^{\infty}|a_n|$ converges, then $\sum_{n=1}^{\infty}a_n$ converges as well.*

Proof. Because $\sum_{n=1}^{\infty}|a_n|$ converges, we know that, given an $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that

$$|a_{m+1}| + |a_{m+2}| + \cdots + |a_n| < \epsilon$$

for all $n > m \geq N$. By the triangle inequality,

$$|a_{m+1} + a_{m+2} + \cdots + a_n| \leq |a_{m+1}| + |a_{m+2}| + \cdots + |a_n|,$$

so the sufficiency of the Cauchy Criterion guarantees that $\sum_{n=1}^{\infty}a_n$ also converges. □

The converse of this theorem is false. In the opening discussion of this chapter, we considered the *alternating harmonic series*

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \cdots$$

Taking absolute values of the terms gives us the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$, which we have seen diverges. However, it is not too difficult to prove that with the alternating negative signs the series indeed converges. This is a special case of the Alternating Series Test.

Alternating Series Test

Theorem 19 (Alternating Series Test)**.** *Let $\{a_n\}$ be a sequence satisfying,*

(i) $a_1 \geq a_2 \geq a_3 \geq \cdots$ *and*

(ii) $\{a_n\} \to 0$.

Then, the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges.

Exercise 18. Proving the Alternating Series Test amounts to showing that the sequence of partial sums

$$s_n = a_1 - a_2 + a_3 + \cdots + (-1)^{n+1}a_n$$

converges. Different characterizations of completeness lead to different proofs.

(a) Prove the Alternating Series Test by showing that $\{s_n\}$ is a Cauchy sequence.

(b) Supply another proof for this result using the Nested Interval Property.

(c) Consider the subsequences $\{s_{2n}\}$ and $\{s_{2n+1}\}$, and show how the Monotone Convergence Theorem leads to a third proof for the Alternating Series Test.

Exercise 19. Give an example of divergent alternating series $\sum_{n=1}^{\infty}(-1)^n a_n$ with $a_n \geq 0$ for all $n \in \mathbb{N}$ and $\{a_n\} \to 0$.

Definition 10 (Absolute Convergence)**.** If $\sum_{n=1}^{\infty} |a_n|$ converges, then we say that the original series $\sum_{n=1}^{\infty} a_n$ *converges absolutely.* If, on the other hand, the series $\sum_{n=1}^{\infty} a_n$ converges but the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ does not converge, then we say that the original series $\sum_{n=1}^{\infty} a_n$ *converges conditionally.*

In terms of this newly defined jargon, we have shown that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

converges conditionally, whereas

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}, \qquad \sum_{n=1}^{\infty} \frac{1}{2^n}, \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n}$$

converges absolutely. In particular, any convergent series with (all but finitely many) positive terms must converge absolutely.

Ratio Test

Theorem 20 (Ratio Test)**.** *Given a series $\sum_{n=1}^{\infty} a_n$ with $a_n \neq 0$, assume that*

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = r,$$

(i) *if $0 \leq r < 1$ then the series converges absolutely;*
(ii) *if $r > 1$ then the series diverges.*

Exercise 20. Prove the Ratio Test using Absolute Convergence Test.
Hint. (a) Let q satisfy $r < q < 1$. Explain why there exists an N such that $n \geq N$ implies $|a_{n+1}| \leq |a_n| q$.
(b) Applying the Absolute Convergence Test with $|a_N| \sum q^n$ to $\sum_{n=N}^{\infty} a_n$.

Exercise 21. Does the series

$$\frac{2019}{1} + \frac{2019 \cdot 2020}{1 \cdot 3} + \frac{2019 \cdot 2020 \cdot 2021}{1 \cdot 3 \cdot 5} + \frac{2019 \cdot 2020 \cdot 2021 \cdot 2022}{1 \cdot 3 \cdot 5 \cdot 7} + \cdots$$

converge?

Root Test

Theorem 21 (Root Test)**.** *Given a series $\sum_{n=1}^{\infty} a_n$, let*

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = L.$$

(i) *If $L < 1$, then the series converges absolutely.*
(ii) *If $L > 1$, then the series diverges.*

Exercise 22. Prove the Root Test.
Hint. (i) Apply the Absolute Convergence Test to $\sum q^n$ for some q with $L < q < 1$. (ii) Show that $\{a_n\}$ does not converge to 0.

Exercise 23. Show that the series

$$\sum_{n=1}^{\infty} \frac{n \cos^2 \frac{n\pi}{3}}{2^n}$$

converges absolutely.
Hint. (a) Use the Root Test; or (b) use the Absolute Convergence Test plus Comparison Test plus Ratio Test. (c) Why we cannot use Ratio Test directly?

Exercise 24 (Abel's test)**.** Abel's Test for convergence states that if the series $\sum_{k=1}^{\infty} x_k$ converges, and if $\{y_k\}$ is a sequence satisfying $y_1 \geq y_2 \geq y_3 \geq \cdots \geq 0$, then the series $\sum_{k=1}^{\infty} x_k y_k$ converges.

(i) Prove the *summation by parts* formula. Let $s_0 = 0$ and $s_n = x_1 + x_2 + \cdots x_n$ for $n \in \mathbb{N}$. Then

$$\sum_{k=m}^{n} x_k y_k = s_n y_{n+1} - s_{m-1} y_m + \sum_{k=m}^{n} s_k (y_k - y_{k+1})$$

Hint. Note that $x_k = s_k - s_{k-1}$.

(ii) Use the Comparison Test to argue that $\sum_{k=m}^{\infty} s_k(y_k - y_{k+1})$ converges absolutely, and show how this leads directly to a proof of Abel's Test.

Exercise 25 (Dirichlet's Test)**.** Dirichlet's Test for convergence states that if the partial sums of $\sum_{k=1}^{\infty} x_k$ are bounded (but not necessarily convergent), and if $\{y_k\}$ is a sequence satisfying $y_1 \geq y_2 \geq y_3 \geq \cdots \geq 0$, with $\lim_{k\to\infty} y_k = 0$, then the series $\sum_{k=1}^{\infty} x_k y_k$ converges.

(i) Point out how the hypothesis of Dirichlet's Test differs from that of Abel's Test, but show that essentially the same strategy can be used to provide a proof.

(ii) Show how the Alternating Series Test can be derived as a special case of Dirichlet's Test.

7.3 Rearrangements

Informally speaking, a rearrangement of a series is obtained by permuting the terms in the sum into some other order. It is important that all of the original terms eventually appear in the new ordering and that no term gets repeated.

Definition 11. Let $\sum_{k=1}^{\infty} a_k$ be a series. A series $\sum_{k=1}^{\infty} b_k$ is called a *rearrangement* of $\sum_{k=1}^{\infty} a_k$ if there exists a one-to-one, onto function $f : \mathbb{N} \to \mathbb{N}$ such that $b_{f(k)} = a_k$ for all $k \in \mathbb{N}$.

We now have all the tools and notation in place to resolve an issue raised at the beginning of the chapter. In Section 1, we constructed a particular rearrangement of the alternating harmonic series that converges to a limit different from that of the original series. This happens because the convergence is *conditional.*

Theorem 22. *If a series converges absolutely, then any rearrangement of this series converges to the same limit.*

Proof. Assume $\sum_{k=1}^{\infty} a_k$ converges absolutely to A, and let $\sum_{k=1}^{\infty} b_k$ be a rearrangement of $\sum_{k=1}^{\infty} a_k$. Let's use

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$$

for the partial sums of the original series and use

$$t_m = \sum_{k=1}^{m} b_k = b_1 + b_2 + \cdots + b_m$$

for the partial sums of the rearranged series. Thus we want to show that $\{t_m\} \to A$

Let $\epsilon > 0$. By hypothesis, $\{s_n\} \to A$, so choose $N_1 \in \mathbb{N}$ such that

$$|s_n - A| < \frac{\epsilon}{2}$$

for all $n \geq N_1$. Because the convergence is absolute, we can choose N_2 so that

$$\sum_{k=m+1}^{n} |a_k| < \frac{\epsilon}{2}$$

for all $n > m \geq N_2$. Now, take $N = \max\{N_1, N_2\}$. We know that the finite set of terms $\{a_1, a_2, \cdots, a_N\}$ must all appear in the rearranged series, and we want to move far enough out in the series $\sum_{n=1}^{\infty} b_n$ so that we have included all of these terms. Thus, choose

$$M = \max\{f(k) \,|\, 1 \leq k \leq N\}$$

It should now be evident that if $m \geq M$, then $\{t_m - S_N\}$ consists of a finite set of terms, the absolute values of which appear in the tail $\sum_{k=1}^{\infty} |a_k|$. Our choice of N_2 earlier then guarantees $|t_m - s_N| < \epsilon/2$, and so

$$\begin{aligned}
|t_m - A| =& |t_m - s_N + s_N - A| \\
\leq& |t_m - s_N| + |s_N - A| \\
<& \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

whenever $m \geq M$. □

8 Double Summations and Products of Infinite Series

Let's consider a double summation over two index variables, for instance, if we are given a grid of real numbers $\{a_{ij} \,|\, i, j \in \mathbb{N}\}$, where $a_{ij} = 1/2^{j-i}$ if $j > i$, $a_{ij} = -1$ if $j = i$, and $a_{ij} = 0$ if $j < i$.

$$\begin{bmatrix}
-1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \cdots \\
0 & -1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \cdots \\
0 & 0 & -1 & \frac{1}{2} & \frac{1}{4} & \cdots \\
0 & 0 & 0 & -1 & \frac{1}{2} & \cdots \\
0 & 0 & 0 & 0 & -1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}$$

We would like to attach a mathematical meaning to the summation

$$\sum_{i,j} a_{ij}$$

whereby we intend to include every term in the preceding array in the total. One natural idea is to temporarily fix i and sum across each row. Summing the sums of the rows, we get

$$\sum_{i,j} a_{ij} = \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} a_{ij} \right) = \sum_{i=1}^{\infty} 0 = 0.$$

We could just as easily have decided to fix j and sum down each column first. In this case, we have

$$\sum_{i,j} a_{ij} = \sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} a_{ij} \right) = \sum_{i=1}^{\infty} \frac{-1}{2^{j-1}} = -2.$$

Changing the order of the summation changes the value of the sum!

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} \neq \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij} \,.$$

Performing the sum over first one of the variables and then the other is referred to as an *iterated* summation.

There are still other ways to reasonably define $\sum\limits_{i,j=1}^{\infty} a_{ij}$. One natural idea is to calculate a kind of partial sum by adding together finite numbers of terms in larger and larger "rectangles" in the array; that is, for $m, n \in \mathbb{N}$, set

$$s_{mn} = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}.$$

The order of the sum here is irrelevant because the sum is finite. Of particular interest to our discussion are the sums s_{nn} (sums over "squares"), which form a legitimate sequence indexed by n and thus can be subjected to our arsenal of theorems and definitions. If the sequence $\{s_{nn}\}$ converges, for instance, we might wish to define

$$\sum_{i,j=1}^{\infty} a_{ij} = \lim_{n\to\infty} s_{nn}.$$

Exercise 26. Using the particular array a_{ij} from the above example, compute $\lim_{n\to\infty} s_{nn}$. How does this value compare to the two iterated values for the sum already computed?

There is a deep similarity between the issue of how to define a double summation and the topic of rearrangements discussed at Subsection 7.3. Both relate to the commutativity of addition in an infinite setting. For rearrangements, the resolution came with the added hypothesis of absolute convergence, and it is not surprising that the same remedy applies for double summations. Under the assumption of absolute convergence, each of the methods discussed for computing the value of a double sum yields the same result.

Exercise 27. Show that if the iterated series

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |a_{ij}|$$

converges (meaning that for each fixed $i \in \mathbb{N}$ the series $\sum_{j=1}^{\infty} a_{ij}$ converges to some real number b_i, and the series $\sum_{i=1}^{\infty} b_i$ converges as well), then the iterated series

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}$$

converges.

Theorem 23. *Let* $\{a_{ij} \,|\, i, j \in \mathbb{N}\}$ *be a doubly indexed array of real numbers. If*

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}|$$

converges, then both $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}$ *and* $\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}$ *converge to the same value. Moreover,*

$$\lim_{n\to\infty} s_{nn} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}\,,$$

where $s_{nn} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}$.

sketch of proof. Define the rectangle partial sum

$$t_{mn} = \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|.$$

Exercise 28. (a) Prove that the sequence $\{t_{nn}\}$ is convergent; (b) then show that $\{s_{nn}\}$ is also convergent (by the Cauchy Criterion).

Set

$$S = \lim_{n\to\infty} s_{nn}.$$

In order to prove the theorem, we must show that the two iterated sums converge to this same limit. We will first show that

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = S$$

Because $\{t_{mn} \,|\, m, n \in \mathbb{N}\}$ is bounded above, we can let

$$B = \sup\{t_{mn} \,|\, m, n \in \mathbb{N}\}.$$

Exercise 29. (a) Let $\epsilon > 0$ be arbitrary and argue that there exists an $N_1 \in \mathbb{N}$ such that $m, n \geq N_1$ implies $B - \frac{\epsilon}{2} < t_{mn} \leq B$.

(b) Now, show that there exists an N such that

$$|s_{mn} - S| < \epsilon$$

for all $m, n \geq N$.

For the moment, consider $m \in \mathbb{N}$ to be fixed and write s_{mn} as

$$s_{mn} = \sum_{j=1}^{\infty} a_{1j} + \sum_{j=1}^{\infty} a_{2j} + \cdots + \sum_{j=1}^{\infty} a_{mj}$$

Our hypothesis guarantees that for each fixed row i, the series $\sum_{j=1}^{\infty} a_{ij}$ converges absolutely to some real number r_i.

Exercise 30. (a) Show that for each $m \geq N$,

$$|r_1 + r_2 + \cdots + r_m - S| < \epsilon.$$

Conclude that the iterated sum $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}$ converges to S.

(b) Finish the proof by showing that the other iterated sum, $\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}$, converges to S as well. Notice that the same argument can be used once it is established that, for each fixed column j, the sum $\sum_{i=1}^{\infty} a_{ij}$ converges to some real number c_j.

$\square$

One final common way of computing a double summation is to sum along diagonals where $i + j$ equals a constant. Given a doubly indexed array $\{a_{ij} \mid i, j \in \mathbb{N}\}$, let

$$d_2 = a_{11}, \quad d_2 = a_{12} = a_{21}, \quad \cdots, \quad d_k = a_{1,k-1} + a_{2,k-2} + \cdots + a_{k-1,1}\,, \quad \cdots.$$

Then, $\sum_{k=2}^{\infty} d_k$ represents another reasonable way of summing over every a_{ij} in the array.

Exercise 31. (a) Assuming the hypothesis – and hence the conclusion – of Theorem 23, show that $\sum_{k=2}^{\infty} d_k$ converges absolutely.

(b) Imitate the strategy in the proof of Theorem 23 to show that $\sum_{k=2}^{\infty} d_k$ converges to $S = \lim_{n\to\infty} s_{nn}$.

Products of Series

Conspicuously missing from the Algebraic Limit Theorem for Series (Theorem 14) is any statement about the product of two convergent series. One way to formally carry out the algebra on such a product is to write

$$\begin{aligned}\left(\sum_{i=1}^{\infty} a_i\right)\left(\sum_{j=1}^{\infty} b_j\right) =&(a_1 + a_2 + a_3 + \cdots)(b_1 + b_2 + b_3 + \cdots) \\ =&a_1 b_1 + (a_1 b_2 + a_2 b_1) + (a_1 b_3 + a_2 b_2 + a_3 b_1) + \cdots \\ =&\sum_{k=2}^{\infty} d_k,\end{aligned}$$

where

$$d_k = a_1 b_{k-1} + a_2 b_{k-2} + \cdots + a_{k-1} b_1.$$

This particular form of the product, is called the *Cauchy product* of two series. Although there is something algebraically natural about writing the product in this form, it may very well be that computing the value of the sum is more easily done via one or the other iterated summation. The question remains, then, as to how the value of the Cauchy product – if it exists – is related to these other values of the double sum. If the two series being multiplied converge absolutely, it is not too difficult to prove that the sum may be computed in whatever way is most convenient.

Exercise 32. Assume that $\sum_{i=1}^{\infty} a_i$ converges absolutely to A, and $\sum_{j=1}^{\infty} b_j$ converges absolutely to B.

(a) Show that the iterated sum $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_i b_j|$ converges so that we may apply Theorem 23.

(b) Let $s_{nn} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i b_j$, and prove that $\lim_{n\to\infty} s_{nn} = AB$. Conclude that

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_i b_j = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_i b_j = \sum_{k=2}^{\infty} d_k = AB,$$

where, as before, $d_k = a_1 b_{k-1} + a_2 b_{k-2} + \cdots + a_{k-1} b_1$.