MAT2002 Ordinary Differential Equations
Mathematical modeling and classification of ODEs

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Overview

Outline

Course Information

General information:

- **Instructor**:
  Prof. Dongdong He, email: hedongdong@cuhk.edu.cn.
- 1. Lecture time for two sessions L01&L02:
     Tuesday and Thursday, 10:30am-11:50am&1:30pm-2:50pm
  2. Lecture zoom ID: 4327853236
  3. Lecture venue:
     Teaching Building B 202
  4. Office hour:
     Tuesday 3:30-5:30pm, office: Research Building A 212
- **Material**:
  Notes on the one white board (using Ipad) and lecture slides on the blackboard.

Course Information

Tutorials will be conducted by TAs.

Teaching Assistant:
Wang Qinghe: 218019035@link.cuhk.edu.cn
Lin Haoxiang: 115010190@link.cuhk.edu.cn
Wei Jun: 220019064@link.cuhk.edu.cn
Also, there will be 3-4 USTFs.

There is no tutorial in the first week.

Tutorial zoom ID and TAs' office hours will be provided later in this week.

Grading scheme

Grading:

- Assignments (20%). There will be about 6 assignments.
- Mid-term exam (30%)
- Final exam (50%)

Prerequisites

MAT1001 Calculus I
MAT1002 Calculus II
MAT2040 Linear Algebra

In addition, some knowledge from physics and mechanics are required.

Reference Books

Required

- William E. Boyce and Richard C. DiPrima, Elementary differential equations and boundary value problems. 10th edition. Wiley

Recommended

- Basic Theory of Ordinary Differential Equations, Hsieh, Po-Fang, Sibuya, Yasutaka, 1999, 1st edition, Springer-Verlag New York.
- Differential Equations and their Applications, by Martin Braun, 4th edition, Springer, 1993.
- Differential Equations and linear algebra, C. Henry Edwards, David E. Penney and David T. Calvis, 4th edition. Pearson.

Syllabus

- Mathematical modeling for order differential equations: falling object, motion of a pendulum, population models.
- First order differential equations: linear equations,integrating factors, separable equations, exact equations.
- Existence and Uniqueness Thoerem for first-order initial value problem.
- Second order linear equations: solutions for linear second order equations with constant coefficients, characteristic equation, Wronskian, method of undetermined coefficients.
- Higher order linear equations: general theory, homogeneous equations
- Systems of ODEs: first-order systems, solving systems using linear algebra, eigenvalues, fundamental solutions
- Laplace transform and its application
- Stability: phase portraits for linear systems, method of linearization, Liapunov's method.
- Periodic solutions: limit cycle, Poincare-Bendixon Theorem (if time allows)

Outline

Motivation for ODEs

Many phenomena in the world around us:

- growth of plants, animals and population;
- movement of people and objects;
- value of stock prices;
- flow of fluids/gases;

are "**dynamic**" in nature. We can associate "**dynamic**" with "**change (in time)**". We want to use mathematical tool to model the change of these quantities, these usually involve differential equations.

Example

Example

(Motion of an object along x axis)

An object with mass m is placed on the x axis with initial location $x(0) = x_0$ and initial velocity $x'(0) = v_0$, a force F is acting on the object along the x axis. By Newton's second law, one has

$$ma = mx'' = F$$

subject to the initial conditions:

$$x(0) = x_0, \quad x'(0) = v_0$$

where $x(t)$ is the location of the object at time t, $a = \frac{F}{m}$ is the acceleration. Thus, the ODE for $x(t)$ is

$$x''(t) = a.$$

Using the initial conditions, one can get

$$x(t) = \frac{1}{2}at^2 + v_0 t + x_0.$$

Definition for differential equation

In calculus, $\frac{d^2y}{dt^2} = f(t)$. If we know $f(t)$ and want to find $y(t)$, we can integrate the equation twice. $\frac{d^2y}{dt^2} = f(t)$ is indeed an ordinary differential equation.

Now we begin with the definition for differential equation.

Definition

A **differential equation** is an equation that involves the **derivatives** of an unknown function.

Remark 1: If the equation only involves the derivative of an unknown function with single variable, then the differential equation is called "**ordinary differential equation**" (ODE), which is the subject of study in this course. Example: $\frac{dy}{dt} = f(t, y)$, where $y = y(t)$

Remark 2: If the equation only involves the partial derivatives of an unknown function with multiple variables, then the differential equation is called "**partial differential equation**" (PDE), which is the subject of another course. Example: $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, where $u = u(t, x)$.

Notation: Let $y = y(t)$ be a function of t. For any $n \in \mathbb{N}$, we write

$$y^{(n)} = \frac{d^n y}{dt^n}, \quad \text{and often} \quad y' = y^{(1)}, \quad y'' = y^{(2)}.$$

Example:
1.

$$y' = \cos t$$

Solution: $y = \sin t + c, c \in \mathbb{R}$(Called: **general solution**). c could be any number.
2.

$$y'' = \cos t$$

General solution: $y = -\cos t + c_1 t + c_2$, where c_1, c_2 could be any numbers.
3.

$$y^{(n)} = 0$$

General solution: $y = a_0 + a_1 t + \cdots + a_{n-1} t^{n-1}$, where $a_i (i = 0, \cdots, n-1)$ could be any numbers.

Observation: there are infinity many solutions for an ODE. In order to get a **particular solution**, we need to add some additions conditions, these usually can be done by specifying some initial conditions.

1.

$$y' = \cos t$$

General solution: $y = \sin t + c, c \in \mathbb{R}$. We can fix c by prescribing $y(0) = y_0$, then we get a particular solution: $y = \sin t + y_0$.

2.

$$y'' = \cos t$$

General solution: $y = -\cos t + c_1 t + c_2$. We can fix c_2 by prescribing $y(0)$ and fix c_1 by prescribing $y'(0)$ to get a particular solution.

3.

$$y^{(n)} = 0$$

General solution: $y = a_0 + a_1 t + \cdots + a_{n-1} t^{n-1}$. $a_0, \cdots, a_{n-1}$ can be fixed by prescribing $y(0), y'(0), \cdots, y^{(n-1)}(0)$ to get a particular solution.

In order to get a particular solution of an ODE, the number of initial conditions need to be equal to the highest-order of derivative in the ODE.

Components for real-world problem involving ODEs

A real-world problem involving ODEs normally has the following 5 components:

(1) the **independent variable** - usually time, denoted by t;

(2) the **dependent variable** (also the **quantity of interest**) - such as distance, price, number of people, denoted by y;

(3) the equation - specifying how the quantity of interest changes with respect to the independent variable;

(4) the **interval of definition** - denoted by $I \subset \mathbb{R}$, which is the **range** for which the solution to the ODE is defined;

(5) the **initial conditions** - specific conditions related to the particular problem we want to model.

For the example of moving object:
The time t is the **independent variable**, the location of the object $x(t)$ is the **independent variable**, $x''(t) = a$ is the equation, the **interval of definition** can be set to be $[0, \infty)$ and **initial conditions** are $x(0) = x_0, \quad x'(0) = v_0$.

IVP

Definition

An ordinary differential equation is an equation involving **ONE** independent variable $t \in I$(I is an interval) and **ONE** dependent variable y of the form

$$F(t, y, y', y'', \ldots, y^{(n)}) = 0.$$

Given constants $t_0, t_1, \ldots, t_{n-1} \in I$ and $y_0, y_1, \ldots, y_{n-1} \in \mathbb{R}$,we call

$$\begin{cases} F(t, y, y', y'', \ldots, y^{(n)}) = 0, \\ y(t_0) = y_0, \frac{dy}{dt}(t_1) = y_1, \ldots, \frac{d^{(n-1)}y}{dt^{n-1}}(t_{n-1}) = y_{n-1}, \end{cases}$$

an **initial value problem** (IVP).

The **order** of an ODE is the **highest order** of derivative in the ODE.

$y' = \cos t$- first order ODE.
$y'' = \cos t$- second order ODE.
$y^{(n)}(t) = \cos t$- nth-order ODE.

Definition related to ODE

Definition

(a) An ODE $F(t, y, y', y'', \dots, y^{(n)}) = 0$ is **linear** is if F is a **linear function** of $y, \frac{dy}{dt}, \dots, \frac{d^n y}{dt^n}$. Otherwise, it is a **non-linear** ODE. The general **linear** ODE of order n is

$$a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \cdots + a_0(t)y(t) = f(t),$$

for some given functions $a_0, a_1, \dots, a_n$ and f.

(b) An ODE is called **autonomous** if the independent variable does not appear explicitly (only in the derivatives), the **autonomous** ODE has the form: $F(y, y', y'', \dots, y^{(n)}) = 0$. Otherwise it is a **non-autonomous** ODE.

Definition related to ODE

Example:
$y'' + 2y' = y$— linear autonomous ODE.
$y'' + 2y' = t + y$— linear non-autonomous ODE.
$y'' + 2y' = y^2$— non-linear autonomous ODE.
$y'' + 2y' = t + y^2$— non-linear non-autonomous ODE.

Next, by using mathematical modeling, we will show some more real-world applications that involve ODEs.

Example 1-Motion of a falling object

Free-body diagram of the forces on a falling object.

Example 1-Motion of a falling object

Example

Consider an object falling in the air with mass $m > 0$. We are interested in the **velocity** v of the object as time progresses. We are think of v as a function of t and derive an equation for the rate of change $\frac{dv}{dt}$. There are two (opposing) forces acting on the object as it falls:

(1) Gravitational force $F_g = mg$, where g is the gravitational constant;

(2) Movement through the air generates **air resistance/drag forces**, which we take as proportional to the velocity (Model assumption). This gives a (upwards) force $F_a = \gamma v$, where $\gamma > 0$ is the drag coefficient.

The **net** force pointing downwards is therefore

$$F = F_g - F_a = mg - \gamma v.$$

Example 1-Motion of a falling object

Example

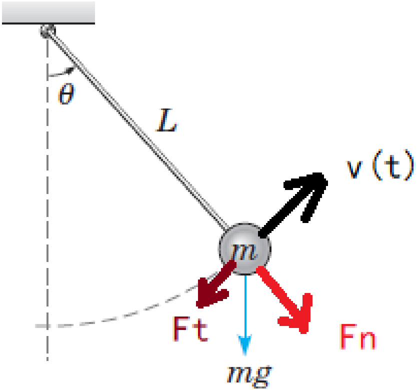
Using **Newton's second law** - which relates net force with the product of mass and acceleration, and recalling acceleration is the rate of change of velocity with respect to time, we are led to

$$mg - \gamma v = F = ma = m\frac{dv}{dt}, \quad \Rightarrow \boxed{m\frac{dv}{dt} = mg - \gamma v}.$$

In the above, we see that

- the independent variable is time t;
- the dependent variable is the velocity v;
- the equation is $mv' = mg - \gamma v$;
- the interval of definition can be taken as $I = [0, \infty)$ - modelling the motion of the object from time $t = 0$ onwards;
- as initial condition we can take $v(0) = 0$.

Example 2-Motion of a pendulum



Example 2-Motion of a pendulum without air resistance

Example

An object of mass $m > 0$ is attached to the lower end of a rigid rod, and the upper end of the rod is attached to the wall and is fixed (cannot move). The object is allowed to rotate in anticlockwise and its trajectory is a circular arc.

We are interested at how the angle θ between the rod and the centreline changes in time as the pendulum rotates. We consider θ as a function of t and derive an equation for $\frac{d\theta}{dt}$. We also look for the force acting on the object. Note that in the gravitational force of the object $W = mg$ can be decomposed into two forces that are perpendicular to each other. A **tangential force** $F_t = mg\sin\theta$ that drives the motion of the pendulum, and a **normal force** $F_n = mg\cos\theta$ that is perpendicular and does not contribute to the motion.

Case that without air resistance:
By Newton's second law and taking into account that the tangential force F_t acting on the opposite direction to the motion of the pendulum, one has

$$mg\sin\theta = F_t = -ma \Rightarrow a = -g\sin\theta.$$

Example 2-Motion of a pendulum without air resistance

Example

The tangential velocity v of the object is given as $v = L\frac{d\theta}{dt}$ ($\frac{d\theta}{dt}$ is the angular velocity), and the acceleration $a = \frac{dv}{dt}$ can be computed as $a = L\frac{d^2\theta}{dt^2}$. Thus, Newton's second law gives

$$-mg\sin\theta = ma = mL\frac{d^2\theta}{dt^2}$$

Thus, we now have the **pendulum equation**

$$\boxed{L\frac{d^2\theta}{dt^2} = -g\sin\theta}.$$

For the interval of definition, we again consider $I = [0, \infty)$, and for the initial conditions we have to prescribe an initial angle $\theta(0) = \theta_0$, and an initial angular velocity $\frac{d\theta}{dt}(0) = \frac{v_0}{L}$.

Example 2-Motion of a pendulum with air resistance

Example

Case that with air resistance:

If there is air resistance force (damping force), which is proportional to the velocity ($F_a = c_0 v = c_0 L \frac{d\theta}{dt} = c\frac{d\theta}{dt} (c = c_0 L)$) and is in the opposite direction to the motion of the pendulum. Assume that both θ and $\frac{d\theta}{dt}$ are positive, then Newton's second law gives

$$-mg\sin\theta - c\frac{d\theta}{dt} = ma = mL\frac{d^2\theta}{dt^2}$$

Thus, we now have the **damped pendulum equation**

$$\boxed{\frac{d^2\theta}{dt^2} + \gamma\frac{d\theta}{dt} + w^2\sin\theta = 0,}$$

where $w^2 = \frac{g}{L}$, $\gamma = \frac{c}{mL} = \frac{c_0}{m} > 0$ is a damping factor taking into account friction forces.

Example 3-Population dynamics

Example

Let us model the growth of the **cows** in a certain region. We are interested in the number of cows after a certain period of time. Setting t as the independent variable and $p(t)$ - the number of cows - as the dependent variable, we now want to derive an ODE for $p(t)$. Note that for physical reasons, only the case $p(t) \geq 0$ makes sense.

If the food are always available and the number of **cows** increases at a rate proportional to the current population, i.e.

$$p' = rp$$

where the proportionality factor r is called the **rate constant** or **growth rate** and r is a positive constant in this case.

Example 3-Population dynamics

Example

However, in reality, the food are limited, it is more reasonable to make following assumptions:

- The rate at which population changes is **proportional** to the population at present time - leading to the equation $\frac{dp}{dt} = h(p)p$, for some non-constant function h;
- When $p(t)$ is small, $h(p(t))$ is positive (more food for everyone, less competition);
- When $p(t)$ is large, $h(p(t))$ is negative (less food for everyone, more competition);
- The function $h(p)$ should decrease as p grows larger.

Example 3-Population dynamics

Example

To obtain the ODE, we simply need to find a suitable function for h. One simple choice is

$$h(p) = r - ap,$$

where $r, a > 0$ are the **reproduction** and **elimination** rates, respectively.
Setting $K := \frac{r}{a}$, which is also known as the **carrying capacity**, the ODE for p now reads as

$$\boxed{\frac{dp}{dt} = (r - ap)p = rp\left(1 - \frac{p}{K}\right)}.$$

The above equation is also called the **Logistic equation**.
Once again, we are interested in the values of p from $t = 0$ onwards, so we set $I = [0, \infty)$. As for initial condition, we set $p(0) = p_0$, where p_0 is the initial number of cows the farmer has.

Classifications of ODEs

ODE	Order	Linear?	Autonomous?
$mv' = mg - \gamma v$	1	✓	✓
$L\theta'' = -g \sin \theta$	2	✗	✓
$\theta'' + \gamma\theta' + w^2 \sin \theta = 0$	2	✗	✓
$p' = rp(1 - p/K)$	1	✗	✓

Equilibrium solution

While one expects the solution y to an ODE is a function depending on t, there is also a **special class** of solutions which don't depend on t.

Definition

For a first order ODE $y' = F(t, y)$ $(t \in I)$, we say that $y = y_*$ (y_* is a constant) is an **equilibrium solution** (or a **stationary solution**) to the ODE if

$$F(t, y_*) = 0, \quad \forall t \in I.$$

Note that the equilibrium solution y_* does not depend on t, i.e., $\frac{dy_*}{dt} = 0$, and so it automatically satisfies the ODE.

Examples

Example

1. For the motion of the falling object, the ODE is $v' = g - \frac{\gamma}{m}v = F(t,v)$. so if $F(t, v_*) = 0$ for some function v_*, we compute to see that

$$0 = F(t, v_*) = g - \frac{\gamma}{m}v_* \Rightarrow v_* = \frac{mg}{\gamma}(\textit{related to the } \underline{\textbf{terminal velocity}}).$$

2. For the population dynamics, the ODE is $p' = rp(1 - p/K) = F(t,p)$. So if $F(t, p_*) = 0$ for some function p_*, we have

$$0 = F(t, p_*) = rp_*(1 - p_*/K) \Rightarrow p_* = 0 \quad or \quad p_* = K.$$

Interpretation: if $p_* = 0$, then we have no cows left (extinction), and if $p_* = K > 0$, then the number of cows is equal to the <u>**carrying capacity**</u> –the maximum number that is supported by the environment.

Goal of this course

Given an ODE with IVP

$$\begin{cases} F(t, y, y', \dots, y^{(n)}) = 0, \\ y(t_0) = y_0, \frac{dy}{dt}(t_1) = y_1, \dots, \frac{d^{(n-1)}y}{dt^{n-1}}(t_{n-1}) = y_{n-1}, \end{cases} \tag{$*$}$$

we want to answer the following questions:

1. Can we find an **explicit formula** for the solution $y(t)$?
2. If not, can we **prove** that there **exists** a solution $y(t)$? If a solution exists, is it a unique solution?
3. Are there stationary (equilibrium) solutions to $(*)$?
4. If a solution $y(t)$ exists, what is its behaviour as t changes?