Lecture 24 MAT 3253

Last time $\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta$

$C : |z| = 1$

$$\int_C \frac{1}{iz} R\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right) dz$$

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\, dx \qquad \deg Q \geq \deg P + 2$$

① $\lim_{b\to\infty} \int_0^b \frac{P(x)}{Q(x)}\, dx + \lim_{a\to-\infty} \int_a^0 \frac{P(x)}{Q(x)}\, dx$

② $\lim_{a\to\infty} \int_{-a}^{a} \frac{P(x)}{Q(x)}\, dx$ principal value

p. v.

p.v. $\int_{-\infty}^{\infty} f(x)\, dx$

arguement principle

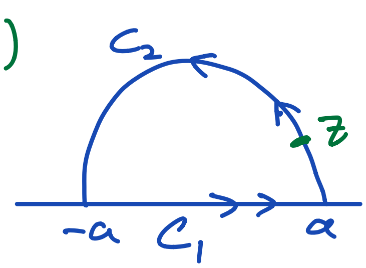
$\int_{-\infty}^{\infty} \frac{x}{1+x^2}\, dx$ ① ∞ $\frac{x}{1+x^2} = O\left(\frac{1}{x}\right)$

p.v. $\int_{-\infty}^{\infty} \frac{x}{1+x^2}\, dx = 0$

(even fn)

<u>Example</u> $\int_0^{\infty} \frac{1}{1+x^2}\, dx$

$= \frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx$



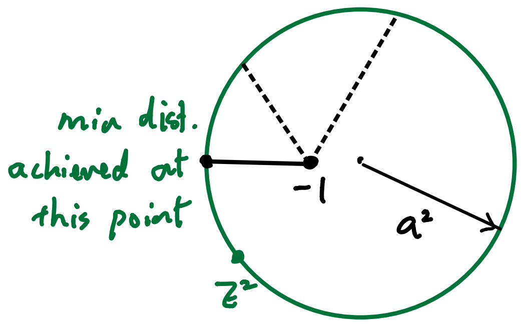
$$\int_{C_1} \frac{1}{1+z^2}\,dz = \int_{-a}^{a} \frac{1}{1+x^2}\,dx$$

$$\left|\int_{C_2} \frac{1}{1+z^2}\,dz\right| \le \frac{1}{a^2-1}\cdot(\pi a)$$

$$= O\left(\frac{1}{a}\right)$$

$$\to 0 \quad \text{as } a\to\infty$$

$z^2+1 = z^2-(-1)$
is the distance
between z^2 and -1

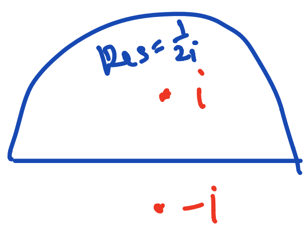


$$|1+z^2| \ge a^2-1$$

$$\frac{1}{1+z^2} \le \frac{1}{a^2-1}$$

$$\int_{C_1}+\int_{C_2} \frac{1}{1+z^2}\,dz \to \text{p.v.}\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$$

$$\frac{1}{1+z^2}$$



$$\frac{1}{(z+i)(z-i)}$$

$$\operatorname{Res}\left(\frac{1}{1+z^2};\, i\right)$$

$$= \lim_{z\to i}\,(z-i)\frac{1}{1+z^2}$$

$$= \left.\frac{1}{z+i}\right|_{z=i} = \frac{1}{2i}$$

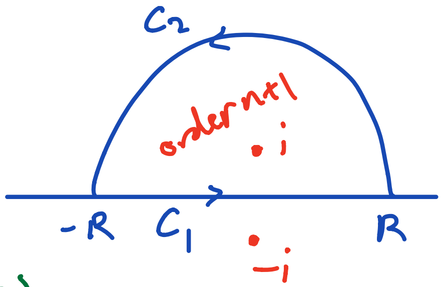
$$\int_{C_1}+\int_{C_2} \frac{1}{1+z^2}\,dz = 2\pi i\,\operatorname{Res}\left(\frac{1}{1+z^2};\, i\right) = 2\pi i\,\frac{1}{2i}$$

$$= \pi$$

$$\int_{0}^{\infty} \frac{1}{1+x^2}\,dx = \underline{\underline{\frac{\pi}{2}}}$$

Example $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^{n+1}}\,dx = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}\cdot\pi$

$n \geq 1$



$$\frac{1}{(1+z^2)^{n+1}} = \frac{1}{(z+i)^{n+1}(z-i)^{n+1}}$$

$$\mathrm{Res}\Big(\overbrace{\frac{1}{(1+z^2)^{n+1}}}^{f(z)}\,;\,i\Big)$$

$$= \lim_{z\to i}\frac{1}{n!}\frac{d^n}{dz^n}(z-i)^{n+1}\frac{1}{(z+i)^{n+1}(z-i)^{n+1}}$$

Reminder: $\displaystyle f(z) = \frac{b_{n+1}}{(z-i)^{n+1}} + \cdots + \frac{b_1}{z-i} + a_0 + a_1(z-i)$

$$(z-i)^{n+1}f(z) = b_{n+1} + \cdots\; b_1(z-i)^n + a_0(z-i)^{n+1} + \cdots$$

$$\frac{d^n}{dz^n}(z-i)^{n+1}f(z) = n!\,b_1 + (n+1)(n)\cdots(2)\,a_0(z-i) + \cdots$$

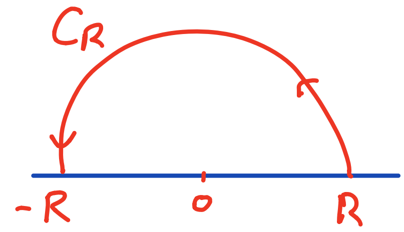
$$\lim_{z\to i}\frac{d^n}{dz^n}(z-i)^{n+1}f(z) = n!\,b_1$$

$$\frac{2\pi i}{n!}\lim_{z\to i}\frac{d^n}{dz^n}(z-i)^{n+1}f(z) = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\pi$$

$$\therefore \int_{-\infty}^{\infty}\frac{dx}{(1+x^2)^{2n+1}} = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\pi$$

$$\int_{-\infty}^{\infty}\frac{\cos x}{1+x^2}\,dx \longrightarrow \int_{\frown}\frac{e^{iz}}{1+z^2}\,dz$$

Jordan lemma



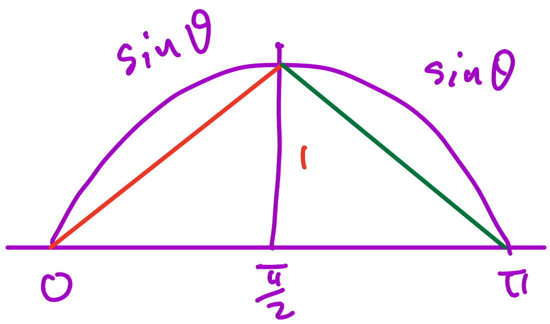
Assume f is analytic on C_R.

For all sufficient large R, $|f(z)| \le M_R$ for z on C_R and $M_R \to 0$ as $R \to \infty$.

Then $\lim_{R\to\infty} \int_{C_R} f(z) e^{iaz}\, dz = 0$ for all real constant $a > 0$.

Proof Jordan inequality

$$\int_0^{\pi} e^{-R\sin\theta}\, d\theta < \frac{\pi}{R} \qquad (R>0)$$



$$\sin\theta \ge \frac{2\theta}{\pi} \quad \text{for} \quad 0 \le \theta \le \frac{\pi}{2}$$

$$\int_0^{\pi/2} e^{-R\sin\theta}\, d\theta \le \int_0^{\pi/2} e^{-R\cdot 2\theta/\pi}\, d\theta \qquad (\text{We need } R>0 \text{ here})$$

$$= \left[\frac{-\pi}{2R} e^{-R2\theta/\pi} \right]_0^{\pi/2}$$

$$= \frac{\pi}{2R}\left(1 - e^{-R}\right)$$

$$< \frac{\pi}{2R}$$

$$\int_0^{\pi} e^{-R\sin\theta}\, d\theta < \frac{\pi}{R}$$

$(a>0)$

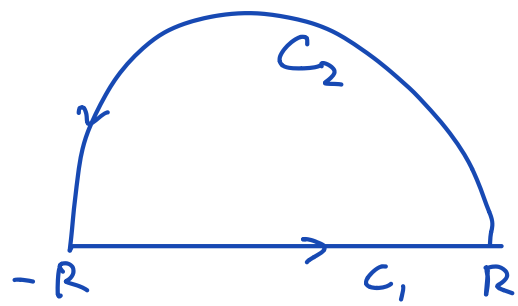
$$\int_{C_R} f(z)e^{iaz}\,dz = \int_0^{\pi} f(Re^{i\theta})\, e^{iaRe^{i\theta}} (Rie^{i\theta})\, d\theta$$

$$\left| \int_0^{\pi} \underbrace{f(Re^{i\theta})}_{\leq M_R} e^{iaR(\cos\theta + i\sin\theta)} R\, ie^{i\theta}\, d\theta \right|$$

$$\leq R M_R \int_0^{\pi} e^{-aR\sin\theta}\, d\theta$$

$$\leq R\, M_R \frac{\pi}{R} = M_R \pi \to 0 \quad \text{as} \quad R \to \infty \qquad \blacksquare$$

__Example__ $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}\,dx$



$$\int_{-\infty}^{\infty} \frac{\sin x}{1+x^2}\,dx = 0$$

Consider $\displaystyle\int_{C_1} + \int_{C_2} \frac{e^{iz}}{1+z^2}\,dz$

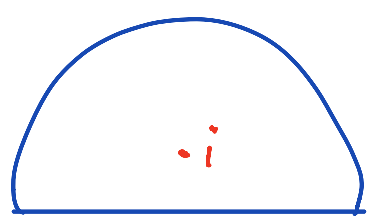
$$\left| \frac{1}{1+(Re^{i\theta})^2} \right| \to 0 \quad \text{as} \quad R \to \infty$$

by Jordan Lemma $\displaystyle\int_{C_2} \frac{e^{iz}}{1+z^2}\,dz \to 0$ as $R \to \infty$

$$\int_{C_1} \frac{e^{iz}}{1+z^2}\,dz = \int_{-R}^{R} \frac{\cos x}{1+x^2}\,dx$$

$$\text{Res}\left(\frac{e^{iz}}{1+z^2} ; i \right)$$

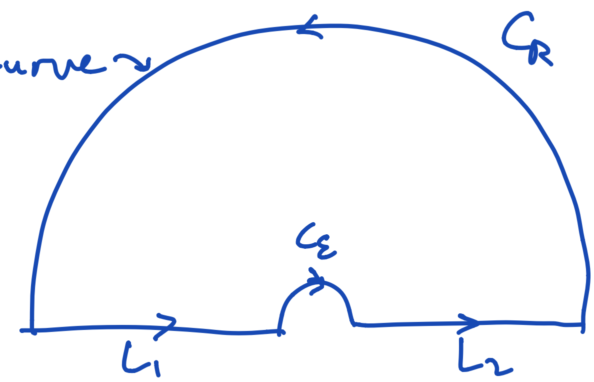
$$= \left. \frac{e^{iz}}{z+i} \right|_{z=i} = \frac{e^{-1}}{2i}$$



$$\therefore \int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}\,dx = 2\pi i \,\mathrm{Res}\left(\frac{e^{iz}}{1+z^2}; i\right)$$
$$= 2\pi i \frac{1}{2ie}$$
$$= \underline{\underline{\frac{\pi}{e}}}$$

<u>Example</u> $\displaystyle\int_0^{\infty} \frac{\sin x}{x} = \frac{\pi}{2} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx$

Consider $\displaystyle\int \frac{e^{iz}}{z}\,dz$ over the curve



By Jordan lemma

$$\int_{C_R} \frac{e^{iz}}{z}\,dz \to 0$$
as $R \to \infty$

$$\int_{L_1} + \int_{L_2} + \int_{C_R} + \int_{C_\varepsilon} \frac{e^{iz}}{z}\,dz = 0$$

Want to show that $\displaystyle\int_{L_1} + \int_{L_2} = \pi i$

sufficient to show $\displaystyle\int_{C_\varepsilon} = -\pi i$

On C_ε, $\displaystyle\frac{e^{iz}}{z} = \frac{1}{z} + \underbrace{i - \frac{z}{2} - \frac{iz^2}{3!} + \dots}_{\text{continuous and bounded}}$

$\left| i - \frac{z}{2} + \dots \right| \leq M$ for $|z| \leq \rho$

ρ is a small constant

$$\int_{C_\varepsilon} i - \frac{z}{2} - \frac{iz^2}{3!} - \dots \, dz \leq M \cdot \pi\varepsilon \longrightarrow 0$$

$$\int_{C_\varepsilon} \frac{1}{z} \, dz = \int_{\pi}^{0} \frac{1}{\varepsilon e^{i\theta}} i\varepsilon e^{i\theta} \, d\theta$$

$$= -\pi i$$

$$\therefore \quad \int_{-\infty}^{\infty} \frac{\sin x}{x} \, dx = \pi$$