1. **Exercise 3.21**

   Solution.(a):

   If $x_2 = 2$, then for model (1), $\hat{y} = 108 + 0.2x_1$ and for model (2), $\hat{y} = 101 + 2.15x_1$.

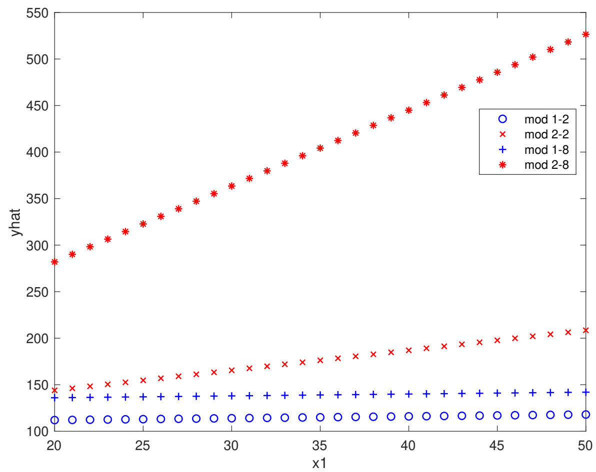
   If $x_2 = 8$, then for model (1), $\hat{y} = 132 + 0.2x_1$ and for model (2), $\hat{y} = 119 + 8.15x_1$.

   For any given x_2, the interaction term in model 2 will affects the slope of the line.

   Matlab codes:

```
clear all; close all;
% data generation
x1 = 20:1:50;
l=length(x1);
x2 = [2,8];
% Define the two models
f1=@(x1,x2) 100+0.2*x1+4*x2;
f2=@(x1,x2) 95+0.15*x1+3*x2+1*x1*x2;
for i=1:l
for j=1:2
yhat1(i,j)=f1(x1(i),x2(j));
yhat2(i,j)=f2(x1(i),x2(j));
end
end
%plot the corresponding figure
figure;
plot(x1,yhat1(:,1),'ob',x1,yhat2(:,1),'xr',x1,yhat1(:,2),'+b',x1,yhat2(:,2),'*r');
xlabel('x1');
ylabel('yhat');
legend('mod 1-2','mod 2-2','mod 1-8','mod 2-8');
```

   The figure of the two models with $x_2 = 2, 8$ are shown below.



Solution.(b):
For $x_2 = 5$, the model (1) becomes $\hat{y} = 120 + 0.2x_1$, as we can see from the model, a unit change in temperature x_1 will cause a different intercept.

The slope is 0.2 regardless of the value of x_2 (specific value of reactoin time).
Solution.(c):
For $x_2 = 5$, the model (2) becomes $\hat{y} = 110 + 5.15x_1$, as we can see from the model, the mean change here is $5 + 0.15 = 5.15$, which is $x_2 + 0.15$.

Thus the result depends on the value of x_2. We can find similar results in $x_2 = 2$, the mean change is 2.15, $x_2 = 8$, the mean change is 8.15.

2. **Exercise 3.22**

   Solution:

   By definition in lecture, we have

$$\begin{aligned}
F_0 &= \frac{SS_R/k}{SS_{Res}/(n-p)} \\
&= \frac{SS_R/((p-1)(SS_T))}{SS_{Res}/((n-p)(SS_T))} \\
&= \frac{R^2(n-p)}{(p-1)(1-R^2)} \\
&= \frac{R^2(n-p)}{k(1-R^2)}
\end{aligned}$$

   (Hint: $R^2 = \frac{SS_R}{SS_T}$; $1 - R^2 = \frac{SS_{Res}}{SS_T}$; $p = k+1$; $SS_T = SS_R + SS_{Res}$)

3. **Exercise 3.23**

   Solution.(a): $F_0 = \frac{0.9(25-3)}{(3-1)(1-0.9)} = 99$ which exceeds the critical value of $F_{0.05,2,22} = 3.44$ (we can obtain this number from look-up table), so H_0 is rejected.

   Solution.(b): The value of R^2 should be surprisingly low.

$$\begin{aligned}
\frac{R^2(n-p)}{k(1-R^2)} &> 3.44 \\
\frac{R^2(22)}{2(1-R^2)} &> 3.44 \\
\frac{R^2}{(1-R^2)} &> 0.312727 \\
R^2 &> 0.312727 - 0.312727R^2 \\
R^2 &> 0.238
\end{aligned}$$

4. **Exercise 3.24**

   Solution 1:
   we denote $\mathbf{1}^T \triangleq [1,1,\cdots,1] \in \mathbb{R}^{1\times n}$, then we can derive $n = \mathbf{1}^T\mathbf{1}$, and $\bar{y} = \frac{1}{n}(\mathbf{1}^T y) = (\mathbf{1}^T\mathbf{1})^{-1}\mathbf{1}^T y$.
   By definition, we have

$$\begin{aligned}
SS_R &= \sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2 \\
&= (\hat{y} - \mathbf{1}\bar{y})^T(\hat{y} - \mathbf{1}\bar{y}) \\
&= (X(X^TX)^{-1}X^Ty - \mathbf{1}(\mathbf{1}^T\mathbf{1})^{-1}\mathbf{1}^Ty)^T(X(X^TX)^{-1}X^Ty - \mathbf{1}(\mathbf{1}^T\mathbf{1})^{-1}\mathbf{1}^Ty) \\
&= y^T(X(X^TX)^{-1}X^T - \mathbf{1}(\mathbf{1}^T\mathbf{1})^{-1}\mathbf{1}^T)^T(X(X^TX)^{-1}X^T - \mathbf{1}(\mathbf{1}^T\mathbf{1})^{-1}\mathbf{1}^T)y \\
&= y^T(X(X^TX)^{-1}X^T - \mathbf{1}(\mathbf{1}^T\mathbf{1})^{-1}\mathbf{1}^T)y
\end{aligned}$$

   Note that $(X(X^TX)^{-1}X^T - \mathbf{1}(\mathbf{1}^T\mathbf{1})^{-1}\mathbf{1}^T)$ is idempotent. Recall that $H = X(X^TX)^{-1}X^T$ is idempotent, we have $HH = H,\ H^T = H$ similar, $\mathbf{1}(\mathbf{1}^T\mathbf{1})^{-1}\mathbf{1}^T$ is also idempotent, then

$$\begin{aligned}
SS_R &= y^THy - y^T(\mathbf{1}(\mathbf{1}^T\mathbf{1})^{-1}\mathbf{1}^T)^T\mathbf{1}(\mathbf{1}^T\mathbf{1})^{-1}\mathbf{1}^Ty \\
&= y^TH^THy - \bar{y}(\mathbf{1}^T\mathbf{1})\bar{y} \quad (\text{recall:}\hat{y} = Hy) \\
&= \hat{y}^T\hat{y} - n\bar{y}^2 \\
&= \sum_{i=1}^{n}\hat{y}_i^2 - n\bar{y}^2
\end{aligned}$$

   Solution 2:

$$
\begin{aligned}
SS_R &= \sum_{i=1}^{n} (\hat{y}_i - \bar{y})^2 \\
&= \sum_{i=1}^{n} \hat{y}_i^2 - 2\sum_{i=1}^{n} \hat{y}_i \bar{y} + n\bar{y}^2 \\
&= \sum_{i=1}^{n} \hat{y}_i^2 - 2\sum_{i=1}^{n} y_i \bar{y} + n\bar{y}^2 \\
&= \sum_{i=1}^{n} \hat{y}_i^2 - 2n\frac{1}{n}\sum_{i=1}^{n} y_i \bar{y} + n\bar{y}^2 \\
&= \sum_{i=1}^{n} \hat{y}_i^2 - 2n\bar{y}^2 + n\bar{y}^2 \\
&= \sum_{i=1}^{n} \hat{y}_i^2 - n\bar{y}^2
\end{aligned}
$$

(Hint: For LS estimators, we have $\sum_{i=1}^{n} \hat{y}_i = \sum_{i=1}^{n} y_i$, the proof is in manuscript week2.)