MAT3253 Tutorial 5

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1 Power Series

- **Abel's Theorem** (on the continuity of power series **on** the circle of convergence)
  let f(z) be defined by a a power series $\sum a_n z^n$ inside its circle of convergence $|z| < 1$. If $\sum a_n$ converges, then $f(z) \to \sum a_n$, as $z \to 1$ in such a way that $\frac{|1-z|}{1-|z|}$ remains bounded.

2 Basic Point Set Topology

1. **Definition of Metric Space**
   - Given any set Y, a function $d : Y \times Y \to R^+ \cup \{0\}$ is a metric if it satisfies:
     - $d(x, y) = 0 \Leftrightarrow x = y.$
     - $d(x, y) = d(y, x)$
     - $d(x, z) \leq d(x, y) + d(y, z)$
   - A metric space if a set together with a metric defined on it.
2. **Definition of Open Subsets of a Metric Space**
   - A ball in a metric space X is defined by: $\{y \in X | d(x, y) < \epsilon\}$, denoted by $B(x, \epsilon)$
   - X is a metric space, a subset U of X is said to be open in X if $\forall x \in U, \exists \epsilon > 0, s.t., B(x, \epsilon) \subset U$
3. *Remark*: As a result, balls are open.
4. **Any subset Y of a metric space X can itself stand as a metric space**, using the metric in X restricted to Y. Hence Y also has its collection of open sets, simply replace X with Y in the definition (for open set) in 2. Hence when speaking of open sets, sometimes one need to make explicit which metric space is referred to.
5. A metric space X is said to be disconnected if there are two *nonempty* and *disjoint* open sets U,V in X, such that there union is X. X is said to be connected otherwise.
6. *Remark*: Contrary to the concept of openness, connectedness is a property of the metric space, instead of a property of subsets of metric space. Hence a set will be connected independent of which metric space it's imbedded in.

3 Differentiability

Theorem 1. A complex function f defined on an open set U is differentiable at a point $z \in U$ if and only if $\exists$ a function $\phi(h)$, a number a, such that the following equation holds for all h sufficiently small: $f(z + h) - f(z) = ha + h\phi(h)$, where $\phi(h) \to 0$ as $h \to 0$.

Remark: As a result of this criterion of differentiability (one that didn't involve a quotient, you can prove the chain rule.