MAT2002 Ordinary Differential Equations Second-order linear equations–Non-homogeneous equations

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Overview

Outline

Non-homogeneous equations

We now turn our attention to ODE of the form

$$y'' + p(t)y' + q(t)y = r(t), \tag{1}$$

for given functions p, q and r that are continuous in an interval I. The corresponding **homogeneous** equation is

$$y'' + p(t)y' + q(t)y = 0. \tag{2}$$

Immediately we have the following observation. Let Z_1 and Z_2 be solutions to the non-homogeneous problem(1). Then, the difference $Z := Z_1 - Z_2$ satisfies

$$Z'' + p(t)Z' + q(t)Z = r - r = 0.$$

That is, the **difference** Z satisfies the **homogeneous** equation (2). If (y_1, y_2) are a fundamental set of solutions to the homogeneous problem (2), then we can write $Z = Z_1 - Z_2$ as

$$Z_1(t) - Z_2(t) = c_1 y_1(t) + c_2 y_2(t)$$

for some constants c_1, c_2.

Non-homogeneous equations

From the above we actually derive a general expression for the solution to the non-homogeneous equation (1). Let $Y(t)$ denote a solution to (1), then **any solution** y to (1) can be expressed as

$$y(t) = Y(t) + c_1 y_1(t) + c_2 y_2(t),$$

where (y_1, y_2) is a fundamental set of solutions to the homogeneous problem (2).

Non-homogeneous equations

Definition 7.1

For a solution expression

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$$

to the ODE

$$y'' + p(t)y' + q(t)y = r(t),$$

we call the function

$$y_c(t) := c_1 y_1(t) + c_2 y_2(t)$$

the **complementary solution**, which is a solution to the homogeneous equation, and the function $Y(t)$ the **particular solution**, which is a solution to the non-homogeneous equation.

Non-homogeneous second order linear ODEs

This gives us a Non-homogeneous second order linear ODEs:

(1) Obtain a fundamental set of solutions (y_1, y_2) to the homogeneous problem (2).

(2) Find a solution $Y(t)$ to the non-homogeneous problem (1).

(3) The general solution to (1) is then given as

$$y(t) = Y(t) + c_1 y_1(t) + c_2 y_2(t).$$

However, several difficulties remain:

- How do we find y_1 and y_2?
- How do we find $Y(t)$?

Remark: The general method for finding the second-order linear ODE with non-constant coefficient $a(t)y'' + b(t)y' + c(t)y = r(t)$ is still missing. We will look at the special cases when a, b, c are real constants and $r(t)$ is in some particular form.

Non-homogeneous second order linear ODEs

In previous lecture, we saw how to find y_1 and y_2 for equations with constant coefficients:

$$ay'' + by' + cy = 0.$$

Therefore, in this section we will show how to obtain a solution Y to the ODE

$$ay'' + by' + cy = r(t)$$

for some specific forms of $r(t)$.

Non-homogeneous second order linear ODEs

One method is called the **method of undetermined coefficients**. Since the $ay'' + by' + cy = 0$ has been completely solved. We only need to find the particular solution $Y(t)$.

The idea is to make a **guess** on what the particular solution $Y(t)$ could look like.

There are only certain classes of functions for $r(t)$ which the particular solution $Y(t)$ could be obtained explicitly.

In particular we consider the non-homogeneous term $r(t)$ to be a mixture of **polynomials**, **exponentials**, **sine** and **cosine**. Although this does not solve the general problem, the method of undetermined coefficients is straightforward to use.

Non-homogeneous second order linear ODEs

Let's look at some examples first.

Example 7.1

Solve

$$y'' - 3y' - 4y = 3e^{2t}.$$

In the standard form (1) we have

$$r(t) = 3e^{2t}.$$

Since the derivative of exponential function is also the exponential function. A possible choice for the particular solution Y would involve exponentials. Before that let us solve the homogeneous problem:

$$y'' - 3y' - 4y = 0$$

and determine the complementary solution.

Non-homogeneous second order linear ODEs

Example 7.1

The characteristic equation to the homogeneous ODE is

$$r^2 - 3r - 4 = (r-4)(r+1) = 0.$$

The roots are $r_1 = 4, r_2 = -1$, and so a general solution to the homogeneous problem is

$$y_c(t) = c_1 e^{4t} + c_2 e^{-t}.$$

Returning to the non-homogeneous problem, assume $Y(t)$ is of the form

$$Y(t) = Ae^{qt}$$

for some coefficients A and q that are **not determined yet**, (hence the name method of undetermined coefficients).

Non-homogeneous second order linear ODEs

Example 7.1

Plugging into the non-homogeneous equations gives

$$Y'' - 3Y' - 4Y = Aq^2e^{qt} - 3Aqe^{qt} - 4Ae^{qt} = A(q^2 - 3q - 4)e^{qt} = 3e^{2t}.$$

Therefore, it makes sense to choose

$$q = 2, \quad A(q^2 - 3q - 4) = 3 \Rightarrow A = -\frac{1}{2} \Rightarrow Y(t) = -\frac{1}{2}e^{2t}.$$

Hence, the general solution y to the ODE $y'' - 3y' - 4 = 3e^{2t}$ can be expressed as

$$y(t) = c_1e^{4t} + c_2e^{-t} - \frac{1}{2}e^{2t}.$$

Remark: In this case, we tried $Y(t) = Ae^{2t}$, where the $r(t)$ is proportional to e^{2t}. But this type of guessing (constant multiplying exponential functions) does not always work.

Non-homogeneous second order linear ODEs

Example 7.2

Solve

$$y'' - 3y' - 4y = 2e^{-t}.$$

Since $r(t)$ is an exponential, if we try $Y(t) = Ae^{-t}$ and determine the value of A. However, it turns out that

$$Y'' - 3Y' - 4Y = A(1 + 3 - 4)e^{-t} = 0.$$

So no choice of A would satisfy the non-homogeneous ODE. What's wrong here?

If you recall, a fundamental set of solutions to the homogeneous ODE $y'' - 3y' - 4y = 0$ is $y_1 = e^{4t}$ and $y_2 = e^{-t}$. That is, the guess function $Y(t) = Ae^{-t}$ actually is a solution to the homogeneous problem, and consequently, it cannot be a solution to the non-homogeneous problem!

Non-homogeneous second order linear ODEs

Example 7.2

In this case, where the assumed form of the particular solution Y is a duplicate of one of the solutions to the homogeneous problem, we can consider a new guess for Y which looks like

$$Y(t) = Ate^{-t},$$

for undetermined constant A.
(This is similar to the fundamental set of solutions $(e^{-\frac{b}{2a}t}, te^{-\frac{b}{2a}t})$ for the ODE $ay'' + by' + cy = 0$ when $b^2 = 4ac$.)
Trying this new guess yields

$$Y'' - 3Y' - 4Y = -5Ae^{-t} = 2e^{-t}.$$

This means that we should take

$$A = -\frac{1}{5} \quad \Rightarrow \quad Y(t) = -\frac{2}{5}te^{-t}.$$

Thus a general solution y to the ODE $y'' - 3y' - 4y = 2e^{-t}$ is

$$y(t) = c_1e^{4t} + c_2e^{-t} - \frac{2}{5}te^{-t}.$$

Non-homogeneous second order linear ODEs

One more example but now $r(t)$ is a polynomial.

Example 7.3

Solve

$$y'' - 3y' - 4y = t^2 + t + 1.$$

We know the complementary solution is $y_c = c_1 e^{4t} + c_2 e^{-t}$. Since $r(t)$ is a polynomial of degree 2, a possible guess is that the particular solution Y is also a polynomial of the same degree, that is $Y(t) = At^2 + Bt + C$ for some undetermined coefficients A, B, C. Then, plugging into the equation gives

$$\begin{aligned} Y'' - 3Y' - 4Y &= 2A - 3(2At + B) - 4(At^2 + Bt + C) \\ &= -4At^2 - (4B + 6A)t + (2A - 3B - 4C) = t^2 + t + 1 \end{aligned}$$

Non-homogeneous second order linear ODEs

Example 7.3

Comparing coefficients immediately gives

$$A = \frac{-1}{4}, \quad B = \frac{1}{8}, \quad C = \frac{-15}{32},$$

and so the general solution y to the ODE $y'' - 3y' - 4y = t^2 + t + 1$ can be expressed as

$$y(t) = c_1 e^{4t} + c_2 e^{-t} - \frac{1}{4}t^2 + \frac{1}{8}t - \frac{15}{32}.$$

What about if $r(t)$ involves the multiplication of exponentials function and polynomials? Indeed, one can try the following method

Case 1: $r(t) = P_n(t)e^{\alpha t}$.

Case 1: $r(t) = P_n(t)e^{\alpha t}$. A possible guess is

$$Y(t) = t^s Q_n(t)e^{\alpha t}, \tag{3}$$

$Q_n(t) = A_0 + A_1 t + \ldots + A_n t^n$ is a polynomial with undetermined coefficients $A_0, \ldots, A_n$, and $s \in \{0, 1, 2\}$ is an exponent determined by the following criterion:

$$s = \begin{cases} 0 & \text{if } \alpha \neq r_1, \alpha \neq r_2, \\ 1 & \text{if } \alpha = r_1 \neq r_2, \\ 2 & \text{if } r_1 = r_2 = \alpha. \end{cases}$$

where r_1 and r_2 are the roots to the characteristic equation

$$ar^2 + br + c = 0.$$

In fact, s is the **multiplicity** of α as a root of the characteristic equation.

Reason

The problem of determining a particular solution to the ODE

$$ay'' + by' + cy = P_n(t)e^{\alpha t}$$

can be done by a substitution. Let

$$Y(t) = e^{\alpha t}u(t),$$

and by substituting this into the ODE we obtain

$$\begin{aligned} &e^{\alpha t}(a[u'' + 2\alpha u' + \alpha^2 u] + b[u' + \alpha u] + cu) = e^{\alpha t}P_n(t) \\ \Rightarrow &\alpha u'' + (2a\alpha + b)u' + (a\alpha^2 + b\alpha + c)u = P_n(t). \end{aligned} \tag{4}$$

To determine a particular solution u, it is reasonable to take

$$\begin{aligned} u(t) = &\begin{cases} A_n t^n + \cdots + A_0 & \text{if } a\alpha^2 + b\alpha + c \neq 0, \\ t(A_n t^n + \cdots + A_0) & \text{if } a\alpha^2 + b\alpha + c = 0,\ 2a\alpha + b \neq 0, \\ t^2(A_n t^n + \cdots + A_0) & \text{if } a\alpha^2 + b\alpha + c = 0,\ 2a\alpha + b = 0, \end{cases} \\ = &t^s(A_n t^n + \cdots + A_0), \quad s = \begin{cases} 0 & \text{if } \alpha \neq r_1, \alpha \neq r_2, \\ 1 & \text{if } \alpha = r_1 \neq r_2, \\ 2 & \text{if } r_1 = r_2 = \alpha. \end{cases} \end{aligned}$$

Reason

If $a\alpha^2 + b\alpha + c \neq 0$, then α is not the root of the characteristic equation, in this case $s = 0$.

If $a\alpha^2 + b\alpha + c = 0, 2a\alpha + b \neq 0$, α is one of the roots of the characteristic equation, but not both, in this case $s = 1$.

If $a\alpha^2 + b\alpha + c = 0, 2a\alpha + b = 0$, α is double root of the characteristic equation, in this case $s = 2$.

Example 7.4

$$y'' - 3y' - 4y = te^{-t},$$

where e^{-t} was a solution to the homogeneous problem, and the non-homogeneous term was $r(t) = te^{-t}$. In this case we have $r_2 = \alpha = -1$ and $r_1 = 4$. Taking $s = 1$, it is suggested to try a particular solution Y of the form

$$Y(t) = t(A_1 t + A_0)e^{-t} = (A_1 t^2 + A_0 t)e^{-t}.$$

$Y'(t) = (-A_1 t^2 + (2A_1 - A_0)t + A_0)e^{-t}$, $Y''(t) = (A_1 t^2 + (A_0 - 4A_1)t + 2A_1 - 2A_0)e^{-t}$. Substituting these into the equation, one can get $(-10A_1 t + 2A_1 - 5A_0)e^{-t} = te^{-t}$. Thus, $-10A_1 = 1, 2A_1 - 5A_0 = 0$. Therefore, $A_1 = -\frac{1}{10}, A_0 = -\frac{1}{25}$. The particular solution is

$$Y(t) = t(-\frac{1}{10}t - \frac{1}{25})e^{-t}$$

Non-homogeneous second order linear ODEs

What about if $r(t)$ involves the multiplication of exponentials function and polynomial as well as sine(cosine) function? Let's look at another example.

Non-homogeneous second order linear ODEs

Example 7.5

This time, solve
$$y'' - 3y' - 4y = 2\sin(t).$$
We know from above that the complementary solution is $y_c = c_1 e^{4t} + c_2 e^{-t}$. Since the non-homogeneous term $r(t) = 2\sin(t)$, a possible solution would involve sine and cosine, so consider
$$Y(t) = a\sin(\alpha t) + b\cos(\beta t)$$
for undetermined coefficients a, b, α, β. Then, plugging the formula into the non-homogeneous equations gives
$$\begin{aligned}
&Y'' - 3Y' - 4Y \\
=& -a\alpha^2\sin(\alpha t) - b\beta^2\cos(\beta t) - 3(a\alpha\cos(\alpha t) - b\beta\sin(\beta t)) \\
&- 4(a\sin(\alpha t) + b\cos(\beta t)) \\
=& \sin(\alpha t)[-a\alpha^2 - 4a] + \cos(\beta t)[-b\beta^2 - 4b] + \cos(\alpha t)[-3a\alpha] + \sin(\beta t)[3b\beta] \\
=& 2\sin(t).
\end{aligned}$$

Non-homogeneous second order linear ODEs

Example 7.5

Since the RHS only involves $\sin(t)$, we can already set

$$\alpha = 1, \quad \beta = 1.$$

This simplifies the above calculation to

$$\sin(t)[-5a + 3b] + \cos(t)[-5b - 3a] = 2\sin(t).$$

Since there is no term involving the cosine on the RHS, we must have

$$-5a + 3b = 2, \quad -5b - 3a = 0 \quad \Rightarrow \quad a = -\frac{5}{17}, \quad b = \frac{3}{17}.$$

Therefore, the general solution y to the ODE $y'' - 3y' - 4y = 2\sin(t)$ can be expressed as

$$y(t) = c_1 e^{4t} + c_2 e^{-t} - \frac{5}{17}\sin(t) + \frac{3}{17}\cos(t).$$

Non-homogeneous second order linear ODEs

Remark

What if we only consider Y as a function of sine? Suppose we have $Y(t) = a\sin(\alpha t)$ for undetermined coefficients a and α. Plugging this into the ODE gives

$$\begin{aligned} Y'' - 3Y' - 4Y &= -a\alpha^2 \sin(\alpha t) - 3a\alpha\cos(\alpha t) - 4a\sin(\alpha t) \\ &= \sin(\alpha t)[-a\alpha^2 - 4a] + \cos(\alpha t)[-3a\alpha] = 2\sin(t). \end{aligned}$$

Again we choose $\alpha = 1$, but now we have

$$-5a\sin(t) - 3a\cos(t) = 2\sin(t).$$

Since the RHS does not contain any cosine, we must have $a = 0$, but if $a = 0$, then $Y(t) = a\sin(t) = 0$. This leads to a contradiction, which means that our **guess** $Y(t) = a\sin(\alpha t)$ is not sufficient. Therefore we need to include a cosine into the guess.

Case 2: $r(t) = e^{\alpha t} P_n(t) \cos(\beta t)$ or $e^{\alpha t} P_n(t) \sin(\beta t)$

Case 2: $r(t) = e^{\alpha t} P_n(t) \cos(\beta t)$ or $e^{\alpha t} P_n(t) \sin(\beta t)$.
Using the Euler formula: $\cos(\beta t) = \frac{1}{2}(e^{\beta i t} + e^{-\beta i t})$, $\quad \sin(\beta t) = \frac{1}{2i}(e^{\beta i t} - e^{-\beta i t})$, the ODE becomes

$$ay'' + by' + cy = \frac{1}{2} P_n(t) \left(e^{(\alpha + \beta i)t} + e^{(\alpha - \beta i)t} \right) \tag{5}$$

$$ay'' + by' + cy = \frac{1}{2i} P_n(t) \left(e^{(\alpha + \beta i)t} - e^{(\alpha - \beta i)t} \right). \tag{6}$$

Case 2: $r(t) = e^{\alpha t} P_n(t) \cos(\beta t)$ or $e^{\alpha t} P_n(t) \sin(\beta t)$

A possible guess for the above two ODEs is

$$Y(t) = t^s (Q_n(t) \cos(\beta t) + R_n(t) \sin(\beta t)) e^{\alpha t}, \tag{7}$$

$Q_n(t) = A_0 + A_1 t + \cdots + A_n t^n, R_n(t) = B_0 + B_1 t + \cdots + B_n t^n$ are polynomials with undetermined coefficients $A_0, \ldots, A_n, B_0, \ldots, B_n$, and $s \in \{0, 1\}$ is an exponent determined by the following:

$$s = \begin{cases} 0 & \text{if } \alpha + i\beta \textit{ is not a root of the characteristic equation}, \\ 1 & \text{if } \alpha + i\beta \textit{ is a root of the characteristic equation}. \end{cases}$$

where the characteristic equation is

$$ar^2 + br + c = 0.$$

Note: both sine and cosine are needed in this case.

Reason

Case 2: $r(t) = e^{\alpha t} P_n(t) \cos(\beta t)$ or $e^{\alpha t} P_n(t) \sin(\beta t)$. The two cases are similar, and so let us consider only the case $r(t) = e^{\alpha t} P_n(t) \sin(\beta t)$. We consider

$$Y(t) = e^{\alpha t}(Q(t)\cos(\beta t) + R(t)\sin(\beta t)),$$

for some functions Q and R, and upon differentiating

$$\begin{aligned}
Y'(t) &= \alpha e^{\alpha t}(Q(t)\cos(\beta t) + R(t)\sin(\beta t)) + e^{\alpha t}\beta(-Q(t)\sin(\beta t) + R(t)\cos(\beta t)) \\
&\quad + e^{\alpha t}(Q'(t)\cos(\beta t) + R'(t)\sin(\beta t)), \\
Y''(t) &= \alpha^2 e^{\alpha t}(Q(t)\cos(\beta t) + R(t)\sin(\beta t)) + 2e^{\alpha t}\alpha\beta(-Q(t)\sin(\beta t) + R(t)\cos(\beta t)) \\
&\quad + 2\alpha e^{\alpha t}(Q'(t)\cos(\beta t) + R'(t)\sin(\beta t)) + \beta^2 e^{\alpha t}(-Q(t)\cos(\beta t) - R(t)\sin(\beta t)) \\
&\quad + 2\beta e^{\alpha t}(-Q'(t)\sin(\beta t) + R'(t)\cos(\beta t)) + e^{\alpha t}(Q''(t)\cos(\beta t) + R''(t)\sin(\beta t)).
\end{aligned}$$

Reason

Plugging the above expression into the ODE yields

$$
\begin{aligned}
& e^{\alpha t} P_n(t)\sin(\beta t) = aY'' + bY' + cY \\
& = e^{\alpha t}\cos(\beta t)[(a\alpha^2 - a\beta^2 + b\alpha + c)Q + (2\alpha a + b)(\beta R + Q') + 2a\beta R' + aQ''] \\
& + e^{\alpha t}\sin(\beta t)[(a\alpha^2 - a\beta^2 + b\alpha + c)R + (2\alpha a + b)(-\beta Q + R') - 2a\beta Q' + aR''].
\end{aligned}
$$

Equating coefficients means that

$$
\begin{aligned}
(a\alpha^2 - a\beta^2 + b\alpha + c)Q + (2\alpha a + b)(\beta R + Q') + 2a\beta R' + aQ'' = 0, \\
(a\alpha^2 - a\beta^2 + b\alpha + c)R + (2\alpha a + b)(-\beta Q + R') - 2a\beta Q' + aR'' = P_n.
\end{aligned}
\tag{8}
$$

Observe that, $\alpha + i\beta$ is a root of the characteristic equation if and only if

$$
a(\alpha + i\beta)^2 + b(\alpha + i\beta) + c = [a\alpha^2 - a\beta^2 + b\alpha + c] + i(2a\alpha + b)\beta = 0.
$$

Using the fact that a complex number is zero if and only if the real and imaginary parts are zero, we have

$$
\alpha + i\beta \text{ is a root} \quad \Leftrightarrow \quad a(\alpha^2 - \beta^2) + b\alpha + c = 0, \quad (2a\alpha + b)\beta = 0.
$$

Reason

As the RHS of (8) are polynomials, it is likely that taking Q and R to be polynomials would give a particular solution. The question is what is the degree.

Case 1: $\alpha + i\beta$ is not a root of the characteristic equation.
Consider the case where $\alpha + i\beta$ is not a root of the characteristic equation. Then, $(a\alpha^2 - a\beta^2 + b\alpha + c)$ and $(2a\alpha + b)\beta$ are not all zeros, then from the second equation of (8) we have that the degree on the LHS would be the degree of R or Q (which ever is higher). This is due to the fact that taking derivatives of a polynomial reduces the degree. Therefore, for convenience, we can take Q and R to have the **same degree** as the polynomial P_n, i.e.,

$$Q(t) = A_n t^n + \cdots + A_0, \quad R(t) = B_n t^n + \cdots + B_0$$

Reason

Case 2: $\alpha + i\beta$ is a root of the characteristic equation, then (8) simplifies to

$$(2\alpha a + b)Q' + 2a\beta R' + aQ'' = 0,$$
$$(2\alpha a + b)R' - 2a\beta Q' + aR'' = P_n,$$

and from the second equation, we see that the degree of the LHS would be the degree of R' or Q' (which ever is higher). This motivates us to take

$$Q(t) = t(A_n t^n + \cdots + A_1 t + A_0), \quad R(t) = t(B_n t^n + \cdots + B_1 t + B_0),$$

in order to match the degree with the RHS.

Example 7.6

Find a particular solution of

$$y'' - 3y' - 4y = -8e^t \cos 2t. \tag{9}$$

We guess our particular solution $Y(t)$ is the product of e^t and *a linear combination of* $\cos 2t$ *and* $\sin 2t$, i.e.

$$Y(t) = Ae^t \cos 2t + Be^t \sin 2t$$

It follows that

$$\begin{aligned} Y'(t) &= [A\cos 2t - 2A\sin 2t]e^t + [B\sin 2t + 2B\cos 2t]e^t \\ &= (A+2B)e^t \cos 2t + (-2A+B)e^t \sin 2t \end{aligned}$$

and

$$\begin{aligned} Y''(t) &= [(A+2B)\cos 2t - 2(A+2B)\sin 2t]e^t + [(-2A+B)\sin 2t + 2(-2A+B)\cos 2t]e^t \\ &= (-3A+4B)e^t \cos 2t + (-4A-3B)e^t \sin 2t \end{aligned}$$

Example 7.6

After substituing for y, y' and y'' in Eq.(9) we obtain:

$$\begin{aligned} & e^t \cos 2t[(-3A+4B)-3(A+2B)-4A] \\ +& e^t \sin 2t[(-4A-3B)-3(-2A+B)-4B] = -8e^t \cos 2t \end{aligned}$$

Hence we derive:

$$\begin{cases} -10A - 2B = -8 \\ 2A - 10B = 0 \end{cases} \Longrightarrow \begin{cases} A = \dfrac{10}{13} \\ B = \dfrac{2}{13} \end{cases}$$

Hence our particular solution is:

$$Y(t) = \frac{10}{13} e^t \cos 2t + \frac{2}{13} e^t \sin 2t.$$

Summary

For

$$ay'' + by' + cy = r(t)$$

the trial function $Y(t)$ vs. $r(t)$ is listed as follows:

$r(t)$	Y(t)	The value for s
$P_n(t)e^{\alpha t}$	$Q_n(t)t^s e^{\alpha t}$	$s = \begin{cases} 0, \alpha \text{ is not a root.} \\ 1, \alpha = r_1 \neq r_2 \\ 2, \alpha = r_1 = r_2 \end{cases}$ r_1, r_2 are roots of $ar^2 + br + c = 0$
$\begin{cases} P_n e^{\alpha t} \sin \beta t \\ P_n e^{\alpha t} \cos \beta t \end{cases}$	$\begin{cases} [Q_n(t) \cos \beta t \\ +R_n(t) \sin \beta t] t^s e^{\alpha t} \end{cases}$	$s = \begin{cases} 0, \text{if } \alpha + i\beta \text{ is not a root of } ar^2 + br + c = 0. \\ 1, \text{if } \alpha + i\beta \text{ is a root of } ar^2 + br + c = 0. \end{cases}$

Non-homogeneous second order linear ODEs

Now we suppose that $g(t)$ is a sum of two terms, $g(t) = g_1(t) + g_2(t)$. And we know that $Y_1(t)$ and $Y_2(t)$ are two solutions of the equations

$$ay'' + by' + cy = g_1(t) \tag{10}$$
$$ay'' + by' + cy = g_2(t) \tag{11}$$

Then how to find the particular solution for the equation

$$ay'' + by' + cy = g(t) = g_1(t) + g_2(t)? \tag{12}$$

Non-homogeneous second order linear ODEs

Theorem 7.2

Suppose Y_1 *is a solution to*

$$ay'' + by' + cy = g_1(t),$$

and Y_2 *is a solution to*

$$ay'' + by' + cy = g_2(t).$$

Then the sum $Y_1 + Y_2$ *is a solution to*

$$ay'' + by' + cy = g_1(t) + g_2(t).$$

Proof. Since Y_1 is a solution to $ay'' + by' + cy = g_1(t)$. and Y_2 is a solution to $ay'' + by' + cy = g_2(t)$. We have

$$aY_1'' + bY_1' + cY_1 = g_1(t) \tag{13}$$
$$aY_2'' + bY_2' + cY_2 = g_2(t) \tag{14}$$

Then Eq.(13)+Eq.(14) follows that

$$\begin{aligned}
&[aY_1'' + bY_1' + cY_1] + [aY_2'' + bY_2' + cY_2] \\
=&a[Y_1'' + Y_2''] + b[Y_1' + Y_2'] + c[Y_1 + Y_2] \\
=&a[Y_1 + Y_2]'' + b[Y_1 + Y_2]' + c[Y_1 + Y_2] \\
=&g_1(t) + g_2(t) = g(t).
\end{aligned}$$

Hence we derive that $Y_1 + Y_2$ is a particular solution to $ay'' + by' + cy = g_1(t) + g_2(t)$. The following example shows this procedure.

Example 7.7

Find a particular solution of

$$y'' - 3y' - 4y = 3e^{2t} + 2e^{-t} + 2\sin t - 8e^t \cos 2t. \tag{15}$$

By splitting up the RHS of Eq.(15) we obtain the four equations:

$$y'' - 3y' - 4y = 3e^{2t}, \quad Y(t) = -\frac{1}{2}e^{2t}$$
$$y'' - 3y' - 4y = 2e^{-t}, \quad Y(t) = -\frac{2}{5}te^{-t}$$
$$y'' - 3y' - 4y = 2\sin t, \quad Y(t) = -\frac{5}{17}\sin t + \frac{3}{17}\cos t$$
$$y'' - 3y' - 4y = -8e^t \cos 2t, \quad Y(t) = \frac{10}{13}e^t \cos 2t + \frac{2}{13}e^t \sin 2t.$$

Example 7.7

Through the previous examples, we can find the particular solution of these equations.

Thus a particular solution to Eq.(15) is a sum of them, i.e.

$$Y(t) = -\frac{1}{2}e^{2t} - \frac{2}{5}te^{-t} - \frac{5}{17}\sin t + \frac{3}{17}\cos t + \frac{10}{13}e^{t}\cos 2t + \frac{2}{13}e^{t}\sin 2t.$$

Outline

Motivation

The method of undetermined coefficients is a straightforward method, but requires that the non-homogeneous term $r(t)$ to be in a special form. If we encounter an ODE

$$y'' - 3y' + 2y = \frac{e^{3t}}{e^t + 1}$$

then the method of undetermined coefficients does not apply. Therefore, we need a more general method that in principle can be applied to any equation. One such method is the **variation of parameters**.

General theory

We now outline a general theory. Consider a general 2nd-order linear ODE

$$y'' + p(t)y' + q(t)y = r(t), \tag{16}$$

and suppose (y_1, y_2) forms a fundamental set of solutions to the homogeneous equation

$$y'' + p(t)y' + q(t)y = 0.$$

But how to find a particular solution to the non-homogeneous equation (16)?

General theory

The idea is as follows. Consider for some functions $u_1(t), u_2(t)$ such that the new function

$$y(t) = u_1(t)y_1(t) + u_2(t)y_2(t) \tag{17}$$

solves the non-homogeneous equation (16). We now determine what equations u_1 and u_2 have to satisfy.
Differentiating (17) yields

$$y' = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'.$$

Since there are two unknown functions $u_1(t)$ and $u_2(t)$, in order to simplify the computations later, let us impose a condition

$$u_1'y_1 + u_2'y_2 = 0.$$

Then the derivative becomes

$$y' = u_1y_1' + u_2y_2'. \tag{18}$$

General theory

Differentiating again leads to

$$y'' = u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2''. \tag{19}$$

Substituting (17)-(19) into the non-homogeneous ODE then gives

$$\begin{aligned} y'' + p(y)y' + q(t)y =& u_1(y_1'' + p(t)y_1' + q(t)y_1) + u_2(y_2'' + p(t)y_2' + q(t)y_2) \\ &+ u_1'y_1' + u_2'y_2' \\ =& u_1'y_1' + u_2'y_2' = r(t). \end{aligned}$$

General theory

Thus, we obtain two conditions for u_1 and u_2:

$$u_1' y_1 + u_2' y_2 = 0, \quad u_1' y_1' + u_2' y_2' = r(t),$$

which can be conveniently summarised in matrix notion

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ r \end{pmatrix}$$

Since the determinant is the Wronskian $W(y_1, y_2)[t]$ which is non-zero since (y_1, y_2) is a fundamental set of solutions, (u_1', u_2') to the above problem can be solved.

General theory

Therefore, we can compute

$$u_1'(t) = -\frac{y_2 r}{W(y_1, y_2)}(t), \quad u_2'(t) = \frac{y_1 r}{W(y_1, y_2)}(t). \tag{20}$$

Integrating gives

$$u_1(t) = -\int \frac{y_2 r}{W(y_1, y_2)}(t)dt + d_1, \quad u_2(t) = \int \frac{y_1 r}{W(y_1, y_2)}(t)dt + d_2, \tag{21}$$

for constants $d_1, d_2 \in \mathbb{R}$, and the general solution to the non-homogeneous equation is

$$y(t) = (c_1 + d_1)y_1 + (c_2 + d_2)y_2 - y_1 \int \frac{y_2 r}{W(y_1, y_2)}(t)dt + y_2 \int \frac{y_1 r}{W(y_1, y_2)}(t)dt.$$

In fact, we can always take $d_1 = d_2 = 0$ in (21).

General theory

Let us summarise with a theorem.

Theorem 7.3

Let $I \subset \mathbb{R}$ be an open interval, p, q, r continuous on I. If (y_1, y_2) is a fundamental set of solutions to the homogeneous equation $y'' + p(t)y' + q(t)y = 0$, then a particular solution to the non-homogeneous equation $y'' + p(t)y' + q(t)y = r(t)$ is

$$Y(t) = -y_1 \int \frac{y_2 r}{W(y_1, y_2)}(t)dt + y_2 \int \frac{y_1 r}{W(y_1, y_2)}(t)dt,$$

and the general solution to the non-homogeneous equation is

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$$

for constants $c_1, c_2 \in \mathbb{R}$.

General theory

Remark

This method is able to treat rather general second order ODEs (since $p(t)$ and $q(t)$ need not be constants). However, it is not easy to find a fundamental set of solutions (if $p(t)$ and $q(t)$ are not constant functions). Furthermore, another difficulty lies in the evaluation of the integrals:

$$-\int \frac{y_2 r}{W(y_1, y_2)}(t)dt, \quad \int \frac{y_1 r}{W(y_1, y_2)}(t)dt$$

which may not be possible if r, y_1, y_2 are complicated functions.

Example

Example 7.8

Solve the following ODE

$$y'' - 3y' + 2y = \frac{e^{3t}}{e^t + 1}.$$

Let us first look at the homogeneous problem

$$y'' - 3y' + 2y = 0,$$

which we know the general solution (complementary solution) is given as

$$y_c(t) = c_1 e^t + c_2 e^{2t}.$$

We now compute for u_1 and u_2, where we use

$$y_1 = e^t, \quad y_2 = e^{2t}, \quad r = \frac{e^{3t}}{e^t + 1}, \quad W(y_1, y_2)[t] = e^{3t}.$$

Example 7.8

From (20), we see

$$u_1'(t) = -\frac{e^{2t}}{e^t+1}, \quad u_2'(t) = \frac{e^t}{e^t+1}.$$

Integrating gives

$$u_1(t) = \ln(e^t+1) - e^t, \quad u_2(t) = \ln(e^t+1).$$

Hence, a particular solution is

$$Y(t) = u_1 y_1 + u_2 y_2 = e^t \ln(e^t+1) + e^{2t}\ln(e^t+1) - e^{2t}.$$

The general solution to the ODE (16) is

$$\begin{aligned} y(t) =& c_1 e^t + c_2 e^{2t} + e^t \ln(e^t+1) + e^{2t}\ln(e^t+1) - e^{2t} \\ =& d_1 e^t + d_2 e^{2t} + e^t \ln(e^t+1) + e^{2t}\ln(e^t+1). \end{aligned}$$

where $d_1 = c_1, d_2 = c_2 - 1$ are arbitary constants.