MAT2006: Elementary Real Analysis
Assignment #5

Deadline Dec. 17

1. Consider the function g defined by the power series

$$g(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots .$$

(a) Is g defined on $(-1, 1)$? Is it continuous on this set? Is g defined on $(-1, 1]$? Is it continuous on this set? What happens on $[-1, 1]$? Can the power series for $g(x)$ possibly converge for any other points $|x| > 1$? Explain.

(b) For what values of x is $g'(x)$ defined? Find a formula for g'.

2. Find suitable coefficients $\{a_n\}$ so that the resulting power series $\sum a_n x^n$ has the given properties, or explain why such a request is impossible.

(a) Converges for every value of $x \in \mathbb{R}$.
(b) Diverges for every value of $x \in \mathbb{R}$.
(c) Diverges for every value of $x \in \mathbb{R} \setminus \{0\}$.
(d) Converges absolutely for all $x \in [-1, 1]$ and diverges off of this set.
(e) Converges conditionally at $x = -1$ and converges absolutely at $x = 1$.
(f) Converges conditionally at both $x = -1$ and $x = 1$.

3. (**Term-by-term Antidifferentiation**).
Assume $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges on $(-R, R)$.

(a) Show that

$$F(x) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

is defined on $(-R, R)$ and satisfies $F'(x) = f(x)$.

(b) Antiderivatives are not unique. If g is an arbitrary function satisfying $g'(x) = f(x)$ on $(-R, R)$, find a power series representation for g.

4. (a) Show that power series representations are unique. If we have

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$$

for all x in an nonempty interval $(-R, R)$, prove that $a_n = b_n$ for all $n = 0, 1, 2, \ldots$.

(b) Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converge on $(-R, R)$, and assume $f'(x) = f(x)$ for all $x \in (-R, R)$ and $f(0) = 1$. Deduce the values of a_n.

5. A series $\sum_{n=0}^{\infty} a_n$ is said to be *Abel-summable* to L if the power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

converges for all $x \in [0,1)$ and $L = \lim_{x\to 1-} f(x)$.

(a) Show that any series that converges to a limit L is also Abel-summable to L.

(b) Show that $\sum_{n=0}^{\infty}(-1)^n$ is Abel-summable and find the sum.

6. (Cauchy's Remainder Theorem). Let f be differentiable $N+1$ times on $(-R,R)$. For each $a \in (-R,R)$, let $S_N(x,a)$ be the partial sum of the Taylor series for f centered at a; in other words, define

$$S_N(x,a) = \sum_{n=0}^{N} c_n (x-a)^n \quad \text{where} \quad c_n = \frac{f^{(n)}(a)}{n!}.$$

Let $E_N(x,a) = f(x) - S_N(x,a)$. Now fix $x \neq 0$ in $(-R,R)$ and consider $E_N(x,a)$ as a function of a.

(a) Find $E_N(x,x)$.

(b) Explain why $E_N(x,a)$ is differentiable with respect to a, and show

$$E_N'(x,a) = -\frac{f^{(N+1)}(a)}{N!}(x-a)^N.$$

(c) Show

$$E_N(x) = E_N(x,0) = \frac{f^{(N+1)}(c)}{N!}(x-c)^N x$$

for some c between 0 and x. This is Cauchy's form of the remainder for Taylor series centered at the origin.

7. Consider $f(x) = 1/\sqrt{1-x}$.

(a) Generate the Taylor series for f centered at zero, and use Lagrange's Remainder Theorem to show the series converges to f on $[0,1/2]$. (The case $x < 1/2$ is more straightforward while $x = 1/2$ requires some extra care.) What happens when we attempt this with $x > 1/2$?

(b) Use Cauchy's Remainder Theorem to show the series representation for f holds on $[0,1)$.

8. Let $f : [a,b] \to \mathbb{R}$ be increasing on the set $[a,b]$. Show that f is integrable on $[a,b]$.

9. For each $n \in \mathbb{N}$ let

$$h_n(x) = \begin{cases} 1/2^n & \text{if } 0 \le x \le \frac{1}{/}2^n \\ 0 & \text{if } 1/2^n < x \le 1 \end{cases}$$

and set $H(x) = \sum_{n=1}^{\infty} h_n(x)$. Show that $H(x)$ is integrable and compute $\int_0^1 H(x)dx$.

10. Let $\{f_n\}_{n=1}^{\infty} \cup \{f\}$ is uniformly bounded on $[0,1]$. Assume that $f_n \to f$ pointwise on $[0,1]$ and uniformly on any set of the form $[0,\alpha]$, where $0 < \alpha < 1$.

If all the functions are integrable, show that $\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)dx$.

11. Assume g is integrable on $[0,1]$ and continuous at 0. Show that

$$\lim_{n\to\infty} \int_0^1 g(x^n)dx = g(0).$$

12. (a) Let $f(x) = |x|$ and define $F(x) = \int_{-1}^x f(t)dt$. Find a piecewise algebraic formula for $F(x)$ for all x. Where is F continuous? Where is F differentiable? Where does $F'(x) = f(x)$?

(b) Repeat part (a) for the function

$$f(x) = \begin{cases} 1 & \text{if } x < 0 \\ 2 & \text{if } x \geq 0. \end{cases}$$

13. Show that if $f : [a,b] \to \mathbb{R}$ is continuous and $\int_a^x f(t)dt = 0$ for all $x \in [a,b]$, then $f(x) = 0$ everywhere on $[a,b]$. Provide an example to show that this conclusion does not follow if f is not continuous.

14 (Integration by parts)**.** Assume $h(x)$ and $k(x)$ have continuous derivatives on $[a,b]$ and derive the familiar integration-by-parts formula

$$\int_a^b h(x)k'(x)dx = h(b)k(b) - h(a)k(a) - \int_a^b h'(x)k(x)dx.$$

15. Given a function f on $[a,b]$, define the total variation of f to be

$$Vf = \sup \left\{ \sum_{k=1}^n |f(x_k) - f(x_{k-1})| \right\}$$

where the supremum is taken over all partitions P of $[a,b]$.

(a) If f is continuously differentiable (f' exists as a continuous function), use the Fundamental Theorem of Calculus to show $Vf \leq \int_a^b |f'(x)|dx$.

(b) Use the Mean Value Theorem to establish the reverse inequality and conclude that $Vf = \int_a^b |f'(x)|dx$.

16. Assume f is integrable on $[a,b]$ and has a jump discontinuity at $c \in (a,b)$.

(a) Show that, in this case, $F(x) = \int_a^x f(t)dt$ is not differentiable at $x = c$.

(b) Construct a continuous monotone function that fails to be differentiable on $\mathbb{Q}$.

— End —