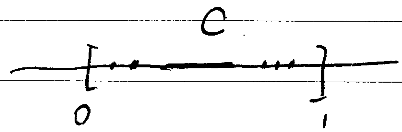
MAT 2006 Tutorial 3 9.28

$\bigcup_{n=1}^{N} A_n$, by induction. $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n\in\mathbb{N}} A_n$ not by induction

1. (a) Proof.



$a = \frac{1}{n}, \frac{1}{n+1}, \frac{1}{n+2}, \cdots$

Suppose for contradiction, that $\forall a \in (0,1)$, $C \cap [a,1]$ is ctb.

$\forall n \in \mathbb{N}$, $C \cap [\frac{1}{n}, 1]$ is ctb

$\bigcup_{n=1}^{\infty} (C \cap [\frac{1}{n},1])$ is ctb $\Rightarrow C \cap (\bigcup_{n=1}^{\infty} [\frac{1}{n},1])$ is ctb.

$\Rightarrow C \cap (0,1]$ is ctb.

$\Rightarrow (C \cap (0,1]) \cup \{0\}$ is ctb

$C \subseteq (C \cap (0,1]) \cup \{0\} \Rightarrow C$ is ctb Contradiction!

(b). Possibly no.

Let $C = [0,1) \Rightarrow \forall a \in (0,1)$, $C \cap [a,1) = [a,1)$ is unctb.

$\Rightarrow A = (0,1) \Rightarrow \alpha = 1$

$C \cap [\alpha,1) = C \cap \{1\} = \phi$.

We can do better!

Proof. $C \cap [\alpha,1]$ is either finite or ctb

$\alpha + \frac{1}{n} > a \Rightarrow \alpha + \frac{1}{n} \notin A$.

$\Rightarrow C \cap [\alpha + \frac{1}{n}, 1]$ is finite or ctb.

$\Rightarrow \bigcup_{n=1}^{\infty} (C \cap [\alpha + \frac{1}{n}, 1]) = C \cap (\alpha, 1]$ is finite or ctb.

$\Rightarrow C \cap [\alpha,1]$ is finite or ctb.

(c). $C = \{\frac{1}{n} \mid n \in \mathbb{N}\}$. $\forall a \in (0,1)$, $C \cap [a,1]$ is finite.

By AP, $\exists N \in \mathbb{N}$. $\frac{1}{N} < a$, $C \cap [a,1]$ can only contains $\frac{1}{N-1}, \frac{1}{N-2}, \cdots, 1$.

2. proof $P(\mathbb{N}) \sim \mathbb{R}$. $P(\mathbb{N}) \sim (0,1)$.

1-1 $f: P(\mathbb{N}) \to (0,1)$ 1-1 $g: (0,1) \to P(\mathbb{N})$

① 1-1 $f: P(\mathbb{N}) \to (0,1)$. $f(A) = 0.d_1 d_2 d_3 \cdots$, where $d_i = \begin{cases} 2, & \text{if } i \in A \\ 3, & \text{if } i \notin A \end{cases}$.

$\forall A \in P(\mathbb{N})$

f is 1-1: If $A \neq B$, WLOG, assume $\exists n \in \mathbb{N}$, s.t $n \in A$, $n \notin B$.

$\Rightarrow$ n^{th} digit of $f(A)$ is 2, of $f(B)$ is 3.

$\Rightarrow$ $f(A) \neq f(B)$.

② 1-1 $g: (0,1) \to P(\mathbb{N})$

(a) $0.2999\cdots \longrightarrow \{2, 29, 299, 2999, \cdots\}$

$0.3000\cdots \longrightarrow \{3, 30, 300, 3000, \cdots\}$

$0.3 \longrightarrow \{3\}$.

Write every decimal # in non-terminating fashion.

$(0.4 \to 0.3999\cdots)$.

$0.20299\cdots \to \{2, 20, 202, 2029, \cdots\}$

$0.020299\cdots \to \{2, 20, 202, 2029, \cdots\}$. $(g: (0,1) \to P(\mathbb{N}))$

(b) Binary expansion: non-terminating expansion

$0.1_{(2)} = \underbrace{0.0111\cdots_{(2)}}$ (V)

$0.a_1 a_2 a_3 \cdots_{(2)} = \sum_{i=1}^{\infty} \frac{a_i}{2^i} \ (10)$.

$0.a_1 a_2 a_3 \cdots_{(2)} \longmapsto \{n \in \mathbb{N} \mid a_n = 1\}$

(c) $0.0020503999\cdots \to \{200, 50200, 3050200, \cdots\}$.

3. proof. $\forall \varepsilon > 0$. $\exists N \in \mathbb{N}$. s.t $n \geq N \Rightarrow |a_n - a| < \varepsilon$

$\big||a_n| - |a|\big| \leq |a_n - a| < \varepsilon$.

$\begin{cases} |a_n| - |a| \leq |a_n - a| \\ |a| - |a_n| \leq |a_n - a| \end{cases}$

Binary expansion: $(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$.

4. Proof (i). $\lim_{n\to\infty} \sqrt[n]{p} = 1$, Let $a_n = p^{\frac{1}{n}} - 1$

① $p=1$, already done!

② $p \neq 1$.

✓ If $p>1$, $(a_n+1)^n = p = 1 + \binom{n}{1}a_n + \binom{n}{2}a_n^2 + \cdots$

$= 1 + na_n + \frac{n(n-1)}{2}a_n^2 + \cdots$

Then $p \geq na_n \Rightarrow 0 < a_n \leq \frac{p}{n}$

By squeeze theorem, $a_n \to 0$ as $n \to \infty$.

$\Rightarrow (p)^{\frac{1}{n}} \to 1$ as $n \to \infty$.

✓ If $p<1$, then $\frac{1}{p} > 1$, we can get $(\frac{1}{p})^{\frac{1}{n}} \to 1$ as $n \to \infty$

$\Rightarrow (p)^{\frac{1}{n}} \to 1$ as $n \to \infty$.

(ii). $\lim_{n\to\infty} \sqrt[n]{n} = 1$, let $a_n = n^{\frac{1}{n}} - 1$, $a_n \geq 0$.

$(a_n+1)^n = n \geq \frac{n(n-1)}{2} a_n^2$

$\Rightarrow a_n^2 \leq \frac{2}{n-1} \quad (n \geq 2)$

$\Rightarrow 0 < a_n < \sqrt{\frac{2}{n-1}}$

By squeeze theorem, $a_n \to 0$ as $n \to \infty$

$\Rightarrow n^{\frac{1}{n}} \to 1$ as $n \to \infty$.

(iii). $\lim_{n\to\infty} \sqrt[2n+1]{n^2+n} = 1$. $(n^2+n)^{\frac{1}{2n+1}} > 1$

$1 < (n^2+n)^{\frac{1}{2n+1}} \leq (2n^2)^{\frac{1}{2n+1}}$ $n^{\frac{1}{n+\frac{1}{2}}}$

$= 2^{\frac{1}{2n+1}} \cdot n^{\frac{2}{2n+1}}$

$\leq 2^{\frac{1}{2n}} \cdot n^{\frac{1}{n}}$

$= \sqrt{2}^{\frac{1}{n}} \cdot n^{\frac{1}{n}} \to 1$ as $n \to \infty$

$\Rightarrow (n^2+n)^{\frac{1}{2n+1}} \to 1$ as $n \to \infty$.