MAT 3253 Lecture 28

Problem 3

(a) Show that $f(z) = \frac{z-i}{z+i}$ maps the upper half-plane to unit disc.

(i) Show that real axis maps to unit circle

(ii) Show that $i \longmapsto 0$
maps to

(iii) Show that upper half-plane is mapped to unit disc

<u>Solution</u>

$z = x+yi$

$$\left| \frac{x+yi-i}{x+yi+i} \right| \overset{?}{=} 1$$

If z is on the real axis then $|f(z)| = 1$

If $y = 0$ $\quad \frac{|x-i|}{|x+i|} = \frac{\sqrt{x^2+1}}{\sqrt{x^2+1}} = 1$

Also want $|f(z)| = 1 \Rightarrow z$ is on the real axis.

$$\left| \frac{z-i}{z+i} \right| = 1 \iff |z-i|^2 = |z+i|^2$$

$(z+i)(\bar{z}-i)$

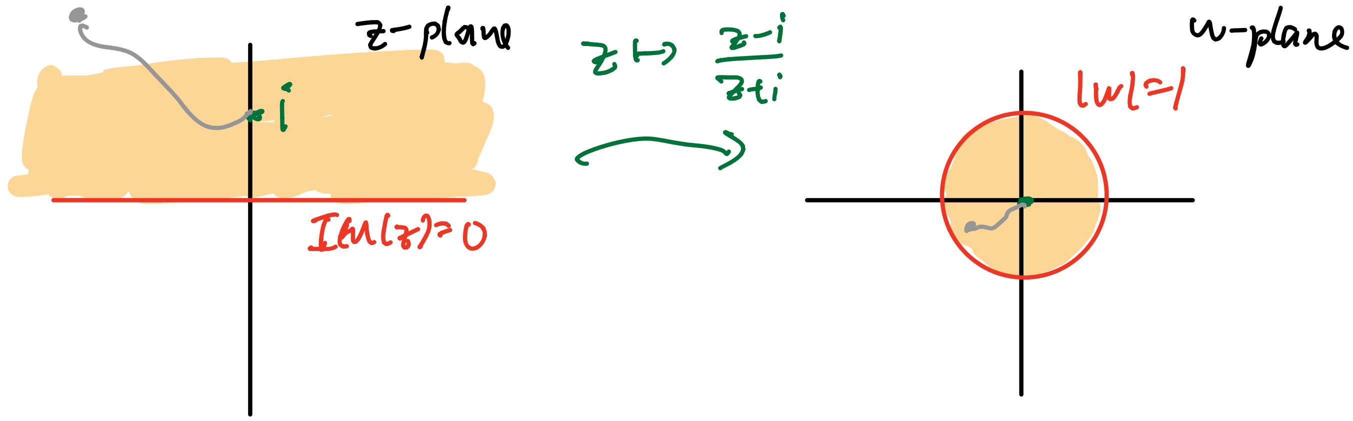
$$\iff \cancel{|z|^2} - zi + \bar{z}i + \cancel{1} = \cancel{|z|^2} + iz - i\bar{z} + \cancel{1}$$

$$\iff 2i\bar{z} = 2iz$$

$$\iff z = \bar{z}$$

$$\iff x+iy = x-iy$$

$$\iff y = 0$$



(ii) $\left.\frac{z-i}{z+i}\right|_{z=i} = 0$

(iii) Continuity argument.

(b) Show $f(z) = \frac{z+2}{z-1}$ maps unit circle to the the $x = -\frac{1}{2}$

$$w = \frac{z+2}{z-1}$$

$$(z-1)w = z+2$$

$$zw - w = z+2$$

$$z(w-1) = w+2$$

$$z = \frac{w+2}{w-1}$$

$$|z| = 1 \iff \left|\frac{w+2}{w-1}\right| = 1$$

$$\iff |w+2|^2 = |w-1|^2$$

$$\iff (w+2)(\bar{w}+2) = (w-1)(\bar{w}-1)$$

$$\iff w + \bar{w} = -1$$

$$\iff \mathrm{Re}(w) = -\frac{1}{2}$$

$f(z) = \dfrac{az+b}{cz+d}$ is one-to-one on $\mathbb{C} \cup \{\infty\}$

$$f(z) = w \Leftrightarrow az+b = w(cz+d)$$

$$(a - wc)z = -b + wd$$

$$z = \frac{wd - b}{-wc + a} \longrightarrow -\frac{d}{c} \text{ when } w \to \infty$$

$$g(w) = \frac{wd-b}{-wc+a}$$

$f\left(-\frac{d}{c}\right) = \infty$ Check $g(\infty) = -\frac{d}{c}$

$f(\infty) = \frac{a}{c}$ Check $g\left(\frac{a}{c}\right) = \infty$

Problem 5

(a) Compute $\int_C x \, dz$ where C is the unit circle.

$$\int_C z \, dz = 0 \quad \text{by Cauchy thm}$$

$$\| $$

$$\int_C x \, dz + i \int_C y \, dz$$

$$\Rightarrow \int_C x \, dz = 0 \quad , \quad \int_C y \, dz \, dz = 0$$

(b) $\int_C \frac{1}{|z|} dz$ where C is the unit circle

$$\|$$

$$\int_C 1 \, dz = 0$$

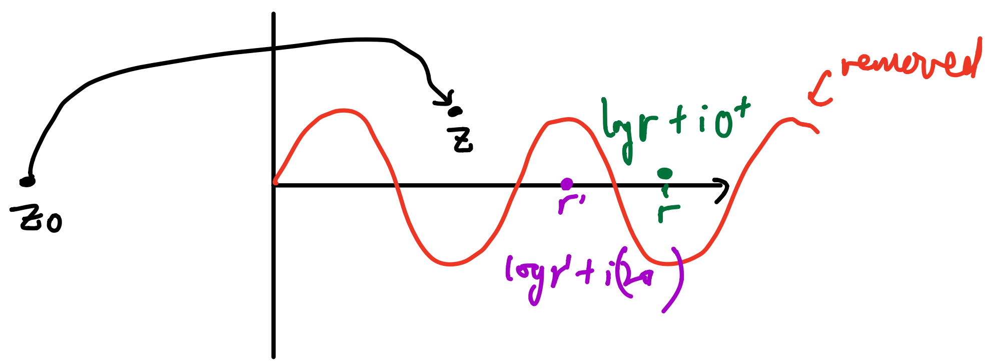
(c) $\int_C z\cos(z^2)\,dz$ $\quad C$ unit circle

$z\cos(z^2)$ is analytic

Cauchy thm $\Rightarrow$ Ans $= 0$

(d) Draw the region $\mathbb{C}\setminus\{x + i\sin x : x \geq 0\}$

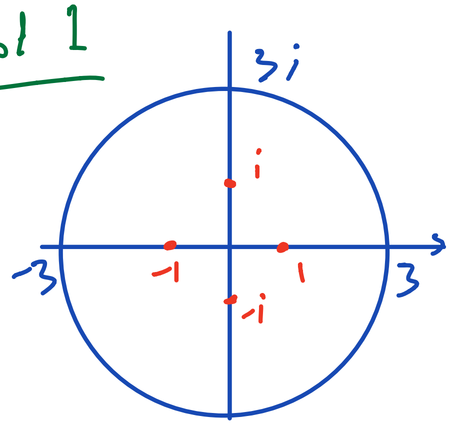
Could you define a branch of Log function in this region ?



Define $\log(z) = \int_{z_0}^{z} \frac{1}{w}\,dw$ is well-defined.

(e) Compute $\int_C \frac{z^2}{z^4-1}\,dz$ $\quad C: |z| = 3$.

Sol 1



Let $f(z) = \frac{z^2}{z^4-1}$

$\mathrm{Res}(f; 1) = \frac{(1)^2}{4(1)^3} = \frac{1}{4}$

$\mathrm{Res}(f; i) = \frac{(i)^2}{4(i)^3} = \frac{-i}{4}$

$\mathrm{Res}(f; -1) = \frac{(-1)^2}{4(-1)^3} = \frac{-1}{4}$

$\mathrm{Res}(f; -i) = \frac{(-i)^2}{4(-i)^3} = \frac{i}{4}$

$$\boxed{\mathrm{Res}\left(\frac{p(z)}{q(z)}; z_0\right) = \frac{p(z_0)}{q'(z_0)}}$$

$\int_C f\,dz = 2\pi i\left(\frac{1}{4} - \frac{i}{4} - \frac{1}{4} + \frac{i}{4}\right) = \underline{0}$

Sol. 2

$$\int_C \frac{z^2}{z^4-1}\,dz = \mathrm{Res}\left(\frac{1}{w^2} f\left(\frac{1}{w}\right);0\right)$$

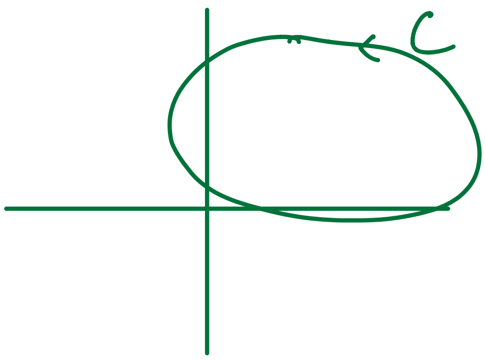
$$= \mathrm{Res}\left(\frac{\left(\frac{1}{w}\right)^2}{w^2\left(\frac{1}{w^4}-1\right)};0\right)$$

$$= \mathrm{Res}\left(\frac{1}{1-w^4};0\right)$$

$$= \underline{\underline{0}}$$

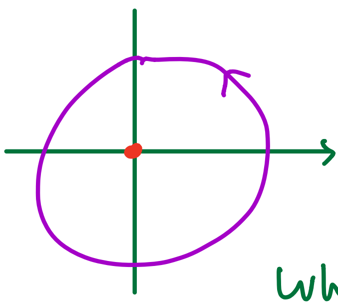
(f) Does $\int_C \frac{e^z}{z^2}\,dz = 0$ (C is simple closed curve)?

Solution



$\int_C \frac{e^z}{z^2}\,dz = 0$

by Cauchy thm



$$e^z = 1 + z + \frac{z^2}{2} + \cdots$$

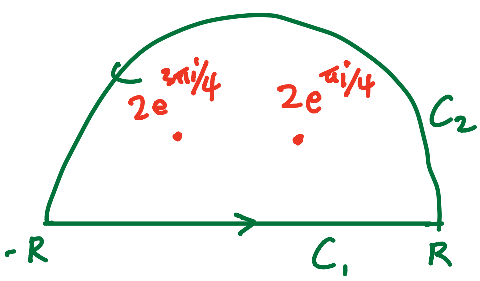
$$\frac{1}{z^2}e^z = \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2} + \cdots$$

When C contains origin, positive orientation

then $\int_C \frac{e^z}{z^2}\,dz = 2\pi i\,\mathrm{Res}(f:0)$

$= 2\pi i$

(g) Compute $\int_{-\infty}^{\infty} \frac{1}{x^4+16}\,dx$



$$\left| \int_{C_2} \frac{1}{z^4+16}\, dz \right| \leq \frac{1}{R^4-16} \cdot \pi R = O\left(\frac{1}{R^3}\right)$$

$$\to 0 \quad \text{as } R \to \infty$$

$$\operatorname{Res}(f; 2e^{i\pi/4}) = \frac{e^{-3\pi i/4}}{32}$$

$$\operatorname{Res}(f; 2e^{i3\pi/4}) = \frac{e^{-\pi i/4}}{32}$$

Sum of residue $= -\dfrac{\sqrt{2}}{32} i$

$$\int_{-\infty}^{\infty} \frac{1}{x^4+16}\, dx = \lim_{R\to\infty} \int_{C_1} f\, dz$$

$$\boxed{f(z) = \frac{1}{z^4+16}}$$

$$= (2\pi i)\left(-\frac{\sqrt{2}}{32} i\right) - \lim_{R\to\infty} \int_{C_2} f\, dz$$

$$= \frac{\pi\sqrt{2}}{16}$$