Stochastic Processes

Lecture 16: Continous time Markov chains

Hailun Zhang@SDS of CUHK-Shenzhen

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Continuous time Markov chains (CTMC)

- A continuous-time Markov chain (CTMC) is a continuous-time stochastic process $X = \{X(t), t \geq 0\}$ that
- takes values in a discrete state space S,
- and has piecewise constant sample paths;
- the times between jumps are exponentially distributed;
- at each jump time, the CTMC jumps from the current state to another state independently of the history.

Parameters: jump matrix and rates for holding times

- $S = \{1, 2, 3\}$
- Jump matrix

$$J = \begin{pmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix}.$$

- Holding time rates

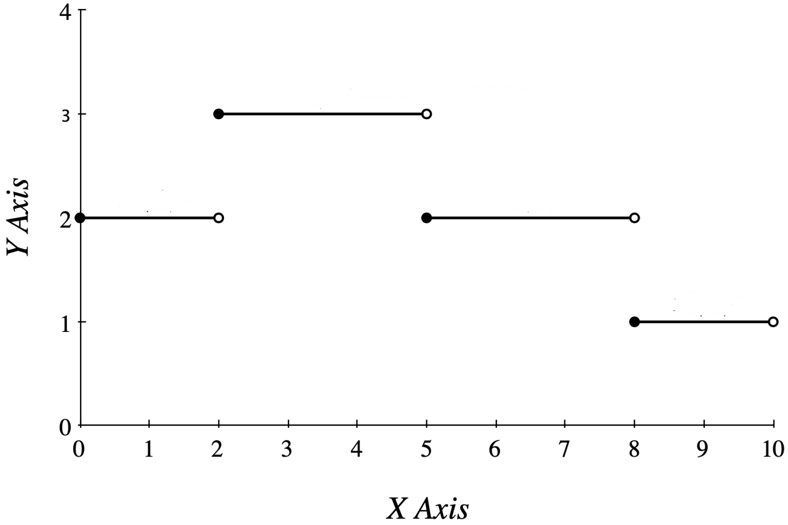
$$\lambda(1) = 3, \quad \lambda(2) = 2, \quad \lambda(3) = 1.$$

- Exponential clocks

Monte-Carlo simulation

- Learn how to simulate the sample path
- Given three coins (in this example, generally $|S|$-sided "non-uniform" dice)
  - First one: a biased coin probabilty 2/3 leading a head
  - Second one: a fair coin
  - Third one: a biased coin probabilty 100% leading a head
- A sequence of i.i.d. exponentially distributed, mean one, random variables

$$u(1), \quad u(2), \ldots u(n), \ldots,$$


4
3
2
1
0
Y Axis
0
1
2
3
4
5
6
7
8
9
10
X Axis

Sample path construction

- $\sigma_0 = 0$, $X(\sigma_0) = i \in S$
- σ_n is the nth jump times.
- Immediately after the nth jump, the state is $Y_n = X(\sigma_n)$.
- The next jump time is

$$\sigma_{n+1} = \sigma_n + \frac{1}{\lambda(Y_n)} u(n+1).$$

- Shortly before σ_{n+1}, there is no jump yet, namely

$$X(\sigma_{n+1}-) = X(\sigma_n).$$

- $X(\sigma_{n+1})$ is determined by the outcome of the Y_n-th coin.

Regular CTMCs

- A CTMC is said to be regular if

$$\mathbb{P}_i\Big\{\lim_{n\to\infty}\sigma_n=\infty\Big\}=1 \qquad \text{for each } i\in S.$$

- Non-regular CTMC:

$$S=\{1,2,\ldots,\},\quad J_{i,i+1}=1 \quad \text{and} \quad \lambda(i)=i^2.$$

Markov property

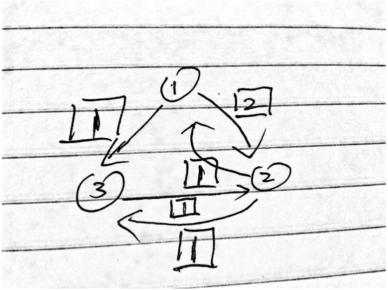
- Markov property:

$$
\begin{aligned}
&\mathbb{P}\Big\{X(t+s)=j|X(t_1)=i_1,\ldots,\\
&\qquad X(t_{n-1})=i_{n-1},X(t)=i\Big\}\\
&=\mathbb{P}\{X(t+s)=j|X(t)=i\}\\
&=\mathbb{P}\{X(s)=j|X(0)=i\}\\
&=P_{ij}(s)
\end{aligned}
$$

- $P(s)$ is an $|S|\times|S|$ matrix, $P(0)=I$.

Transition rate diagram

- The transition rate diagram



- Competing clocks

$$\lambda_{ij} = \lambda(i) J_{ij}, \quad j \neq i.$$

Generator matrix G

- Let

$$G = \begin{pmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix}.$$

- off-diagonals are non-negative: $G_{ij} = \lambda_{ij}$
- diagonals are strictly negative: $G_{ii} = -\sum_{j \neq i} G_{ij}$
- row sums are zero.
- The three representations of the *input* for a CTMC are equivalent.

Three equivalent forms of inputs

- State $S = \{1, 2, 3\}$
- Generator

$$G = \begin{pmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix} = \begin{pmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{pmatrix}.$$

- Jump matrix plus holding time rates
- Transition rate diagram

One machine, one repair person

- On times are iid exponentially distributed with mean 6 hours.
- Repair times are iid exponentially distributed with mean 1 hour.

Two machines, one repair person

- On times are iid exponentially distributed with mean 6 hours.
- Repair times are iid exponentially distributed with mean 1 hour.

Three machines, two repair persons

- On times are iid exponentially distributed with mean 6 hours.
- Repair times are iid exponentially distributed with mean 1 hour.

Three machines, John & Jay repair

- On times are i.i.d. exponentially distributed with mean 6 hours.
- John's repair times are i.i.d. exponentially distributed with mean 2 hour.
- Jay's repair times are i.i.d. exponentially distributed with mean 1 hour.

Chapman-Kolmogorov equation

- Note

$$
\begin{aligned}
&P_{ij}(t+s) = \mathbb{P}\{X(t+s) = j | X(0) = i\} \\
&= \sum_{k \in S} \mathbb{P}\{X(t+s) = j, X(t) = k | X(0) = i\} \\
&= \sum_{k \in S} \mathbb{P}\{X(t+s) = j | X(t) = k, X(0) = i\} \mathbb{P}\{X(t) = k | X(0) = i\} \\
&= \sum_{k \in S} P_{kj}(s) P_{ik}(t).
\end{aligned}
$$

- Thus,

$$P(t+s) = P(t)P(s), \quad P(2t) = (P(t))^2$$

Computing $P(t)$

- Suppose

$$P(0.1) = \begin{pmatrix} 0.7486327 & 0.1607327 & 0.0906346 \\ 0.0783127 & 0.8310527 & 0.0906346 \\ 0.0041073 & 0.0865273 & 0.9093654 \end{pmatrix} \tag{1}$$

- Compute

$$\begin{aligned} &\mathbb{P}\{X(.4) = 3, X(.2) = 1, X(.1) = 3 | X(0) = 2\} \\ &= (P(0.1))^2_{1,3} P_{3,1}(0.1) P_{2,3}(.1) \\ &= (0.164840)(0.0041073)(0.0906346). \end{aligned}$$

How to obtain (1)?

- From

$$P(t+s) = P(t)P(s),$$

one has

$$P'(s) = P'(0+)P(s), \quad s \geq 0,$$

where $P'(0+)$ exists and

$$P'(0+) = G. \tag{2}$$

- Solving $P'(s) = GP(s)$, one has

$$P(s) = e^{sG} = \sum_{k=0}^{\infty} \frac{s^k G^k}{k!}.$$