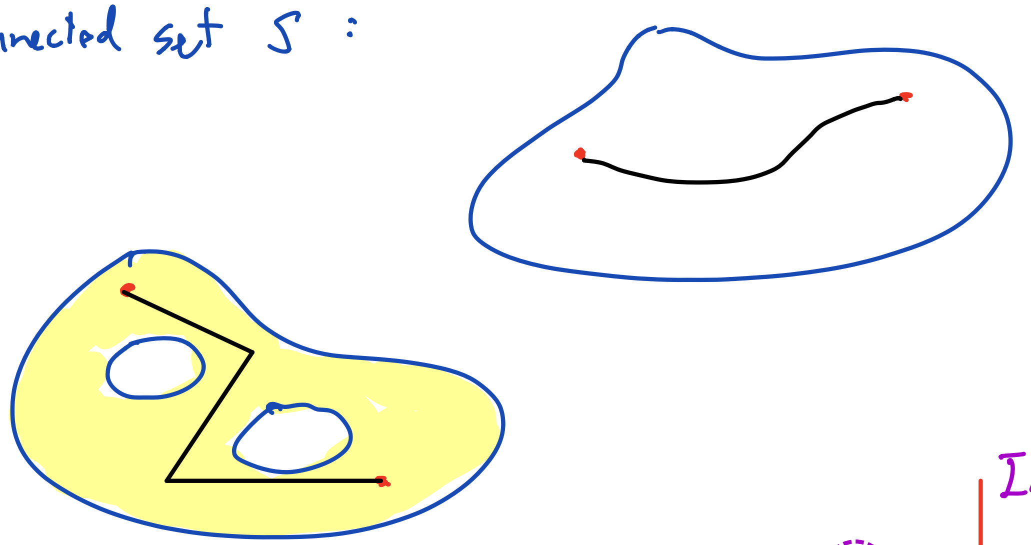
MAT 3253 Lecture 5

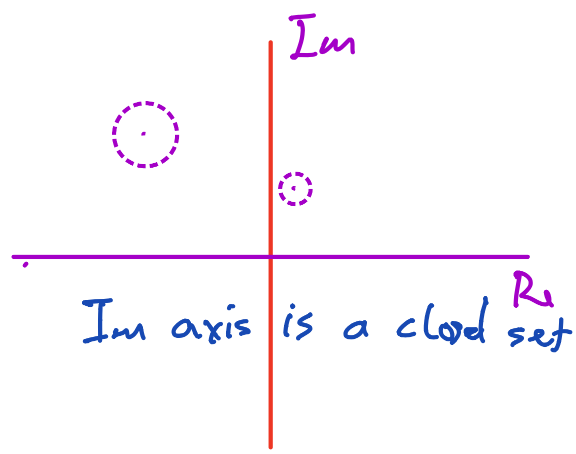
Function : Domain $\xrightarrow{f}$ Co-domain

Def

In complex analysis, a domain/region is an open and connected set in $\mathbb{C}$.

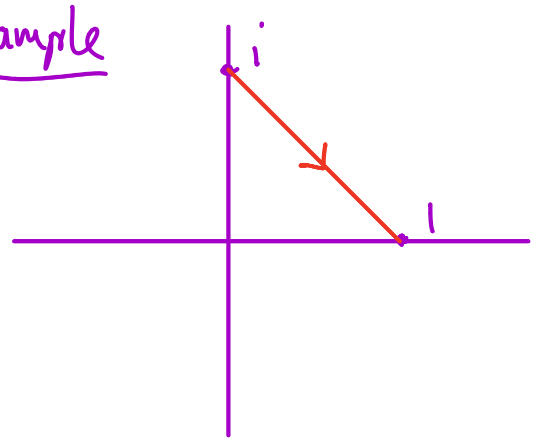
Connected set S :





Functions: Domain \ Co-domain	$\mathbb{R}$	$\mathbb{C}$
Subset of $\mathbb{R}$	$\sin(x)$, x^2, $\frac{1}{x}$	paths: $[0,1] \to \mathbb{C}$, $t \in [0,1]$, $t \mapsto \cos 2\pi t + i \sin 2\pi t$
Subset of $\mathbb{C}$	$\mathrm{Re}(z)$, $\mathrm{Im}(z)$, $\lvert z \rvert$, $\mathrm{Arg}(z)$	$f(z) = z^2$, $f(z) = 1/z$, $f(z) = \frac{z+1}{z-i}$

Example



① Use x as the parameter

$$f(x) = x + i(1-x)$$

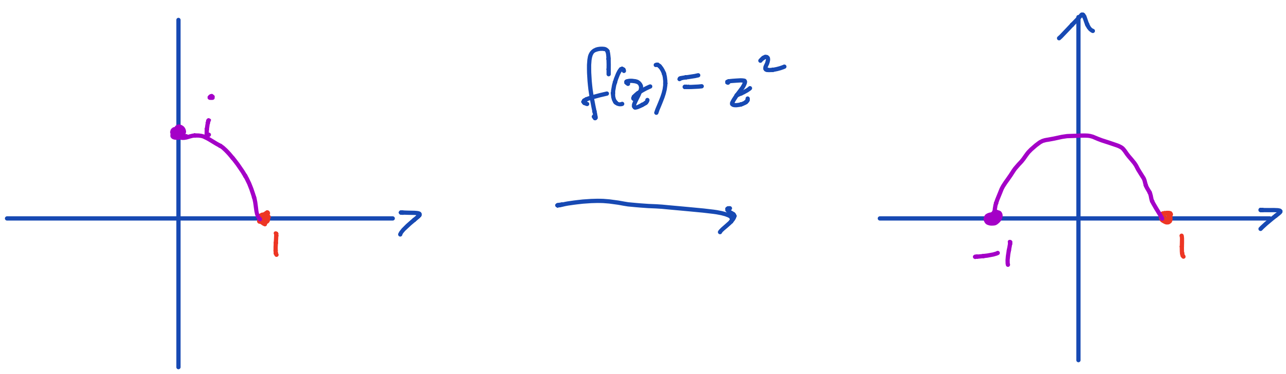
$$x \in [0,1]$$

② Use y as the parameter

$$g(y) = (1-y) + iy$$

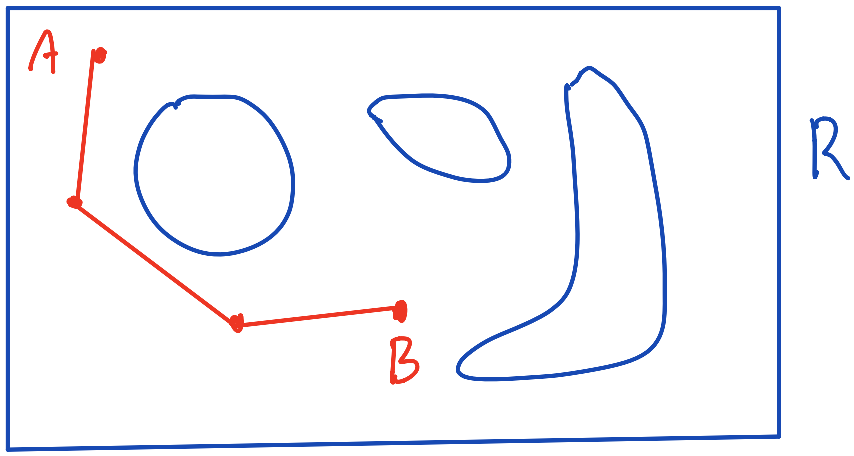
y from 1 to 0

To Visualize a complex function



Polygonal curve / piece-wise linear

Suppose ① R is an open set, and ② A and B are points in R that are connected by a path.



Then $\exists$ a polygonal path from A to B, with finitely many linear parts.

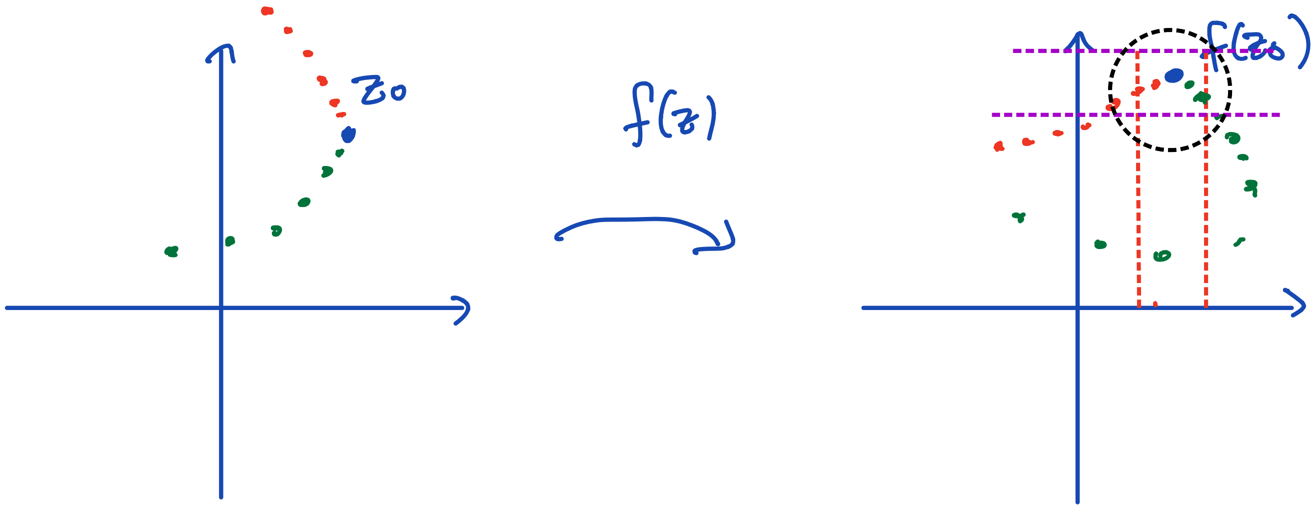
Def A function $f: \mathbb{C} \to \mathbb{C}$ is said to be continuous at z_0, if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$$|z - z_0| < \delta \implies |f(z) - f(z_0)| < \varepsilon$$

A function f is continuous in a domain D if f is continuous at every point in D.

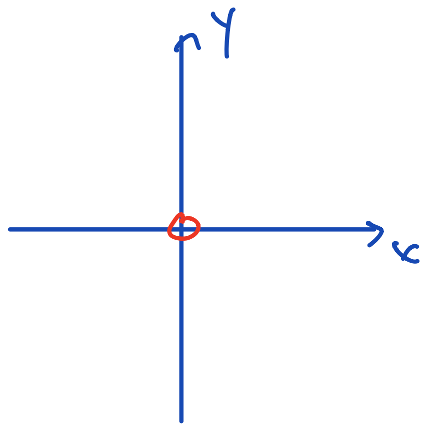
Theorem A complex function f is continuous iff the real and imaginary parts are continuous.

Suppose real and imaginary part are continuous



e.g. $f(z) = \frac{1}{z}$

Domain of f is $\mathbb{C} \setminus \{0\}$



$\frac{1}{z}$ is continuous fn on $\mathbb{C} \setminus \{0\}$

$$\frac{1}{z} = \frac{1}{x+iy}$$

$$= \frac{1}{x+iy} \cdot \frac{x-iy}{x-iy}$$

$$= \frac{x-iy}{x^2+y^2}$$

$$= \underbrace{\frac{x}{x^2+y^2}}_{\text{Continuous}} + i\underbrace{\left(-\frac{y}{x^2+y^2}\right)}_{\text{Continuous}}$$

$\therefore$ $f(z) = \frac{1}{z}$ is continuous

Differentiable of $\underset{\overset{\shortparallel}{u(x,y)+i\,v(x,y)}}{f(x+iy)}$ as a vector field

means the vector-valued function $(u(x,y),\ v(x,y))$ is differentiable as defined in multivariable calculus.

Example

$$f(x,y) = f(z) = x^2+i(x+y)$$
$$= \langle x^2,\ x+y\rangle$$

$$u(x,y) = x^2$$
$$v(x,y) = x+y$$

2×2

$$\begin{bmatrix} u(x+\Delta x, y+\Delta y) \\ v(x+\Delta x, y+\Delta y) \end{bmatrix} \doteq \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} + \begin{bmatrix} \frac{\partial u}{\partial x}(x,y) & \frac{\partial u}{\partial y}(x,y) \\ \frac{\partial v}{\partial x}(x,y) & \frac{\partial v}{\partial y}(x,y) \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}$$

$$\begin{bmatrix} (x+\Delta x)^2 \\ (x+\Delta x+y+\Delta y) \end{bmatrix} \doteq \begin{bmatrix} x^2 \\ x+y \end{bmatrix} + \begin{bmatrix} 2x & 0 \\ y & x \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}$$

off

Complex differentiable at z_0

complex multiplication

$$f(\underbrace{z_0}_{\in \mathbb{C}} + \underbrace{\Delta z}_{\in \mathbb{C}}) \doteq f(z_0) + \underbrace{w_0}_{\text{some complex number}} \cdot \Delta z$$

$$\lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} = w_0$$

complex division

Example $f(z) \triangleq z^3 = (x+iy)^3$

$$= \underbrace{(x^3 - 3xy^2)}_{u(x,y)} + i\,\underbrace{(3x^2y - y^3)}_{v(x,y)}$$

$x_0 = 1$, $y_0 = 1$

$$\begin{bmatrix} u(1+\Delta x, 1+\Delta y) \\ v(1+\Delta x, 1+\Delta y) \end{bmatrix} \doteq \begin{bmatrix} -2 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 & -6 \\ 6 & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}$$

$\frac{\partial u}{\partial x} = 3x^2 - 3y^2$, $\frac{\partial u}{\partial y} = -6xy$, $\frac{\partial v}{\partial x} = 6xy$, $\frac{\partial v}{\partial y} = 3x^2 - 3y^2$

in general

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \sim \begin{bmatrix} 3x^2 - 3y^2 & -6xy \\ 6xy & 3x^2 - 3y^2 \end{bmatrix}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

CR equation