Student ID: Name:

MAT2006: Elementary Real Analysis
Mid-term Test

Two hours, closed book.

Question 1. [20 marks] State the following theorems (proofs are not required).

(a) The Least Upper Bound Property;

Every nonempty set of real numbers that is bounded above has a least upper bound. [3']

(b) The Archimedean Property;

For any $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $x < n$. [2']

(c) The Nested Interval Property;

Let $I_1 \supset I_2 \supset I_3 \supset \cdots$ be a sequence of nested closed intervals, then

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset. \qquad [3']$$

(d) The Monotone Convergence Theorem;

If a sequence is monotone and bounded, then it converges. [3']

(e) The Bolzano–Weierstrass Theorem;

Every bounded sequence contains a convergent subsequence. [3']

(f) The Cauchy Criterion for sequences;

A sequence converges if and only if it is a Cauchy sequence. [3']

(g) The Heine–Borel Theorem.

A set $K \subset \mathbb{R}$ is compact if and only if it is closed and bounded. [3']

Question 2. [15 marks]

(i) Write down the sup, inf, max and min for the sets

$$A = (0, 1]; \qquad B = \left\{ \frac{1}{n} \, : \, n \in \mathbb{N} \right\}.$$

(ii) For the sequence $x_n = (-1)^n$. Write down

$$\limsup_{n\to\infty} x_n \qquad \text{and} \qquad \liminf_{n\to\infty} x_n.$$

(iii) Assume $\{x_n\}$ and $\{y_n\}$ are two bounded sequences. Show that

$$\limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n \geq \limsup_{n\to\infty}(x_n + y_n).$$

Solution. (i) [4']

$$\sup A = 1, \qquad \inf A = 0, \qquad \max A = 1, \qquad \min A \quad \text{does not exist.}$$

$$\sup B = 1, \qquad \inf B = 0, \qquad \max B = 1, \qquad \min B \quad \text{does not exist.}$$

(ii) [2']

$$\limsup_{n\to\infty} x_n = 1, \qquad \liminf_{n\to\infty} x_n = -1.$$

(iii) For $n, m \in \mathbb{N}$ and $n \geq m$, we have

$$x_n \leq \sup\{x_n\}_{n=m}^{\infty}, \qquad y_n \leq \sup\{y_n\}_{n=m}^{\infty}. \qquad [2']$$

which implies

$$x_n + y_n \leq \sup\{x_n\}_{n=m}^{\infty} + \sup\{y_n\}_{n=m}^{\infty}, \qquad \forall n \geq m. \qquad [1']$$

This means that the right hand side of the last equation is an upper bound of the set

$$\{x_n + y_n\}_{n=m}^{\infty},$$

and therefore

$$\sup\{x_n\}_{n=m}^{\infty} + \sup\{y_n\}_{n=m}^{\infty} \geq \sup\{x_n + y_n\}_{n=m}^{\infty} \qquad \forall m \in \mathbb{N}. \qquad [2']$$

Taking the limits when $m \to \infty$ to the above inequality, we have

$$\begin{aligned} &\lim_{m\to\infty} \sup\{x_n\}_{n=m}^{\infty} + \lim_{m\to\infty} \sup\{y_n\}_{n=m}^{\infty} \\ &= \lim_{m\to\infty} \left(\sup\{x_n\}_{n=m}^{\infty} + \lim_{m\to\infty} \sup\{y_n\}_{n=m}^{\infty} \right) \\ &\geq \lim_{m\to\infty} \sup\{x_n + y_n\}_{n=m}^{\infty} \qquad [3'] \end{aligned}$$

which, by the definition of upper limits [1'], is

$$\limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n \geq \limsup_{n\to\infty}(x_n + y_n),$$

as desired.

Question 3.[10 marks] Using the Heine–Borel theorem to prove that any bounded infinite set must have a limit point.

Proof. Assume A is a bounded infinite set. Suppose, by contradiction, that A does not have a limit point. [1'] Then, by the definition, A is closed. [1'] Since A is also bounded, the Heine–Borel theorem, A is a compact set. [1'] For every $x \in A$, since A has no limit point, in particular, x is not a limit point of A. Therefore, there exists a neighborhood $V_{\epsilon_x}(x)$ of x, such that

$$(*) \qquad A \cap V_{\epsilon_x}(x) = \{x\}. \quad [2']$$

It is clearly that

$$\bigcup_{x \in A} V_{\epsilon_x}(x) \supset A,$$

that is $\{V_{\epsilon_x}(x)\}_{x \in A}$ form an open cover of A. [1'] By the compactness of A, it has a finite subcover [1'] – there exists $N \in \mathbb{N}$ such that

$$\bigcup_{n=1}^{N} V_{\epsilon_{x_n}}(x_n) \supset A.$$

But from $(*)$, we have

$$A \cap \left(\bigcup_{n=1}^{N} V_{\epsilon_{x_n}}(x_n) \right) = \bigcup_{n=1}^{N} \left(A \cap V_{\epsilon_{x_n}}(x_n) \right) = \{x_1, \cdots, x_N\}. \quad [1']$$

Thus

$$A = \{x_1, \cdots, x_N\}, \quad [1']$$

which implies A is a finite set and thus is a contradiction with A is infinite. [1']

Therefore, A must have a limit point. $\square$

Question 4.[15 marks] Suppose the series $\sum_{n=1}^{\infty} a_n$ converges.

(i) Assume $a_n \geq 0$ for each $n \in \mathbb{N}$. Show that $\sum_{n=1}^{\infty} a_n^2$ also converges.

(ii) If we don't assume $a_n \geq 0$, does $\sum_{n=1}^{\infty} a_n^2$ still converge? If so, provide a proof. If not, give an example.

(iii) Assume $a_n \geq 0$ and $a_{n+1} \leq a_n$ for each $n \in \mathbb{N}$. Show that $\lim_{n\to\infty} na_n = 0$.

Proof. (i) The series $\sum a_n$ converges implies $\{a_n\} \to 0$ [2']. Hence, there exists $N \in \mathbb{N}$ such that $|a_n| < 1$ for each $n \geq N$ [1']. From the hypothesis $a_n \geq 0$, we have $0 \leq a_n < 1$ for $n \geq N$ and $a_n^2 < a_n$ for $n \geq N$ [1']. By the comparison Test [2'], the series $\sum a_n^2$ also converges.

(ii) No [1']. If we set $a_n = \frac{(-1)^n}{\sqrt{n}}$ [1']. The series $\sum a_n$ converges, by the Alternating Series Test [1']. But

$$\sum_{n=1}^{\infty} a_n^2 = \sum_{n=1}^{\infty} \frac{1}{n}$$

diverges, this is the harmonic series. [1']

(iii) Let $b_n = a_n$. Proof by contradiction. Suppose $b_n = na_n$ does not converge to 0 [1']. Then, there exists $\epsilon_0 > 0$ and a subsequence $n_k a_{n_k}$ such that

$$|b_{n_k} - 0| = |n_k a_{n_k}| = n_k a_{n_k} \geq \epsilon_0. \quad [1']$$

where we have made use of the fact that each a_n, and hence each a_{n_k}, is positive.

Now, whenever $j \geq n_k$, we have

$$a_j \geq a_{n_k} \geq \frac{\epsilon_0}{n_k}. \quad [1']$$

Therefore,

$$\begin{aligned} |a_{m+1} + a_{m+2} + \cdots + a_n| =& a_{m+1} + a_{m+2} + \cdots + a_n \\ \geq& \frac{n-m}{n_k}\epsilon_0 \end{aligned}$$

Thus, for any $N \in \mathbb{N}$, choose a n_k such that $n_k \geq 2N$. Then, for the particular $n = n_k$ and $m = N$, we have

$$|a_{N+1} + \cdots + a_{n_k}| \geq \frac{n_k - N}{n_k}\epsilon_0 \geq \frac{1}{2}\epsilon_0. \quad [1']$$

That is the series $\sum a_n$ doesnot meet the Cauchy criterion for the series convergence [1'], and so $\sum a_n$ diverges, which is a contradiction with the assumption. Therefore, $na_n \to 0$ as $n \to \infty$. □

Question 5.[20 marks]

Consider the following seven sets.

$$\emptyset; \qquad \mathbb{R}; \qquad \mathbb{Q}; \qquad \mathbb{I}; \qquad [0,1]; \qquad (0,1]; \qquad C \quad \text{(the Cantor set)}.$$

(i) Among the above sets, point out the finite, the countable, and the uncountable sets.

(ii) Among the above sets, point out the open, the closed, and the compact sets.

(iii) Show that any bounded open interval is F_σ.

(iv) Using the Baire Category Theorem show that $\mathbb{I}$ is not F_σ.

(v) Using part (iv), provide an example of "the countable intersection of F_σ sets is not F_σ."

Solution.

(i) Finite set: $\emptyset$. Countable set: $\mathbb{Q}$; Uncountable sets: $\mathbb{R}$, $\mathbb{I}$, $[0,1]$, $(0,1]$, C. [3']

(ii) Open sets: $\emptyset$, $\mathbb{R}$; Closed sets: $\emptyset$, $\mathbb{R}$, $[0,1]$, C; Compact sets: $\emptyset$, $[0,1]$, C [3'].

(iii) Let (a,b) be an bounded open interval, might be empty when $a \geq b$. Then

$$(a,b) = \bigcup_{n=1}^{\infty} \left[a + \frac{1}{n}, b - \frac{1}{n}\right]. \qquad [2']$$

(Note that when $b - a < 2$, first few of or all of the above closed intervals might be empty set, which is also closed.) Thus, any bounded open interval can be written as a union of countable many of closed set, thus is a F_σ set. [1']

(iv) Assume, for contradiction, $\mathbb{I}$ is F_σ, that is

$$\mathbb{I} = \bigcup_{n=1}^{\infty} I_n,$$

where each of I_n is a closed set. [1'] Recall that $\mathbb{Q}$ is a F_σ set, since it is countable, we may write

$$\mathbb{Q} = \bigcup_{n=1}^{\infty} Q_n,$$

where each of Q_n is closed. [1'] Note that $\mathbb{I}$ doesnot contain any open interval, since for otherwise, if $a < b$ and $(a,b) \subset \mathbb{I}$, there will be no rational number exists on (a,b), which is a contradiction with the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$ [1']. It then follows from $I_n \subset \mathbb{I}$ that I_n, and as its own closure, dosenot contain any nonempty open interval, thus each I_n is nowhere dense. Similarly, each of Q_n is also nowhere dense [2']. Then

$$\mathbb{R} = \mathbb{Q} \cup \mathbb{I} = \left(\bigcup_{n=1}^{\infty} Q_n\right) \cup \left(\bigcup_{n=1}^{\infty} I_n\right),$$

and the right-hand side is a countable union of nowhere dense sets [1'], which is a contradiction of Baire's Theorem [1']. Thus, $\mathbb{I}$ is not F_σ.

(v) Since $\mathbb{Q}$ is countable, we may write $\mathbb{Q} = \{q_1, q_2, q_3, \dots\}$. Note that

$$\mathbb{I} = \mathbb{Q}^c = \left(\bigcup_{n=1}^{\infty} \{q_n\}\right)^c = \bigcap_{n=1}^{\infty} \{q_n\}^c,$$

where we have made use of the De Morgan law. [2'] Note that $\{q_n\}^c = (-\infty, q_n) \cup (q_n, \infty)$, as a union of two F_σ sets is F_σ. [1'] The fact that $\mathbb{I}$ is not F_σ provides a desired example. [1']

Question 6. [20 marks]

(i) Let A' denote the derived set of A, that is the set of all limit points of A. Show that $(A')' \subset A'$, that is A' is closed.

(ii) Let $\{x_n\}$ be a bounded sequence and we may regard it as a set of real numbers, denoted by A and assume A is infinite. Let $E := A'$ be the set of limits points of A. Show that $s = \sup E$ exists and that ~~s is a limit point of E~~ $s \in E$.

(iii) We have shown that $\limsup_{n\to\infty} x_n = \sup E$. Prove that $\max E$ exists and that $\limsup_{n\to\infty} x_n = \max E$.

(iv) For a bounded nonempty set B, denote by $-B = \{-x \,|\, x \in B\}$. Show that $-\inf B = \sup(-B)$ and that $-\min B = \max(-B)$. Use this and part (iii) to show that $\liminf_{n\to\infty} x_n = \min E$.

(v) We have shown that $\{x_n\}$ converges if and only if $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n$. Using this to show that: if every convergent subsequence of $\{x_n\}$ converge to the same limit, then $\{x_n\}$ converges.

Proof. (i) Let $y \in (A')'$, that is ℓ is a limit point of A', the derived set of A. By the definition of limit point, for any $\epsilon > 0$, $V_\epsilon^0(y) \cap A' \neq \emptyset$. [1'] That is, there exists $\ell \in A'$, such that $\ell \neq y$ and $\ell \in V_\epsilon(y)$ [1']. Now take $\epsilon' = \min\{|y-\ell|, |y+\epsilon-\ell|, |y-\epsilon-\ell|\} > 0$ [1']. By $\ell \in A'$, there exists a $x \in A$ such that $x \in V_{\epsilon'}(\ell) \subset V_\epsilon^0(y)$ [1']. That is $A \cap V_\epsilon^0(y) \neq \emptyset$. Therefore, by definition, y is also a limit point of A, that is $y \in A'$ [1']. Thus A' is closed.

(ii) Since A is bounded and infinite, according to Problem 3, A must have a limit point [1']. That is E is nonempty [1']. For any $y \in E = A'$, there exists a sequence in A, not equal to y, converges to y. Thus, by carefully choosing, there exists a subsequence $\{x_{n_k}\}$ converges to y [1']. Since $\{x_n\}$ is bounded, assume $|x_n| \leq M$ for all $n \in \mathbb{N}$. Then by the Order Limit Theorem, the limit of x_{n_k}, y, satisfies $|y| \leq M$. That is E is a bounded set [1']. By the Least Upper Bound Property [1'], $\sup E$ exists.

By definition of $s = \sup E$, for any $n \in \mathbb{N}$, there exists $z_n \in E$ such that $s - \frac{1}{n} < z_n \leq s$, that is $z_n \to s$ as $n \to \infty$. There are two cases, Case (1), if $z_n \neq s$ for each $n \in \mathbb{N}$, then s is a limit point of E, and since E is closed by part (i), $s \in E$. [2'] Case (2) if $z_{n_0} = s$ for some n_0, then $s = z_{n_0} \in E$.

(iii) By part (ii), $s = \sup E \in E$, thus s is not only supremum, it is as well as the maximum of E, that is $s = \sup E = \max E$. [1']

Thus $\limsup_{n\to\infty} x_n = \max E$.

(iv) Let $s = \sup(-B)$, then for all $x \in B$, $-x \leq s$, that is $x \geq -s$, which means $-s$ is a lower bound of B. Assume ℓ is also a lower bound of B, that is $x \leq \ell$, then $-x \geq -\ell$ for all $-x \in (-B)$, that is $-\ell$ is an upper bound of $(-B)$, thus $s = \sup(-B) \leq -\ell$, that is $-s \geq \ell$. Thus, by definition, $-s$ is the greatest lower bound of B, or, $\inf B = -s = -\sup(-B)$. [2']

Suppose $M = \max(-B)$ exists, then $M \in B$ and $-x \leq M$ for all $x \in B$. Thus, $-M \in B$ and $x \geq -M$, which means that $\min B = -M = -\max(-B)$. [1']

Let $y_n = -x_n$, then it is clearly that

$$\inf_{n\geq m} \{x_n\} = -\sup_{n\geq m} y_n,$$

by taking limit $m \to \infty$, we have

$$\liminf_{n\to\infty} x_n = -\limsup_{n\to\infty}(-x_n) = -\max(-E) = \min E.$$

Here, we have made use of the facts we just proved. [2']

(v) By the assumption, the set A has only one limit point, that is E is a set contains only one point, say $E = \{s\}$. [1'] Thus $\max E = \min E = s$. Therefore,

$$\limsup_{n\to\infty} x_n = \max E = s = \min E = \liminf_{n\to\infty} x_n,$$

and hence $\{x_n\}$ converges. [1'] $\square$