MAT 3253 Lecture 26

Morera theorem

Theorem Consider continuous function $f(z)$ in D.
If $\int_C f(z)\, dz = 0$ for all closed smooth in D
then f is analytic in D.

Proof Show that f has an anti-derivative.

$\int_C f(z)\, dz = 0 \quad \forall$ closed curve C

$\int f(z)\, dz$ is independent of path

Define $F(z) = \int_{z_0}^{z} f(z)\, dz$. It is well-defined

$F(z)$ is an anti-derivative of $f(z)$ $\blacksquare$

Theorem Given a domain D. Then the followings are equivalent

(i) $\int_C f(z)\, dz = 0 \quad \forall$ closed curves
and for all analytic functions f

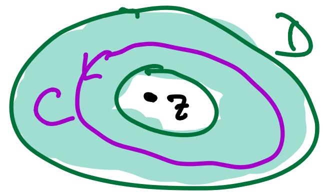
(ii) D is simply connected

Proof (ii) ⇒ (i) Cauchy thm

(i) ⇒ (ii) Suppose D is not simply connected

Fact: D is simply connected iff $w(C;z)=0$ for all closed curve $C \subseteq D$, $z \notin D$.

$$w(C;z) \neq 0$$
$$\|$$
$$\frac{1}{2\pi i}\int_C \frac{1}{w-z}\,dw \neq 0$$



$\blacksquare$

Suppose D is multiply connected

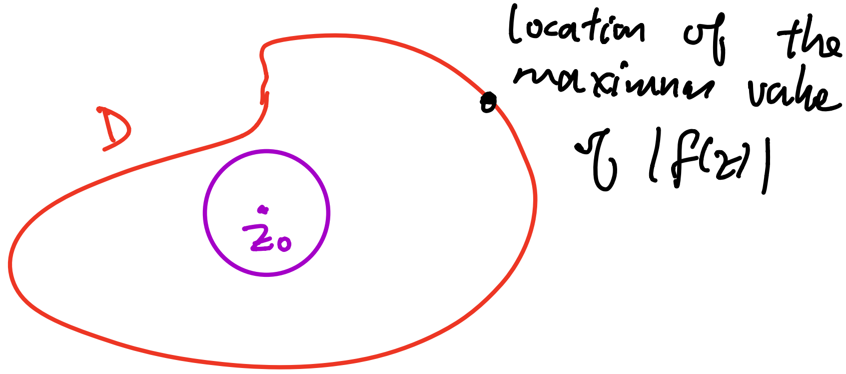
* $\int_C f(z)\,dz = 0$ $\forall$ closed curve $C \subseteq D$ but not for all analytic f.

* $\int_C f(z)\,dz = 0$ $\forall$ analytic f must fail for some closed curve C

Theorem Maximum modulus principle

Consider a analytic function $f(z)$

If the maximum value $|f(z)|$ in a domain occurs in the interior of D, then f must be a constant function.



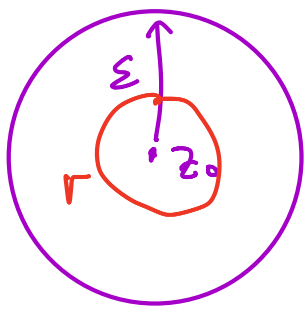
Proof Suppose the modulus of $f(z)$ attains maximum at an interior z_0 in D.

We can find a sufficiently small $\varepsilon > 0$ s.t. $D(z_0; \varepsilon) \subseteq D$.

$$|f(z)| \leq |f(z_0)| \quad \forall\, z \text{ in } D(z_0; \varepsilon).$$

If $g(x)$ is continuous and ≥ 0 $x \in [a,b]$.

$$\int_a^b g(x) = 0 \implies g(x) = 0 \quad \forall x$$



Pick $r < \varepsilon$

$|z - z_0| = r$

$z = z_0 + re^{i\theta} \qquad 0 \leq \theta \leq 2\pi$

$$f(z_0) \overset{CIF}{=} \frac{1}{2\pi \cancel{i}} \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{\cancel{re^{i\theta}}} \cancel{r i e^{i\theta}}\, d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta$$

$$|f(z_0)| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + re^{i\theta})|\, d\theta$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi} f(z_0)\, d\theta = |f(z_0)|$$

$$\frac{1}{2\pi}\int_0^{2\pi} |f(z_0)|\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + re^{i\theta})|\, d\theta$$

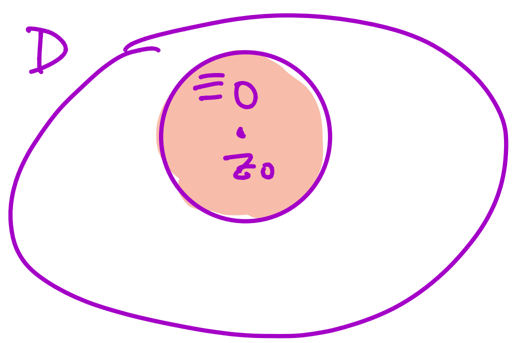
$$\begin{cases} \int_0^{2\pi} |f(z_0)| - |f(z_0+re^{i\theta})|\, d\theta = 0 \\ \underbrace{|f(z_0)| - |f(z_0+re^{i\theta})|}_{g(\theta)} \geq 0 \end{cases}$$

$$\therefore\ |f(z_0)| = |f(z_0 + re^{i\theta})| \quad \forall\, \theta$$

CR eqn

$$\therefore\ |f(z_0)| = |f(z)| \quad \forall\, z \in D(z_0; \varepsilon).$$

$$f(z) = \text{Constant}$$



By identity theorem

f is constant in D.

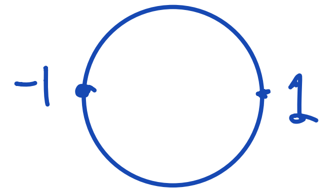
<u>Example</u> Find maximum of $|z^2 - z|$ in $|z| \leq 1$.

Maximum value occurs at some point z $|z| = 1$.

① $z = \cos\theta + i\sin\theta \qquad \theta \in [0, 2\pi]$

② $|z^2 - z| = |z|\,|z-1|$

$$\max_{|z|=1} |z-1| = |(-1)-1| = 2$$



max occurs at $z=-1$

<u>Def</u> A function $f(z)$ one-to-one (injective) in domain D $f(z_1) \neq f(z_2)$ for two $z_1 \neq z_2$ in D.

A function $f(z)$ is <u>locally one-to-one at z_0</u> if there is an open disc $D(z_0;\delta)$ for $\delta > 0$ s.t. $f(z)$ is one-to-one inside $D(z_0;\delta)$.

<u>Example</u> e^z is not one-to-one but e^z is locally one-to-one at every point z.

<u>Theorem</u> Suppose $f(z)$ is an analytic function and z_0 is a point in the domain of f. Then the followings are equivalent.

(i) $f(z)$ preserves angle at z_0 (geometric)

(ii) $f''(z_0) \neq 0$ (analytic)

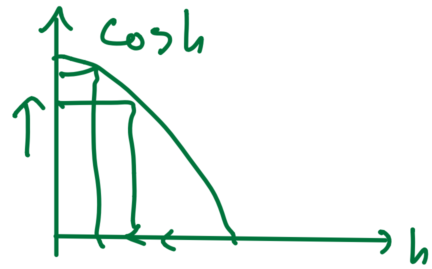
(iii) $f(z)$ is locally one-to-one at z_0. (algebraic)

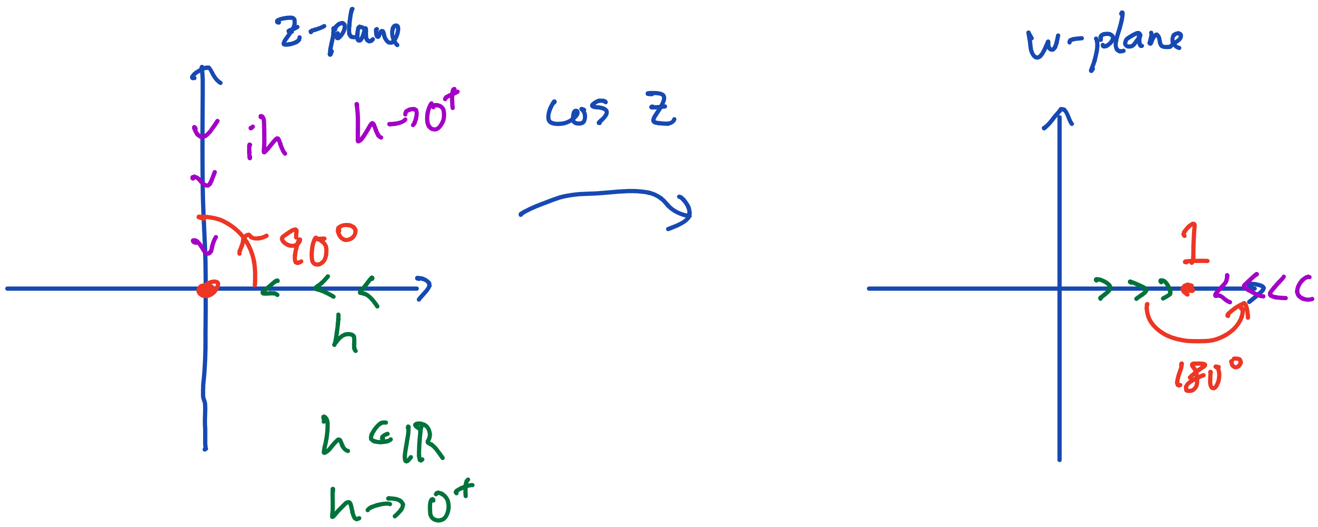
Example $f(z)=\cos(z) = \dfrac{e^{iz}+e^{-iz}}{2}$

$z_0 = 0$

$\cos(0) = 1$

$\sin(0) = 0$





$$\cos(ih) = 1 - \frac{(ih)^2}{2} + \frac{(ih)^4}{4!} - \dots$$

$$= \underbrace{1 + \frac{h^2}{2} + \frac{h^4}{4!} + \dots}_{\to 1 \text{ from above.}}$$

real valued

$$\cos z - 1 = \frac{z^2}{2} + \frac{z^4}{4!} + \dots$$