MAT3253 Homework 10

Due date: 9 Apr.

Question 1. (Brown&Churchill Ex.52.1) Let C denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$. Evaluate each of these integrals:

(a) $\displaystyle\int_C \frac{e^{-z}}{z - (\pi i/2)}\, dz$;

(b) $\displaystyle\int_C \frac{\cos z}{z(z^2 + 8)}\, dz$;

(c) $\displaystyle\int_C \frac{z}{2z + 1}\, dz$;

(d) $\displaystyle\int_C \frac{\cosh z}{z^4}\, dz$;

(e) $\displaystyle\int_C \frac{\tan(z/2)}{(z - x_0)^2}\, dz \quad (-2 < x_0 < 2)$.

Question 2. (Brown&Churchill Ex.52.4) Let C be any simple closed contour, described in the positive sense in the z plane, and write

$$g(z) = \int_C \frac{s^3 + 2s}{(s - z)^3}\, ds.$$

Show that $g(z) = 6\pi i z$ when z is inside C and that $g(z) = 0$ when z is outside.

Question 3. (Brown&Churchill Ex.52.7) Let C be the unit circle $z = e^{i\theta}$ $(-\pi \le \theta \le \pi)$. First show that for any real constant a,

$$\int_C \frac{e^{az}}{z}\, dz = 2\pi i.$$

Then write this integral in terms of θ to derive the integration formula

$$\int_0^{\pi} e^{a\cos\theta} \cos(a \sin\theta)\, d\theta = \pi.$$

Question 4. (Brown&Churchill Ex.52.8, Legendre polynomials)

(a) With the aid of the binomial formula, show that for each value of n, the function

$$P_n(z) = \frac{1}{n!2^n} \frac{d^n}{dz^n}(z^2-1)^n \quad (n = 0, 1, 2, 3 \ldots)$$

is a polynomial of degree n.

(b) Write down the polynomial in part (a) for $n = 0, 1, 2, 3$.

(c) Let C denote any positively oriented simple closed contour surrounding a fixed point z. With the aid of the integral representation for the nth derivative of a function, show that the polynomials in part (a) can be expressed in the form

$$P_n(z) = \frac{1}{2^{n+1}\pi i} \int_C \frac{(s^2-1)^n}{(s-z)^{n+1}}\, ds \quad (n = 0, 1, 2, \ldots).$$

(d) Point out how the integrand in the representation for $P_n(z)$ in part (c) can be written $(s+1)^n/(s-1)$ if $z = 1$. Then apply the Cauchy integral formula to show that

$$P_n(1) = 1 \qquad (n = 0, 1, 2, \ldots).$$

Similar, show that

$$P_n(-1) = (-1)^n \qquad (n = 0, 1, 2, \ldots).$$

Question 5. (Bak&Newman Chapter 5 ex.8) Suppose f is entire and $|f(z)| \leq A + B|z|^{3/2}$ for some constants A and B. Show that f is a linear polynomial.

Question 6. (Bak&Newman Chapter 5 ex.11) A *real* polynomial is a polynomial whose coefficients are all real. Prove that a real polynomial of odd degree must have a real zero. (Use the property that the complex roots of a real polynomial can be grouped into conjugate pairs.)

Question 7. Prove that there is only one entire function whose restriction to the real axis is the real exponential function. (Show that if there are two entire functions $f(z)$ and $g(z)$ such that $f(x) = g(x) = e^x$ for all $x \in \mathbb{R}$, then $f(z) = g(z)$ for all $z \in \mathbb{C}$.)