MAT 3253 Lecture 6

Review

A vector function $\vec{f}(x,y) \in \mathbb{R}^2$ is called **real differentiable** at (x_0,y_0) if $\exists$ 2×2 matrix M s.t.

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \frac{\left\| \vec{f}(x_0+\Delta x, y_0+\Delta y) - \underbrace{\left(\vec{f}(x_0,y_0) + M\cdot \begin{bmatrix}\Delta x\\ \Delta y\end{bmatrix}\right)}_{\text{linear approx}} \right\|}{\sqrt{\Delta x^2 + \Delta y^2}} = 0$$

$$\vec{f}(x_0+\Delta x, y_0+\Delta y) \doteq f(x_0,y_0) + M\cdot \begin{bmatrix}\Delta x\\ \Delta y\end{bmatrix}$$

* necessary condition $\vec{f} = (u(x,y), v(x,y))$

u_x, u_y, v_x, v_y exists at (x_0,y_0)

* sufficient condition

① u_x, u_y, v_x, v_y exists in a neighborhood of (x_0,y_0)

② u_x, u_y, v_x, v_y are continuous at (x_0,y_0)

<u>Def</u> A complex function $f(z) = f(x+iy)$ is called <u>complex differentiable</u> at $z_0 = x_0+iy_0$

if $\quad f(z_0+\Delta z) \doteq f(z_0) + f'(z_0)\cdot \Delta z$

or equivalently $\lim_{h\to 0} \dfrac{f(z_0+h)-f(z_0)}{h}$ exists

and we write this limit as $f'(z_0)$.

A necessary condition for complex differentiable at z_0

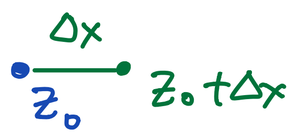
Theorem Suppose $f(z)$ is complex differentiable at z_0 in the domain of $f(z)$.

$f(x,y) = u(x,y) + i\,v(x,y)$.

Then $u_x = v_y$ and $u_y = -v_x$.

(Cauchy - Riemann eqn)

Proof $\Delta z = \Delta x$



$$\lim_{\Delta x\to 0} \frac{u(x_0+\Delta x, y_0) + i\,v(x_0+\Delta x, y_0) - u(x_0,y_0) - i\,v(x_0,y_0)}{\Delta x}$$

$$= \lim_{\Delta x\to 0} \frac{u(x_0+\Delta x, y_0) - u(x_0,y_0)}{\Delta x} + i \lim_{\Delta x\to 0} \frac{v(x_0+\Delta x, y_0) - v(x_0,y_0)}{\Delta x}$$

$$= \frac{\partial u}{\partial x}(x_0,y_0) + i\,\frac{\partial v}{\partial x}(x_0,y_0).$$

$$\lim_{\Delta y \to 0} \frac{u(x_0, y_0+\Delta y) + i v(x_0, y_0+\Delta y) - u(x_0,y_0) - i v(x_0,y_0)}{i \Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{u(x_0, y_0+\Delta y) - u(x_0,y_0)}{i\Delta y} + \lim_{\Delta y \to 0} \frac{i v(x_0, y_0+\Delta y) - i v(x_0,y_0)}{i\Delta y}$$

$$= -i \frac{\partial u}{\partial y}(x_0,y_0) + \frac{\partial v}{\partial y}(x_0,y_0)$$

$$\therefore \quad u_x = v_y \quad \text{and} \quad u_y = -v_x \qquad \blacksquare$$

$$\vec{f}(x_0+\Delta x, y_0+\Delta y) \doteq \vec{f}(x_0,y_0) + \underbrace{\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}}\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}$$

is in the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ if f is complex differentiable

A sufficient condition $\quad f(z) = u(x,y) + i v(x,y)$

<u>Theorem</u> A complex function f is complex differentiable at z_0 if

1. u_x, u_y, v_x, v_y exists in a neighborhood of z_0
2. Cauchy-Riemann equations are satisfied at z_0
3. u_x, u_y, v_x, v_y are continuous functions at z_0

Example $\quad f(z) = az+b \qquad a, b \in \mathbb{C}$

$$f'(z) = a \qquad \text{for any } z \in \mathbb{C}$$

Example $\quad f(z) = \bar{z} = x - iy$

$$u(x,y) = x \quad , \quad v(x,y) = -y$$

$$u_x = 1 \qquad\qquad v_y = -1$$

$\therefore$ $f(z) = \bar{z}$ is not complex differentiable at any point z in $\mathbb{C}$.

Example $\quad f(z) = z^2$

Method 1 $\quad (x+iy)^2 = \underbrace{x^2 - y^2}_{u(x,y)} + i\,\underbrace{2xy}_{v(x,y)}$

$$u_x = 2x$$
$$u_y = -2y$$
$$v_x = 2y$$
$$v_y = 2x$$

1. Partial derivatives exists everywhere

2. $\begin{cases} u_x = 2x = v_y \\ u_y = -2y = -v_y \end{cases}$

3. All partial derivatives are continuous everywhere

$\therefore$ $f(z) = z^2$ is complex differentiable at all $z \in \mathbb{C}$

Method 2 by first principle

$$\lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$$

$$\frac{(z+h)^2 - z^2}{h} = \frac{2zh + h^2}{h} = 2z + h$$

$$\lim_{h\to 0} 2z+h = 2z$$

$f'(z) = 2z$ for all $z \in \mathbb{C}$.

Example $f(z) = |z|^2 = x^2+y^2$

$u(x,y) = x^2+y^2$ $\quad v(x,y) = 0$

$u_x = 2x$ $\quad v_x = 0$

$u_y = 2y$ $\quad v_y = 0$

$$\left.\begin{aligned} 2x = u_x = v_y = 0 \\ 2y = u_y = -v_x = 0 \end{aligned}\right\} \text{only solution is } (x,y) = (0,0)$$

1. Partial derivatives exists in a neighborhood of $z=0$.
2. CR satisfied at $z=0$
3. Partial derivatives are continuous at $z=0$

$\therefore$ $|z|^2$ is complex differentiable at $z=0$.

x^2+y^2 is real differentiable for all (x,y).

Example $f(z) = \frac{1}{z}$ for $z \in \mathbb{C} \setminus \{0\}$

By first principle

Suppose $z \neq 0$

$$\frac{\frac{1}{z+h} - \frac{1}{z}}{h} = \frac{1}{h}\left(\frac{\cancel{z} - (\cancel{z}+h)}{(z+h)\cdot z} \right)$$

$$= \frac{-h}{h(z+h)\cdot z}$$

$$= \frac{-1}{z(z+h)}$$

$$\lim_{h \to 0} \left(\frac{-1}{z(z+h)} \right) = -\frac{1}{z^2}$$

$\therefore$ $f(z) = \frac{1}{z}$ is complex differentiable in $\mathbb{C} \setminus \{0\}$.

Def A function f is <u>analytic</u> at a point z_0 if there is a neighborhood of z_0 s.t. f is complex differentiable at every point z in the neighborhood.

Def A function is <u>entire</u> if it is complex differentiable at every point $z \in \mathbb{C}$.

$f(z) = |z|^2$ is not analytic

$\frac{1}{z}$ is not entire