MAT2006: Elementary Real Analysis
Assignment #4

Deadline: Dec 5

1. Let

$$g_a = \begin{cases} x^a \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Find a particular (potentially noninteger) value for a so that

(a) g_a is differentiable on $\mathbb{R}$ but such that $g'(a)$ is unbounded on $[0, 1]$.

(b) g_a is differentiable on $\mathbb{R}$ with g_a' continuous but not differentiable at zero.

(c) g_a is differentiable on $\mathbb{R}$ and g_a' is differentiable on $\mathbb{R}$, but such that g_a'' is not continuous at zero.

2. Recall that a function $f : (a, b) \to \mathbb{R}$ is increasing on (a, b) if $f(x) \leq f(y)$ whenever $x < y$ in (a, b). A familiar mantra from calculus is that a differentiable function is increasing if its derivative is positive, but this statement requires some sharpening in order to be completely accurate.

Show that the function

$$g(x) = \begin{cases} x/2 + x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is differentiable on $\mathbb{R}$ and satisfies $g'(0) > 0$. Now, prove that g is not increasing over any open interval containing 0.

In the next section we will see that f is indeed increasing on (a, b) if and only if $f'(x) \geq 0$ for all $x \in (a, b)$.

3. A fixed point of a function f is a value x where $f(x) = x$. Show that if f is differentiable on an interval with $f'(x) \neq 1$, then f can have at most one fixed point.

4. Let $f(x) = x \sin(1/x^4) e^{-1/x^2}$ and $g(x) = e^{-1/x^2}$. Using the familiar properties of these functions, compute the limit as x approaches zero of $f(x)$, $g(x)$, $f(x)/g(x)$, and $f'(x)/g'(x)$. Explain why the results are surprising but not in conflict with the content of L'Hosipital's Rule.

5. (i) Assume $f(x)$ is continuous on $[a, b]$ and differentiable in (a, b), $f(a) < 0$, $f(b) < 0$, and there exists one $c \in (a, b)$ such that $f(c) > 0$. Show that there exists $\xi \in (a, b)$ such that

$f(\xi) + f'(\xi) = 0$.
Hint. Consider $F(x) = e^x f(x)$.

(ii) Assume $g(x)$ is continuous on $[0,1]$ and differentiable in $(0,1)$. Show that there exists $\xi \in (0,1)$ such that $g'(\xi)g(1-\xi) = g(\xi)g'(1-\xi)$.

6. Let

$$f_n(x) = \frac{nx}{1+nx^2}.$$

(a) Find the pointwise limit of $\{f_n\}$ for all $x \in (0,\infty)$.
(b) Is the convergence uniform on $(0,\infty)$?
(c) Is the convergence uniform on $(0,1)$?
(d) Is the convergence uniform on $(1,\infty)$?

7. (i) Define a sequence of functions on $\mathbb{R}$ by

$$f_n(x) = \begin{cases} 1 & \text{if } x = 1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$

and let f be the pointwise limit of f_n.

Is each f_n continuous at zero? Does $f_n \to f$ uniformly on $\mathbb{R}$? Is f continuous at zero?

(ii) Repeat this exercise using the sequence of functions

$$g_n(x) = xf_n(x) = \begin{cases} x & \text{if } x = 1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n} \\ 0 & \text{otherwise.} \end{cases}$$

(iii) Repeat the exercise once more with the sequence

$$h_n(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n} \\ x & \text{if } x = 1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n-1} \\ 0 & \text{otherwise.} \end{cases}$$

In each case, explain how the results are consistent with the content of the Continuous Limit Theorem.

8. For each $n \in \mathbb{N}$ and $x \in [0,\infty)$, let

$$g_n(x) = \frac{x}{1+x^n}, \qquad h_n(x) = \begin{cases} 1 & \text{if } x \geq 1/n \\ nx & \text{if } 0 \leq x < 1/n. \end{cases}$$

Answer the following questions for the sequences $\{g_n\}$ and $\{h_n\}$:

(a) Find the pointwise limit on $[0,\infty)$.
(b) Explain how we know that the convergence cannot be uniform on $[0,\infty)$.
(c) Choose a smaller set over which the convergence is uniform and supply an argument to show that this is indeed the case.

9. Assume $f_n \to f$ on a set A. The Continuous Limit Theorem is an example of a typical type of question which asks whether a trait possessed by each f_n is inherited by the limit function. Provide an example to show that all of the following propositions are false if the convergence is only assumed to be pointwise on A. Then go back and decide which are true under the stronger hypothesis of uniform convergence.

(a) If each f_n is uniformly continuous, then f is uniformly continuous.

(b) If each f_n is bounded, then f is bounded.

(c) If each f_n has a finite number of discontinuities, then f has a finite number of discontinuities.

(d) If each f_n has fewer than M discontinuities (where $M \in \mathbb{N}$ is fixed), then f has fewer than M discontinuities.

(e) If each f_n has at most a countable number of discontinuities, then f has at most a countable number of discontinuities.

10. Assume $f_n \to f$ pointwise on $[a, b]$ and the limit function f is continuous on $[a, b]$. If each f_n is increasing (but not necessarily continuous), show $f_n \to f$ uniformly.

11 (Dini's Theorem)**.** Assume $f_n \to f$ pointwise on a compact set K and assume that for each $x \in K$ the sequence $f_n(x)$ is increasing. Follow these steps to show that if f_n and f are continuous on K, then the convergence is uniform.

(a) Set $g_n = f - f_n$ and translate the preceding hypothesis into statements about the sequence $\{g_n\}$.

(b) Let $\epsilon > 0$ be arbitrary, and define $K_n = \{x \in K \mid g_n(x) \geq \epsilon\}$. Argue that $K_1 \supset K_2 \supset K_3 \supset \cdots$, and use this observation to finish the argument.

12 (Cantor Function)**.** Review the construction of the Cantor set $C \subset [0, 1]$.

(a) Define $f_0(x) = x$ for all $x \in [0, 1] = C_0$. Now, let

$$f_1(x) = \begin{cases} (3/2)x & \text{for } 0 \leq x \leq 1/3 \\ 1/2 & \text{for } 1/3 < x < 2/3 \\ (3/2)x - 1/2 & \text{for } 2/3 \leq x \leq 1. \end{cases}$$

Sketch f_0 and f_1 over $[0, 1]$ and observe that f_1 is continuous, increasing, and constant on the middle third $(1/3, 2/3) = [0, 1] \setminus C_1$.

(b) Construct f_2 by imitating this process of flattening out the middle third of each nonconstant segment of f_1. Specifically, let

$$f_2(x) = \begin{cases} (1/2)f_1(3x) & \text{for } 0 \leq x \leq 1/3 \\ f_1(x) & \text{for } 1/3 < x < 2/3 \\ (1/2)f_1(3x - 2) + 1/2 & \text{for } 2/3 \leq x \leq 1. \end{cases}$$

If we continue this process, show that the resulting sequence $\{f_n\}$ converges uniformly on $[0, 1]$.

(c) Let $f = \lim_{n \to \infty} f_n$. Prove that f is a continuous, increasing function on $[0, 1]$ with $f(0) = 0$ and $f(1) = 1$ that satisfies $f'(x) = 0$ for all x in the open set $[0, 1] \setminus C$. Recall that the "length" of the Cantor set C is 0. Somehow, f manages to increase from 0 to 1 while remaining constant on a set of "length 1."

13. Let

$$g_n(x) = \frac{nx + x^2}{2n}$$

and set $g(x) = \lim_{n\to\infty} g_n(x)$. Show that g is differentiable in two ways:

(a) Compute $g(x)$ by algebraically taking the limit as $n \to \infty$ and then find $g'(x)$.

(b) Compute $g_n'(x)$ for each $n \in \mathbb{N}$ and show that the sequence of derivatives $\{g_n'\}$ converges uniformly on every interval $[-M, M]$. Then conclude $g'(x) = \lim_{n\to\infty} g_n'(x)$.

(c) Repeat parts (a) and (b) for the sequence $f_n(x) = (nx^2 + 1)/(2n + x)$.

14. Provide an example or explain why the request is impossible. Let's take the domain of the functions to be all of $\mathbb{R}$.

(a) A sequence $\{f_n\}$ of nowhere differentiable functions with $f_n \to f$ uniformly and f everywhere differentiable.

(b) A sequence $\{f_n\}$ of differentiable functions such that $\{f_n'\}$ converges uniformly but the original sequence $\{f_n\}$ does not converge for any $x \in \mathbb{R}$.

(c) A sequence $\{f_n\}$ of differentiable functions such that both $\{f_n\}$ and $\{f_n'\}$ converge uniformly but $f = \lim f_n$ is not differentiable at some point.

15. Decide whether each proposition is true or false, providing a short justification or counterexample as appropriate.

(a) If $\sum_{n=1}^{\infty} g_n$ converges uniformly, then $\{g_n\}$ converges uniformly to zero.

(b) If $0 \le f_n \le g_n$ and $\sum_{n=1}^{\infty} g_n$ converges uniformly, then $\sum_{n=1}^{\infty} f_n$ converges uniformly.

(c) If $\sum_{n=1}^{\infty} f_n$ converges uniformly on A, then there exist constants M_n such that $|f_n(x)| \le M_n$ for all $x \in A$ and $\sum_{n=1}^{\infty} M_n$ converges.

16. (a) Prove that

$$h(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

is continuous on $[-1, 1]$.

(b) The series

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$$

converges for every x in the half-open interval $[-1, 1)$ but does not converge when $x = 1$. For a fixed $x_0 \in (-1, 1)$, explain how we can still use the Weierstrass M-Test to prove that f is continuous at x_0.

17. Let

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{x+n} = \frac{1}{x} - \frac{1}{x+1} + \frac{1}{x+2} - \frac{1}{x+3} + \cdots .$$

Show f is defined for all $x > 0$. Is f continuous on $(0, \infty)$? How about differentiable?

18. Let

$$f(x) = \sum_{k=1}^{\infty} \frac{\sin kx}{k^3}.$$

(a) Show that $f(x)$ is differentiable and that the derivative $f'(x)$ is continuous.

(b) Can we determine if f is twice-differentiable?

19. Consider the function

$$f(x) = \sum_{k=1}^{\infty} \frac{\sin(x/k)}{k}.$$

Where is f defined? Continuous? Differentiable? Twice-differentiable?

20. Let $\{r_1, r_2, r_3, \dots\}$ be an enumeration of the set of rational numbers. For each $r_n \in \mathbb{Q}$, define

$$u_n(x) = \begin{cases} 1/2^n & \text{for } x > r_n \\ 0 & \text{for } x \leq r_n. \end{cases}$$

Now, let $h(x) = \sum_{n=1}^{\infty} u_n(x)$. Prove that h is a monotone function defined on all of $\mathbb{R}$ that is continuous at every irrational point.

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