Question 1. Prove by induction the following is true.

$$\sum_{i=1}^{n} ix^i = \frac{x-(n+1)x^{n+1}+nx^{n+2}}{(1-x)^2}$$

Question 2. If sequence $\{a_n\}$ is defined as follows

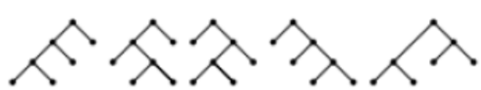
$$a_n = \begin{cases} 1 & \text{for } n = 0 \\ 1 + \sum_{i=0}^{n-1} a_i & \text{for } n > 0 \end{cases}$$

Use induction to prove that $a_n = 2^n$ for any integer $n \geq 0$.

Question 3. Define $F_0 = 0, F_1 = 1, F_{n+2} = F_{n+1} + F_n$ for $n = 0, 1, 2, \ldots$ Prove that for any $n \geq 0$ we have $F_n \leq ((1+\sqrt{5})/2)^{n-1}$.

Question 4. Show that the equation $8u^4 + 4v^4 + 2w^4 = x^4$ has no non-zero integer solution.

Question 5. A full binary tree is a rooted binary tree where every vertex has either two children or no children. Let B_n be the number of full binary trees with n leaves, e.g. $B_4 = 5$. Find a recurrence relation of B_n. Explain your answer.



Question 6. A chain letter works as follows: One person sends a copy of the letter to five friends, each of whom sends a copy to five friends, each of whom sends a copy to five friends, and so forth. Suppose they only send to friends who have not received copies. How many people will have received copies of the letter after the twentieth repetition of this process?

Question 7. A runner targets herself to improve her time on a certain course by 3 seconds a day. If on day 0 she runs the course in 3 minutes, how fast must she run it on day 14 to stay on target?

Question 8. Show that if r, s, a_0, and a_1 are numbers with $r \neq s$, then there exists unique numbers C and D so that $C + D = a_0, Cr + Ds = a_1$.

Question 9. Let $a_0, a_1, a_2, \ldots$ be the sequence defined by the explicit formula $a_n = C \cdot 2^n + D$ for all integers $n \geq 0$, where C and D are real numbers. Show that for any choice of C and $D, a_k = 3a_{k-1} - 2a_{k-2}$ for all integers $k \geq 2$.

Question 10. Compute $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n$ for small values of n (up to about 5 or 6). Conjecture explicit formulas for the entries in this matrix, and prove your conjecture using mathematical induction.