MAT2002 Ordinary Differential Equations
Review for the first 3 weeks

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Definition of ODE and IVP

Definition R.1

An ordinary differential equation is an equation involving **ONE** independent variable $t \in I$(I is an interval) and **ONE** dependent variable y of the form

$$F(t, y, y', y'', \ldots, y^{(n)}) = 0.$$

Given constants $t_0, t_1, \ldots, t_{n-1} \in I$ and $y_0, y_1, \ldots, y_{n-1} \in \mathbb{R}$,we call

$$\begin{cases} F(t, y, y', y'', \ldots, y^{(n)}) = 0, \\ y(t_0) = y_0, \frac{dy}{dt}(t_1) = y_1, \ldots, \frac{d^{(n-1)}y}{dt^{n-1}}(t_{n-1}) = y_{n-1}, \end{cases}$$

an **initial value problem** (IVP).

The **order** of an ODE is the **highest order** of derivative in the ODE.

Definition related to ODE

Definition R.2

(a) An ODE $F(t, y, y', y'', \dots, y^{(n)}) = 0$ is linear is if F is a **linear function** of $y, \frac{dy}{dt}, \dots, \frac{d^n y}{dt^n}$. Otherwise, it is a **non-linear** ODE. The general **linear** ODE of order n is

$$a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \cdots + a_0(t)y(t) = f(t),$$

for some given functions $a_0, a_1, \dots, a_n$ and f.

(b) An ODE is called **autonomous** if the independent variable does not appear explicitly (only in the derivatives), the **autonomous** ODE has the form: $F(y, y', y'', \dots, y^{(n)}) = 0$. Otherwise it is a **non-autonomous** ODE.

Real-world mathematical modelling

(a) Motion of a falling object. Two force acting on the object: gravitational force and air resistance force (assume to be proportional to the velocity). Need to use Newton's second law.

(b) Motion of a pendulum. Need to use Newton's second law.

(c) Modeling the growth of cows.

1st-order linear ODE

$$\begin{cases} \frac{dy}{dt} = p(t)y + q(t), \\ y(t_0) = y_0, \end{cases} \tag{1}$$

General solution:

$$y(t) = \frac{1}{\mu(t)}\left[\int \mu(t)q(t)dt + c\right], \quad \text{where} \quad \mu(t) = \exp\left(-\int p(t)dt\right). \tag{2}$$

In order to use the initial condition more easily, the general solution can be rewritten as

$$y(t) = \frac{1}{\mu(t)}\left[\int_{t_0}^{t} \mu(t)q(t)dt + c\right], \quad \text{where} \quad \mu(t) = \exp\left(-\int_{t_0}^{t} p(t)dt\right). \tag{3}$$

Using the initial condition, one can get $c = y_0$. The solution to IVP is

$$y(t) = \frac{1}{\mu(t)}\left[\int_{t_0}^{t} \mu(t)q(t)dt + y_0\right], \quad \text{where} \quad \mu(t) = \exp\left(-\int_{t_0}^{t} p(t)dt\right). \tag{4}$$

1st-order non-linear ODE-separable equation

For 1st-order non-linear ODE, a general method is still missing. We can only solve some special type of 1st-order non-linear ODEs.

Definition R.3

(Separable equation). A first order ODE $y' = f(t, y)$ is **separable** if it can be written in the form

$$M(t) + N(y)\frac{dy}{dt} = 0 \tag{5}$$

for some functions M and N.

Suppose there exist functions m and n such that

$$m' = M, \quad n' = N.$$

Then (5) can be written as

$$\frac{d}{dt}m(t) + \frac{d}{dt}n(y(t)) = 0.$$

Integrating yields the general (implicit) solution

$$\boxed{m(t) + n(y(t)) = c}, \quad c \in \mathbb{R} \tag{6}$$

1st-order non-linear ODE-Exact equation

Definition R.4

(Exact equation). A first order ODE $M(t,y) + N(t,y)\frac{dy}{dt} = 0$ is an **exact equation** if there exists a function $\Psi(t,y)$ such that

$$\boxed{\frac{\partial \Psi}{\partial t}(t,y) = M(t,y), \quad \frac{\partial \Psi}{\partial y}(t,y) = N(t,y)}. \tag{7}$$

The general solution $y(t)$ to the ODE is given implicitly as $\Psi(t, y(t)) = c,\ c \in \mathbb{R}$.

Remark:

$M(t,y) + N(t,y)\frac{dy}{dt} = 0$ (M_y and N_t are continuous) is exact $\Leftrightarrow$ $M_y = N_t$.

1st-order non-linear ODE-(non-Exact equation)

If $M(t,y) + N(t,y)\frac{dy}{dt} = 0$ is non-exact equation. Try to look for factor $\mu(t,y)$ s.t. $\mu M(t,y) + \mu N(t,y)\frac{dy}{dt} = 0$ is exact $\Leftrightarrow (\mu M)_y = (\mu N)_t$.

(1) $\mu(t,y) = \mu(t)$. Compute $K(t,y) = \frac{M_y - N_t}{N}(t,y)$;
If $K(t,y) = K(t)$, take $\mu(t) = e^{\int K(t)dt}$.

(2) $\mu(t,y) = \mu(y)$. Compute $H(t,y) = \frac{N_t - M_y}{M}(t,y)$;
If $H(t,y) = H(y)$, take $\mu(y) = e^{\int H(y)dy}$.

(3) $\mu(t,y) = \mu(ty) = \mu(z), z = ty$. Compute $L(t,y) = \frac{N_t - M_y}{tM - yN}$;
If $L(t,y) = L(ty) = L(z)$ $(z = ty)$, take $\mu(z) = e^{\int L(z)dz}$.

For 1st order nonlinear non-exact ODE, there is no general method.

Existence and Uniqueness

Next, we will address the theoretical fundamental question, is the 1st order nonlinear ODE $y' = f(t, y), y(t_0) = y_0$ has a solution? If it is, is the solution the only solution?

Existence and Uniqueness for first order linear ODE

Theorem R.5

(Existence and Uniqueness for first order linear ODEs).
Suppose functions p and q are **continuous** *on $(\alpha, \beta) \subset \mathbb{R}$ (α, β are some real numbers). Then, for any $t_0 \in (\alpha, \beta), y_0 \in \mathbb{R}$, there* **exists** *a* **unique** *function $y(t)$ satisfying*

$$
\begin{aligned}
\frac{dy}{dt} &= p(t)y + q(t), \quad \forall t \in (\alpha, \beta), \\
y(t_0) &= y_0.
\end{aligned}
$$

And the solution is defined throughout the interval (α, β).

The solution **globally** exists in the interval (α, β) in which p and q are continuous.

Existence and Uniqueness for first order general ODE

Theorem R.6

Consider the initial value problem (IVP)

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0.$$

Let R be a closed rectangle

$$R = \{(t, y) \| t - t_0| \leq a, \quad |y - y_0| \leq b\} (a > 0, b > 0).$$

Assume that both $f(t, y)$ and $\frac{\partial f}{\partial y}$ are continuous on R.
Then the IVP has a unique solution $y = y(t)$ defined on the interval $(t_0 - h, t_0 + h)$, where $h = \min\left(\frac{b}{M}, a\right)$ and $M = \max_{(t,y)\in R} |f(t, y)|$.

The solution only **locally** exists in the interval $[t_0 - a, t_0 + a]$.

Existence and uniqueness of the solution

Proof.

IVP is equivalent to $y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))ds. - - - - - - - - (**)$
Indeed, it can be easily shown that (*) and (**) are equivalent.
One method of showing that the integral equation (**) has a unique solution is known as the **Picard's iterative method**. First, we define the following successive iterations $\{\phi_n(t)\}_{n=0}^{\infty}$.

$$\phi_0(t) = y_0.$$

$$\phi_1(t) = y_0 + \int_{t_0}^{t} f(s, \phi_0(s))ds.$$

$$\vdots$$

$$\phi_{n+1}(t) = y_0 + \int_{t_0}^{t} f(s, \phi_n(s))ds$$

$\square$

Existence and uniqueness of the solution

Proof.

We have used the following four steps to prove the theorem

- Show all $\{\phi_n(t)\}_{n=0}^{\infty}$ satisfy $|\phi_n(t) - y_0| \leq b$, $\forall t \in (t_0 - h, t_0 + h)$ (We need $(t, \phi_n(t)) \in R$ for $t \in (t_0 - h, t_0 + h)$ in order to show $\{\phi_n(t)\}_{n=0}^{\infty}$ is uniformly convergent)
- Show the sequence $\{\phi_n(t)\}_{n=0}^{\infty}$ is uniformly convergent.
- Show the limit function of $\{\phi_n(t)\}_{n=0}^{\infty}$ is the solution of the integration equation (**).
- Show the uniqueness of the solution.

$\square$