MAT3253 Homework 5

Due date: 5 Mar.

Question 1. (Brown&Churchill 26.1) Show that $u(x, y)$ is harmonic in some domain and find a harmonic conjugate $v(x, y)$ when
(a) $u(x, y) = 2x(1 - y)$; (c) $u(x, y) = \sinh x \sin y$.

Question 2. (Brown&Churchill 26.5) Let the function

$$f(z) = u(r, \theta) + iv(r, \theta)$$

be analytic in a domain D that does not include the origin. Using the Cauchy–Riemann equations in polar coordinates (see Question 2 in Homework 4) and assuming continuity of partial derivatives, show that throughout D the function $u(r, \theta)$ satisfies the partial differential equation

$$r^2 u_{rr}(r, \theta) + r u_r(r, \theta) + u_{\theta\theta}(r, \theta) = 0,$$

which is the *polar form of Laplace's equation.* Show that the same is true for the function $v(r, \theta)$.

Question 3. (Brown&Churchill 29.1) Show that
(a) $\exp(2 \pm 3\pi i) = -e^2$; (b) $\exp((2 + \pi i)/4) = \sqrt{\frac{e}{2}}(1 + i)$;
(c) $\exp(z + \pi i) = -\exp(z)$.

Question 4. (Brown&Churchill 29.3) Show that the function $f(z) = \exp(\bar{z})$ is not analytic anywhere.

Question 5. (Bak&Newman Chapter 3 Ex.14) Find all solutions of
(b) $e^z = i$; (d) $e^z = 1 + i$.

Question 6. (Brown&Churchill 33.1) Show that
(a) $(1 + i)^i = \exp\left(-\frac{\pi}{4} + 2n\pi\right) \exp\left(i\frac{\ln 2}{2}\right)$, for $n \in \mathbb{Z}$;
(b) $(-1)^{1/\pi} = e^{(2n+1)i}$, for $n \in \mathbb{Z}$.

Question 7. (Bak&Newman Chapter 3 Ex.19) Find all solutions of the equation

$$e^{e^z} = 1.$$