STA 3007 Exercise 1. 118010350

Q1. (a) F. Wilcoxon signed rank test is based on the distributions of $X_1, \dots, X_n$ are symmetric about a common median.

(b) F. Under $H_0: \theta = 0$, then the Wilcoxon signed rank statistic has a symmetric distribution.

(c) T. Based on sign test, a $100(1-\alpha)\%$ confidence interval of θ is $(\theta_L, \theta_U) = (X_{(n+1-b_{\alpha/2})}, X_{(b_{\alpha/2})})$

Q2. (a). $B = \sum_{i=1}^{8} I\{x_i > 0\} = 2$. Under $H_0: \theta = 0$, $B \sim Bin(8, 0.5)$.

$Pr(B \le 2) = 0.1445 > 0.05$, accept $H_0: \theta = 0$

i.e. there is not sufficient evidence for $\theta < 0$ at 5% significance level.

(b)

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
X_i	-11	6	-20	-9	-18	-22	16	-28
ψ_i	0	1	0	0	0	0	1	0
R_i	3	1	6	2	5	7	4	8

$T^+ = \sum_{i=1}^{8} R_i \psi_i = 1 + 4 = 5$.

$\frac{1}{2^8} = \frac{1}{256}$

Outcomes	T^+	Pr
$B=0$	0	$\frac{1}{256}$
(1)	1	$\frac{1}{256}$
(2)	2	$\frac{1}{256}$
(1,2) (3)	3	$\frac{2}{256}$
(1,3) (4)	4	$\frac{2}{256}$
(1,4) (2,3) (5)	5	$\frac{3}{256}$

$Pr(T^+ \le 5) = \frac{10}{256} = 0.0391 < 0.05$, reject $H_0: \theta = 0$.

i.e. there is sufficient evidence for $\theta < 0$ at 5% significance level.

Q3. (a) $B = \sum_{i=1}^{14} I\{Z_i > 0\} = 10$, Under $H_0: \theta = 0$, $B \sim Bin(14, 0.5)$.

The p-value is $Pr(B \ge 10) = 0.0898$

Use the continuity correction, $Pr(B \ge 10) = Pr(B > 9.5)$

$B^* = \frac{B - 0.5n}{0.5\sqrt{n}} = \frac{9.5 - 0.5 \times 14}{0.5 \cdot \sqrt{14}} = 1.3363$, $B^* \sim N(0,1)$

The approximation p-value is $Pr(B^* \ge 1.3363) = 0.0907$

(b) $\hat{\theta} = median\{Z_i, 1 \le i \le 14\} = \frac{1}{2}(Z_{(7)} + Z_{(8)}) = 80$

For $B \sim Bin(14, 0.5)$, $Pr(B \ge 11) = 0.0287$ $Pr(B \ge 12) = 0.0065$.

To achieve 95% confidence level, take $\alpha/2 = 0.0065$, $\alpha = 0.013$

A $100(1-\alpha)\% = 98.7\%$ confidence interval of θ is

$(\theta_L, \theta_U) = (Z_{(3)}, Z_{(12)}) = (-45, 325)$

(c) $T^+ = \sum_{i=1}^{14} R_i \psi_i = 10.5 \times 2 + 14 + 9 + 12 + 2 + 4 + 5 + 13 + 7 = 87$

$E_0(T^+) = \frac{n(n+1)}{4} = 52.5$, $Var_0(T^+) = \frac{n(n+1)(2n+1)}{24} - \frac{1}{48}\sum_{j=1}^{g} t_j(t_j-1)(t_j+1) = 253.625$

$T^* = \frac{T^+ - E_0(T^+)}{\sqrt{Var_0(T^+)}} = \frac{87 - 52.5}{\sqrt{253.625}} = 2.1663$, $T^* \sim N(0,1)$

The approximate p-value is $Pr(T^* \geq 2.1663) = 0.0151$

(d) $M = \frac{n(n+1)}{2} = 105$, $\hat{\theta} = median\{\frac{X_i + X_j}{2}, i \leq j\} = W_{(53)} = 122.5$.

$C_{0.05} \approx E_0(T^+) - Z_{0.025} \cdot \sqrt{Var_0(T^+)} = 52.5 - 1.96 \times \sqrt{253.625} = 21.2858$.

Round $C_{0.05} = 21$, then $(\theta_L, \theta_U) = (W_{(21)}, W_{(85)}) = (5, 237.5)$.

A approximate 95% confidence interval is $(\theta_L, \theta_U) = (5, 237.5)$

(e) Compare: The p-value based on sign test is larger than that based on wilcoxon signed rank test.

Comment: wilcoxon signed rank test provide a strict way to test the median of statistics, and sign test is a relatively rough way when compared to wilcoxon.

(f) Compare: The confidence interval of wilcoxon signed rank test is narrower than that of sign test

Comment: wilcoxon signed rank test give us a more accurate confidence interval than sign test in the same confidence level