The Chinese University of Hong Kong, Shenzhen
iDDA · Institute for Data and Decision Analytics

Andre Milzarek · Summer Semester 2020

MAT 3007 – Optimization

Exercise Sheet 2

Exercise E2.1 (Multiple Choice – LPs and the Simplex Method):
We consider an LP in its standard form:

$$\min_x \; c^\top x \qquad \text{s.t.} \qquad Ax = b, \quad x \geq 0, \tag{1}$$

where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$ are given. Decide whether the following statements are true or false and explain your answer (i.e., give a short proof or a counterexample).

a) We apply the two-phase simplex method to solve problem (1). Let (x^*, y^*) be a solution of the auxiliary problem generated in phase I. Then, (x^*, y^*) can have more than m non-zero entries.

☐ True. ☐ False.

b) Phase I of the two-phase simplex method can indicate unboundedness of problem (1).

☐ True. ☐ False.

c) Let x^* be an optimal solution of (1) and suppose that x^* has strictly less than m positive components. Then, there exists another optimal solution that is a basic feasible solution and that has m positive entries.

☐ True. ☐ False.

d) Suppose that x^* is the only basic feasible solution of (1). Then, x^* is an optimal solution.

☐ True. ☐ False.

e) Basic solutions of (1), that are not extreme points of the set $\{x : Ax = b, x \geq 0\}$, need to be infeasible.

☐ True. ☐ False.

f) An iteration of the simplex method may move the current BFS by a positive distance while leaving the objective function unchanged.

☐ True. ☐ False.

Exercise E2.2 (Degeneracy Example):
Use the simplex method to solve the following LP:

$$\begin{array}{rll} \text{minimize} & -2x_1 - 3x_2 + x_3 + 12x_4 & \\ \text{subject to} & -2x_1 - 9x_2 + x_3 + 9x_4 + x_5 & = 0 \\ & \frac{1}{3}x_1 + x_2 - \frac{1}{3}x_3 - 2x_4 + x_6 & = 0 \\ & x_1, x_2, x_3, x_4, x_5, x_6 & \geq 0. \end{array}$$

Assignment A2.1 (Application of the Simplex Method): (approx. *20* points)
Consider the following linear program:

$$\begin{array}{lll} \text{maximize} & x_1 + 2x_2 + 3x_3 + 8x_4 & \\ \text{subject to} & x_1 - x_2 + x_3 & \le 2 \\ & x_3 - x_4 & \le 1 \\ & 2x_2 + 3x_3 + 4x_4 & \le 8 \\ & x_1, x_2, x_3, x_4 & \ge 0. \end{array}$$

Apply the simplex method to solve this problem. For each step, clearly mark what is the current basis, the current basic solution, and the corresponding objective function value.

Assignment A2.2 (Two-Phase Simplex Method: I): (approx. *20* points)
Apply the two-phase simplex method to solve the following linear program:

$$\begin{array}{lll} \text{minimize} & x_1 - x_2 + x_3 & \\ \text{subject to} & 2x_1 - x_2 + x_3 & \le -1 \\ & x_1 - x_2 - x_3 & \le 4 \\ & x_2 - x_4 & = 0 \\ & x_1, x_2, x_3, x_4 & \ge 0. \end{array}$$

Assignment A2.3 (Two-Phase Simplex Method: II): (approx. *20* points)
Use the two-phase simplex method to completely solve the linear optimization problem:

$$\begin{array}{lll} \text{minimize} & 2x_1 + 3x_2 + x_4 - 2x_5 & \\ \text{subject to} & x_1 + 3x_2 + 4x_4 + x_5 & = 2 \\ & x_1 + 3x_2 - 3x_4 + x_5 & = 2 \\ & -x_1 - 4x_2 + 3x_3 & = 1 \\ & x_1, x_2, x_3, x_4, x_5 & \ge 0. \end{array}$$

Assignment A2.4 (Simplex Tableau): (approx. *20* points)
While solving a linear program in standard form, we arrive at the simplex tableau given in Table 1 with basic variables x_1, x_2, x_4. The entries α, β, δ and η in the tableau are unknown parameters. For each of the following statements, find suitable conditions for the parameter values that will make the statement true (sufficient conditions are enough).

B	0	0	δ	0	-2	η	0
4	0	0	α	1	1	-2	β
1	1	0	-2	0	1	-2	10
2	0	1	-1	0	-1	1	1

Table 1: Simplex tableau with unknown parameter

a) The current solution is feasible but not optimal.

b) The LP is unbounded (i.e., the optimal value is $-\infty$).

c) The basic solution is feasible and after one simplex iteration the basis changes to $B = \{4, 5, 2\}$.

d) The basic solution is feasible and we reach an optimal solution after one simplex iteration.

e) The current solution has the optimal objective value and there are multiple sets of basic indices that achieve the same objective value.

Assignment A2.5 (Conditions for a Unique Optimum): (approx. *20* points)
Let x^* be a basic feasible solution associated with some basic indices B. Prove the following:

a) If the reduced cost of every non-basic variable is positive, then x^* is the unique optimal solution.

Hint: Let $y \in \{x \in \mathbb{R}^n : Ax = b,\, x \geq 0\}$ be given with $y \neq x^*$. First, show that there exists $\ell \in N = B^{\mathsf{C}}$ such that $y_\ell > 0$. You can then mimic the proof of the stopping criterion based on the reduced costs (lecture 05; slide 15).

b) If x^* is the unique optimal solution and if x^* is nondegenerate, then the reduced cost of every non-basic variable is positive.

Hint: The construction of the simplex method via basic directions can be helpful (lecture 05; slide 8–12).

Information: Exercises and Assignments

- Problems that are marked as "Assignment A?.?" (type A) need to be submitted and will be graded and taken into account for the final grade.
- Problems that are labeled as "Exercise E?.?" (type E) are additional exercises that are intended to specifically practice different aspects of new topics or mathematical concepts and notations. Those problems will **not** be considered for grading and will generally **not** be corrected (even if you hand in your solutions)!
- Hand in your solutions **in time** before the specified deadline via Blackboard. Please hand in your (written, scanned, or copied) solutions as a pdf-document.

Sheet 2 is due on **Jun, 26th**. Submit your solutions before **Jun, 26th**, **11:00 am**.