MAT2006: Elementary Real Analysis
Assignment #5

Reference Solutions

1. Consider the function g defined by the power series

$$g(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots .$$

(a) Is g defined on $(-1, 1)$? Is it continuous on this set? Is g defined on $(-1, 1]$? Is it continuous on this set? What happens on $[-1, 1]$? Can the power series for $g(x)$ possibly converge for any other points $|x| > 1$? Explain.

(b) For what values of x is $g'(x)$ defined? Find a formula for g'.

Solution. (a) Note that $g(x)$ is a power series with coefficient $a_n = \frac{(-1)^{n+1}}{n}$, $(n > 1)$. The radius of convergence R is given by

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|} = 1,$$

thus the power series is convergent on $(-1, 1)$ and g is defined on $(-1, 1)$. According to the theory of power series, $g(x)$ is continuous on $(-1, 1)$.

When $x = 1$, the power series becomes an alternating series with $|a_n|$ decreasing to zero, by the Alternating Series Test, the series converges at $x = 1$. Thus $g(x)$ is defined on $(-1, 1]$, and it is continuous on $(-1, 1]$ according to the Abel theorem.

Since at $x = -1$, the series is harmonic series which diverges, thus g is not well defined at $x = 1$.

The power series diverges for each $|x| > 1$, since the term of the series doesnot converge to zero.

(b) According to the theory of power series, $g(x)$ can be differentiated term by term on $(-1, 1)$, and we have

$$g'(x) = 1 - x + x^2 - x^3 + \cdots = \frac{1}{1-x}, \qquad |x| < 1. \qquad \square$$

2. Find suitable coefficients $\{a_n\}$ so that the resulting power series $\sum a_n x^n$ has the given properties, or explain why such a request is impossible.

(a) Converges for every value of $x \in \mathbb{R}$.

(b) Diverges for every value of $x \in \mathbb{R}$.

(c) Diverges for every value of $x \in \mathbb{R} \setminus \{0\}$.

(d) Converges absolutely for all $x \in [-1, 1]$ and diverges off of this set.

(e) Converges conditionally at $x = -1$ and converges absolutely at $x = 1$.

(f) Converges conditionally at both $x = -1$ and $x = 1$.

Solution. (a) $a_n = 1/n^n$. For this case, $\limsup_{n\to\infty} \sqrt[n]{1/n^n} = 0$ and thus the radius of convergence is ∞.

(b) Note possible. Every power series must converge at $x = 0$.

(c) $a_n = n^n$. For this case, $\limsup_{n\to\infty} \sqrt[n]{n^n} = \infty$ and thus the radius of convergence is 0.

(d) $a_n = \frac{1}{n^2}$. For this case, $\limsup_{n\to\infty} \sqrt[n]{1/n^2} = 1$ and thus the radius of convergence is 1, the power series diverges out of $[-1, 1]$ and converges on $(-1, 1)$. The convergence at $x = \pm 1$ is obvious.

(e) Not possible, since $\sum_{n=1}^{\infty} |a_n x^n| = \sum_{n=1}^{\infty} |a_n|$ at $x = 1$ and $x = -1$.

(f) Let

$$a_n = \begin{cases} (-1)^m/m & \text{if } n = 2m \\ 0 & \text{if } n = 2m+1. \end{cases}$$

Note that

$$\sum_{n=1}^{\infty} a_n x^n = \sum_{m=1}^{\infty} \frac{(-x^2)^m}{m}.$$

This series converges at $x = \pm 1$ but

$$\sum_{n=1}^{\infty} |a_n x^n| = \sum_{m=1}^{\infty} \frac{(x^2)^m}{m}.$$

diverges at $x = \pm 1$. □

3. (**Term-by-term Antidifferentiation**).
Assume $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges on $(-R, R)$.

(a) Show that

$$F(x) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

is defined on $(-R, R)$ and satisfies $F'(x) = f(x)$.

(b) Antiderivatives are not unique. If g is an arbitrary function satisfying $g'(x) = f(x)$ on $(-R, R)$, find a power series representation for g.

Proof. (a) Since the radius of convergence of the power series $f(x)$ is R, we have

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|} = R.$$

And for the power series $F(x)/x$, its radius of convergence R' is given by

$$\frac{1}{R'} = \limsup_{n\to\infty} \sqrt[n]{\frac{|a_n|}{n+1}} = \limsup_{n\to\infty} \sqrt[n]{|a_n|} \lim_{n\to\infty} \sqrt[n]{\frac{1}{n+1}} = \frac{1}{R}\cdot 1 = \frac{1}{R},$$

hence $R' = R$ and $F(x)$ is defined on $(-R, R)$. According to the theory of power series, in the circle of convergence, the power series can be differentiated term-by-term, thus

$$F'(x) = \sum_{n=0}^{\infty} \left(\frac{a_n}{n+1} x^{n+1} \right)' = \sum_{n=0}^{\infty} a_n x^n = f(x).$$

(b) If $g'(x) = f(x)$, we know that $g(x) = F(x) + c$, where c is an arbitrary constant, therefore, the power series of g has the form

$$g(x) = c + \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} = \sum_{n=0}^{\infty} \frac{a_{n-1}}{n} x^n$$

where $a_{-1} = c$ is an arbitrary constant. □

4. (a) Show that power series representations are unique. If we have

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$$

for all x in an nonempty interval $(-R, R)$, prove that $a_n = b_n$ for all $n = 0, 1, 2, \dots$.

(b) Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converge on $(-R, R)$, and assume $f'(x) = f(x)$ for all $x \in (-R, R)$ and $f(0) = 1$. Deduce the values of a_n.

Solution. (a) Since $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ both converges on $(-R, R)$, the Algebraic Limit Theorem yields

$$\sum_{n=0}^{\infty} (a_n - b_n) x^n = \sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} b_n x^n = 0 := h(x).$$

on $(-R, R)$. Then, the differentiable limit theorem yields that

$$a_n - b_n = \frac{h^{(n)}(0)}{n!} = 0, \qquad \forall n \in \mathbb{N}.$$

(b) The power series can be differentiated term-by-term, we have

$$f'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} (n+1) a_{n+1} x^n.$$

If $f'(x) = f(x)$, that is

$$\sum_{n=1}^{\infty}(n+1)a_{n+1}x^n = \sum_{n=0}^{\infty} a_n x^n,$$

it follows from part (a) that

$$(n+1)a_{n+1} = a_n, \qquad \forall n \in \mathbb{N}.$$

It also follows from $f(0) = 1$ that $a_0 = 1$. Thus $a_n = \frac{1}{n!}$ by induction. (This is the Taylor expansion of e^x). $\square$

5. A series $\sum_{n=0}^{\infty} a_n$ is said to be *Abel-summable* to L if the power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

converges for all $x \in [0, 1)$ and $L = \lim_{x \to 1-} f(x)$.

(a) Show that any series that converges to a limit L is also Abel-summable to L.

(b) Show that $\sum_{n=0}^{\infty}(-1)^n$ is Abel-summable and find the sum.

Proof. (a) Assume $\sum_{n=1}^{\infty} a_n = L$. Then the power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges absolutely for all $|x| < 1$, and thus converges for all $[0, 1]$. The Abel Theorem then yields that this power series converges uniformly on $[0, 1]$, and thus $f(x)$ is continuous on $[0, 1]$ according to the Continuous Limit Theorem. As a consequence, $\lim_{x \to 1-} f(x) = f(1) = L$, which is exactly to say that the series $\sum_{n=0}^{\infty} a_n$ is Abel-summable to L.

(b) The power series $\sum_{n=1}^{\infty}(-1)^n x^n = \frac{1}{1+x}$ for $|x| < 1$. Note that

$$\lim_{x \to 1-} \frac{1}{x+1} = \frac{1}{2}.$$

Thus $\sum_{n=0}^{\infty}(-1)^n$ is Abel-summable and its Abel sum is $1/2$. $\square$

6. (Cauchy's Remainder Theorem). Let f be differentiable $N + 1$ times on $(-R, R)$. For each $a \in (-R, R)$, let $S_N(x, a)$ be the partial sum of the Taylor series for f centered at a; in other words, define

$$S_N(x, a) = \sum_{n=0}^{N} c_n (x-a)^n \quad \text{where} \quad c_n = \frac{f^{(n)}(a)}{n!}.$$

Let $E_N(x, a) = f(x) - S_N(x, a)$. Now fix $x \neq 0$ in $(-R, R)$ and consider $E_N(x, a)$ as a function of a.

(a) Find $E_N(x, x)$.

(b) Explain why $E_N(x, a)$ is differentiable with respect to a, and show

$$E_N'(x, a) = -\frac{f^{(N+1)}(a)}{N!}(x-a)^N.$$

(c) Show

$$E_N(x) = E_N(x,0) = \frac{f^{(N+1)}(c)}{N!}(x-c)^N x$$

for some c between 0 and x. This is Cauchy's form of the remainder for Taylor series centered at the origin.

Solution. (a) Let $x = a$, we have

$$E_N(x,x) = f(x) - S_N(x,x) = f(x) - c_0 = f(x) - f(a) = f(x) - f(x) = 0.$$

(b) Note that, by understanding that the prime denotes the derivative with respect to a,

$$E'_N(x,a) = (f(x) - S_N(x,a))' = -S'_N(x,a)$$

and $S_N(x,a)$ is differentiable with respect to a since it is a polynomial in a. Now,

$$\begin{aligned} E'_N(x,a) = -S'_N(x,a) &= -\sum_{n=0}^{N} \frac{f^{(n+1)}(a)}{n!}(x-a)^n + \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!} n(x-a)^{n-1} \\ &= -\sum_{n=0}^{N} \frac{f^{(n+1)}(a)}{n!}(x-a)^n + \sum_{n=0}^{N-1} \frac{f^{(n+1)}(a)}{n!}(x-a)^n \\ &= -\frac{f^{(N+1)}(a)}{N!}(x-a)^N. \end{aligned}$$

(c) Now, $E_N(x,a)$ as a function of a is continuous on $[0,x]$ (or $[x,0]$), and differentiable on $(0,x)$ (or $(x,0)$). The Mean Value Theorem yields that

$$E_N(x,x) - E_N(x,0) = E'(x,c)x$$

where c is in between 0 and x. That is

$$E_N(x) = E_N(x,0) = E_N(x,x) - E'(x,c)x = 0 + \frac{f^{(N+1)}(c)}{N!}(x-c)^N x. \qquad \square$$

7. Consider $f(x) = 1/\sqrt{1-x}$.

(a) Generate the Taylor series for f centered at zero, and use Lagrange's Remainder Theorem to show the series converges to f on $[0, 1/2]$. (The case $x < 1/2$ is more straightforward while $x = 1/2$ requires some extra care.) What happens when we attempt this with $x > 1/2$?

(b) Use Cauchy's Remainder Theorem to show the series representation for f holds on $[0, 1)$.

Proof. Solution (a) By induction, we have

$$f^{(n)}(x) = (\frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} \cdots \frac{2n-1}{2})(1-x)^{-\frac{1}{2}-n}, \qquad \forall n \in \mathbb{N}.$$

Thus

$$f^{(n)}(0) = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n}, \qquad \forall n \geq 1.$$

Thus the Taylor series of f centered at 0 is

$$f(x) = 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n n!} x^n := \sum_{n=0}^{N} a_n x^n + E_N(x).$$

By Lagrange's Remainder Theorem, we have

$$E_N(x) = \frac{f^{(N+1)}(\xi)}{(N+1)!} x^{N+1} = \frac{1 \cdot 3 \cdot 5 \cdots (2N+1)}{2^{N+1}(N+1)!} (1-\xi)^{-\frac{2N-1}{2}} x^{N+1}.$$

By the Ratio Test,

$$\lim_{N \to \infty} |\frac{E_{N+1}(x)}{E_N(x)}| = \lim_{N \to \infty} \frac{2N+3}{2N+4} \frac{|x|}{|1-\xi|} = \frac{|x|}{|1-\xi|}.$$

When $|x| \leq 1/2$, we have $|1-\xi| > |1-x| \geq \frac{1}{2}$ and thus $\frac{|x|}{|1-\xi|} < 1$. This implies that the series $\sum_{N=1}^{\infty} E_N(x)$ converges absolutely, and thus further implies that $E_N(x) \to 0$. Therefore, the Taylor series converges on $[-1/2, 1/2]$.

When $x > 1/2$, the Lagrange's Remainder Theorem fails to draw a conclusion on the convergence.

(b) By Cauchy's Remainder Theorem, we have

$$E_N(x) = \frac{f^{(N+1)}(c)}{N!} (x-c)^N x = \frac{1 \cdot 3 \cdot 5 \cdots (2N+1)}{2^{N+1} N!} (1-c)^{-\frac{2N-1}{2}} (x-c)^N x$$

where c is in between 0 and x. Apply the Ration Test again, we have

$$\lim_{N \to \infty} |\frac{E_{N+1}(x)}{E_N(x)}| = \lim_{N \to \infty} \frac{2N+3}{2N+4} \frac{|x-c|}{|1-c|} = \frac{|x-c|}{|1-c|}.$$

Since c is in between 0 and x, we have

$$\frac{|x-c|}{|1-c|} < 1 \qquad \forall x \in [-1, 1).$$

This implies that the series $\sum_{N=1}^{\infty} E_N(x)$ converges absolutely, and thus further implies that $E_N(x) \to 0$ when $x \in [-1, 1)$. Therefore, the Taylor series converges on $[-1, 1)$. □

8. Let $f : [a, b] \to \mathbb{R}$ be increasing on the set $[a, b]$. Show that f is integrable on $[a, b]$.

Proof. Let $\epsilon > 0$. There exists $n \in \mathbb{N}$ such that $\frac{f(b)-f(a)}{n} < \frac{\epsilon}{b-a}$. Denote

$$y_n = f(a) + \frac{k}{n}(b-a), \qquad k = 0, 1, \ldots, n.$$

Let $x_0 = a$ and $x_n = b$. Since $f(x)$ is increasing on $[a, b]$, it has only jump discontinuity, then there exists a unique x_k such that

$$\lim_{x \to x_k^-} f(x) \leq y_k \leq \lim_{x \to x_k^+} f(x), \qquad 1 \leq k \leq n-1.$$

By the fact $f(x)$ is increasing, $\{x_k\}$ is also increasing in k, and they form a partition of $[a, b]$ denoted by P_ϵ. Now, on each $[x_{k-1}, x_k]$, by $f(x)$ is increasing, we also have

$$m_k = \inf_{x_{k-1} \leq x \leq x_k} f(x) = y(x_{k-1}) \geq y_{k-1}$$

and

$$M_k = \sup_{x_{k-1} \leq x \leq x_k} f(x) = f(x_k) \leq y_k.$$

Now,

$$M_k - m_k \leq y_k - y_{k-1} = \frac{f(b) - f(a)}{n} < \frac{\epsilon}{b-a}$$

Therefore,

$$U(f, P_\epsilon) - L(f, P_\epsilon) = \sum_{k=1}^{n} (M_k - m_k)(x_k - x_{k-1}) < \frac{\epsilon}{b-a} \sum_{k=1}^{n} x_k - x_{k-1} = \epsilon.$$

Thus, $f(x)$ is integrable on $[a, b]$. □

Method II.. Recall that the monotone function $f(x)$ has only jump discontinuities and at most countable many of them. That is D_f is at most a countable set, and hence a measure zero set. By Lebesgue's criterion, f is integrable on $[a, b]$. □

9. For each $n \in \mathbb{N}$ let

$$h_n(x) = \begin{cases} 1/2^n & \text{if } 0 \leq x \leq 1/2^n \\ 0 & \text{if } 1/2^n < x \leq 1 \end{cases}$$

and set $H(x) = \sum_{n=1}^{\infty} h_n(x)$. Show that $H(x)$ is integrable and compute $\int_0^1 H(x)dx$.

Proof. Note that each $h_n(x)$ is integrable, since it has only one discontinuity. Note that $|h_n(x)| \leq \frac{1}{2^n}$ for all $n \in \mathbb{N}$ and for all $x \in [0, 1]$, and the series $\sum \frac{1}{2^n}$ converges. By the Weierstrass M-test,

$$H(x) = \sum_{n=1}^{\infty} h_n(x)$$

converges uniformly on $[0, 1]$, and hence it can be integrated term by term,

$$\int_0^1 H(x)dx = \sum_{n=1}^{\infty} \int_0^1 h_n(x)dx = \sum_{n=1}^{\infty} \int_0^{1/2^n} \frac{1}{2^n} dx = \sum_{n=1}^{\infty} \frac{1}{4^n} = \frac{\frac{1}{4}}{1 - \frac{1}{4}} = \frac{1}{3}.$$ □

10. Let $\{f_n\}_{n=1}^{\infty} \cup \{f\}$ is uniformly bounded on $[0,1]$. Assume that $f_n \to f$ pointwise on $[0,1]$ and uniformly on any set of the form $[0,\alpha]$, where $0 < \alpha < 1$.

If all the functions are integrable, show that $\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)dx$.

Proof. Let $\epsilon > 0$. By the hypothesis, there exists an $M > 0$ such that

$$|f_n(x)| \leq M, \quad |f(x)| \leq M, \qquad \forall n \in \mathbb{N} \quad \forall x \in [0,1].$$

Then there exists $0 < \alpha < 1$ such that $1 - \alpha < \frac{\epsilon}{3M}$. Since $f_n(x) \to f(x)$ uniformly on $[0,\alpha]$, we have

$$\lim_{n\to\infty} \int_0^{\alpha} f_n(x)dx = \int_0^{\alpha} f(x)dx.$$

Then, there exists a $N \in \mathbb{N}$ such that

$$\left| \int_0^{\alpha} f_n(x)dx - \int_0^{\alpha} f(x)dx \right| < \frac{\epsilon}{3}, \qquad \forall n \geq N.$$

Now,

$$\begin{aligned}
\left| \int_0^1 f_n(x)dx - \int_0^1 f(x)dx \right| &= \left| \int_0^{\alpha} f_n(x)dx + \int_{\alpha}^1 f_n(x)dx - \int_0^{\alpha} f(x)dx - \int_{\alpha}^1 f(x)dx \right| \\
&\leq \left| \int_0^{\alpha} f_n(x)dx - \int_0^{\alpha} f(x)dx \right| + \left| \int_{\alpha}^1 f_n(x)dx \right| + \left| \int_{\alpha}^1 f(x)dx \right| \\
&< \frac{\epsilon}{3} + \int_{\alpha}^1 |f_n(x)|dx + \int_{\alpha}^1 |f(x)|dx \\
&\leq \frac{\epsilon}{3} + 2M(1-\alpha) \\
&< \frac{\epsilon}{3} + \frac{2}{3}\epsilon = \epsilon.
\end{aligned}$$

Therefore, we have

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)dx.$$ □

11. Assume g is integrable on $[0,1]$ and continuous at 0. Show that

$$\lim_{n\to\infty} \int_0^1 g(x^n)dx = g(0).$$

Proof. Let α be such that $0 < \alpha < 1$ and let $\epsilon > 0$. Since g is continuous at 0, there exists $\delta < 0$ such that

$$|g(y) - g(0)| < \epsilon, \qquad |y| < \delta.$$

Then, there exists $N \in \mathbb{N}$ such that $\alpha^N < \delta$. Now,

$$x^n \leq \alpha^n \leq \alpha^N < \delta \qquad \forall x \in [0,\alpha] \quad \forall n \geq N.$$

and

$$|g(x^n) - g(0)| < \epsilon \forall n \geq N \quad \forall x \in [0, \alpha].$$

Define

$$f(x) = \begin{cases} g(0) & \text{if } 0 \leq x < 1, \\ g(1) & \text{if } x = 1. \end{cases}$$

It is clear that $g(x_n)$ converges to $f(x)$ pointwise on $[0, 1]$ and uniformly on $[0, \alpha]$ for arbitrary $\alpha \in (0, 1)$. By the previous problem, we have

$$\lim_{n\to\infty} \int_0^1 g(x^n)dx = \int_0^1 f(x)dx = \int_0^1 g(0)dx = g(0). \qquad \square$$

12. (a) Let $f(x) = |x|$ and define $F(x) = \int_{-1}^{x} f(t)dt$. Find a piecewise algebraic formula for $F(x)$ for all x. Where is F continuous? Where is F differentiable? Where does $F'(x) = f(x)$?

(b) Repeat part (a) for the function

$$f(x) = \begin{cases} 1 & \text{if } x < 0 \\ 2 & \text{if } x \geq 0. \end{cases}$$

Solution. (a) When $x \leq 0$, we have

$$F(x) = \int_{-1}^{x} f(t)dt = \int_{-1}^{x} (-t)dt = -\left.\frac{t^2}{2}\right|_{t=-1}^{t=x} = \frac{1 - x^2}{2}.$$

When $x > 0$, we have

$$F(x) = \int_{-1}^{x} f(t)dt == \int_{-1}^{0} f(t)dt + \int_{0}^{x} f(t)dt = \frac{1}{2} + \int_0^x tdt = \frac{1}{2} + \frac{x^2}{2} = \frac{1 + x^2}{2}.$$

Note that $F(x)$ is continuous at $x = 0$, and hence $F(x)$ is continuous on $\mathbb{R}$.

It is clear that $F(x)$ is differentiable with $F'(x) = f(x)$ on $\mathbb{R} \setminus \{0\}$. When $x < 0$, note that

$$\frac{F(x) - F(0)}{x - 0} = \frac{\frac{1-x^2}{2} - \frac{1}{2}}{x} = -\frac{x}{2} \to 0$$

as $x \to 0^-$. Similarly, when $x > 0$

$$\frac{F(x) - F(0)}{x - 0} = \frac{\frac{1+x^2}{2} - \frac{1}{2}}{x} = \frac{x}{2} \to 0$$

as $x \to 0^+$. Thus $F(x)$ is differentiable at $x = 0$ and $F'(0) = f(0)$. To summarize, $F(x)$ is differentiable on $\mathbb{R}$ with $F'(x) = f(x)$.

(b) When $x \leq 0$, we have

$$F(x) = \int_{-1}^{x} f(t)dt = \int_{-1}^{x} dt = x + 1.$$

When $x > 0$, we have

$$F(x) = \int_{-1}^{x} f(t)dt == \int_{-1}^{0} f(t)dt + \int_{0}^{x} f(t)dt = 1 + \int_{0}^{x} 2dt = 1 + 2x.$$

Note that $F(x)$ is continuous at $x = 0$, and hence $F(x)$ is continuous on $\mathbb{R}$.

It is clear that $F(x)$ is differentiable with $F'(x) = f(x)$ on $\mathbb{R} \setminus \{0\}$. When $x < 0$, note that

$$\frac{F(x) - F(0)}{x - 0} = \frac{(1 + x) - 1}{x} = 1 \to 1$$

as $x \to 0^-$. Similarly, when $x > 0$

$$\frac{F(x) - F(0)}{x - 0} = \frac{(1 + 2x) - 1}{x} = 2 \to 2$$

as $x \to 0^+$. Thus $F(x)$ is not differentiable at $x = 0$. To summarize, $F(x)$ is differentiable on $\mathbb{R} \setminus \{0\}$ with $F'(x) = f(x)$ there.

Note, this result verifies the Fundamental Theorem of Calculus. □

13. Show that if $f : [a, b] \to \mathbb{R}$ is continuous and $\int_a^x f(t)dt = 0$ for all $x \in [a, b]$, then $f(x) = 0$ everywhere on $[a, b]$. Provide an example to show that this conclusion does not follow if f is not continuous.

Proof. Suppose there exists $x_0 \in (a, b)$ such that $f(x_0) \neq 0$. Without loss of generality, we may assume $f(x_0) > 0$. By the continuity of f, there exists a δ such that

$$|f(x) - f(x_0)| < \frac{f(x_0)}{2}, \qquad \forall x \in [a, b] \cap V_\delta(x_0).$$

We may choose δ small enough so that $V_\delta(x_0) \subset [a, b]$. Thus,

$$\int_{a}^{x_0+\delta} f(t)dt = \int_{a}^{x_0-\delta} f(t)dt + \int_{x_0-\delta}^{x_0+\delta} f(t)dt = 0 + \int_{x_0-\delta}^{x_0+\delta} f(t)dt \geq \frac{f(x_0)}{2} 2\delta = f(x_0)\delta > 0,$$

which is a contradiction. Thus, we must have $f(x) = 0$ for all $x \in (a, b)$. By the continuity of f on $[a, b]$, we have $f(x) = 0$ identically on $[a, b]$.

Example. Define

$$f(x) = \begin{cases} 0 & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

Then $\int_{-1}^{x} f(t)dt = 0$ for all $x \in [-1, 1]$, but $f(x)$ is not identically zero. □

14 (Integration by parts)**.** Assume $h(x)$ and $k(x)$ have continuous derivatives on $[a, b]$ and derive the familiar integration-by-parts formula

$$\int_a^b h(x)k'(x)dx = h(b)k(b) - h(a)k(a) - \int_a^b h'(x)k(x)dx.$$

Proof. By the product rule,

$$[h(x)k(x)]' = h'(x)k(x) + h(x)k'(x).$$

Therefore, by the Fundamental Theorem of Calculus,

$$h(b)k(b) - h(a)k(a) = \int_a^b [h(x)k(x)]'dx = \int_a^b h'(x)k(x)dx + \int_a^b h(x)k'(x)dx$$

which yields the desired formula. □

15. Given a function f on $[a, b]$, define the total variation of f to be

$$Vf = \sup\left\{\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|\right\}$$

where the supremum is taken over all partitions P of $[a, b]$.

(a) If f is continuously differentiable (f' exists as a continuous function), use the Fundamental Theorem of Calculus to show $Vf \leq \int_a^b |f'(x)|dx$.

(b) Use the Mean Value Theorem to establish the reverse inequality and conclude that $Vf = \int_a^b |f'(x)|dx$.

Proof. For a partition $P : a = x_0 < x_1 < \cdots < x_n = b$, by the Fundamental Theorem of Calculus, and the triangle inequality, we have

$$|f(x_k) - f(x_{k-1})| = \left|\int_{x_{k-1}}^{x_k} f'(x)dx\right| \leq \int_{x_{k-1}}^{x_k} |f'(x)|dx.$$

hence

$$\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \leq \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} |f'(x)|dx = \int_a^b |f'(x)|dx.$$

Since the partition is arbitrary, we have

$$Vf = \sup\left\{\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|\right\} \leq \int_a^b |f'(x)|dx$$

For the partition P, by the Mean Value Theorem, we have

$$|f(x_k) - f(x_{k-1})| = |f'(\xi_k)||x_k - x_{k-1}| \geq m_k(x_k - x_{k-1})$$

where $\xi_k \in (x_{k-1}, x_k)$ and $m_k = \inf\{|f'(x)|, x_{k-1} \leq x \leq x_k\}$. Thus,

$$\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \geq \sum_{k=1}^{n} m_k(x_k - x_{k-1}) = L(|f'|, P)$$

Taking sup to both sides among the set of all partitions, we have

$$Vf = \sup \sum_{k=1}^{n} |f(x_k) - f(x_{k-1}) \geq L(|f'|).$$

Since f' is continuous, thus is integrable on $[a,b]$, then the function $|f'|$ is also integrable on $[a,b]$, and thus

$$L(|f'|) = \int_a^b |f'(x)|dx,$$

which, together with the last inequality, yield

$$Vf \geq \int_a^b |f'(x)|dx.$$

Combining this with part (a), we have

$$Vf = \int_a^b |f'(x)|dx. \qquad \square$$

16. Assume f is integrable on $[a,b]$ and has a jump discontinuity at $c \in (a,b)$.

(a) Show that, in this case, $F(x) = \int_a^x f(t)dt$ is not differentiable at $x = c$.

(b) Construct a continuous monotone function that fails to be differentiable on $\mathbb{Q}$.

Proof. (a) Denote

$$\lim_{x \to c^-} f(x) = L, \qquad \lim_{x \to c^+} = R,$$

then $L \neq R$. Let $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - R| < \epsilon \qquad \forall c < x < x + \delta.$$

Now, when $c < x < c + \delta$, we have

$$\left| \frac{F(x) - F(x)}{x - c} - R \right| = \frac{1}{x - c} \left| \int_c^x [f(t) - R] \right| \leq \frac{1}{x - c} \int_c^x |f(t) - R|dx < \frac{\epsilon}{x - c} \int_c^x dt = \epsilon.$$

Therefore, we have

$$\lim_{x \to c^+} \frac{F(x) - F(x)}{x - c} = R,$$

and similarly,

$$\lim_{x \to c^-} \frac{F(x) - F(x)}{x - c} = L.$$

Since $L \neq R$, the above left- and right-hand side limits are not equal to each other, thus the limit doesnot exist and so $F(x)$ is not differentiable at c.

(b) Let $\mathbb{Q} = \{r_1, r_2, \dots\}$. For each $n \in \mathbb{N}$ define

$$f_n(x) = \begin{cases} 0 & \text{if } x < r_n \\ \frac{1}{2^n} & \text{if } x \geq r_n \end{cases}$$

Now, by the Weierstrass M-text, the series

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

converges uniformly on $\mathbb{R}$. Since each $f_n(x)$ is integrable, so is $F(x)$. And define

$$F(x) = \int_0^x f(t)dt.$$

Note that $f(x)$ has jump discontinuity at each $x \in \mathbb{Q}$, we must have F is not differentiable at each $x \in \mathbb{Q}$. $\square$

— End —