# MAT 3253 lecture 19

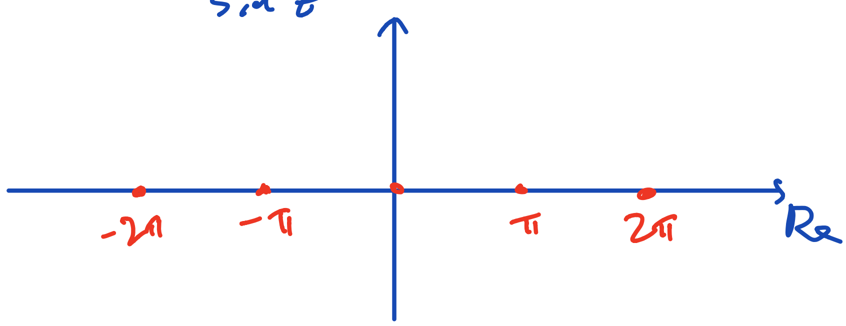
* If the zeros of an analytic function $f$ have a cluster point, then $f$ is equal to zero function in the domain

* If $z_0$ is a pole of $f$, then $\frac{1}{f}$ has a zero at $z_0$.

<u>Def</u> A function $f$ is called meromorphic if the singularity points are all poles.

i.e. there are finitely many or a sequence of point $z_i$, $i = 1, 2, 3 \ldots$, that are poles of $f$.

e.g. $f(z) = \dfrac{1}{\sin z}$



**Theorem** (Riemann's thm on removable singularity)

Suppose $f$ is analytic in some domain containing a punctured disc $\{z : 0 < |z - z_0| < \varepsilon\}$

$$= D(z_0; \varepsilon) \setminus \{z_0\}$$

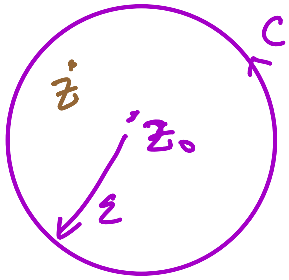
If $f$ is bounded in $D(z_0; \varepsilon) \setminus \{z_0\}$ then we can re-defined $f$ at $z_0$ so that $f$ is analytic in $D(z_0; \varepsilon)$.

( There exists an analytic $\tilde{f}$ in $D(z_0; \varepsilon)$,
$\tilde{f}(z) = f(z)$ for all $z \in D(z_0; \varepsilon) \setminus \{z_0\}$ )

**Example** $f(z) = \dfrac{z^2+1}{z+i} = \dfrac{(z+i)(z-i)}{(z+i)}$

$$= \begin{cases} z - i & \text{if } z \neq -i \\ \text{undefined} & \text{otherwise} \end{cases}$$

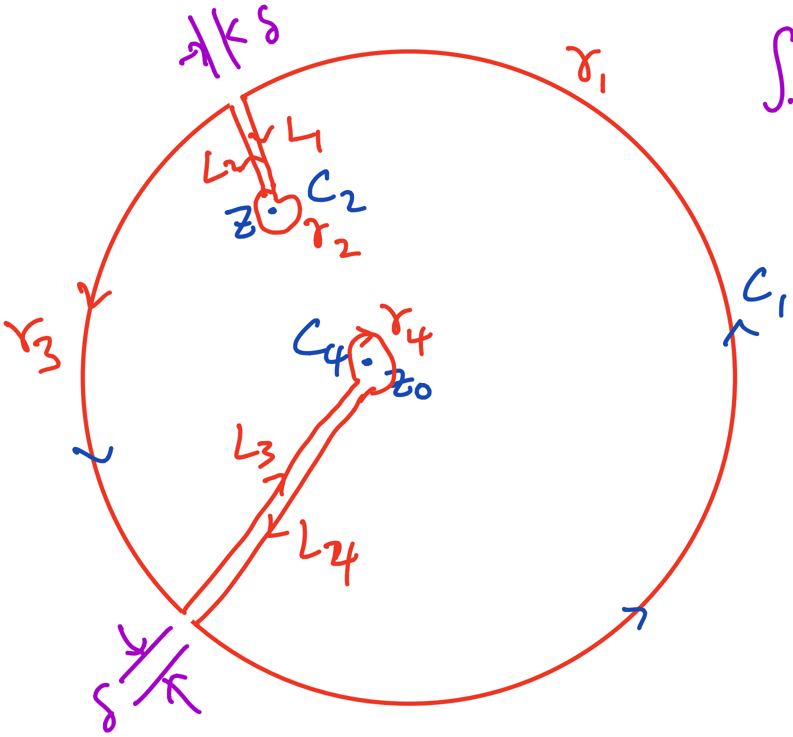
**Proof** Define $\tilde{f}(z) \triangleq \frac{1}{2\pi i} \int_C \frac{f(w)}{w-z} \, dw$




want to show

(1) $\tilde{f}(z) = f(z)$ $0 < |z - z_0| < \varepsilon$

(2) $\tilde{f}(z)$ is analytic.




$$\int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} + \int_{\gamma_4} + \int_{L_1} + \int_{L_2} + \int_{L_3} + \int_{L_4} = 0$$

Take limit as $\delta \to 0$

$\gamma_1 + \gamma_3 \to C_1$

$\gamma_2 \to C_2$

$\gamma_4 \to C_4$

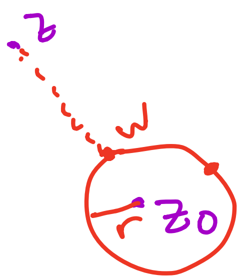
$$\int_{C_1} \frac{f(w)}{w-z}\,dw = \int_{C_2} \frac{f(w)}{w-z}\,dw + \int_{C_4} \frac{f(w)}{w-z}\,dw$$

Cauchy integral formula $\Rightarrow \int_{C_2} \frac{f(w)}{w-z}\,dw = 2\pi i f(z)$

$f(z)$ is bounded around $z_0$. $|f(z)| \leq M$

$|w-z| > |z-z_0| - r$ $\quad$ $r$ is the radius $C_4$

$$\frac{1}{|w-z|} < \frac{1}{|z-z_0|-r}$$




$$\left| \int_{C_4} \frac{f(w)}{w-z}\,dw \right| \leq \frac{M}{|z-z_0|-r}\, 2\pi r$$

$$= O(r)$$

$$\Rightarrow \left| \int_{C_4} \frac{f(w)}{w-z}\,dw \right| = 0 \Rightarrow \int_{C_4} \frac{f(w)}{w-z}\,dw = 0$$

$\therefore \quad \tilde{f}(z) = f(z) \quad \forall\, z \in D(z_0; \varepsilon) \setminus \{z_0\}.$

Need to show $\tilde{f}(z)$ is analytic at $z_0$.

$$\tilde{f}(z_0) \triangleq \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z_0}\,dw$$

$$\tilde{f}(z_0+h) \triangleq \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z_0-h}\,dw$$

$$\frac{f(z_0+h)-f(z_0)}{h} = \frac{1}{h\,2\pi i}\int_C f(w)\left[\frac{1}{w-z_0-h}-\frac{1}{w-z_0}\right]dw$$

$$= \frac{1}{h\,2\pi i}\int_C f(w)\,\frac{h}{(w-z_0)(w-z_0-h)}\,dw$$

$$\left(\tilde{f}(z)\right)' \overset{?}{=} \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}\,dw$$

$$\overset{?}{=} \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z)^2}\,dw$$

$$\left|\frac{f(z_0+h)-f(z_0)}{h} - \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z_0)^2}\,dw\right|$$

$$= \frac{1}{2\pi}\left|\int_C \frac{f(w)}{(w-z_0)(w-z_0-h)} - \frac{f(w)}{(w-z_0)^2}\,dw\right|$$

$$= \frac{1}{2\pi}\left|\int_C \frac{f(w)\,h}{(w-z_0)^2(w-z_0-h)}\,dw\right|$$

↳ ML inequality

$= O(r)$ $\qquad$ $r$ is the radius of $C$.

$\to 0$ as $r \to 0$ $\qquad \blacksquare$

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Suppose $z_0$ is a pole

$$|f(z)| \to \infty \quad \text{if} \quad z \to z_0$$

$$\left|\frac{1}{f(w)}\right| \to 0 \quad \text{if} \quad z \to z_0$$

$\frac{1}{f(z)}$ is bounded around $z_0$

$\frac{1}{f(z_0)}$ has a removable singularity at $z_0$

$\frac{1}{f(z)}$ can be expressed as a power series

$$= a_m(z-z_0)^m + a_{m+1}(z-z_0)^{m+1} + \cdots$$

$a_m \neq 0$

$m \geq 1$

$$f(z_0) = \frac{1}{a_m(z-z_0)^m + a_{m+1}(z-z_0)^{m+1} + \cdots}$$

$$= \frac{1}{(z-z_0)^m}\left[\frac{1}{a_m + a_{m+1}(z-z_0) + \cdots}\right]$$

$\neq 0$

We say that $z_0$ is a pole of <u>order $m$</u>.

<u>Example</u> $f(z) = \frac{(z-1)(z-2)}{(z-3)(z+i)^2}$

Two simple zeros at 1 and 2

Pole of order 1 at $z=3$

Pole of order 2 at $z=-i$.

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<u>Example</u> $f(z) = \frac{1}{z(z-1)} - \frac{2}{z(z-2)}$

$$= \frac{(z-2) - 2(z-1)}{z(z-1)(z-2)}$$

$$= \frac{-z}{z(z-1)(z-2)}$$

Pole at $z = 1$ with order 1

Pole at $z = 2$ with order 1