MAT3253 Complex Variables
Lecture Notes

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Contents

This is a set of notes for MAT3253 Complex Variables. The followings are the reference books.

References

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[BrownChurchill] J. W. Brown and R. V. Churchill, *Complex Variables and Applications*, 8th edition, McGraw-Hill, New York, 2009.

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[Rudin2] W. Rudin, *Real and Complex Analysis*, 3rd edition, McGraw-Hill, New York, 1987.

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1 Lecture 1 (Complex numbers)

<u>Summary:</u>

- Construction of complex field using pairs of real numbers.
- Construction of complex field using 2×2 matrices.

Complex numbers were first invented to solve algebraic equations. As a vector space, the set of complex numbers is an extension of the real numbers of dimension 2. It is also equipped with a multiplication operator that extends the multiplication of real numbers.

We start with the axioms of a complex field.

Definition 1.1. A number system $(F, +, \cdot)$ is called a field if

1. (closed) $a + b \in F$ for all $a, b \in F$.

2. (associative) $(a+b)+c = a+(b+c)$, for all $a, b, c \in F$.

3. (commutative) $a+b = b+a$, for all $a, b \in F$.

4. (existence of zero) $\exists 0 \in F$ such that $0+a = a+0 = a$, for all $a \in F$.

5. (additive inverse) for all $a \in F$, $\exists a' \in F$ such that $a + a' = 0$.

6. (closed) $a \cdot b \in F$ for all $a, b \in F$.

7. (associative) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$, for all $a, b, c \in F$.

8. (commutative) $a \cdot b = b \cdot a$, for all $a, b \in F$.

9. (existence of one) $\exists 1 \in F$ such that $1 \cdot a = a \cdot 1 = a$, for all $a \in F$.

10. (multiplicative inverse) for all $a \in F \setminus \{0\}$, $\exists a'' \in F$ such that $a \cdot a'' = 1$.

11. (distributive) $a \cdot (b+c) = a \cdot b + a \cdot c$, for all $a, b, c \in F$.

A subset K of a field F is called a subfield of F if the elements in K satisfy the all axioms of a field, and F is called an extension of K. Examples of fields include the rational numbers $\mathbb{Q}$ and the real numbers $\mathbb{R}$. A field F containing $\mathbb{R}$ as a subfield and a special element I that satisfies $I^2 + 1 = 0$ is called a complex field. Complex field is denoted by $\mathbb{C}$.

Using this terminology, we say that the complex field is obtained by extending $\mathbb{R}$ so that the equation $x^2 + 1 = 0$ has a solution.

Remark. The definition of complex field above is not 100% accurate. To be precise, we should say that $\mathbb{C}$ is *generated* by the real numbers in $\mathbb{R}$ and the special number I. The term "generated" roughly means that all numbers in $\mathbb{C}$ can be obtained from "mixing" real numbers and the number I by addition, subtraction, multiplication and division.

Remark. In general, a field needs not be infinite. (None of the axioms require that there are infinitely many elements in a field.) We can construct number systems consisting of finitely many elements satisfying the axioms of field. To construct an example of a field of size 3, we can label the elements by 0, 1, 2, and define the addition and multiplication by the following tables

$+$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$\cdot$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

The addition and multiplication are addition and multiplication modulo 3.

The special number I in a complex number is usually called the imaginary unit. However, the calculations with complex numbers is very concrete and not imaginary. We provide two constructions of complex field below. In the first construction a complex number is a pair of real numbers. In the second one a complex number is a 2×2 matrix over the real numbers.

Construction of complex field (I)

Let

$$F_1 \triangleq \{(a, b) : a, b \in \mathbb{R}\}. \tag{1.1}$$

A "complex number" is thus regarded as a point on a plane, called the complex plane or Argand plane. The addition and multiplication operators are defined by

$$\begin{aligned}(a, b) + (c, d) &\triangleq (a + b,\, c + d),\\ (a, b) \cdot (c, d) &\triangleq (ac - bd,\, ad + bc).\end{aligned}$$

The additive and multiplicative identities are $(0, 0)$ and $(1, 0)$, respectively. The real numbers are embedded in F_1 by $x \mapsto (x, 0)$. Real-number calculation can be carried out in F_1. By identifying x_1 with $(x_1, 0)$ and x_2 with $(x_2, 0)$, the sum and product of x_1 and x_2 are respectively

$$\begin{aligned}(x_1, 0) + (x_2, 0) &= (x_1 + x_2, 0), \text{ and}\\ (x_1, 0) \cdot (x_2, 0) &= (x_1 x_2, 0).\end{aligned}$$

The special number I in this representation is $I = (0, 1)$. We can check that

$$I^2 = (0, 1) \cdot (0, 1) = ((0)(0) - (1)(1),\, (0)(1) + (1)(0)) = (1, 0).$$

F_1 is therefore a complex field.

Construction of complex field (II)

Let

$$F_2 \triangleq \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} : a, b \in \mathbb{R} \right\}. \tag{1.2}$$

Addition and multiplication are performed using the usual matrix addition and multiplication. The additive and multiplicative identities are the zero matrix and identity matrix, respectively. The "imaginary unit" I is $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. The field F_2 contains $\mathbb{R}$ as a subfield because the subset

$$\left\{ \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} : x \in \mathbb{R} \right\} \tag{1.3}$$

can be identified with the set of real numbers. Real numbers are represented as diagonal matrices with equal diagonal entries. The matrix addition and multiplication reduces to real-number addition and multiplication when restricted to matrices in (1.3),

$$\begin{bmatrix} x_1 & 0 \\ 0 & x_1 \end{bmatrix} + \begin{bmatrix} x_2 & 0 \\ 0 & x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 & 0 \\ 0 & x_1 + x_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 & 0 \\ 0 & x_1 \end{bmatrix} \cdot \begin{bmatrix} x_2 & 0 \\ 0 & x_2 \end{bmatrix} = \begin{bmatrix} x_1 x_2 & 0 \\ 0 & x_1 x_2 \end{bmatrix}.$$

We can use F_2 as a numerical model for calculating complex numbers.

The two constructions are essentially the same (meaning that F_1 and F_2 are isomorphic). The first construction emphasizes that a complex number is a pair of real numbers. The second construction emphasizes that complex multiplication is the same as multiplying by matrix in a special form. One can check that

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} ac - bd & -(ad + bc) \\ ad + bc & ac - bd \end{bmatrix}.$$

We will write $a + bi$ as a notation for (a, b) or $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. Using the $a + bi$ notation, the multiplication of complex numbers can be written as

$$(a + bi)(c + di) = ac - bd + i(ad + bc).$$

The complex numbers as a vector space has dimension 2 over $\mathbb{R}$. We can pick 1 and i as a basis. Using the first construction method, a complex number (x, y) can be written as

$$(x, y) = x(1, 0) + y(0, 1),$$

with $(1,0)$ and $(0,1)$ serving as the standard basis vectors. If we using the second construction method, we can use $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ as a basis,

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

2 Lecture 2 (Basic notions and operations of complex numbers)

Summary

- Complex conjugate, modulus
- Complex division
- Polar form of complex numbers, DeMoivre formula

Definition 2.1. For a complex number $z = a + bi$ in $\mathbb{C}$, define the real and imaginary part of z as

$$\mathrm{Re}(z) \triangleq a \quad \text{and} \quad \mathrm{Im}(z) \triangleq b.$$

Define the complex conjugate of z by

$$\bar{z} \triangleq z^* \triangleq a - bi.$$

The modulus of z is defined as

$$|z| = \sqrt{a^2 + b^2}.$$

The modulus of z is also called the absolute value or the radius.

Geometrically, the complex conjugate of z is the reflection of z along the real axis. The modulus is the distance between the origin and the point z in the complex plane. The next proposition says that the complex conjugate, as a mapping from $\mathbb{C}$ to $\mathbb{C}$, is compatible with complex addition and multiplication.

Proposition 2.2.

(i) $(z^*)^* = z$ *for any* $z \in \mathbb{C}$.

(ii) Given any two complex numbers z_1 *and* z_2 *in* $\mathbb{C}$,

$$(z_1 + z_2)^* = z_1^* + z_2^* \quad \textit{and} \quad (z_1 z_2)^* = z_1^* z_2^*.$$

Part (ii) in Prop. 2.2 says that the reflection of the sum (resp. product) of two complex numbers is the same as the sum (resp. product) of the two points obtained by reflection. The proof is simple and is omitted. Using Prop. 2.2, we can show that the modulus is a multiplicative function.

Proposition 2.3. *For any two complex numbers* $z_1, z_2 \in \mathbb{C}$, $|z_1 z_2| = |z_1|\,|z_2|$.

Proof. Use the fact that $|z|^2 = z\bar{z}$ for any $z \in \mathbb{C}$, and complex multiplication is commutative

$$|z_1 z_2|^2 = (z_1 z_2)(z_1 z_2)^* = z_1 z_2 z_1^* z_2^* = z_1 z_1^* z_2 z_2^* = |z_1|^2\,|z_2|^2.$$

□

The relationship between the real part, imaginary part and complex conjugate are

$$\mathrm{Re}(z) = \frac{z + z^*}{2} \quad \text{and} \quad \mathrm{Im}(z) = \frac{z - z^*}{2i}. \tag{2.1}$$

We can use complex conjugate to perform division in complex numbers. Suppose we want to divide $z_1 = a + bi$ by $z_2 = c + di$, where c and d are not zero. We multiply and divide by the conjugate of z_2,

$$\frac{z_1}{z_2} = \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di} = \frac{ac + bd}{c^2 + d^2} + i\frac{bc - ad}{c^2 + d^2}. \tag{2.2}$$

We can also do complex division using the 2×2 representation of complex numbers. Division is the same as taking matrix inverse,

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & -d \\ d & c \end{bmatrix}^{-1} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & d \\ -d & c \end{bmatrix} \frac{1}{c^2 + d^2} = \frac{1}{c^2 + d^2} \begin{bmatrix} ac + bd & ad - bc \\ bc - ad & ac + bd \end{bmatrix}.$$

The answer is the same as in (2.2). We note that $c^2 + d^2$ is the determinant of the matrix $\begin{bmatrix} c & -d \\ d & c \end{bmatrix}$ and is the same as the square of the absolute value of $c + di$.

Definition 2.4. The argument of a nonzero complex number z is defined as the angle from the positive real axis to the straight line from 0 to z. We write $\arg(z)$ to denote the argument function. The argument of $z = 0$ is not defined. The argument of a nonzero complex number is defined only up to integral multiples of 2π.

Definition 2.5. The points in the complex plane with modulus equal to 1 is called the unit circle.

A complex number $z = x + iy$ can be written in polar form

$$z = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta),$$

where r is the modulus of z and θ is an argument of z. Note that $\cos\theta + i\sin\theta$ lies on the unit circle for any θ. There are more than one way to write a complex number in polar form, because we can always add $2\pi k$ to θ, for any integer k, and get the same point on the complex plane.

Using the polar form, complex multiplication can calculated in terms of the modulus and the argument.

Proposition 2.6. *Given $z_1 = r_1\cos\theta_1 + ir_1\sin\theta_1$ and $z_2 = r_2\cos\theta_2 + ir_2\sin\theta_2$ in polar form, their product can be computed by*

$$z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)].$$

Proof. The proof follows from the definition of complex multiplication and basic trigonometric identities,

$$\begin{aligned}
z_1z_2 &= r_1r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\\
&= r_1r_2\big[\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)\big]\\
&= r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)].
\end{aligned}$$

□

The polar form suggests that the operation of complex multiplication can be decomposed into two parts. Using the second construction of complex numbers (1.2), the 2×2 matrix corresponding to a complex number $a + bi$ can be factorized as

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = r \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

where $r = \sqrt{a^2 + b^2}$ and $\theta = \arg(a + bi)$. The matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is a rotation matrix. The matrix-vector product

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}$$

is the point obtained by rotating (x, y) counter-clockwise by angle θ. The geometric meaning of multiplication by $a + bi = r(\cos\theta + i\sin\theta)$ is thus, (i) first rotate by θ counter-clockwise, then (ii) scale up (or down) by a factor of r.

Using the polar form, complex conjugate and complex division are computed by

$$(r\cos\theta + ir\sin\theta)^* = (r\cos(-\theta) + ir\sin(-\theta))$$
$$(r_1\cos\theta_1 + ir_1\sin\theta_1)/(r_2\cos\theta_2 + ir_2\sin\theta_2) = (r_1/r_2)(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)),$$

provided that $r_2 \neq 0$.

Theorem 2.7 (DeMoivre formula)**.** *For any $n \in \mathbb{Z}$ and $\theta \in \mathbb{R}$, we have*

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta). \tag{2.3}$$

Proof. The formula is obviously true when $n = 1$. When $n = 2$, it follows directly from Prop. 2.6,

$$(\cos\theta + i\sin\theta)^2 = \cos(\theta + \theta) + i\sin(\theta + \theta) = \cos(2\theta) + i\sin(2\theta).$$

We apply mathematical induction to establish (2.3) for all positive integers n and for all real numbers θ.

For negative n, we first note that

$$\begin{aligned} (\cos\theta + i\sin\theta)^{-1} &= \frac{1}{\cos\theta + i\sin\theta} \\ &= \frac{1}{\cos\theta + i\sin\theta} \cdot \frac{\cos\theta - i\sin\theta}{\cos\theta - i\sin\theta} \\ &= \cos\theta - i\sin\theta \\ &= \cos(-\theta) + i\sin(-\theta). \end{aligned}$$

Hence for positive integer m, we have

$$\begin{aligned}(\cos\theta + i\sin\theta)^{-m} &= ((\cos\theta + i\sin\theta)^{-1})^m \\ &= (\cos(-\theta) + i\sin(-\theta))^m \\ &= \cos(-m\theta) + i\sin(-m\theta).\end{aligned}$$

$\square$

Using the matrix representation of complex numbers, the DeMoivre's formula can be stated as

$$\begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}^n = \begin{bmatrix}\cos(n\theta) & -\sin(n\theta) \\ \sin(n\theta) & \cos(n\theta)\end{bmatrix}.$$

Geometrically speaking, this says that rotating n times by an angle θ is the same as rotating once by angle $n\theta$.

Example 2.1. Compute $(-1 + i\sqrt{3})^8$.

The complex number $-1 + i\sqrt{3}$ in polar form is

$$2(-1/2 + i\sqrt{3}/2) = 2(\cos(2\pi/3) + i\sin(2\pi/3)).$$

Hence, with the use of DeMoivre's formula, we get

$$(-1 + i\sqrt{3})^8 = 2^8(\cos(8\cdot 2\pi/3) + i\sin(8\cdot 2\pi/3)) = 256(\cos(4\pi/3) + i\sin(4\pi/3)).$$

In Cartesian form, the answer is $128(-1 - i\sqrt{3})$.

Example 2.2. Compute $(-1 + i)^{20}$.

Express $-1 + i$ in polar form $\sqrt{2}(\cos(3\pi/4) + i\sin(3\pi/4))$. By DeMoivre's formula,

$$\begin{aligned}(-1 + i)^{20} &= 20^{20/2}(\cos(20\cdot 3\pi/4) + i\sin(20\cdot 3\pi/4)) \\ &= 1024(\cos\pi + i\sin\pi) \\ &= -1024.\end{aligned}$$

Example 2.3. Express $\sin(5\theta)$ as a polynomial in $\sin(\theta)$.

By DeMoivre's formula,

$$\begin{aligned}&(\cos(5\theta) + i\sin(5\theta)) = (\cos\theta + i\sin\theta)^5 \\ &= \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta + 5\cos^4\theta\sin\theta + i\sin^5\theta.\end{aligned}$$

Equating the imaginary parts, we obtain

$$\begin{aligned}
\sin(5\theta) &= 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta \\
&= 5(1-\sin^2\theta)^2\sin\theta - 10(1-\sin^2\theta)\theta\sin^3\theta + \sin^5\theta \\
&= 5\sin\theta - 10\sin^3\theta + 5\sin^5\theta - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta \\
&= 5\sin\theta - 20\sin^3\theta + 16\sin^5\theta.
\end{aligned}$$

3 Lecture 3 (n-th roots of complex number)

Summary

- Complex division
- Principal argument
- Extracting the n-th roots of a complex number

Dividing a complex number $a+ib$ by $c+di$, where a, b, c, and d are arbitrary real numbers, means finding a complex number $w = x+iy$ such that $(x+iy)(c+di) = a+bi$. The problem can be reduced to a system of linear equations. By equating real and imaginary partsin

$$(x+iy)(c+di) = (cx-dy) + i(xd+yc) = a+bi,$$

we obtain

$$\begin{bmatrix} c & -d \\ d & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}.$$

The solution can be obtained by multiplying both sides by the inverse of $\begin{bmatrix} c & -d \\ d & c \end{bmatrix}$,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{c^2+d^2} \begin{bmatrix} c & d \\ -d & c \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

A faster method is to apply the following trick using complex conjugate

$$\frac{a+bi}{c+di} = \frac{a+bi}{c+di} \cdot \frac{c-di}{c-di} = \frac{(ac+bd)+i(bc-ad)}{c^2+d^2}.$$

Definition 3.1. Given a nonzero complex number z, the principal argument of z is the *unique* angle θ_0 (in radian) in $(-\pi, \pi]$ such that $z = |z|(\cos(\theta_0) + i\sin(\theta_0))$.

We follow the notation in [BrownChurchill] and denote the principal argument of z by $\mathrm{Arg}(z)$. In general, the argument function is multi-valued; it can be equal to $\mathrm{Arg}(z) + 2k\pi$ for any integer k.

Example 3.1. Compute the square roots of $4 + i$.

Write $4+i$ as $\sqrt{17}(\cos\phi + i\sin\phi)$, where $\phi = \tan^{-1}(1/4)$. A complex number w is called a square root of $4 + i$ if $w^2 = 4 + i$. By DeMoivre's formula, the modulus of w must be $\sqrt{\sqrt{17}}$. If we denote the argument of w by θ, then $2\theta = \phi + 2k\pi$ for some integer k. This gives

$$\theta = \frac{\phi}{2} + k\pi,$$

and we can take $k = 0, 1$ (because adding an integral multiple of 2π to the argument gives the same complex number.) We can write the answer as

$$\sqrt{4+i} = (17)^{1/4}(\cos(\phi/2 + k\pi) + i\sin(\phi/2 + k\pi)), \qquad \text{for } k = 0, 1,$$

or

$$\sqrt{4+i} = \pm(17)^{1/4}(\cos(\phi/2) + i\sin(\phi/2)).$$

Example 3.2. Compute the cube roots of unit.

Method 1. It amounts to solving $z^3 - 1 = 0$. After factorizing the polynomial into

$$(z-1)(z^2 + z + 1) = 0,$$

the solutions are 1, and the two roots of $z^2 + z + 1$, namely $-\frac{1}{2} \pm i\sqrt{\frac{3}{2}}$.

Method 2. We find all complex numbers with unit modulus and argument θ such that $3\theta = 0$. There are three possible values for θ, and they are 0, $2\pi/3$ and $-2\pi/3$. The cube roots of unity are

$$1,\ \cos(2\pi/3) + i\sin(2\pi/3),\ \cos(2\pi/3) - i\sin(2\pi/3).$$

Example 3.3. Compute the cube roots of i.

The principal argument of i is $\pi/2$. We want to find the values of θ such that

$$3\theta = \frac{\pi}{2} + 2\pi k, \qquad \text{for } k \in \mathbb{Z}.$$

There are three choices for θ, namely, $\pi/6 + 2\pi k/3$, for $k = 0, 1, 2$. The cube roots of i are

$$\cos(\frac{\pi}{6} + \frac{2\pi k}{3}) + i\sin(\frac{\pi}{6} + \frac{2\pi k}{3}).$$

for $k = 0, 1, 2$.

In general, there are n solutions when taking the n-th root of a nonzero number. The method is the same as in the above examples. If we plot the n solutions in the complex plane, they form a regular n-gon with the origin as the center.

4 Lecture 4 (Complex plane as metric space and topological space)

Summary

- Point at infinity
- Concepts from metric space

In a 3-dimensional space, identify the points in the horizontal plane as the complex numbers. A point in the 3-D space has coordinates (ξ, η, ζ). A complex number $x + iy$ is thus located at $(x, y, 0)$. Put a sphere of radius 1 on the x-y plane, touching the x-y plane at the origin. The equation of the sphere is

$$\xi^2 + \eta^2 + (\zeta - \frac{1}{2})^2 = (\frac{1}{2})^2.$$

If we draw a straight line connecting $(0, 0, 1)$ and a point $(x, y, 0)$ on the horizontal plane, there is a unique intersection point on the sphere. This gives a one-to-one correspondence between the points on the complex plane and the points on the sphere, except the north pole. This mapping is called the stereographical projection. The totality of all complex numbers can be represented as the points on a punctured sphere. The picture can be completed by adjoining an extra point to the complex numbers.

Definition 4.1. The extended complex numbers as a set is defined as $\mathbb{C} \cup \{\infty\}$, where ∞ is a symbol called the point at infinity. The symbol ∞ corresponds to the north pole in the stereographic projection. The sphere in the stereographical projection is called the *Riemann sphere*.

Remark. The importance of the Riemann sphere is that it is a compact set, and compact set has nice topological properties.

In this lecture we study complex numbers as points on a metric space, with the metric induced by the complex absolute value; the distance between two complex numbers z_1 and z_2 is $|z_1 - z_2|$. We readily check that the triangular inequality is satisfied.

Proposition 4.2 (Triangle inequality)**.**

$$|z_1 + z_2| \le |z_1| + |z_2|$$

for any two complex numbers z_1 and z_2.

Proof. Take the square of the left-hand side,

$$\begin{aligned} |z_1 + z_2|^2 &= (z_1 + z_2)(z_1^* + z_2^*) \\ &= |z_1|^2 + 2\mathrm{Re}(z_1 z_2^*) + |z_2|^2 \\ &\le |z_1|^2 + 2|\mathrm{Re}(z_1 z_2^*)| + |z_2|^2. \end{aligned}$$

It is sufficient to prove

$$|\mathrm{Re}(z_1 z_2^*)| \le |z_1|\,|z_2|, \tag{4.1}$$

because it will immediately give $|z_1 + z_2|^2 \le (|z_1|^2 + |z_2|)^2$.

To prove (4.1), suppose $z_1 = a + bi$ and $z_2 = c + di$, and write $\mathrm{Re}(z_1 z_2^*) = ac + bd$. We want to prove $(ac + bd)^2 \le (a^2 + b^2)(c^2 + d^2)$. This inequality holds for any real numbers a, b, c and d because

$$(a^2 + b^2)(c^2 + d^2) - (ac + bd)^2 = (ad - bc)^2 \ge 0.$$

□

Notation: a sequence of complex numbers $z_1, z_2, z_3, \ldots$ is denoted by $(z_k)_{k=1}^{\infty}$ or $\{z_k\}$.

Definition 4.3. Given a complex sequence $(z_n)_{n=1}^{\infty}$, we say that z_n converges to w if

$$\forall \epsilon > 0\ \exists N,\ s.t.\ |z_n - w| < \epsilon,\ \forall n \ge N.$$

It is equivalent to requiring that $(|z_n - w|)_{n=1}^{\infty}$ as a real sequence is converging to 0 as $n \to \infty$. We write $z_n \to w$ if z_n converges to w, and

$$\lim_{n\to\infty} z_n = w.$$

Example 4.1.

$$\frac{1}{n} + \frac{i}{n^2} \to 0.$$

$$(0.5)^n(\cos n + i \sin n) \to 0.$$

Example 4.2. Compute $\lim_{n\to\infty} \frac{n}{n+i}$.

We can first make a guess that the limit should be 1, because when n is large, adding i to n has negligible effect. To make the argument rigorous, we write

$$\frac{n}{n+i} = \frac{n+i-i}{n+i} = 1 - \frac{i}{n+i}.$$

Then

$$\left|\frac{n}{n+i} - 1\right| = \left|\frac{i}{n+i}\right| = \frac{1}{\sqrt{n^2+1}} \to 0, \qquad \text{as } n \to \infty.$$

Therefore $n/(n+i) \to 1$ as $n \to \infty$.

Example 4.3. The sequence $((2i)^n)_{n=1}^{\infty}$ does not converge to any complex number in $\mathbb{C}$. However, if we look at the projection of $(2i)^n$ on the Riemann sphere, it is converging to the point at infinity ∞. Hence we can say that $(2i)^n \to \infty$.

More generally, we say that a sequence of complex numbers $(z_n)_{n=1}^{\infty}$ converges to the point at infinity if $z_n^{-1} \to 0$.

Definition 4.4. A sequence $(z_k)_{k=1}^{\infty}$ is called a Cauchy sequence if for all $\epsilon > 0$, there exists an integer N such that

$$|z_m - z_n| \le \epsilon \quad \text{whenever } m, n \ge N.$$

The basic property of Cauchy sequence for real numbers extends to the complex case.

Theorem 4.5. *A complex sequence $(z_k)_{k=1}^{\infty}$ converges if and only if $(z_k)_{k=1}^{\infty}$ is Cauchy.*

We have the following relationship between convergence of complex sequence and real sequences.

Theorem 4.6. *A complex sequence $(z_k)_{k=1}^{\infty}$ converges if and only if both $(\mathrm{Re}(z_k))_{k=1}^{\infty}$ and $(\mathrm{Im}(z_k))_{k=1}^{\infty}$ converge.*

The notion of infinite series for complex numbers is the same as in calculus.

Definition 4.7. An infinite series of complex numbers $\sum_{k=1}^{\infty} z_k$ converges if the sequence of partial sums

$$\Big(\sum_{k=1}^{n} z_k\Big)_{n=1}^{\infty}$$

is convergent.

Proposition 4.8. *Given a sequence of complex numbers $(z_k)_{k=1}^{\infty}$, if the real infinite series $\sum_{k=1}^{\infty} |z_k|$ converges, then $\sum_{k=1}^{\infty} z_k$ also converges.*

The proof can be done by consider the real and imaginary parts of z_k, and reduced to the real case.

Definition 4.9. We say that a series $(z_k)_{k=1}^{\infty}$ is absolutely convergent if $\sum_{k=1}^{\infty} |z_k|$ is convergent.

Example 4.4. (Complex geometric series) Evaluate $\sum_{k=1}^{\infty}(0.5i)^k$.

We can check that this is absolutely convergent. Because $|0.5i| = 0.5 < 1$,

$$\sum_{k=1}^{\infty} |0.5i|^k = \sum_{k=1}^{\infty}(0.5)^k$$

is a geometric series with common ratio strictly less than 1, and hence is convergent.

For any finite n, we have

$$\sum_{k=1}^{n}(0.5i)^k = \frac{(0.5i)^{n+1} - 0.5i}{0.5i - 1}.$$

We take limit as $n \to \infty$,

$$\sum_{k=1}^{\infty}(0.5i)^k = \lim_{n\to\infty} \frac{(0.5i)^{n+1} - 0.5i}{0.5i - 1} = \frac{-0.5i}{0.5i - 1} = \frac{-1 + 2i}{5}.$$

Definition 4.10. An open disc centered at z_0 with radius r is defined as

$$D(z_o; r) \triangleq \{z \in \mathbb{C} : |z - z_0| < r\}.$$

A circle centered at z_0 with radius r is

$$C(z_o; r) \triangleq \{z \in \mathbb{C} : |z - z_0| \le r\}.$$

A set S in the complex plane is said to be open if for any $z \in S$, we can find $\delta > 0$ such that $D(z; \delta_0) \subseteq S$.

It can be shown that an open disc is indeed open.

Definition 4.11. The boundary of a set S, denoted by ∂S, is defined as

$$\{z \in \mathbb{C} : \forall \delta > 0,\ D(z;\delta) \cap S \neq \emptyset \text{ and } D(z;\delta) \cap S^c \neq \emptyset\}.$$

A set is said to be a closed set if the complement is open.

A set is bounded if it is contained in $D(0; M)$ for some larger M.

A set is compact if it is closed and bounded.

5 Lecture 5 (Complex function)

Summary

- Domain of a function
- Continuous function
- Complex differential function

In complex analysis, a domain/region is an open and connected set in $\mathbb{C}$.

In MAT3253, we can understand "connected" as "path-connected", i.e., any two points in the set are connected by a path. The following is a useful fact for two-dimensional region.

Proposition 5.1. *Suppose R is an open set, and A and B are points in R that are connected by a path, then there exists a polygonal path from A to B with finitely many linear parts.*

The above proposition means that when we consider connectedness, it is sufficient to consider piece-wise linear paths.

Definition 5.2. A function $f : \mathbb{C} \to \mathbb{C}$ is said to be continuous at z_0 if for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$|z - z_0| < \delta \ \Rightarrow \ |f(z) - f(z_0)| < \epsilon.$$

A function f is said to be continuous in a domain D if f is continuous at every point in D.

By consider the real and imaginary part separately, we can prove the following

Theorem 5.3. *A complex function f is continuous if and only if the real and imaginary parts are continuous.*

Example 5.1. Show that $f(z) = 1/z$ is continuous in the domain $\mathbb{C} \setminus \{0\})$.

Suppose $z = x + iy$ and $z \neq 0$.

$$\frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$$

The real part of $f(z)$ is $x/(x^2+y^2)$ and the imaginary part is $-y/(x^2+y^2)$. Both of them are continuous functions in the domain $\mathbb{C} \setminus \{0\})$. Hence $f(z)$ is continuous by the previous theorem.

A complex function $f(z)$ can be interpreted as a two-dimensional vector field,

$$f(x+iy) = u(x,y) + iv(x,y).$$

When we say that $f(x+iy)$ is real differentiable, we mean that the vector-valued function $(u(x,y), v(x,y))$ is differentiable as in multivariable calculus. By the definition of differentiability, if $(u(x,y), v(x,y)$ is differentiable at a point (x_0, y_0), we can approximate the effect of a small change in x and y by linear function,

$$\begin{bmatrix} u(x_0+\Delta x, y_0+\Delta y) \\ v(x_0+\Delta x, y_0+\Delta y) \end{bmatrix} \approx \begin{bmatrix} u(x_0,y_0) \\ v(x_0,y_0) \end{bmatrix} + \begin{bmatrix} u_x(x_0,y_0) & u_y(x_0,y_0) \\ v_x(x_0,y_0) & v_y(x_0,y_0) \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}. \tag{5.1}$$

The entries in the 2×2 matrix are the partial derivatives of u and v evaluated at (x_0, y_0). The symbol "$\approx$" means that the higher-order terms are converging to zero faster than the linear term. More precisely, it means that the limit

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\|\text{Difference between L.H.S and R.H.S. of (5.1)}\|}{\sqrt{\Delta x^2 + \Delta y^2}} = 0.$$

Example 5.2. The function $f(z)$ defined by $x^2 + i(x+y)$ is real differentiable. Partial derivatives of the real part $u(x,y) = x^2$ and imaginary part $v(x,y) = x+y$ exist, and we have

$$\begin{bmatrix} (x+\Delta x)^2 \\ x+\Delta x+y+\Delta y \end{bmatrix} \approx \begin{bmatrix} x^2 \\ x+y \end{bmatrix} + \begin{bmatrix} 2x & 0 \\ y & x \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} \tag{5.2}$$

for any x and y.

Definition 5.4. A complex function f is said to be complex differentiable at z_0 if the limit

$$\lim_{\Delta z \to 0} \frac{f(z_0+\Delta) - f(z_0)}{\Delta z} \tag{5.3}$$

exists. This is equivalent to requiring that

$$f(z_0 + \Delta z) \approx f(z_0) + w_0 \Delta z,$$

were w_0 is a complex constant and is the limit in (5.3). The limit in (5.3) is denoted by $f'(z_0)$.

We can understand this by interpreting complex multiplication as matrix multiplication. If a function f is complex differentiable at a point z_0, then the 2×2 matrix in (5.1) must be in the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, so that we can realize the matrix multiplication in (5.1) by complex multiplication.

Example 5.3. Consider the function

$$f(z) = z^3 = (x + iy)^3 = (x^3 - 3xy^2) + i(3x^2 y - y^3).$$

The real and imaginary parts are $u(x, y) = x^3 - 3xy^2$ and $v(x, y) = 3x^2 y - y^3$, respectively. Suppose we fix a base point $(x_0, y_0) = (1, 1)$. The linear approximation in (5.1) at $(1, 1)$ can be written as

$$\begin{bmatrix} u(1 + \Delta x, 1 + \Delta y) \\ v(1 + \Delta x, 1 + \Delta y) \end{bmatrix} \approx \begin{bmatrix} -2 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 & -6 \\ 6 & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}.$$

For general $z = x + iy$, the 2×2 derivative matrix is

$$\begin{bmatrix} 3x^2 - 3y^2 & -6xy \\ 6xy & 3x^2 - 3y^2 \end{bmatrix}.$$

We can use complex arithemtic to realize the linear approximation

$$f(z + \Delta x) = f(z) + (3x^2 - 3y^2 + i(6xy)) \cdot \Delta z,$$

and the complex derivative turns out to be equal to $3x^2 - 3y^2 + i(6xy) = 3z^2$.

The function in Example 5.2 is real differentiable everywhere but not complex differentiable in general. In fact it is complex differentiable only at $(x, y) = (0, 0)$. However, the function in Example 5.3 is complex differentiable at all points in $\mathbb{C}$, and the complex derivative is $3z^2$.

6 Lecture 6 (Analytic functions)

Summary

- Cauchy-Riemann equation
- Definition of analytic function

We recall two basic results from multivariable calculus.

Theorem 6.1. *Suppose $\vec{f}(x,y) = (u(x,y), v(x,y))$ be a two-dimensional vector field.*

1. *A necessary condition for $\vec{f}$ to be real differentiable at a point (x_0, y_0) is that all partial derivatives u_x, u_y, v_x and v_y exists in a neighborhood of (x_0, y_0).*
2. *A sufficient condition for $\vec{f}$ to be real differentiable at (x_0, y_0) is (i) partial derivatives u_x, u_y, v_x and v_y exists in a neighborhood of (x_0, y_0), and (ii) the partial derivatives u_x, u_y, v_x are continuous at (x_0, y_0).*

We first derive an important necessary condition for complex differentiability.

Theorem 6.2 (Cauchy-Riemann equations)**.** *Suppose $f(x+iy) = u(x,y) + iv(x,y)$ is complex differentiable at z_0 (see Definition 5.4), where z_0 is in the domain of $f(z)$. Then*

$$u_x = v_y, \quad \text{and} \quad v_y = -v_x.$$

Proof. The limit in computing complex derivative does not depend on how we approach the point z_0. We can approach z_0 horizontally or vertically, and the results must be the same if the function is complex differentiable.

Let $\Delta z = \Delta x$ and take $\Delta x \to 0$.

$$\begin{aligned}
&\lim_{\Delta x \to 0} \frac{u(x_0+\Delta x, y_0) + iv(x_0+\Delta x, y_0) - u(x_0,y_0) - iv(x_0,y_0)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{u(x_0+\Delta x, y_0) - u(x_0,y_0)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x_0+\Delta x, y_0) - v(x_0,y_0)}{\Delta x} \\
&= \frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}(x_0,y_0).
\end{aligned}$$

Next suppose $\Delta z = i\Delta y$ and take $\Delta y \to 0$.

$$\begin{aligned}
&\lim_{\Delta y \to 0} \frac{u(x_0, y_0 + \Delta y) + iv(x_0, y_0 + \Delta y) - u(x_0, y_0) - iv(x_0, y_0)}{i\Delta y} \\
&= \lim_{\Delta y \to 0} \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{i\Delta y} + i \lim_{\Delta y \to 0} \frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{i\Delta y} \\
&= -i\frac{\partial u}{\partial x}(x_0, y_0) + \frac{\partial v}{\partial x}(x_0, y_0).
\end{aligned}$$

By equating real and imaginary parts of the two limits, we get $u_x(x_0, y_0) = v_y(x_0, y_0)$ and $u_y(x_0, y_0) = -v_x(x_0, y_0)$. □

Similar to Theorem 6.1, we have the following sufficient condition for complex differentiability.

Theorem 6.3. *A complex function f is complex differentiable at z_0 if*

1. *The partial derivatives u_x, u_y, v_x, v_y exists in a neighborhood of z_0.*
2. *Cauchy-Riemann equations are satisfied at z_0.*
3. *u_x, u_y, v_x, v_y are continuous at z_0.*

If the above conditions hold, the complex derivative of f is given by $u_x + iv_x = v_y - iu_y$.

Example 6.1. The function $f(z) = az + b$, for any $a, b \in \mathbb{C}$ is complex differentiable at any $z \in \mathbb{C}$. The complex derivative is $f'(z) = a$.

Example 6.2. The conjugate function $f(z) = z^*$ is not complex differentiable anywhere. It is because $u_x = 1$ and $v_y = -1$, and $u_x \neq v_y$ at any point in $\mathbb{C}$.

Example 6.3. Consider the square function $f(z) = z^2$. The real and imaginary parts are $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$, respectively. We check that the partial derivatives

$$u_x = 2x,\ u_y = -2y,\ v_x = 2y,\ v_y = 2x$$

exist and are continuous at every point in $\mathbb{C}$. This check conditions 1 and 3 in Theorem 6.3. Furthermore, the Cauchy-Riemann equalities are satisfied everywhere, because

$$u_x = v_y = 2x, \quad \text{and} \quad u_y = -v_x = -2y.$$

By Thoeorem 6.3, $f(z) = z^2$ is differentiable, and the complex derivative is $u_x + iv_x = 2z$.

Example 6.4. The function $f(z) = |z|^2 = x^2 + y^2$ has zero imaginary part. As a real-valued function it is real differentiable. However it is complex differentiable only at $z = 0$. We see this by computing the partial derivatives

$$u_x = 2x, \qquad v_x = 0,$$
$$u_y = 2y, \qquad v_y = 0.$$

The Cauchy-Riemann equations are satisfied only at $z = 0$. Therefore it is not complex differentible if $z \neq 0$. By Theorem 6.3, it is indeed complex differentiable at $z = 0$.

Example 6.5. The function $f(z) = 1/z$ is defined in the domain $\mathbb{C} \setminus \{0\}$. It is complex differentiable everywhere in the domain because, for $z \neq 0$,

$$\begin{aligned} \frac{\frac{1}{z+h} - \frac{1}{z}}{h} &= \frac{1}{h}\Big(\frac{z-(z+h)}{(z+h)z}\Big) \\ &= -\frac{1}{z(z+h)}. \end{aligned}$$

When $h \to 0$, the limit of $-\frac{1}{z(z+h)}$ is $-1/z^2$. Therefore $f(z) = 1/z$ is complex differentiable for $z \in \mathbb{C} \setminus \{0\}$, and the complex derivative is $-1/z^2$.

The function in Example 6.4 is complex differentiable only at one time, and is considered as pathological. The main theorems in complex analysis usually require that the function is complex differentiable in a domain (the interior is nonempty).

Definition 6.4. A function f is said to be analytic/holomorphic/regular at a point z_0 if there is a neighborhood of z_0 such that f is complex differentiable at every point in the neighborhood. A function is said to be entire if it is complex differentiable at every point in $\mathbb{C}$.

For example, the function in Example 6.1 and 6.3 is entire. The function $1/z$ in Example 6.5 is analytic in the domain of definition. Example 6.2 and 6.4 are not analytic anywhere.

7 Lecture 7 (Conformal property, Harmonic conjugate)

Summary

- Angle-preserving property of analytic functions
- Harmonic functions
- Harmonic conjugate

Complex differentiable functions are very special. This lecture investigates two such special properties.

Conformal property

Suppose $f(z)$ is complex differentiable at a given point z_0 in $\mathbb{C}$, and suppose $f'(z_0)$ is nonzero. The function f is *conformal*, or *angle-reserving*. It is based on the fact that multiplication by a nonzero complex constant can be interpreted geometrically as a rotation.

Draw two parametric curves $\gamma_1(t)$ and $\gamma_2(t)$ through z_0. By "parametric curve" it means a smooth map from an interval in $\mathbb{R}$ to to $\mathbb{C}$. Let the range of the parameter t in $\gamma_1(t)$ be (a, b), where $a < 0 < b$, and $\gamma_1(0) = z_0$. Likewise, let the range of t in $\gamma_2(t)$ be (a', b'), where $a' < 0 < b'$ and $\gamma_2(0) = z_0$. In the domain of f, the angle between the lines from $\gamma_1(0)$ to $\gamma_1(\Delta t)$ and the line from $\gamma_2(0)$ to $\gamma_2(\Delta 2)$ is

$$\arg\Big(\frac{\gamma_2(\Delta t) - \gamma_2(0)}{\gamma_1(\Delta t) - \gamma_1(0)}\Big),$$

which converges to

$$\arg(\gamma_2'(0)/\gamma_1'(0))$$

as $\Delta t \to 0$. In the range of f, the angle between the lines from $f(\gamma_1(0))$ to $f(\gamma_1(\Delta t))$ and from $f(\gamma_2(0))$ to $f(\gamma_2(\Delta t))$ is

$$\arg\Big(\frac{f(\gamma_2(\Delta t)) - f(\gamma_2(0))}{f(\gamma_1(\Delta t)) - f(\gamma_1(0))}\Big).$$

Since f is assumed to be complex differentiable, when we take limit as Δt approaches 0, and write the limit as

$$\lim_{\Delta t \to 0} \arg\Big(\frac{f'(z_0)\cdot(\gamma_2(\Delta t) - \gamma_2(0))}{f'(z_0)\cdot(\gamma_1(\Delta t) - \gamma_1(0))}\Big).$$

Because $f'(z_0)$ is non-zero, the limit is the same as $\arg(\gamma_2'(0)/\gamma_1'(0))$. Therefore the angle between $\gamma_1(t)$ and $\gamma_2(t)$ at the point z_0 is the same as the angle between the images $f(\gamma_2(t))$ and $f(\gamma_1(t))$ at the point $f(z_0)$. If we draw some perpendicular grid lines in the domain of $f(z)$, then the images of these lines will intersect at 90 degrees.

The conformal property explains why the conjugate function $f(z) = \bar{z}$ is not complex differentiable anywhere. The conjugate function is a reflection geometrically, and reflection reverses the orientation of angles.

Harmonic functions

Definition 7.1. A function $u(x, y)$ in two variables is called a harmonic function if it satisfies the Laplace equation

$$u_{xx} + u_{yy} = 0.$$

Proposition 7.2. *Write $z = x + iy$ and suppose $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain. If $u(x, y)$ and $v(x, y)$ are twice differentiable, the second-order partial derivatives are continuous, then the $u(x, y)$ and $v(x, y)$ are harmonic functions.*

Proof. If $f(z)$ is analytic, then it satisfies the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$. Take partial derivatives again, we obtain $u_{xx} = v_{yx}$ and $u_{yy} = -v_{xy}$. Since we assume the order of the partial derivatives can be exchanged,

$$u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0.$$

Similarly, we can show that $v(x, y)$ is harmonic. □

Remark. The assumption that $u(x, y)$ and $v(x, y)$ are twice differentiable can be relaxed. We will show that this conditions are automatically satisfied if $f(z)$ is analytic.

In the rest of this lecture we assume that all functions are in C^2, i.e., the second partial derivatives exist and are continuous.

Definition 7.3. Given a harmonic function $u(x, y)$, a function $v(x, y)$ is called a harmonic conjugate of $u(x, y)$ if $u(x, y)$ and $v(x, y)$ satisfy the Cauchy-Riemann equations.

Suppose $u(x, y)$ is harmonic function in a simply connected domain D, then we can find a harmonic conjugate of $u(x, y)$ using path integral. The differential form

$$v_x dx + v_y dy = -u_y dx + u_x dy$$

is exact, because

$$\frac{\partial}{\partial x} u_x - \frac{\partial}{\partial y}(-u_y) = u_{xx} + u_{yy} = 0.$$

By Green's theorem, the line integral

$$\int -u_y dx + u_x dy$$

only depends on the start point and the end point of the path. We can define a function $v(x, y)$ by first fixing a point (x_0, y_0) in the domain D, and let

$$v(\tilde{x}, \tilde{y}) = \int_{(x_0,y_0)}^{(\tilde{x},\tilde{y})} -u_y dx + u_x dy \tag{7.1}$$

for any point $(\tilde{x}, \tilde{y})$ in D. This is well-defined because the line integral is independent of path. (Here $\tilde{x}$ and $\tilde{y}$ are fixed real constants, and x and y are the dummy variables used in the integral.)

We can check that the function $v(x, y)$ so defined is a harmonic conjugate of $u(x, y)$. Let $(\tilde{x}, \tilde{y})$ be any point in D. The partial derivative of v with respect to x at this point is

$$\begin{aligned}
v_x(\tilde{x}, \tilde{y}) &= \lim_{\Delta x \to 0} \frac{v_x(\tilde{x} + \Delta x, \tilde{y}) - v_x(\tilde{x}, \tilde{y})}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_{(\tilde{x},\tilde{y})}^{(\tilde{x}+\Delta x,\tilde{y})} -u_y dx + u_x dy \\
&= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_{(\tilde{x},\tilde{y})}^{(\tilde{x}+\Delta x,\tilde{y})} -u_y dx \\
&= -u_y(\tilde{x}, \tilde{y}).
\end{aligned}$$

In the last step we have used the assumption that u_y is continuous at $(\tilde{x}, \tilde{y})$.

Likewise, by consider the partial derivative of v with respect to y, we get

$$\begin{aligned}
v_y(\tilde{x}, \tilde{y}) &= \lim_{\Delta y \to 0} \frac{v_x(\tilde{x}, \tilde{y} + \Delta y) - v_x(\tilde{x}, \tilde{y})}{\Delta y} \\
&= \lim_{\Delta y \to 0} \frac{1}{\Delta y} \int_{(\tilde{x},\tilde{y})}^{(\tilde{x},\tilde{y}+\Delta y)} -u_y dx + u_x dy \\
&= \lim_{\Delta y \to 0} \frac{1}{\Delta y} \int_{(\tilde{x},\tilde{y})}^{(\tilde{x},\tilde{y}+\Delta y)} u_x dy \\
&= u_x(\tilde{x}, \tilde{y}).
\end{aligned}$$

Again, we have used the continuity of u_x in the last step.

This proves that the Cauchy-Riemann equations are satisfied. In summary, we can conclude that a harmonic conjugate of $u(x, y)$ can be written as in (7.1) when D is simply connected. We note that $v(x, y)$ is defined up to a constant, because we can always add an integration constant to (7.1).

Example 7.1. Find a harmonic conjugate of the function

$$u(x, y) = -2x^2 + x^3 + 2y^2 - 3xy^2.$$

The function $u(x, y)$ is defined on the whole complex plane. We check that it is harmonic:

$$\begin{aligned} u_x &= -4x + 3x^2 - 3y^2 \\ u_{xx} &= -4 + 6x \\ u_y &= 4y - 6xy \\ u_{yy} &= 4 - 6x. \end{aligned}$$

Hence $u_{xx} + u_{yy}$ is identically equal to zero.

<u>METHOD 1</u> Using path integral, we can obtain the function v by

$$v(\tilde{x}, \tilde{y}) := \int_{(0,0)}^{(\tilde{x},\tilde{y})} (6xy - 4y)dx + (-4x + 3x^2 - 3y^2)dy.$$

for any $(\tilde{x}, \tilde{y}) in \mathbb{R}^2$. To calculate the path integral, we can simply take the direct path from $(0, 0)$ to $(\tilde{x}, \tilde{y})$. Parametrize this path by

$$\begin{cases} x = t\tilde{x} \\ y = t\tilde{y} \end{cases}$$

for $0 \leq t \leq 1$. We then calculate the path integral

$$\begin{aligned} v(\tilde{x}, \tilde{y}) &= \int_0^1 (6t^2\tilde{x}\tilde{y} - 4t\tilde{y})\tilde{x} + (-4t\tilde{x} + 3t^2\tilde{x}^2 - 3t^2\tilde{y}^2)\tilde{y}\, dt \\ &= \int_0^1 -8\tilde{x}\tilde{y}t + (9\tilde{x}^2\tilde{y} - 3\tilde{y}^3)t^2 \\ &= -4\tilde{x}\tilde{y} + 3\tilde{x}^2\tilde{y} - \tilde{y}^3 + C \end{aligned}$$

where C is a constant.

METHOD 2. The second method is ad hoc and is the same as in Calculus II. We first integrate

$$v_x = -u_y = -4y + 6xy$$

with respect to x and get

$$v(x, y) = \int -4y + 6xy\, dx = -4xy + 3x^2 y + C(y),$$

where $C(y)$ is a constant that may involve y. Differentiate the above with respect to y,

$$v_y = -4x + 3x^2 + C'(y).$$

After comparing with u_x, we see that $C'(y) = -3y^2$, and hence $C(y) = -y^3 + C$ for some constant C. The answer is

$$v(x, y) = -4xy + 3x^2 y - y^3 + C.$$

8 Lecture 8 (Complex exponential function)

Summary

- Complex exponential function (first definition)
- Complex log function
- Complex powers

We want an analytic function that extends the real exponential function, i.e., we want a complex function so that if we restrict the input to a real number, the output is the same as the result obtained from the real exponential function.

We start with a function $u(x, y) = e^x \cos(y)$. It is easily checked that this function is harmonic, and $u(x, 0) = e^x$. Using the method of path integral, we consider the path integral

$$\int_{(0,0)}^{(\tilde{x},\tilde{y})} -u_y dx + u_x dy = \int_{(0,0)}^{(\tilde{x},\tilde{y})} (e^x \sin y) dx + (e^x \cos y) dy.$$

This path integral is conservative, because we can find a "potential function"

$$v(x, y) = e^x \sin y$$

such that

$$v_x = e^x \sin y, \quad \text{and} \quad v_y = e^x \cos y.$$

Therefore

$$u(x, y) + iv(x, y) = e^x \cos y + ie^x \sin y$$

is an analytic function, and it is defined and analytic on the whole complex plane.

Definition 8.1. We define the complex exponential function by

$$\exp(z) \triangleq e^x(\cos y + i \sin y).$$

We often write e^z as a short-hand notation.

Definition 8.1 is adopted in [BakNewman] and [BrownChurchill] as the definition of complex exponential function.

We note that the complex exponential function has a complex period $2\pi i$; that is, for any $z \in \mathbb{C}$,

$$e^{z+2\pi ki} = e^z, \quad \text{for } k \in \mathbb{Z}.$$

As a result, the inverse function of e^z is multi-valued. Given a complex number $r(\cos\theta + i\sin\theta)$ in polar form, the complex log function of $r(\cos\theta + i\sin\theta)$ can take values

$$\log r + i(\theta + 2\pi k)$$

for $k \in \mathbb{Z}$. More formally we have the following

Definition 8.2. For nonzero $w \in \mathbb{C}$, the complex log function is defined as

$$\log(w) = \log|w| + i(\arg(w) + 2\pi k)$$

where $k = 0, \pm 1, \pm 2, \ldots$ Here the function $\arg(w)$ is the multi-valued argument function. If we take the principal argument, we have a uniquely defined function

$$\log(w) = \log|w| + i\mathrm{Arg}(w),$$

and it is called the principal complex log function.

Example 8.1. Compute 2^i by calculating $e^{i\log 2}$.

$$\begin{aligned}\exp(i\log 2) &= \exp(i(\log 2 + i2\pi k))\\ &= \exp(-2\pi k + i\log 2)\\ &= e^{-2\pi k}(\cos(\log 2) + i\sin(\log 2)),\end{aligned}$$

where k can take any integer as its value.

Example 8.2. Compute i^i by calculating $e^{i\log i}$.

$$\begin{aligned}\exp(i\log i) &= \exp(i(\log 1 + i(\frac{\pi}{2} + 2\pi k)))\\ &= \exp(-(\frac{\pi}{2} + 2\pi k)),\end{aligned}$$

where $k \in \mathbb{Z}$. If we take $k = 0$, then we can say that the principal value of i^i is $e^{-\pi/2}$, which is a real number.

If we want to define the log function as an analytic function, we have to take a branch cut. Usually we take the negative real axis

$$\{x + 0i : x \leq 0\}$$

as the branch cut. We can see why we cannot define a complex log function on the whole complex plane by trying to get a harmonic conjugate of

$$u(x, y) = \log(\sqrt{x^2 + y^2}).$$

We can calculate

$$u_x = \frac{x}{x^2 + y^2} \quad \text{and } u_y = \frac{y}{x^2 + y^2}.$$

The path integral

$$\int -\frac{y}{x^2 + y^2}dx + \frac{x}{x^2 + y^2}dy$$

is not conservative for paths in the punctured plane $\mathbb{C} \setminus \{(0, 0)\}$.

If we take

$$D = \mathbb{C} \setminus \{x + 0i : x \leq 0\}$$

as the domain, then we can define the complex log function as

$$Log(z) = \log(|z|) + i\text{Arg}(z).$$

where $\text{Arg}(z)$ denotes the principal argument function. The complex derivative of $Log(z)$ is

$$Log'(z) = u_x + iv_x = u_x - iu_y = \frac{x - iy}{x^2 + y^2} = \frac{1}{z}.$$

9 Lecture 9 (Euler's formula)

Summary

- Complex exponential function (second definition)
- Complex sine and cosine function
- Euler's formula

For real number x, the meaning of e^x can be defined in several ways.

- First define the constant e by $e = \lim_{n\to\infty}(1 + 1/n)^n$. For integer a, define e^a by $\underbrace{e \cdot e \cdots e}_{a \text{ factors}}$. For integer b, let $e^{1/b}$ be the number y such that $y^b = e$. For rational number a/b, define $e^{a/b}$ by $(e^a)^{(1/b)}$. For irrational number x, we approximate x by taking a sequence of rational numbers $(a_k/b_k)_{k=1}^{\infty}$ that converges to x, and defined e^x by
$$e^x \triangleq \lim_{k\to\infty} e^{a_k/b_k}.$$
- First define the log function by
$$\log(y) \triangleq \int_1^y \frac{1}{y}\, dy$$
for $y > 0$, and define e^x be the inverse function of $\log(y)$. That is, given x_0, let e^{x_0} be the number y_0 such that
$$x_0 = \int_1^{y_0} \frac{1}{y}\, dy.$$
The number y_0 is uniquely determined because the function $log(y)$ is monotonic.

- Given a real number x, define e^x by power series

$$e^x \triangleq \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

All three definitions are equivalent. We will use the third one to extend e^x to complex numbers. The definition of complex exponential function using power series is adopted in more advanced texts such as [Rudin2] and [Ahlfors].

Definition 9.1. Given a complex number z, define e^z by power series

$$e^z \triangleq \exp(z) \triangleq \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots .$$

It is obvious that when z is a real number the above definition reduces to the real exponential function. It is also easy to see $\exp(0) = 1$. However it is not obvious from this definition that e^z is never equal to zero.

Remark. In this lecture we first neglect all the convergence issue. Assume that the series are convergent for all z, and furthermore, assume that the convergence is absolute. This is analogous to the real power series. Moreover, we will assume in this lecture that the convergence is uniform, so that we can re-arrange the order of terms and compute derivative term-wise. We shall return to these questions in the next lecture.

We establish below a fundamental property of the function $\exp(z)$.

Theorem 9.2. *For any complex numbers z_1 and z_2,*

$$\exp(z_1 + z_2) = \exp(z_1)\exp(z_2).$$

Proof. We use a fact about product of infinite series: Given two absolutely convergent series $\sum_k a_k$ and $\sum_k b_k$, the product $(\sum_k a_k)(\sum_k b_k)$ is equal to $\sum_k c_k$, where c_k is defined by the convolution

$$c_k \triangleq a_0 b_k + a_1 b_{k-1} + a_2 b_{k-2} + \cdots + a_{k-1} b_1 + a_k b_0.$$

Apply this fact to $a_k = z_1^k/k!$ and $b_k = z_2^k/k!$. For $k \geq 0$, the coefficient c_k is

$$\begin{aligned} c_k &= \sum_{\ell=0}^{k} \frac{z_1^\ell}{\ell!} \frac{z_2^{k-\ell}}{(k-\ell)!} \\ &= \frac{1}{k!} \sum_{\ell=0}^{k} \binom{k}{\ell} z_1^k z_2^{k-\ell} \\ &= \frac{1}{k!}(z_1 + z_2)^k. \end{aligned}$$

Hence

$$\exp(z_1)\exp(z_2) = \sum_{n=1}^{\infty} \frac{(z_1+z_2)^n}{n!} \triangleq \exp(z_1+z_2).$$

□

Using this theorem we can derive some immediate corollaries.

Theorem 9.3. *For any $z \in \mathbb{C}$,*

- $e^{-z} = (e^z)^{-1}$*;*
- $e^z \neq 0$*;*
- $e^{nz} = (e^z)^n$ *for integer n.*

Proof. In the previous theorem, let $z_1 = z$ and $z_2 = -z$,

$$e^z e^{-z} = e^{z-z} = e^0 = 1.$$

Therefore $(e^z)^{-1} = e^{-z}$. We see that the value of e^z cannot equal to 0 because it is the reciprocal of some complex number.

For positive integer n, we can prove $e^{nz} = (e^z)^n$ by repeatedly applying the previous theorem. For negative integer n, we apply the first part in this theorem. □

Definition 9.4. Let the complex sine and complex cosine function be defined by

$$\begin{aligned} \sin(z) &\triangleq \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \\ \cos(z) &\triangleq \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}. \end{aligned}$$

Since the coefficients of the sine and cosine power series are the same as in the real case, $\sin(z)$ and $\cos(z)$ reduce to the real sine and cosine function when z is a real number. The main relationship between the exponential and the sinusoidal functions is recorded in the next theorem.

Theorem 9.5 (Euler's formula)**.**

$$e^{iz} = \cos(z) + i\sin(z). \tag{9.1}$$

Proof. The power series expansion of e^{iz} is

$$e^{iz} = 1 + iz - \frac{z^2}{2!} - \frac{iz^3}{3!} + \frac{z^4}{4!} + \frac{iz^5}{5!} - \frac{z^6}{6!} + \cdots$$

Assuming that we can exchange the order of adding the terms in the powers, we can separate the terms without i and the terms with i. By direct comparison, we can re-arrange the sum to $\cos(z) + i\cos(z)$. □

In particular when z is a real number ϕ, we have

$$e^{i\phi} = \cos\phi + i\sin\phi. \tag{9.2}$$

Using Theorem 9.5, we can express sin and cos in terms of exp.

Theorem 9.6. *For any $z \in \mathbb{C}$,*

$$\begin{aligned}\cos(z) &= \frac{e^{iz} + e^{iz}}{2} \\ \sin(z) &= \frac{e^{iz} - e^{iz}}{2i}.\end{aligned}$$

Proof. We use the properties that $\cos(z)$ is an even function and $\sin(z)$ is an odd function. This can be easily seen because the complex cosine function is even because all the terms in the power series that defines $\cos(z)$ in Def. 9.4 have even power. The terms in the power series that defines $\sin(z)$ all have odd powers. Hence

$$e^{-iz} = \cos(z) - i\sin(z) \tag{9.3}$$

By adding (9.3) to (9.1), we get

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

after some re-arrangement of terms. Similarly, by subtracting (9.3) from (9.1), we get

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}.$$

□

The result in Theorem 9.6 holds for any complex numbers. In particular, when we restrict z to a real number ϕ, we can get

$$\cos(\phi) = \frac{e^{i\phi} + e^{i\phi}}{2}$$
$$\sin(\phi) = \frac{e^{i\phi} - e^{i\phi}}{2i}.$$

We end this lecture by showing that the definition of exp by power series is the same as that given in previous lecture.

Theorem 9.7. *For any complex number $z = x + iy$,*

$$e^{x+iy} = e^x(\cos y + i \sin y).$$

Proof. Apply Theorem 9.2 to $z_1 = x$ and $z_2 = iy$,

$$e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y).$$

The last equality follows from (9.2). □

10 Lecture 10 (Convergence of complex power series)

Summary

- Ratio test
- Region of convergence

- Root test / Hadamard formula for radius of convergence

In Definition 9.1 and 9.4 we define complex exponential function and sinusoidal functions by power series,but the issue of convergence is neglected in Lecture 9. We discuss some convergence criteria in this lecture.

A power series centered at $z_0 \in \mathbb{C}$ has the form

$$\sum_{k=0}^{k} a_k (z - z_0)^k,$$

where a_k's are the coefficients. The coefficients may take complex values in general. For the ease of notation, we assume $z_0 = 0$ in this lecture.

Theorem 10.1 (Limit ratio test). *Consider a sequence of complex number $(b_k)_{k=1}^{\infty}$. Suppose*

$$\lim_{k\to\infty} \left| \frac{b_{k+1}}{b_k} \right| = L.$$

- *If $L > 1$, then $\sum_k b_k$ is divergent.*
- *If $L < 1$, then $\sum_k b_k$ is convergent.*
- *If $L = 1$, there is no conclusion.*

Proof. First suppose $L > 1$. There exists a sufficiently large N such that

$$\frac{|b_{k+1}|}{|b_k|} > 1$$

for all $k \geq N$. In particular, we have

$$|b_N| < |b_{N+1}| < |b_{N+2}| < \cdots$$

By the n-th term test, the series $\sum_k b_k$ must be divergent.

Suppose $L < 1$. Let M be any number such that $L < M < 1$. There exists a sufficiently large N such that

$$\frac{|b_{k+1}|}{|b_k|} < M$$

for all $k \geq N$. This gives

$$\begin{aligned} |b_{N+1}| &< |b_N| M \\ |b_{N+2}| &< |b_{N+1}| M < |b_N| M^2 \\ &\vdots \quad \vdots \end{aligned}$$

In general, we have $|b_{N+k}| < |b_N|M^k$, for $k \geq 1$. By comparing with $\sum_{k=N}^{\infty} |b_N|M^{k-N}$, which is a convergent geometric series, the power series $\sum_{k=N}^{\infty} |b_k|$ is convergent. Hence $\sum_{k=0}^{\infty} b_k$ converges absolutely. $\square$

Example 10.1. $\sum_{n=0}^{\infty} n!z^n$ diverges for all $z \neq 0$, because for any $z \neq 0$, the ratio

$$\left|\frac{(n+1)!z^{n+1}}{n!z^n}\right| = (n+1)|z|$$

diverges to ∞ as $n \to \infty$.

Example 10.2. $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges for all $z \in \mathbb{C}$. We can apply the ratio test by calculating

$$\frac{\frac{z^n}{(n+1)!}}{\frac{z^n}{n!}} = \frac{|z|}{n+1}.$$

For any fixed $z \in \mathbb{C}$, this ratio approach 0 as $n \to \infty$. Hence it converges for any z.

Example 10.3. Consider the series

$$\sum_{n=0}^{\infty} \frac{1}{2n+1} z^n.$$

The ratio of the moduli of two consecutive terms is

$$\frac{2n-1}{2n+1}|z|.$$

It converge to a complex number with modulus strictly less 1 if and only if $|z| < 1$. Hence the series converges inside the unit circle.

Theorem 10.2. *Suppose $\sum_{k=0}^{\infty} a_k z^k$ converges at $z_1 \neq 0$, then it converges for all z with $|z| < |z_1|$.*

Proof. Fix $\epsilon > 0$. Since $|a_k z_1^k| \to 0$ as $k \to \infty$, there is a sufficiently large N such that

$$|a_k z_1^k| < \epsilon, \qquad \forall k \geq N.$$

Hence, for all $k \geq 0$, we have

$$|a_k z_1^k| \leq \max(\epsilon, |a_0|, |a_1 z_1|, |a_2 z_2^2|, \ldots, |a_{N-1} z_1^{N-1}|) \triangleq C.$$

For each k, re-write $|a_k z^k|$ as

$$|a_k z^k| = |a_k z_1^k| \frac{|z|^k}{|z_1|^k} \leq C\rho^k,$$

where ρ is defined as $\rho \triangleq |z/z_1| < 1$. Because $\sum_k C\rho^k$ is a convergent geometric series, we can apply comparison test and conclude that $\sum_k a_k z^k$ is convergent absolutely for $|z| < |z_1|$. □

Corollary 10.3. *If $\sum_k a_k z^k$ diverges at z_1, then $\sum_k a_k z^k$ diverges whenever $|z| > |z_1|$.*

Proof. We prove by contradiction. Suppose $\sum_k a_k z^k$ converges for some point $z = z_2$ with modulus $|z_2|$ strictly larger than $|z_1|$. By the previous theorem, the power series must converge (absolutely) at $z = z_1$. This contradicts the assumption that $\sum_k a_k z^k$ diverges at z_1. □

Radius of convergence

Given a power series $\sum_k a_k z^k$, there are only two logical possibilities: it diverges for all $z \neq 0$, or it converges for some $z \neq 0$. In the former case, we have divergence on the whole complex plane, and we do not need to consider it as far as convergence is concerned.

Suppose $\sum_k a_k z^k$ converges at some point $z_1 \in \mathbb{C}$ other than the origin. Let

$$R \triangleq \sup \left\{ |z| : \sum_k a_k z^k \text{ converges } \right\} \tag{10.1}$$

We have $R > 0$. By Theorem 10.2 and its corollary,

$$\sum_k a_k z^k \text{ converges whenever } |z| < R,$$
$$\sum_k a_k z^k \text{ diverges whenever } |z| > R.$$

Notation: If $\sum_k a_k z^k$ converges for all z, then we write $R = \infty$, and say that the radius of convergence is infinite. If $\sum_k a_k z^k$ converges only at $z = 0$, then $R = 0$.

Definition 10.4. Given a power series $\sum_{k=0}^{\infty} a_k (z - z_0)^k$ with center z_0, the value of R in (10.1) is called the radius of convergence of $\sum_{k=0}^{\infty} a_k (z - z_0)^k$. The region $|z - z_0| < R$ is called the region of convergence.

Example 10.4. The complex functions $\exp(z)$, $\sin(z)$ and $\cos(z)$ all have radius of convergence $R = \infty$. The radius of convergence in Example 10.3 is 1.

Theorem 10.5 (Hadamard formula for radius of convergence)**.** *Given a complex power series* $\sum_{k=0}^{\infty} a_k z^k$*, the radius of convergence can be computed by*

$$R = \frac{1}{\limsup_k |a_k|^{1/k}}.$$

Proof. Suppose z is a complex number with $|z| < R$. Pick a real number ρ between $|z|$ and R, i.e., $|z| < \rho < R$. By the definition of R in the theorem,

$$\frac{1}{\rho} > \limsup |a_k|^{1/k}.$$

By the property of limsup, there exists a sufficiently large N such that for all $k \geq N$,

$$|a_k|^{1/k} < 1/\rho \qquad \Rightarrow \qquad |a_k| < 1/\rho^k.$$

Apply this upper bound to get

$$|a_k z_k| < \frac{|z|^k}{\rho^k}.$$

Since $|z|^k/\rho^k < 1$, the geometric series $\sum_k |z|^k/\rho^k$ is convergent. By comparison test, $\sum_k a_k z^k$ is convergent.

Now suppose that $|z| > R$. Pick a real number ρ such that $|z| > \rho > R$. Taking reciprocal,

$$\frac{1}{\rho} < \limsup |a_k|^{1/k}.$$

By the defining property of limsup,

$$\frac{1}{\rho} < |a_k|^{1/k} \quad \text{for infinitely many } k.$$

Hence $|a_k z^k| > \left|\frac{z^k}{\rho^k}\right| > 1$ for infinitely many k. By the nth-term test, the power series diverges when $|z| > R$. □

As in real analysis, absolute convergence of an infinite series implies that we can rearranging the order of summation [Rudin1, Theorem 3.55]. The same is true for absolutely convergent complex series. We state this property formally below

Theorem 10.6. *Let $\sum_{k=0}^{\infty} w_k$ be an absolutely convergent complex power series. For any bijective mapping b from $\{0, 1, 2, \ldots\}$ to itself, the series $\sum_{k=0}^{\infty} c_{b(k)}(z - z_0)^{b(k)}$ obtained by re-arranging the order of summation is convergent and*

$$\sum_{k=0}^{\infty} c_{b(k)}(z - z_0)^{b(k)} = \sum_{k=0}^{\infty} c_k (z - z_0)^k.$$

Proof. The proof is basically the same for the real case and is omitted. □

We now give a complete proof of the Euler's formula in Theorem 9.5.

Proof of Theorem 9.5. In view of Theorem 10.6, we just need to verify that $\sum_n (iz)^n/n!$ is absolutely convergent. Given any $z \in \mathbb{C}$, let z_1 be any complex number with modulus strictly larger than $|z|$. Since the radius of convergence of the complex exponential function is infinity, $\sum_n z_1^n/n!$ is convergent. By Theorem 10.2, the series $\sum_n (iz)^n/n!$ converges absolutely, and hence we can re-arrange the order of summation in $\sum_n (iz)^n/n!$ to get

$$\begin{aligned} e^{iz} &\triangleq \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!} + i \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!} \\ &= \cos(z) + i \sin(z). \end{aligned}$$

□

11 Lecture 11 (Properties of power series)

Summary

- Uniform convergence
- The uniqueness of coefficients of power series
- Term-wise differentiation of power series
- Analyticity of power series

Definition 11.1. An infinite series of functions $\sum_{k=0}^{\infty} f_k(z)$ is said to converge uniformly to $g(z)$ in a region $R \subseteq \mathbb{C}$ if for all $\epsilon > 0$, there exists $N(\epsilon)$ such that

$$\Big|\sum_{k=0}^{n} f_k(z) - g(z)\Big| < \epsilon$$

for all $n \geq N(\epsilon)$ and for all $z \in R$. The number $N(\epsilon)$ could be a function of ϵ but should not depend on z.

The following theorem gives a sufficient condition for uniform convergence.

Theorem 11.2. *Suppose a complex power series $\sum_{k=0}^{\infty} c_k(z-z_0)^k$ converges absolutely at $z = z_1$, then $\sum_{k=0}^{\infty} c_k(z-z_0)^k$ converges uniformly in the closed disc $S: |z-z_0| \leq |z_1 - z_0|$.*

It is tacitly assumed that z_1 is in the interior of the region of convergence. Very often when we talk about uniform convergence, the region in which the function is converging uniformly is a compact set. We note that the region S in the theorem is compact.

Proof. Let $g(z) \triangleq \sum_k c_k(z-z_0)^k$ be the limit function. From Theorem 10.2 in Lecture 10, $g(z)$ converges (absolutely) in $|z-z_0| < |z_1 - z_0|$. We repeat the same proof idea to show that it is also convergent on the boundary of S.

$$\begin{aligned}\sum_{k=0}^{\infty} |c_k||z-z_0|^k &= \sum_{k=0}^{\infty} |c_k||z_1-z_0|^k \frac{|z-z_0|^k}{|z_1-z_0|^k} \\ &\leq \sum_{k=0}^{\infty} |c_k||z_1-z_0|^k.\end{aligned}$$

In the last step we use the assumption that $z \in S$. Since it is assumed that $\sum_k c_k(z-z_0)^k$ is absolutely convergent at $z = z_1$, the series $\sum_{k=0}^{\infty} |c_k||z_1-z_0|^k$ is finite and hence by comparison test, $\sum_{k=0}^{\infty} c_k(z_1-z_0)^k$ is absolutely convergent for all $z \in S$.

For $k \geq 0$, let $M_k \triangleq |c_k||z_1-z_0|^k$. By assumption, $\sum_{k=0}^{\infty} M_k$ converges. Given any $\epsilon > 0$, there exists an integer $N(\epsilon)$ such that

$$\sum_{k=n+1}^{\infty} M_k \leq \epsilon, \qquad \text{for all } n \geq N(\epsilon).$$

So, for this choice of $N(\epsilon)$, we have

$$\begin{aligned}\Big|\sum_{k=0}^{\infty} c_k(z-z_0)^k - g(z)\Big| &= \Big|\sum_{k=n+1}^{\infty} c_k(z-z_0)^k\Big| \\ &\le \sum_{k=n+1}^{\infty} |c_k||z-z_0|^k \\ &\le \sum_{k=n+1}^{\infty} M_k \le \epsilon\end{aligned}$$

for all $z \in S$. This proves that $\sum_k c_k(z-z_0)^k$ is uniformly convergent to $g(z)$ for $z \in S$. $\square$

Similar to power series with real variable, complex power series has many nice properties, including

(i) term-wise differentiability from uniform convergence [Rudin1, Theorem 7.17],

(ii) exchanging the order of limit and summation [Rudin1, Theorem 7.11].

The proof of in the complex case is similar to the real case, and will not be repeated in the lecture note. In property (ii), Theorem 7.11 in [Rudin1] requires uniform convergence of power series. For term-wise differentiation, we need the following property.

Theorem 11.3. *Given a power series $\sum_n c_n(z-z_0)^n$, the series $\sum_n nc_n(z-z_0)^{n-1}$ obtained by term-wise differentiation has the same radius of convergence as $\sum_n c_n(z-z_0)^n$.*

Proof. By Theorem 10.5, the radius of convergence of $\sum_n c_n(z-z_0)^n$ is given by

$$R = \frac{1}{\limsup_n |c_n|^{1/n}}.$$

We want to show that the radius of convergence of $\sum_n nc_n(z-z_0)^{n-1}$ is R.

To simplify calculations, we note that the radius of convergence of $\sum_n nc_n(z-z_0)^{n-1}$ is the same as the radius of convergence of $\sum_n nc_n(z-z_0)^n$, and the latter can be computed by

$$\frac{1}{\limsup_n |nc_n|^{1/n}} = \frac{1}{\limsup_n \sqrt[n]{n}|c_n|^{1/n}} = \frac{1}{\limsup_n |c_n|^{1/n}} = R.$$

In the second equality, we have used the fact that $\lim_{n\to\infty} \sqrt[n]{n} = 1$. $\square$

From Theorem 11.3 and Theorem 10.2, we obtain the following

Theorem 11.4 (Analyticity of power series)**.** *Let R be the radius of convergence of a complex power series $\sum_n c_n(z-z_0)^n$. The function $f(z)$ defined by the power series $\sum_n c_n(z-z_0)^n$ is complex differentiable at every point in the open disc $D(z_0;R)=\{z\in\mathbb{C}: |z-z_0|<R\}$, i.e., it is analytic in the region of convergence.*

Proof. Let ρ to be a positive real number strictly less than R (in case $R=\infty$, we just can pick any positive real number). Pick two complex numbers z_1 and z_2 such that

$$\rho=|z_1-z_0|<|z_2-z_0|<R.$$

By Theorem 11.3, the power series $\sum_n nc_n(z_2-z_0)^{n-1}$ converges (because z_2 is inside the region of convergence). By Theorem 10.2, $\sum_n nc_n(z_1-z_0)^{n-1}$ converges absolutely at $z=z_1$, and by Theorem 11.2, $\sum_n nc_n(z-z_0)^{n-1}$ converges uniformly in the $D(z_0,\rho)$. We can now adapt the proof of Theorem 7.17 of [Rudin1] to conclude that

$$f'(z)=\sum_{n=0}^{\infty} nc_n(z-z_0)^{n-1}. \tag{11.1}$$

Since this holds for any z in the open disc $D(z_0;\rho)$ and ρ can be arbitrarily close to R, (11.1) holds for all z in the open disc $D(z_0;R)$. □

By repeatedly apply Theorem 11.4, we can differentiate arbitrarily many times.

Corollary 11.5. *For any positive integer j, the function $f(z)$ defined by a power series $\sum_{n=0}^{\infty} c_n(z-z_0)^n$ can be differentiated j-th time. The j-th derivative is*

$$f^{(j)}(z)=\sum_{n=j}^{\infty} n(n-1)(n-2)\cdots(n-j+1)c_n(z-z_0)^{n-j},$$

and the region of convergence is the same as that of $\sum_{n=0}^{\infty} c_n(z-z_0)^n$.

Theorem 11.6 (Uniqueness of coefficients)**.** *Suppose $(z_j)_{j=1}^{\infty}$ is a sequence of complex numbers converging to z_0 (with $z_j\neq z_0$ for all j). If two power series $\sum_{k=0}^{\infty} a_k(z-z_0)$ and $\sum_{k=0}^{\infty} b_k(z-z_0)$ takes the same values at $z=z_j$, for $j=0,1,2,\ldots$, then $a_k=b_k$ for all k.*

Proof. By assumption we have $\sum_{k=0}^{\infty} a_k(z_j-z_0)$ and $\sum_{k=0}^{\infty} b_k(z_j-z_0)$ for $j=1,2,3,\ldots$.

Take limit as $j \to \infty$ and exchange the order of limit and summation,

$$\begin{aligned}
\lim_{j\to\infty} \sum_{k=0}^{\infty} a_k (z_j - z_0)^k &= \lim_{j\to\infty} \sum_{k=0}^{\infty} b_k (z_j - z_0)^k \\
\sum_{k=0}^{\infty} a_k \lim_{j\to\infty} (z_j - z_0)^k &= \sum_{k=0}^{\infty} b_k \lim_{j\to\infty} (z_j - z_0)^k \\
\Rightarrow \qquad a_0 &= b_0.
\end{aligned}$$

Subtract a_0 and b_0 from both sides, we get

$$(z_j - z_0) \sum_{k=1}^{\infty} a_k (z_j - z_0)^{k-1} = (z_j - z_0) \sum_{k=1}^{\infty} b_k (z_j - z_0)^{k-1}$$

for all $j = 1, 2, \ldots$. Since $z_j \neq z_0$ for all j, we can repeat the same argument as in the previous paragraph to

$$\sum_{k=0}^{\infty} a_{k+1} (z_j - z_0)^k = \sum_{k=0}^{\infty} b_{k+1} (z_j - z_0)^k$$

and obtain $a_1 = b_1$. The theorem can then be proved by induction. □

Corollary 11.7. *If $\sum_{k=0}^{\infty} a_k (z - z_0)^k = \sum_{k=0}^{\infty} b_k (z - z_0)^k$ for all z in an open disc $D(z_0; r)$ with center z_0 and radius r (where r is smaller than the radius of convergence), then $a_k = b_k$ for all k.*

Proof. For $j = 1, 2, 3, \ldots$, take z_j to be any complex number one the circle $C(z_0; r/j)$ with center z_0 and radius r/j, and apply the previous theorem. □

From this result we also see that a power series with nonzero coefficients and positive radius of convergence defines a nonzero analytic function.

Corollary 11.8. *If $\sum_{k=0}^{\infty} a_k (z - z_0)^k = 0$ for all z in an open disc $D(z_0; r)$ with center z_0 and radius r, then $a_k = 0$ for all k.*

We illustrate an application of the uniqueness of coefficients by solving a famous recurrence relation.

Example 11.1. Solve the recurrence relation

$$x_{n+2} = x_{n+1} + x_n$$

for $n \geq 0$ with a given initial condition $x_0 = 1$ and $x_1 = 1$. The solutions are the Fibonacci numbers $F_0 = 1,\ F_1 = 1,\ F_2 = 2,\ F_3 = 3,\ F_4 = 5,\ F_5 = 8, \ldots$. The purpose of this example is to find a closed form expression for the Fibonacci numbers, using the method of generating function.

Let $g(z)$ be a function defined by a power series whose coefficients are the Fibonacci numbers,

$$g(z) \triangleq \sum_{n=0}^{\infty} F_n z^n.$$

The radius of convergence is positive because F_n's grow slower than 2^n. Hence $g(1/2)$ is a convergent series. The radius of convergence is at least equal to $1/2$. From

$$\begin{aligned} g(z) &= F_0 + F_1 z + F_2 z^2 + F_3 z^3 + F_4 z^4 + \cdots \\ zg(z) &= \phantom{F_0 + {}} F_0 z + F_1 z^2 + F_2 z^3 + F_3 z^4 + \cdots \\ z^2 g(z) &= \phantom{F_0 + F_1 z + {}} F_0 z^2 + F_1 z^3 + F_2 z^4 + \cdots \end{aligned}$$

we can get

$$g(z) - zg(z) - z^2 g(z) = F_0 + (F_1 - F_0) z = 1.$$

Therefore, as a function, $g(z)$ can be computed by

$$g(z) = \frac{1}{1 - z - z^2}$$

for z in a sufficient small disk $D(0; r)$ (we can take $r = 1/2$ in this example).

We next apply the method of partial fraction to expand $g(z)$ as

$$g(z) = \frac{A}{1 - \phi z} + \frac{B}{1 - \mu z}$$

where A and B are some numbers, and ϕ and μ are the reciprocal roots of $1 - z - z^2$. It is more convenient to consider the "reciprocal polynomial" $w^2 - w - 1$. The roots are $(1 \pm \sqrt{5})/2$. Let $\phi = (1 + \sqrt{5})/2$ and $\mu = (1 - \sqrt{5})/2$. Using linear algebra, we can find $A = \phi/\sqrt{5}$ and $B = -\mu/\sqrt{5}$ by solving a 2×2 linear system. We thus get

$$g(z) = \frac{\phi}{\sqrt{5}} \frac{1}{1 - \phi z} - \frac{\mu}{\sqrt{5}} \frac{1}{1 - \mu z}.$$

Expand both term by geometric series,

$$g(z) = \frac{\phi}{\sqrt{5}} \sum_{n \geq 0} \phi^n z^n - \frac{\mu}{\sqrt{5}} \sum_{n \geq 0} \mu^n z^n.$$

By the uniqueness of coefficients, we can equate the n-th coefficients with F_n,

$$F_n = \frac{1}{\sqrt{5}} \Big((\frac{1+\sqrt{5}}{2})^{n+1} - (\frac{1-\sqrt{5}}{2})^{n+1} \Big).$$

Since $\frac{1-\sqrt{5}}{2}$ has absolute value strictly less than 1, the second term converges to zero exponentially fast. We can see that the ratio of two consecutive Fibonacci numbers approaches the limit ϕ, which is the Golden ratio.

12 Lecture 12 (Contour integral)

Summary

- Definition of integral of complex function of a real variable
- Definition of contour integral
- Independence of parameterization

Definition 12.1. Given a continuous function $f : \mathbb{R} \to \mathbb{C}$ mapping real numbers to complex numbers, with $u(t)$ and $v(t)$ as the real and imaginary parts,

$$f(t) = u(t) + iv(t)$$

for $t \in [a, b]$, we define the integral of f as

$$\int_a^b f(t)\, dt \triangleq \int_a^b u(t)\, dt + i \int_a^b v(t)\, dt.$$

The real and imaginary parts are the limits of Riemann sums. Given a partition of $[a, b]$ into subintervals $[t_{k-1}, t_k]$, for $k = 1, 2, \ldots, n$, with

$$a = t_0 < t_1 < t_2 < \cdots < t_n = b,$$

the corresponding Riemann sums are

$$\sum_{k=1}^{n} u(t_k^*)(t_k - t_{k-1}), \quad \text{and} \quad \sum_{k=1}^{n} v(t_k^*)(t_k - t_{k-1}),$$

where t_k^* is any point in the subinterval $[t_{k-1}, t_k]$. The limit is taken with $n \to \infty$ and $\max_k(t_k - t_{k-1}) \to 0$.

Example 12.1.

$$\int_0^1 e^{\pi i t}\, dt = \int_0^1 \cos(\pi t)\, dt + i \int_0^1 \sin(\pi t)\, dt = \frac{2i}{\pi}.$$

Review of work integral and flow integral

Given a vector field $\mathbf{F}(x, y) = (M(x, y), N(x, y))$, the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C M\, dx + N\, dy.$$

over a curve C has the physical interpretation of work done. It is the work done by the force $\mathbf{F}$ on a particle moving along the curve C.

On the other hand, if the vector field $\mathbf{F}(x, y) = (M(x, y), N(x, y))$ is regarded as the flow of some fluid, then the flux through a curve C is given by

$$\int_C \mathbf{F} \cdot \hat{\mathbf{n}}\, ds = \int_C M\, dy - N\, dx.$$

Here $\hat{\mathbf{n}}$ signifies a unit normal vector along the curve.

Definition 12.2. A parametric a curve C represented by

$$z : [a, b] \to \mathbb{C}$$
$$z(t) = x(t) + iy(t)$$

is said to be smooth if

(i) $x(t)$ and $y(t)$ are continuous on $[a, b]$,

(ii) $x(t)$ and $y(t)$ are differentiable on $[a, b]$, and

(iii) the vector $(x'(t), y'(t))$ is not equal to the zero vector for $t \in [a, b]$.

We note that the condition in (iii) means that the tangent vector is well-defined for all points on the curve. We write $z'(t) = x'(t) + iy'(t)$ as a tangent vector at t. To emphasize the direction/orientation of the curve from $t = a$ to $t = b$, we sometime use the notation "contour". If a curve can be divided into infinitely many parts and each part is smooth, then we call it a piece-wise smooth curve.

Remark. In the rest of the lecture notes, we shall not distinguish smooth curve and piece-wise smooth curve. All piece-wise smooth curve can be treated as a concatenation of smooth curves. All results for smooth curves also hold for piece-wise smooth curve.

Definition 12.3. Given a continuous complex-valued function $f : \mathbb{C} \to \mathbb{C}$ and a smooth curve C, the complex integral (or contour integral) of f over C is defined as

$$\int_C f(z)\,dz \triangleq \int_a^b f(z(t)) \cdot z'(t)\,dt \tag{12.1}$$

with the right-hand side defined as in Definition 12.1.

It is important to note that the integral depends on the direction of the curve C.

Physical interpretation Given a complex function $f(z) = u(x, y) + iv(x, y)$, define a vector field $\mathbf{F}(x, y) = (M(x, y), N(x, y))$ by

$$M(x, y) = u(x, y) \quad \text{and} \quad N(x, y) = v(x, y).$$

Then

$$\begin{aligned}\int_C f(z)\,dz &= \int_C (u + iv)(x' + iy')dt \\ &= \int_C (u + iv)(dx + idy)\end{aligned}$$

After formally expanding the product in the integrand, we see that

$$\int_C f(z)\,dz = \int_C M\,dx + N\,dy) + i\int_C M\,dy - N\,dx.$$

The real and imaginary parts can be interpreted as the work integral and the flux integral, respectively.

In order for Definition 12.3 to make sense, we need to first show that the right-hand side of (12.1) does not depend on the parameterization of curve.

Suppose C has two parameterizations

$$z(t) \text{ for } a \le t \le b$$
$$w(t) \text{ for } c \le t \le d.$$

We can find a monotonically increasing $\lambda : [c, d] \to [a, b]$ such that

$$w(t) = z(\lambda(t)).$$

The main step is the chain rule $w'(t) = z'(\lambda(t))\lambda'(t)$.

$$\begin{aligned}
\int_c^d f(w(t))w'(t)\,dt &= \int_c^d f(z(\lambda(t))z'(\lambda(t))\lambda'(t)\,dt \\
&= \int_c^d [u(z(\lambda(t))) + iv(z(\lambda(t)))][x'(\lambda(t)) + iy'(\lambda(t))]\lambda'(t)\,dt \\
&= \int_c^d u(z(\lambda(t)))x'(\lambda(t)) - v(z(\lambda(t))y'(\lambda(t))\lambda'(t)\,dt \\
&\quad + i\int_c^d u(z(\lambda(t)))y'(\lambda(t)) + v(z(\lambda(t))x'(\lambda(t))\lambda'(t)\,dt.
\end{aligned}$$

Substitute $\tau = \lambda(t)$, $d\tau = \lambda'(t)\,dt$.

$$\begin{aligned}
\int_c^d f(w(t))w'(t)\,dt &= \int_a^b u(z(\tau))x'(\tau) - v(z(\tau))y'(\tau)\,d\tau + \\
&\quad + i\int_a^b u(z(\tau))y'(\tau) + v(z(\tau))x'(\tau)\,d\tau \\
&= \int_a^b f(z(\tau))z'(\tau)\,d\tau.
\end{aligned}$$

This proves that the definition of complex integral is independent of path.

Example 12.2. Integrate $f(z) = \bar{z}$ over the curve $C:\ z(t) = 0 + it$ for $0 \le t \le 2$.

$$\begin{aligned}
\int_C \bar{z}\,dz &= \int_0^2 (0 + it)^*(i)\,dt \\
&= \int_0^2 (-it)i\,dt \\
&= \left[\frac{t^2}{2}\right]_0^2 = 2.
\end{aligned}$$

Example 12.3. Integrate $f(z) = z^2$ over the curve $C: \; z(t) = \cos(t) + i\sin(t)$ for $0 \le t \le \pi$. The derivative of $z(t)$ is $z'(t) = -\sin(t) + i\cos(t)$.

$$\begin{aligned}
\int_C z^2\,dz &= \int_0^\pi (\cos t + i\sin t)^2(-\sin t + i\cos t)\,dt \\
&= i\int_0^\pi (\cos 2t + i\sin 2t)(\cos t + i\sin t)\,dt \\
&= i\int_0^\pi (\cos 3t + i\sin 3t)\,dt \\
&= i\Big[\frac{\sin 3t}{3} - i\frac{\cos 3t}{3}\Big]_0^\pi \\
&= \frac{2}{3}.
\end{aligned}$$

Definition 12.4. The negation of a curve C is the curve with the same locus with reverse direction. The negation of C is denoted by $-C$. Given two contours C_1 and C_2, with the end point of C_1 identical to the start point of C_2, then the concatenated contour (first traveling along C_1 and then along C_2) is denoted by $C_1 + C_2$.

Theorem 12.5. *For any continuous function $f(z)$ and $g(z)$, constant a, and smooth curves C, C_1 and C_2,*

$$\begin{aligned}
\int_{-C} f(z)\,dz &= -\int_C f(z)\,dz \\
\int_{C_1+C_2} f(z)\,dz &= \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz \\
\int_C f(z) + g(z)\,dz &= \int_C f(z)\,dz + \int_C g(z)\,dz \\
\int_C af(z)\,dz &= a\int_C f(z)\,dz.
\end{aligned}$$

The proof is omitted.

13 Lecture 13 (Independence of path)

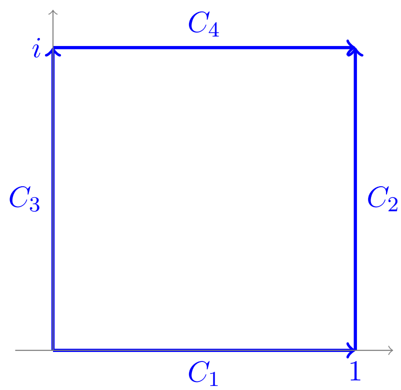
Summary

- Triangle inequality for complex integral

- ML inequality
- Fundamental theorem of calculus for complex functions

A motivating example.

Example 13.1. Consider the function $f(z) = x^2 + iyx$. Compute the complex integral from 0 to $1+i$ (i) from 0 to 1 and then from 1 to $1+i$, (ii) from 0 to i and then from i to $1+i$.



Let C_1, C_2, C_3 and C_4 be directed path as indicated above. Parameterize C_1 and C_2,

$$\begin{aligned} C_1 : z(x) &= x + 0i, \quad 0 \le x \le 1 \\ z'(x) &= 1, \\ C_2 : z(y) &= 1 + iy, \quad 0 \le y \le 1 \\ z'(x) &= i. \end{aligned}$$

The complex integral along $C_1 + C_2$ is

$$\begin{aligned} \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz &= \int_0^1 x^2\,dx + \int_0^1 (1+iy)i\,dy \\ &= \frac{1}{3} + i - \frac{1}{2} \\ &= i - \frac{1}{6}. \end{aligned}$$

Next, parameterize C_3 and C_4 as

$$\begin{aligned} C_3 : z(x) &= iy, \quad 0 \le y \le 1 \\ z'(y) &= i, \\ C_4 : z(x) &= x + i, \quad 0 \le x \le 1 \\ z'(x) &= 1. \end{aligned}$$

The complex integral along $C_3 + C_4$ is

$$\begin{aligned} \int_{C_3} f(z)\,dz + \int_{C_4} f(z)\,dz &= \int_0^1 0\,dy + \int_0^1 (x^2 + ix)\,dx \\ &= \frac{1}{3} + \frac{i}{2}. \end{aligned}$$

It is obvious that the two values obtained from the two different paths are different.

Theorem 13.1 (Triangle inequality for complex integral)**.** *Suppose $g(t)$ is a continuous complex function from $[a, b]$ to $\mathbb{C}$. Then*

$$\left| \int_a^b g(t)\,dt \right| \le \int_a^b |g(t)|\,dt \tag{13.1}$$

Proof. Let α denote the complex integral $\alpha \triangleq \int_a^b g(t)\,dt$. If $\alpha = 0$, then (13.1) is obviously true, because the left-hand side is zero. Hence, we can suppose $\alpha \ne 0$. Write α in polar form $\alpha = re^{i\theta}$. (θ is defined because $\alpha \ne 0$.)

Then,

$$e^{-i\theta} \int_a^b g(t)\,dt = r$$

is a real number. We can re-write it as

$$\int_a^b e^{-i\theta} g(t)\,dt.$$

Let $u(t)$ and $v(t)$ be the real and imaginary parts of $e^{-i\theta} g(t)$, respectively. We get

$$\int_a^b u(t)\,dt = r, \quad \int_a^b v(t)\,dt = 0.$$

But

$$\begin{aligned} u(t) &\le \sqrt{u^2(t)} \\ &\le \sqrt{u^2(t)+v^2(t)} \\ &= |g(t)| \end{aligned}$$

By the monotonic property for real functions,

$$\Big|\int_a^b g(t)\,dt\Big| = r = \int_a^b u(t)\,dt \le \int_a^b |g(t)|\,dt.$$

$\square$

Example 13.2.

$$\Big|\int_a^b e^{it}\,dt\Big| \le \int_a^b |e^{it}|\,dt = \int_a^b \,dt = b-a.$$

Definition 13.2. The length of a smooth curve C, represented by $z(t)$, $a \le t \le b$, is defined as $\int_a^b |z'(t)|\,dt$.

Theorem 13.3 **(ML inequality).** *If $|f(z)| \le M$ for z on a smooth curve C and the length of C is equal to L, then*

$$\Big|\int_C f(z)\,dz\Big| \le ML.$$

Proof. Apply the triangle inequality to

$$\Big|\int_C f(z)\,dz\Big| = \Big|\int_a^b f(z(t))z'(t)\,dt\Big|.$$

This yields

$$\Big|\int_C f(z)\,dz\Big| \le \int_a^b |f(z(t))|\cdot|z'(t)|\,dt \le M\int_a^b |z'(t)|\,dt = ML.$$

$\square$

Theorem 13.4 **(Fundamental theorem of calculus for complex function).** *Suppose $f(z)$ is a continuous complex function in a region R. Then the followings are equivalent:*

(i) $f(z)$ is the derivative of a function $F(z)$ in R; (The function $F(z)$ is necessarily analytic)

(ii) for any smooth contour C in R from z_1 to z_2,

$$\int_C f(z)\,dz = F(z_2) - F(z_1).$$

Proof. ((i) $\Rightarrow$ (ii)) Suppose C is a smooth curve in R parameterized by $z(t)$ for $a \leq t \leq b$. We have $z(a) = z_1$ and $z(b) = z_2$.

Let $g(t) = F(z(t))$. By chain rule for complex functions,

$$g'(t) = F'(z(t))z'(t) = f(z(t))z'(t).$$

The problem then reduces to fundamental theorem of calculus for real functions,

$$\begin{aligned}\int_C f(z)\,dz &= \int_a^b f(z(t))z'(t)\,dt \\ &= \int_a^b g'(t)\,dt \\ &= g(b) - g(a) \\ &= F(z_2) - F(z_1).\end{aligned}$$

((ii) $\Rightarrow$ (i)) Fix a base point z_0 in the R. Define $F(z) \triangleq \int_C f(z)\,dz$, where C is a smooth path from z_0 to z. Since it is assumed that the integral is independent of path, the integral only depends on the start point and end point. We can write

$$F(z) = \int_{z_0}^{z} f(w)\,dw.$$

We want to show that $F'(z) = f(z)$.

Consider

$$\begin{aligned}\frac{F(z+h) - F(z)}{h} - f(z) &= \frac{1}{h}\int_z^{z+h} f(w)\,dw - f(z) \\ &= \frac{1}{h}\int_z^{z+h} (f(w) - f(z))\,dw.\end{aligned}$$

Because f is continuous at z, given any $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(w) - f(z)| < \epsilon \quad \text{whenever} \quad |w - z| < \delta.$$

Since the integral is independent of path, we can take the line segement from z to $z + h$ as the path. The length is equal to $|h|$. When $|h|$ is smaller than δ, the modulus of $f(w) - f(z)$ is upper bounded by ϵ. We then apply ML inequality (Theorem 13.3)to get an upper bound,

$$\Big|\frac{F(z+h) - F(z)}{h} - f(z)\Big| \le \frac{1}{|h|}\epsilon|h| \le \epsilon$$

for all $|h| \le \delta$. Hence,

$$\lim_{h \to 0} \frac{F(z+h) - F(z)}{h} = f(z).$$

□

Suppose C is a piece-wise smooth curve $C = C_1 + C_2 + \cdots + C_n$, so that C_k is a differentiable curve from z_{k-1} to z_k, for $k = 1, 2, \ldots, n$. To apply Theorem 13.4 for piece-wise smooth curve, we can write

$$\begin{aligned} \int_C f(z)\,dz &= \sum_{k=1}^{\infty} \int_{C_k} f(z)\,dz \\ &= \sum_{k=1}^{n} F(z_k) - F(z_{k-1}) \\ &= F(z_n) - F(z_0). \end{aligned}$$

Example 13.3. Compute $\int_C z^3 - z\,dz$ for a curve C from z_0 to z_1. Since $z^4/4 - z^2/2$ is an anti-derivative of $z^3 - z$ exists, we can compute the integral by

$$\int_C z^3 - z\,dz = \Big[\frac{z^4}{4} - \frac{z^2}{2}\Big]_{z_0}^{z_1} = \frac{z_1^4}{4} - \frac{z_1^2}{2} - \frac{z_0^4}{4} + \frac{z_0^2}{2}.$$

14 Lecture 14 (Cauchy-Goursat theorem)

Summary

- Closed curve
- Cauchy-Goursat theorem for rectangle

Definition 14.1. A smooth (or piece-wise smooth) curve is said to be closed if the start point is the same as the end point.

As in multi-variable calculus, independence of path is equivalent to the condition that the integral over any closed curve is zero.

Theorem 14.2. *$\int_C f\,dz$ is path independent for any piece-wise smooth curve C in a domain, if and only if $\oint_C f\,dz = 0$ for any closed curve C in the domain.*

The proof is the same as in multi-variable calculus and is omitted.

Definition 14.3. A curve is called simple if there is no self-intersection.

The figure-8 curve, which looks like ∞, is a typical example of non-simple curve.

Remark. The famous Jordan curve theorem says that a simple closed curve C divides the plane into two components. One component is bounded and the other is unbounded, and the curve is the boundary of each component. Actually, we only need to assume that the curve is continuous (not necessarily smooth) in the Jordan curve theorem.

We need the following theorem called "Cantor's intersection theorem" from point-set topology.

Theorem 14.4. *Suppose K_1, K_2, $K_3, \ldots$ are nonempty compact sets (in some topological space). If K_1, K_2, $K_3, \ldots$ is a decreasing sequence, i.e.,*

$$K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots$$

then $\cap_{j=1}^{\infty} K_j$ is not empty.

The main theorem in this lecture is the following

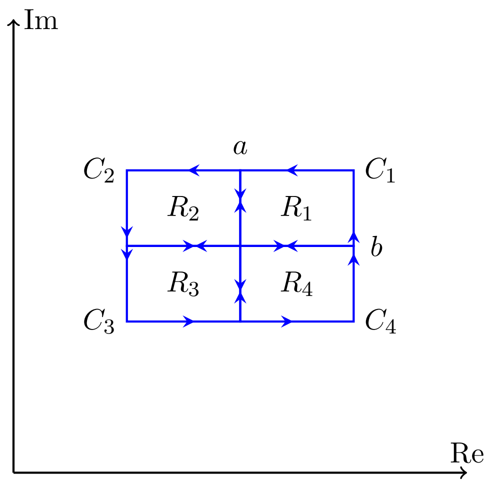
Theorem 14.5 (Cauchy-Goursat for rectangle)**.** *Suppose $f(z)$ is analytic in a domain D and R is a rectangle contained inside D, with sides parallel to the real and imaginary axes. If C is the boundary of R, Then*

$$\oint_C f(z)\,dz = 0.$$

Remark. The function $f(z)$ in Theorem 14.5 is analytic at all points in R. Actually, $f(z)$ is analytic in an open neighborhood containing R. It is important to note that we only need to assume the first-order derivative of $f(z)$ exists. We do not need to assume that $f'(z)$ is continuous.

Proof. We can assume that the orientation of C is counter-clockwise. Suppose the width and height of R are a and b, respectively.

Suppose $\int_C f(z)\,dz = I$. We want to show that $|I| = 0$.



Divide rectangle R into four equal parts, R_1, R_2, R_3 and R_4. Let C_i be the boundary of R_i, for $i = 1, 2, 3, 4$, all in the counter-clockwise direction. Due to cancellations of internal lines, we have

$$I = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \int_{C_3} f(z)\,dz + \int_{C_4} f(z)\,dz.$$

By triangle inequality,

$$|I| = \left|\int_{C_1} f(z)\,dz\right| + \left|\int_{C_2} f(z)\,dz\right| + \left|\int_{C_3} f(z)\,dz\right| + \left|\int_{C_4} f(z)\,dz\right|.$$

Suppose $R^{(1)}$ is the rectangle among R_1 to R_4 that has the largest integral in magnitude,

$$\left|\int_{\partial R^{(1)}} f(z)\,dz\right| = \max_{j=1,2,3,4} \left|\int_{C_j} f(z)\,dz\right|$$

(Here $\partial R^{(1)}$ means the boundary of $R^{(1)}$.)

This implies

$$\frac{|I|}{4} \leq \Big| \int_{\partial R^{(1)}} f(z)\, dz \Big|.$$

Recursively, for $k \geq 1$, divide $R^{(k)}$ into four equal parts, and let $R^{(k+1)}$ be the sub-rectangle that has the largest integral in absolute value. Thus,

$$R \supset R^{(1)} \supset R^{(2)} \supset R^{(3)} \supset R^{(4)} \supset \cdots$$

and

$$\frac{|I|}{4^k} \leq \Big| \int_{\partial R^{(k)}} f(z)\, dz \Big|$$

for $k = 1, 2, 3, \ldots$ The perimeter of $R^{(k)}$ is $L/2^k$, for $k \geq 1$.

By the Theorem 14.4, there is a point z_0 that lies inside $R^{(k)}$ for all k. Since f is assumed to be analytic in R, it is complex differentiable at the point z_0. Suppose that the derivative is $f'(z_0)$. For any $h \in \mathbb{C}$ such that $z + h \in R$, by the definition of complex derivative, we have

$$f(z_0 + h) = f(z_0) + f'(z_0)h + \epsilon h \tag{14.1}$$

where $|\epsilon| \to 0$ as $|h| \to 0$.

Put $z = z_0 + h$ in (14.1). For any $k = 1, 2, \ldots$, the distance between a point z on the boundary of $R^{(k)}$ and z_0 can be bounded by

$$\begin{aligned} |z - z_0| &\leq \text{ diagonal of } R^{(k)} \\ &\leq \sqrt{\Big(\frac{a}{2^k}\Big)^2 + \Big(\frac{b}{2^k}\Big)^2} \\ &\leq \frac{\max(a, b)\sqrt{2}}{2^k}. \end{aligned}$$

We now fix $\delta > 0$, and choose a sufficiently large integer k, such that $|\epsilon| < \delta$ for for all z on the boundary of $R^{(k)}$. This k certainly exists because $\frac{\max(a,b)\sqrt{2}}{2^k} \to 0$ as $k \to \infty$.

The integral of f over the boundary of $R^{(k)}$ is

$$\int_{\partial R^{(k)}} f(z)\, dz = \int_{\partial R^{(k)}} f(z_0) + f'(z_0)(z - z_0) + \epsilon(z - z_0)\, dz.$$

Because $f(z_0) + f'(z_0)(z - z_0)$ is a linear function in z, it has an anti-derivative. By Theorem 13.4, $\int_{\partial R^{(k)}} f(z_0) + f'(z_0)(z - z_0)\, dz$ is zero. So,

$$\int_{\partial R^{(k)}} f(z)\, dz = \int_{\partial R^{(k)}} \epsilon(z - z_0)\, dz.$$

To finish the proof, we upper bound $|I|$ using ML inequality (Theorem 13.3)

$$\begin{aligned}|I| &\le 4^k \Big| \int_{\partial R^{(k)}} \epsilon(z - z_0)\, dz \Big| \\ &\le 4^k \frac{\delta \max(a,b)\sqrt{2}}{2^k} \cdot \frac{2(a+b)}{2^k} = 2\sqrt{2}\max(a,b)(a+b)\delta.\end{aligned}$$

Here $2(a+b)$ is the perimeter of the original rectangle R. Since δ can be arbitrarily small, and a and b are constant, we conclude that $|I| = 0$. Hence $I = 0$. □

There are several versions of Cauchy theorem in the literature. Another version is for simple closed curve C.

Theorem 14.6. *If $f(z)$ is analytic in a domain D and C is a simple closed curve in D, such that $f'(z)$ exists in the interior of C (the interior is well-defined by the Jordan curve theorem), then*

$$\oint_C f(z)\, dz = 0.$$

In Theorem 14.6, the condition that $f'(z)$ exists in the interior of C is essential (compare with the remark after Theorem 14.5). The following is a counter-example.

Example 14.1. The function $f(z) = 1/z$ is analytic in the punctured plane $\mathbb{C}\backslash\{0\}$. Consider the circle C_r with radius r and center equal to the origin. The integral $\int_{C_r} 1/z\, dz = 2\pi i$, which is not zero. We can compute the integral using the definition of integral.

$$\begin{aligned}\int_{C_r} 1/z\, dz &= \int_0^{2\pi} r^{-1} e^{-i\theta}(rie^{i\theta})\, d\theta \\ &= \int_0^{2\pi} i\, d\theta \\ &= 2\pi i.\end{aligned}$$

However, if C is a circle in the complex plane that does not contain the origin, then $\int_C 1/z\, dz = 0$ by Theorem 14.6.

15 Lecture 15 (Closed-curve theorem)

Summary

- Existence of local primitive in a disc
- Closed curve theorem

The closed-curve theorem is the analog of Green's theorem in multi-variable calculus. A difference between the closed-curve theorem and Green's theorem is that we only need to assume that the function is complex differentiable in the closed-curved theorem, no need to assume continuity of derivative.

Definition 15.1. Given a complex function $f(z)$ defined on a domain D, a function $F(z)$ such that $F'(z) = f(z)$ for all points $z \in D$ is called a primitive or a primitive function, or an anti-derivative of $f(z)$.

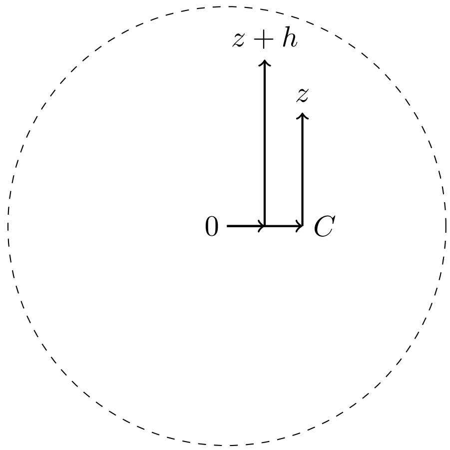
Example 14.1 indicates that in many cases it is impossible to have a primitive function on the largest domain on which the function $f(z)$ is defined. The anti-derivative of $1/z$ is the log function, which is multi-valued. We need to pick a branch cut to make the log function analytic. But once we make a branch cut, the domain is strictly smaller than the puncture plane $\mathbb{C} \setminus \{0\}$.

The best thing we can have is the existence of local primitive.

Theorem 15.2. *If $f(z)$ is analytic in an open disc, then f has a primitive in the open disc.*

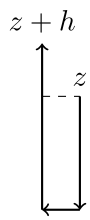
Proof. Without loss of generality assume that open disc is centered at the origin. For each z in the disc, consider a polygonal path C that starts at the origin, goes to the right until it reaches $\mathrm{Re}(z)$, and then moves up to the point z. Define a function $F(z)$ by

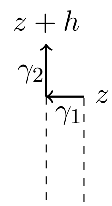
$$F(z) \triangleq \int_C f(w)\, dw.$$



We want to show that $F'(z) = f(z)$ for all z in the open disc.

Add a complex number h to z so that $z+h$ is inside the disc. The value $F(z+h)$ is computed by integrating on a piece-wise linear path from 0 to $\mathrm{Re}(z+h)$ and then from $\mathrm{Re}(z+h)$ to $z+h$. The difference $F(z+h) - F(z)$ is the integral with the path shown on the left below.





By Cauchy-Goursat theorem for rectangle, the integral of $f(z)$ along the rectangle is zero. Hence the path can be simplified to the path on the right, consisting of a horizontal path γ_1 and a upward vertical path γ_2. We can now write

$$F(z+h) - F(z) = \int_{\gamma_1+\gamma_2} f(z)\, dz.$$

Since $f(z)$ is analytic, it must be continuous at z,

$$\lim_{w\to z} f(w) = f(z).$$

Next we use the fact $\int_{\gamma_1+\gamma_2} f(z)\, dw = f(z)h$, to write

$$\Big|\frac{F(z+h)-F(z)}{h} - f(z)\Big| = \Big|\frac{1}{h}\int_{\gamma_1+\gamma_2} f(w)\, dw - \frac{1}{h}\int_{\gamma_1+\gamma_2} f(z)\, dw\Big|$$
$$= \frac{1}{|h|}\Big|\int_{\gamma_1+\gamma_2} f(w) - f(z)\, dw\Big|.$$

Fix an arbitrarily small and positive ϵ. Find a sufficiently small $\delta > 0$ such that $|f(w) - f(z)| < \epsilon$ for all w with $|w - z| < \delta$. (This is possible by the continuity of f at z.) Then for $|h| < \delta$, by ML inequality,

$$\Big|\frac{F(z+h)-F(z)}{h} - f(z)\Big| \leq \frac{1}{|h|}\epsilon \cdot (\text{ length of } \gamma_1 + \text{ length of } \gamma_2).$$

By the total length of γ_1 and γ_2 is less than $2|h|$ (because the length of each γ_1 and γ_2 is less than $|h|$). Therefore

$$\Big|\frac{F(z+h)-F(z)}{h} - f(z)\Big| \leq 2\epsilon.$$

Since ϵ is arbitrarily small, this proves that $\frac{F(z+h)-F(z)}{h}$ tends to $f(z)$ as $h \to 0$. □

Theorem 15.3 (Closed-curve theorem for open disc)**.** *If f is analytic in an open disc, then the integral of f over any closed curve C in the disc is zero.*

Proof. By the previous theorem, $f(z)$ has a primitive $F(z)$ in the open disc. By the fundamental theorem of calculus for complex function Theorem 13.4, we have $\int_C f(z)\, dz$ for all C that lies inside the disc. □

Remark. If f is entire, then the radius of the open disc in Theorem 15.3 can be taken to be infinity.

Remark. The closed-curve theorem also holds for other simple shapes such as triangles, semi-circle, parallelogram, or a sector of a circle.

When a function failed to be complex differentiable at some point in an open disc, then the conclusion in Theorem 15.3 may or may not hold. The following is an example

Example 15.1. Let C_r be a circle with radius r centered at the origin.

$$\int_{C_r} z^n\, dz = \begin{cases} 0 & \text{if } n = 0, 1, 2, 3, \dots, \\ 2\pi i & \text{if } n = -1, \\ 0 & \text{if } n = -2, -3, -4, \dots. \end{cases}$$

When $n \geq 0$, z^n has a primitive throughout $\mathbb{C}$. By Theorem 13.4, $\int_{C_r} z^n \, dz = 0$. When $n = -1$, the integral was evaluated in Example 14.1. When $n \leq -2$, the function z^n is not defined at the origin. However, $z^{n+1}/(n+1)$ is a primitive of z^n for z in the punctured plane $\mathbb{C} \setminus \{0\}$. By Theorem 13.4, $\int_{C_r} z^n \, dz = 0$.

Example 15.2. We can apply Theorem 15.2 to define a branch of the log function as a primitive function of $1/z$. Let D be the set

$$D = \mathbb{C} \setminus \{x + iy \in \mathbb{C} : x \leq 0, y = 0\}.$$

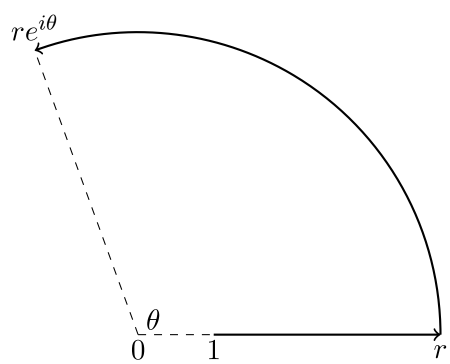
Take $z = 1$ to be a base point. Every point in D can be reached by a piece-wise linear path γ from 1 to $i\mathrm{Im}(z)$, and then from $i\mathrm{Im}(z)$ to z. The proof of Theorem 15.2 go through by defining the primitive of $1/z$ as

$$Log(z) \triangleq \int_\gamma \frac{1}{z} \, dz$$

for z in the domain defined above. Since a primitive exists, the integral is independent of path. We can write

$$Log(z) \triangleq \int_1^z \frac{1}{z} \, dz.$$

Suppose $z = re^{i\theta}$ with $-\pi < \theta < \pi$. The calculation of $Log(z)$ is easy when we take the path that first goes from 1 to r horizontally, and then travel along an arc from r to $re^{i\theta}$.



The integral from 1 to r is

$$\int_1^r \frac{1}{z} \, dz = \int_1^r \frac{1}{x} \, dx = \ln r.$$

The integral from r to $re^{i\theta}$ is

$$\begin{aligned}\int_r^{re^{i\theta}} \frac{1}{z}\,dz &= \int_0^\theta r^{-1}e^{-i\theta}(ire^{i\theta})\,d\theta \\ &= i\theta.\end{aligned}$$

Therefore the function

$$Log(z) = \ln(r) + i\theta$$

is a primitive of $1/z$ in the domain D. This the same as the function obtained from the inverse of the exponential function.

16 Lecture 16 (Cauchy integral formula)

Summary

- Cauchy theorem for multiply connected region
- Cauchy integral formula
- Taylor series expansion of analytic functions

Definition 16.1. A connected region is simply connected if every closed curve in the region can be continuously shrink to a single point, without leaving the region. A connected region that is not simply connected is said to be multiply connected.

A multiply connected region is a connected region with some "holes" inside. The "hole" is drawn in order to contain all points where the function is not defined.

A stronger version of Cauchy theorem is

Theorem 16.2. *Suppose D is a simply connected region and f is analytic in D. Then*

$$\oint_C f(z)\,dz = 0$$

for all closed and smooth curve C in D.

Proof sketch. We first pick a base point, say z_0, in D. For any other point z in D, we can connect z and z_0 by a piece-wise linear path C, with each piece parallel to either the real or imaginary axis. This can always be done because D is connected. Define a function $F(z)$

by $\int_C f(z)\,dz$. This is well-defined (does not depend on the choice of piece-wise linear path) because D is simply connected. Then we show as in Theorem 15.2 that $F(z)$ is a primitive of $f(z)$ in D. Therefore the integral of any closed curve in D is zero, by Theorem 14.2. $\square$

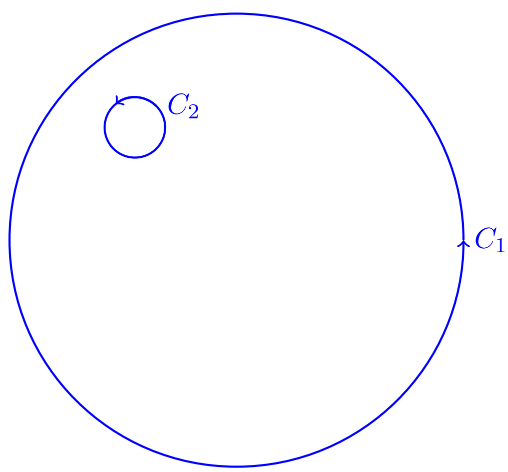
For multiply connected region, we can apply Cauchy theorem as follows.

Theorem 16.3. *Consider a complex function f that is analytic inside a domain D. Circles C_1 and C_2 are in D so that C_2 is contained inside the outer circle C_1, both with positive orientation. If $f(z)$ is analytic in the region between C_1 and C_2, then*

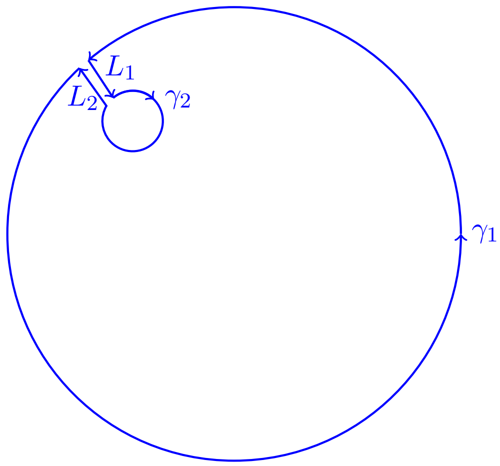
$$\int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz = 0.$$

Remark. Theorem 16.3 can be extended to the case with two or more holes, but we shall prove it for the case with a single hole.

Remark. Actually, the function $f(z)$ in Theorem 16.3 is analytic in a region that contains C_1, C_2 and the area between C_1 and C_2.



Proof. Draw a narrow "river" that connects the exterior to the inner circle as follows.



Note that the orientation of γ_2 is opposite to that of C_2. The resulting contour encloses a simply connected region. By Theorem 16.2,

$$\int_{\gamma_1} + \int_{\gamma_2} + \int_{L_1} + \int_{L_2} = 0$$

as the gap between L_1 and L_2 approaches 0, the integrals over L_1 and L_2 cancel each other. The curve γ_1 becomes the circle C_1 and the curve γ_2 becomes the circle $-C_2$. Hence

$$\int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz = 0.$$

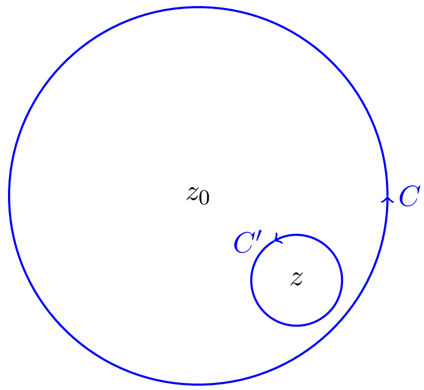
□

Theorem 16.4 (Cauchy integral formula)**.** *Consider a circle C with radius r and center at z_0. If $f(z)$ is analytic in a region that contain C and its interior, then*

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w-z}\,dw.$$

for all z in the interior of C.

Proof. In this prove we fix a complex number z that lies in the interior of the circle C, and draw a small circle C' with z as the center. The radius of C' is ρ, and ρ should be chosen so that C' lies completely inside C.



By Theorem 16.3, it suffices to prove

$$\int_{C'} \frac{f(w)}{w-z}\,dw = 2\pi i f(z).$$

The variable w represents a point on the circle C'.

Write

$$f(w) = f(z) + [f(w) - f(z)]$$

and decompose the complex integral into two parts

$$\int_{C'} \frac{f(w)}{w-z}\,dw = \int_{C'} \frac{f(z)}{w-z}\,dw + \int_{C'} \frac{f(w)-f(z)}{w-z}\,dw. \tag{16.1}$$

The first integral on the RHS of (16.1) equals

$$\int_{C'} \frac{f(z)}{w-z}\,dw = f(z)\int_{C'} \frac{1}{w-z}\,dw = 2\pi i f(z)$$

using a calculation similar to Example 14.1.

For the second integral on the RHS of (16.1), we let $g(w) = f(w) - f(z)$ and write the integral as

$$\int_{C'} \frac{g(w)}{w-z}\,dw.$$

Since C' is a compact set and $g(w)$ is a continuous function, the maximum value of $|g(w)|$ on C' is well-defined,

$$m_\rho \triangleq \sup_{z\in C'} |g(w)| = \max_{z\in C'} |g(w)|.$$

By ML inequality (Theorem 13.3)

$$\Big|\int_{C'} \frac{g(w)}{w-z}\,dw\Big| \leq \frac{m_\rho}{\rho} 2\pi\rho = 2\pi m_\rho.$$

Since the function g is continuous, m_ρ approaches 0 as ρ approaches 0. Therefore

$$\int_{C'} \frac{f(w)-f(z)}{w-z}\,dw = \int_{C'} \frac{g(w)}{w-z}\,dw = 0.$$

The RHS of (16.1) is thus equal to $2\pi i f(z)$. This finishes the derivation of the Cauchy's integral formula. □

Example 16.1. Calculate $\int_C \frac{1}{z^2-1}\,dz$ for (i) C is the circle $C(1;1)$, (ii) C is the circle $C(-1;1)$, (iii) C is the circle $C(0;2)$. All contours have positive orientation.

The integrate can be factorized as

$$\frac{1}{z^2-1} = \frac{1}{(z-1)(z+1)}.$$

(i) The circle $C(1;1)$ with radius 1 and center $z=1$ does not enclose the point $z=-1$. The fraction $\frac{1}{z+1}$ is analytic inside $C(1,1)$.

$$\int_{C(1,;1)} \frac{\frac{1}{z+1}}{z-1}\,dz = 2\pi i\Big(\frac{1}{z+1}\Big)\Big|_{z=1} = \pi i.$$

(ii) The fraction $\frac{1}{z-1}$ is analytic inside $C(-1;1)$, because the point $z=1$ is not inside the circle $C(-1,1)$ with radius 1 and center $z=-1$.

$$\int_{C(-1;1)} \frac{\frac{1}{z-1}}{z+1}\,dz = 2\pi i\Big(\frac{1}{z-1}\Big)\Big|_{z=-1} = -\pi i.$$

(iii) Apply Theorem 16.3 on $C(0;2)$ as the outer circle and $C(1;1)$ and $C(-1;1)$ as the inner circle. The function is analytic in the area outside $C(1;1)$ and $C(-1;1)$. Therefore

$$\int_{C(0;2)} \frac{\frac{1}{z+1}}{z-1}\,dz = \pi i + (-\pi i) = 0.$$

Theorem 16.5 (Taylor expansion)**.** *Suppose $f(z)$ is analytic in a domain that contains the closure of the open disc $D(z_0;r)$. Then $f(z)$ has Taylor series expansion*

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$$

for z inside $D(z_0; r)$. Moreover, for $n = 0, 1, 2, 3, \ldots$, the coefficient a_n is given by

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(w)}{(w - z_0)^{n+1}}\, dw$$

with the integral taken over the circle $|z - z_0| = r$.

In Theorem 16.5, the function $f(z)$ is required to be defined on every point on the boundary of $D(z_0; r)$. Hence f should be analytic in a neighborhood that contain $D(z_0; r)$ in its interior.

Proof. By Theorem 16.3, we can take the circle C centered at z_0 as the curve in the Cauchy's integral formula,

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(w)}{w - z}\, dw.$$

The proof idea is to expand $1/(w - z)$ using geometric series,

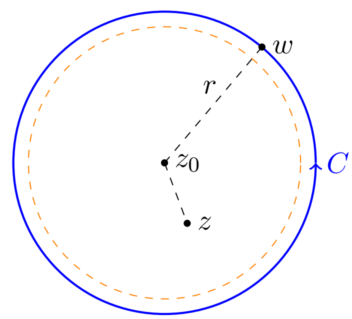
$$\begin{aligned}
\frac{1}{w - z} &= \frac{1}{w - z_0 - (z - z_0)} \\
&= \frac{1}{w - z_0}\Big(\frac{1}{1 - \frac{z - z_0}{w - z_0}}\Big) \\
&= \frac{1}{w - z_0} + \frac{z - z_0}{(w - z_0)^2} + \frac{(z - z_0)^2}{(w - z_0)^3} + \frac{(z - z_0)^3}{(w - z_0)^4} + \cdots \qquad (16.2)
\end{aligned}$$

This geometric series converges because $|z - z_0| < |w - z_0|$ for all z in the open disc $D(z_0; r)$. If we can justify the exchange of infinite summation an contour integral, then we get

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i} \oint_C \sum_{n=0}^{\infty} f(w) \frac{(z - z_0)^n}{(w - z_0)^{n+1}}\, dw \\
&= \sum_{n=0}^{\infty} \Big(\frac{1}{2\pi i} \oint_C \frac{f(w)}{(w - z_0)^{n+1}}\Big)(z - z_0)^n.
\end{aligned}$$

This is a Taylor series centered at z_0.

The remaining part of the proof is to make this argument rigorous. In fact, we will make use of uniform convergence implicitly. Consider a smaller circle $C(z_0; \rho)$ with radius $\rho < r$ (the dashed line in the following figure), and let z be a complex inside the smaller circle, i.e., $|z - z_0| < \rho$.



Let n be a positive integer. Consider the partial sum on the right-hand side of (16.2) up to degree n. The remainder is

$$\sum_{k=n+1}^{\infty} \frac{(z-z_0)^k}{(w-z_0)^{k+1}} = \frac{(z-z_0)^{n+1}}{(w-z_0)^{n+2}} \frac{1}{1-\frac{z-z_0}{w-z_0}} = \frac{1}{(w-z)} \frac{(z-z_0)^{n+1}}{(w-z_0)^{n+1}}$$

We can thus write $1/(w-z)$ as

$$\frac{1}{w-z} = \frac{1}{w-z_0} + \frac{z-z_0}{(w-z_0)^2} + \cdots + \frac{(z-z_0)^n}{(w-z_0)^{n+1}} + \frac{1}{(w-z)} \frac{(z-z_0)^{n+1}}{(w-z_0)^{n+1}}.$$

Multiply each term by $f(w)$ and integrate over the curve C. (Since this is a finite sum, there is no problem in exchanging finite summation and integration.)

$$\frac{1}{2\pi i} \int_C \frac{f(w)}{w-z}\, dw = \sum_{k=0}^{n} \Big(\frac{1}{2\pi i} \int_C \frac{f(w)\, dw}{(w-z_0)^{k+1}} \Big) (z-z_0)^k + \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-z)} \frac{(z-z_0)^{n+1}}{(w-z_0)^{n+1}}\, dw.$$

The above equation holds for any $n \geq 0$. The last summation can be bounded by observing

$$\begin{aligned} |z-z_0| &< \rho \\ |w-z| &\geq \rho - |z-z_0| \\ |w-z_0| &= r, \end{aligned}$$

and $|f(z)|$ is bounded by some constant M for all z on the curve C. By ML inequality (Theorem 13.3),

$$\Big| \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-z)} \frac{(z-z_0)^{n+1}}{(w-z_0)^{n+1}}\, dw \Big| \leq \frac{1}{2\pi} \frac{M}{\rho - |z-z_0|} \frac{\rho^{n+1}}{r^{n+1}} \cdot 2\pi r = \frac{M\rho}{\rho - |z-z_0|} \Big(\frac{\rho}{r} \Big)^n.$$

Take $n \to \infty$, the modulus of the remainder term approach 0. (This step depends on the assumption that $\rho < r$). Consequently, the Taylor series converges for all z in the disk $D(z_0;\rho)$. Since ρ can be any number less than r, we conclude that the Taylor series converges for all z in the disk $D(z_0;r)$. □

Since we can differentiate a Taylor series arbitrarily many times, Theorem 16.5 says that the function $f(z)$ can be differentiated arbitrarily many times at $z = z_0$. Apply Theorem 16.5 to each point z_0 in the domain of f, we obtain the following important theorem.

Theorem 16.6. *If a function f is complex differentiable once for all points in a domain, then f is infinitely differentiable.*

Remark. The property in Theorem 16.6 is certainly false in real analysis. There are examples of real functions that can be differentiated once but not twice. There are also examples of real functions that can be differentiated twice but not differentiated three times, and so on.

Remark. Theorem 16.5 implies that we can always expand an analytic function f as a power series at any point z_0 in the domain of f. In some books, this property is taken as the definition of analytic functions, i.e., some people define analytic function as a function that can be expanded as an a power series at any point z_0 in the domain. A function that can be differentiated once at any point in the domain is called holomorphic instead. Using these terminologies, holomorphic function is analytic by Theorem 16.5. Conversely, analytic function is holomorphic, because Taylor series can be differentiated term-wise. In the remainder of this lecture notes, "holomorphic" and "analytic" will be treated as synonyms.

17 Lecture 17 (Applications of the Cauchy integral formula)

Summary

- Integral formula for higher derivatives
- Liouville theorem
- A proof of fundamental theorem of algebra
- Zeros of a function

Theorem 17.1. *With the same assumption in Theorem 16.5, the n-th derivative of f at z_0 exists in $D(z_0; r)$, and is given by*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(w)}{(w - z_0)^{n+1}}\, dw. \tag{17.1}$$

Proof. (Proof 1) From Theorem 16.5 we know that the Taylor series

$$f(z) = \sum_{k=0} a_n (z - z_0)^n \tag{17.2}$$

converges at $z = z_0$, with coefficient a_n equal to

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(w)}{(w - z_0)^{n+1}}\, dw.$$

The curve C is a small circle centered at z_0 so that $f(z)$ is analytic inside C. By substituting $z = z_0$ into (17.1), we get

$$f(z_0) = a_0 = \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z_0}\, dw,$$

which is the same as Cauchy's integral formula in Theorem 16.4. Since Taylor series can be differentiated term-wise, we can differentiate both sides of (17.2) and substitute $z = z_0$. This gives

$$f'(z_0) = a_1 = \frac{1}{2\pi i} \int_C \frac{f(w)}{(w - z_0)^2}\, dw.$$

Repeat this process recursively, we obtain (17.1).

(Proof 2) We can also derive (17.1) directly from Cauchy integral formula. We first consider the first derivative $f'(z_0)$. Using Cauchy integral formula, we can express the fraction $(f(z_0 + h) - f(z_0))/h$ as

$$\frac{f(z_0 + h) - f(z_0)}{h} = \frac{1}{h} \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z_0 - h} - \frac{f(w)}{w - z_0}\, dw$$

where C is a the circle $C(z_0, r)$. The integral on the right can be simplified to

$$\frac{1}{2\pi i} \int_C \frac{f(w)}{(w - z_0 - h)(w - z_0)}\, dw.$$

Then,

$$\begin{aligned}
\frac{f(z_0 + h) - f(z_0)}{h} - \frac{1}{2\pi i} \int_C \frac{f(w)}{(w - z_0)^2}\, dw &= \frac{1}{2\pi i} \int_C \frac{f(w)}{(w - z_0 - h)(w - z_0)} - \frac{f(w)}{(w - z_0)^2}\, dw \\
&= \frac{h}{2\pi i} \int_C \frac{f(w)}{(w - z_0 - h)(w - z_0)^2}\, dw
\end{aligned}$$

By ML inequality (Theorem 13.3), it can be bounded by

$$\Big| \int_C \frac{f(w)}{(w - z_0 - h)(w - z_0)^2}\, dw \Big| \le \frac{1}{2\pi} \frac{M|h|}{(r - |h|)r^2} 2\pi r$$

where M denote the maximum of $f(w)$ on the circle $C(z_0; r)$. As h approach 0, it approaches 0. Therefore

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = \frac{1}{2\pi i} \int_C \frac{f(w)}{(w - z_0)^2} \, dw.$$

Higher-order derivatives can be derived by induction. ◻

Based on (17.1) one can estimate the modulus of the n-th derivative.

Theorem 17.2 (Cauchy estimate)**.** *If $|f(z)| \leq M$ for z on the circle $C(z_0; r) = \{z : |z - z_0| = r\}$ and f is analytic in a neighborhood of the disc $|z - z_0| \leq r$, then*

$$|f^{(n)}(z_0)| \leq M \frac{n!}{r^n}$$

for $n = 0, 1, 2, 3, \ldots$

Proof.

$$|f^{(n)}(z_0)| = \left| \frac{n!}{2\pi i} \int_C \frac{f(w)}{(w - z_0)^{n+1}} \, dw \right| \leq \frac{n!}{2\pi} \frac{M}{r^{n+1}} 2\pi r = \frac{n! M}{r^n}.$$

◻

Using the Cauchy estimate, we can derive the following theorem

Theorem 17.3 (Liouville theorem)**.** *If a complex analytic function f is bounded on the whole complex plane, then it must be a constant function.*

We remark that $\sin(z)$ is *not* a counter-example, because $|\sin(z)|$ becomes very large when $\mathrm{Im}(z) \gg 0$.

Proof. Suppose $f(z) \leq M$ for all $z \in \mathbb{C}$. Then for any fixed complex number $z_0 \in \mathbb{C}$,

$$|f'(z_0)| \leq \frac{M}{r}$$

by Theorem 17.2. Since this is true for any $r > 0$, we can take $r \to \infty$. This gives $f'(z_0) = 0$. Because this is true for any $z_0 \in \mathbb{C}$, f' is identically zero. This means that f is a constant function. ◻

One can prove the fundamental theorem of algebra using Liouville theorem.

Theorem 17.4 (Fundamental theorem of algebra)**.** *Suppose* $p(z) = c_0 + c_1 z + \cdots + c_d z^d$ *is a polynomial of degree* $d \geq 1$*. Then* $p(z)$ *has a root in* $\mathbb{C}$*.*

Proof. We prove this by contradiction. Suppose $p(z) \neq 0$ for all $z \in \mathbb{C}$. Then $1/p(z)$ is a well-defined and analytic for all $z \in \mathbb{C}$, i.e., $1/p(z)$ is an entire function.

Without loss of generality, assume the leading coefficient c_d is equal to 1,

$$p(z) = c_0 + c_1 z + \cdots + z^d.$$

The ratio

$$\frac{p(z)}{z^d} = \frac{c_0}{z^d} + \frac{c_1}{z^{d-1}} + \cdots + 1$$

approaches 1 as $|z| \to \infty$. Hence, there exists a large real number N such that

$$|\frac{p(z)}{z^d}| > \frac{1}{2}$$

for all z with modulus larger than N. In other words, we have

$$|\frac{1}{p(z)}| < \frac{2}{|z^d|} < \frac{2}{N^d}$$

for all $|z| > N$.

On the other hand, the closed disc $\{z : |z| \leq N\}$ is compact and $1/p(z)$ is continuous in this compact set. So $|1/p(z)|$ is bounded by some number M for all $|z| \leq N$.

Therefore $|1/p(z)|$ is bounded by some constant for all $z \in \mathbb{C}$. By Liouville theorem, $1/p(z)$ is a constant function. But then $p(z)$ is constant, and it contradicts the assumption that $d \geq 1$. □

Definition 17.5. A point z_0 is called a zero of function f if $f(z_0) = 0$.

Suppose f is analytic in a domain D and $z_0 \in D$ is a zero of f. By Theorem 16.5 we can write f as a Taylor series

$$f(z) = c_1(z - z_0) + c_2(z - z_0)^2 + \cdots + c_n(z - z_0)^n + \cdots$$

for z in some neighborhood $D(z_0; r)$ of z_0. One of the following possibilities should occur

(a) $c_k = 0$ for all $k = 1, 2, 3, \ldots$. In this case $f(z)$ is identically equal to 0 in $D(z_0; r)$.

(b) The coefficients $c_1, c_2, \ldots$ are not all zero. The smallest integer m such that $c_m \neq 0$ is called the order of the zero at z_0. We can write $f(z)$ as

$$f(z) = (z - z_0)^m \big[c_m + c_{m+1}(z - z_0) + c_{m+2}(z - z_0)^2 + \cdots\big].$$

Because $c_m \neq 0$ by assumption, the power series inside the square bracket is nonzero for z in a sufficiently small neighborhood of z_0. We say that z_0 is an isolated zero.

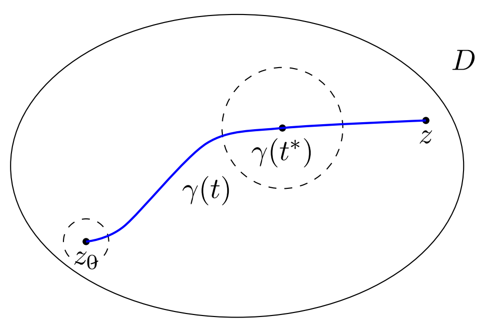
The following is a special property of analytic functions in general.

Theorem 17.6. *Suppose f is analytic in a domain D and $\{z_1, z_2, z_3, \ldots\} \subset D$ is a set of zeros of f. If $\{z_1, z_2, z_3, \ldots\}$ contains an accumulation point in D, then f is identically zero throughout the domain D.*

Proof. Let $z_0 \in D$ be an accumulation point of $\{z_1, z_2, z_3, \ldots\}$. The point z_0 must be a zero of f, because f is continuous. However, z_0 cannot be an isolated zero. Since exactly one of the condition in (a) or (b) before the theorem is true, the coefficients in the Taylor expansion

$$f(z) = c_1(z - z_0) + c_2(z - z_0)^2 + \cdots$$

must be all zero. i.e., $c_k = 0$ for all $k = 1, 2, 3, \ldots$. So $f(z)$ is identically zero in some open disc centered at z_0.



Let z be any point in the domain D. Since D is connected, we can draw a smooth path $\gamma(t)$ from z_0 to z, with $\gamma(0) = z_0$ and $\gamma(1) = z$. We know that $f(\gamma(t))$ is a continuous function of t, and $f(\gamma(t))$ is zero for all t closed to 0, i.e., $f(\gamma(t)) = 0$ for $0 \le t \le \epsilon$ for some ϵ between 0 and 1.

We start from z_0 and travel along the curve $\gamma(t)$ until $f(\gamma(t))$ is not zero. If $f(\gamma(t)) = 0$ for all $0 \le t \le 1$, then we will stop at $t = 1$, and $f(\gamma(1)) = f(z) = 0$. If $f(\gamma(t))$ is not zero for some t between 0 and 1, we can let t^* be the largest parameter such that $f(\gamma(t)) = 0$ for all t from 0 to t^*. More rigorously, we should define t^* using supremum

$$t^* \triangleq \sup\{\alpha : f(\gamma(t)) = 0 \text{ for } 0 \le t \le \alpha\}.$$

If $t^* < 1$, then the point $\gamma(t^*)$ is a non-isolated zero (because $f(\gamma(t)) = 0$ for all $t < t^*$). The Taylor series centered at $\gamma(t^*)$ defines a zero function within the radius of convergence. This means that t^* cannot be the supremum, as we can increase t^* slightly. This is a contradiction to the definition of t^*.

Therefore we must have $\alpha = 1$, and thus $f(z) = f(\gamma(1)) = 0$. Because this is true for any z in D, we have $f(z) = 0$ for all $z \in D$. □

The previous theorem can be stated in the following form. It is called the identity theorem, because it give a criterion under which two functions are identical.

Theorem 17.7 (Identity theorem)**.** *If f and g are analytic functions in D and $\{z : f(z) = g(z)\}$ has an accumulation point in D, then $f(z) = g(z)$ for all $z \in D$.*

Proof. Take $h(z) = f(z) - g(z)$ as the function in Theorem 17.6. By assumption, the zeros of the function $h(z)$ has an accumulation point in D. Hence $h(z) = 0$ for all $z \in D$. □

Theorem 17.7 is usually applied to the situation when two analytic functions $f(z)$ and $g(z)$ agree on a small open disc. Then $f(z)$ and $g(z)$ have the same value for all z in the domain.

In other words, suppose we know the values of a function $f(z)$ in a small open disc. Then, if the function is analytic, the values of $f(z)$ at other points in the domain are uniquely determined. No matter how small the open disc is, the value of an analytic function $f(z)$ is completed determined by the values of $f(z)$ inside this open disc. This is a special feature of analytic functions.

18 Lecture 18 (Integration of multi-valued function)

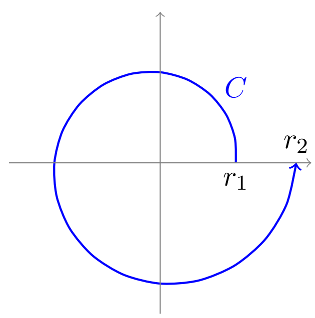
Summary

- Evaluation of integral using branch cut
- Evaluation of integral using local primitive

In Theorem 15.2 we show that we can always find a primitive of a function that is analytic in a disc. In general, a global primitive needs not exist. In this lecture illustrates how to evaluate complex integral of multi-valued function.

Consider the square root function $f(z) = \sqrt{z}$. It is a multi-valued function because for each nonzero complex number there are two choices for the square root. There are two branches for the square root function $\sqrt{z}$. We can draw a branch cut to select a particular branch and make the function single-valued.

As a numerical demonstration, we compute $\int_C \sqrt{z}\,dz$ for over a curve C shown as follows.

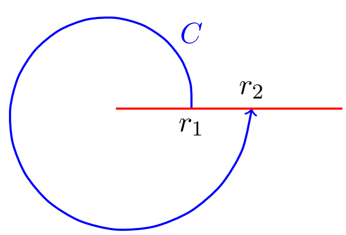


The curve start at the point $z = r_1$ and end at the point $z = r_2$, where r_1 and r_2 are positive real numbers. The curve revolve around the origin once. The function $\sqrt{z}$ is multi-valued. We take the positive branch at the beginning when $z = r_1$. Thus, the value of $\sqrt{r_1}$ is taking to be positive. The values of $\sqrt{z}$ varies continuously as we move along the curve. At the terminal point, value of $\sqrt{r_1}$ is negative. Now the question is well specified. We give two solutions below.

<u>Solution 1</u>. Parameterize the curve by a smooth function $\gamma(t)$ for t from 0 to 1, so that $\gamma(0) = r_1$ and $\gamma(1) = r_1$. We take a branch cut at the positive real axis, i.e., the argument is from 0 to 2π. For argument in this range, the square root function is single-valued:

$$\sqrt{re^{i\theta}} = \sqrt{r}e^{i\theta/2}$$

for $0 < \theta < 2\pi$.



A primitive function for this branch of square root function is

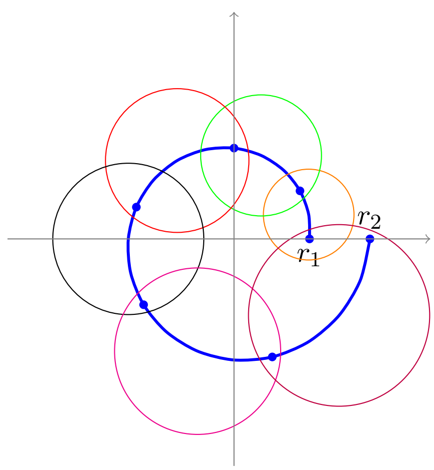
$$G(z) = \frac{2}{3} z^{3/2} = \frac{2}{3} |z|^{3/2} e^{i3\arg(z)/2},$$

where arg(z) takes values in the open interval $(0, 2\pi)$. The integral can be evaluated as follows,

$$\begin{aligned}\int_C \sqrt{z}\, dz &= \lim_{\epsilon\to 0^+} \frac{2}{3}\Big[G(\gamma(2\pi-\epsilon)) - G(\gamma(\epsilon))\Big] \\ &= \frac{2}{3}\Big[r_2^{3/2} e^{i3\pi} - r_1^{3/2} e^0\Big] \\ &= \frac{2}{3}(-r_2^{3/2} - r_1^{3/2}).\end{aligned}$$

In the calculation we take limit as ϵ approaching to zero from above. Taking limit is necessary because the function is not defined on the branch cut.

Solution 2. We divide the curve into pieces. Cover each piece by a circle and define a local primitive function in each circle. We arrange the circles so that adjacent circles are overlapping, and the two local primitive functions on the adjacent circles agree with each other on the overlapping area.



An example of covering the path is shown in the figure above. Each circle covers a portion of the curve, but does not cover the origin. The argument of the complex numbers in the first circle (in orange color) is between two constants, say a_1 and b_1. In this example

a_1 is negative, and b_1 is approximately equal to $\pi/4$ in radian. For argument θ in the range (a_1, b_1), we define a local primitive function

$$F_1(z) = F_1(re^{i\theta}) = \frac{2}{3} r^{3/2} e^{i3\theta/2},$$

for $a_1 < \theta < b_1$. Since we restrict the domain of $F_1(z)$ to be the interior of the orange circle, $F_1(z)$ is a single-valued function.

A point in the second circle (shown in green color) has argument in a range (a_2, b_2), where a_2 is a number less than b_1, and b_2 is larger than $\pi/2$. We define a local primitive function F_2 for points within the green circle,

$$F_2(z) = F_2(re^{i\theta}) = \frac{2}{3} r^{3/2} e^{i3\theta/2},$$

for $a_2 < \theta < b_2$. The two functions F_1 and F_2 agree with each other on the overlapping area of the orange and green circle.

Similarly we define six local primitive functions. In the last circle, the range of the argument is from a_6 to b_6, where b_6 is a number slightly larger than 2π, and a_6 is a number larger than $3\pi/2$. The formula for computing $F_6(z)$ is

$$F_6(z) = F_2(re^{i\theta}) = \frac{2}{3} r^{3/2} e^{i3\theta/2},$$

for θ in the range (a_6, b_6).

Suppose that the curve is divided into six pieces. The range of t is divided into

$$[0, t_1],\ [t_1, t_2],\ [t_2, t_3],\ [t_3, t_4],\ [t_4, t_5],\ [t_5, 2\pi].$$

The parameters t_1 is chosen such that $\gamma(t_1)$ is contained in the first and second circles. For $j = 2, 3, 4, 5$, the value of t_j is chosen such that $\gamma(t_j)$ is contained in the j-th and $(j+1)$-th circle.

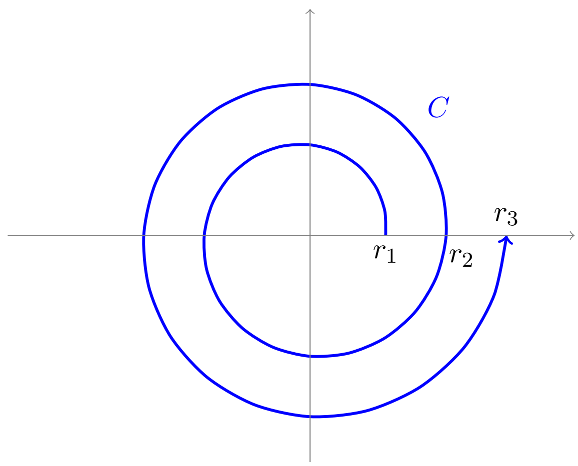
The integral can be calculated by

$$\begin{aligned}
\int_C \sqrt{z}\, dz = {} & [F_1(\gamma(t_1)) - F_1(\gamma(0))] + [F_2(\gamma(t_2)) - F_2(\gamma(t_1))] + [F_3(\gamma(t_3)) - F_3(\gamma(t_2))] \\
& + [F_4(\gamma(t_4)) - F_4(\gamma(t_3))] + [F_5(\gamma(t_5)) - F_5(\gamma(t_4))] + [F_6(\gamma(2\pi)) - F_6(\gamma(t_5))] \\
= {} & F_6(r_2) - F_1(r_1).
\end{aligned}$$

In the last circle, the value of $F_6(2\pi)$ is $\frac{2}{3}r_2^{3/2}e^{i3(2\pi)/2} = -\frac{2}{3}r_2^{3/2}$. Therefore the answer is

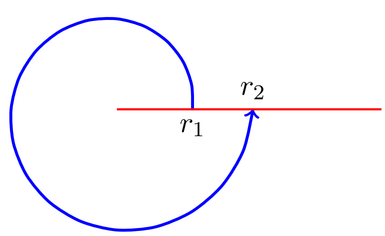
$$\frac{2}{3}(-r_2^{3/2} - r_1^{3/2}).$$

As the second example, we compute $\int_C \sqrt{z}\,dz$ for C shown below. The values of r_1, r_2 and r_3 are real, and $r_1 < r_2 < r_3$.

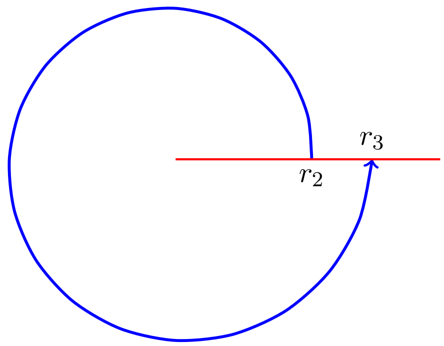


As in the first example, at the initial point $z = r_1$, we take $\sqrt{r_1}$ to be a positive number. The value of $\sqrt{z}$ changes continuously as we travel along the curve C. When the curve intersects the real axis at $z = r_2$, the value of $\sqrt{r_2}$ should be negative. At the end point $z = r_3$, the value of $\sqrt{r_3}$ is positive.

Solution 1. Divide the path into two parts. In each part, we take the positive real axis as the branch cut. The first part goes from $z = r_1$ to $z = r_2$. The calculation is the same as in the previous example.



The second part goes from $z = r_2$ to $z = r_3$.



In the second part we should use the primitive function

$$F(z) = -\frac{2}{3} r^{3/2} e^{i3\theta/2}$$

for $0 < \theta < 2\pi$.

Combining the two parts, we obtain the answer

$$\int_C \sqrt{z}\, dz = \frac{2}{3}\big(- r_2^{3/2} - r_1^{3/2}\big) + \frac{2}{3}\big(r_3^{3/2} + r_2^{3/2}\big) = \frac{2}{3}\big(r_3^{3/2} - r_1^{3/2}\big).$$

Solution 2. As we go from $z = r_1$ to $z = r_3$ long the curve, the argument of the point on the curve varies from 0 to 4π. By covering the curves by discs and using local primitive functions in each disc, we can compute the integral by

$$\int_C \sqrt{z}\, dz = \frac{2}{3} r_3^{3/2} e^{i3(4\pi)/2} - \frac{2}{3} r_1^{3/2} e^{i0} = \frac{2}{3}\big(r_3^{3/2} - r_1^{3/2}\big).$$

We note that in the example, the first method divides the path into two parts, the second method divides the path into more than two parts. In both methods, a local primitive function is selected for each patch, so that all the intermediate values are canceled.

19 Lecture 19 (Singularity)

Summary

- Removable singularity
- Pole
- Essential singularity

- Riemann's theorem on removable singularity

- Order of a pole

From Theorem 17.6, we know that an analytic function with a sequence of zeros converging to another zero in the domain must be the zero function. This is the reason why we mainly study functions with isolated zeros. For singular points, we have a similar notion of isolated singularity.

Definition 19.1. A point where a function failed to be analytic is called a singular point. A singular point z_0 is called an isolated singular point if the function is analytic in a neighborhood of z_0, except the point z_0.

Example 19.1. The function $f(z) = 1/(z^2+1)$ has two isolated singular points located at $z = i$ and $z = -i$.

Example 19.2. The function $\frac{1}{\sin(\pi z)}$ has a non-isolated singular point at $z = 0$. It is because this function is not defined, and hence singular, at $z = 1/n$, for $n = 1, 2, 3, 4, \ldots$. The sequence $\{1/n\}_{n=1}^{\infty}$ converges to $z = 0$.

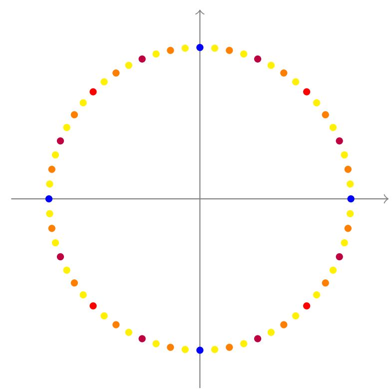
Example 19.3. The power series

$$f(z) \sum_{n=0}^{\infty} z^{2^n} = z + z^2 + z^4 + z^8 + z^{16} + \cdots$$

converges for $|z| < 1$. (This can be shown by comparison test.) This series diverges at (2^n)-th roots of unity, for all positive integer n. For example,

$$\begin{aligned}
f(-1) &= -1 + 1 + 1 + 1 + \cdots = \infty, \\
f(i) &= i - 1 + 1 + 1 + \cdots = \infty, \\
f(-i) &= -i - 1 + 1 + 1 + \cdots = \infty, \\
f(e^{2\pi i/8}) &= e^{2\pi i/8} + e^{2\pi i/4} + e^{2\pi i/2} + 1 + 1 + \cdots = \infty.
\end{aligned}$$

The set of singular points of $f(z)$ on the unit circle is dense, and hence are not isolated. This is an example of *natural boundary*, meaning that we cannot extend the domain of this function. (We shall discuss analytic continuation in later lecture.) The singular points on the unit circle are plotted below.



Isolated singularity can classified into three categories.

Definition 19.2. An isolated singular point z_0 of $f(z)$ is called

(i) a removable singularity if $f(z)$ is bounded in a neighborhood of z_0;

(ii) a pole if $|f(z)| \to \infty$ as $z \to z_0$;

(iii) an essential singularity otherwise.

Example 19.4. The function $f(z) = \frac{z^2-1}{z+1}$ has a removable singularity at $z = 1$, because

$$f(z) = \frac{(z+1)(z-1)}{z-1} = z+1$$

for all complex number z that is close to but not equal to 1, and $z+1$ is bounded in any open disc with finite radius centered at $z = 1$.

Example 19.5. The function $f(z) = \frac{1}{z(z+1)}$ has two poles located at $z = 0$ and $z = -1$.

Example 19.6. The function $f(z) = e^{1/z}$ has an essential singularity at $z = 0$. We can see that it is an essential singular point by approaching $z = 0$ from the right and from the left. For real variable x, if we take $x \to 0$ from the right, the value of $f(z)$ tends to positive infinity. Hence it is not bounded near $z = 0$. On the other hand if we take x approaching 0 from the negative real axis, then $f(z)$ tends to 0. The modulus is not approaching infinity.

The next theorem justifies the name "removable singularity".

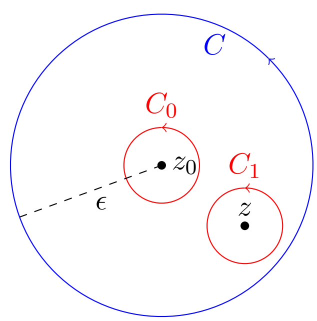
Theorem 19.3 (Riemann's theorem on removable singularity)**.** *Suppose f is analytic in a domain that contains a punctured disc $D(z_0;\epsilon)\setminus\{z_0\}$ (which equals $\{z:\, 0<|z-z_0|<\epsilon\}$). If f is bounded in $D(z_0;\epsilon)\setminus\{z_0\}$, the we can re-defined f at z_0 so that f is analytic in $D(z_0;\epsilon)$*

More formally, the theorem is saying that there exists an analytic function $\tilde{f}$ defined on the open disc $D(z_0;\epsilon)$ such that $\tilde{f}(z)=f(z)$ for all $z\in D(z_0;\epsilon)\setminus\{z_0\}$[1]. For instance, in Example 19.4, we can re-define $f(1)$ by $f(1)=2$. The new function is the same as the function $z+1$, which is an entire function.

Proof. Suppose that $|f(z)|$ is upper bounded by M for z in the punctured disc $D(z_0;\epsilon)\setminus\{z_0\}$. Define

$$\tilde{f}(z) \triangleq \frac{1}{2\pi i}\oint_C \frac{f(w)}{w-z}\,dw,$$

where C is the circle $|z-z_0|=\epsilon$, with counter-clockwise orientation, and z is any point inside the circle. We want to show that (i) $\tilde{f}(z)=f(z)$ for all $z\in D(z_0;\epsilon)\setminus\{z_0\}$, and (ii) $\tilde{f}$ is analytic in $D(z_0;\epsilon)$.



Let z be a point in the puncture disc $0<|z-z_0|<\epsilon$. Draw a small circle C_0 of radius ϵ_0 with center at z_0, such that C_0 is inside the circle C but does not contain z. Draw another small circle C_1 of radius ϵ_1 with center at z_1 such that C_1 is inside the circle C but does not contain z_0.

By Cauchy's theorem for multiply connected region (Theorem 16.3), we have

$$2\pi i\tilde{f}(z)=\int_C \frac{f(w)}{w-z}\,dw=\int_{C_0}\frac{f(w)}{w-z}\,dw+\int_{C_1}\frac{f(w)}{w-z}\,dw. \tag{19.1}$$

[1]The symbol $\tilde{f}$ is pronounced as "tilde f" or "twiddle f"

Since $f(z)$ is analytic inside the circle C_1, we can apply Cauchy integral formula (Theorem 16.4) to obtain

$$\int_{C_1} \frac{f(w)}{w-z}\, dw = 2\pi i f(z).$$

We next prove that $\int_{C_0} \frac{f(w)}{w-z}\, dw = 0$. For complex number w on C_0, the distance $|w-z|$ is lower bounded by $|z-z_0| - \epsilon_0$. Moreover, it is assumed that $f(z)$ is upper bounded by M for z insider the circle C, and hence is bounded by M for w on the circle C_0. By ML inequality (Theorem 13.3)

$$\Big| \int_{C_0} \frac{f(w)}{w-z}\, dw \Big| \leq \frac{M}{|z-z_0| - \epsilon_0} 2\pi\epsilon_0,$$

which approaches 0 as $r \to 0$. Therefore $\int_{C_1} \frac{f(w)}{w-z}\, dw = 0$. The equality in (19.1) becomes $2\pi i \tilde{f}(z) = 2\pi i f(z)$ for all $z \in D(z_0; \epsilon) \setminus \{z_0\}$. This proves part (i).

For part (ii), we just need to show that $\tilde{f}(z)$ is analytic at z_0, because $f(z)$ is equal to $f(z)$ that is analytic in $D(z_0; \epsilon) \setminus \{z_0\}$. We first use the definition of $\tilde{f}$ at z_0 to write

$$\frac{\tilde{f}(z_0+h) - f(z_0)}{h} = \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-z_0-h)(w-z_0)}\, dw$$

where C is the circle $|z-z_0| = \epsilon$. We then show that it converges as $h \to 0$,

$$\begin{aligned}
\Big| \frac{\tilde{f}(z_0+h) - \tilde{f}(z_0)}{h} - \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-z_0)^2} dw \Big| &= \frac{1}{2\pi} \int_C \frac{f(w)h}{(w-z_0)^2(w-z_0-h)}\, dw \\
&\leq \frac{1}{2\pi} \frac{M|h|}{\epsilon^2(\epsilon - |h|)} 2\pi\epsilon.
\end{aligned}$$

For fixed ϵ, it converges to 0 as $|h|$ approaches 0. Therefore $f'(z_0)$ exists and is equal to

$$f'(z_0) = \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-z_0)^2}\, dw.$$

This proves part (ii). $\square$

In view of Theorem 19.3, we can ignore any removable singularity, because we can always re-define the function appropriately so that it is no longer a singularity. This leads to the following definition.

Definition 19.4. A complex function is called a meromorphic function if all the singularity points are poles.

Example 19.7. The complex function $\frac{1}{\sin z}$ is not defined at 0, $\pm\pi$, $\pm 2\pi$, $\pm 3\pi$, etc. There are countably many singularity points, and each singular point is a pole of order 1.

Suppose z_0 is a pole of $f(z)$. By definition we have $|f(z)| \to \infty$ as $z \to z_0$, and $f(z)$ is nonzero in a small neighborhood of z_0. The reciprocal function $g(z) = 1/f(z)$ is thus bounded in a neighborhood of z_0. The singular point z_0 is a removable singularity of $g(z)$. By Theorem 19.3, we can re-defined $g(z)$ at $z = z_0$ by $g(z_0) = 0$ so that we can expand $g(z)$ as a power series near z_0,

$$g(z) = \sum_{n=1}^{\infty} a_n (z - z_0)^n.$$

Note that the constant term is zero because $g(z_0) = 0$. The coefficients a_n's cannot be all zero, otherwise $g(z)$ is a zero function within the region of convergence. The smallest integer m such that a_m is not zero is the order of zero z_0, and is defined as the order of the pole of $f(z)$ at z_0,

$$a_1 = a_2 = \cdots = a_{m-1} = 0, \text{ but } a_m \neq 0.$$

To compute the order more efficiently, we note that the order of a zero at z_0 of a function $f(z)$ is the smallest positive integer m such that

$$\lim_{z \to z_0} \frac{f(z)}{(z - z_0)^m} = c \neq 0.$$

Similarly, the order of a pole at z_0 of a function $f(z)$ is the smallest positive integer m such that

$$\lim_{z \to z_0} (z - z_0)^m f(z) = c \neq 0.$$

We can take this as the definition of the order of a pole.

Definition 19.5. The order of a pole z_0 of $f(z)$ is the smallest positive integer m such that $\lim_{z \to z_0} (z - z_0)^m f(z)$ is a nonzero constant. A pole of order 1 is called a simple pole. A pole of order 2 is called a double pole.

Example 19.8. Consider a rational function $f(z) = \frac{(z-1)(z-2)}{(z-3)(z+i)^2}$. This function $f(z)$ has two zeros of order 1 at $z = 1$ and $z = 2$. There is a simple pole at $z = 3$, because

$$f(z) = \frac{1}{z-3} \Big[\frac{(z-1)(z-2)}{(z+i)^2} \Big]$$

and $\frac{(z-1)(z-2)}{(z+i)^2}$ is nonzero when evaluated at $z = 3$. We can formally check this by calculating

$$\lim_{z \to 3}(z-3)f(z) = \lim_{z \to 3} \frac{(z-1)(z-2)}{(z+i)^2} = \frac{(3-1)(3-2)}{(3+i)^2} \neq 0.$$

Another pole of $f(z)$ is located at $z = -i$. This is a double pole because

$$\lim_{z \to -i}(z+i)f(z) = \lim_{z \to -i} \frac{(z-1)(z-2)}{(z-3)(z+i)} = \infty,$$

but

$$\lim_{z \to -i}(z+i)^2 f(z) = \lim_{z \to -i} \frac{(z-1)(z-2)}{z-3} = \frac{(-i-1)(-i-2)}{-i-3} \neq 0.$$

Example 19.9. The function

$$f(z) = \frac{1}{z(z-1)} - \frac{2}{z(z-2)}$$

has two poles, located at $z = 1$ and $z = 2$. We can see this by simplying $f(z)$ to

$$f(z) = \frac{(z-2) - 2(z-1)}{z(z-1)(z-2)} = \frac{-z}{z(z-1)(z-2)}$$

The singularity at $z = 0$ is removable. The pole at $z = 1$ and $z = 2$ are simple.

20 Lecture 20 (Laurent series)

Summary

- Convergence region of Laurent series
- Computing the coefficients of Laurent series by complex integral

Definition 20.1. A Laurent series is an infinite sum in the form

$$\sum_{n=-\infty}^{\infty} a_n z^n. \tag{20.1}$$

We say that it converges if both $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=1}^{\infty} a_{-n} z^{-n}$ converge. Sometime we write a Laurent series in the form

$$\sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^{-n}.$$

More generally, a Laurent series at z_0 is an expression in the form

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} b_n(z-z_0)^{-n}.$$

The first summation $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is called the analytic part and the second summation $\sum_{n=1}^{\infty} b_n(z-z_0)^{-n}$ is called the principal part. If the principal part is zero, a Laurent series reduces to a Taylor series.

Remark. Suppose the coefficients a_n's in (20.1) are conjugate symmetric, i.e., $a_{-n} = a_n^*$. If we substitute z by $e^{2\pi i t}$, for some real number t, then

$$\sum_{n=-\infty}^{\infty} a_n e^{2\pi i n t}$$

is a real function with period 1. In fact we can re-write it as

$$\begin{aligned} &a_0 + \sum_{n=1}^{\infty}(a_n e^{2\pi i n t} + a_{-n} e^{-2\pi i n t}) \\ &= a_0 + \sum_{n=1}^{\infty} \mathrm{Re}(a_n)\cos(2\pi n t) - \mathrm{Im}(a_n)\sin(2\pi n t). \end{aligned}$$

It is the same as a Fourier series.

Example 20.1. Let $f(z)$ denote the function $\frac{1}{z(1-z)}$. Using geometric series, we can expand it at $z = 0$,

$$\begin{aligned} \frac{1}{z(1-z)} &= \frac{1}{z}(1 + z + z^2 + z^3 + \cdots) \\ &= \frac{1}{z} + 1 + z + z^2 + z^3 + \cdots \end{aligned}$$

It converges for $0 < |z| < 1$. The principal part is $1/z$.

If we expand it at $z = 1$ using geometric series, we obtain

$$\begin{aligned} \frac{1}{z(1-z)} &= -\frac{1}{z-1}\frac{1}{1+z-1} \\ &= -\frac{1}{z-1}(1 - (z-1) + (z-1)^2 - (z-1)^3 + \cdots) \\ &= -\frac{1}{z-1} + 1 - (z-1) + (z-1)^2 - \cdots \end{aligned}$$

The principal part is $-1/(z-1)$. It converges for $0 < |z-1| < 1$.

If we expand it at $z = 2$, the Laurent series at $z = 2$ is

$$\begin{aligned} \frac{1}{z(1-z)} &= \frac{1}{z} + \frac{1}{1-z} \\ &= \frac{1}{2}\frac{1}{1+\frac{z-2}{2}} - \frac{1}{1+(z-2)} \\ &= \sum_{n=0}^{\infty}(-1)^n\big[\frac{(z-2)^n}{2^{n+1}} - (z-2)^n\big] \end{aligned}$$

This is a Taylor series centered at $z = 2$. The principal part is zero, and it converges for all $|z-2| < 1$.

The next example shows that we can also compute Laurent series at essential singularity.

Example 20.2. The power series expansion of e^z is

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

By substituting z by $1/z$ we get the Laurent series of $e^{1/z}$,

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots$$

converging whenever $z \neq 0$. This function has an essential singularity at $z = 0$ (see Example 19.6).

Theorem 20.2. *A Laurent series* $\sum_{n=0}^{\infty} a_n z^n$ *in general converges in an annulus* $R_1 < |z| < R_2$.

Proof. Using Hadamard's formula for radius of convergence, the analytic part $\sum_{n=0}^{\infty} a_n z^n$ converges if

$$|z| < \frac{1}{\limsup |a_n|^{1/n}} \triangleq R_2.$$

For the principal part we make a substitution $u = 1/z$,

$$\sum_{n=1}^{\infty} b_n z^{-n} = \sum_{n=1}^{\infty} b_n u^n.$$

It converges whenever

$$|u| < \frac{1}{\limsup |b_n|^{1/n}}$$

or

$$|z| > \limsup |b_n|^{1/n} \triangleq R_1.$$

Combining the two parts, a Laurent series converges if $R_1 < |z| < R_2$. $\square$

Theorem 20.3. *A function f analytic in an annulus $R_1 < |z| < R_2$ can be expanded as a Laurent series (in powers of z).*

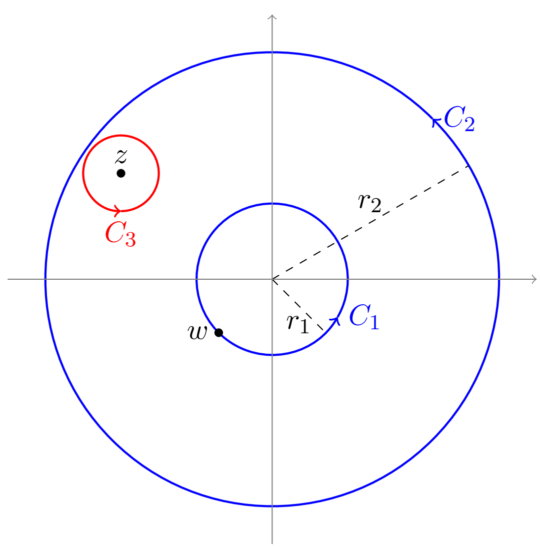
Remark. For Laurent series it is not true in general that $a_n = f^{(n)}(0)/n!$, because the function f need not be defined at $z = 0$. Instead, the coefficients can be computed using integrals.

Proof. Pick two real numbers r_1 and r_2 such that

$$R_1 < r_1 < r_2 < R_2.$$

Draw a circle C_1 with radius r_1 centered at origin, and a circle C_2 with radius r_2 centered at the origin, with counter-clockwise orientation. The choice of r_1 and r_2 ensures that function $f(z)$ is well-defined on C_1 and C_2.

Let z be a complex number in the area between C_1 and C_2. Draw a positively-oriented circle C_3 centered at z with radius r_3 such that C_3 is within the area between C_1 and C_2. The notation is illustrated in the following figure.



By Cauchy's integral formula (Theorem 16.4),

$$f(z) = \frac{1}{2\pi i} \int_{C_3} \frac{f(w)}{w - z} \, dw. \tag{20.2}$$

Then, by applying Cauchy's theorem for multiply connected region (Theorem 16.3), we get

$$f(z) = \frac{1}{2\pi i} \int_{C_3} \frac{f(w)}{w - z} \, dw = \frac{1}{2\pi i} \int_{C_2} \frac{f(w)}{w - z} \, dw - \frac{1}{2\pi i} \int_{C_1} \frac{f(w)}{w - z} \, dw. \tag{20.3}$$

The first integral can be expanded as a convergent Taylor series (See the proof of Theorem 16.5)

$$\frac{1}{2\pi i} \int_{C_2} \frac{f(w)}{w - z} \, dw = \sum_{n=0}^{\infty} a_n z^n, \tag{20.4}$$

and the n-th coefficient can be computed by

$$a_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(w)}{w^{n+1}} \, dw.$$

We note that the derivation of (20.4) does not require that $f(z)$ is analytic throughout the region enclosed by C_1. But we do need to assume $f(z)$ is analytic inside C_3 in (20.2).

For the second integral in (20.3), we consider the integral

$$\int_{C_1} \frac{f(w)}{z - w} \, dw$$

and write

$$\frac{1}{z - w} = \frac{1}{z(1 - w/z)} = \frac{1}{z}\Big(1 + \frac{w}{z} + \frac{w^2}{z^2} + \frac{w^3}{z^3} + \cdots\Big)$$

which converges for $|w| < |z|$. Since the location of z is outside C_1, we have convergence for all $w \in C_1$. Ignoring convergence issue for the moment, we obtain

$$\int_{C_1} \frac{f(w)}{w - z} \, dw = \sum_{n=1}^{\infty} \Big(\int_{C_1} f(w) w^{n-1} \, dw \Big) \frac{1}{z^n}. \tag{20.5}$$

This is the required principal part.

To make the argument rigorous, we can avoid the exchange of infinite sum and integral by introducing a remainder term.

$$\sum_{n=1}^{\infty} \frac{w^n}{z^n} = \sum_{n=1}^{N-1} \frac{w^n}{z^n} + \sum_{n=N}^{\infty} \frac{w^n}{z^n} = \sum_{n=1}^{N-1} \frac{w^n}{z^n} + \frac{w^N}{z^{N-1}(z - w)}.$$

The tail of the infinite summation starting from the N-th term converges to the remainder term if $|w| < |z|$. Since it is now a finite sum, we can exchange the order of summation and integral to get

$$\int_{C_1} \frac{f(w)}{w-z}\,dw = \sum_{n=1}^{N-1} \Big(\int_{C_1} f(w) w^{n-1} \Big) z^{-n} + \int_{C_1} \frac{f(w)}{z-w} \frac{w^N}{z^N}\,dw.$$

The next step is to show that the remainder term approach zeros, as N approach infinity. For $w \in C_1$ and z outside C_1, we have

$$|w| = r_1, \quad \text{and } |z - w| \geq |z| - r_1.$$

Furthermore, $|f(w)|$ is bounded by some constant M for $w \in C_1$, because C_1 is a compact set and $|f(w)|$ is a continuous function on C_1. By ML inequality (Theorem 13.3),

$$\Big| \int_{C_1} \frac{f(w)}{z-w} \frac{w^N}{z^N}\,dw \Big| \leq \frac{M}{|z| - r_1} \frac{r_1^N}{|z|^N} 2\pi r_1.$$

Because $\frac{r_1}{|z|} < 1$, $\frac{r_1^N}{|z|^N} \to 0$ as $N \to \infty$. This proves the convergence in (20.5).

Since r_1 can be any number larger than R_1, and r_2 can be any number less than R_2 (satisfying $r_1 < r_2$), we have a convergent Laurent series for any z in the annulus $R_1 < |z| < R_2$. □

Corollary 20.4. *With the same notation in the proof of Theorem 20.3, the coefficients in the Laurent series*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^{-n}$$

can be obtained by

$$a_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(w)}{w^{n+1}}\,dw \quad (n = 0, 1, 2, 3, \ldots)$$

and

$$b_n = \frac{1}{2\pi i} \int_{C_1} f(w) w^{n-1}\,dw \quad (n = 1, 2, 3, 4, \ldots).$$

21 Lecture 21 (Residue theorem)

Summary

- Uniqueness of coefficients in Laurent series
- Residue theorem
- Calculation of the residue at a pole

In Corollary 20.4 the two formula for computing the coefficients of Laurent series can be combined into one. We can integrate over any closed curve inside the annulus, winding around the origin once in the counter-clockwise orientation. If we write the Laurent series as

$$\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n,$$

then

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z_0)^{n+1}}\,dw \quad (\text{for } n = 0, \pm 1, \pm 2, \ldots)$$

We next show that the value of the coefficients have no other choice; it must be equal to the above formula. For notational convenience we suppose $z_0 = 0$. Suppose $f(z)$ is expanded as the following Laurent series

$$f(z) = \sum_{n=0}^{\infty} a'_n z^n + \sum_{n=1}^{\infty} b'_n z^{-n},$$

where a'_n and b'_n are some coefficients, and the region of convergence is an annulus with center $z = 0$. We divide the right-hand side into three parts

$$f(z) = \sum_{n=0}^{\infty} a'_n z^n + \frac{b_1}{z} + \sum_{n=2}^{\infty} b'_n z^{-n}. \tag{21.1}$$

The first part has anti-derivative

$$\sum_{n=0}^{\infty} \frac{a'_n}{n+1} z^{n+1}.$$

We use the property that Taylor series can be differentiated term-wise. The third part also has anti-derivative. To derive the anti-derivative, we can make a substitute $u = 1/z$. Let $g(u)$ denote the Taylor series

$$g(u) = \sum_{n=2}^{\infty} b'_n u^{n-2}.$$

(Note $n = 2, 3, 4, \ldots$ and $n - 2 \geq 0$.) We can differentiate termwise to obtain

$$\frac{d}{du} \sum_{n=2}^{\infty} \frac{b'_n}{n-1} u^{n-1} = g(u).$$

Using chain rule, we get

$$\frac{d}{dz} \sum_{n=2}^{\infty} \frac{b'_n}{n-1} (\frac{1}{z})^{n-1} = g(\frac{1}{z})(\frac{-1}{z^2}) = \sum_{n=2}^{\infty} b'_n z^{-n+2}(-1/z^2) = -\sum_{n=2}^{\infty} b'_n z^{-n}.$$

This shows that $\sum_{n=2}^{\infty} b'_n z^{-n}$ has an anti-derivative.

Integrate both sides of (21.1) over a close curve C inside the annulus of convergence. The first and the third term on the right side of (21.1) becomes zero, because they have anti-derivative (Theorem 13.4). Therefore

$$\int_C f(z)\, dz = \int_C \frac{b'_1}{z}\, dw = 2\pi i b_1.$$

The coefficient b_1 is uniquely determined by

$$b'_1 = \frac{1}{2\pi i} \int_C f(z)\, dz, \tag{21.2}$$

where C is a close curve traveling inside the annulus once counter-clockwise. The integral only depends on $f(z)$ and hence the coefficient b'_1 is uniquely determined.

For other coefficients in the Laurent series we can multiply by z^n, for $n \geq 1$, to get b_{n+1}

$$\int_C f(z) z^n\, dz = \int_C \frac{b'_{n+1}}{z}\, dw = 2\pi i b'_{n+1},$$

and divide by z^n, for $n \geq 1$, to get a_{n-1},

$$\int_C f(z)/z^n\, dz = \int_C \frac{a'_{n-1}}{z}\, dw = 2\pi i a'_{n-1}.$$

As a result, all coefficients are uniquely determined by $f(z)$ and the annulus of convergence.

Remark. The coefficients in a Laurent series not only depend on the function $f(z)$, they also depend on the annulus in which the function $f(z)$ is analytic. Different convergence regions will give different coefficients, because the curve C must be chosen inside the annulus.

After establishing the uniqueness of the coefficients in a Laurent series, we can now define residue as follows

Definition 21.1. Consider a complex function $f(z)$ that is analytic in a domain except some isolated singular points. The residue of $f(z)$ at a point z_0 is defined as the coefficient b_1 in the Laurent series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n},$$

with convergence in a small open disk $D(z_0; \epsilon) \setminus \{z_0\}$ centered at z_0. There are several notation for residue, e.g. $\operatorname{Res}(f; z_0)$, $\operatorname{Res}_{z_0}(f)$, and $\operatorname*{Res}_{z=z_0}(f)$.

(In view of the remark before Definition 21.1, it is important to state the region of convergence in the definition.)

When $f(z)$ is analytic at z_0, then $\operatorname{Res}(f; z_0) = 0$, because the principal part is zero.

Example 21.1. The residue of $e^{1/z}$ at $z = 0$ is 1, because the coefficient of $1/z$ in

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \cdots$$

is 1.

For pole with smaller order, the residue can be computed efficiently. If z_0 is a pole of $f(z)$ with order m, then

$$f(z) = \frac{b_m}{(z - z_0)^m} + \cdots + \frac{b_1}{z - z_0} + a_0 + a_1(z - z_0) + \cdots$$
$$(z - z_0)^m f(z) = b_m + \cdots + b_1 (z - z_0)^{m-1} + a_0 (z - z_0)^m + \cdots .$$

We can extract the coefficient b_1 by

$$b_1 = \operatorname{Res}(f; z_0) = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d}{dz} \big[(z - z_0)^m f(z)\big].$$

In particular, for pole with order 1,

$$\operatorname{Res}(f; z_0) = \lim_{z \to z_0} (z - z_0) f(z),$$

and for pole with order 2,

$$\operatorname{Res}(f; z_0) = \lim_{z \to z_0} \frac{d}{dz} (z - z_0)^2 f(z).$$

Theorem 21.2 (Residue theorem)**.** *Suppose f is analytic in a domain D except for some isolated singularities. If C is a simple closed curve enclosing singular points $z_1, z_2, \ldots, z_k$ in the interior, then*

$$\int_C f(z)\,dz = 2\pi i \sum_{j=1}^{k} \mathrm{Res}(f, z_j).$$

Proof. For $j = 1, 2, \ldots, k$, we draw a small circle C_j centered at z_j so that the circle contains does not contain the other singular points. By Cauchy theorem for multiply connected region (Theorem 16.3),

$$\int_C f(z)\,dz = \sum_{j=1}^{k} \int_{C_j} f(z)\,dz.$$

Since the residue of f at z_0 is equal to the integral $2\pi i \int_{C_j} f(z)\,dz$ (see Equation (21.2)), we obtain

$$\int_C f(z)\,dz = 2\pi i \sum_{j=1}^{k} \mathrm{Res}(f, z_j).$$

□

Example 21.2. Compute

$$\int_C \frac{dz}{z(z-1)(z-2)}$$

with C being the contour $|z| = 1.5$ with counter-clockwise orientation.

The contour C contains two poles at $z = 0$ and $z = 1$. The residues at these two poles are

$$\mathrm{Res}\left(\frac{1}{z(z-1)(z-2)}; 0\right) = \lim_{z\to 0} z \frac{1}{z(z-1)(z-2)} = \frac{1}{2}$$
$$\mathrm{Res}\left(\frac{1}{z(z-1)(z-2)}; 1\right) = \lim_{z\to 1} (z-1) \frac{1}{z(z-1)(z-2)} = -1.$$

Apply residue theorem,

$$\int_C \frac{dz}{z(z-1)(z-2)} = 2\pi i(\frac{1}{2} - 1) = -\pi i.$$

Example 21.3. Evaluate

$$\int_C \frac{dz}{z(z-1)^2}$$

over the contour $C : |z| = 2$ with counter-clockwise orientation.

The contour C encloses the simple pole at $z = 0$ and the double pole at $z = 1$.

$$\operatorname{Res}\Big(\frac{1}{z(z-1)^2};0\Big) = \lim_{z\to 0} z\frac{1}{z(z-1)^2} = 1$$
$$\operatorname{Res}\Big(\frac{1}{z(z-1)^2};1\Big) = \lim_{z\to 1} \frac{d}{dz}\Big[(z-1)^2\frac{1}{z(z-1)^2}\Big] = -1.$$

By residue theorem, the integral is equal to $2\pi i(1 + (-1)) = 0$.

22 Lecture 22 (Winding number, argument principle)

Summary

- Winding number
- A more general form of residue theorem
- Argument principle

Theorem 22.1. *If γ is a closed piece-wise smooth curve not passing through a point z_0, then*

$$\frac{1}{2\pi i}\int_\gamma \frac{1}{z-z_0}\,dz$$

is an integer.

Proof. Represent the curve by a parameterization

$$\gamma : [0, 1] \to \mathbb{C} \setminus \{z_0\}.$$

Write $\gamma(t) - z_0$ in polar form

$$\gamma(t) - z_0 = r(t) + e^{i\theta(t)}.$$

The distance between $\gamma(t)$ and z_0 is given by $r(t)$. The angle $\theta(t)$ is measured with respect to the given point z_0. Since it is assumed that $\gamma(t)$ does not pass through z_0, we have $r(t) > 0$ for all t.

Differentiating $\gamma(t)$ once to get

$$\gamma'(t) = [r'(t) + ir(t)\theta'(t)]e^{i\theta(t)}.$$

Using the definition of complex integral, we compute

$$\begin{aligned}
\frac{1}{2\pi i}\int_\gamma \frac{1}{z-z_0}\,dz &= \frac{1}{2\pi i}\int_0^1 \frac{[r'(t)+ir(t)\theta'(t)]e^{i\theta(t)}}{r(t)e^{i\theta(t)}}\,dt \\
&= \frac{1}{2\pi i}\int_0^1 r'(t)/r(t)\,dt + \frac{1}{2\pi}\int_0^1 \theta'(t)\,dt \\
&= \frac{1}{2\pi i}[\log(r(1))-\log(r(0))] + \frac{1}{2\pi}[\theta(1)-\theta(0)].
\end{aligned}$$

The first term is equal to zero, because $r(0) = r(1)$. The second term is precisely the number of time the curve γ goes around the point z_0. $\square$

Based on the previous theorem, we can make the following definition

Definition 22.2. The winding number of a closed curve γ around a point z_0 is defined as

$$n(\gamma; z_0) \triangleq \frac{1}{2\pi i}\int_\gamma \frac{1}{z-z_0}\,dz.$$

Other notation for the winding number includes: $w(\gamma, z_0)$, $ind(\gamma, z_0)$.

The Jordan curve theorem (see Remark 14), the winding number of a simple closed curve about a fixed point z_0 is either 0, or ± 1, depending on whether the point z_0 is inside or outside the curve, and whether the orientation of the curve is positive or negative. For closed curve in general (which need not be simple), the winding number could be any integer.

Theorem 22.3 (Generalized residue theorem)**.** *Suppose f is analytic in a simply connected domain except k isolated singular points z_1, $z_2, \ldots, z_k$. For any closed and piecewise smooth curve C, not intersecting any one of the k singular points, we have*

$$\int_C f(z)\,dz = 2\pi i \sum_{j=1}^{k} n(C; z_j)\,\mathrm{Res}(C; z_j). \tag{22.1}$$

We note that when C is a simple closed path with positive orientation, the generalized residue theorem reduces to the residue theorem (Theorem 21.2). The winding numbers in 22.1 are some weighting factor for the corresponding residue. The weight factor is 0 if the singular point is outside the curve.

Proof. For each$j = 1, 2, \ldots, k$, expand the function $f(z)$ using Laurent series at z_j. Denote the principal part by $p_j((z - z_j)^{-1})$, for $j = 1, 2, \ldots k$. We mote that $p_j((z - z_j)^{-1})$ is analytic except at the point z_j.

Consider the function

$$g(z) \triangleq f(z) - \sum_{j=1}^{k} p_j((z - z_j)^{-1})$$

obtained by subtracting all the principal parts $p_1((z - z_1)^{-1})$ to $p_k((z - z_k)^{-1})$ from $f(z)$. For any point other than the k singular points, the sum $\sum_j p_j((z - z_j)^{-1})$ is analytic. At the point z_j, $g(z)$ can be written as

$$g(z) = f(z) - p_j((z - z_j)^{-1}) - \sum_{\substack{\ell=1 \\ \ell \neq j}}^{k} p_\ell((z - z_\ell)^{-1}).$$

But $f(z) - p_j((z - z_j)^{-1})$ is analytic at z_j. Hence $g(z)$ is analytic at z_j. So, $g(z)$ is analytic at all points in the domain. By Cauchy theorem (Theorem 14.6),

$$\int_C g(z)\, dz = 0.$$

This yields

$$\int_C g(z)\, dz = \sum_{j=1}^{k} \int_C p_j((z - z_j)^{-1}).$$

For each $j = 1, 2, \ldots$, the integral $\int_C p_j((z - z_j)^{-1})$ only depends on the term with degree -1 (see the proof of the uniqueness of the coefficients of Laurent series),

$$\begin{aligned}
\int_C p_j((z - z_j)^{-1}) &= 2\pi i \operatorname{Res}(f; z_j) \frac{1}{2\pi i} \int_C \frac{1}{z - z_j}\, dz \\
&= 2\pi i \operatorname{Res}(f; z_j) n(\gamma; z_j).
\end{aligned}$$

This proves (22.1). □

We next go back to the simpler case of simple closed curve in the next theorem

Theorem 22.4 (Argument principle)**.** *Suppose C is the boundary of a simply connected region and f is analytic inside C and on the boundary of C. Then*

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)}\, dz = \text{ no. of zeros inside } C \text{ (counted with multiplicity).}$$

Proof. Suppose z_1, $z_2, \dots, z_k$ are the zeros of f inside C. For each $j = 1, 2, \dots, k$, suppose the order of zero at z_j is m_j. We note that f'/f is analytic except at z_j, for $j = 1, 2, \dots, k$. Let C_j be a small circle centered at z_j, so that C_j lies inside C and the circle C_1, $C_2, \dots, C_k$ do not overlap. By Cauchy theorem for multiply connected region (Theorem 16.3),

$$\int_C \frac{f'}{f}\, dz = \sum_{j=1}^{k} \int_{C_j} \frac{f'}{f}\, dz.$$

For $j = 1, 2, \dots, k$, We can write $f(z)$ as

$$\begin{aligned} f(z) &= a_{m_j}(z - z_j)^{m_j} + a_{m_j+1}(z - z_j)^{m_j+1} + a_{m_j+2}(z - z_j)^{m_j+2} + \cdots \\ f'(z) &= m_j a_{m_j}(z - z_j)^{m_j-1} + (m_j + 1)a_{m_j+1}(z - z_j)^{m_j} + (m_j + 2)a_{m_j+2}(z - z_j)^{m_j+1} + \cdots . \end{aligned}$$

Here a_{m_j} is a nonzero coefficient. Factor out $a_{m_j}(z - z_j)^{m_j}$ from $f(z)$.

$$f(z) = a_{m_j}(z - z_j)^{m_j} \big[1 + \frac{a_{m_j+1}}{a_{m_j}}(z - z_j) + \frac{a_{m_j+2}}{a_{m_j}}(z - z_j)^2 + \cdots \big]$$

We see that the expression inside the square bracket is an analytic function in a small neighborhood of z_j. Moreover, the value of this analytic function at z_j is 1. Denote the expression inside the square bracket by $h(z)$. We have $h(z) \neq 0$ in a small neighborhood of z_j and $h(z_j) = 1$. We can take the reciprocal of $h(z)$ in a small enough neighborhood of z_j, and get

$$\begin{aligned} \frac{f'}{f} &= \frac{m_j a_{m_j}(z - z_j)^{m_j-1} + (m_j + 1)a_{m_j+1}(z - z_j)^{m_j} + \cdots}{a_{m_j}(z - z_j)^{m_j}} \frac{1}{h(z)} \\ &= \frac{m_j}{z - z_j} \frac{1}{h(z)} + \text{an analytic function.} \end{aligned}$$

We use the property that $h^{-1}(z)$ can be expanded as a Taylor series centered at z_j with constant term 1,

$$\frac{1}{h(z)} = c_0 + c_1(z - z_j) + c_2(z - z_j)^2 + \cdots$$

The constant term c_0 must equal 1. Substitute the above Taylor series expansion to the expression about f'/f, we get

$$\frac{f'}{f} = \frac{m_j}{z - z_j} + \text{ another analytic function.}$$

We adjust the size of C_j so that C_j lies within the region in which $h(z)$ is nonzero. With this choice of C_j the integral of the analytic function in the previous equation along C_j is zero. Meanwhile, the integral of $1/(z-z_j)$ around C_j is $2\pi i$. We finally obtain

$$\int_{C_j} \frac{f'}{f}\,dz = 2\pi i m_j.$$

Summing it over all $j = 1, 2, \ldots, k$, we get

$$\frac{1}{2\pi i}\int_C \frac{f'}{f}\,dz = \frac{1}{2\pi i}\sum_{j=1}^{k}\int_{C_j}\frac{f'}{f}\,dz = m_1 + m_2 + \cdots + m_k.$$

This is the number zeros inside C, counted with multiplicity. $\square$

23 Lecture 23 (Rouche's theorem)

Summary

- A connection between principle argument and winding number
- Rouche theorem

In this lecture we derive Rouche's theorem using the principle argument. We will use the following connection between the argument principle and the winding number.

Lemma 23.1. *Suppose C is a simple closed curve parameterized by $\gamma(t)$, for $a \le t \le b$. Then*

$$\frac{1}{2\pi i}\int_C \frac{f'}{f}\,dz = n(f(\gamma(t)); 0).$$

That is, the number of zeros inside C is exactly the same as the number of time the curve $f(\gamma(t))$ goes around the origin.

Proof. Consider the curve C' that is parameterized by $f(\gamma(t))$, for t in $[a,b]$. and let $g(t) = f(\gamma(t))$. The winding number of C' around 0 is

$$\begin{aligned} n(C';0) &= \frac{1}{2\pi i}\int_{C'} \frac{1}{z}\,dz \\ &= \frac{1}{2\pi i}\int_a^b \frac{g'(t)}{g(t)}\,dt \\ &= \frac{1}{2\pi i}\int_a^b \frac{f'(\gamma(t))\gamma'(t)}{f(\gamma(t))}\,dt. \end{aligned}$$

On the other hand, we have

$$\frac{1}{2\pi i}\int_C \frac{f'}{f}\,dz = \frac{1}{2\pi i}\int_a^b \frac{f'(\gamma(t))}{f(\gamma(t))}\gamma'(t)\,dz.$$

$\square$

Theorem 23.2 (Rouché's theorem)**.** *Suppose C is a simple closed curve with positive orientation. If f and g are functions analytic in a neighborhood containing C and*

$$|f(z)| > |g(z)|$$

for all $z \in C$, then the number of zeros of $f+g$ inside C is the same as the number of zeros of f inside C.

We note that the assumption $|f(z)| > |g(z)|$ implies (i) $f(z) \neq 0$ for all $z \in C$, and (ii) $f(z) + g(z) \neq 0$ for all $z \in C$.

Proof. Parameterize the curve C by $\gamma(t)$, for $t \in [0,1]$. The main idea of proof is that the curve $g(\gamma(t))/f(\gamma(t))$, for $t \in [0,1]$ lies inside the circle $|z-1| = 1$ with radius 1 and center at $z = 1$. The winding number of the curve $g(\gamma(t))/f(\gamma(t))$ around the origin is thus zero.

By the argument principle (Theorem 22.4), the number of zeros of $f+g$ inside C can be computed by

$$\frac{1}{2\pi i}\int_C \frac{f'+g'}{f+g}\,dz.$$

Using product rule for differentiation, we can re-write the integrand as

$$\frac{f'+g'}{f+g} = \frac{f'}{f'} + \frac{\left(1+\frac{g}{f}\right)'}{1+\frac{g}{f}}.$$

Therefore

$$\frac{1}{2\pi i}\int_C \frac{f'+g'}{f+g}\,dz = \frac{1}{2\pi i}\int_C \frac{f'}{f'}\,dz + \frac{1}{2\pi i}\int_C \frac{\left(1+\frac{g}{f}\right)'}{1+\frac{g}{f}}\,dz$$

If we let $h(z) = 1 + g(z)/f(z)$, then the second integral on the right is equal to the number of time the curve $h \circ \gamma$ goes around the origin (see Lemma 23.1) But we have just shown that the curve lies completely inside the right half plane $\{x+iy \in \mathbb{C} : x > 0\}$. Hence the winding number $n(h\circ\gamma, 0)$ is equal to 0. This proves

$$\frac{1}{2\pi i}\int_C \frac{f'+g'}{f+g}\,dz = \frac{1}{2\pi i}\int_C \frac{f'}{f'}\,dz.$$

The integral on the right-hand side is the number of zeros of f inside the curve C, by the argument principle. $\square$

Example 23.1. Using Rouche's theorem, we can show that the polynomial $z^{100} + 3z^3 + 1$ has exactly 3 complex roots inside the unit circle. For $|z| = 1$, we check that

$$|3z^3| = 3, \quad \text{but } |z^{100} - 1| < 2.$$

Apply Rouche's theorem with $f(z) = 3z^3$ and $g(z) = z^{100} - 1$. The number roots of $z^{100} + 3z^3 + 1$ inside the unit circle is the same as the number of roots of $3z^3$ inside the unit circle. Since $3z^3$ has a triple root at $z = 0$, $z^{100} + 3z^3 + 1$ has three roots inside the unit circle.

Example 23.2. Consider a polynomial of degree n with leading coefficient equal to 1,

$$h(z) = z^n + c_1 z^{n-1} + c_2 z^{n-2} + \cdots + c_n.$$

Show that there is some point z on the unit circle such that $|h(z)| \geq 1$.

Suppose on the contrary that $|h(z)| < 1$ for all z on the unit circle. Apply the Rouche theorem with $f(z) = z^n$ and $g(z) = -h(z)$. We can check that the condition $|f(z)| > |g(z)|$ is satisfied on the unit circle, i.e.,

$$|f(z)| = |z^n| = 1 > |-h(z)| = |g(z)| \quad \text{for all } z \text{ with } |z| = 1.$$

By Rouche theorem, the function f and $f + g$ have the same number of zeros inside the unit circle. On one hand, $f(z) = z^n$ has exactly n zeros, namely, n repeated roots at $z = 0$, inside the unit circle. However

$$f(z) + g(z) = c_1 z^{n-1} + c_2 z^{n-2} + \cdots + c_n$$

has at most $n - 1$ zeros. The polynomial $f + g$ cannot have n zeros inside the unit circle. This contradiction shows that there must be some point z with $|z| = 1$ such that $|h(z)| \geq 1$.

24 Lecture 24 (Evaluation of real integrals)

<ins>Summary</ins>

- Trigonometric integral

- Integral of rational function
- Jordan lemma

The integral of a rational function $R(\cos\theta, \sin\theta)$ in $\cos\theta$ and $\sin\theta$, for θ from 0 to 2π, can be transformed to complex integral,

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta = \int_{|z|=1} \frac{1}{iz} R\Big(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\Big)\, dz.$$

The complex integral on the right is computed on the unit circle, with counter-clockwise orientation. Recall that $z = e^{i\theta}$ when z is on the unit circle, and $z'(\theta) = iz$. The integrand on the right is a rational function in z, and can be evaluated using residue theorem.

Example 24.1. Show that

$$\int_0^{2\pi} \frac{1}{1 + a\cos\theta}\, d\theta = \frac{2\pi}{\sqrt{1-a^2}} \quad \text{for } a \in \mathbb{R},\ |a| < 1$$

Let C be the unit circle with clock-wise orientation. Parameterize C by $z(\theta)$ for $0 \leq \theta \leq 2\pi$. The trigonometric integral can be written as

$$\begin{aligned} \int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta} &= \int_C \frac{dz}{iz(1 + a(\frac{z+z^{-1}}{2}))} \\ &= \frac{2}{i} \int_C \frac{dz}{az^2 + 2z + a}. \end{aligned}$$

The denominator is a quadratic function. Let α and β be the roots

$$\alpha = \frac{-1 + \sqrt{1-a^2}}{a}, \quad \beta = \frac{-1 - \sqrt{1-a^2}}{a}.$$

Both α and β are real roots. We can show that α is inside the unit circle, while β is outside the unit circle. For example, when $a > 0$, we have $\beta < -1$ and $0 > \alpha = 1/\beta > -1$. Likewise, we can show that when $a < 0$, we have $\beta > 1$ and $0 < \alpha < 1$. When $a = 0$, there is only one root at $z = 0$. We may assume $0 < |a| < 1$ in the following calculations.

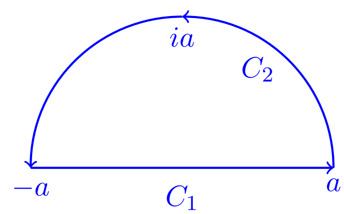


Figure 1: Boundary of a semi-circle

By the residue theorem (Theorem 21.2), we can compute

$$\begin{aligned}\int_0^{2\pi} \frac{1}{1+a\cos\theta}\,d\theta &= 4\pi \operatorname{Res}\big(\frac{1}{a(z-\alpha)(z-\beta)};\alpha\big)\\ &= 4\pi\frac{1}{a}\Big(\frac{1}{z-\beta}\Big)\Big|_{z=\alpha}\\ &= \frac{4\pi}{a(\alpha-\beta)}\\ &= \frac{2\pi}{\sqrt{1-a^2}}.\end{aligned}$$

Consider a real integral in the form

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx$$

where $P(x)$ and $Q(x)$ are polynomial function and $\deg Q(x) \geq 2 + \deg P(x)$. The function $Q(x)$ in the denominator is assumed to be analytic on the real axis.

We can use complex integral to compute the principal value

$$\lim_{a\to\infty}\int_{-a}^{a} \frac{P(x)}{Q(x)}\,dx$$

using contour shown in Figure 1.

The first part C_1 is a line segment from $-a$ to a. The real integral $\int_{-a}^{a} P(x)/Q(x)\,dx$ is the same as the complex integral $\int_{C_1} P(z)/Q(z)\,dz$.

The second part C_2 is a semi-circle in the upper half plane from a to $-a$. The assumption that $\deg Q \geq 2 + \deg P$ implies that the integral $\int_{C_2} P(z)/Q(z)\,dz$ approaches 0 as $a \to \infty$.

Example 24.2. Show that
$$\int_0^\infty \frac{1}{1+x^2}\,dx = \frac{\pi}{2}. \tag{24.1}$$

Since the integrand is an even function, it is equivalent to

$$\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx = \pi.$$

Referring to the contour in Fig. 1, when $a>1$, the pole at $z=i$ is inside the contour, and the residue is
$$\operatorname{Res}\big(\frac{1}{1+z^2};i\big) = \Big(\frac{1}{z+i}\Big)\Big|_{z=i} = \frac{1}{2i}.$$

The integral along C_1 is

$$\int_{C_1} \frac{1}{1+z^2}\,dz = \int_{-a}^{a} \frac{1}{1+x^2}\,dx.$$

When z is a point on C_2, the function value is upper bounded by $\frac{1}{a^2-1}$. By ML inequality (Theorem 13.3),
$$\Big|\int_{C_2} \frac{1}{1+z^2}\,dz\Big| \leq \frac{1}{a^2-1}(\pi a) = O(1/a).$$

The modulus converges to zero as $a\to\infty$. Hence, the integral to be computed can be written as

$$\begin{aligned}
\lim_{a\to\infty}\int_{-a}^{a} \frac{1}{1+x^2}\,dx &= \lim_{a\to\infty}\int_{C_1}\frac{1}{1+z^2}\,dz \\
&= 2\pi i \operatorname{Res}\big(\frac{1}{1+z^2};i\big) - \lim_{a\to\infty}\int_{C_2}\frac{1}{1+z^2}\,dz \\
&= \pi - 0 = \pi.
\end{aligned}$$

This proves (24.1).

Example 24.3. Derive

$$\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^{n+1}}\,dx = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\pi$$

for integer $n\geq 1$.

We use the same contour as in Fig. 1. Let $f(z)$ denote the complex function $\frac{1}{(1+z^2)^{n+1}}$. By residue theorem (Theorem 21.2), we obtain

$$\int_{C_1} + \int_{C_2} = 2\pi i \operatorname{Res}(f;i).$$

We can use similar analysis as in the previous example to show that the integral over C_2 tends to 0 as the radius approach infinity. Hence, what we need to show is

$$2\pi i \operatorname{Res}(f; i) = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\pi. \tag{24.2}$$

In this example, the pole at $z = i$ has order $n+1$. We first calculate

$$\begin{aligned}\frac{d^n}{dz^n}(z-i)^{n+1}f(z) &= \frac{d^n}{dz^n}\frac{1}{(z+i)^{n+1}} \\ &= \frac{(-1)^n(n+1)(n+2)\cdots(2n)}{(z+i)^{2n+1}}\end{aligned}$$

Then we take limit as z tends to i and multiply by $2\pi i/n!$,

$$\begin{aligned}\frac{2\pi i}{n!}\lim_{z\to i}\frac{d^n}{dz^n}(z-i)^{n+1}f(z) &= \frac{2\pi i}{n!}\frac{(-1)^n(n+1)(n+2)\cdots(2n)}{(2i)^{2n+1}} \\ &= \frac{\pi}{n!}\frac{(n+1)(n+2)\cdots(2n)}{2^{2n}} \\ &= \pi\frac{(2n)!}{2^2\cdot 4^2\cdot 6^2\cdots(2n)^2} \\ &= \pi\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}.\end{aligned}$$

This proves (24.2) and completes the derivation.

Lemma 24.1 (Jordan lemma)**.** *Consider the contour C_R shown in Fig. 2 Assume f is analytic on the contour C_R for all sufficiently large R. If $|f(z)| \le M_R$ for z on the semi-circle C_R and $M_R \to 0$ as $R \to$, then*

$$\lim_{R\to\infty}\int_{C_R} f(z)e^{iaz}\,dz = 0 \quad \textit{for any real constant } a > 0.$$

Proof. We first prove the following Jordan inequality, which is an inequality for real integral,

$$\int_0^\pi e^{-R\sin\theta}\,d\theta < \frac{\pi}{R} \quad \text{for } R > 0. \tag{24.3}$$

For θ in the range $0 \le \theta \le \pi/2$, the value $\sin\theta$ is larger than or equal to $2\theta/\pi$. We can see this by comparing the curve of $\sin\theta$ for $0 \le \theta \le \pi/2$ and the line segment from the origin

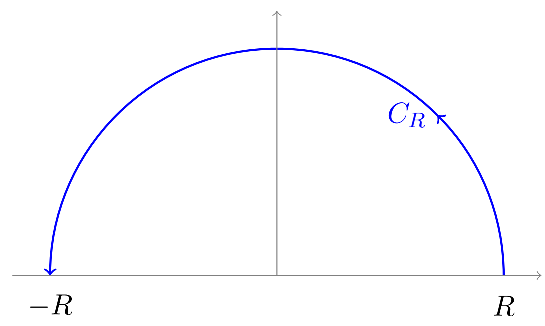


Figure 2: The semi-circular contour in Jordan lemma.

to the point $(\pi/2, 1)$. This yields

$$\begin{aligned}\int_0^{\pi/2} e^{-R\sin\theta}\,d\theta &\leq \int_0^{\pi/2} e^{-R(2\theta/\pi)}\,d\theta \\ &= \Big[\frac{-\pi}{2R}e^{-R2\theta/\pi}\Big]_0^{\pi/2} \\ &= \frac{\pi}{2R}(1-e^R) \\ &< \frac{\pi}{2R}.\end{aligned}$$

By using the symmetry of the graph of sine function, we can see that the same argument apply to the second part of the interval from $\pi/2$ to π,

$$\int_{\pi/2}^{\pi} e^{-R\sin\theta}\,d\theta < \frac{\pi}{2R}.$$

This proves (24.3).

For real constant $a > 0$, we have

$$\int_{C_R} f(z)e^{iaz}\,dz = \int_0^{\pi} f(Re^{i\theta})e^{iaR(\cos\theta + i\sin\theta)}(Rie^{i\theta})\,d\theta.$$

By triangle inequality for complex integral (Theorem (13.1)),

$$\Big|\int_0^{\pi} f(Re^{i\theta})e^{iaR(\cos\theta+i\sin\theta)}(Rie^{i\theta})\,d\theta\Big| \leq RM_R\int_0^{\pi} e^{-aR\sin\theta}\,d\theta.$$

By Jordan inequality, the integral on the right-hand side is less than or equal to $\pi/(aR)$. As a result, the modulus of $\int_{C_R} f(z)e^{iaz}\,dz$ is upper bounded by $\pi M_R/a$. Since a is a constant and M_R approaches 0, the upper bound approaches 0 as R approaches ∞. $\square$

Jordan lemma is useful in evaluating integral of type

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos(x)\, dx \quad \text{or} \quad \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \sin(x)\, dx$$

where $P(x)$ and $Q(x)$ are polynomials and $\deg Q - \deg P \geq 1$.

Example 24.4. Derive

$$\int_{-\infty}^{\infty} \frac{\cos(bx)}{1+x^2}\, dx = \frac{\pi}{e^b} \quad \text{for } b > 0.$$

Since $(\sin x)/(1+x^2)$ is an odd function, establishing the above equation is the same as showing

$$p.v. \int_{-\infty}^{\infty} \frac{e^{ibx}}{1+x^2}\, dx = \frac{\pi}{e^b} \quad \text{for } b > 0.$$

Let C_1 and C_2 be the contours in Figure 1. We compute the complex integral

$$\int_{C_1+C_2} \frac{e^{iz}}{1+z^2}\, dz$$

along the closed path formed by C_1 and C_2.

By Lemma 24.1,

$$\Big| \int_{C_2} \frac{e^{iz}}{1+z^2}\, dz \Big| \to 0, \qquad \text{as } a \to \infty.$$

Hence

$$\lim_{a\to\infty} \int_{C_1+C_2} \frac{e^{iz}}{1+z^2}\, dz = \int_{-\infty}^{\infty} \frac{e^{ibx}}{1+x^2}\, dx.$$

On the other hand, the residue of $\frac{e^{ibz}}{1+z^2}$ at $z = i$ is

$$\operatorname{Res}\big(\frac{e^{ibz}}{1+z^2}; i\big) = \frac{e^{ibz}}{z+i}\Big|_{z=i} = \frac{e^{-b}}{2i}.$$

By residue theorem (Theorem 21.2), we get

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}\, dx = \int_{-\infty}^{\infty} \frac{e^{ibx}}{1+x^2}\, dx = 2\pi i \frac{e^{-b}}{2i} = \frac{\pi}{e^b}.$$

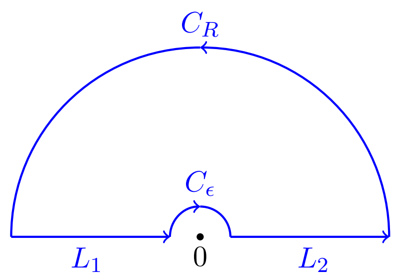
Example 24.5. Show that

$$\int_0^{\infty} \frac{\sin x}{x}\, dx = \frac{\pi}{2}.$$

Since $\frac{\sin x}{x}$ is an even function and $\frac{\cos x}{x}$ is odd, it is sufficient to prove

$$p.v. \quad \int_{-\infty}^{\infty} \frac{e^{ix}}{x}\, dx = \pi i.$$

The function $\frac{e^{iz}}{z}$ has a pole at the origin. In order to avoid the pole, we consider the following indented contour.



The outer semi-circle has radius R and the inner semi-circle has radius ϵ.

The function $\frac{e^{iz}}{z}$ is analytic inside the contour. By Cauchy theorem (Theorem 14.6), we have

$$\int_{L_1+L_2+C_R+C_\epsilon} \frac{e^{iz}}{z}\, dz = 0.$$

By Jordan lemma (Lemma 24.1), the integral of $\frac{e^{iz}}{z}$ along C_R approaches 0 as $R \to \infty$. The integrals on the real axis approaches $\int_{-\infty}^{\infty} e^{iz}/z\, dz$ as $R \to \infty$ and $\epsilon \to 0$. The problem reduces to proving

$$\int_{C_\epsilon} \frac{e^{iz}}{z}\, dz = -\pi i.$$

The integrand can be represented by Laurent series

$$\frac{e^{iz}}{z} = \frac{1}{z} + i - \frac{z}{2} - \frac{iz^2}{6} + \cdots$$

For small enough $\epsilon > 0$, the analytic part $+i - \frac{z}{2} - \frac{iz^2}{6} + \cdots$ is bounded (because it converges and is continuous at $z = 0$). By ML inequality (Theorem 13.3),

$$\Big| \int_{C_\epsilon} +i - \frac{z}{2} - \frac{iz^2}{6} + \cdots\, dz \Big| \to 0$$

as $\epsilon \to 0$. The integral of $1/z$ on C_ϵ is equal to

$$\int_{C_\epsilon} \frac{1}{z}\, dz = \int_{-\pi}^{0} \frac{i\epsilon e^{i\theta}}{\epsilon e^{i\theta}}\, d\theta = -\pi i.$$

This proves that

$$i\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \int_{-\infty}^{\infty} \frac{e^{iz}}{z}\,dz = \lim_{\substack{R\to 0\\ \epsilon\to 0}} \int_{L_1+L_2} \frac{e^{iz}}{z}\,dz = -\lim_{\epsilon\to 0}\int_{C_\epsilon} \frac{e^{iz}}{z}\,dz = i\pi.$$

25 Lecture 25 (Keyhole contour, analytic at infinity)

Summary

- Evaluating real integral in the form $\int_0^\infty P(x)/Q(x)\,dx$.
- Being analytic at the point at infinity
- Residue at the point at infinity.

We can use the keyhole contour in Fig. 3 to evaluate real integral

$$\int_0^\infty \frac{P(x)}{Q(x)}\,dx,$$

where $\deg Q \geq \deg P + 2$ and $Q(x) \neq 0$ for $x \geq 0$. We demonstrate the procedure using the following example:

$$\text{Evaluate } \int_0^\infty \frac{1}{x^3+1}\,dx.$$

The first step is to multiply the function to be integrated by a log function, and consider the complex integral

$$\int_C \frac{\log z}{z^3+1}\,dz,$$

over the keyhole contour C as shown in Fig. 3. For the complex log function we take the nonnegative real axis as the branch cut, i.e., for complex number in polar form $re^{i\theta}$, with $0 < \theta < 2\pi$, the log function is evaluated as

$$\log r + i\theta \quad \text{for } 0 < \theta < 2\pi.$$

The contour C consists of four parts. The outer circle C_R has radius R and positive orientation. The inner circle C_ϵ has radius ϵ and negative orientation. The distance between L_1 and L_2 is 2δ. When $R \to \infty$, $\epsilon \to 0$ and $\epsilon \to 0$, the integrals along L_1 and L_2 have limits

$$\int_{L_1} \frac{\log z}{z^3+1}\,dz \to \int_0^\infty \frac{\log x}{x^3+1}\,dx$$

$$\int_{L_2} \frac{\log z}{z^3+1}\,dz \to -\int_0^\infty \frac{\log x + 2\pi i}{x^3+1}\,dx.$$

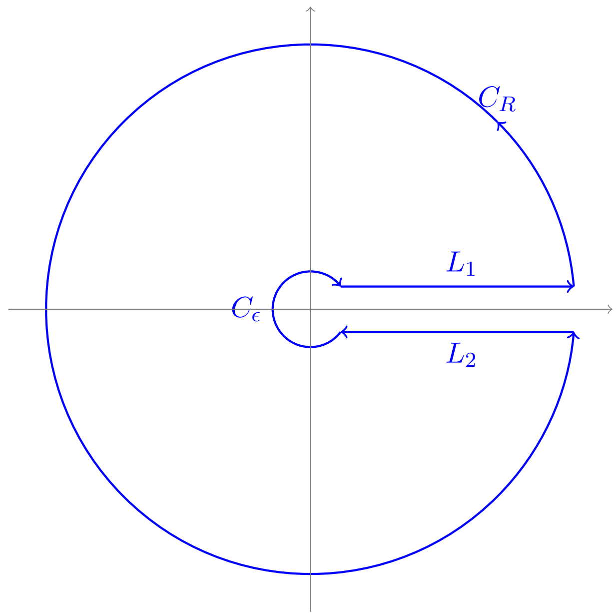


Figure 3: Keyhole contour

The integral over C_ϵ has modulus upper bounded by

$$\Big| \int_{C_\epsilon} \frac{\log z}{z^3+1}\, dz \Big| \le 2\pi\epsilon M_\epsilon \max_{|z|=\epsilon} |\log z|$$

where M_ϵ denotes the maximum of $1/(z^3+1)$ on the circle $|z| = \epsilon$. Since it is assumed that $Q(z)$ is defined at $z = 0$, M_ϵ can be upper bounded by another constant independent of ϵ. In this example M_ϵ is approach 1 as ϵ approaches 0, and hence we can say that $M_\epsilon < 2$ for all sufficiently small ϵ. The modulus of $\log(z)$ is no more than the modulus of $\log(\epsilon) + i2\pi$. Hence, as $\epsilon \to 0$, the modulus of the integral of C_ϵ is upper bounded by a constant times $\epsilon|\log\epsilon|$, which decreases to zero as $\epsilon \to 0$.

For complex number z on C_R, the modulus $|\log(z)/(z^3+1)|$ is upper bounded by a constant times $\frac{\log R}{R^3-1}$. The integral over C_R approaches 0 in the order of $O(R\frac{\log R}{R^3})$.

Therefore,

$$\lim_{\substack{\epsilon\to 0\\ \delta\to 0\\ R\to\infty}} \int_{L_1+L_2+C_R+C_\epsilon} \frac{\log(z)}{z^3+1}\, dz = -2\pi i \int_0^\infty \frac{1}{x^3+1}\, dx.$$

We can re-write the above equation as

$$\int_0^\infty \frac{1}{x^3+1}\,dx = -\lim_{\substack{\epsilon\to 0\\ \delta\to 0\\ R\to\infty}} \frac{1}{2\pi i}\int_{L_1+L_2+C_R+C_\epsilon} \frac{\log(z)}{z^3+1}\,dz.$$

The polynomial z^3+1 has three roots, namely, -1, $e^{\pi i/3}$ and $e^{5\pi i/3}$. By residue theorem (Theorem 21.2), we can compute the integral by

$$\int_0^\infty \frac{1}{x^3+1}\,dx = -\Big[\operatorname{Res}\big(\frac{\log(z)}{z^3+1};-1\big) + \operatorname{Res}\big(\frac{\log(z)}{z^3+1};e^{\pi i/3}\big) + \operatorname{Res}\big(\frac{\log(z)}{z^3+1};e^{5\pi i/3}\big)\Big].$$

Since the pole of $\log(z)/(z^3+1)$ are all simple roots, we can evaluate the residues at -1, $e^{\pi i/3}$ and $e^{5\pi i/3}$ by

$$\begin{aligned}
\operatorname{Res}\big(\frac{\log(z)}{z^3+1};-1\big) &= \frac{\log(z)}{3z^2}\Big|_{-1} = \pi i\frac{1}{3}\\
\operatorname{Res}\big(\frac{\log(z)}{z^3+1};e^{\pi i/3}\big) &= \frac{\log(z)}{3z^2}\Big|_{e^{\pi i/3}} = \frac{\pi i}{3}\frac{e^{-2\pi i/3}}{3}\\
\operatorname{Res}\big(\frac{\log(z)}{z^3+1};e^{5\pi i/3}\big) &= \frac{\log(z)}{3z^2}\Big|_{e^{5\pi i/3}} = \frac{5\pi i}{3}\frac{e^{-10\pi/3}}{3}.
\end{aligned}$$

(See Question 5 in Homework 14.)

Adding the three residues, we get

$$\frac{i\pi}{3}\big[1+\frac{-1-\sqrt{3}i}{6}+5\frac{-1+\sqrt{3}i}{6}\big] = -\frac{2\sqrt{3}}{9}\pi.$$

Hence, the answer is

$$\int_0^\infty \frac{1}{x^3+1}\,dx = \frac{2\sqrt{3}}{9}\pi.$$

To understand the behavior of a function $f(z)$ at the point at infinity, we make a change of variable $w = 1/z$. The new variable $1/z$ is called the local parameter at ∞.

Definition 25.1. Given a complex function $f(z)$, make a change of variable and define a new function $g(w) = f(1/w)$. We say that the function $f(z)$ is analytic at $z=\infty$ if $g(w)$ is analytic at $w=0$. The point at infinity is said to be a removable singularity (resp. pole, or essential singularity) if $g(w)$ has a removable regularity (resp. pole, or essential singularity) at $w=0$.

Example 25.1. The function $f(z) = z$ has a simple pole at $z = \infty$, because $g(w) = f(1/w) = 1/w$ has a simple pole at $w = 0$.

Example 25.2. The function $f(z) = 1/z$ has a simple zero at $z = \infty$, because $g(w) = f(1/w) = w$ has a simple zero at $w = 0$.

Example 25.3. The function $f(z) = e^z$ has an essential singularity at $z = \infty$, because

$$g(w) = \exp(1/w) = 1 + \frac{1}{w} + \frac{1}{2w^2} + \cdots$$

has an essential singularity at $w = 0$.

Definition 25.2. Suppose $f(z)$ has finitely many singular points in the complex plane, so that f converges in the domain $R < |z|$ for some R. The residue at ∞ of $f(z)$ is defined as

$$\operatorname{Res}(f;\infty) \triangleq \frac{1}{2\pi i}\int_{C_0} f(z)\,dz$$

where C_0 is a circle containing all singular points in the interior, with *clockwise orientation.*

The assumption that $f(z)$ has finitely many singular points is the same as assuming that the point at infinity is an isolated singular point.

By making a change of variable $w = 1/z$, $dw = -1/z^2\,dz$, we get

$$\frac{1}{2\pi i}\int_{C_0} f(z)\,dz = \frac{1}{2\pi i}\int_{C} \frac{-1}{w^2} f\Big(\frac{1}{w}\Big)\,dw = -\operatorname{Res}\Big(\frac{1}{w^2} f\Big(\frac{1}{w}\Big);0\Big).$$

where C is the image of C_0 under the transformation $w = 1/z$. In the w-plane, the function $\frac{-1}{w^2} f\big(\frac{1}{w}\big)$ has a isolated singularity at $w = 0$. This proves the following property.

Theorem 25.3. *Suppose f has finitely many singular points and γ is a contour with positive orientation, containing all singluar points in the interior. Then*

$$\frac{1}{2\pi i}\int_{\gamma} f(z)\,dz = \operatorname{Res}\Big(\frac{1}{w^2} f\Big(\frac{1}{w}\Big);0\Big).$$

Example 25.4. Evaluate

$$\int_{|z|=2} \frac{4z+1}{z(z-1)}\,dz.$$

There are two simple poles at $z = 0$ and $z = 1$. Both of them are inside the contour. Using the previous theorem, we can calculate the complex integral by

$$\begin{aligned}\int_{|z|=2} \frac{4z+1}{z(z-1)}\, dz &= 2\pi i \operatorname{Res}\Big(\frac{1}{w^2}\frac{4(1/w)+1}{(1/w)((1/w)-1)}; 0\Big) \\ &= 2\pi i \operatorname{Res}\Big(\frac{4+w}{w(1-w)}; 0\Big) \\ &= 2\pi i \cdot 4 = 8\pi i.\end{aligned}$$

26 Lecture 26 (Further properties of analytic functions)

Summary

- Morera theorem
- Maximum modulus principle
- Schwarz lemma
- Locally one-to-one transformation

Morera theorem is a partial converse to Cauchy theorem.

Theorem 26.1 (Morera theorem)**.** *Consider a continuous function f in a domain D. If*

$$\int_C f(z)\, dz = 0$$

for all closed curves in D, then f is analytic in D.

Proof. The theorem is proved by exhibiting an anti-derivative of f. Pick a point z_0 in D. For each other point $z \in D$, we can find a path from z_0 to z, because the domain D is assumed to be connected. Since $\int_C f\, dz$ is zero for all closed path C, the integral from z_0 to z is independent of path. We can define a function F by

$$F(z) \triangleq \int_{z_0}^{z} f(z)\, dz,$$

with the integral taken over any smooth curve from z_0 to z. The function $F(z)$ so defined is differentiable and $F'(z) = f(z)$ for all $z \in D$. (See the proof of Theorem 13.4.) □

Morera theorem implies that a continuous but non-differentiable function cannot have an anti-derivative.

Remark. In compare to real analysis, we note that a continuous real function in one variable always has anti-derivative.

In the next theorem we use the following characterization of simply connected region: D is simply connected if and only if $w(C; z) = 0$ for all closed curves in D and $z \neq D$.

Theorem 26.2. *Given a domain D, the followings are equivalent:*

- $\int_C f(z)\,dz = 0$ *for* <u>*all*</u> *closed curves C in D and for* <u>*all*</u> *analytic functions f defined on D;*

- *D is simply connected.*

Proof. ($\Leftarrow$) This is Cauchy theorem (Theorem 16.2).

($\Rightarrow$) Suppose D is not simply connected. We can find a point $z_0 \notin D$ and a closed curved C_0 in D such that $w(C_0; z_0) \neq 0$. The function $\frac{1}{z-z_0}$ is analytic in D. The integral

$$\frac{1}{2\pi i}\int_{C_0} \frac{1}{z-z_0}\,dz = w(C_0; z_0) \neq 0$$

gives a contradiction. □

Theorem 26.3 (Maximum modulus principle)**.** *Consider an analytic function $f(z)$. If the maximum value of $|f(z)|$ in a domain occurs in the interior of D, then f must be a constant function.*

Remark. An equivalent form of the maximum modulus principle is: if f is a nonconstant analytic function, then the maximum value of $|f(z)|$ over a domain D occurs on the boundary of D.

Proof. Suppose the modulus of $f(z)$ attains maximum at an interior point z_0 of D. We can find a sufficiently small $\epsilon > 0$ such that the open disc $D(z_0; \epsilon)$ lies inside the domain D. By assumption, we have

$$|f(z)| \leq |f(z_0)| \quad \text{for all } z \in D(z_0; \epsilon).$$

Pick any real number r that is between 0 and ϵ. By Cauchy integral formula (Theorem 16.4),

$$f(z_0) = \frac{1}{2\pi i}\int_{|z-z_0|=r} \frac{f(z)}{z-z_0}\,dz.$$

Represent the circle $|z - z_0| = r$ by $z = z_0 + re^{i\theta}$, for $0 \le \theta \le 2\pi$. Then

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} ire^{i\theta}\, d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta. \end{aligned}$$

Take modulus on both sides,

$$\begin{aligned} |f(z_0)| &\le \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + re^{i\theta})|\, d\theta \\ &\le \frac{1}{2\pi}\int_0^{2\pi} |f(z_0)|\, d\theta \\ &\le |f(z_0)|. \end{aligned}$$

Hence all of the above equalities hold and

$$|f(z_0)| = \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + re^{i\theta})|\, d\theta.$$

We claim that $|f(z_0)| = |f(z_0 + re^{i\theta})|$ for all θ. Otherwise, if $|f(z_0 + re^{i\theta})| < |f(z_0)|$ for some θ, then $|f(z_0 + re^{i\theta})| < |f(z_0)|$ for θ in some nonempty interval, and this yields

$$\frac{1}{2\pi}\int_0^{2\pi} |f(z_0)| - |f(z_0 + re^{i\theta})|\, d\theta > 0$$

and

$$|f(z_0)| > \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + re^{i\theta})|\, d\theta,$$

a contradiction.

Since the radius r can be any positive real number less than ϵ, we obtain

$$|f(z_0)| = |f(z_0 + re^{i\theta})|$$

for all $r \in (0, \epsilon)$ and $\theta \in [0, 2\pi]$. Thus, the modulus $|f(z)|$ is constant in $D(z_0; \epsilon)$. This implies $f(z)$ is a constant in $D(z_0; \epsilon)$. By Identity Theorem (Theorem 17.7), $f(z)$ is constant in the whole domain D. □

Example 26.1. Find the maximum value of $|z^2 - z|$ in $|z| \le 1$.

By the maximum modulus principle, we only need to consider the boundary $|z| = 1$. On the boundary the function to be maximized is

$$|z^2 - z| = |z| \cdot |z - 1| = |z - 1|.$$

The maximum value occurs at the point on the unit circle that is farthest away from $z = 1$. The maximum value is thus $|(-1)^2 - (-1)| = 2$.

We prove the Schwarz lemma as a consequence of the maximum modulus principle.

Theorem 26.4 (Schwarz lemma)**.** *Suppose f is analytic in a domain that contains the unit disc. Assume $|f(z)| \leq 1$ for $|z| \leq 1$ and $f(0) = 0$. Then*

(i) $|f(z)| \leq |z|$ *for all* $|z| \leq 1$, *and*

(ii) $|f'(0)| \leq 1$.

If equality holds in (i) for some z or equality holds in (ii), then $f(z) = e^{i\theta} z$ for some θ.

The condition in the lemma is saying that $f(z)$ is mapping the unit disc into the unit disc, with a fixed point at $z = 0$.

Proof. Define the function

$$g(z) \triangleq \begin{cases} \frac{f(z)}{z} & \text{for } 0 < |z| \leq 1 \\ f'(0) & \text{if } z = 0. \end{cases}$$

By construction, $g(z)$ is analytic, and for any $0 < r < 1$, we have $|g(z)| \leq \frac{1}{r}$ for z in the circle $|z| = r$.

We claim that $|g(z)| \leq 1$ for $|z| \leq 1$. Suppose on the contrary that $|g(z_0)| > 1$ for some z_0 inside the open disc $|z| < 1$. Pick a real number $r \in (0, 1)$ such that $|g(z_0)| > \frac{1}{r}$ and $|z_0| < r$. Since $|g(z)| \leq \frac{1}{r}$ for $|z| = r$, this violates the maximum modulus principle.

From the above claim, we have $|f(z)| \leq |z|$ for $|z| < 1$ and $|f'(0)| \leq 1$.

If $|g(z_0)| = 1$ for some z_0 in the unit disc, then, by the maximum modulus principle, $g(z)$ is a constant function. In particular, the modulus of $g(z)$ is as constant, and the constant must be equal to 1, i.e., $|g(z)| = 1$ for $|z| < 1$. Hence

$$|g(z)| = e^{i\theta} \quad \text{for some } \theta.$$

This gives $f(z) = e^{i\theta} \cdot z$ for all z in the unit disc. $\square$

Definition 26.5. A function $f(z)$ is one-to-one (or injective) in a domain D if $f(z_1) \neq f(z_2)$ for any two distinct z_1, $z_2 \in D$. A function $f(z)$ is called locally one-to-one at z_0 if there is a disc $D(z_0; \delta)$, for some $\delta > 0$, such that $f(z)$ is one-to-one in $D(z_0; \delta)$.

Remark. A "locally one-to-one" function is also called a locally invertible function.

Example 26.2. The complex exponential function e^z is locally one-to-one at every point $z_0 \in \mathbb{C}$, but not one-to-one in the complex plane $\mathbb{C}$. The square function $f(z) = z^2$ is locally one-to-one at every nonzero point z_0, but is not locally one-to-one at $z_0 = 0$.

The following two theorems give the relationship between locally one-to-one function and complex derivative.

Theorem 26.6. *Suppose $f(z)$ is a complex function that is analytic at z_0 and $f'(z_0) = 0$. Then f cannot be locally one-to-one at z_0.*

The idea of proof is to see that the function $f(z) - f(z_0)$ behaves like the function $(z - z_0)^k$, where k is the order of zero of $f(z) - f(z_0)$ at z_0.

Proof. We can divide the class of analytic functions with $f'(z_0) = 0$ into two categories. The first category consists of constant functions, and the second category consists of nonconstant functions. If $f(z)$ is a constant function, then it is obvious that $f(z)$ cannot be locally one-to-one. In the following proof, we can assume that $f(z)$ is not a constant function.

Since $f(z)$ is analytic at z_0, we can expand $f(z)$ at z_0 (Theorem 16.5).

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$$

We know that the constant term is $a_0 = f(z_0)$. By assumption $a_1 = 0$. Let k be the smallest integer $k \geq 2$ such that $a_k \neq 0$. (Such a k must exist because it is assumed that $f(z)$ is nonconstant.)

We can now write

$$\begin{aligned} f(z) - f(z_0) &= a_k(z - z_0)^k + a_{k+1}(z - z_0)^{k+1} + a_{k+2}(z - z_0)^{k+2} \cdots \\ &= (z - z_0)^k [a_k + a_{k+1}(z - z_0) + a_{k+2}(z - z_0)^2 + \cdots] \end{aligned}$$

Denote the function in the square bracket above by $g(z)$. We have $g(z_0) = a_k \neq 0$. The function $g(z)$ is analytic in the disc $D(z_0; \delta)$. Pick a branch of $g(z)^{1/k}$. We can solve for the Taylors series of $g(z)^{1/k}$ algebraically. Suppose $g(z)^{1/k}$ has Taylor series expansion $\sum_{j=0}^{\infty} c_j(z-z_0)^j$.

$$a_k + a_{k+1}(z-z_0) + a_{k+2}(z-z_0)^2 + \cdots = \left(\sum_{j=0}^{\infty} c_j(z-z_0)^j\right)^k = c_0^k + kc_0^{k-1}c_1(z-z_0) + \cdots$$

We can pick c_0 to be any k-th root of a_k, and then recursively solve for c_1, c_2, etc.

After fixing $g(z)^{1/k}$, the function $f(z)$ can be expressed as $f(z) = [(z-z_0)g(z)^{1/k}]^k$. Let $h(z)$ be the function $(z-z_0)g(z)^{1/k}$, so that

$$f(z) = h(z)^k.$$

The function $h(z)$ is mapping z_0 to 0. Let ϵ be a positive real number such that $D(0;\epsilon)$ lies inside the image of $D(z_0;\delta)$ under the mapping $h(z)$. (We can always find a suitable value of ϵ because the image of $D(z_0;\delta)$ under h is open and contains 0 in the interior.) The transformation induced by $f(z)$ is to first map the point in $h^{-1}(D(0;\epsilon))$, which contains the point $z = z_0$, continuously to the open disc $D(0;\epsilon)$, and then apply the map z^k.

Since the map z^k is k-to-1, the function $f(z)$ when restricted to $h^{-1}(D(0;\epsilon))$ is mapping (at least) k points to one point. Hence $f(z)$ cannot be locally one-to-one. □

Remark. From the proof of Theorem 26.6, we see that the function $f(z)$ is not conformal at z_0. The angle between two lines intersecting at z_0 is expanded by a factor of $k \geq 2$.

Theorem 26.7. *If $f(z)$ is analytic at z_0 and $f'(z_0) \neq 0$, then f is locally one-to-one at z_0.*

Proof. We need to show that $f(z)$ is one-to-one in an open disc $D(z_0;\delta)$ for some $\delta > 0$.

Let $f(z_0) = w_0$. The function $f(z) - w_0$ has a zero at $z = z_0$. We take δ to be a positive real number such that $f(z) - w_0$ has *only one* zero in $D(z_0;\delta)$. Such a positive real number δ exists. Otherwise, we can find a sequence of complex numbers z_1, z_2, $z_3, \ldots$ converging to z_0 with $f(z_j) = 0$ for $j \geq 1$. By Identity Theorem (Theorem 17.7), $f(z)$ is identically zero, and it implies $f'(z_0) = 0$. This contradicts the assumption that $f'(z_0)$ is nonzero.

Consider the circle C defined by $|z - z_0| = \delta$. By our choice of δ, there is exactly one zero of $f(z) - w_0$ inside C. Let C' be the image of C under the function $f(z)$. By appealing to Lemma 23.1, the winding number of C' about the point w_0 is exactly one,

$$1 = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)-w_0}\,dz = \frac{1}{2\pi i}\int_{C'} \frac{1}{w-w_0}\,dw = n(C';w_0).$$

The winding number is a locally constant inside C'. We have $n(C'; w') = 1$ for w' in a small open disc $D(w_0; \epsilon)$ centered at w_0. Therefore

$$1 = \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z) - f(z_1)}\, dz$$

for all z_1 in an open disc $D(z_0; \delta')$ contained in $f^{-1}(D(w_0; \epsilon))$. This means that there is exactly one z in $D(z_0; \delta')$ such that $f(z) = f(z_1)$. □

In Lecture 7 we see that the non-vanishing of $f'(z_0)$ implies that the function f preserves angle at $z = z_0$. In Theorem 26.6, we see that the transformation f fails to preserve angle if $f'(z_0) = 0$. We summarize Theorem 26.6 and 26.7 in the following

Theorem 26.8. *Suppose $f(z)$ is an analytic function and z_0 is a point in the domain of f. The followings are equivalent:*

(i) *$f(z)$ preserves angle at z_0,*

(ii) *$f'(z_0) \neq 0$,*

(iii) *$f(z)$ is locally one-to-one at z_0.*

27 Lecture 27 (Fractional linear transformations)

Summary

- Fractional linear transformation (Möbius transformation)
- Mapping properties of fractional linear transformation
- Examples of fractional linear transformation that maps half-plane to unit disc, and from unit disc to itself.

In this lecture we study a class of conformal mappings from the Riemann sphere to Riemann sphere.

Definition 27.1. There are two definitions of conformal mappings in the literature. The first definition requires that a function f to be analytic and $f'(z) \neq 0$ for all z in the domain of f. A conformal mapping is therefore locally one-to-one by Theorem 26.8. This definition is adopted in [BrownChurchill]. In another definition, which is adopted in [BakNewman], a conformal function is defined as a complex function that is one-to-one and analytic.

Definition 27.2. A one-to-one analytic function from the Riemann sphere to itself is called an automorphism of the Riemann sphere. In general, given a domain $D \subseteq \mathbb{C}$, an automorphism of D is a one-to-one analytic function from D to itself.

Definition 27.3. A fractional linear transformation (or Möbius transformation, or bilinear transformation) is a function in the form

$$f(z) = \frac{az+b}{cz+d} \tag{27.1}$$

where a, b, c and d are complex numbers with $ad - bc \neq 0$. When $c \neq 0$ and $z = -d/c$, we define $f(-d/c)$ as the point at infinity. When z is the point at infinity, we define $f(z)$ as the limit of $f(z)$ as $|z| \to \infty$, i.e. $f(\infty) = a/c$ if $c \neq 0$.

Remark. If we multiply all the coefficients a, b, c and d in (27.1) by a nonzero constant k, the resulting expression induces the same function, because

$$\frac{kaz+kb}{kcz+kd} = \frac{az+b}{cz+d}.$$

Therefore, there are only three degrees of freedom in specifying a fractional linear transformation.

Theorem 27.4. *A fractional linear transformation is an automorphism on the Riemann sphere.*

Proof. The condition $ad - bc \neq 0$ ensures that the function $f(z) = (az+b)/(cz+d)$ has non-vanishing first derivative. For complex number $z \in \mathbb{C}$,

$$f'(z) = \frac{a(cz+d) - (az+b)c}{(cz+d)^2} = \frac{ad-bc}{(cz+d)^2} \neq 0.$$

To see that the function $f(z)$ in (27.1) is analytic at ∞, we make a change of variable $w = 1/z$ and consider

$$g(w) = f(1/w) = \frac{a/w+b}{c/w+d} = \frac{bw+a}{dw+c}.$$

We can check that the function $g(w)$ is analytic at $w = 0$ from

$$g'(w) = \frac{bc-ad}{(dw+c)^2} \neq 0, \quad \text{when } w = 0.$$

Finally, for the special point $-d/c$, we apply the map $1/z$ after $f(z)$ and consider $h(z) = 1/f(z)$. This function is analytic at $-d/c$ because

$$h'(z) = \frac{d}{dz}\frac{cz+d}{az+b} = \frac{c(az+b)-(cz+d)a}{(az+b)^2} = \frac{cb-ad}{(az+b)^2}.$$

A fractional linear transformation is easily seen to be one-to-one on the Riemann sphere. For any $w \in \mathbb{C}$, we can solve

$$\begin{aligned} \frac{az+b}{cz+d} &= w \\ az+b &= cwz+dw \\ (a-cw)z &= dw-b \\ \Rightarrow \quad z &= \frac{dw-b}{a-cw}. \end{aligned}$$

□

The following two properties are straightforward.

(i) The inverse function of a fractional linear transformation is another fractional linear transformation.

(ii) Composition of two fractional linear transformations is a fractional linear transformation.

Example 27.1.

If $b = 0$, $c = 0$ and $d = 1$, a fractional linear transformation reduces to a rotation followed by a dilation $f(z) = az = re^{i\theta}z$.

If $a = 1$, $c = 0$ and $d = 1$, a fractional linear transformation is a translation $f(z) = z+b$.

If $a = 0$, $b = 1$, $c = 1$ and $d = 0$, a fractional linear transformation becomes the inversion function $f(z) = 1/z$.

A function in the form $f(z) = az + b$ is called an affine transformation. It maps ∞ to ∞ and $\mathbb{C}$ onto $\mathbb{C}$. Any fractional linear transformation can be written as the composition of some affine transformations and the inversion function $1/z$. Let $g(z)$ be the affine transformation $g(z) = cz + d$, and $h(z) = 1/z$. Then

$$h(g(z)) = \frac{1}{cz+d}.$$

We can apply another affine transformation $\ell(z) = \alpha z + \beta$ and pick $\alpha = b - \frac{ad}{c}$ and $\beta = \frac{a}{c}$ to make

$$\ell(h(g(z)) = \frac{\alpha}{cz+d} + \beta = \frac{az+b}{cz+d}.$$

Any affine transformation preserves the shape of a geometric figure. The next theorem shows that the mapping $1/z$ preserves circles and straight lines.

Theorem 27.5. *The inversion function $f(z) = 1/z$ maps circles and straight lines to circles and straight lines.*

We illustrate this theorem using examples.

Example 27.2. Find the image of the circle $|z-2i| = 1$ under the transformation $f(z) = 1/z$.

Let $w = 1/z$.

$$\begin{aligned} & \left|\frac{1}{w} - 2i\right| = 1 \\ \Leftrightarrow \quad & |1 - 2wi|^2 = |w|^2 \\ \Leftrightarrow \quad & 3|w|^2 - 2wi + 2\bar{w}i = -1. \end{aligned}$$

By completing square, the last equation is equivalent to

$$\left|w + \frac{2i}{3}\right|^2 = \frac{1}{9},$$

which is the equation of a circle with radius $1/3$ and center $-2i/3$.

Example 27.3. Consider a circle $C: \ |z - z_0| = |z_0|$ that passes through the origin. Find the image of C under the transformation $f(z) = 1/z$.

Define a variable $w = 1/z$.

$$\begin{aligned} & \left|\frac{1}{w} - z_0\right| = |z_0| \\ \Leftrightarrow \quad & |1 - wz_0|^2 = |wz_0|^2 \\ \Leftrightarrow \quad & 1 - wz_0 - \bar{w}z_0 = 0 \\ \Leftrightarrow \ & \mathrm{Re}(wz_0) = 1/2. \end{aligned}$$

This equation defines a straight line in the w-plane. Indeed, if we write $w = u + iv$ and $z_0 = x_0 + iy_0$, then

$$\mathrm{Re}(wz_0) = 1/2 \iff ux_0 - vy_0 = 0.5.$$

By adjusting the values of x_0 and y_0, we can obtain the equation of any straight line in the w-plane.

Since the inverse of a Möbius transformation is a Möbius transformation, the previous example show that a Möbius transformation maps a straight line to a circle that passes through the origin.

Theorem 27.6. *We can find a unique Möbius transformation that takes any three distinct points in the Riemann sphere to another three distinct points in the Riemann sphere.*

This depends on the following fact.

Fact: Given any three distinct points z_1, z_2 and z_3 in $\mathbb{C} \cup \{\infty\}$, there is a fractional linear transformation f that maps z_1 to 0, z_2 to 1, and z_3 to ∞. When z_1, z_2 and z_3 are in $\mathbb{C}$, this is given by

$$f(z) = \frac{z - z_1}{z - z_3} \frac{z_2 - z_3}{z_2 - z_1}. \tag{27.2}$$

Example 27.4. Find a Möbius function that maps 0 to 0, 1 to i, and ∞ to 2.

We first find a Möbius transformation that maps 0 to 0, i to 1, and 2 to ∞. By (27.2), we can take

$$g(z) = \frac{z}{z-2} \cdot \frac{i-2}{i} = \frac{(1+2i)z}{z-2}.$$

The inverse of $g(z)$ is the answer

$$f(z) = g^{-1}(z) = \frac{2z}{z - 1 - 2i}.$$

Proof of Theorem 27.6. Let z_1, z_2 and z_3 be three distinct points in $\mathbb{C} \cup \{\infty\}$, and w_1, w_2 and w_3 be another three distinct points in $\mathbb{C} \cup \{\infty\}$. Using the fact mentioned after the statement of Theorem 27.6, we can find a fractional linear transformation $g(z)$ such that

$$g(z_1) = 0, \qquad g(z_2) = 1, \qquad g(z_3) = \infty,$$

and another fractional linear transformation $h(z)$ such that

$$h(w_1) = 0, \qquad h(w_2) = 1, \qquad h(w_3) = \infty.$$

The required fractional linear transformation is then given by the function composition $h^{-1} \circ g$.

To prove uniqueness, we first show that the fractional linear function $f(z) = 1/z$ is the only fractional linear function that maps 0 to 0, 1 to 1 and ∞ to ∞. Suppose $f(z) =$

$(az+b)/(cz+d)$ is a fractional linear function such that $f(0)=0$, $f(1)=1$ and $f(\infty)=\infty$. From the assumption $f(\infty)=\infty$, we can assume $c=0$ and $d=1$. Hence $f(z)$ takes the form $az+b$ for some complex numbers a and b. Then $f(0)=0$ and $f(1)=1$ imply $a=1$ and $b=0$.

Now suppose f and $\tilde{f}$ are fractional linear transformations that satisfy the condition in Theorem 27.6. The function $\ell = f^{-1}\circ\tilde{f}$ is fractional linear transformation that satisfies $\ell(z_1)=z_1$, $\ell(z_2)=z_2$, $\ell(z_3)=z_3$. The function $g\circ\ell\circ g^{-1}$ is a Möbius mapping 0 to 0, 1 to 1 and ∞ to ∞, and hence must be the identity function,

$$g\circ\ell\circ g^{-1} = id.$$

This gives

$$\ell = g^{-1}\circ id\circ g = id.$$

So ℓ is the identity function, and hence $\tilde{f}=f$. □

Example 27.5. Find a Möbius transformation that maps

$$0 \mapsto 0,$$
$$\infty \mapsto i,$$
$$-1 \mapsto 2.$$

Since $0\mapsto 0$, the Möbius transformation must have the form

$$h(z) = \frac{z}{cz+d}.$$

From the condition $\infty \mapsto 1$, we must take $c=-i$, i.e.,

$$h(z) = \frac{z}{-iz+d}$$

for some coefficient d. We use the last condition $-1\mapsto 2$ to solve for d,

$$\frac{-1}{i+d} = 2 \ \Rightarrow d = \frac{-1-2i}{2}.$$

The Möbius transformation is

$$h(z) = \frac{z}{-iz+\frac{-1-2i}{2}} = \frac{-2z}{2iz+1+2i}.$$

Example 27.6. Find a Möbius transformation that maps

$$\begin{aligned} 0 &\mapsto -i, \\ 1 &\mapsto 1, \\ -1 &\mapsto -1. \end{aligned}$$

Let $f(z)$ be a Möbius transformation that satisfies the requirements $f(0) = i$, $f(1) = 1$ and $f(-1) = -1$. From the condition $f(0) = -i$, we can assume $d = 1$ and $b = -i$, i.e., $f(z)$ has the form

$$f(z) = \frac{az - i}{cz + 1}.$$

This function has two fixed points at $z = 1$ and $z = -1$. The equation $f(z) = z$ should have two roots at 1 and -1.

$$\begin{aligned} \frac{az - i}{cz + 1} &= z \\ cz^2 + (1 - a)z - i &= 0 \\ z^2 + \frac{1 - a}{c} z + \frac{i}{c} &= 0. \end{aligned}$$

Comparing with the equation $z^2 - 1 = 0$, which has roots at 1 and -1, we get

$$\frac{1 - a}{c} = 0, \quad \text{and } \frac{i}{c} = -1.$$

This gives $a = 1$ and $c = -i$. The required Möbius transformation is

$$f(z) = \frac{z - i}{-iz + 1}.$$

We note that this function $f(z)$ maps ∞ to i. Indeed, it is mapping the whole real axis plus the point at infinity to the unit circle. Since $f(i) = 0$, we can see that it is mapping the upper half-plane to the interior of the unit circle.

Example 27.7. For any complex number a with $|a| < 1$, the fractional linear transformation

$$f(z) = \frac{z - a}{1 - \bar{a}z}$$

is an automorphism of the unit disc, and maps the point a to the origin.

Since a is inside the unit circle, the complex number $\bar{a}^{-1}$ lies outside the unit circle. The denominator of $f(z)$ never vanishes for $|z| \leq 1$. We next prove that

$$\left|\frac{z-a}{1-\bar{a}z}\right| \leq 1 \iff |z| \leq 1.$$

Consider the condition

$$|f(z)| = \left|\frac{z-a}{1-\bar{a}z}\right| \leq 1$$

It is equivalent to

$$|z-a|^2 \leq |1-\bar{a}z|^2.$$

Expanding both sides, we get

$$|z|^2 - \bar{a}z - a\bar{z} + |a|^2 \leq 1 - \bar{a}z - a\bar{z} + |a|^2|z|^2,$$

which can be simplified to

$$|z|^2(1-|a|^2) \leq (1-|a|^2).$$

Therefore, $|f(z)| \leq 1$ iff $|z| \leq 1$.