(**Exercise 3.25 and 3.26 are optional**)

- **Exercise 3.25:**(More examples can refer to Example 3.6 and 3.7 in the textbook)

  Solutions:

  (a). Let

$$\mathbf{T} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{\beta} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix}, \quad \boldsymbol{C} = \begin{pmatrix} 0 \\ \beta \\ \beta \\ \beta \\ \beta \end{pmatrix}.$$

  Use $\mathbf{T}\boldsymbol{\beta} = \boldsymbol{C}$ to test.

  (b). Similarly, Let

$$\mathbf{T} = \begin{pmatrix} 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}, \quad \boldsymbol{\beta} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix}, \quad \boldsymbol{C} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

  Use $\mathbf{T}\boldsymbol{\beta} = \boldsymbol{C}$ to test.

  (c). Let

$$\mathbf{T} = \begin{pmatrix} 0 & 1 & -2 & -4 & 0 \\ 0 & 1 & 2 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\beta} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix}, \quad \boldsymbol{C} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

  Use $\mathbf{T}\boldsymbol{\beta} = \boldsymbol{C}$ to test.

- **Exercise 3.26:**

  Solutions:

  (a). In this case, consider a new variable $z = \left\{ \begin{array}{ll} 0 & \text{if sample 1} \\ 1 & \text{if sample 2} \end{array} \right\}$. Then write the model as $y_i = \beta_0 + \beta_1 x_i + (\gamma_0 - \beta_0) z + (\gamma_1 - \beta_1) x_i z + \varepsilon_i$

  (b). Denote that $\gamma_0 - \beta_0 = v_1$, $\gamma_1 - \beta_1 = v_2$. Then we want to test $H_0 : v_2 = 0$. We use $\mathbf{T} = (0\ \ 0\ \ 0\ \ 1)$, $\boldsymbol{\beta} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ v_1 \\ v_2 \end{pmatrix}$ and $\boldsymbol{C} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ to test.

(c). Similar to (b), this is test of $v_1 = 0$, $v_2 = 0$. We let $\mathbf{T} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $\boldsymbol{\beta} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ v_1 \\ v_2 \end{pmatrix}$ and $\boldsymbol{C} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ to test.

(d). Test of $\beta_1 = c$, $v_2 = 0$. Use $\mathbf{T} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $\boldsymbol{\beta} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ v_1 \\ v_2 \end{pmatrix}$ and $\boldsymbol{C} = \begin{pmatrix} c \\ 0 \end{pmatrix}$ to test.

- **Exercise 3.27:**

  Solutions:

$$\begin{aligned} \mathrm{Var}(\hat{\boldsymbol{y}}) = \mathrm{Cov}(\hat{\boldsymbol{y}}) &= \mathrm{Cov}(\boldsymbol{H} \cdot \boldsymbol{y}) \\ &= \boldsymbol{H} \cdot \mathrm{Cov}(\boldsymbol{y}) \cdot \boldsymbol{H}^T \\ &= \boldsymbol{H} \cdot \sigma^2 \boldsymbol{I} \cdot \boldsymbol{H}^T \\ &= \sigma^2 \cdot \boldsymbol{H} \quad (\text{since } H = H^T \text{ and } HH = H) \end{aligned}$$

  Note that here since $\hat{\boldsymbol{y}}$ is a random vector, better to use $\mathrm{cov}(\cdot)$ instead.

- **Exercise 3.28:**

  Solutions:

$$\begin{aligned} \mathbf{HH} &= \mathbf{X}\left(\mathbf{X}^T\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{X}\left(\mathbf{X}^T\mathbf{X}\right)^{-1}\mathbf{X}^T \\ &= \mathbf{X}\left(\mathbf{X}^T\mathbf{X}\right)^{-1}\mathbf{X}^T \\ &= \mathbf{H} \end{aligned}$$

  and

$$\begin{aligned} (\mathbf{I} - \mathbf{H})(\mathbf{I} - \mathbf{H}) &= \mathbf{I} - \mathbf{H} - \mathbf{H} + \mathbf{HH} \\ &= \mathbf{I} - \mathbf{H} - \mathbf{H} + \mathbf{H} \\ &= \mathbf{I} - \mathbf{H} \end{aligned}$$

- **Exercise 3.36:**

  Solutions:

Recall that

$$SS_T = \sum_{i=1}^{n} (y_i - \bar{y})^2$$
$$SS_R = \sum_{i=1}^{n} (\hat{y}_i - \bar{y})^2$$
$$SS_{Res} = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2$$
$$SS_T = SS_R + SS_{Res}$$

and

$$R^2 = \frac{SS_R}{SS_T} = 1 - \frac{SS_{Res}}{SS_T}.$$

Since given the data samples, then SS_T is fixed, we only need to show that the sum of squares for regression for model B, SS_{R_B} is greater than the sum of squares for regression for model SS_{R_A}, i.e.

$$SS_{R_B} \geq SS_{R_A} \Rightarrow R_B^2 \geq R_A^2$$

We prove this using partitioning SSR into sequential sums of squares.

- Consider i parameters in $\boldsymbol{\beta}_1$ and j parameters in $\boldsymbol{\beta}_2$.
- Then model B is using $(i \times j)$ parameters of which the first i are the same as model A. Then SS_{R_B} equals

  $$\begin{aligned} R\left(\beta_{i1}, \beta_{i2}, \ldots, \beta_{ii}, \beta_{j1}, \beta_{j2}, \ldots, \beta_{jj} | \beta_0\right) = R\left(\beta_{i1}, \beta_{i2}, \ldots, \beta_{ii} | \beta_0\right) \\ + R\left(\beta_{j1}, \beta_{j2}, \ldots, \beta_{jj} | \beta_0, \beta_{i1}, \beta_{i2}, \ldots, \beta_{ii}\right), \end{aligned}$$

  Since the second term on the right is a **sum of squares**, it must be greater than or equal to zero. Thus, $SS_{R_B} \geq SS_{R_A}$, which is equivalent to $R_B^2 \geq R_A^2$.

- **Exercise 3.38:**

  Solutions:

$$\begin{aligned} \sum_{i=1}^{n} \text{Var}\left(\widehat{y}_i\right) &= \sum_{i=1}^{n} \mathbf{x}_i^T \left(\mathbf{X}^T\mathbf{X}\right)^{-1} \mathbf{x}_i \left(\sigma^2\right) \\ &= \sigma^2 \left(\sum_{i=1}^{n} h_{ii}\right) \\ &= \sigma^2 (\text{rank of } \mathbf{X}) \\ &= p\sigma^2 \end{aligned}$$