MAT 3280 Lecture 27

Def A function $f(z)$ is <u>conformal</u> if

① f analytic and $f'(z)$ is not zero.
(locally one-to-one)

② f analytic and one-to-one.

Example e^z conformal according ①

e^z is not one-to-one function

Affine function / transformation

$f(z) = z + b$ translation

$f(z) = e^{i\theta} z$ rotation

$f(z) = r \cdot z$ dilation $r > 0$.

CUHK $\longrightarrow$ CUHK

$\{ az + b : a, b \in \mathbb{C},\ a \neq 0 \}$

$a = 1$ $z + b$

$b = 0$ az

* Composition of two affine trans. is affine

$c(az+b)+d$ is affine

* inverse of affine trans. is affine.

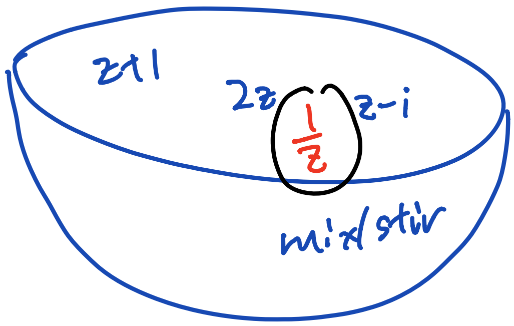
<u>Def</u> A fractional linear transformation / Möbius transformation is a function in form

$$f(z) = \frac{az+b}{cz+d} \qquad a,b,c,d \in \mathbb{C}$$

$$ad-bc \neq 0$$

function

lambda : z : 1/z



$\dfrac{kaz+kb}{kcz+kd}$ represents the same function as $\dfrac{az+b}{cz+d}$.

$f(z) = \dfrac{az+b}{cz+d}$ defines a one-to-one function on the $\mathbb{C} \cup \{\infty\}$

When $z = -\dfrac{d}{c}$, $f(-\frac{d}{c}) \triangleq \infty$

$z = \infty$, $f(\infty) \triangleq \lim\limits_{z \to \infty} \dfrac{az+b}{cz+d} = \dfrac{a}{c}$

Example $f(z) \frac{\overset{a}{z} + \overset{b}{1}}{\underset{c}{z} - \underset{d}{1}}$ $\quad ad-bc = 1\cdot(-1) - 1\cdot(1) = -2$

$$f(\infty) = \frac{1\cdot\infty + 1}{1\cdot\infty - 1} = \frac{1}{1} = 1$$

$$f(1) = \frac{1+1}{1-1} = \infty$$

Theorem When $ad-bc \neq 0$, $f(z) = \frac{az+b}{cz+d}$ is analytic and $f'(z) \neq 0$ $\forall z$.

Proof

$$f'(z) = \frac{a(cz+d) - (az+b)c}{(cz+d)^2} = \frac{ad-bc}{(cz+d)^2} \neq 0$$

if $(cz+d) \neq 0$

$f(\infty)$?? $\quad w = \frac{1}{z}$

$$g(w) = f\left(\frac{1}{w}\right) = \frac{a\frac{1}{w} + b}{c\frac{1}{w} + d} = \frac{a+bw}{c+dw}$$

$$g(0) = \frac{a}{c}, \quad g'(w) = \frac{bc-ad}{(c+dw)^2} \neq 0$$

when $w=0$

$f'(-\frac{d}{c})$ see the lecture notes. $\square$

* Composition of two Mobius transforms is a Mobius transformation.

$$\frac{A\left(\frac{az+b}{cz+d}\right) + B}{C\left(\frac{az+b}{cz+d}\right) + D}$$ is a fractional linear trans.

$*$ Inverse of Mobius trans. is Mobius trans.

$$f(z) = \frac{1\cdot z + 0}{0\cdot z + 1} \quad \text{is the identity fn}$$

Fact: A Mobius transformation $\dfrac{az+b}{cz+d}$ $(ad-bc\neq 0)$

is a composition of $\ell \circ g \circ h$ $\quad \frac{1}{z}$

affine transformation

take $h(z) = cz+d$

$g(z) = 1/z$

$$g(h(z)) = \frac{1}{cz+d}$$

$$\ell(z) = \alpha z + \beta$$

$$\ell(g(h(z))) = \frac{az+b}{cz+d} = \alpha\left(\frac{1}{cz+d}\right) + \beta$$

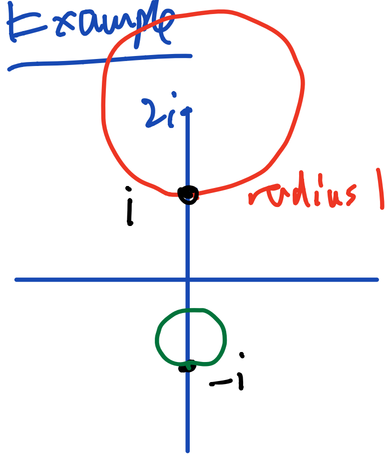
$$= \frac{\alpha + \beta cz + \beta d}{cz+d}$$

$\alpha = b - \dfrac{ad}{c}$

$\beta = \dfrac{a}{c}$

<u>Theorem</u> The inversion function $f(z) = \dfrac{1}{z}$ maps circles and straight lines to circles and straight lines.

Example



$|z-2i|=1$

Find the image of this circle under the trans. $1/z$

$$w = 1/z$$

$$\left|\frac{1}{w}-2i\right| = 1$$

$$\Leftrightarrow \quad |1-2iw|^2 = |w|^2$$

$$\Leftrightarrow \quad 1-2iw+2i\bar{w}+4|w|^2 = |w|^2$$

$$\Leftrightarrow \quad 3|w|^2-2iw+2i\bar{w} = -1$$

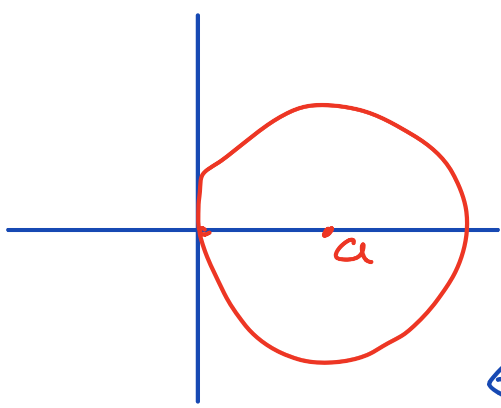
$$\Leftrightarrow \quad |w|^2-\frac{2iw}{3}+\frac{2i\bar{w}}{3} = -\frac{1}{3}$$

$$\Leftrightarrow \quad \left|w+\frac{2i}{3}\right|^2 = \frac{1}{3}$$

Example

$|z-a| = a \qquad a>0$

$$w=\frac{1}{z}$$



$$\left|\frac{1}{w}-a\right| = a$$

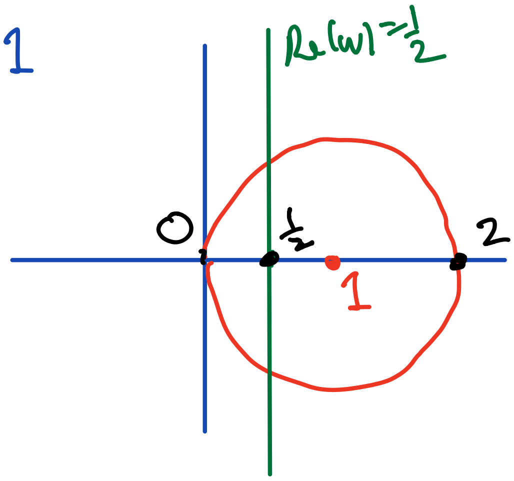
$$\Leftrightarrow \quad |1-wa|^2 = |aw|^2$$

$$\Leftrightarrow \quad 1-wa-\bar{w}a+|wa|^2 = |wa|^2$$

$$\Leftrightarrow \quad wa+\bar{w}a = 1$$

$$\mathrm{Re}(wa) = \frac{1}{2}$$

$a = 1$



$\mathrm{Re}(w) = \frac{1}{2}$

<u>Theorem</u> We can find a unique Mobius trans that takes any three points in Riemann sphere to another three points in Riemann sphere.

Fact: Suppose $z_1, z_2, z_3 \in \mathbb{C} \cup \{\infty\}$ there is a Mobius Trans. s.t.

$$z_1 \mapsto 0 \qquad z_2 \mapsto 1 \qquad z_3 \mapsto \infty$$

$$\frac{z - z_1}{z - z_3} \cdot \frac{z_2 - z_3}{z_2 - z_1}$$

<u>Example</u>

$$\left.\begin{aligned} 0 &\mapsto 0 \\ 1 &\mapsto i \\ \infty &\mapsto 2 \end{aligned}\right\} \text{ we want this}$$

$$\left.\begin{aligned} 0 &\mapsto 0 \\ i &\mapsto 1 \\ 2 &\mapsto \infty \end{aligned}\right\} \quad g(z) = \frac{z}{z-2} \cdot \frac{i-2}{i}$$

$$f(z) = g^{-1}(z) = \frac{2z}{z - 1 - 2i}$$

want $\begin{cases} z_1 \mapsto w_1 \\ z_2 \mapsto w_2 \\ z_3 \mapsto w_3 \end{cases}$

$z_1 \mapsto 0$, $z_2 \mapsto 1$, $z_3 \mapsto \infty$: $g(z)$

$w_1 \mapsto 0$, $w_2 \mapsto 1$, $w_3 \mapsto \infty$: $h(z)$

$f(z) = h^{-1}(g(z))$ is a Mobius transformation that satisfies $f(z_i) = w_i$ for $i = 1, 2, 3$.

Such $f(z)$ is unique.

Fact: $f(z) = \dfrac{1 \cdot z + 0}{0 \cdot z + 1}$ is the only Mobius trans that maps $0 \mapsto 0$, $1 \mapsto 1$, $\infty \mapsto \infty$.

Suppose $\tilde{f}(z_i) = w_i$ $i = 1, 2, 3$.

$\ell = \tilde{f}^{-1} \circ f$

$\ell(z_1) = z_1$

$\ell(z_2) = z_2$

$\ell(z_3) = z_3$

Then $g \circ \ell \circ g^{-1}$ maps 0 to 0, 1 to 1, ∞ to ∞

and thus $g \circ \ell \circ g^{-1}$ must be the identity function id

$g \circ \ell \circ g^{-1} = id$

$\ell = g^{-1} \circ id \circ g = id$

$\Rightarrow \tilde{f}^{-1} \circ f = id \Rightarrow f = \tilde{f}$