MAT2002 Ordinary Differential Equations
High-order linear equations

Dongdong He

The Chinese University of Hong Kong (Shenzhen)

March 9, 2021

Overview

1. Higher order linear equations
   - General theory
   - Homogeneous equation with constant coefficients
   - Non-homogeneous equations
   - Method of undetermined coefficients
   - Variation of parameters

Outline

Review: second-order linear equations

The theory for higher order linear equations is analogous to that of the second order case. Let us give a brief review:

- For a general second order equation

$$y'' + p(t)y' + q(t)y = g(t).$$

  If there is an interval I such that p, q and g are continuous, then for $t_0 \in I$ and given initial conditions $x_0, x_1 \in \mathbb{R}$, the IVP with $y(0) = x_0, y'(0) = x_1$ has exactly one solution in I.

- Given two linearly independent solutions y_1, y_2 to the homogeneous equation

$$y'' + p(t)y' + q(t)y = 0,$$

  they form a fundamental set of solutions if any solution ϕ to the homogeneous ODE can be written as a linear combination of y_1 and y_2. This is equivalent to the Wronskian $W(y_1, y_2)[t_*] = y_2'(t_*)y_1(t_*) - y_1'(t_*)y_2(t_*) \neq 0$ for some $t_* \in I$.

- Abel's theorem states that $W(y_1, y_2)[t] = ce^{-\int p(t)dt}$ for some constant c not depending on t.

Review: second-order linear equations

- For homogeneous equations with constant coefficients:

$$ay'' + by' + cy = 0,$$

finding two solutions y_1 and y_2 related to the roots of the characteristic equation

$$ar^2 + br + c = 0.$$

- For non-homogeneous equations we have two methods:
  1. Method of undetermined coefficients: if $g(t)$ is a sum or product of exponentials, polynomials, cosine and sine.
  2. Variation of parameters: for more general linear equations where the solution is of the form $y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$.

General theory

The general nth order linear ODE is of the form

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \cdots + p_1(t)y' + p_0(t)y = g(t),$$

and for an IVP we provide initial conditions

$$y(t_0) = x_0, \quad y'(t_0) = x_1, \quad \ldots, \quad y^{(n-1)}(t_0) = x_{n-1}.$$

We first state the existence and uniqueness theorem.

Theorem 8.1

(Existence and Uniqueness.) *Let $I \subset \mathbb{R}$ be an open interval and suppose $g, p_0, p_1, \ldots, p_{n-1}$ are continuous functions in I. For $t_0 \in I$ and $x_0, \ldots, x_{n-1} \in \mathbb{R}$, there is exactly one solution to the IVP*

$$\begin{cases} y^{(n)} + p_{n-1}(t)y^{(n-1)} + \cdots + p_1(t)y' + p_0(t)y = g(t), \\ y(t_0) = x_0, \quad y'(t_0) = x_1, \quad \ldots, \quad y^{(n-1)}(t_0) = x_{n-1}. \end{cases}$$

General theory

Definition 8.2

We say that the functions $f_1(t), \ldots, f_n(t)$ are **linearly independent** on the interval I if

$$
\begin{aligned}
& \alpha_1 f_1(t) + \cdots + \alpha_n f_n(t) = 0, \quad \forall t \in I \\
\Rightarrow \quad & \alpha_1 = \cdots = \alpha_n = 0.
\end{aligned}
$$

Otherwise, we say that the functions $f_1(t), \ldots, f_n(t)$ are **linearly dependent**.

General theory

Example 8.3

Given functions $f_1(t) = 1, f_2(t) = t, f_3(t) = t^2$ defined on the interval $I = \mathbb{R}$, suppose there are constants $\alpha_1, \alpha_2, \alpha_3$ such that

$$\alpha_1 f_1(t) + \alpha_2 f_2(t) + \alpha_3 f_3(t) = \alpha_1 + \alpha_2 t + \alpha_3 t^2 = 0 \quad \forall t \in I. \tag{1.1}$$

Then, in order for the above equality to hold for all $t \in I = \mathbb{R}$, it must be true at any three distinct points in I. It is convenient to choose $t = 0, t = 1, t = -1$, leading to three equations

$$\alpha_1 = 0, \quad \alpha_1 + \alpha_2 + \alpha_3 = 0, \quad \alpha_1 - \alpha_2 + \alpha_3 = 0.$$

The first equation gives $\alpha_1 = 0$, and the second and third equations then give $\alpha_2 = \alpha_3 = 0$, thus there does not exist a set of non-zero constants $(\alpha_1, \alpha_2, \alpha_3)$ for which the condition (1.1) is satisfied, which then implies that f_1, f_2, f_3 are linearly independent in $I = \mathbb{R}$.

General theory

Similar to the second order case, we have the following principle of superposition:

Theorem 8.4

(Principle of superposition.) *Let* $y_1, \ldots, y_n$ *be solutions to the homogeneous equation*

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \cdots + p_1(t)y' + p_0(t)y = 0,$$

then, for any constants $c_1, \ldots, c_n \in \mathbb{R}$*, the function*

$$\phi(t) = c_1 y_1(t) + \cdots + c_n y_n(t)$$

is also a solution to the above homogeneous equation.

General theory

We also have an analogue to the Wronskian:

Definition 8.5

Given functions $f_1, \dots, f_n$ that are differentiable up to order $n-1$, we define the Wronskian W as

$$W(f_1, \dots, f_n)[t] = \det \begin{pmatrix} f_1 & f_2 & \dots & f_n \\ f_1' & f_2' & \dots & f_n' \\ \vdots & \vdots & \vdots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \dots & f_n^{(n-1)} \end{pmatrix} [t].$$

General theory

The natural question is: given n solutions $y_1, \ldots, y_n$ to the homogeneous equation

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \cdots + p_1(t)y' + p_0(t)y = 0.$$

Can every solution ϕ to the homogeneous equation be expressed as a linear combination of $y_1, \ldots, y_n$? Similarly with the case of the second-order ODE, one has the following theorem.

Theorem 8.6

If $p_0, \ldots, p_{n-1}$ are continuous functions in I, and $y_1, \ldots, y_n$ are solutions to the above homogeneous equation, then every solution ϕ to the homogeneous equation can be expressed as a linear combination of $y_1, \ldots, y_n$ if and only if $W(y_1, \ldots, y_n)[t_0] \neq 0$ for some $t_0 \in I$. In this case, we call $(y_1, \ldots, y_n)$ a **fundamental set of solutions (FSS)** *to the homogeneous equation.*

General theory

One can easily show that

$$W(y_1, \ldots, y_n)[t_0] \neq 0 \Rightarrow (y_1, \ldots, y_n) \text{ are linearly independent.}$$

Again, the converse is also true if $y_1, \ldots, y_n$ are solutions to the homogeneous ODE.

Theorem 8.7

Let $y_1, \ldots, y_n$ be linearly independent solutions to the homogeneous equation

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \cdots + p_1(t)y' + p_0(t)y = 0,$$

for $t \in I$. Then, the Wronskian $W(y_1, \ldots, y_n)[t]$ is non-zero in I.

Proof. Suppose the conclusion is not true, that is, there is at least one point $t_0 \in I$ where the Wronskian is zero. Then, consider the equation

$$\alpha_1 y_1(t) + \cdots + \alpha_n y_n(t) = 0,$$

for constants $\alpha_1, \ldots, \alpha_n$.

General theory

Differentiating repeatedly leads to

$$\begin{aligned} \alpha_1 y_1'(t) + \cdots + \alpha_n y_n'(t) &= 0, \\ &\vdots \\ \alpha_1 y_1^{(n-1)}(t) + \cdots + \alpha_n y_n^{(n-1)}(t) &= 0. \end{aligned}$$

In particular we obtain after substituting $t = t_0$

$$\begin{pmatrix} y_1(t_0) & y_2(t_0) & \dots & y_n(t_0) \\ y_1'(t_0) & y_2'(t_0) & \dots & y_n'(t_0) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(t_0) & y_2^{(n-1)}(t_0) & \dots & y_n^{(n-1)}(t_0) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

General theory

Since the Wronskian is zero at $t = t_0$, there exists a non-zero solution $(\alpha_1^*, \ldots, \alpha_n^*)$ to the above matrix problem. Defining the function

$$\phi(t) = \alpha_1^* y_1(t) + \cdots + \alpha_n^* y_n(t),$$

where thanks to the principle of superposition, ϕ is also a solution to the homogeneous equation. Furthermore, at $t = t_0$, ϕ satisfies the initial conditions

$$\phi(t_0) = 0, \quad \phi'(t_0) = 0, \ldots, \phi^{(n-1)}(t_0) = 0.$$

But the solution $z(t) = 0$ for $t \in I$ is also a solution to the IVP with zero initial conditions. Consequently, by the Uniqueness of solutions to IVP we find that $\phi(t) = 0$ for $t \in I$. Thus, we have found non-zero constants $\alpha_1^*, \ldots, \alpha_n^*$ such that

$$\alpha_1^* y_1(t) + \cdots + \alpha_n^* y_n(t) = 0 \quad \forall t \in I.$$

This contradicts with the linear independence of $y_1, \ldots, y_n$.

General theory

Finally, we state an analogous result to Abel's theorem:

Theorem 8.8

(Abel's theorem). *Let $y_1, \ldots, y_n$ be solutions to the homogeneous equation*

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \cdots + p_1(t)y' + p_0(t)y = 0,$$

for $t \in I$. Then,

$$W(y_1, \ldots, y_n)[t] = ce^{-\int p_{n-1}(t)dt}$$

for a constant c not dependent on $t \in I$.

General theory

Proof. The idea is to derive an equation satisfied by the Wronskian. From properties of matrix determinants, we see that

$$\frac{d}{dt}\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \frac{d}{dt}(ad - bc) = ad' + a'd - bc' - b'c = \begin{vmatrix} a' & b' \\ c & d \end{vmatrix} + \begin{vmatrix} a & b \\ c' & d' \end{vmatrix}.$$

General theory

Hence, we can deduce

$$
\frac{d}{dt}W[t] = \begin{vmatrix} y_1' & y_2' & \cdots & y_n' \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} + \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1'' & y_2'' & \cdots & y_n'' \\ y_1'' & y_2'' & \cdots & y_n'' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}
$$

$$
+ \cdots + \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_n^{(n-2)} \\ y_1^{(n)} & y_2^{(n)} & \cdots & y_n^{(n)} \end{vmatrix} = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_n^{(n-2)} \\ y_1^{(n)} & y_2^{(n)} & \cdots & y_n^{(n)} \end{vmatrix}.
$$

(Noting that in the first $n-1$ determinants, there is always two identical rows, hence the first $n-1$ determinants are zero and only the last determinant is nonzero.)

General theory

Using that for each $1 \leq k \leq n$,

$$y_k^{(n)} = -p_{n-1} y_k^{(n-1)} - \cdots - p_1 y_k' - p_0 y_k,$$

then applying elementary row operations we find that

$$\frac{d}{dt} W[t] = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_n^{(n-2)} \\ -p_{n-1} y_1^{(n-1)} & -p_{n-1} y_2^{(n-1)} & \cdots & -p_{n-1} y_n^{(n-1)} \end{vmatrix} = -p_{n-1} W[t].$$

Thus,

$$W(y_1, \ldots, y_n)[t] = c e^{-\int p_{n-1}(t) dt}$$

for a constant c not dependent on $t \in I$.

Homogeneous equation with constant coefficients

Our aim is to study, for constants $a_n \neq 0$, $a_{n-1}, \ldots, a_0 \in \mathbb{R}$, the equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0.$$

From the theory of second order equations, we consider a trial function $\phi = e^{rt}$ for $r \in \mathbb{R}$. Substituting this into the above equation gives the **characteristic equation**

$$a_n r^n + \cdots + a_1 r + a_0 = 0.$$

The **characteristic polynomial** is

$$Z(r) = a_n r^n + \cdots + a_1 r + a_0.$$

From the fundamental theorem of algebra, every polynomial with real coefficients of degree n has n complex roots. Hence

$$Z(r) = a_n (r - r_1)(r - r_2) \ldots (r - r_n),$$

where $r_1, \ldots, r_n$ are complex numbers, it is possible that some roots are repeated.

Homogeneous equation with constant coefficients

Definition 8.9

Let $P_k(x)$ be a polynomial of degree k in the variable x. A root r has **multiplicity** $m \in \mathbb{N}, m \geq 1$, if there is another polynomial $S_{k-m}(x)$ of degree $k - m$ such that $S_{k-m}(r) \neq 0$ and

$$P_k(x) = S_{k-m}(x)(x - r)^m.$$

We will discuss the following several cases.

Case 1: real and distinct roots

Case 1. If the roots of $Z(r) = 0$ are all **real** and **distinct**, then we have the solutions

$$y_1(t) = e^{r_1 t}, \quad \ldots, \quad y_n(t) = e^{r_n t}.$$

They are linearly independent solutions and form a fundamental set of solutions.
Exercise. Show the above n solutions form a fundamental set of solutions.
Hint.

$$\begin{aligned} W(e^{r_1 t}, e^{r_2 t}, \ldots, e^{r_n t})(t) = & \begin{vmatrix} e^{r_1 t} & e^{r_2 t} & \cdots & e^{r_n t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} & \cdots & r_n e^{r_n t} \\ \vdots & \vdots & \ddots & \vdots \\ r_1^{n-1} e^{r_1 t} & r_2^{n-1} e^{r_2 t} & \cdots & r_n^{n-1} e^{r_n t} \end{vmatrix} \\ = & e^{(r_1 + \cdots + r_n)t} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ r_1 & r_2 & \cdots & r_n \\ \vdots & \vdots & \ddots & \vdots \\ r_1^{n-1} & r_2^{n-1} & \cdots & r_n^{n-1} \end{vmatrix} \\ = & e^{(r_1 + \cdots + r_n)t} \prod_{1 \le i < j \le n} (r_j - r_i) \neq 0. \end{aligned}$$

(The results of the determinant for the vandermonde matrix is used.)

Example 8.10

Find the general solution to

$$y^{(4)} - 7y''' + 6y'' + 30y' - 36y = 0 \tag{1.2}$$

Solution.

The characteristic equation for Eq.(1.2) is:

$$r^4 - 7r^3 + 6r^2 + 30r - 36 = 0. \tag{1.3}$$

Since the factors of $a_0 = -36$ are $\pm 1, \pm 2, \pm 3, \pm 6, \pm 9, \pm 12$. By testing these possible roots, we find that -2 and 3 are actual roots. Hence we could factorize Eq.(1.3) as:

$$(r-3)(r+2)(r^2 - 6r + 6) = 0$$

Hence $r_1 = -2, r_2 = 3, r_3 = 3 - \sqrt{3}, r_4 = 3 + \sqrt{3}$. The general solution is given by:

$$y = c_1 e^{-2t} + c_2 e^{3t} + c_3 e^{(3-\sqrt{3})t} + c_4 e^{(3+\sqrt{3})t}.$$

$\square$

Case 2: some roots are complex

Case 2. If some roots are complex, they must appear in pairs, i.e. $\lambda \pm i\mu$.
In this case, we could replace the complex-valued solutions $e^{(\lambda+i\mu)t}$ and $e^{(\lambda-i\mu)t}$ by the real-valued solutions:

$$e^{\lambda t}\cos\mu t, \qquad e^{\lambda t}\sin\mu t.$$

Example 8.11

Find the general solution to

$$y^{(4)} - y = 0. \tag{1.4}$$

Solution.

The characteristic equation for Eq.(1.4) is:

$$r^4 - 1 = 0. \tag{1.5}$$

We derive that $r = 1, -1, \pm i$. And we take the real and imaginary part of the solution e^{it} to form the real-valued solutions:

$$e^{it} = \cos t + i \sin t \implies Re(e^{it}) = \cos t, \quad Im(e^{it}) = \sin t.$$

Hence $\{e^t, e^{-t}, \cos t, \sin t\}$ forms a fundamental set of solutions. The general solution is given by:

$$y = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t.$$

$\square$

Case 3: Some roots are repeated

Case 3: Some roots are repeated

Subcase 1: If one of the real root r_1 is repeated with multipicity s, then the corresponding linearly independent solutions corresponding to root r_1 are:

$$e^{r_1 t}, te^{r_1 t}, t^2 e^{r_1 t}, \ldots, t^{s-1} e^{r_1 t}.$$

Subcase 2: If the complex root $r_1 = \lambda + i\mu$ is repeated with multipicity s, then the corresponding conjugate of $\bar{r_1} = \lambda - i\mu$ is also the root with multipicity s. In this case, we could replace the complex-valued solutions $e^{(\lambda+i\mu)t}, \ldots, t^{s-1}e^{(\lambda+i\mu)t}$ and $e^{(\lambda-i\mu)t}, \ldots, t^{s-1}e^{(\lambda-i\mu)t}$ by the real valued solutions as follows:

$$e^{\lambda t}\cos\mu t, te^{\lambda t}\cos\mu t, t^2 e^{\lambda t}\cos\mu t, \ldots, t^{s-1}e^{\lambda t}\cos\mu t - \text{from real parts}$$
$$e^{\lambda t}\sin\mu t, te^{\lambda t}\sin\mu t, t^2 e^{\lambda t}\sin\mu t, \ldots, t^{s-1}e^{\lambda t}\sin\mu t - \text{from imaginary parts}$$

These are linearly independent solutions corresponding to the repeated roots $r_1 = \lambda + i\mu$ and $\bar{r_1} = \lambda - i\mu$.

Example 8.12

Find the general solution of

$$y^{(4)} + 2y'' + y = 0 \tag{1.6}$$

Solution.

The characteristic equation for Eq.(1.6) is:

$$r^4 + 2r^2 + 1 = (r^2 + 1)(r^2 + 1) = 0. \tag{1.7}$$

We derive that $r = i, i, -i, -i$. Hence the fundamental solution is:

$$e^{it}, te^{it}, e^{-it}, te^{-it}.$$

We take the real and imaginary part of $\{e^{it}, te^{it}\}$ or $\{e^{-i}, te^{-it}\}$ to form real-valued solution: Real part: $\cos t, t\cos t$. Imaginary part: $\sin t, t\sin t$. The general solution is given by:

$$y = c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \sin t.$$

Non-homogeneous equations

Consider the non-homogeneous equation

$$a_n y^n + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = g(t). \tag{1.8}$$

If Y_1 and Y_2 are both solutions to the non-homogeneous problem, then $Y_1 - Y_2$ is a solution to the corresponding homogeneous equation

$$a_n y^n + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0. \tag{1.9}$$

Given a fundamental set of solutions $(y_1, \ldots, y_n)$ to the corresponding homogeneous equation, we see that a general solution to the non-homogeneous equation (1.8) is

$$y(t) = c_1 y_1(t) + \cdots + c_n y_n(t) + Y(t),$$

where $Y(t)$ is a particular solution to the non-homogeneous equation (1.8), $c_1 y_1(t) + \cdots + c_n y_n(t)$ is the complementary solution (solution to the homogeneous equation).

Method of undetermined coefficients

Similar to second order equations, we now find a particular solution Y to the non-homogeneous equation (1.8) if $g(t)$ is a sum/product of exponentials, cosine, sine and polynomials. But the **main difference** is that the multiplicity of roots to the characteristic equation can be **greater** than two. There, **higher powers** of t need to be multiplied to get the solution to the non-homogeneous equation.

We again investigate the cases:

(1) $g(t) = e^{\alpha t} P_m(t)$,

(2) $g(t) = e^{\alpha t} P_m(t) \cos(\beta t)$, or $g(t) = e^{\alpha t} P_m(t) \sin(\beta t)$.

Remember the characteristic equation for the corresponding homogeneous equation

$$a_n y^n + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0.$$

is

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0 - - - - - (*).$$

Method of undetermined coefficients

The possible particular solutions can be used are

(1) $Y(t) = t^s e^{\alpha t} Q_m(t)$,where

$$Q_m(t) = A_m t^m + \cdots + A_1 t + A_0$$

for undetermined coefficients $A_m, \ldots, A_0$, and

$$s = \begin{cases} 0, & \text{if } \alpha \text{ is not a root of the characteristic equation (*).} \\ m, & \text{if } \alpha \text{ is a root of the characteristic equation (*) with multiplicity } m \end{cases}$$

(2) $Y(t) = t^s e^{\alpha t}[Q_m(t)\cos(\beta t) + R_m(t)\sin(\beta t)]$, where $Q_m = A_m t^m + \cdots + A_1 t + A_0, R_m = B_m t^m + \cdots + B_1 t + B_0$ are polynomials of degree m with undetermined coefficients $A_m, \ldots, A_0, B_m, \ldots, B_0$, and

$$s = \begin{cases} 0, & \text{if } \alpha + i\beta \text{ is not a root of the characteristic equation (*).} \\ m, & \text{if } \alpha + i\beta \text{ is a root of the characteristic equation(*) with multiplicity } m. \end{cases}$$

Method of undetermined coefficients

Example 1

Solve

$$y''' - 3y'' + 3y' - y = 4e^t.$$

For the homogeneous equation, the associated characteristic equation is

$$r^3 - 3r^2 + 3r - 1 = (r-1)^3 = 0,$$

and so $r_1 = r_2 = r_3 = 1$, i.e., a repeated eigenvalue of multiplicity three. So we set

$$y_1 = e^t, \quad y_2 = te^t, \quad y_3 = t^2e^t,$$

and the complementary solution (to the homogeneous equation) is

$$y_c(t) = c_1e^t + c_2te^t + c_3t^2e^t.$$

Since $g(t) = 4e^t$ and so $\alpha = 1$ is a root of the characteristic equation with multiplicity 3.

Method of undetermined coefficients

Example 1

Therefore we have to consider $s = 3$ and a trial solution

$$Y(t) = At^3 e^t.$$

Computing gives

$$Y''' - 3Y'' + 3Y' - Y = 6Ae^t = 4e^t \Rightarrow A = \frac{2}{3},$$

and so the general solution to the non-homogeneous ODE is

$$y(t) = c_1 e^t + c_2 te^t + c_3 t^2 e^t + \frac{2}{3} t^3 e^t.$$

Example 2

Find the general solution to the ODE

$$y''' - 3y'' + 4y' - 2y = t^2 e^{2t} \tag{1.10}$$

For its homogeneous part, the characteristic equation is given by:

$$r^3 - 3r^2 + 4r - 2 = 0. \implies (r-1)(r^2 - 2r + 2) = 0.$$

Thus $r_1 = 1, r_2 = 1 + i, r_3 = 1 - i$. We take the real and imaginary part of the solution $e^{(1+i)t}$:

$$Re(e^{(1+i)t}) = Re(e^t(\cos t + i \sin t)) = e^t \cos t$$

$$Im(e^{(1+i)t}) = Im(e^t(\cos t + i \sin t)) = e^t \sin t$$

Hence the solution to the homogeneous part is:

$$y_c = c_1 e^t + c_2 e^t \cos t + c_3 e^t \sin t$$

Then we want to find the particular solution. $\alpha = 2, \beta = 0$, α is the not the root of the characteristic equation.

Example 2

We guess the form of $Y(t)$ to be:

$$Y(t) = (At^2 + Bt + C)e^{2t}$$

It follows that

$$\begin{aligned} Y' &= (2At + B)e^{2t} + 2(At^2 + Bt + C)e^{2t} \\ &= [2At^2 + (2A + 2B)t + (B + 2C)]e^{2t} \end{aligned}$$

$$\begin{aligned} Y'' &= [4At + (2A + 2B)]e^{2t} + 2[2At^2 + (2A + 2B)t + (B + 2C)]e^{2t} \\ &= [4At^2 + (8A + 4B)t + (2A + 4B + 4C)]e^{2t} \end{aligned}$$

$$\begin{aligned} Y''' &= [8At + (8A + 4B)]e^{2t} + 2[4At^2 + (8A + 4B)t + (2A + 4B + 4C)]e^{2t} \\ &= [8At^2 + (24A + 8B)t + (12A + 12B + 8C)] \end{aligned}$$

Example 2

We plug the above formulas to Eq.(1.10) to obtain:

$$\left((8A-12A+8A-2A)t^2+((24A+8B)-3(8A+4B)+4(2A+2B)-2B)t\right)e^{2t}$$
$$+\left((12A+12B+8C)-3(2A+4B+4C)+4(B+2C)-2C\right)e^{2t}=t^2e^{2t}.$$

It follows that

$$\begin{aligned} 8A-12A+8A-2A&=1\\ (24A+8B)-3(8A+4B)+4(2A+2B)-2B&=0\\ (12A+12B+8C)-3(2A+4B+4C)+4(B+2C)-2C&=0 \end{aligned} \Longrightarrow \begin{cases} A=\dfrac{1}{2}\\ B=-2\\ C=\dfrac{5}{2} \end{cases}$$

The particular solution is given by:

$$Y(t)=\left(\frac{1}{2}t^2-2t+\frac{5}{2}\right)e^{2t}.$$

The general solution is obtained:

$$y=y_c+Y(t)=c_1e^t+c_2e^t\cos t+c_3e^t\sin t+\left(\frac{1}{2}t^2-2t+\frac{5}{2}\right)e^{2t}.$$

Method of undetermined coefficients

Example 3

Solve

$$y^{(4)} + 2y'' + y = 3\sin t.$$

The characteristic equation corresponding to the homogeneous equation is

$$r^4 + 2r^2 + 1 = (r^2 + 1)(r^2 + 1) = 0$$

and so $r_1 = r_3 = i$, $r_2 = r_4 = -i$,i.e., a repeated pair of complex conjugate roots (multiplicity is two). Then we see that

$$y_1 = \cos t, \quad y_2 = \sin t, \quad y_3 = t\cos t, \quad y_4 = t\sin t,$$

and the complementary solution to the homogeneous equation is

$$y_c(t) = c_1\cos t + c_2\sin t + c_3 t\cos t + c_4 t\sin t.$$

For the non-homogeneous term $g(t) = 3\sin t$, we have $\alpha = 0, \beta = 1$, $\alpha + i\beta = i$ is the root with multiplicity 2. Thus, $s = 2$.

Method of undetermined coefficients

Example 3

Thus we consider a trial solution

$$Y(t) = At^2 \sin t + Bt^2 \cos t.$$

Then,

$$Y^{(4)} + 2Y'' + Y = -8A \sin t - 8B \cos t = 3 \sin t \Rightarrow B = 0, \quad A = -\frac{3}{8}.$$

Hence, the general solution to the non-homogeneous equation is

$$y(t) = c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \sin t - \frac{3}{8} t^2 \sin t.$$

Variation of parameters

Similar to second order equations, there is also a method to treat rather general high order equations

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \cdots + p_1(t)y' + p_0(t)y = g(t), \quad t \in I.$$

Suppose we have a fundamental set of solutions to $y_1, \ldots, y_n$ to the homogeneous equation. Then, the complementary solution is

$$y_c(t) = c_1 y_1(t) + \cdots + c_n y_n(t).$$

Now, we consider a trial solution for the non-homogeneous equation of the form

$$Y(t) = u_1(t)y_1(t) + \cdots + u_n(t)y_n(t)$$

for unknown functions $u_1, \ldots, u_n$. Differentiating gives

$$Y'(t) = u_1(t)y_1'(t) + \cdots + u_n(t)y_n'(t) + u_1'(t)y_1(t) + \cdots + u_n'(t)y_n(t).$$

Variation of parameters

As before we set the constraint

$$u_1'(t)y_1(t) + u_2'(t)y_2(t) + \cdots + u_n'(t)y_n(t) = 0,$$

so that the expression for Y' simplifies to

$$Y'(t) = u_1(t)y_1'(t) + u_2(t)y_2'(t) + \cdots + u_n(t)y_n'(t).$$

Computing Y'' and setting

$$u_1'(t)y_1'(t) + \cdots + u_n'(t)y_n'(t) = 0$$

leads to the simplified expression for the second derivative

$$Y''(t) = u_1(t)y_1''(t) + \cdots + u_n(t)y_n''(t).$$

Variation of parameters

Repeating this procedure (differentiating and then setting the sum of terms involving the derivatives of $u_1, \ldots, u_n$ to zero) leads to the $n-1$ equations

$$u_1'(t)y_1^{(m)}(t) + \cdots + u_n'(t)y_n^{(m)}(t) = 0 \quad \forall 1 \leq m \leq n-2,$$

as well as a simplified expression for $Y^{(m)}$:

$$Y^{(m)}(t) = u_1(t)y_1^{(m)}(t) + \cdots + u_n(t)y_n^{(m)}(t), \quad m = 1, \ldots, n-1,$$
$$Y^{(n)}(t) = u_1(t)y_1^{(n)}(t) + \cdots + u_n(t)y_n^{(n)}(t) + u_1'(t)y_1^{(n-1)}(t) + \cdots + u_n'(t)y_n^{(n-1)}(t).$$

So if Y is a particular solution to the non-homogeneous equation, substituting all the expressions for Y and its derivative into the equation, and using that $y_1, \ldots, y_n$ solve the homogeneous equation, we are lead to

$$u_1'(t)y_1^{(n-1)}(t) + \cdots + u_n'(t)y_n^{(n-1)}(t) = g(t).$$

Variation of parameters

Collecting all the expressions involving the first derivative of $u_1, \ldots, u_n$, we obtain

$$\begin{pmatrix} y_1 & y_2 & \cdots & y_{n-1} & y_n \\ y_1' & y_2' & \cdots & y_{n-1}' & y_n' \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_{n-1}^{(n-2)} & y_n^{(n-2)} \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_{n-1}^{(n-1)} & y_n^{(n-1)} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \\ \vdots \\ u_{n-1}' \\ u_n' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ g(t) \end{pmatrix}.$$

Thus, the derivatives of the unknown functions $u_1, \ldots, u_n$ can be found by inverting the matrix of derivatives. The determinant of the matrix is the Wronskian, which is non-zero thanks to the fact that $(y_1, \ldots, y_n)$ forms a fundamental set of solutions. Setting $M(t)$ as the matrix, we solve

$$M(t) \begin{pmatrix} u_1' \\ u_2' \\ \vdots \\ u_{n-1}' \\ u_n' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ g(t) \end{pmatrix}.$$

Variation of parameters

To invert $M(t)$, we use Cramers rule, by setting

$$M_i(t) = \begin{pmatrix} y_1 & \dots & 0 & \dots & y_n \\ y_1' & \dots & 0 & \dots & y_n' \\ \vdots & & \vdots & & \vdots \\ y_1^{(n-2)} & \dots & 0 & \dots & y_n^{(n-2)} \\ y_1^{(n-1)} & \dots & 1 & \dots & y_n^{(n-1)} \end{pmatrix},$$

i.e., replace the ith column of $M(t)$ with the vector $(0, \dots, 0, 1)^T$. Then Cramer's rule gives

$$u_i'(t) = \frac{g(t) \det M_i(t)}{\det M(t)},$$

and by integrating we get an expression for $u_i(t)$. The particular solution to the non-homogeneous equation is therefore

$$Y(t) = y_1(t) \int \frac{g(t) \det M_1(t)}{\det M(t)} dt + \dots + y_n(t) \int \frac{g(t) \det M_n(t)}{\det M(t)} dt.$$

Variation of parameters

However, in general the evaluation of the integrals can be difficult, but we can always use Abel's theorem to simplify, since

$$\det M(t) = W(y_1, \ldots, y_n)[t] = ce^{-\int p_{n-1}(t)dt}.$$

We finish with two examples.

Example 4

Solve

$$y''' + y' = \sec^2(t) \text{ for } t \in (-\pi/2, \pi/2).$$

The characteristic equation for the homogeneous problem is $r^3 + r = 0$ and so $r_1 = 0$, $r_2 = i$ and $r_3 = -i$. Hence the complementary solution is

$$y_c(t) = c_1 + c_2 \cos t + c_3 \sin t.$$

By variation of parameters we look for a particular solution of the form

$$Y(t) = u_1 y_1 + u_2 y_2 + u_3 y_3 = u_1(t) + u_2(t) \cos t + u_3(t) \sin t,$$

with

Variation of parameters

Example 4

$$\begin{aligned} u_1' + u_2' \cos t + u_3' \sin t &= 0, \\ -u_2' \sin t + u_3' \cos t &= 0, \\ -u_2' \cos t - u_3' \sin t &= \sec^2(t), \end{aligned}$$

or equivalently

$$M(t) \begin{pmatrix} u_1' \\ u_2' \\ u_3' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \sec^2(t) \end{pmatrix}, \quad M(t) = \begin{pmatrix} 1 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 0 & -\cos t & -\sin t \end{pmatrix}.$$

Computing the determinant of M, we see that $\det M(t) = 1$. Now, define

$$M_1(t) = \begin{pmatrix} 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 1 & -\cos t & -\sin t \end{pmatrix}, \quad M_2(t) = \begin{pmatrix} 1 & 0 & \sin t \\ 0 & 0 & \cos t \\ 0 & 1 & -\sin t \end{pmatrix}$$

$$M_3(t) = \begin{pmatrix} 1 & \cos t & 0 \\ 0 & -\sin t & 0 \\ 0 & -\cos t & 1 \end{pmatrix},$$

Variation of parameters

Example 4

it is easy to compute that

$$\det M(t) = 1, \quad \det M_1(t) = 1, \quad \det M_2(t) = -\cos t, \quad \det M_3(t) = -\sin t,$$

and so

$$\begin{aligned}
u_1 &= \int \sec^2(t)dt = \tan(t), \\
u_2 &= \int -\sec^2(t)\cos(t)dt = -\ln(|\sec(t) + \tan(t)|), \\
u_3 &= \int -\sec^2(t)\sin(t)dt = -\sec(t).
\end{aligned}$$

Hence, the particular solution is

$$\begin{aligned}
Y(t) &= \tan(t) - \cos(t)\ln(|\sec(t) + \tan(t)|) - \sin(t)\sec(t) \\
&= -\cos(t)\ln(|\sec(t) + \tan(t)|).
\end{aligned}$$

Variation of parameters

Example 5

Find the general solution to equation:

$$y''' - 3y'' + 4y' - 2y = \frac{e^t}{\cos t}, \quad t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \tag{1.11}$$

(Hint: you may use the formula $\int \frac{1}{\cos t} dt = \frac{1}{2} \ln \left|\frac{1+\sin t}{1-\sin t}\right| + C$.)
It is easy to verify that the general solution to the corresponding homogeneous ODE is:

$$y_c = c_1 e^t + c_2 e^t \cos t + c_3 e^t \sin t.$$

The formula for the particular solution is:

$$Y(t) = \sum_{i=1}^{3} \left[\int \frac{g(s) \det M_i(s)}{\det M(s)} ds \right] y_i$$

Variation of parameters

Example 5

$$\det M(t) = \begin{vmatrix} e^t & e^t\cos t & e^t\sin t \\ e^t & e^t(\cos t - \sin t) & e^t(\sin t + \cos t) \\ e^t & e^t(-2\sin t) & e^t(2\cos t) \end{vmatrix}$$

$$= e^{3t}\begin{vmatrix} 1 & \cos t & \sin t \\ 1 & \cos t - \sin t & \sin t + \cos t \\ 1 & -2\sin t & 2\cos t \end{vmatrix} = e^{3t}$$

$$\det M_1(t) = \begin{vmatrix} 0 & e^t\cos t & e^t\sin t \\ 0 & e^t(\cos t - \sin t) & e^t(\sin t + \cos t) \\ 1 & e^t(-2\sin t) & e^t(2\cos t) \end{vmatrix} = e^{2t}.$$

$$\det M_2(t) = \begin{vmatrix} e^t & 0 & e^t\sin t \\ e^t & 0 & e^t(\sin t + \cos t) \\ e^t & 1 & e^t(2\cos t) \end{vmatrix} = -e^{2t}\cos t.$$

Variation of parameters

Example 5

$$\det M_3(t) = \begin{vmatrix} e^t & e^t \cos t & 0 \\ e^t & e^t(\cos t - \sin t) & 0 \\ e^t & e^t(-2\sin t) & 1 \end{vmatrix} = -e^{2t} \sin t$$

It follows that

$$\begin{aligned} Y(t) &= e^t \int \frac{\frac{e^t}{\cos t} e^{2t}}{e^{3t}} dt + e^t \cos t \int \frac{\frac{e^t}{\cos t} \cdot (-e^{2t} \cos t)}{e^{3t}} dt + e^t \sin t \int \frac{\frac{e^t}{\cos t} \cdot (-e^{2t} \sin t)}{e^{3t}} dt \\ &= e^t \int \frac{1}{\cos t} dt + e^t \cos t \int (-1) dt + e^t \sin t \int (-\tan t) dt \\ &= \frac{e^t}{2} \ln \left| \frac{1 + \sin t}{1 - \sin t} \right| - te^t \cos t + e^t \sin t \ln |\cos t| \end{aligned}$$

Variation of parameters

Example 5

Since $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$, we obtain our particular solution:

$$Y(t) = \frac{e^t}{2}\ln\left(\frac{1+\sin t}{1-\sin t}\right) - te^t\cos t + e^t\sin t\ln(\cos t)$$

Hence our general solution is given by:

$$\begin{aligned} y &= y_c(t) + Y(t) \\ &= c_1e^t + c_2e^t\cos t + c_3e^t\sin t + \frac{e^t}{2}\ln\left(\frac{1+\sin t}{1-\sin t}\right) - te^t\cos t + e^t\sin t\ln(\cos t) \end{aligned}$$