21 Lecture 21 (Residue theorem)

Summary

- Uniqueness of coefficients in Laurent series
- Residue theorem
- Calculation of the residue at a pole

In Corollary 20.4 the two formula for computing the coefficients of Laurent series can be combined into one. We can integrate over any closed curve inside the annulus, winding around the origin once in the counter-clockwise orientation. If we write the Laurent series as

$$\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n,$$

then

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z_0)^{n+1}}\,dw \quad (\text{for } n = 0, \pm 1, \pm 2, \dots)$$

We next show that the value of the coefficients have no other choice; it must be equal to the above formula. For notational convenience we suppose $z_0 = 0$. Suppose $f(z)$ is expanded as the following Laurent series

$$f(z) = \sum_{n=0}^{\infty} a'_n z^n + \sum_{n=1}^{\infty} b'_n z^{-n},$$

where a'_n and b'_n are some coefficients, and the region of convergence is an annulus with center $z = 0$. We divide the right-hand side into three parts

$$f(z) = \sum_{n=0}^{\infty} a'_n z^n + \frac{b_1}{z} + \sum_{n=2}^{\infty} b'_n z^{-n}. \tag{21.1}$$

The first part has anti-derivative

$$\sum_{n=0}^{\infty} \frac{a'_n}{n+1} z^{n+1}.$$

We use the property that Taylor series can be differentiated term-wise. The third part also has anti-derivative. To derive the anti-derivative, we can make a substitute $u = 1/z$. Let $g(u)$ denote the Taylor series

$$g(u) = \sum_{n=2}^{\infty} b'_n u^{n-2}.$$

(Note $n = 2, 3, 4, \ldots$ and $n - 2 \geq 0$.) We can differentiate termwise to obtain

$$\frac{d}{du}\sum_{n=2}^{\infty}\frac{b'_n}{n-1}u^{n-1} = g(u).$$

Using chain rule, we get

$$\frac{d}{dz}\sum_{n=2}^{\infty}\frac{b'_n}{n-1}(\frac{1}{z})^{n-1} = g(\frac{1}{z})(\frac{-1}{z^2}) = \sum_{n=2}^{\infty}b'_n z^{-n+2}(-1/z^2) = -\sum_{n=2}^{\infty}b'_n z^{-n}.$$

This shows that $\sum_{n=2}^{\infty} b'_n z^{-n}$ has an anti-derivative.

Integrate both sides of (21.1) over a close curve C inside the annulus of convergence. The first and the third term on the right side of (21.1) becomes zero, because they have anti-derivative (Theorem 13.4). Therefore

$$\int_C f(z)\,dz = \int_C \frac{b'_1}{z}\,dw = 2\pi i b_1.$$

The coefficient b_1 is uniquely determined by

$$b'_1 = \frac{1}{2\pi i}\int_C f(z)\,dz, \tag{21.2}$$

where C is a close curve traveling inside the annulus once counter-clockwise. The integral only depends on $f(z)$ and hence the coefficient b'_1 is uniquely determined.

For other coefficients in the Laurent series we can multiply by z^n, for $n \geq 1$, to get b_{n+1}

$$\int_C f(z)z^n\,dz = \int_C \frac{b'_{n+1}}{z}\,dw = 2\pi i b'_{n+1},$$

and divide by z^n, for $n \geq 1$, to get a_{n-1},

$$\int_C f(z)/z^n\,dz = \int_C \frac{a'_{n-1}}{z}\,dw = 2\pi i a'_{n-1}.$$

As a result, all coefficients are uniquely determined by $f(z)$ and the annulus of convergence.

Remark. The coefficients in a Laurent series not only depend on the function $f(z)$, they also depend on the annulus in which the function $f(z)$ is analytic. Different convergence regions will give different coefficients, because the curve C must be chosen inside the annulus.

After establishing the uniqueness of the coefficients in a Laurent series, we can now define residue as follows

Definition 21.1. Consider a complex function $f(z)$ that is analytic in a domain except some isolated singular points. The *residue* of $f(z)$ at a point z_0 is defined as the coefficient b_1 in the Laurent series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n},$$

with convergence in a small open disk $D(z_0; \epsilon) \setminus \{z_0\}$ centered at z_0. There are several notation for residue, e.g. $\operatorname{Res}(f; z_0)$, $\operatorname{Res}_{z_0}(f)$, and $\operatorname*{Res}_{z=z_0}(f)$.

(In view of the remark before Definition 21.1, it is important to state the region of convergence in the definition.)

When $f(z)$ is analytic at z_0, then $\operatorname{Res}(f; z_0) = 0$, because the principal part is zero.

Example 21.1. The residue of $e^{1/z}$ at $z = 0$ is 1, because the coefficient of $1/z$ in

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2! z^2} + \cdots$$

is 1.

For pole with smaller order, the residue can be computed efficiently. If z_0 is a pole of $f(z)$ with order m, then

$$f(z) = \frac{b_m}{(z - z_0)^m} + \cdots + \frac{b_1}{z - z_0} + a_0 + a_1(z - z_0) + \cdots$$
$$(z - z_0)^m f(z) = b_m + \cdots + b_1 (z - z_0)^{m-1} + a_0 (z - z_0)^m + \cdots .$$

We can extract the coefficient b_1 by

$$b_1 = \operatorname{Res}(f; z_0) = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d}{dz} \big[(z - z_0)^m f(z) \big].$$

In particular, for pole with order 1,

$$\operatorname{Res}(f; z_0) = \lim_{z \to z_0} (z - z_0) f(z),$$

and for pole with order 2,

$$\operatorname{Res}(f; z_0) = \lim_{z \to z_0} \frac{d}{dz} (z - z_0)^2 f(z).$$

Theorem 21.2 (Residue theorem). *Suppose f is analytic in a domain D except for some isolated singularities. If C is a simple closed curve enclosing singular points $z_1, z_2, \ldots, z_k$ in the interior, then*

$$\int_C f(z)\,dz = 2\pi i \sum_{j=1}^{k} \operatorname{Res}(f, z_j).$$

Proof. For $j = 1, 2, \ldots, k$, we draw a small circle C_j centered at z_j so that the circle contains does not contain the other singular points. By Cauchy theorem for multiply connected region (Theorem 16.2),

$$\int_C f(z)\,dz = \sum_{j=1}^{k} \int_{C_j} f(z)\,dz.$$

Since the residue of f at z_0 is equal to the integral $2\pi i \int_{C_j} f(z)\,dz$ (see Equation (21.2)), we obtain

$$\int_C f(z)\,dz = 2\pi i \sum_{j=1}^{k} \operatorname{Res}(f, z_j).$$

□

Example 21.2. Compute

$$\int_C \frac{dz}{z(z-1)(z-2)}$$

with C being the contour $|z| = 1.5$ with counter-clockwise orientation.

The contour C contains two poles at $z = 0$ and $z = 1$. The residues at these two poles are

$$\operatorname{Res}\left(\frac{1}{z(z-1)(z-2)}; 0\right) = \lim_{z\to 0} z \frac{1}{z(z-1)(z-2)} = \frac{1}{2}$$
$$\operatorname{Res}\left(\frac{1}{z(z-1)(z-2)}; 1\right) = \lim_{z\to 1} (z-1) \frac{1}{z(z-1)(z-2)} = -1.$$

Apply residue theorem,

$$\int_C \frac{dz}{z(z-1)(z-2)} = 2\pi i(\frac{1}{2} - 1) = -\pi i.$$

Example 21.3. Evaluate

$$\int_C \frac{dz}{z(z-1)^2}$$

over the contour $C : |z| = 2$ with counter-clockwise orientation.

The contour C encloses the simple pole at $z = 0$ and the double pole at $z = 1$.

$$\operatorname{Res}\left(\frac{1}{z(z-1)^2}; 0\right) = \lim_{z\to 0} z\frac{1}{z(z-1)^2} = 1$$
$$\operatorname{Res}\left(\frac{1}{z(z-1)^2}; 1\right) = \lim_{z\to 1} \frac{d}{dz}\left[(z-1)^2\frac{1}{z(z-1)^2}\right] = -1.$$

By residue theorem, the integral is equal to $2\pi i(1 + (-1)) = 0$.

22 Lecture 22 (Winding number, argument principle)

Summary

- Winding number
- A more general form of residue theorem
- Argument principle

Theorem 22.1. *If γ is a closed piece-wise smooth curve not passing through a point z_0, then*

$$\frac{1}{2\pi i}\int_\gamma \frac{1}{z - z_0}\,dz$$

is an integer.

Proof. Represent the curve by a parameterization

$$\gamma : [0, 1] \to \mathbb{C} \setminus \{z_0\}.$$

Write $\gamma(t) - z_0$ in polar form

$$\gamma(t) - z_0 = r(t) + e^{i\theta(t)}.$$

The distance between $\gamma(t)$ and z_0 is given by $r(t)$. The angle $\theta(t)$ is measured with respect to the given point z_0. Since it is assumed that $\gamma(t)$ does not pass through z_0, we have $r(t) > 0$ for all t.

Differentiating $\gamma(t)$ once to get

$$\gamma'(t) = [r'(t) + ir(t)\theta'(t)]e^{i\theta(t)}.$$

Using the definition of complex integral, we compute

$$\begin{aligned}
\frac{1}{2\pi i}\int_\gamma \frac{1}{z-z_0}\,dz &= \frac{1}{2\pi i}\int_0^1 \frac{[r'(t)+ir(t)\theta'(t)]e^{i\theta(t)}}{r(t)e^{i\theta(t)}}\,dt \\
&= \frac{1}{2\pi i}\int_0^1 r'(t)/r(t)\,dt + \frac{1}{2\pi}\int_0^1 \theta'(t)\,dt \\
&= \frac{1}{2\pi i}[\log(r(1)) - \log(r(0))] + \frac{1}{2\pi}[\theta(1)-\theta(0)].
\end{aligned}$$

The first term is equal to zero, because $r(0) = r(1)$. The second term is precisely the number of time the curve γ goes around the point z_0. $\square$

Based on the previous theorem, we can make the following definition

Definition 22.2. The *winding number* of a closed curve γ around a point z_0 is defined as

$$n(\gamma; z_0) \triangleq \frac{1}{2\pi i}\int_\gamma \frac{1}{z-z_0}\,dz.$$

Other notation for the winding number includes: $w(\gamma, z_0)$, $ind(\gamma, z_0)$.

The Jordan curve theorem (see Remark 14), the winding number of a simple closed curve about a fixed point z_0 is either 0, or ± 1, depending on whether the point z_0 is inside or outside the curve, and whether the orientation of the curve is positive or negative. For closed curve in general (which need not be simple), the winding number could be any integer.

Theorem 22.3 (Generalized residue theorem)**.** *Suppose f is analytic in a simply connected domain except k isolated singular points z_1, $z_2, \ldots, z_k$. For any closed and piecewise smooth curve C, not intersecting any one of the k singular points, we have*

$$\int_C f(z)\,dz = 2\pi i \sum_{j=1}^{k} n(C; z_j)\,\mathrm{Res}(C; z_j). \tag{22.1}$$

We note that when C is a simple closed path with positive orientation, the generalized residue theorem reduces to the residue theorem (Theorem 21.2). The winding numbers in 22.1 are some weighting factor for the corresponding residue. The weight factor is 0 if the singular point is outside the curve.

Proof. For each$j = 1, 2, \ldots, k$, expand the function $f(z)$ using Laurent series at z_j. Denote the principal part by $p_j((z - z_j)^{-1})$, for $j = 1, 2, \ldots k$. We mote that $p_j((z - z_j)^{-1})$ is analytic except at the point z_j.

Consider the function

$$g(z) \triangleq f(z) - \sum_{j=1}^{k} p_j((z - z_j)^{-1})$$

obtained by subtracting all the principal parts $p_1((z - z_1)^{-1})$ to $p_k((z - z_k)^{-1})$ from $f(z)$. For any point other than the k singular points, the sum $\sum_j p_j((z - z_j)^{-1})$ is analytic. At the point z_j, $g(z)$ can be written as

$$g(z) = f(z) - p_j((z - z_j)^{-1}) - \sum_{\substack{\ell=1 \\ \ell \neq j}}^{k} p_\ell((z - z_\ell)^{-1}).$$

But $f(z) - p_j((z - z_j)^{-1})$ is analytic at z_j. Hence $g(z)$ is analytic at z_j. So, $g(z)$ is analytic at all points in the domain. By Cauchy theorem (Theorem 14.6),

$$\int_C g(z)\, dz = 0.$$

This yields

$$\int_C g(z)\, dz = \sum_{j=1}^{k} \int_C p_j((z - z_j)^{-1}).$$

For each $j = 1, 2, \ldots$, the integral $\int_C p_j((z - z_j)^{-1})$ only depends on the term with degree -1 (see the proof of the uniqueness of the coefficients of Laurent series),

$$\begin{aligned}
\int_C p_j((z - z_j)^{-1}) &= 2\pi i \operatorname{Res}(f; z_j) \frac{1}{2\pi i} \int_C \frac{1}{z - z_j}\, dz \\
&= 2\pi i \operatorname{Res}(f; z_j) n(\gamma; z_j).
\end{aligned}$$

This proves (22.1). $\square$

We next go back to the simpler case of simple closed curve in the next theorem

Theorem 22.4 (Argument principle)**.** *Suppose C is the boundary of a simply connected region and f is analytic inside C and on the boundary of C. Then*

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)}\, dz = \text{ no. of zeros inside } C \text{ (counted with multiplicity).}$$

Proof. Suppose z_1, $z_2, \dots, z_k$ are the zeros of f inside C. For each $j = 1, 2, \dots, k$, suppose the order of zero at z_j is m_j. We note that f'/f is analytic except at z_j, for $j = 1, 2, \dots, k$. Let C_j be a small circle centered at z_j, so that C_j lies inside C and the circle C_1, $C_2, \dots, C_k$ do not overlap. By Cauchy theorem for multiply connected region (Theorem 16.3),

$$\int_C \frac{f'}{f}\,dz = \sum_{j=1}^{k} \int_{C_j} \frac{f'}{f}\,dz.$$

For $j = 1, 2, \dots, k$, We can write $f(z)$ as

$$f(z) = a_{m_j}(z - z_j)^{m_j} + a_{m_j+1}(z - z_j)^{m_j+1} + a_{m_j+2}(z - z_j)^{m_j+2} + \cdots$$
$$f'(z) = m_j a_{m_j}(z - z_j)^{m_j-1} + (m_j + 1)a_{m_j+1}(z - z_j)^{m_j} + (m_j + 2)a_{m_j+2}(z - z_j)^{m_j+1} + \cdots.$$

Here a_{m_j} is a nonzero coefficient. Factor out $a_{m_j}(z - z_j)^{m_j}$ from $f(z)$.

$$f(z) = a_{m_j}(z - z_j)^{m_j}\big[1 + \frac{a_{m_j+1}}{a_{m_j}}(z - z_j) + \frac{a_{m_j+2}}{a_{m_j}}(z - z_j)^2 + \cdots\big]$$

We see that the expression inside the square bracket is an analytic function in a small neighborhood of z_j. Moreover, the value of this analytic function at z_j is 1. Denote the expression inside the square bracket by $h(z)$. We have $h(z) \neq 0$ in a small neighborhood of z_j and $h(z_j) = 1$. We can take the reciprocal of $h(z)$ in a small enough neighborhood of z_j, and get

$$\begin{aligned}\frac{f'}{f} &= \frac{m_j a_{m_j}(z - z_j)^{m_j-1} + (m_j + 1)a_{m_j+1}(z - z_j)^{m_j} + \cdots}{a_{m_j}(z - z_j)^{m_j}} h^{-1}(z) \\ &= \frac{m_j}{z - z_j} + \text{an analytic function.}\end{aligned}$$

(We use the property that $h^{-1}(z)$ can be expanded as a Taylor series centered at z_j with constant term 1.) By adjusting the size of C_j, so that C_j lies within the region in which $h(z)$ is nonzero, we obtain

$$\int_{C_j} \frac{f'}{f}\,dz = 2\pi i m_j.$$

Summing it over all $j = 1, 2, \dots, k$, we get

$$\frac{1}{2\pi i}\int_C \frac{f'}{f}\,dz = \frac{1}{2\pi i}\sum_{j=1}^{k}\int_{C_j} \frac{f'}{f}\,dz = m_1 + m_2 + \cdots + m_k.$$

This is the number zeros inside C, counted with multiplicity. ∎

23 Lecture 23 (Rouche's theorem)

Summary

- A connection between principle argument and winding number
- Rouche theorem

In this lecture we derive Rouche's theorem using the principle argument. We will use the following connection between the argument principle and the winding number.

Lemma 23.1. *Suppose C is a simple closed curve parameterized by $\gamma(t)$, for $a \leq t \leq b$. Then*

$$\frac{1}{2\pi i}\int_C \frac{f'}{f}\,dz = n(f(\gamma(t));0).$$

That is, the number of zeros inside C is exactly the same as the number of time the curve $f(\gamma(t))$ goes around the origin.

Proof. Consider the curve C' that is parameterized by $f(\gamma(t))$, for t in $[a,b]$. and let $g(t) = f(\gamma(t))$. The winding number of C' around 0 is

$$\begin{aligned} n(C';0) &= \frac{1}{2\pi i}\int_{C'} \frac{1}{z}\,dz \\ &= \frac{1}{2\pi i}\int_a^b \frac{g'(t)}{g(t)}\,dt \\ &= \frac{1}{2\pi i}\int_a^b \frac{f'(\gamma(t))\gamma'(t)}{f(\gamma(t))}\,dt. \end{aligned}$$

On the other hand, we have

$$\frac{1}{2\pi i}\int_C \frac{f'}{f}\,dz = \frac{1}{2\pi i}\int_a^b \frac{f'(\gamma(t))}{f(\gamma(t))}\gamma'(t)\,dz.$$

□

Theorem 23.2 (Rouché's theorem)**.** *Suppose C is a simple closed curve with positive orientation. If f and g are functions analytic in a neighborhood containing C and*

$$|f(z)| > |g(z)|$$

for all $z \in C$, then the number of zeros of $f+g$ inside C is the same as the number of zeros of f inside C.

We note that the assumption $|f(z)| > |g(z)|$ implies (i) $f(z) \neq 0$ for all $z \in C$, and (ii) $f(z) + g(z) \neq 0$ for all $z \in C$.

Proof. Parameterize the curve C by $\gamma(t)$, for $t \in [0, 1]$. The main idea of proof is that the curve $g(\gamma(t))/f(\gamma(t))$, for $t \in [0, 1]$ lies inside the circle $|z - 1| = 1$ with radius 1 and center at $z = 1$. The winding number of the curve $g(\gamma(t))/f(\gamma(t))$ around the origin is thus zero.

By the argument principle (Theorem 22.4), the number of zeros of $f + g$ inside C can be computed by

$$\frac{1}{2\pi i} \int_C \frac{f' + g'}{f + g}\, dz.$$

Using product rule for differentiation, we can re-write the integrand as

$$\frac{f' + g'}{f + g} = \frac{f'}{f'} + \frac{\left(1 + \frac{g}{f}\right)'}{1 + \frac{g}{f}}.$$

Therefore

$$\frac{1}{2\pi i} \int_C \frac{f' + g'}{f + g}\, dz = \frac{1}{2\pi i} \int_C \frac{f'}{f'}\, dz + \frac{1}{2\pi i} \int_C \frac{\left(1 + \frac{g}{f}\right)'}{1 + \frac{g}{f}}\, dz$$

If we let $h(z) = 1 + g(z)/f(z)$, then the second integral on the right is equal to the number of time the curve $h \circ \gamma$ goes around the origin (see Lemma 23.1) But we have just shown that the curve lies completely inside the right half plane $\{x + iy \in \mathbb{C} : x > 0\}$. Hence the winding number $n(h \circ \gamma, 0)$ is equal to 0. This proves

$$\frac{1}{2\pi i} \int_C \frac{f' + g'}{f + g}\, dz = \frac{1}{2\pi i} \int_C \frac{f'}{f'}\, dz.$$

The integral on the right-hand side is the number of zeros of f inside the curve C, by the argument principle. □

Example 23.1. Using Rouche's theorem, we can show that the polynomial $z^{100} + 3z^3 + 1$ has exactly 3 complex roots inside the unit circle. For $|z| = 1$, we check that

$$|3z^3| = 3, \quad \text{but } |z^{100} - 1| < 2.$$

Apply Rouche's theorem with $f(z) = 3z^3$ and $g(z) = z^{100} - 1$. The number roots of $z^{100} + 3z^3 + 1$ inside the unit circle is the same as the number of roots of $3z^3$ inside the unit circle. Since $3z^3$ has a triple root at $z = 0$, $z^{100} + 3z^3 + 1$ has three roots inside the unit circle.

Example 23.2. Consider a polynomial of degree n with leading coefficient equal to 1,

$$h(z) = z^n + c_1 z^{n-1} + c_2 z^{n-2} + \cdots + c_n.$$

Show that there is some point z on the unit circle such that $|h(z)| \geq 1$.

Suppose on the contrary that $|h(z)| < 1$ for all z on the unit circle. Apply the Rouche theorem with $f(z) = z^n$ and $g(z) = -h(z)$. We can check that the condition $|f(z)| > |g(z)|$ is satisfied on the unit circle, i.e.,

$$|f(z)| = |z^n| = 1 > |-h(z)| = |g(z)| \quad \text{for all } z \text{ with } |z| = 1.$$

By Rouche theorem, the function f and $f + g$ have the same number of zeros inside the unit circle. On one hand, $f(z) = z^n$ has exactly n zeros, namely, n repeated roots at $z = 0$, inside the unit circle. However

$$f(z) + g(z) = c_1 z^{n-1} + c_2 z^{n-2} + \cdots + c_n$$

has at most $n - 1$ zeros. The polynomial $f + g$ cannot have n zeros inside the unit circle. This contradiction shows that there must be some point z with $|z| = 1$ such that $|h(z)| \geq 1$.