MAT3253 Homework 2

Due date: 29 Jan.

Question 1. (Bak & Newman Ex.1.15) Describe the sets whose points satisfy the following relations. Which of the sets are regions?

a. $|z - i| \leq 1$.

b. $\left|\frac{z-1}{z+1}\right| = 1$

c. $|z - 2| > |z - 3|$.

d. $|z| < 1$ and $\text{Im}(z) > 0$.

e. $\frac{1}{z} = \bar{z}$.

f. $|z|^2 = \text{Im}z$.

g. $|z^2 - 1| < 1$. (Hint: use polar coordinates.)

Question 2. (Bak & Newman Ex.1.17) Let $Arg(w)$ denote that value of the argument between $-\pi$ and π (inclusive). Show that

$$Arg\Big(\frac{z-1}{z+1}\Big) = \begin{cases} \pi/2 & \text{if } \text{Im}(z) > 0 \\ -\pi/2 & \text{if } \text{Im}(z) < 0 \end{cases}$$

where z is a point on the unit circle $|z| = 1$.

Question 3. (Brown & Churchill) by recalling the corresponding result for series of real numbers, show that

$$\text{if } \sum_{n=1}^{\infty} z_n = S \text{ and } \sum_{n=1}^{\infty} w_n = T, \quad \text{then } \sum_{n=1}^{\infty}(z_n + w_n) = S + T.$$

Question 4. (a) Show that

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z}$$

for any complex numbers z and positive integer n.

(b) For any complex number with modulus strictly less than 1, prove that

$$\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}.$$

(c) Use part (b) to derive the following identity for $0 \leq r < 1$ and $\theta \in \mathbb{R}$

$$\sum_{k=0}^{\infty} r^k \cos(k\theta) = \frac{1 - r\cos\theta}{1 + r^2 - 2r\cos\theta}.$$