MAT3253 Homework 7

Due date: 19 Mar.

Question 1. (Bak&Newman Ex.2.14) Find the radius of convergence of

(a) $\sum_{n=0}^{\infty} \frac{(-1)^n z^n}{n!}$; (b) $\sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$;

(c) $\sum_{n=0}^{\infty} \frac{n! z^n}{n^n}$; (d) $\sum_{n=0}^{\infty} \frac{2^n z^n}{n!}$.

Question 2. (Bak&Newman Ex.2.23) Find the domain of convergence of

(a) $\sum_{n=0}^{\infty} n(z-1)^n$; (b) $\sum_{n=0}^{\infty} \frac{(-1)^n (z+1)^n}{n!}$;

(c) $\sum_{n=0}^{\infty} n^2(2z-1)^n$.

Question 3. Let n be a positive integer. Define the Bessel function (of the first kind) of order n by

$$J_n(z) \triangleq \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!} \left(\frac{z}{2}\right)^{n+2k}.$$

Show that $J_n(z)$ is defined for all complex number z and it satisfies the differential equation

$$z^2 \frac{d^2 w}{dz^2} + z \frac{dw}{dz} + (z^2 - n^2) w = 0.$$

Question 4. Consider the following recurrence relation:

$$x_{n+1} = 2x_n + 3^n$$

for $n \geq 0$, with initial condition $x_0 = 0$.

(a) Compute the first 5 numbers in the sequence $(x_n)_{n=0}^{\infty}$.

(b) Show that the generating function

$$g(z) \triangleq \sum_{n=0}^{\infty} x_n z^n$$

has positive radius of convergence. (Show that $g(z)$ converges for some nonzero z. For example, you can show that $x_n \leq 5^{n-1}$ by induction.)

(c) Derive an expression for x_n, for $n \geq 0$.

Question 5. (Brown&Churchill Ex.39.5) Suppose that a function $f(z)$ is analytic at a point $z_0 = z(t_0)$ lying on a smooth arc $z = z(t)$, $(a \leq t \leq b)$. Show that if $w(t) = f(z(t))$, then

$$w'(t) = f'(z(t))z'(t)$$

when $t = t_0$.

Hint: Write $f(z) = u(x, y) + iv(x, y)$ and $z(t) = x(t) + iy(t)$, so that

$$w(t) = u(x(t), y(t)) + iv(x(t), y(t)).$$

Then apply the chain rule in calculus for functions of two real variables to write

$$w'(t) = (u_x x' + u_y y') + i(v_x x' + v_y y'),$$

and use Cauchy-Riemann equations.

Question 6. (Brown&Churchill Ex.39.6) Let $y(x)$ be a real-valued function defined on the interval $0 \leq x \leq 1$ by means of the equations

$$y(x) = \begin{cases} x^3 \sin(\pi/x) & \text{when } 0 < x \leq 1, \\ 0 & \text{when } x = 0. \end{cases}$$

(a) Show that the equation

$$z(x) = x + iy(x) \qquad (0 \leq x \leq 1)$$

represents an arc C that intersects the real axis at the points $z = 1/n$, $(n = 1, 2, \ldots)$ and z =0.

(b) Verify that the arc C in part (a) is, in fact, a smooth arc.

Hint: To establish the continuity of $y(x)$ at $x = 0$, observe that

$$0 \leq \left| x^3 \sin(\pi/x) \right| \leq x^3$$

when $x > 0$. A similar remark applies in finding $y'(0)$ and showing that $y'(x)$ is continuous at $x = 0$.

Question 7. (Brown&Churchill Ex.42.2,4) For the functions f and contour C below, use parametric representations for C to evaluate

$$\int_C f(z)\, dz.$$

(a) $f(z) = z - 1$ and C is the arc from $z = 0$ to $z = 2$ consisting of the semicircle $z = 1 + e^{i\theta}$, $(\pi \leq \theta \leq 2\pi)$.

(b) $f(z) = z - 1$ and C is the arc from $z = 0$ to $z = 2$ consisting of the segment $z = x$ $(0 \leq x \leq 2)$ of the real axis.

(c) $f(z)$ is defined by means of the equations

$$f(z) = \begin{cases} 1 & \text{when } y < 0, \\ 4y & \text{when } y > 0, \end{cases}$$

and C is the arc from $z = -1 - i$ to $z = 1 + i$ along the curve $y = x^3$.