MAT2006: Elementary Real Analysis
Mid-term Test

Two hours, closed book.

Question 1. [20 marks] State the following theorems (proofs are not required).

(a) The Least Upper Bound Property;

(b) The Archimedean Property;

(c) The Nested Interval Property;

(d) The Monotone Convergence Theorem;

(e) The Bolzano–Weierstrass Theorem;

(f) The Cauchy Criterion for sequences;

(h) The Heine–Borel Theorem.

Question 2. [15 marks]

(i) Write down the sup, inf, max and min for the sets

$$A = (0, 1]; \qquad B = \left\{ \frac{1}{n} \, : \, n \in \mathbb{N} \right\}.$$

(ii) For the sequence $x_n = (-1)^n$. Write down

$$\limsup_{n\to\infty} x_n \qquad \text{and} \qquad \liminf_{n\to\infty} x_n.$$

(iii) Assume $\{x_n\}$ and $\{y_n\}$ are two bounded sequences. Show that

$$\limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n \geq \limsup_{n\to\infty} (x_n + y_n).$$

Question 3.[10 marks] Using the Heine–Borel theorem to prove that any bounded infinite set must have a limit point.

Question 4.[15 marks] Suppose the series $\sum_{n=1}^{\infty} a_n$ converges.

(i) Assume $a_n \geq 0$ for each $n \in \mathbb{N}$. Show that $\sum_{n=1}^{\infty} a_n^2$ also converges.

(ii) If we don't assume $a_n \geq 0$, does $\sum_{n=1}^{\infty} a_n^2$ still converge? If so, provide a proof. If not, give an example.

(iii) Assume $a_n \geq 0$ and $a_{n+1} \leq a_n$ for each $n \in \mathbb{N}$. Show that $\lim_{n \to \infty} n a_n = 0$.

Question 5.[20 marks]

Consider the following seven sets.

$$\emptyset; \qquad \mathbb{R}; \qquad \mathbb{Q}; \qquad \mathbb{I}; \qquad [0,1]; \qquad (0,1]; \qquad C \quad \text{(the Cantor set)}.$$

(i) Among the above sets, point out the finite, the countable, and the uncountable sets.

(ii) Among the above sets, point out the open, the closed, and the compact sets.

(iii) Show that any bounded open interval is F_σ.

(iv) Using the Baire Category Theorem show that $\mathbb{I}$ is not F_σ.

(v) Using part (iv), provide an example of "the countable intersection of F_σ sets is not F_σ."

Question 6. [20 marks]

(i) Let A' denote the derived set of A, that is the set of all limit points of A. Show that $(A')' \subset A'$, that is A' is closed.

(ii) Let $\{x_n\}$ be a bounded sequence and we may regard it as a set of real numbers. Let $E := A'$ be the set of limits points of A. Show that $s = \sup E$ exists and that s is a limit point of E.

(iii) We have shown that $\limsup_{n\to\infty} x_n = \sup E$. Prove that $\max E$ exists and that $\limsup_{n\to\infty} x_n = \max E$.

(iv) For a set B, denote by $-B = \{-x \mid x \in B\}$. Show that $-\inf B = \sup(-B)$ and that $-\min B = \max(-B)$. Use this and part (iii) to show that $\liminf_{n\to\infty} x_n = \min E$.

(v) We have shown that $\{x_n\}$ converges if and only if $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n$. Using this to show that: if every convergent subsequence of $\{x_n\}$ converge to the same limit, then $\{x_n\}$ converges.

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