Chapter 7. Sequences and Series of Functions *

1 Discussion: The Power of Power Series

In 1689, Jacob Bernoulli published his *Tractatus de Seriebus Infinitis* summarizing what was known about infinite series toward the end of the 17th century. Full of clever calculations and conclusions, this publication was also notable for one particular question that it didn't answer; namely, what is the precise value of the series (a.k.a the Basel Problem)

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots .$$

Geometric series are the most prominent class of examples that can be readily summed.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots \tag{1.1}$$

for all $|x| < 1$. Geometric series were part of mathematical folklore long before Bernoulli; however, what was relatively novel in Bernoulli's time was the idea of operating on infinite series such as (1.1) with tools from the budding theory of calculus. For instance, what happens if we take the derivative on each side of equation (1.1)? The left side is $1/(1-x)^2$. But what about the right side? Adopting a 17th century mindset, a natural way to proceed is to treat the infinite series as a polynomial, albeit of infinite degree. Differentiation across equation (1.1) in this fashion gives

$$\frac{1}{(1-x)^2} = 0 + 1 + 2x + 3x^2 + 4x^3 + \cdots . \tag{1.2}$$

Is this valid for $x \in (-1, 1)$. (We shall see the answer is affirmative later.)

*Lecture notes for CUHKSZ course MAT2006: Elementary Real Analysis.

Manipulations of this sort can be used to create a wide assortment of new series representations for familiar functions. Substituting $-x^2$ for x in (1.1) gives

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots \tag{1.3}$$

for all $x \in (-1, 1)$.

Once again closing our eyes to the potential danger of treating an infinite series as though it were a polynomial, let's see what happens when we take antiderivatives. Using the fact that

$$(\arctan(x))' = \frac{1}{1+x^2}, \qquad \arctan(0) = 0.$$

Now, integrating (1.3) gives

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots . \tag{1.4}$$

Plugging $x = 1$ into equation (1.4) yields the striking relationship

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots . \tag{1.5}$$

The constant π, which arises from the geometry of circles, has somehow found its way into an equation involving the reciprocals of the odd integers. Is this a valid formula? Can we really treat the infinite series in (1.3) like a finite polynomial? Even if the answer is yes there is still another mystery to solve in this example. Plugging $x = 1$ into equations (1.1), (1.2), or (1.3) yields mathematical gibberish, so is it prudent to anticipate something meaningful arising from equation (1.4) at this same value?

Let us return to the Bernoulli's plea for help, the Basel Problem – it was answered by Leonard Euler. In 1735, Euler announced

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6}.$$

Recall the power series of $\sin x$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots ,$$

Factoring out x and dividing yields a power series with leading coefficient equal to 1:

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots ,$$

and regarding this as "a polynomial of degree of infinity". Notice that this "polynomial" has a constant term 1 and its zeros are $\pm\pi, \pm 2\pi, \pm 3\pi, \ldots$. Euler's intuition is to factorize the "polynomial" as usual ones:

$$\begin{aligned}
&1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots \\
=&\left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right)\left(1 - \frac{x}{2\pi}\right)\left(1 + \frac{x}{2\pi}\right)\left(1 - \frac{x}{3\pi}\right)\left(1 + \frac{x}{3\pi}\right)\cdots \\
=&\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)\cdots \\
=&1 - \left(1 + \frac{1}{4} + \frac{1}{9} + \cdots\right)\frac{x^2}{\pi^2} + \left(\frac{1}{4} + \frac{1}{9} + \cdots\right)\frac{x^4}{\pi^4} + \cdots .
\end{aligned}$$

Comparing the coefficients of x^2, we have

$$\frac{1}{3!} = \frac{1}{\pi^2}\left(1 + \frac{1}{4} + \frac{1}{9} + \cdots\right),$$

which solves the Basel Problem.

Numerical approximations of each side of this equation confirmed for Euler that, despite the audacious leaps in his argument, he had landed on solid ground. By our standards, this derivation falls well short of being a proper proof, and we will have to tend to this in the upcoming chapters. The takeaway of this discussion is that the hard work ahead is worth the effort. Infinite series representations of functions are both useful and surprisingly elegant, and can lead to remarkable conclusions when they are properly handled.

The evidence so far suggests power series are quite robust when treated as if they were finite in nature. Term-by-term differentiation produced a valid conclusion in equation (1.2), and taking antiderivatives fared similarly well in (1.4). We will see that these manipulations are not always justified for infinite series of more general types of functions. What is it about power series in particular that makes them so impervious to the dangers of the infinite? Of the many unanswered questions in this discussion, this last one is probably the most central, and the most important to understanding series of functions in general.

2 Uniform Convergence of a Sequence of Functions

Because convergence of infinite series is defined in terms of the associated sequence of partial sums, the results from our study of sequences will be immediately applicable to the questions we have raised about both power series and more general infinite series of functions.

2.1 Pointwise Convergence

Definition 1. For each $n \in \mathbb{N}$, let f_n be a function defined on a set $A \subset \mathbb{R}$. The sequence $\{f_n\}$ of functions converges pointwise on A to a function f if, for all $x \in A$, the sequence of real numbers $\{f_n(x)\}$ converges to $f(x)$.

In this case, we write $f_n \to f$, or $\lim_{n\to\infty} f_n(x) = f(x)$. This latter expression is helpful if there is any confusion as to whether x or n is the limiting variable.

Example 2.1. (i) Consider $f_n(x) = (x^2 + nx)/n$ on all of $\mathbb{R}$. We have

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x^2 + nx}{n} = \lim_{n\to\infty} \frac{x^2}{n} + x = x.$$

Thus, $\{f_n\}$ converges pointwise to $f(x) = x$ on $\mathbb{R}$.

(ii) Let $g_n(x) = x^n$ on the set $[0, 1]$. We have

$$\lim_{n\to\infty} g_n(x) = g(x) = \begin{cases} 0 & \text{for } 0 \leq x < 1 \\ 1 & \text{for } x = 1. \end{cases}$$

(iii) Consider $h_n(x) = x^{1+\frac{1}{2n-1}}$ on the set $[-1, 1]$. For a fixed $x \in [-1, 1]$ we have

$$\lim_{n\to\infty} h_n(x) = x \lim_{n\to\infty} x^{\frac{1}{2n-1}} = |x|.$$

Parts (ii) and (iii) of the above Example are our first indication that there is some difficult work ahead of us. The central theme of this chapter is analyzing which properties the limit function inherits from the approximating sequence. In part (iii) we have a sequence of differentiable functions converging pointwise to a limit that is not differentiable at the origin. In part (ii), we see an even more fundamental problem of a sequence of continuous functions converging to a limit that is not continuous.

2.2 Uniform Convergence

Definition 2 (Uniform Convergence)**.** Let $\{f_n\}$ be a sequence of functions defined on a set $A \subset \mathbb{R}$. Then, $\{f_n\}$ *converges uniformly* on A to a limit function f defined on A if, for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \epsilon$ whenever $n \geq N$ and $x \in A$.

The use of the adverb *uniformly* here should be reminiscent of its use in the phrase "uniformly continuous" from Chapter 5. In both cases, the term "uniformly" is employed to express the fact that the response (δ or N) to a prescribed ϵ can be chosen to work simultaneously for all values of x in the relevant domain.

Example 2.2. (i) Let

$$g_n(x) = \frac{1}{n(1+x^2)}.$$

For any fixed $x \in \mathbb{R}$, we can see that $\lim_{n\to\infty} g_n(x) = 0$ so that $g(x) = 0$ is the pointwise limit of the sequence $\{g_n\}$ on $\mathbb{R}$. Notice that

$$|g_n(x) - g(x)| = \left|\frac{1}{n(1+x^2)}\right| \leq \frac{1}{n}.$$

Thus, given $\epsilon > 0$, we can choose $N > 1/epsilon$ (which does not depend on x), and it follows that

$$|g_n(x) - g(x)| < \epsilon \qquad \forall n \geq N, \quad \forall x \in \mathbb{R}.$$

By the definition, $g_n \to g$ uniformly on $\mathbb{R}$.

(ii) Recall that $f_n(x) = (x^2 + nx)/n$ converges pointwise on $\mathbb{R}$ to $f(x) = x$. Now,

$$|f_n(x) - f(x)| = \frac{x^2}{n},$$

and notice that in order to force $|f_n(x) - f(x)| < \epsilon$, we are going to have to choose

$$N > \frac{x^2}{\epsilon}.$$

Although this is possible to do for each $x \in \mathbb{R}$, there is no way to choose a single value of N that will work for all values of x at the same time.

On the other hand, we can show that $f_n \to f$ uniformly on the set $[-b, b]$. By restricting our attention to a bounded interval, we may now assert that

$$\frac{x^2}{n} < \frac{b^2}{n}.$$

Given $\epsilon > 0$, then, we can choose

$$N > \frac{b^2}{\epsilon}$$

independently of $x \in [-b, b]$.

2.3 Cauchy Criterion

Recall that the Cauchy Criterion for convergent sequences of real numbers was an equivalent characterization of convergence which, unlike the definition, did not make explicit mention of the limit. The usefulness of the Cauchy Criterion suggests the need for an analogous characterization of uniformly convergent sequences of functions. As with all statements about uniformity, pay special attention to the relationship between the response variable ($N \in \mathbb{N}$) and the domain variable ($x \in A$).

Theorem 1 (Cauchy Criterion for Uniform Convergence)**.** *A sequence of functions* $\{f_n\}$ *defined on a set* $A \subset \mathbb{R}$ *converges uniformly on* A *if and only if for every* $\epsilon > 0$ *there exists an* $N \in \mathbb{N}$ *such that* $|f_n(x) - f_m(x)| < \epsilon$ *whenever* $m, n \geq N$ *and* $x \in A$*.*

Proof. Exercise. □

2.4 Continuity of the Limit Function

The stronger assumption of uniform convergence is precisely what is required to remove the flaws from our attempted proof that the limit of continuous functions is continuous.

Theorem 2 (Continuous Limit Theorem)**.** *Let* $\{f_n\}$ *be a sequence of functions defined on* $A \subset \mathbb{R}$ *that converges uniformly on* A *to a function* f*. If each* f_n *is continuous at* $c \in A$*, then* f *is continuous at* c*.*

Proof. Fix $c \in A$ and let $\epsilon > 0$. Choose N so that

$$|f_N(x) - f(x)| < \frac{\epsilon}{3}$$

for all $x \in A$. Because f_N is continuous, there exists a $\delta > 0$ for which

$$|f_N(x) - f_N(c)| < \frac{\epsilon}{3}, \qquad \forall |x - c| < \delta,$$

which implies that

$$\begin{aligned} |f(x) - f(c)| =& |f(x) - f_N(x) + f_N(x) - f_N(c) + f_N(c) - f(c)| \\ \leq& |f(x) - f_N(x)| + |f_N(x) - f_N(c)| + |f_N(c) - f(c)| \\ < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

Thus, f is continuous at $c \in A$. □

Exercises

Exercise 1. Let
$$f_n(x) = \frac{nx}{1+nx^2}.$$

(a) Find the pointwise limit of $\{f_n\}$ for all $x \in (0,\infty)$.
(b) Is the convergence uniform on $(0,\infty)$?
(c) Is the convergence uniform on $(0,1)$?
(d) Is the convergence uniform on $(1,\infty)$?

Exercise 2. (i) Define a sequence of functions on $\mathbb{R}$ by

$$f_n(x) = \begin{cases} 1 & \text{if } x = 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$

and let f be the pointwise limit of f_n.

Is each f_n continuous at zero? Does $f_n \to f$ uniformly on $\mathbb{R}$? Is f continuous at zero?

(ii) Repeat this exercise using the sequence of functions

$$g_n(x) = xf_n(x) = \begin{cases} x & \text{if } x = 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n} \\ 0 & \text{otherwise.} \end{cases}$$

(iii) Repeat the exercise once more with the sequence

$$h_n(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n} \\ x & \text{if } x = 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n-1} \\ 0 & \text{otherwise.} \end{cases}$$

In each case, explain how the results are consistent with the content of the Continuous Limit Theorem.

Exercise 3. For each $n \in \mathbb{N}$ and $x \in [0,\infty)$, let

$$g_n(x) = \frac{x}{1+x^n}, \qquad h_n(x) = \begin{cases} 1 & \text{if } x \geq 1/n \\ nx & \text{if } 0 \leq x < 1/n. \end{cases}$$

Answer the following questions for the sequences $\{g_n\}$ and $\{h_n\}$:

(a) Find the pointwise limit on $[0,\infty)$.
(b) Explain how we know that the convergence cannot be uniform on $[0,\infty)$.
(c) Choose a smaller set over which the convergence is uniform and supply an argument to show that this is indeed the case.

Exercise 4. Assume $f_n \to f$ on a set A. The Continuous Limit Theorem (Theorem 2) is an example of a typical type of question which asks whether a trait possessed by each f_n is inherited by the limit function. Provide an example to show that all of the following propositions are false if the convergence is only assumed to be pointwise on A. Then go back and decide which are true under the stronger hypothesis of uniform convergence.

(a) If each f_n is uniformly continuous, then f is uniformly continuous.

(b) If each f_n is bounded, then f is bounded.

(c) If each f_n has a finite number of discontinuities, then f has a finite number of discontinuities.

(d) If each f_n has fewer than M discontinuities (where $M \in \mathbb{N}$ is fixed), then f has fewer than M discontinuities.

(e) If each f_n has at most a countable number of discontinuities, then f has at most a countable number of discontinuities.

Exercise 5. Let f be uniformly continuous on all of $\mathbb{R}$, and define a sequence of functions by $f_n(x) = f(x + \frac{1}{n})$. Show that $f_n \to f$ uniformly. Give an example to show that this proposition fails if f is only assumed to be continuous and not uniformly continuous on $\mathbb{R}$.

Exercise 6. Let $\{g_n\}$ be a sequence of continuous functions that converges uniformly to g on a compact set K. If $g(x) \neq 0$ on K, show $\{1/g_n\}$ converges uniformly on K to $1/g$.

Exercise 7. Assume $\{f_n\}$ and $\{g_n\}$ are uniformly convergent sequences of functions defined on A.

(a) Show that $\{f_n + g_n\}$ is a uniformly convergent sequence of functions.

(b) Give an example to show that the product $\{f_n g_n\}$ may not converge uniformly.

(c) Prove that if there exists an $M > 0$ such that $|f_n(x)| \leq M$ and $|g_n| \leq M$ for all $n \in \mathbb{N}$ and $x \in A$, then $\{f_n g_n\}$ does converge uniformly.

Exercise 8. Assume $f_n \to f$ pointwise on $[a, b]$ and the limit function f is continuous on $[a, b]$. If each f_n is increasing (but not necessarily continuous), show $f_n \to f$ uniformly.

Exercise 9 (Dini's Theorem)**.** Assume $f_n \to f$ pointwise on a compact set K and assume that for each $x \in K$ the sequence $f_n(x)$ is increasing. Follow these steps to show that if f_n and f are continuous on K, then the convergence is uniform.

(a) Set $g_n = f - f_n$ and translate the preceding hypothesis into statements about the sequence $\{g_n\}$.

(b) Let $\epsilon > 0$ be arbitrary, and define $K_n = \{x \in K \mid g_n(x) \geq \epsilon\}$. Argue that $K_1 \supset K_2 \supset K_3 \supset \cdots$, and use this observation to finish the argument.

Exercise 10 (Cantor Function)**.** Review the construction of the Cantor set $C \subset [0, 1]$.

(a) Define $f_0(x) = x$ for all $x \in [0,1] = C_0$. Now, let

$$f_1(x) = \begin{cases} (3/2)x & \text{for } 0 \le x \le 1/3 \\ 1/2 & \text{for } 1/3 < x < 2/3 \\ (3/2)x - 1/2 & \text{for } 2/3 \le x \le 1. \end{cases}$$

Sketch f_0 and f_1 over $[0,1]$ and observe that f_1 is continuous, increasing, and constant on the middle third $(1/3, 2/3) = [0,1] \setminus C_1$.

(b) Construct f_2 by imitating this process of flattening out the middle third of each nonconstant segment of f_1. Specifically, let

$$f_2(x) = \begin{cases} (1/2)f_1(3x) & \text{for } 0 \le x \le 1/3 \\ f_1(x) & \text{for } 1/3 < x < 2/3 \\ (1/2)f_1(3x-2) + 1/2 & \text{for } 2/3 \le x \le 1. \end{cases}$$

If we continue this process, show that the resulting sequence $\{f_n\}$ converges uniformly on $[0,1]$.

(c) Let $f = \lim_{n\to\infty} f_n$. Prove that f is a continuous, increasing function on $[0,1]$ with $f(0) = 0$ and $f(1) = 1$ that satisfies $f'(x) = 0$ for all x in the open set $[0,1] \setminus C$. Recall that the "length" of the Cantor set C is 0. Somehow, f manages to increase from 0 to 1 while remaining constant on a set of "length 1."

Remark. The Cantor function is the most used example of a uniformly continuous but not *absolutely continuous* function. The "absolutely continuous functions" are discussed in MAT3006, Real Analysis.

3 Arzela–Ascoli Theorem*

Recall that the Bolzano–Weierstrass Theorem states that every bounded sequence of real numbers has a convergent subsequence. An analogous statement for bounded sequences of functions is not true in general.

3.1 The set of continuous functions as a metric space.

Consider the set of continuous functions defined on a bounded closed interval $[a,b]$ and denoted it by $C[a,b]$. That is

$$C[a,b] = \{f \mid f \text{ is continuous on } [a,b]\}.$$

For each of $f \in C[a,b]$, by the Extreme Value Theorem, we may define the *norm* of f by

$$\|f\| = \max_{a \le x \le b} |f(x)|.$$

Norms are generalizations of absolute value of real numbers or length of n-vectors. They have similar properties.

(i) (positive definite) $\|f\| \ge 0$ for all $f \in C[a,b]$ and $\|f\| = 0$ if and only if $f \equiv 0$.

(ii) (triangle inequality) $\|f+g\| \le \|f\| + \|g\|$

(iii) (linear) $\|kf\| = |k|\,\|f\|$ for all $k \in \mathbb{R}$ and $f \in C[a,b]$.

A space with norm naturally induces a *metric*, (or a *distance function*), by

$$d(f,g) = \|f-g\|.$$

(Just like for two real numbers x and y, we define their distance as $d(x,y) = |x-y|$). To show this is indeed a distance function, we need to verify, for each $f, g, h \in C[a,b]$, that

(i) (positive definite) $d(f,g) \ge 0$ and $d(f,g) = 0$ if and only if $f = g$.

(ii) (symmetry) $d(f,g) = d(g,f)$.

(iii) (triangle inequality) $d(f,g) \le d(f,h) + d(h,g)$.

A set with metric is called a *metric space.*

For any metric space, we may define the convergence of sequences. Let $\{f_n\}$ be a sequence in $C[a,b]$, we say $\{f_n\}$ converges to a limit f if for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$d(f_n, f) < \epsilon \qquad \forall n \ge N.$$

Exercise 11. (Completeness of $C[a,b]$).

(i) Show that the above definition of a sequence is convergent in $C[a,b]$ is indeed the uniform convergence of series of functions. Hence, if $\{f_n\} \subset C[a,b]$ converges, it must converge to a $f \in C[a,b]$.

(ii) A sequence $\{f_n\} \subset C[a,b]$ is called a *Cauchy sequence* if for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $d(f_n, f_m) < \epsilon$ whenever $n > m \ge N$.

Show that $C[a,b]$ with the above given metric is *complete*, namely the following Cauchy Criterion holds: Any convergent sequence of $C[a,b]$ is a Cauchy sequence.

As a metric space, we may define the boundedness for a subset of $C[a, b]$.

Definition 3. A set $\mathcal{F} \subset C[a, b]$ is called *bounded* or, more precisely, *uniformly bounded*, if there exists $M > 0$ such that $\|f\| < M$ for each $f \in \mathcal{F}$.

Remark. Note that, $\|f\| < M$ for each $f \in \mathcal{F}$ is equivalent to

$$|f(x)| \leq M \qquad \forall f \in \mathcal{F}, \quad \forall x \in [a, b].$$

Thus M is a bound for each $f_n(x)$, this bound doesnot depend on n – this is why the term "uniformly bounded."

Exercise 12. (i) Provide an example of a sequence $\{f_n\} \subset C[a, b]$, which is bounded for each $f_n(x)$ but $\{f_n\}$ not uniformly bounded.

(ii) A sequence $\{f_n\} \subset C[a, b]$ is called *pointwise bounded* if for each $x \in [a, b]$, $\{f_n(x)\}$ is a bounded sequence of $\mathbb{R}$. Provide an example of pointwise bounded but not uniformly bounded sequence of $C[a, b]$.

Exercise 13. Note that $f_n(x) = \sin(nx)$ is a uniformly bounded sequence of $C[0, 2\pi]$. Show that $\{f_n\}$ doesnot have any convergent subsequence in $C[0, 2\pi]$.

The above exercise shows that Bolzano–Weierstrass Theorem doesnot apply to the metric space $C[a, b]$ directly. We may impose a further condition to grantee the existence of a convergent subsequence in $C[a, b]$. The extra condition we shall use is the following one.

Definition 4 (Equicontinuity)**.** A sequence of functions $\{f_n\}$ defined on a set $E \subset \mathbb{R}$ is called *equicontinuous* if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|f_n(x) - f_n(y)| < \epsilon$ for all $n \in \mathbb{N}$ and $|x - y| < \delta$ in E.

Exercise 14. (i) What is the difference between saying that a sequence of functions $\{f_n\}$ is equicontinuous and just asserting that each f_n in the sequence is individually uniformly continuous?

(ii) Give a qualitative explanation for why the sequence $g_n(x) = x^n$ is not equicontinuous on $[0, 1]$. Is each g_n uniformly continuous on $[0, 1]$?

3.2 Arzela–Ascoli Theorem

Theorem 3. *If $\{f_n\} \subset C[a, b]$ is a sequence of functions which is uniformly bounded and equicontinuous, then $\{f_n\}$ contains a convergent subsequence in $C[a, b]$.*

Remark. We emphasize again that a convergent sequence of $C[a,b]$ means that $\{f_n\}$ uniformly converges to some $f \in C[a,b]$.

Proof. **Step 1.** Constructing a subsequence that converges at rational numbers.

Exercise 15. Let f_n be a (uniformly) bounded sequence of $C[a,b]$. Write $A = \mathbb{Q} \cap [a,b] = \{x_1, x_2, x_3, \dots\}$.

(i) Take $x = x_1$. Show that $\{f_n(x_1)\}$ contains a convergent subsequence $\{f_{n_k}(x_1)\}$. To indicate that the subsequence of functions $\{f_{n_k}\}$ is generated by considering the values of the functions at x_1, we will use the notation $f_{n_k} = f_{1,k}$.

(ii) Take $x = x_2$. Show that $\{f_{1,n}(x_2)\}$ contains a convergent subsequence $\{f_{1,n_k}(x_2)\}$. As previously, we denote $f_{1,n_k} = f_{2,k}$.

(iii) Carefully construct a nested family of subsequences $f_{m,k}$. Show that $\{f_{k,k}\}$ is a subsequence of $\{f_n\}$ that converges at every point of A.

Step 2.

Exercise 16. To simplify notations, let us denote $f_{k,k}$ constructed in the previous step by g_k. Let $\epsilon > 0$. By equicontinuity, there exists a $\delta > 0$ such that

$$|g_k(x) - g_k(y)| < \frac{\epsilon}{3}, \qquad \forall k \in \mathbb{N}, \quad \forall |x - y| < \delta.$$

Here, $x, y \in [a,b]$.

(i) Let δ to be the above fixed one. Show that there exists a finite collection of rational numbers $\{r_1, r_2, \dots, r_m\}$ such that $\bigcup_{i=1}^m V_\delta(r_i) \supset [a,b]$.

(ii) Explain why there must exist an $N \in \mathbb{N}$ such that

$$|g_s(r_i) - g_t(r_i)| < \frac{\epsilon}{3}$$

or all $s, t \geq N$ and r_i in the finite subset of $[a,b]$ just described. Why does having the set $\{r_1, r_2, \dots, r_m\}$ be finite matter?

Step 3. Playing with the Cauchy Criterion.

Now, for any $x \in [a,b]$, $x \in V_\delta(r_i)$ for some $i = 1, 2, \dots, m$. Therefore, whenever $s, t \geq N$, one has

$$\begin{aligned}
|g_s(x) - g_t(x)| =& |g_s(x) - g_s(r_i) + g_s(r_i) - g_t(r_i) + g_t(r_i) - g_t(x)| \\
\leq& |g_s(x) - g_s(r_i)| + |g_s(r_i) - g_t(r_i)| + |g_t(r_i) - g_t(x)| \\
<& \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}$$

This means that $\{g_k\}$ is a Cauchy sequence in $C[a,b]$, thus it converges uniformly to a limit $g \in C[a,b]$. $\square$

4 Uniform Convergence and Differentiation

Example 4.1. (i) Recall the example of sequence of functions

$$h_n(x) = x^{1+\frac{1}{2n-1}}.$$

We have $h_n \to h(x) = |x|$ uniformly for $x \in [-1, 1]$.

(ii) Let

$$f_n(x) = \frac{\sin nx}{\sqrt{n}}.$$

Then, $f_n(x) \to f = 0$ uniformly for $x \in \mathbb{R}$.

The above example (i) imposes some significant restrictions on what we might hope to be true regarding differentiation and uniform convergence. If $h_n \to h$ uniformly and each h_n is differentiable, we should not anticipate that $h_n' \to h'$ because in this example $h'(x)$ does not even exist at $x = 0$. In the above example (ii), $f_n \to f$ uniformly with $\{f_n\}$ and f all differentiable, but the sequence f_n' diverges at every point of the domain.

The key assumption necessary to be able to prove any facts about the derivative of the limit function is that the sequence of derivatives be *uniformly convergent*. This may sound as though we are assuming what it is we would like to prove, and there is some validity to this complaint. The more hypotheses a proposition has, the more difficult it is to apply. The content of the next theorem is that if we are given a pointwise convergent sequence of differentiable functions, and if we know that the sequence of derivatives converges uniformly to *something*, then the limit of the derivatives is indeed the derivative of the limit.

Theorem 4 (Differentiable Limit Theorem)**.** *Let $f_n \to f$ pointwise on the closed interval $[a, b]$, and assume that each f_n is differentiable. If $\{f_n'\}$ converges uniformly on $[a, b]$ to a function g, then the function f is differentiable and $f' = g$.*

Proof. Fix $c \in [a, b]$ and let $\epsilon > 0$. We want to argue that $f'(c)$ exists and equals $g(c)$. That is to show that there exists $\delta > 0$ such that

$$\left| \frac{f(x) - f(c)}{x - c} - g(c) \right| < \epsilon, \qquad \forall 0 < |x - c| < \delta.$$

To this end, we shall use the triangle inequality to get

$$\left| \frac{f(x) - f(c)}{x - c} - g(c) \right| \le \left| \frac{f(x) - f(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c} \right| + \left| \frac{f_n(x) - f_n(c)}{x - c} - f_n'(c) \right| + |f_n'(c) - g(c)|$$

Our intent is to first find an f_n that forces the first and third terms on the right-hand side to be less than $\epsilon/3$. Once we establish which f_n we want, we can then use the differentiability of f_n to produce a δ that makes the middle term less than $\epsilon/3$ for all x satisfying $0 < |x-c| < \delta$.

Let's start by choosing an N_1 such that

$$|f'_m(c) - g(c)| < \frac{\epsilon}{3} \qquad \forall m \geq N_1. \tag{4.1}$$

We now invoke the uniform convergence of $f'_n(x)$ to assert that there exists an N_2 such that $m, n \geq N_2$ implies

$$|f'_m(x) - f'_n(x)| < \frac{\epsilon}{3} \qquad \forall x \in [a, b] \tag{4.2}$$

Set $N = \max\{N_1, N_2\}$.

The function f_N is differentiable at c, and so there exists a $\delta > 0$ for which

$$\left| \frac{f_N(x) - f_N(c)}{x - c} - f'_N(c) \right| < \frac{\epsilon}{3} \qquad \forall 0 < |x - c| < \delta. \tag{4.3}$$

Fix an x satisfying $0 < |x - c| < \delta$, let $m \geq M$, and apply the Mean Value Theorem to $f_m - f_N$ on the interval $[c, x]$, (If $x < c$ the argument is the same.) By MVT, there exists an $\alpha \in (c, x)$ such that

$$\frac{[f_m(x) - f_N(x)] - [f_m(c) - f_N(c)]}{x - c} = f'_m(\alpha) - f'_N(\alpha),$$

which, togeter with (4.2), imply that

$$\left| \frac{f_m(x) - f_m(c)}{x - c} - \frac{f_N(x) - f_N(c)}{x - c} \right| < \frac{\epsilon}{3}.$$

Because $f_m \to f$ we can take the limit as $m \to \infty$, and the Order Limit Theorem asserts that

$$\left| \frac{f(x) - f(c)}{x - c} - \frac{f_N(x) - f_N(c)}{x - c} \right| \leq \frac{\epsilon}{3}. \tag{4.4}$$

Finally, the inequalities in (4.1), (4.2), and (4.4), together imply that for x satisfying $0 < |x - c| < \delta$,

$$\begin{aligned} \left| \frac{f(x) - f(c)}{x - c} - g(c) \right| &\leq \left| \frac{f(x) - f(c)}{x - c} - \frac{f_N(x) - f_N(c)}{x - c} \right| \\ &\quad + \left| \frac{f_N(x) - f_N(c)}{x - c} - f'_N(c) \right| + |f'_N(c) - g(c)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \quad \square \end{aligned}$$

The hypothesis in the Differentiable Limit Theorem is unnecessarily strong. We actually do not need to assume that $f_n(x) \to f(x)$ at each point in the domain because the assumption that the sequence of derivatives $\{f_n'\}$ converges uniformly is nearly strong enough to prove that $\{f_n\}$ converges, uniformly in fact. Two functions with the same derivative may differ by a constant, so we must assume that there is at least one point x_n where $f_n(x_0) \to f(x_0)$.

Theorem 5. *Let $\{f_n\}$ be a sequence of differentiable functions defined on the closed interval $[a,b]$, and assume $\{f_n'\}$ converges uniformly on $[a,b]$. If there exists a point $x_0 \in [a,b]$ where $f_n(x_0)$ is convergent, then $\{f_n\}$ converges uniformly on $[a,b]$.*

Sketch of proof. Given $x \in [a,b]$, by the triangle inequality, we have

$$|f_n(x) - f_m(x)| \leq |[f_n(x) - f_n(x_0)] - [f_m(x) - f_m(x_0)]| + |f_n(x_0) - f_m(x_0)|.$$

Given any $\epsilon > 0$, the convergence of $\{f_n(x_0)\}$ grantees that there exists $N_1 > 0$ such that

$$|f_n(x_0) - f_m(x_0)| < \frac{\epsilon}{2} \qquad \forall n > m \geq N_1.$$

Moreover, the MVT shows that

$$\begin{aligned} |[f_n(x) - f_n(x_0)] - [f_m(x) - f_m(x_0)]| =& |[f_n(x) - f_m(x)] - [f_n(x_0) - f_m(x_0)]| \\ =& |f_n'(\xi) - f_m'(\xi)| \cdot |x - x_0| \\ \leq& |f_n'(\xi) - f_m'(\xi)|(b-a), \end{aligned}$$

where $\xi \in [x_0, x]$ (similar if $x < x_0$) and may depend on n, m. Since $\{f_n'\} \to f'$ uniformly, there must exist $N_2 > 0$ such that

$$|f_n'(x) - f_m'(x)| < \frac{\epsilon}{2(b-a)} \qquad \forall n > m \geq N_2.$$

Set $N = \max\{N_1, N_2\}$. Then

$$|f_n(x) - f_m(x)| \leq \epsilon \qquad \forall n > m \geq N \quad \forall x \in [a,b].$$

which shows $\{f_n\}$ converges uniformly. $\square$

Combining the last two results produces a stronger version of the Differentiable Limit Theorem.

Theorem 6. *Let $\{f_n\}$ be a sequence of differentiable functions defined on the closed interval $[a,b]$, and assume $\{f_n'\}$ converges uniformly to a function g on $[a,b]$. If there exists a point $x_0 \in [a,b]$ for which $f_n(x_0)$ is convergent, then $\{f_n\}$ converges uniformly. Moreover, the limit function $f = \lim_{n\to\infty} f_n$ is differentiable and satisfies $f' = g$.*

Exercises

Exercise 17. Consider the sequence of functions

$$h_n = \sqrt{x^2 + \frac{1}{n}}.$$

(a) Compute the pointwise limit of h_n and then prove that the convergence is uniform on $\mathbb{R}$.

(b) Note that each h_n is differentiable. Show $g(x) = \lim_{n\to\infty} h_n'(x)$ exists for all x, and explain how we can be certain that the convergence is not uniform on any neighborhood of zero.

Exercise 18. Consider the sequence of functions

$$f_n(x) = \frac{x}{1 + nx^2}.$$

(a) Find the points on $\mathbb{R}$ where each $f_n(x)$ attains its maximum and minimum value. Use this to prove $\{f_n\}$ converges uniformly on $\mathbb{R}$. What is the limit function?

(b) Let $f = \lim_{n\to\infty} f_n$. Compute $f_n'(x)$ and find all the values of x for which $f'(x) = \lim_{n\to\infty} f_n'(x)$.

Exercise 19. Let

$$g_n(x) = \frac{nx + x^2}{2n}$$

and set $g(x) = \lim_{n\to\infty} g_n(x)$. Show that g is differentiable in two ways:

(a) Compute $g(x)$ by algebraically taking the limit as $n \to \infty$ and then find $g'(x)$.

(b) Compute $g_n'(x)$ for each $n \in \mathbb{N}$ and show that the sequence of derivatives $\{g_n'\}$ converges uniformly on every interval $[-M, M]$. Then conclude $g'(x) = \lim_{n\to\infty} g_n'(x)$.

(c) Repeat parts (a) and (b) for the sequence $f_n(x) = (nx^2 + 1)/(2n + x)$.

Exercise 20. Provide an example or explain why the request is impossible. Let's take the domain of the functions to be all of $\mathbb{R}$.

(a) A sequence $\{f_n\}$ of nowhere differentiable functions with $f_n \to f$ uniformly and f everywhere differentiable.

(b) A sequence $\{f_n\}$ of differentiable functions such that $\{f_n'\}$ converges uniformly but the original sequence $\{f_n\}$ does not converge for any $x \in \mathbb{R}$.

(c) A sequence $\{f_n\}$ of differentiable functions such that both $\{f_n\}$ and $\{f_n'\}$ converge uniformly but $f = \lim f_n$ is not differentiable at some point.

5 Series of Functions

Definition 5. For each $n \in \mathbb{N}$, let f_n and f be functions defined on a set $A \subset \mathbb{R}$. The infinite series

$$\sum_{n=1}^{\infty} f_n(x) = f_1(x) + f_2(x) + f_3(x) + \cdots$$

converges pointwise on A to $f(x)$ if the sequence $\{s_k(x)\}$ of partial sums defined by

$$s_k(x) = f_1(x) + f_2(x) + \cdots + f_k(x)$$

converges pointwise to $f(x)$. The series *converges uniformly* on A to f if the sequence $\{s_k(x)\}$ converges uniformly on A to $f(x)$.

In either case, we write $f = \sum_{n=1}^{\infty} f_n$ or $f(x) = \sum_{n=1}^{\infty} f_n(x)$, always being explicit about the type of convergence involved.

If we have a series $\sum_{n=1}^{\infty} f_n$ where the functions f_n are continuous, then the Algebraic Continuity Theorem guarantees that the partial sums – because they are finite sums – will be continuous as well. A corresponding observation is true if we are dealing with differentiable functions. As a consequence, we can immediately translate the results for sequences in the previous sections into statements about the behavior of infinite series of functions.

Theorem 7 (Term-by-term Continuity Theorem)**.** *Let f_n be continuous functions defined on a set $A \subset \mathbb{R}$, and assume $\sum_{n=1}^{\infty} f_n$ converges uniformly on A to a function f. Then, f is continuous on A.*

Theorem 8 (Term-by-term Differentiability Theorem)**.** *Let f_n be differentiable functions defined on an interval A, and assume $\sum_{n=1}^{\infty} f_n'$ converges uniformly to a limit $g(x)$ on A. If there exists a point $x_0 \in A$ where $\sum_{n=1}^{\infty} f_n(x_0)$ converges, then the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly to a differentiable function $f(x)$ satisfying $f'(x) = g(x)$ on A. In other words,*

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \qquad \text{and} \qquad f'(x) = \sum_{n=1}^{\infty} f_n'(x).$$

In the vocabulary of infinite series, the Cauchy Criterion takes the following form.

Theorem 9 (Cauchy Criterion for Uniform Convergence of Series)**.** *A series $\sum_{n=1}^{\infty} f_n$ converges uniformly on $A \in \mathbb{R}$ if and only if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that*

$$|f_{m+1}(x) + f_{m+2}(x) + \cdots + f_n(x)| < \epsilon$$

whenever $n > m \geq N$ and $x \in A$.

The benefits of uniform convergence over pointwise convergence suggest the need for some ways of determining when a series converges uniformly. The following corollary to the Cauchy Criterion is the most common such tool. In particular, it will be quite useful in our upcoming investigations of power series.

Theorem 10 (Weierstrass M-Test)**.** *For each $n \in \mathbb{N}$, let f_n be a function defined on a set $A \subset \mathbb{R}$, and let $M_n > 0$ be a real number satisfying*

$$|f_n(x)| \leq M_n \qquad \forall x \in A.$$

If $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n$ converges uniformly on A.

Proof. Using the triangle inequality and the Cauchy Criterion. $\square$

Exercises

Exercise 21. Decide whether each proposition is true or false, providing a short justification or counterexample as appropriate.

(a) If $\sum_{n=1}^{\infty} g_n$ converges uniformly, then $\{g_n\}$ converges uniformly to zero.

(b) If $0 \leq f_n \leq g_n$ and $\sum_{n=1}^{\infty} g_n$ converges uniformly, then $\sum_{n=1}^{\infty} f_n$ converges uniformly.

(c) If $\sum_{n=1}^{\infty} f_n$ converges uniformly on A, then there exist constants M_n such that $|f_n(x)| \leq M_n$ for all $x \in A$ and $\sum_{n=1}^{\infty} M_n$ converges.

Exercise 22. (a) Show that

$$g(x) = \sum_{n=1}^{\infty} \frac{\cos(2^n x)}{2^n}$$

is continuous on all of $\mathbb{R}$.

(b) The function g was cited previously as an example of a continuous nowhere differentiable function. What happens if we try to use the Differentiable Limit Theorem to explore whether g is differentiable?

Exercise 23. Define

$$g(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{1 + x^{2n}}.$$

Find the values of x where the series converges and show that we get a continuous function on this set.

Exercise 24. (a) Prove that

$$h(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

is continuous on $[-1, 1]$.

(b) The series

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$$

converges for every x in the half-open interval $[-1, 1)$ but does not converge when $x = 1$. For a fixed $x_0 \in (-1, 1)$, explain how we can still use the Weierstrass M-Test to prove that f is continuous at x_0.

Exercise 25. Let

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{x+n} = \frac{1}{x} - \frac{1}{x+1} + \frac{1}{x+2} - \frac{1}{x+3} + \cdots .$$

Show f is defined for all $x > 0$. Is f continuous on $(0, \infty)$? How about differentiable?

Exercise 26. Let

$$f(x) = \sum_{k=1}^{\infty} \frac{\sin kx}{k^3}.$$

(a) Show that $f(x)$ is differentiable and that the derivative $f'(x)$ is continuous.
(b) Can we determine if f is twice-differentiable?

Exercise 27. Consider the function

$$f(x) = \sum_{k=1}^{\infty} \frac{\sin(x/k)}{k}.$$

Where is f defined? Continuous? Differentiable? Twice-differentiable?

Exercise 28. Let

$$h(x) = \sum_{n=1}^{\infty} \frac{1}{x^2 + n^2}.$$

(a) Show that h is a continuous function defined on all of $\mathbb{R}$.
(b) Is h differentiable? If so, is the derivative function h' continuous?

Exercise 29. Let $\{r_1, r_2, r_3, \dots\}$ be an enumeration of the set of rational numbers. For each $r_n \in \mathbb{Q}$, define

$$u_n(x) = \begin{cases} 1/2^n & \text{for } x > r_n \\ 0 & \text{for } x \le r_n. \end{cases}$$

Now, let $h(x) = \sum_{n=1}^{\infty} u_n(x)$. Prove that h is a monotone function defined on all of $\mathbb{R}$ that is continuous at every irrational point.

6 Power Series

Consider the power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots .$$

The first objective is to determine the points $x \in \mathbb{R}$ for which the resulting series on the right-hand side converges. This set clearly contains $x = 0$ and it takes a very predictable form.

Theorem 11. *If a power series $\sum_{n=0}^{\infty} a_n x^n$ converges at some point $x_0 \in \mathbb{R}$, then it converges absolutely for any x satisfying $|x| < |x_0|$.*

Proof. If $\sum_{n=0}^{\infty} a_n x_0^n$ converges, then the sequence of terms $\{a_n x_0^n\}$ is bounded. (In fact, it converges to 0.) Let $M > 0$ satisfy $|a_n x_0^n| \le M$ for all $n \in \mathbb{N}$. If $x \in \mathbb{R}$ satisfies $|x| < |x_0|$, then

$$|a_n x^n| = |a_n x_0^n| \left| \frac{x}{x_0} \right|^n \le M \left| \frac{x}{x_0} \right|^n$$

Note that

$$\sum_{n=0}^{\infty} M \left| \frac{x}{x_0} \right|^n$$

is a geometric series with ratio $|x/x_0| < 1$ and so converges. By the Comparison Test, $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely. □

The main implication of Theorem 11 is that the set of points for which a given power series converges must necessarily be $\{0\}$, $\mathbb{R}$, or a bounded interval centered around $x = 0$. Because of the strict inequality in Theorem 11, there is some ambiguity about the endpoints of the interval, and it is possible that the set of convergent points may be of the form $(-R, R)$, $[-R, R)$, $(-R, R]$, or $[-R, R]$.

The value of $\mathbb{R}$ is referred to as the radius of convergence of a power series, and it is customary to assign R the value 0 or ∞ to represent the set $\{0\}$ or $\mathbb{R}$, respectively.

Theorem 12 (Cauchy–Hadamard)**.** *Consider the power series*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Then the radius of convergence R of this power series is given by

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}.$$

Proof. We shall show first that the power series $\sum a_n x^n$ converges for $|x| < R$, and then that it diverges for $|x| > R$.

First suppose $|x| < R$. Let $t = 1/R$ not be zero or ∞. For any $\epsilon > 0$, there exists only a finite number of n such that $\sqrt[n]{|a_n|} \geq t + \epsilon$. Now $|a_n| \leq (t+\epsilon)^n$ for all but a finite number of a_n, so the series $\sum a_n x^n$ converges if $|x| < 1/(t+\epsilon)$. This proves the first part.

Conversely, for $\epsilon > 0$, $|a_n| \geq (t-\varepsilon)^n$ for infinitely many a_n, so if $|x| = 1/(t-\epsilon) > R$, we see that the series cannot converge because its n-th term does not tend to 0. $\square$

The limit involved in the Ratio Test is usually easier to compute, and when that limit exists, it shows that the radius of convergence is finite,

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|.$$

Example 6.1. Determine the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$ when

$$(a)\quad a_n = \frac{1}{n}; \qquad (b)\quad a_n = \frac{1}{n^2}; \qquad (c)\quad a_n = n; \qquad (d)\quad a_n = n^n; \qquad (e)\quad a_n = \frac{1}{n!}.$$

Of more interest to us here is the investigation of the properties of functions defined in this way. Are they continuous? Are they differentiable? If so, can we differentiate the series term-by-term? What happens at the endpoints?

Uniform Convergence

The positive answers to the preceding questions, and the usefulness of power series in general, are largely due to the fact that they converge uniformly on compact sets contained in their domain of convergent points. As we are about to see, a complete proof of this fact requires a fairly delicate argument attributed to the Norwegian mathematician Niels Henrik Abel. A significant amount of progress, however, can be made with the Weierstrass M-Test.

Theorem 13. *If a power series $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely at a point x_0, then it converges uniformly on the closed interval $[-c, c]$, where $c = |x_0|$.*

Proof. This proof requires a straightforward application of the Weierstrass M-Test. □

For many applications, Theorem 13 is good enough. For instance, because any $x \in (-R, R)$ is contained in the interior of a closed interval $[-c, c] \subset (-R, R)$, it now follows that a power series that converges on an open interval is necessarily continuous on this interval.

But what happens if we know that a series converges at an endpoint of its interval of convergence? Does the good behavior of the series on $(-R, R)$ necessarily extend to the endpoint $x = R$? If the convergence of the series at $x = R$ is absolute convergence, then we can again rely on Theorem 13 to conclude that the series converges uniformly on the set $[-R, R]$. The remaining interesting open question is what happens if a series converges conditionally at a point $x = R$. We may still use Theorem 11 to conclude that we have pointwise convergence on the interval $(-R, R]$, but more work is needed to establish uniform convergence on compact sets containing $x = R$.

Abel's Theorem

We should remark that if the power series $g(x) = \sum_{n=0}^{\infty} a_n x^n$ converges conditionally at $x = R$, then it is possible for it to diverge when $x = -R$. The series

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$$

with $R = 1$ is an example. To keep our attention fixed on the convergent endpoint, we will prove uniform convergence on the set $[0, R]$.

Lemma 14 (Abel's Lemma)**.** *Let b_n satisfy $b_1 \geq b_2 \geq b_3 \geq \cdots \geq 0$, and let $\sum_{n=1}^{\infty} a_n$ be a series for which the partial sums are bounded. In other words, assume there exists $A > 0$ such that*

$$|a_1 + a_2 + \cdots + a_n| \leq A$$

for all $n \in \mathbb{N}$. Then for all $n \in \mathbb{N}$,

$$|a_1 b_1 + a_2 b_2 + \cdots + a_n b_n| \leq A b_1 .$$

Proof. Let $s_n = a_1 + a_2 + \cdots + a_n$. Using the summation-by-parts formula, we can write

$$\begin{aligned}\left|\sum_{k=1}^{n} a_k b_k\right| = &\left|s_n b_{n+1} + \sum_{k=1}^{n} s_k(b_k - b_{k+1})\right| \\ \leq &A b_{n+1} + \sum_{k=1}^{n} A(b_k - b_{k+1}) \\ = &A b_1. \quad \square\end{aligned}$$

It is worth observing that if A were an upper bound on the partial sums of $\sum |a_n|$ (note the absolute value bars), then the proof of Lemma 14 would be a simple exercise in the triangle inequality. The point of the matter is that because we are only assuming conditional convergence, the triangle inequality is not going to be of any use in proving Abel's Theorem, but we are now in possession of an inequality that we can use in its place.

Theorem 15 (Abel's Theorem)**.** *Let $g(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series that converges at the point $x = R > 0$. Then the series converges uniformly on the interval $[0, R]$. A similar result holds if the series converges at $x = -R$.*

Proof. To apply Lemma 14, we set

$$g(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (a_n R^n) \left(\frac{x}{R}\right)^n .$$

Let $\epsilon > 0$. By the Cauchy Criterion for Uniform Convergence of Series, we need to show that

$$\left|(a_{m+1}R^{m+1})\left(\frac{x}{R}\right)^{m+1} + (a_{m+2}R^{m+2})\left(\frac{x}{R}\right)^{m+2} + \cdots + (a_n R^n)\left(\frac{x}{R}\right)^{n}\right| < \epsilon.$$

Because we are assuming that $\sum_{n=0}^{\infty} a_n R^n$ converges, the Cauchy Criterion for convergent series of real numbers guarantees that there exists an N such that

$$\left|a_{m+1}R^{m+1} + a_{m+2}R^{m+2} + \cdots + a_n R^n\right| < \frac{\epsilon}{2}$$

whenever $n > m \geq N$. But now, for any fixed $m \in \mathbb{N}$, we can apply Abel's Lemma to the sequences obtained by omitting the first m terms. Using $\epsilon/2$ as a bound on the partial sums of $\sum_{j=1}^{\infty} a_{m+j} R^{m+j}$ and observing that $(x/R)^{m+j}$ is monotone decreasing, an application of Abel's Lemma gives

$$\left|(a_{m+1}R^{m+1})\left(\frac{x}{R}\right)^{m+1} + (a_{m+2}R^{m+2})\left(\frac{x}{R}\right)^{m+2} + \cdots + (a_n R^n)\left(\frac{x}{R}\right)^{n}\right| \leq \frac{\epsilon}{2}\left(\frac{x}{R}\right)^{m+1} < \epsilon. \quad \square$$

The Success of Power Series

Theorem 16. *If a power series converges pointwise on the set $A \subset \mathbb{R}$, then it converges uniformly on any compact set $K \subset A$.*

Proof. A compact set contains both a maximum x_1 and a minimum x_0, which by hypothesis must be in A. Abel's Theorem implies the series converges uniformly on the interval $[x_0, x_1]$ and thus also on K. □

This fact leads to the desirable conclusion that a power series is continuous at every point at which it converges. To make an argument for differentiability, we would like to appeal to Theorem 8; however, this result has a slightly more involved set of hypotheses. In order to conclude that a power series $\sum_{n=0}^{\infty} a_n x^n$ is differentiable, and that term-by-term differentiation is allowed, we need to know beforehand that the differentiated series $\sum_{n=1}^{\infty} n a_n x^{n-1}$ converges uniformly.

Theorem 17. *If $\sum_{n=0}^{\infty} a_n x^n$ converges for all $x \in (-R, R)$, then the differentiated series $\sum_{n=1}^{\infty} n a_n x^{n-1}$ converges at each $x \in (-R, R)$ as well. Consequently, the convergence is uniform on compact sets contained in $(-R, R)$.*

Proof. Exercise. □

We should point out that it is possible for a series to converge at an endpoint $x = R$ but for the differentiated series to diverge at this point. The series $\sum_{n=1}^{\infty} x^n/n$ has this property when $x = -1$. On the other hand, if the differentiated series does converge at the point $x = R$, then Abel's Theorem applies and the convergence of the differentiated series is uniform on compact sets that contain R.

With all the pieces in place, we summarize the impressive conclusions of this section.

Theorem 18. *Assume*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

converges on an interval $A \subset \mathbb{R}$. The function f is continuous on A and differentiable on any open interval $(-R, R) \subset A$. The derivative is given by

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

Moreover, f is infinitely differentiable on $(-R, R)$, and the successive derivatives can be obtained via term-by-term differentiation of the appropriate series.

Proof. The details for why f is continuous have been discussed. Theorem 17 justifies the application of the Term-by-term Differentiability Theorem (Theorem 8), which verifies the formula for f'.

A differentiated power series is a power series in its own right, and Theorem 17 implies that, although the series may no longer converge at a particular endpoint, the radius of convergence does not change. By induction, then, power series are differentiable an infinite number of times. $\square$

Exercise

Exercise 30. Consider the function g defined by the power series

$$g(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots .$$

(a) Is g defined on $(-1,1)$? Is it continuous on this set? Is g defined on $(-1,1]$? Is it continuous on this set? What happens on $[-1,1]$? Can the power series for $g(x)$ possibly converge for any other points $|x| > 1$? Explain.

(b) For what values of x is $g'(x)$ defined? Find a formula for g'.

Exercise 31. Find suitable coefficients $\{a_n\}$ so that the resulting power series $\sum a_n x^n$ has the given properties, or explain why such a request is impossible.

(a) Converges for every value of $x \in \mathbb{R}$.
(b) Diverges for every value of $x \in \mathbb{R}$.
(c) Diverges for every value of $x \in \mathbb{R} \setminus \{0\}$.
(d) Converges absolutely for all $x \in [-1,1]$ and diverges off of this set.
(e) Converges conditionally at $x = -1$ and converges absolutely at $x = 1$.
(f) Converges conditionally at both $x = -1$ and $x = 1$.

Exercise 32. (**Term-by-term Antidifferentiation**).
Assume $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges on $(-R,R)$.

(a) Show that

$$F(x) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

is defined on $(-R,R)$ and satisfies $F'(x) = f(x)$.

(b) Antiderivatives are not unique. If g is an arbitrary function satisfying $g'(x) = f(x)$ on $(-R,R)$, find a power series representation for g.

Exercise 33. Recall that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots, \qquad \forall |x| < 1.$$

Using the above formula to find values for $\sum_{n=1}^{\infty} n/2^n$ and $\sum_{n=1}^{\infty} n^2/2^n$.

Exercise 34. (a) Show that power series representations are unique. If we have

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$$

for all x in an nonempty interval $(-R, R)$, prove that $a_n = b_n$ for all $n = 0, 1, 2, \ldots$.

(b) Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converge on $(-R, R)$, and assume $f'(x) = f(x)$ for all $x \in (-R, R)$ and $f(0) = 1$. Deduce the values of a_n.

Exercise 35. A series $\sum_{n=0}^{\infty} a_n$ is said to be *Abel-summable* to L if the power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

converges for all $x \in [0, 1)$ and $L = \lim_{x \to 1-} f(x)$.

(a) Show that any series that converges to a limit L is also Abel-summable to L.

(b) Show that $\sum_{n=0}^{\infty} (-1)^n$ is Abel-summable and find the sum.

7 Taylor Series

Theorem 19 (Taylor's Formula)**.** *Let*

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

be defined on some nontrivial interval centered at zero. Then,

$$a_n = \frac{f^{(n)(0)}}{n!}.$$

The Taylor series of $\sin x$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots.$$

Theorem 20 (Lagrange's Remainder Theorem)**.** *Let f be differentiable $N+1$ times on $(-R, R)$, define $a_n = f^{(n)(0)}/n!$ for $n = 0, 1, 2, \ldots, N$, and let*

$$S_N(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_N x^N.$$

Given $x \neq 0$ in $(-R, R)$, there exists a point c with $c \in (-x, x)$, such that the error function (the remainder) $E_N(x) = f(x) - S_N(x)$ satisfies

$$E_N(x) = \frac{f^{N+1}(c)}{(N+1)!} x^{N+1}.$$

Proof. The Taylor coefficients are chosen so that $f^{(n)}(0) = S^{(n)}(0)$ for all $0 \leq n \leq N$, which implies that the error function $E_N(x) = f(x) - S_N(x)$ satisfies

$$E_N^{(n)}(0) = 0, \qquad \forall 0 \leq n \leq N.$$

For simplicity, we assume $x > 0$ and the case when $x < 0$ is similar. Applying the Cauchy Mean Value Theorem to $E_N(x)$ and x^{N+1} on the interval $[0, x]$ implies there exists $x_1 \in (0, x)$ such that

$$\frac{E_N(x)}{x^{N+1}} = \frac{E_N'(x_1)}{(N+1)x_1^N}.$$

Applying the Cauchy Mean Value Theorem to $E_N'(x)$ and $(N+1)x^N$ on the interval $[0, x_1]$ implies there exists $x_2 \in (0, x_1)$ such that

$$\frac{E_N(x)}{x^{N+1}} = \frac{E_N'(x_1)}{(N+1)x_1^N} = \frac{E_N''(x_2)}{(N+1)N x_2^{N-1}}.$$

Continuing in this manner we find

$$\frac{E_N(x)}{x^{N+1}} = \frac{E_N^{(N+1)}(x_{N+1})}{(N+1)!},$$

where $x_{N+1} \in (0, x_N) \subset \cdots \subset (0, x)$. Now set $c = x_{N+1}$. Because $S_N^{(N+1)}(x) = 0$, we have $E_N^{(N+1)} = f^{(N+1)}(x)$ and it follows that

$$E_N(x) = \frac{f^{(N+1)(c)}}{(N+1)!} x^{N+1}$$

as desired. $\square$

Taylor series at $a \neq 0$

Throughout this chapter we have focused our attention on series expansions centered at zero, but there is nothing special about zero other than notational simplicity. If f is defined in some neighborhood of $a \in \mathbb{R}$ and infinitely differentiable at a, then the Taylor series expansion around a takes the form

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n, \quad \text{where} \quad c_n = \frac{f^{(n)}(a)}{n!}.$$

Setting $E_N(x) = f(x) - S_N(x)$ as usual, Lagrange's Remainder Theorem in this case says that there exists a value c between a and x where

$$E_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!}(x-a)^{N+1}.$$

A Counterexample

Lagrange's Remainder Theorem is extremely useful for determining how well the partial sums of the Taylor series approximate the original function, but it leaves unresolved the central question of whether or not the Taylor series necessarily converges to the function that generated it. The appearance of $f^{(N+1)}(c)$ in the error formula makes any general statement impossible. Let

$$g(x) = \begin{cases} e^{-1/x^2} & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

Computing the Taylor coefficients for this function, it's clear that $a_0 = g(0) = 0$. To compute a_1 we write

$$a_1 = g'(0) = \lim_{x \to 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x \to 0} \frac{e^{-1/x^2}}{x} = \lim_{x \to 0} \frac{1/x}{e^{1/x^2}},$$

where both numerator and denominator tend to ∞ as x approaches zero. Applying the ∞/∞ version of L'Hospital's Rule we see

$$a_1 = \lim_{x \to 0} \frac{1/x}{e^{1/x^2}} = \lim_{x \to 0} \frac{-1/x^2}{-2/x^3 e^{1/x^2}} = \lim_{x \to 0} \frac{x}{2e^{1/x^2}} = 0.$$

We can show that

$$g^{(n)}(0) = 0, \qquad \forall n \in \mathbb{N}.$$

The implications of this example are highly significant. The function g is infinitely differentiable, and every one of its Taylor coefficients is equal to zero. By default, then, its Taylor series converges uniformly on all of $\mathbb{R}$ to the zero function. But other than at $x = 0$, $g(x)$ is never equal to zero. The Taylor series for $g(x)$ converges, but it does not converge to $g(x)$ except at the center point $x = 0$. The unmistakable conclusion is that not every infinitely differentiable function can be represented by its Taylor series.

Exercise

Exercise 36. Using only Lagrange's Remainder Theorem (and no references to Abel's Theorem) prove

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \log 2.$$

Exercise 37. (Cauchy's Remainder Theorem). Let f be differentiable $N+1$ times on $(-R, R)$. For each $a \in (-R, R)$, let $S_N(x, a)$ be the partial sum of the Taylor series for f centered at a; in other words, define

$$S_N(x, a) = \sum_{n=0}^{N} c_n (x - a)^n \quad \text{where} \quad c_n = \frac{f^{(n)}(a)}{n!}.$$

Let $E_N(x, a) = f(x) - S_N(x, a)$. Now fix $x \neq 0$ in $(-R, R)$ and consider $E_N(x, a)$ as a function of a.

(a) Find $E_N(x, x)$.

(b) Explain why $E_N(x, a)$ is differentiable with respect to a, and show

$$E'_N(x, a) = -\frac{f^{(N+1)}(a)}{N!}(x - a)^N.$$

(c) Show

$$E_N(x) = E_N(x, 0) = \frac{f^{(N+1)}(c)}{N!}(x - c)^N x$$

for some c between 0 and x. This is Cauchy's form of the remainder for Taylor series centered at the origin.

Exercise 38. Consider $f(x) = 1/\sqrt{1 - x}$.

(a) Generate the Taylor series for f centered at zero, and use Lagrange's Remainder Theorem to show the series converges to f on $[0, 1/2]$. (The case $x < 1/2$ is more straightforward while $x = 1/2$ requires some extra care.) What happens when we attempt this with $x > 1/2$?

(b) Use Cauchy's Remainder Theorem to show the series representation for f holds on $[0, 1)$.

8 The Weierstrass Approximation Theorem

Theorem 21 (Weierstrass Approximation Theorem)**.** *Let f be a continuous function on $[a, b]$. Given $\epsilon > 0$, there exists a polynomial $p(x)$ satisfying*

$$|f(x) - p(x)| < \epsilon \qquad \forall x \in [a, b].$$

A restatement of the Weierstrass Approximation Theorem (WAT) without all the symbols is that every continuous function on a closed interval can be uniformly approximated by a polynomial.

Exercise 39. Assuming WAT, show that if f is continuous on $[a, b]$, then there exists a sequence p_n of polynomials such that $p_n \to f$ uniformly on $[a, b]$.

In the lagrange of topology, any $f \in C[a, b]$ is a limit point of a sequence of polynomials, thus the set of polynomials P is dense in $C[a, b]$, or in other words, $\overline{P} = C[a, b]$.

There are several proofs of the WAT, we shall present Lebesgue's proof in what follows. The idea is first show that a continuous function on $[a, b]$ can be approximated uniformly by polygonal functions (piecewise linear functions) and each polygonal function can be approximated by polynomials.

Interpolation.

Definition 6. A continuous function $\phi : [a, b] \to \mathbb{R}$ is *polygonal* if there is a partition

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

of $[a, b]$ such that ϕ is linear on each subinterval $[x_{i-1}, x_i]$, where $i = 1, 2, \ldots, n$.

Theorem 22. *Let $f : [a, b] \to \mathbb{R}$ be continuous. Given $\epsilon > 0$, there exists a polygonal function ϕ satisfying*

$$|f(x) - \phi(x)| < \epsilon.$$

Approximating the Absolute Value Function

Taylor series expansion for $\sqrt{1-x}$

$$\sqrt{1-x} = \sum_{n=0}^{\infty} a_n x^n, \tag{8.1}$$

where

$$a_0 = 1, \qquad a_n = -\frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2 \cdot 4 \cdot 6 \cdots (2n)} \quad \forall n \geq 1.$$

The Taylor expansion in (8.1) is valid when $x \in (-1, 1)$, which can be shown by estimating the error function with the Cauchy's Remainder Theorem

$$E_N(x) = \sqrt{1-x} - \sum_{n=0}^{N} a_n x^n = \frac{f^{(N+1)}(c)}{N!}(x-c)^N x$$

for some c between 0 and x.

Indeed (8.1) also holds at $x = \pm 1$.

Exercise 40. (a) Let

$$c_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}$$

for all $n \geq 1$. Show that $c_n < \frac{2}{\sqrt{2n+1}}$.

(b) Use (a) to show that $\sum_{n=0}^{\infty} a_n$ converges (absolutely, in fact) where a_n is the sequence of Taylor coefficients in (8.1).

(c) Carefully explain how this verifies that equation (8.1) holds for all $x \in [-1, 1]$.

Exercise 41. (a) Use the fact that $|a| = \sqrt{a^2}$ to prove that, given $\epsilon > 0$, there exists a polynomial $q(x)$ satisfying

$$||x| - q(x)| < \epsilon, \qquad \forall x \in [-1, 1].$$

(b) Generalize this conclusion to an arbitrary interval $[a, b]$.

Proving WAT

Earlier we suggested that proving WAT for the special case of the absolute value function was the key to the whole proof. Now it is time to fill in the details.

Exercise 42. (a) Fix $a \in [-1, 1]$ and sketch

$$h_a(x) = \frac{|x-a| + (x-a)}{2}$$

over $[-1, 1]$. Note that h_a is polygonal and satisfies $h_a(x) = 0$ for all $x \in [-1, a]$.

(b) Explain why we know $h_a(x)$ can be uniformly approximated with a polynomial on $[-1, 1]$.

(c) Let ϕ be a polygonal function that is linear on each subinterval of the partition

$$-1 = a_0 < a_1 < a_2 < \cdots < a_n = 1$$

Show there exist constants $b_0, b_1, \ldots, b_{n-1}$ so that

$$\phi(x) = \phi(-1) + b_0 h_{a_0}(x) + b_1 h_{a_1}(x) + \cdots + b_{n-1} h_{a_{n-1}}(x)$$

for all $x \in [-1, 1]$.

(c) Complete the proof of WAT for the interval $[-1, 1]$, and then generalize to an arbitrary interval $[a, b]$.

Exercise 43. (a) Find a counterexample which shows that WAT is not true if we replace the closed interval $[a, b]$ with the open interval (a, b).

(b) What happens if we replace $[a, b]$ with the closed set $[a, \infty)$. Does the theorem still hold?

Exercise 44. Is there a countable subset of polynomials $\mathcal{C}$ with the property that every continuous function on $[a, b]$ can be uniformly approximated by polynomials from $\mathcal{C}$?