MAT 3007 – Optimization

Duality Theory

Lecture 08

June 23th

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Repetition

Dual Linear Programs

In the last lecture, we introduced the dual problems of LPs. For example, for the LP in the standard form:

$$\begin{aligned} \text{minimize}_x \quad & c^\top x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

The dual problem is:

$$\begin{aligned} \text{maximize}_y \quad & b^\top y \\ \text{subject to} \quad & A^\top y \leq c \end{aligned}$$

$\rightsquigarrow$ If min and max can be exchanged, the primal and dual problem should have the same optimal value.

General Dual Problems

Primal			Dual		
min	$c^\top x$		max	$b^\top y$	
s.t.	$a_i^\top x \geq b_i,$	$i \in M_1,$	s.t.	$y_i \geq 0,$	$i \in M_1$
	$a_i^\top x \leq b_i,$	$i \in M_2,$		$y_i \leq 0,$	$i \in M_2$
	$a_i^\top x = b_i,$	$i \in M_3,$		y_i free,	$i \in M_3$
	$x_j \geq 0,$	$j \in N_1,$		$A_j^\top y \leq c_j,$	$j \in N_1$
	$x_j \leq 0,$	$j \in N_2,$		$A_j^\top y \geq c_j,$	$j \in N_2$
	x_j free,	$j \in N_3,$		$A_j^\top y = c_j,$	$j \in N_3$

- $a_i^\top$ is the ith row of A, A_j is the jth column of A.
- Each primal constraint corresponds to a dual variable.
- Each primal variable corresponds to a dual constraint.
- Equality constraints always correspond to free variables.

To Memorize: Table & Signs

Primal	minimize	maximize	Dual
constraints	$\geq b_i$	≥ 0	variables
	$\leq b_i$	≤ 0	
	$= b_i$	free	
variables	≥ 0	$\leq c_j$	constraints
	≤ 0	$\geq c_j$	
	free	$= c_j$	

1. Each primal constraint is associated with a dual variable; each primal variable is associated with a dual constraint.
2. Equality constraints correspond to free variables, vice versa.

Consider the following problem:

$$\begin{array}{lcccl} \text{minimize} & x_1 & +2x_2 & +3x_3 & \\ \text{subject to} & -x_1 & +3x_2 & & =5 \\ & 2x_1 & -x_2 & +3x_3 & \geq 6 \\ & & & x_3 & \leq 4 \\ & x_1 \geq 0 & x_2 \leq 0 & x_3 \text{ free} & \end{array}$$

The dual is:

$$\begin{array}{lcccl} \text{maximize} & 5y_1 & +6y_2 & +4y_3 & \\ \text{subject to} & -y_1 & +2y_2 & & \leq 1 \\ & 3y_1 & -y_2 & & \geq 2 \\ & & +3y_2 & +y_3 & =3 \\ & y_1 \text{ free} & y_2 \geq 0 & y_3 \leq 0 & \end{array}$$

Recall the support vector machine problem. The primal problem is:

$$\begin{array}{rll} \text{minimize}_{w,b,t} & \sum_{i=1}^m t_i & \\ \text{subject to} & y_i(x_i^\top w + b) + t_i \geq 1, & \forall\ i = 1, ..., m \\ & t_i \geq 0 & \forall\ i = 1, ..., m \end{array}$$

with variables $w \in \mathbb{R}^n$, $b \in \mathbb{R}$, and $t \in \mathbb{R}^m$.

⇝ Associate u_i to each of the constraints.

- The dual problems has $n + m + 1$ constraints.
- Defining the data matrix $X = \begin{pmatrix} x_1 & x_2 & \ldots & x_m \end{pmatrix} \in \mathbb{R}^{n\times m}$, the primal constraints can be written compactly:

$$\begin{pmatrix} \mathrm{diag}(y)X^\top & y & I \end{pmatrix} \begin{pmatrix} w \\ b \\ t \end{pmatrix} \geq \mathbb{1}.$$

Example Continued

The dual constraint becomes

$$\begin{pmatrix} X\mathrm{diag}(y) \\ y^\top \\ I \end{pmatrix} u = \begin{pmatrix} X\mathrm{diag}(y)u \\ y^\top u \\ u \end{pmatrix} \begin{array}{lll} \Longrightarrow & = 0 & (w \text{ free}) \\ \Longrightarrow & = 0 & (b \text{ free}) \\ \Longrightarrow & \leq \mathbb{1} & (t \geq 0) \end{array}$$

with additional constraints $u \geq 0$.

The dual problem is

$$\begin{array}{rl} \text{maximize}_u & \sum_{i=1}^m u_i \\ \text{subject to} & X\mathrm{diag}(y)u = 0 \\ & y^\top u = 0 \\ & 0 \leq u_i \leq 1, \qquad \forall\ i = 1, \ldots, m \end{array}$$

SVM - Primal Problem:

$$\begin{aligned} \text{minimize}_{w,b,t} \quad & \mathbb{1}^\top t \\ \text{subject to} \quad & \mathrm{diag}(y)X^\top w + yb + t \geq \mathbb{1} \\ & t \geq 0 \end{aligned}$$

SVM - Dual Problem:

$$\begin{aligned} \text{maximize}_{u} \quad & \mathbb{1}^\top u \\ \text{subject to} \quad & X\mathrm{diag}(y)u = 0 \\ & y^\top u = 0 \\ & u \geq 0,\ u \leq \mathbb{1} \end{aligned}$$

- X is the data matrix, y are the labels (± 1).

Duality Theory

Theorem: Duality under Transformations

If we transform a linear program to an equivalent one (e.g., by replacing free variables, adding slack variables, etc.), then the dual of the two problems will be equivalent.

Theorem: Dual^2

If we transform the primal to its dual and transform the resulting dual to its dual, then we will obtain a problem equivalent to the primal problem, i.e., the dual of dual is the primal.

By construction, the primal and the dual problem should have the same optimal value.

- In the following, we will formally prove this claim.
- Due to the transformation invariance, we only need to discuss the standard form. The next theorems also apply to general cases as well.

Primal		Dual	
min	$c^\top x$	max	$b^\top y$
s.t.	$Ax = b,\ x \geq 0$	s.t.	$A^\top y \leq c$

Weak Duality Theorem

If x is feasible for the primal problem and y is feasible for the dual, then we have:

$$b^\top y \leq c^\top x.$$

In the latter situation (primal $\rightsquigarrow$ min, dual $\rightsquigarrow$ max):

- Any dual feasible solution will give a lower bound on the primal optimal value.
- Any primal feasible solution will give an upper bound on the dual optimal value.
- The optimal value of the primal is larger than that of the dual.

Proof:

Assume x is a feasible point of the primal problem and let y be a feasible dual variable. Then we have:

$$b^\top y = (Ax)^\top y = x^\top (A^\top y) \le c^\top x.$$

The last inequality follows from $x \ge 0$ and $A^\top y \le c$. $\square$

Corollary: Feasibility and Boundedness

- If the primal problem is unbounded (i.e., the optimal value is $-\infty$), then the dual problem must be infeasible.
- If the dual problem is unbounded (i.e., the optimal value is $+\infty$), then the primal problem must be infeasible.

Optimality via Duality

Corollary: Optimality and Duality

Let x and y be feasible points of the primal and dual problem, respectively. If $c^\top x = b^\top y$, then x and y must be optimal solutions of the primal and dual, respectively.

Optimality conditions for LP. If x, y satisfy:

1. x is primal feasible.
2. y is dual feasible.
3. The objective values coincide, i.e., $c^\top x = b^\top y$.

Then, x and y are optimal solutions of the primal and dual problems respectively.

$\leadsto$ The reverse is also true (see the next theorem)!

Strong Duality Theorem

If a linear program has an optimal solution, so does its dual and the optimal values of the primal and dual problems are equal.

- We present a constructive proof: for a given primal optimal solution, we construct a dual optimal solution and show that their objective values are equal.
- We use the simplex method in our proof.
- In the proof, we will see that the simplex method actually also finds the dual optimal solution when it terminates.

Remark

The dual optimal solution actually comes as a by-product when we use the simplex algorithm. The term

$$c_B^\top A_B^{-1}$$

will be the dual optimal solution (if the primal has an optimal solution). Thus, when solving the primal problem, the dual is solved automatically.

This is not a coincidence. Nearly all LP alg. (simplex method, interior point method, or ellipsoid method) solve both primal and dual problems simultaneously.

Discussion: Continued

Based on the strong duality theorem, the pair (x, y) is optimal for the primal and dual, respectively, if and only if:

- x is primal feasible.
- y is dual feasible.
- They achieve the same objective values.

Solving LPs is equivalent to solving the following linear system:

- $Ax = b, \quad x \geq 0.$
- $A^\top y \leq c.$
- $b^\top y = c^\top x.$

Later, we will show how this perspective helps when solving LPs.

Which of the following states are possible for an LP and its dual?

P | D	Finite Optimum	Unbounded	Infeasible
Finite Optimum	?	?	?
Unbounded	?	?	?
Infeasible	?	?	?

Example: Primal and Dual are Infeasible

Primal:

$$\begin{aligned} \text{minimize} \quad & x_1 + 2x_2 \\ \text{subject to} \quad & x_1 + x_2 &= 1 \\ & 2x_1 + 2x_2 &= 3 \end{aligned}$$

Dual:

$$\begin{aligned} \text{maximize} \quad & y_1 + 3y_2 \\ \text{subject to} \quad & y_1 + 2y_2 &= 1 \\ & y_1 + 2y_2 &= 2 \end{aligned}$$

The only possible cases for LPs are:

P | D	Finite Optimum	Unbounded	Infeasible
Finite Optimum	✓		
Unbounded			✓
Infeasible		✓	✓

- If both the primal and dual are feasible, then both problems must have optimal solutions. (This is one way to quickly determine if an LP is bounded).
- By strong duality, their optimal values must be the same.

We studied the relation between primal and dual optimal values.

Next, we study the relation between primal and dual optimal solutions.

Primal		Dual	
min	$c^\top x$	max	$b^\top y$
s.t.	$Ax = b,\ x \geq 0$	s.t.	$A^\top y \leq c$

Theorem: Complementarity Conditions

Let x and y be feasible points of the primal and dual problem, respectively. Then, x and y are optimal solutions if and only if:

$$\begin{aligned} x_i > 0 \quad &\Longrightarrow \quad A_i^\top y = c_i \\ A_i^\top y < c_i \quad &\Longrightarrow \quad x_i = 0. \end{aligned}$$

Or in other words,

$$x_i \cdot (c_i - A_i^\top y) = 0, \quad \forall\ i.$$

Example

Primal:

$$\begin{array}{rrrrcl} \text{minimize} & 13x_1 & +10x_2 & +6x_3 & & \\ \text{s.t.} & 5x_1 & +x_2 & +3x_3 & = & 8 \\ & 3x_1 & +x_2 & & = & 3 \\ & x_1, & x_2, & x_3 & \geq & 0 \end{array}$$

$\rightsquigarrow$ Optimal solution: $(1, 0, 1)$.

Dual problem:

$$\begin{array}{rrrcl} \text{maximize} & 8y_1 & +3y_2 & & \\ \text{s.t.} & 5y_1 & +3y_2 & \leq & 13 \\ & y_1 & +y_2 & \leq & 10 \\ & 3y_1 & & \leq & 6 \end{array}$$

$\rightsquigarrow$ Optimal solution: $(2, 1)$.

Verify the complementarity conditions:

$$x_1 \cdot (13 - 5y_1 - 3y_2) = 0, \;\; x_2 \cdot (10 - y_1 - y_2) = 0, \;\; x_3 \cdot (6 - 3y_1) = 0.$$

Sometimes, we write the dual problem (equivalently) as:

Primal		Dual	
min	$c^\top x$	max	$b^\top y$
s.t.	$Ax = b,\ x \geq 0$	s.t.	$A^\top y + s = c,\ s \geq 0$

We call s the dual slack variables. Then, the complementarity conditions can be written as:

$$x_i \cdot s_i = 0 \quad \forall\ i.$$

We sometimes call the complementarity conditions the complementary slackness condition.

Consider the primal-dual pair:

Primal			Dual		
min	$c^\top x$		max	$b^\top y$	
s.t.	$a_i^\top x \geq b_i,$	$i \in M_1,$	s.t.	$y_i \geq 0,$	$i \in M_1$
	$a_i^\top x \leq b_i,$	$i \in M_2,$		$y_i \leq 0,$	$i \in M_2$
	$a_i^\top x = b_i,$	$i \in M_3,$		y_i free,	$i \in M_3$
	$x_j \geq 0,$	$j \in N_1,$		$A_j^\top y \leq c_j,$	$j \in N_1$
	$x_j \leq 0,$	$j \in N_2,$		$A_j^\top y \geq c_j,$	$j \in N_2$
	x_j free,	$j \in N_3,$		$A_j^\top y = c_j,$	$j \in N_3$

Theorem: Complementarity Conditions

Let x and y be feasible points of the primal and dual problem. Then x and y are optimal if and only if:

$$y_i \cdot (a_i^\top x - b_i) = 0, \quad \forall\ i \quad \text{and} \quad x_j \cdot (A_j^\top y - c_j) = 0, \quad \forall\ j.$$

Recall that the optimality conditions for LPs were given by:

1. x is primal feasible.
2. y is dual feasible.
3. $c^\top x = b^\top y$.

Now with the complementarity conditions, we can write an equivalent set of conditions:

1. x is primal feasible.
2. y is dual feasible.
3. The complementarity conditions are satisfied.

If we have an optimal solution of the primal problem, then we may be able to quickly find a dual optimal solution.

If the primal problem is of the standard form and x is the optimal basic solution with basis B and with $x_i > 0$ for $i \in B$.

Then by the complementarity conditions, we know that the optimal solution y of the dual problem has to satisfy:

$$A_i^\top y = c_i \quad i \in B$$

i.e., $y = (A_B^{-1})^\top c_B$ ($\rightsquigarrow$ see our earlier remark).

Example

Primal:

$$\begin{array}{rrrrcl} \text{minimize} & 13x_1 & +10x_2 & +6x_3 & & \\ \text{s.t.} & 5x_1 & +x_2 & +3x_3 & = & 8 \\ & 3x_1 & +x_2 & & = & 3 \\ & x_1, & x_2, & x_3 & \geq & 0 \end{array}$$

$\rightsquigarrow$ Optimal solution $(1, 0, 1)$.

Dual problem:

$$\begin{array}{rrrcl} \text{maximize} & 8y_1 & +3y_2 & & \\ & 5y_1 & +3y_2 & \leq & 13 \\ & y_1 & +y_2 & \leq & 10 \\ & 3y_1 & & \leq & 6 \end{array}$$

Because x_1 and x_3 are positive, the first and third constraint have to be tight for the dual optimal solution.

Thus the optimal solution of the dual must be $(2, 1)$.

If the primal optimal solution is obtained from using the simplex tableau and the initial problem was constructed from adding slack variables, then one can find the optimal dual solution $(A_B^{-1})^\top c_B$ from the simplex tableau when it finishes.

When the initial tableau is constructed from adding slack variables (thus it is naturally a canonical form), we can write the initial tableau as follows:

$c^\top$	0_m	0
A	I_m	b

$c^\top$	0_m	0
A	I_m	b

Suppose after some iterations, the simplex method reaches an optimal solution with basis B. Then the tableau becomes:

$c^\top - c_B^\top A_B^{-1} A$	$-c_B^\top A_B^{-1}$	$-c_B^\top A_B^{-1} b$
$A_B^{-1} A$	A_B^{-1}	$A_B^{-1} b$

Therefore, the final reduced costs corresponding to the original identity matrix part is the optimal dual solution.

Consider the production planning problem:

$$
\begin{array}{rcccccl}
\text{minimize} & -x_1 & -2x_2 & & & & \\
\text{subject to} & x_1 & & +s_1 & & & = 100 \\
 & & 2x_2 & & +s_2 & & = 200 \\
 & x_1 & +x_2 & & & +s_3 & = 150 \\
 & x_1, & x_2, & s_1, & s_2, & s_3 & \geq 0
\end{array}
$$

Dual:

$$
\begin{array}{rcccl}
\text{maximize} & 100y_1 & +200y_2 & +150y_3 & \\
\text{subject to} & y_1 & & +y_3 & \leq -1 \\
 & & 2y_2 & +y_3 & \leq -2 \\
 & y_1, & y_2, & y_3 & \leq 0
\end{array}
$$

The initial tableau for the primal:

B	-1	-2	0	0	0	0
3	1	0	1	0	0	100
4	0	2	0	1	0	200
5	1	1	0	0	1	150

Final tableau:

B	0	0	0	1/2	1	250
1	1	0	0	-1/2	1	50
3	0	0	1	1/2	-1	50
2	0	1	0	1/2	0	100

What is the dual optimal solution?

- $(y_1, y_2, y_3) = (0, -1/2, -1)$, with objective value -250.

If the problem is not derived from adding slack variables (therefore the initial top row is not in the form $(c^\top, 0)$), then this method may not give the right answer.

- In that case, one can just compute $(A_B^{-1})^\top c_B$.

Questions?