Sheet 4 118010350

A 4.1 (a) Linear program in standard form:

$$\text{minimize } -3x_1-4x_2-3x_3-6x_4$$
$$\text{subject to } 2x_1+x_2-x_3+x_4-S_1=12$$
$$x_1+x_2+x_3+x_4=8$$
$$-x_2+2x_3+x_4+S_2=10$$
$$x_1,x_2,x_3,x_4,S_1,S_2\geq 0$$

$$A=\begin{bmatrix}2&1&-1&1&-1&0\\1&1&1&1&0&0\\1&-1&2&1&0&1\end{bmatrix},\quad b=\begin{bmatrix}12\\8\\10\end{bmatrix}$$

Choose the basic indices $\{1,5,6\}$, we can find an initial basic feasible solution: $x_1=8, x_2=0, x_3=0, x_4=0, S_1=4, S_2=10$.

Construct the simplex tableau:

B	0	-1	0	-3	0	0	-24
1	1	1	1	1	0	0	8
5	0	$\boxed{1}$	3	1	1	0	4
6	0	-1	2	1	0	1	10

Step 1:

B	0	0	3	-2	1	0	-28
1	1	0	-2	0	-1	0	4
2	0	1	3	$\boxed{1}$	1	0	4
6	0	0	5	2	1	1	14

Step 2:

B	0	2	9	0	3	0	-36
1	1	0	-2	0	-1	0	4
4	0	1	3	1	1	0	4
6	0	-2	-1	0	-1	1	6

Thus, the optimal solution is $x_1=4, x_2=0, x_3=0, x_4=4$.
and the optimal value is $3x_1+4x_2+3x_3+6x_4=36$.

(b) ① "optimal".
Since all the reduced cost is nonnegative, the solution is optimal

② "unique".
Since the solution is $x_1=4, x_2=0, x_3=0, x_4=4$,
then we suppose there exists another optimal solution
$x_1=4-a, x_2=a, x_3=b, x_4=4-b$ and $0\leq a\leq 4, 0\leq b\leq 4$

Then
$$2x_1+x_2-x_3+x_4\geq 12 \qquad 2(4-a)+a-b+(4-b)\geq 12$$
$$x_1+x_2+x_3+x_4=8 \quad\Leftrightarrow\quad (4-a)+a+b+(4-b)=8$$
$$-x_2+2x_3+x_4\leq 10 \qquad -a+2b+(4-b)\leq 10$$

Then we get the constraint $12-a-2b\geq 12$, contradiction.
From ① and ②, the solution is the unique optimal solution

(c) The dual problem of the linear program:

$$\text{minimize } 12y_1+8y_2+10y_3$$
$$\text{subject to } 2y_1+y_2\geq 3$$
$$y_1+y_2-y_3\geq 4$$
$$-y_1+y_2+2y_3\geq 3$$
$$y_1+y_2+y_3\geq 6$$
$$y_1\leq 0,\ y_2\text{-free},\ y_3\geq 0.$$

We can easily find a feasible solution of the dual:
$y_1=-3, y_2=9, y_3=0$, and $12y_1+8y_2+10y_3=36$.
Since $c^Tx=b^Ty$, and x, y are feasible solutions,
then by Collery, y is the optimal solution of the dual.
Since the primal problem has the unique solution,

then by $c_B^TA_B^{-1}$, we also get the unique solution of the dual problem.

(d) By solving the primal problem and the dual problem,
we have the unique solutions $x^*=(4,0,0,4)$, $y^*=(-3,9,0)$
We consider the standard form of the primal: $c=(-3,-4,-3,-6,0,0)$

① $\tilde{c}=(-3,-4,-1,-6,0,0)$
We can compute $\tilde{c}_N^T-\tilde{c}_B^TA_B^{-1}A_N=(2,11,3)$, which contains no negative components, thus the optimal value will not change.

② $\tilde{c}=(-3,-4,-12,-6,0,0)$
We can compute $\tilde{c}_N^T-\tilde{c}_B^TA_B^{-1}A_N=(2,0,3)$, which contains no negative components, thus the optimal value will not change.

③ $\tilde{c}=(-1,-4,-3,-6,0,0)$
We can compute $\tilde{c}_N^T-\tilde{c}_B^TA_B^{-1}A_N=(2,13,5)$, which contains no negative components, thus the optimal solution will not change.
Since $c_j\in B$, then the optimal value will change.

④ $\tilde{c}=(-7,-4,-3,-6,0,0)$
We can compute $\tilde{c}_N^T-\tilde{c}_B^TA_B^{-1}A_N=(2,1,-1)$, which contains negative components, thus the optimal value will change.

(e). Let $e_i=(0,1,0)$. Suppose $\Delta b=\lambda\cdot e_i$, $\lambda\in\mathbb{R}$
Then the new solution is $\hat{x}_B=A_B^{-1}(b+\Delta b)$
$$=A_B^{-1}(b+\lambda\cdot e_i)=A_B^{-1}b+\lambda A_B^{-1}\cdot e_i=x_B^*+\lambda A_B^{-1}e_i$$
Since the current basis is still optimal,
then $\hat{x}_B=x_B^*+\lambda A_B^{-1}e_i\geq 0$.
By solving the inequality, we have $\begin{bmatrix}4\\4\\0\end{bmatrix}+\lambda\begin{bmatrix}-1\\2\\4\end{bmatrix}\geq 0$.

Thus, $-2\leq\lambda\leq 0$

A4.2. (a). Suppose the sales quotas for Special Risk Insurance, Mortage Insurance, Long-Term Care Insurance are x_1, x_2, x_3.
The linear program:

$$\text{maximize } 3300x_1+2000x_2+5000x_3$$
$$\text{subject to } 2x_1+x_2+x_3\leq 250$$
$$3x_1+x_2+2x_3\leq 150$$
$$x_1+2x_2+4x_3\leq 160$$
$$x_1,x_2,x_3\geq 0.$$

Using MATLAB, Solving the LP, we have the optimal solution is $x_1=28, x_2=0, x_3=33$, the optimal value is 2.574×10^5.

(b) The dual problem of the linear program:

$$\text{minimize } 250y_1+150y_2+160y_3$$
$$\text{subject to } 2y_1+3y_2+y_3\geq 3300$$
$$y_1+y_2+2y_3\geq 2000$$
$$y_1+2y_2+4y_3\geq 5000$$
$$y_1,y_2,y_3\geq 0$$

The dual solution can be obtained by $c_B^T\cdot A_B^{-1}$, from the final simplex tableau, we know that $B=\{1,3,4\}$.
Then $c_B^TA_B^{-1}=(0,820,840)$, Thus, the optimal solution of the dual problem is $y_1=0, y_2=820, y_3=840$.

Interpretation: ① y_1, y_2, y_3: The price per working hour of underwriting, administration, claims.
② Objective function: The company wants to produce all the insurances at the lowest price.
③ Constraints: The company wants to make the value

of each insurance larger than the known worth price.

Reasons: Here y_1, y_2, y_3 are the shadow price of underwriting, administration, claims. Then we can compute the worth prices for three kinds of insurances.
① Special Risk: $2\times 0+3\times 820+1\times 840=3300$.
② Mortage: $1\times 0+1\times 820+2\times 840=2500$.
③ Long-Term Care: $1\times 0+2\times 820+4\times 840=5000$.
Since the worth price of Mortage is the lowest, then the company rather produce it by itself than agree to sell.

(c). Let $e_i=(0,0,1)$, Suppose $\Delta b=\lambda\cdot e_i$, $\lambda\in\mathbb{R}$.
Then the new solution is $\hat{x}_B=x_B^*+\lambda A_B^{-1}e_i$.
Since the current basis is still optimal, then we have
$\hat{x}_B=x_B^*+\lambda A_B^{-1}e_i\geq 0$.
By solving the inequality, we have $\begin{bmatrix}28\\33\\0\end{bmatrix}+\lambda\begin{bmatrix}-0.2\\0.3\\0.1\end{bmatrix}\geq 0$.

Thus, $0\leq\lambda\leq 140$.

(d) Suppose we change the profit of special risk to $(3300+\lambda)$
Let $e_i=(1,0,0)$, then $\Delta c=\lambda\cdot e_i$. $(-3300-\lambda)$ in standard)
We can get $r_N^T=(500,820,840)$.
Since we need to have $r_N^T-\lambda e_i^TA_B^{-1}A_N\geq 0$,
then $\begin{bmatrix}500\\820\\840\end{bmatrix}-\lambda\cdot\begin{bmatrix}0\\-0.4\\0.2\end{bmatrix}\geq 0$.

Thus, $-2050\leq\lambda\leq 500$