MAT 3253 Lecture 17

Corollary If f is analytic in an open disc $D(z_0, r)$, then the n^{th} derivative of f exists in $D(z_0, r)$ and

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(w)}{(w-z_0)^{n+1}} \, dw$$

If f is complex differentiable once in a domain, then f can be differentiated arbitrarily many times.

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(w)}{(w-z_0)^{n+1}} \, dw$$

$$f(z_0) = a_0 = \frac{1}{2\pi i} \oint \frac{f(w)}{(w_0 - z_0)} \, dw$$

$$f'(z_0) = a_1$$

$$f''(z_0) = 2! \, a_2$$

$$f^{(n)}(z_0) = n! \, a_n$$

$$= \frac{n!}{2\pi i} \oint_C \frac{f(w)}{(w-z_0)^{n+1}} \, dw$$

Cauchy estimate

If $\quad |f(z)| \le M \quad$ for $C : |z - z_0| = r$

and f is analytic inside the circle $|z - z_0| = r$,

$$|f^{(n)}(z_0)| \le \frac{M\, n!}{r^n} \qquad \text{for } n = 0, 1, 2, \ldots$$

$$|f^{(n)}(z_0)| = \left| \frac{n!}{2\pi i} \int_C \frac{f(w)}{(w - z_0)^{n+1}}\, dw \right|$$

$$\le \frac{n!}{2\pi} \frac{M}{r^{n+1}} \cdot 2\pi r$$

$$= \frac{n!\, M}{r^n}$$

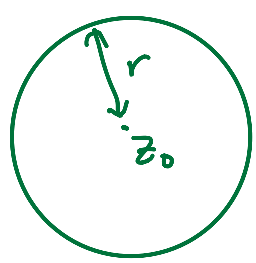
Liouville theorem

A complex analytic that is bounded on the whole complex plane, then it must be a constant function.

$\sin(z)$ can be very large if $\operatorname{Im}(z) \gg 0$.

Proof $\quad$ Suppose $f(z) \le M \quad \forall z \in \mathbb{C}$

$$|f'(z_0)| \le \frac{M}{r}$$



Take $r \to \infty$, $\quad f'(z_0) = 0 \quad \forall z_0 \in \mathbb{C}$

$\Rightarrow f$ is a constant

Theorem Fundamental thm of algebra

Suppose $p(z) = c_0 + c_1 z + c_2 z^2 + \dots + c_d z^d$.
is a polynomial of degree d. $d \geq 1$.

Then $p(z)$ has a root in $\mathbb{C}$.

Proof Suppose $p(z) \neq 0$ for all $z \in \mathbb{C}$.

$\frac{1}{p(z)}$ is well-defined an analytic, for all $z \in \mathbb{C}$

WLOG $c_d = 1$.

$$p(z) = c_0 + c_1 z + \dots + z^d.$$

$$\frac{p(z)}{z^d} = \frac{c_0}{z^d} + \frac{c_1}{z^{d+1}} + \dots + 1$$

$$\frac{p(z)}{z^d} \to 1 \quad \text{as} \quad |z| \to \infty$$

$$\exists N > 0 \quad \text{s.t.} \quad \left|\frac{p(z)}{z^d}\right| > \frac{1}{2} \quad \forall |z| > N$$

$$\left|\frac{1}{p(z)}\right| < \frac{2}{|z^d|} < \frac{2}{N^d} \quad \forall |z| > N$$

$\{z \in \mathbb{C} : |z| \leq N\}$ is compact

$\frac{1}{p(z)}$ is continuous in the closed disc $\{z \in \mathbb{C} : |z| \leq N\}$

$\Rightarrow \left|\frac{1}{p(z)}\right| \leq M \quad \forall |z| \leq N.$

$\therefore \quad \frac{1}{p(z)}$, by Liouville theorem, is constant

Then $p(z)$ is constant.

This contradicts the assumption $d \geq 1$. $\boxed{}$

Gauss (1799)

(1777 - 1855)

Solution of polynomial using $+, -, \times, \div, \sqrt[n]{\ }$

<u>Theorem</u> Polynomial deg ≥ 5 in general cannot be solved using radicals.

Abel (1802 - 1829) Schubert (1797 - 1828)

<u>Def</u> A point z_0 is called a <u>zero</u> of f if $f(z_0) = 0$

Suppose f is analytic in domain D, $z_0 \in D$.

$f(z_0) = 0$

$$f(z) = c_1 (z - z_0) + c_2 (z - z_0)^2 + \dots + c_n (z - z_0)^n + \dots$$

for $z \in D(z_0 ; r)$

(1) $c_k = 0 \quad \forall k = 1, 2, 3 \dots$

$$\Rightarrow f(z) \equiv 0 \quad \text{for} \quad z \in D(z_0 ; r)$$

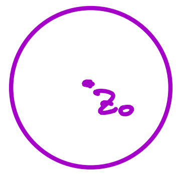
② The coefficients are not all zero.

The smallest m s.t. $c_m \neq 0$ is called the order of z_0

$$c_0 = c_1 = c_2 = \cdots c_{m-1} = 0 \; . \quad c_m \neq 0 \; .$$

z_0 is called a zero of order m.

$$f(z) = (z-z_0)^m \Big[\underbrace{c_m + c_{m+1}(z-z_0) + \cdots\cdots}_{\text{not zero for } 0 < |z-z_0| < \varepsilon} \Big]$$



isolated zero

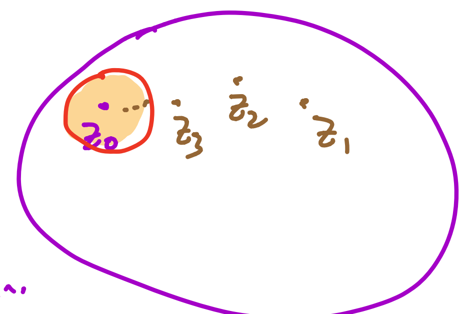
Theorem Suppose f is analytic in a domain D and $\{z_1, z_2, z_3 \cdots\} \subseteq D$ is a set of zeros.

If $\{z_1, z_2, z_3 \ldots\}$ has cluster point in D then f is identically equal to 0 in D.

Proof Let $z_0 \in D$ is a cluster point of $\{z_1, z_2 \ldots\}$

z_0 is not an isolated zero

z_0 cannot have finite order



D

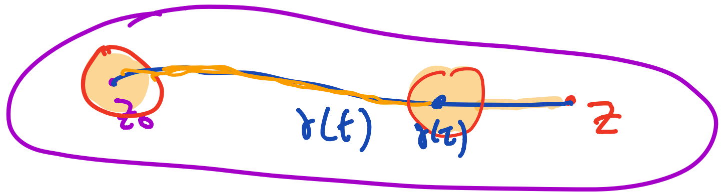
$$f(z) = c_0 + c_1(z-z_0) + \cdots + c_n(z-z_0)^n + \cdots$$

$$c_k = 0 \quad \forall k$$

$\Rightarrow f(z) \equiv 0$ in the circle in red color.

$\gamma(0) = z_0$

$\gamma(1) = z$



$\gamma(t) = 0$ for $0 \le t \le \varepsilon$

$f(\gamma(t))$ is continuous

Let $\tau = \sup \{ t : 0 \le t \le 1, \ f(\gamma(t)) = 0 \}$

$\Rightarrow \tau > 0$

If $\tau < 1$,

$\gamma(\tau)$ is not an isolated zero.

τ cannot be the sup.

$f(z) = 0 \quad \forall z \in D$. $\blacksquare$

<u>Theorem</u> (Identity theorem)

If f, g analytic in D

and $\{ z : f(z) = g(z) \}$ has a cluster point

then $f = g$ in D.

Proof: Take $f - g$ in the previous theorem.