Solution to 2019 Final Examination

Question 1 [15 marks]

(a) Since X_1 and X_2 are independent and continuous,

$$P_1 = \Pr\left(X_1 > 0, |X_1| < |X_2|, X_2 < 0\right) = \Pr\left(0 < X_1 < -X_2\right) = \int_0^\infty F_2(-x) f_1(x)dx,$$

$$P_2 = \Pr\left(X_1 < 0, |X_1| > |X_2|, X_2 > 0\right) = \Pr\left(0 < X_2 < -X_1\right) = \int_0^\infty F_1(-x) f_2(x)dx,$$

where $F_1(x)$ and $F_2(x)$ are the cdf's of X_1 and X_2 respectively. Calculate

$$F_1(-x) = \int_{-\infty}^{-x} f_1(t)dt = \int_{-1}^{-x} (1+t)dt = 0.5(1+t)^2\Big|_{-1}^{-x} = 0.5(1-x)^2 \quad \text{for } -1 \le -x \le 0,$$

$$F_2(-x) = \int_{-\infty}^{-x} f_2(t)dt = \int_{-\infty}^{-x} e^{2t}dt = 0.5e^{2t}\Big|_{-\infty}^{-x} = 0.5e^{-2x} \quad \text{for } -x \le 0$$

It follows that

$$P_1 = \int_0^\infty F_2(-x) f_1(x)dx = \int_0^\infty 0.5e^{-2x}e^{-2x}dx = 0.5\int_0^\infty e^{-4x}dx = \frac{0.5}{4} = 0.125$$

$$P_2 = \int_0^\infty F_1(-x) f_2(x)dx = \int_0^1 0.5(1-x)^2(1-x)dx = 0.5\int_0^1 (1-x)^3 dx = \frac{0.5}{4} = 0.125$$

(b) Since $\{T^+ = 0\} = \{X_1 < 0, X_2 < 0\} = \{S = 0\}$, $\{T^+ = 3\} = \{X_1 > 0, X_2 > 0\} = \{S = 3\}$, and $T^+ \in \{0,1,2,3\}$, it suffices to prove $\Pr(T^+ = 1) = \Pr(S = 1)$. The assumption $X_2 \sim -X_1$ implies $f_2(-x) = f_1(x)$ and $F_2(-x) = 1 - F_1(x)$ for all $x \in \mathbb{R}$. Hence

$$P_1 = \int_0^\infty F_2(-x) f_1(x)dx = \int_0^\infty \left[1 - F_1(x)\right] f_1(x)dx = -\frac{1}{2}\left[1 - F_1(x)\right]^2\Big|_0^\infty$$

$$= 0.5\left[1 - F_1(0)\right]^2 = 0.5\Pr\left(X_1 > 0\right)^2 = 0.5(0.5)^2 = 0.125$$

Similarly, $f_1(-x) = f_2(x)$ and $F_1(-x) = 1 - F_2(x)$ imply $P_2 = 0.125$. It follows that

$$\Pr\left(T^+ = 1\right) = P_1 + P_2 = 0.25 = \Pr\left(X_1 > 0\right)\Pr\left(X_2 < 0\right) = \Pr\left(X_1 > 0, X_2 < 0\right) = \Pr(S = 1)$$

(c) Part (b) indicates that $T^+ \sim S$ is possible without symmetric distributions of X_1 and X_2, and part (a) demonstrates such an example with $F_1(0) = F_2(0) = 0.5$ and

$$f_2(-x) = e^{-2x} I_{\{-x<0\}} + (1+x) I_{\{0 \le -x \le 1\}} = (1+x) I_{\{-1 \le x \le 0\}} + e^{-2x} I_{\{x>0\}} = f_1(x)$$

Question 2 [15 marks]

(a) The ranks for W and scores for C are listed below with $m=11$, $n=10$, $N=21$:

Sample	Y	Y	X	X	X	X	X	X	X	X	
Rank	1	2	3	4	5	6	7	8	9	10	
Score	1	2	3	4	5	6	7	8	9	10	
Sample	X	Y	Y	Y	Y	Y	Y	Y	Y	X	X
Rank	11	12	13	14	15	16	17	18	19	20	21
Score	11	10	9	8	7	6	5	4	3	2	1

Calculate $W=1+2+12+13+\cdots+19=127$, $C=1+2+10+9+\cdots+3=55$,

$$\mathrm{E}_0[W]=\frac{n(N+1)}{2}=\frac{10(22)}{2}=110,\quad \mathrm{Var}_0(W)=\frac{11(10)(22)}{12}=201.67,$$

$$\mathrm{E}_0[C]=\frac{10(22)^2}{4(21)}=57.62 \text{ and } \mathrm{Var}_0(C)=\frac{11(10)(22)(21^2+3)}{48(21^2)}=50.76$$

Then the Lepage test statistic is

$$D=\left(W^*\right)^2+\left(C^*\right)^2=\frac{(127-110)^2}{201.67}+\frac{(55-57.62)^2}{50.76}=1.568<1.833=\chi^2_{2,0.4}$$

Thus H_0 is accepted at the 40% level of significance, which shows little evidence for the two samples to have different location and/or dispersion parameters.

(b) The two-sample Kolmogorov-Smirnov test statistic is

$$J=\frac{mn}{d}D=\frac{11(10)}{1}(0.6182)=68 \Rightarrow J^*=\frac{J}{\sqrt{mnN}}=\frac{68}{\sqrt{11(10)(21)}}=1.4148$$

Thus the approximate p-value of the Kolmogorov-Smirnov test is

$$\Pr\left(J^*\geq 1.4148\right)\approx 2e^{-2\times 1.4148^2}=2e^{-4.0035}=0.0365<0.05$$

This shows sufficient evidence at the 5% level of significance that the two samples have different distributions.

(c) Comments on the results of parts (a) and (b):

1) Part (a) indicates that the two samples may have similar medians and variances, but part (b) shows their distributions are significantly different. Thus there is sufficient evidence for general differences between the distributions of the two samples, but not in their location/dispersion parameters.

2) If the location-scale parameter model were correct, then the results of part (a) would imply that the two samples should have the same or similar distributions. But that would contradict the result of part (b), which is valid without additional conditions. Therefore the location-scale parameter model is not appropriate.

Question 3 [20 marks]

(a) $(n_1,\ldots,n_5)=(6,5,7,4,6) \Rightarrow N=28$. The Jonckheere-Terpstra test statistic is

$$J=U_{12}+U_{13}+U_{14}+U_{15}+U_{23}+U_{24}+U_{25}+U_{34}+U_{35}+U_{45}$$
$$=26+36+20+32+28+11+18+16+16+10=213$$

$$\mathrm{E}_0[J]=\frac{N^2-n_1^2-\cdots-n_5^2}{4}=\frac{28^2-6^2-5^2-7^2-4^2-6^2}{4}=\frac{622}{4}=155.5 \quad \text{and}$$

$$\mathrm{Var}_0(J)=\frac{28^2(56+3)-6^2(12+3)-\cdots-6^2(12+3)}{72}=\frac{7307}{12}=608.92 \Rightarrow$$

$$J^*=\frac{213-155.5}{\sqrt{608.92}}=2.330>2.326=z_{0.01} \Rightarrow \text{Reject } H_0 \text{ at } \alpha=0.01$$

Thus there is very strong evidence for $\tau_1 \le \tau_2 \le \tau_3 \le \tau_4 \le \tau_5$ at the 1% level.

(b) First calculate $U_{vu}=n_u n_v-U_{uv}$ for $1\le u<v\le 5$ to obtain U_{uv} for all $u\ne v$:

		v				
	u	1	2	3	4	5
	1	–	26	36	20	32
	2	4	–	28	11	18
U_{uv}	3	6	7	–	16	16
	4	4	9	12	–	10
	5	4	12	26	14	–

Then for known peak $p=3$, the Mack-Wolfe statistic is calculated by

$$A_3=U_{12}+U_{13}+U_{23}+U_{43}+U_{53}+U_{54}=26+36+28+12+26+14=142$$

and $(n_1,\ldots,n_5)=(6,5,7,4,6) \Rightarrow N_1=6+5+7=18,\ N_2=7+4+6=17 \Rightarrow$

$$\mathrm{E}_0[A_3]=\frac{N_1^2+N_2^2-n_1^2-\cdots-n_5^2-n_3^2}{4}=\frac{18^2+17^2-6^2-\cdots-6^2-7^2}{4}=100.5$$

$$\mathrm{Var}_0(A_3)=\frac{2(18^3+17^3)+3(18^2+17^2)-6^2(12+3)-\cdots-6^2(12+3)-7^2(14+3)}{72}$$
$$+\frac{7(18)(17)-7^2(28)}{6}=\frac{20082}{72}+\frac{770}{6}=\frac{1629}{4}=407.25$$

$$\Rightarrow A_3^*=\frac{142-100.5}{\sqrt{407.25}}=2.056>1.96=z_{0.025} \Rightarrow \text{Reject } H_0 \text{ at } \alpha=0.025$$

Thus there is strong evidence for $\tau_1 \le \tau_2 \le \tau_3 \ge \tau_4 \ge \tau_5$ at the 2.5% level.

(c) From the values of U_{uv} for all $u \neq v$ in part (b), calculate

$U_{\bullet 1} = U_{21} + \cdots + U_{51} = 4+6+4+4 = 18$, $U_{\bullet 2} = 26+7+9+12 = 54$,

$U_{\bullet 3} = 36+28+12+26 = 102$, $U_{\bullet 4} = 20+11+16+14 = 61$, $U_{\bullet 5} = 32+18+16+10 = 76$

Then by the formulae for $\mathrm{E}_0[U_{\bullet q}]$ and $\mathrm{Var}_0(U_{\bullet q})$,

$$\mathrm{E}_0[U_{\bullet 1}] = \frac{6(28-6)}{2} = 66, \quad \mathrm{Var}_0(U_{\bullet 1}) = \frac{6(22)(29)}{12} = 319 \;\Rightarrow\; U_{\bullet 1}^* = \frac{18-66}{\sqrt{319}} = -2.69$$

Similarly, $U_{\bullet 2}^* = -0.21$, $U_{\bullet 3}^* = 1.51$, $U_{\bullet 4}^* = 0.85$, $U_{\bullet 5}^* = 0.76$. Thus $U_{\bullet 3}^* = 1.51 > U_{\bullet q}^*$ for $q = 1,2,4,5 \Rightarrow \hat{p} = 3$. Consequently,

$$A_{\hat{p}}^* = A_3^* = 2.056 > 1.957 = a_{\hat{p},0.0971}^* > a_{\hat{p},0.1}^* \quad \text{with} \quad \alpha = 0.10$$

by the results in part (b) and the output of R. This shows moderate evidence for umbrella alternatives with unknown peak at the 10% level.

(d) Part (a) shows that the achieved level of significance for ordered alternatives is just below 1%. Part (b) shows the achieved level for umbrella alternatives with $p = 3$ is between 1% and 2.5% (closer to 2.5%), while part (c) shows the level close to 10%. So according to the level of significance, the data provide stronger evidence for ordered alternatives than umbrella alternatives.

The test results in parts (a) – (c) are consistent, not contradictive, since ordered alternatives are special cases of umbrella alternatives (with $p = k$). In this problem, it is possible that $\tau_1 < \tau_2 < \tau_3 = \tau_4 = \tau_5$, which are consistent with both ordered and umbrella alternatives. The tests suggest that strict inequalities are likely to hold in treatments 1 – 3, but not in treatments 3 – 5.

Question 4 [15 marks]

(a) By the formulae of $\mathrm{E}[r_{ij}]$ and $\mathrm{Var}(r_{ij})$, and the independence of $\{r_{ij}\}$ over i,

$$\mathrm{E}[R_j] = \sum_{i=1}^{n} \mathrm{E}[r_{ij}] = \frac{p(s+1)}{2} \quad \text{and} \quad \mathrm{Var}(R_j) = \sum_{i=1}^{n} \mathrm{Var}(r_{ij}) = \frac{p(s+1)(s-1)}{12}$$

These together with $p(s-1) = \lambda(k-1)$ for BIBD imply

$$\mathrm{E}[D] = \frac{12}{\lambda k(s+1)} \sum_{j=1}^{k} \mathrm{E}\left[\left(R_j - \frac{p(s+1)}{2}\right)^2\right] = \frac{12}{\lambda k(s+1)} \sum_{j=1}^{k} \mathrm{E}\left[\left(R_j - \mathrm{E}[R_j]\right)^2\right]$$

$$= \frac{12}{\lambda k(s+1)} \sum_{j=1}^{k} \mathrm{Var}(R_j) = \frac{12}{\lambda k(s+1)} \frac{kp(s+1)(s-1)}{12} = \frac{p(s-1)}{\lambda} = k-1$$

(b) This block design is BIBD with $k = n = 5$, $s = p = 4$ and $\lambda = 3$. The in-block ranks $\{r_{ij}\}$ and $R_1, \ldots, R_5$ are shown in the table below:

Block	Treatment j				
i	1	2	3	4	5
1	3	1	2	0	4
2	2	0	1	4	3
3	3	1	2	4	0
4	0	2	1	3	4
5	2	1	0	4	3
R_j	10	5	6	15	14

Then the Durbin-Skillings-Mack test statistic D is calculated by

$$D = 12\frac{10^2 + 5^2 + 6^2 + 15^2 + 14^2}{3(5)(4+1)} - \frac{3(4+1)4^2}{3} = 13.12 \in (11.14, 13.28)$$

Thus $\chi^2_{4,0.025} = 11.14 < D < 13.28 = \chi^2_{4,0.01}$, which shows strong evidence (at the level close to 1%) that the treatment effects $\tau_1, \ldots, \tau_5$ are different.

(c) The Skillings-Mack multiple two-sided all-treatment comparison procedure decides $R_u \neq R_v$ for $u < v$ at $\alpha = 0.1$ if

$$|R_u - R_v| \geq q_{0.1}\sqrt{\frac{(s+1)(ps - p + \lambda)}{12}} = 3.479\sqrt{\frac{5(16-4+3)}{12}} = 8.70$$

Therefore, $|R_2 - R_4| = 15 - 5 = 10 > 8.70 \Rightarrow \tau_2 \neq \tau_4$, $|R_2 - R_5| = 9 > 8.70 \Rightarrow \tau_2 \neq \tau_5$, $|R_3 - R_4| = 9 > 8.70 \Rightarrow \tau_3 \neq \tau_4$, and $|R_u - R_v| \leq 8 < 8.70 \Rightarrow \tau_u = \tau_v$ for all other pairs $(u,v) = (1,2)$, (1,3), (1,4), (1,5), (2,3), (3,5), (4,5) with $u < v$.

Question 5 [20 marks]

(a) Because of the ties in $(X_1,\dots,X_n)$, it is more convenient to rearrange the ranks $(S_1,\dots,S_n)$ of $(Y_1,\dots,Y_n)$ in increasing order. The ranks $(R_1,\dots,R_n)$ of $(X_1,\dots,X_n)$ are then given as follows:

S_i	1	2	3	4	5	6	7	8	9	10
R_i	2	1	4	4	7	8.5	6	4	10	8.5

The number of pairs $\{u<v: R_u<R_v\}$ is $8+8+5+5+3+1+2+2+0=34$, out of total $10(9)/2=45$ pairs, and the number of pairs $\{u<v: R_u=R_v\}$ is $3+1=4$.

Thus $K=34-(45-34-4)=34-7=27$ and

$$g=2,\ \begin{Bmatrix} t_1=3 \\ t_2=2 \end{Bmatrix} \Rightarrow \mathrm{Var}_0(K)=\frac{10(9)(25)-3(2)(11)-2(1)(9)}{18}=\frac{361}{3}=120.33$$

$$\Rightarrow K^*=\frac{K}{\sqrt{\mathrm{Var}_0(K)}}=\frac{27}{\sqrt{120.33}}=2.461>2.236=z_{0.01}$$

Thus H_0 is rejected in favor of $\tau>0$ with very strong evidence at the 1% level.

(b) τ is estimated by $\hat{\tau}=2K/n(n-1)=2(27)/90=0.6$. Calculate $C_{10}=7-1=6 \Rightarrow$

$$\sum_{i=1}^{n}\left(C_i-\bar{C}\right)^2=\sum_{i=1}^{n}C_i^2-\frac{4}{n}K^2=5\times 7^2+5^2+4^2+3^2+1+6^2-\frac{4}{10}\times 27^2=40.4 \Rightarrow$$

$$\hat{\sigma}^2=\frac{2}{10(10-1)}\left[\frac{2(10-2)}{10(10-1)^2}(40.4)+1-0.6^2\right]=0.03196$$

An approximate 95% confidence interval of τ is then given by

$$\hat{\tau}\pm z_{0.025}\hat{\sigma}=0.6\pm 1.96\sqrt{0.03196}=0.6\pm 0.3504=(0.2496, 0.9504)$$

(c) Calculate $A=3(9-1)+2(4-1)=24+6=30$ and

$$\sum_{i=1}^{n}(R_i-S_i)^2=1^2+1^2+1^2+0+2^2+1.5^2+1^2+4^2+1^2+1.5^2=33.5 \Rightarrow$$

$$r_s=\frac{1}{\sqrt{10(99)-30}\sqrt{10(99)}}\left[10(99)-6(33.5)-\frac{30}{2}\right]=\frac{774}{\sqrt{950400}}=0.7939$$

and hence $r_s^*=r_s\sqrt{n-1}=0.7939\sqrt{10-1}=2.382>2.236=z_{0.01}$. Thus H_0 is again rejected in favor of $r>0$ with very strong evidence at the 1% level. Both Kendall and Spearman tests strongly support a positive correlation between (X_i, Y_i).

(d) Since $\mathrm{E}[Q_{is}Q_{it}] = 4\delta - 1 - 2\tau$ for $1 \le s \ne t \ne i \le n$ and there are $(n-1)(n-2)$ pairs (s,t) such that $1 \le s \ne t \ne i \le n$ for each $i \in \{1,\ldots,n\}$,

$$\mathrm{E}[T] = \sum_{i=1}^{n} \sum_{1 \le s \ne t \ne i \le n} \mathrm{E}[Q_{is}Q_{it}] = n(n-1)(n-2)(4\delta - 1 - 2\tau)$$

It is also known that $\mathrm{E}[K] = n(n-1)\tau/2$. Take

$$\hat{\delta} = \frac{T}{4n(n-1)(n-2)} + \frac{K}{n(n-1)} + \frac{1}{4}$$

Then

$$\mathrm{E}[\hat{\delta}] = \frac{\mathrm{E}[T]}{4n(n-1)(n-2)} + \frac{\mathrm{E}[K]}{n(n-1)} + \frac{1}{4} = \frac{4\delta - 1 - 2\tau}{4} + \frac{\tau}{2} + \frac{1}{4} = \delta$$

Thus $\hat{\delta}$ is an unbiased estimator of δ based on T and K.

Question 6 [15 marks]

(a) The ordered values $S_{(1)} \le S_{(2)} \le \cdots \le S_{(N)}$ of $S_{ij} = (Y_j - Y_i)/(x_j - x_i)$, $1 \le i < j \le 7$, are calculated and shown below:

i	1	2	3	4	5	6	7	8	9	10	11
$S_{(i)}$	–2.60	–2.10	–1.60	–1.17	–1.15	–1.10	–1.00	–0.97	–0.95	–0.86	–0.68
i	12	13	14	15	16	17	18	19	20	21	
$S_{(i)}$	–0.66	–0.55	–0.50	–0.40	–0.20	–0.15	–0.03	0.30	0.70	1.30	

where $N = n(n-1)/2 = 7(6)/2 = 21$. Thus an estimate of the slope β related to the Theil test is $\hat{\beta} = S_{(11)} = -0.68$.

Next, calculate $A_i = Y_i - \hat{\beta} x_i$, $i = 1,\ldots,7$, and order $A_1,\ldots,A_7$ to $A_{(1)} \le \cdots \le A_{(7)}$:

i	1	2	3	4	5	6	7
$A_{(i)}$	6.32	7.18	7.28	7.70	8.24	8.26	9.16

Then an estimate of the intercept α is $\hat{\alpha} = A_{(4)} = 7.70$.

(b) Take $\alpha = 2(0.0345) = 0.069$. Then $1 - \alpha = 0.931 = 93.1\%$ and $k_{0.0345} = 0.619 \Rightarrow$ $C_\alpha = Nk_{\alpha/2} - 2 = Nk_{0.0345} - 2 = 21(0.619) - 2 = 13 - 2 = 11 \Rightarrow$

$$M = \frac{N - C_\alpha}{2} = \frac{21-11}{2} = 5 \quad \text{and} \quad Q = \frac{N + C_\alpha}{2} = \frac{21+11}{2} = 16$$

Thus an exact 93.1% confidence interval of β is given by

$$\left(S_{(M)}, S_{(Q+1)}\right) = \left(S_{(5)}, S_{(17)}\right) = (-1.15, -0.15)$$

(c) The results of parts (a) and (b) suggest $\beta < 0$. Hence we test $H_0 : \beta = 0$ against $H_1 : \beta < 0$. Under H_0, $D_i = Y_i - \beta_0 x_i = Y_i$, $i = 1,\ldots,7$. Therefore, $D_j - D_i = Y_j - Y_i$ and so the number of pairs (i, j) such that $D_j - D_i > 0$ is the same as the number of $\{i : S_{(i)} > 0\}$, which is 3 by the result in part (a). It follows that the Theil test statistic is $C = 3 - (21 - 3) = -15$. Calculate

$$C^* = \frac{C}{\sqrt{\mathrm{Var}_0(C)}} = \frac{-15}{\sqrt{7(6)(14+5)/18}} = -2.2528 < -1.96 = -z_{0.025}$$

This shows that by the approximate rejection rule, there is sufficient evidence at the 2.5% level to support a negative slope $\beta < 0$ for the regression line.