# Chapter 5. Limits and Continuity *

## 1 Discussion: Examples of Dirichlet and Thomae

Given a function $f$ with domain $A \subset \mathbb{R}$, we want to define continuity at a point $c \in A$ to mean that if $x \in A$ is chosen near $c$, then $f(x)$ will be near $f(c)$. Symbolically, we will say $f$ is continuous at $c$ if

$$\lim_{x \to c} f(x) = f(c).$$

The problem is that, at present, we only have a definition for the limit of a sequence, and it is not entirely clear what is meant by $\lim_{x \to c} f(x)$. The subtleties that arise as we try to fashion such a definition are well-illustrated via a family of examples, all based on an idea of the prominent German mathematician, Peter Lejeune Dirichlet. Dirichlet's idea was to define a function $g$ in a piecewise manner based on whether or not the input variable $x$ is rational or irrational. Specifically, let

$$g(x) = \begin{cases} 1 & \text{if} \quad x \in \mathbb{Q}, \\ 0 & \text{if} \quad x \notin \mathbb{Q}. \end{cases}$$

Does it make sense to attach a value to the expression $\lim_{x \to 1} g(x)$. One idea is to consider a sequence $\{x_n\} \to 1$. Using our notion of the limit of a sequence, we might try to define $\lim_{x \to 1} g(x)$ as simply the limit of the sequence $g(x_n)$. But notice that this limit depends on how the sequence $\{x_n\}$ is chosen. If each $x_n$ is rational, then

$$\lim_{n \to \infty} g(x_n) = 1.$$

On the other hand, if $\{x_n\}$ is irrational for each $n$, then

$$\lim_{n \to \infty} g(x_n) = 0.$$

*Lecture notes for CUHKSZ course MAT2006: Elementary Real Analysis.

This unacceptable situation demands that we work harder on our definition of functional limits. Generally speaking, we want the value of $\lim_{x\to c} g(x)$ to be independent of how we approach $c$. In this particular case, the definition of a functional limit that we agree on should lead to the conclusion that

$$\lim_{x\to 1} g(x) \quad \text{doest not exist.}$$

Postponing the search for formal definitions for the moment, we should nonetheless realize that Dirichlet's function is not continuous at $c = 1$. In fact, the real significance of this function is that there is nothing unique about the point $c = 1$. Because both $\mathbb{Q}$ and $\mathbb{I}$ (the set of irrationals) are dense in the real line, it follows that for any $z \in \mathbb{R}$ we can find sequences $\{x_n\} \subset \mathbb{Q}$ and $\{y_n\} \subset \mathbb{I}$ such that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = z.$$

Because

$$\lim_{n\to\infty} g(x_n) \neq \lim_{n\to\infty} g(y_n),$$

the same line of reasoning reveals that $g(x)$ is not continuous at $z$. In the jargon of analysis, Dirichlet's function is a *nowhere-continuous* function on $\mathbb{R}$.

What happens if we adjust the definition of $g(x)$ in the following way? Define a new function $h$ on $\mathbb{R}$ by setting

$$h(x) = \begin{cases} x & \text{if} \quad x \in \mathbb{Q}, \\ 0 & \text{if} \quad x \notin \mathbb{Q}. \end{cases}$$

If we take $c$ different from zero, then just as before we can construct sequences $\{x_n\} \to c$ of rationals and $\{y_n\} \to c$ of irrationals so that

$$\lim_{n\to\infty} h(x_n) = c \quad \text{and} \quad \lim_{n\to\infty} h(y_n) = 0.$$

Thus, $h$ is not continuous at every point $c \neq 0$.

If $c = 0$, however, then these two limits are both equal to $h(0) = 0$. In fact, it appears as though no matter how we construct a sequence $\{z_n\}$ converging to zero, it will always be the case that $\lim_{n\to\infty} h(z_n) = 0$. This observation goes to the heart of what we want functional limits to entail. To assert that

$$\lim_{x\to c} f(x) = L$$

should imply that

$$h(x_n) \to L \qquad \text{for all sequences} \quad \{x_n\} \to c.$$

For reasons not yet apparent, it is beneficial to fashion the definition for functional limits in terms of neighborhoods constructed around $c$ and $L$. We will quickly see, however, that this topological formulation is equivalent to the sequential characterization we have arrived at here.

To this point, we have been discussing continuity of a function at a particular point in its domain. This is a significant departure from thinking of continuous functions as curves that can be drawn without lifting the pen from the paper, and it leads to some fascinating questions. In 1875, K.J. Thomae discovered the function

$$t(x) = \begin{cases} 1 & \text{if} \quad x = 0, \\ 1/n & \text{if} \quad x = m/n \in \mathbb{Q} \setminus \{0\} \text{ is in lowest terms with } n > 0, \\ 0 & \text{if} \quad x \notin \mathbb{Q}. \end{cases}$$

If $c \in \mathbb{Q}$, then $t(c) > 0$. Because the set of irrationals is dense in $\mathbb{R}$, we can find a sequence $\{y_n\}$ in $\mathbb{I}$ converging to $c$. The result is that

$$\lim_{n \to \infty} t(y_n) = 0 \neq t(c),$$

and Thomae's function fails to be continuous at any rational point.

The twist comes when we try this argument on some irrational point in the domain such as $c = \sqrt{2}$. All irrational values get mapped to zero by $t$, so the natural thing would be to consider a sequence $\{x_n\}$of rational numbers that converges to 2. The closer a rational number is chosen to a fixed irrational number, the larger its denominator must necessarily be. As a consequence, Thomae's function has the bizarre property of being continuous at every irrational point on $\mathbb{R}$ and discontinuous at every rational point.

**Exercise 1.** Assume $\{x_n\} \subset \mathbb{Q}$ and $\{x_n\} \to \sqrt{2}$. Show that $\lim_{n \to \infty} t(x_n) = 0$.

Is there an example of a function with the opposite property? In other words, does there exist a function defined on all of $\mathbb{R}$ that is continuous on $\mathbb{Q}$ but fails to be continuous on $\mathbb{I}$? Can the set of discontinuities of a particular function be arbitrary? If we are given some set $A \subset \mathbb{R}$, is it always possible to find a function that is continuous only on the set $A^c$? In each of the examples in this section, the functions were defined to have erratic oscillations around points in the domain. What conclusions can we draw if we restrict our attention to functions that are somewhat less volatile? One such class is the set of so-called monotone functions, which are either increasing or decreasing on a given domain. What might we be able to say about the set of discontinuities of a monotone function on $\mathbb{R}$?

# 2 Limits of Functions

Consider a function $f : A \to \mathbb{R}$. Recall that a limit point $c$ of $A$ is a point with the property that every $\epsilon$-neighborhood $V_\epsilon(c)$ intersects $A$ at some point other than $c$. Equivalently, $c$ is a limit point of $A$ if and only if $c = \lim_{n\to\infty} x_n$ for some sequence $\{x_n\} \subset A$ with $x_n \neq c$. It is important to remember that limit points of $A$ do not necessarily belong to the set $A$ unless $A$ is closed.

If $c$ is a limit point of the domain of $f$, then, intuitively, the statement

$$\lim_{x\to c} f(x) = L$$

is intended to convey that values of $f(x)$ get arbitrarily close to $L$ as $x$ is chosen closer and closer to $c$. The issue of what happens when $x = c$ is irrelevant from the point of view of functional limits. In fact, $c$ need not even be in the domain of $f$.

The structure of the definition of functional limits follows the "challenge-response" pattern established in the definition for the limit of a sequence. Recall that given a sequence $\{a_n\}$, the assertion $\lim_{n\to\infty} a_n = L$ implies that for every $\epsilon$-neighborhood $V_\epsilon(L)$ centered at $L$, there is a point in the sequence – call it $a_N$ – after which all of the terms an fall in $V_\epsilon(L)$. Each $\epsilon$-neighborhood represents a particular challenge, and each $N$ is the respective response. For functional limit statements such as $\lim_{x\to c} f(x) = L$, the challenges are still made in the form of an arbitrary $\epsilon$-neighborhood around $L$, but the response this time is a $\delta$-neighborhood centered at $c$.

**Definition 1** (Limit of a Function)**.** Let $f : A \to \mathbb{R}$, and let $c$ be a limit point of the domain $A$. We say that $\lim_{x\to c} f(x) = L$ provided that, for all $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $0 < |x - c| < \delta$ (and $x \in A$) it follows that $|f(x) - L| < \epsilon$.

**Definition 2** (Limit of a Function: Topological Version)**.** Let $f : A \to \mathbb{R}$, and let $c$ be a limit point of the domain $A$. We say that $\lim_{x\to c} f(x) = L$ provided that, for every $\epsilon$-neighborhood $V_\epsilon(L)$ of $L$, there exists a $\delta$-neighborhood $V_\delta^0(c)$ around $c$ with the property that for all $x \in V_\delta(c)$ different from $c$ (with $x \in A$) it follows that $f(x) \in V_\epsilon(L)$.

The parenthetical reminder "$(x \in A)$" present in both versions of the definition is included to ensure that $x$ is an allowable input for the function in question. When no confusion is likely, we may omit this reminder with the understanding that the appearance of $f(x)$ carries with it the implicit assumption that $x$ is in the domain of $f$. On a related note, there is no reason to discuss functional limits at isolated points of the domain. Thus, functional limits will only be considered as $x$ tends toward a limit point of the function's domain.

**Example 2.1.** Show that $\lim_{x\to 2} g(x) = 4$, where $g(x) = x^2$.

For any $\epsilon > 0$, choose $\delta = \min\{1, \frac{\epsilon}{5}\}$.

**Sequential Criterion for Functional Limits**

**Theorem 1** (Sequential Criterion for Functional Limits)**.** *Given a function $f : A \to \mathbb{R}$ and a limit point $c$ of $A$, the following two statements are equivalent:*

(i) $\lim_{x\to c} f(x) = L$;

(ii) *For all sequences $\{x_n\} \subset A$ satisfying $x_n \neq c$ and $\{x_n\} \to c$, it follows that $f(x_n) \to L$.*

*Proof.* ($\Rightarrow$) Let's first assume that $\lim_{x\to c} f(x) = L$. To prove (ii), we consider an arbitrary sequence $\{x_n\}$, which converges to $c$ and satisfies $x_n \neq c$. Our goal is to show that the image sequence $f(x_n)$ converges to $L$. This is most easily seen using the topological formulation of the definition.

Let $\epsilon > 0$. Because we are assuming (i), which implies that there exists $V_\delta(c)$ with the property that all $x \in V_\delta(c)$ different from $c$ satisfy $f(x) \in V_\delta(L)$. All we need to do then is argue that our particular sequence $\{x_n\}$ is eventually in $V_\delta(c)$. But we are assuming that $\{x_n\} \to c$. This implies that there exists a point $x_N$ after which $x_n \in V_\delta(c)$. It follows that $n \geq N$ implies $f(x_n) \in V_\epsilon(L)$, as desired.

($\Rightarrow$) We shall prove this direction by contradiction. Thus, we assume that statement (ii) is true, and (i) is not true. To say that

$$\lim_{x\to c} f(x) \neq L$$

means that there exists at least one particular $\epsilon_0 > 0$ for which no $\delta$ is a suitable response. In other words, no matter what $\delta > 0$ we try, there will always be at least one point

$$x \in V_\delta(c) \quad \text{with } x \neq c \quad \text{for which} \quad f(x) \notin V_{\epsilon_0}(L).$$

Now consider $\delta_n = 1/n$. From the preceding discussion, it follows that for each $n \in \mathbb{N}$ we may pick an $x_n \in V_\delta(c)$ with $x_n \neq c$ and $f(x_n) \notin V_{\epsilon_0}(L)$. But now notice that the result of this is a sequence $\{x_n\} \to c$ with $x_n \neq c$, where the image sequence $f(x_n)$ certainly does not converge to $L$.

Because this contradicts (ii), which we are assuming is true for this argument, we may conclude that (i) must also hold. □

The above theorem has several useful corollaries. In addition to the previously advertised benefit of granting us some short proofs of statements about how functional limits interact with algebraic combinations of functions, we also get an economical way of establishing that certain limits do not exist.

**Theorem 2** (Algebraic Limit Theorem for Functional Limits)**.** *Let $f$ and $g$ be functions defined on a domain $A \subset \mathbb{R}$, and assume $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$ for some limit point $c$ of $A$. Then,*

$$\begin{aligned} &\text{(i)} && \lim_{x\to c} kf(x) = kL, \quad \textit{for all } k \in \mathbb{R}, \\ &\text{(ii)} && \lim_{x\to c}[f(x)+g(x)] = L + M, \\ &\text{(iii)} && \lim_{x\to c}[f(x)g(x)] = LM, \\ &\text{(iv)} && \lim_{x\to c}[f(x)/g(x)] = L/M, \quad \textit{provided } M \neq 0. \end{aligned}$$

*Proof.* These follow from Theorem 1 and the Algebraic Limit Theorem for sequences. □

**Corollary 3** (Divergence Criterion for Functional Limits)**.** *Let $f$ be a function defined on $A$, and let $c$ be a limit point of $A$. If there exist two sequences $\{x_n\}$ and $\{y_n\}$ in $A$ with $x_n \neq c$ and $y_n \neq c$ and*

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = c \qquad \textit{but} \qquad \lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} f(y_n),$$

*then the functional limit $\lim_{x\to c} f(x)$ does not exist.*

**Example 2.2.** Assuming the familiar properties of the sine function, let's show that $\lim_{x\to 0} \sin(1/x)$ does not exist.

Take $x_n = 1/(2n\pi)$ and $y_n = 1/(2n\pi + \pi/2)$.

**Exercise 2** (Infinite Limits)**.** *Definition*: $\lim_{x\to c} f(x) = \infty$ means that for all $M > 0$ we can find a $\delta > 0$ such that whenever $0 < |x - c| < \delta$, it follows that $f(x) > M$.

(i) Show $\lim_{x\to 0} 1/x^2 = \infty$ in the sense described in the previous definition.

(ii) Now, construct a definition for the statement $\lim_{x\to\infty} = L$. Show $\lim_{x\to\infty} 1/x = 0$.

(iii) What would a rigorous definition for $\lim_{x\to\infty} = \infty$ look like? Give an example of such a limit.

**Exercise 3** (Right and Left Limits)**.** Introductory calculus courses typically refer to the right-hand limit of a function as the limit obtained by "letting $x$ approach $c$ from the right-hand side."

(i) Give a proper definition in the style of Definition 1 for the right-hand and left-hand limit statements:

$$\lim_{x\to c^+} f(x) = L, \qquad \lim_{x\to c^-} f(x) = M.$$

(ii) Prove that $\lim_{x\to c} f(x) = L$ if and only if both the right and left-hand limits equal $L$.

**Exercise 4** (Squeeze Theorem)**.** Let $f$, $g$, and $h$ satisfy $f(x) \leq g(x) \leq h(x)$ for all $x$ in some common domain $A$. If

$$\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$$

at some limit point $c$ of $A$, show that $\lim_{x \to c} g(x) = L$ as well.

# 3 Continuous Functions

**Definition 3** (Continuity)**.** A function $f : A \to \mathbb{R}$ is continuous at a point $c \in A$ if, for all $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $|x - c| < \delta$ (and $x \in A$) it follows that $|f(x) - f(c)| < \epsilon$.

If $f$ is continuous at every point in the domain $A$, then we say that $f$ is continuous on $A$.

**Theorem 4** (Characterizations of Continuity)**.** *Let $f : A \to \mathbb{R}$, and let $c \in A$. The function $f$ is continuous at $c$ if and only if any one of the following three conditions is met*

(i) *For all $\epsilon > 0$, there exists a $\delta > 0$ such that $|x - c| < \delta$ (and $x \in A$) implies $|f(x) - f(c)| < \epsilon$;*

(ii) *For all $V_\epsilon(f(c))$, there exists a $V_\delta(c)$ with the property that $x \in V_\delta(c)$ (and $x \in A$) implies $f(x) \in V_\epsilon(f(c))$;*

(iii) *If $\{x_n\} \to c$ (with $x_n \in A$), then $f(x_n) \to f(c)$.*

*If $c$ is a limit point of $A$, then the above conditions are equivalent to*

(iv) $\lim_{x \to c} f(x) = f(c)$.

*Proof.* Statement (i) is just Definition 3, and statement (ii) is the standard rewording of (i) using topological neighborhoods in place of the absolute value notation. Statement (iii) is equivalent to (i) via an argument nearly identical to that of Theorem 1, with some slight modifications for when $x_n = c$. Finally, statement (iv) is seen to be equivalent to (i) by considering Definition 3 and observing that the case $x = c$ (which is excluded in the definition of functional limits) leads to the requirement $f(c) \in V_\epsilon(f(c))$, which is trivially true. $\square$

As a general rule, the sequential characterization of continuity is typically the most useful for demonstrating that a function is *not* continuous at some point.

**Corollary 5** (Criterion for Discontinuity). *Let $f : A \to \mathbb{R}$, and let $c \in A$ be a limit point of $A$. If there exists a sequence $\{x_n\} \subset A$ where $\{x_n\} \to c$ but such that $f(x_n)$ does not converge to $f(c)$, we may conclude that $f$ is not continuous at $c$.*

The sequential characterization of continuity is also important for the other reasons that it was important for functional limits. In particular, it allows us to bring our catalog of results about the behavior of sequences to bear on the study of continuous functions. The next theorem should be compared to Theorem 2 as well as to Algebraic Limit Theorem for Sequences.

**Theorem 6** (Algebraic Continuity Theorem). *Assume $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ are continuous at a point $c \in A$. Then,*

(i) *$kf(x)$ is continuous at $c$ for all $k \in \mathbb{R}$;*

(ii) *$f(x) + g(x)$ is continuous at $c$;*

(iii) *$f(x)g(x)$ is continuous at $c$; and*

(iv) *$f(x)/g(x)$ is continuous at $c$, provided the quotient is defined.*

**Example 3.1.** All polynomials are continuous on $\mathbb{R}$. In fact, rational functions (i.e., quotients of polynomials) are continuous wherever they are defined.

**Example 3.2.** In Example 2.2, we saw that the oscillations of $sin(1/x)$ are so rapid near the origin that $\lim_{x\to 0} sin(1/x)$ does not exist. Now, consider the function

$$g(x) = \begin{cases} x\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

To investigate the continuity of $g$ at $c = 0$, we can estimate

$$|g(x) - g(0)| = |x\sin(1/x)| \leq |x|.$$

Given $\epsilon > 0$, set $\delta = \epsilon$, so that whenever $|x - 0| = |x| < \delta$ it follows that $|g(x) - g(0)| < \epsilon$. Thus, $g$ is continuous at the origin.

**Example 3.3.** The function $f(x) = \sqrt{x}$ is continuous on $A = [0, \infty)$.

Let $\epsilon > 0$. We need to argue that $|f(x) - f(c)|$ can be made less than $\epsilon$ for all values of $x$ in some $\delta$ neighborhood around $c$. If $c = 0$, this reduces to the statement $\sqrt{x} < \epsilon$, which happens as long as $x < \epsilon^2$. Thus, if we choose $\delta = \epsilon^2$, we see that $|x - 0| < \delta$ implies $|f(x) - 0| < \epsilon$.

For a point $c \in A$ different from zero, we need to estimate $|\sqrt{x} - \sqrt{c}|$. This time, write

$$|\sqrt{x} - \sqrt{c}| = \frac{|x - c|}{\sqrt{x} + \sqrt{c}} \leq \frac{|x - c|}{\sqrt{c}}.$$

In order to make this quantity less than $\epsilon$, it suffices to pick $\delta = \sqrt{c}\epsilon$. Then, $|x - c| < \delta$ implies $|\sqrt{x} - \sqrt{c}| < \epsilon$ as desired.

Although we have now shown that both polynomials and the square root function are continuous, the Algebraic Continuity Theorem does not provide the justification needed to conclude that a function such as $h(x) = \sqrt{2x^2 + 1}$ is continuous. For this, we must prove that compositions of continuous functions are continuous.

**Theorem 7** (Composition of Continuous Functions)**.** *Given $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$, assume that the range $f(A)$ is contained in the domain $B$ so that the composition $g \circ f(x) = f(g(x))$ is defined on $A$. If $f$ is continuous at $c \in A$, and if $g$ is continuous at $f(c) \in B$, then $g \circ f$ is continuous at $c$.*

**Exercise 5.** Assume $h : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and let $K = \{x \,|\, h(x) = 0\}$. Show that $K$ is a closed set.

**Exercise 6.** Observe that if $a$ and $b$ are real numbers, then

$$\max\{a, b\} = \frac{(a + b) + |a - b|}{2}.$$

(i) Show that if $f_1, f_2, \ldots, f_n$ are continuous functions, then

$$g(x) = \max\{f_1(x), f_2(x), \ldots, f_n(x)\}$$

is a continuous function.

(ii) Let's explore whether the result in (i) extends to the infinite case. For each $n \in \mathbb{N}$, define $f_n$ on $\mathbb{R}$ by

$$f_n(x) = \begin{cases} 1 & \text{if } |x| > 1/n \\ n|x| & \text{if } |x| \leq 1/n. \end{cases}$$

Now explicitly compute $h(x) = \sup\{f_1(x), f_2(x), f_3(x), \ldots\}$.

**Exercise 7.** Let $F \subset \mathbb{R}$ be a nonempty closed set and define $g(x) = \inf\{|x - a| \,:\, a \in F\}$. Show that $g$ is continuous on all of $\mathbb{R}$ and $g(x) \neq 0$ for all $x \notin F$.

# 4 Continuous Functions on Compact Sets

Given a function $f : A \to \mathbb{R}$ and a subset $B \subset A$, the notation $f(B)$ refers to the range of $f$ over the set $B$; that is,

$$f(B) = \{f(x) \,|\, x \in B\}.$$

The adjectives open, closed, bounded, compact, perfect, and connected are all used to describe subsets of the real line. An interesting question is to sort out which, if any, of these properties are preserved when a particular set $B$ is mapped to $f(B)$ via a continuous function. For instance, if $B$ is open and $f$ is continuous, is $f(B)$ necessarily open? The answer to this question is no. If $f(x) = x^2$ and $B$ is the open interval $(-1, 1)$, then $f(B)$ is the interval $[0, 1)$, which is not open.

The corresponding conjecture for closed sets also turns out to be false. Consider the function

$$g(x) = \frac{1}{1 + x^2}$$

and the closed set $B = [0, \infty)$. But $f(B) = (0, 1]$ is not closed. However, continuous functions always map compact set to compact set.

**Theorem 8** (Preservation of Compact Sets)**.** *Let $f : A \to \mathbb{R}$ be continuous on $A$. If $K \subset \mathbb{R}$ is compact, then $f(K)$ is compact as well.*

*Proof.* Let $\{y_n\}$ be an arbitrary sequence contained in the range set $f(K)$. To prove this result, we must find a subsequence $\{y_{n_k}\}$, which converges to a limit also in $f(K)$.

To assert that $\{y_n\} \subset f(K)$ means that, for each $n \in \mathbb{N}$, we can find (at least one) $x_n \in K$ with $f(x_n) = y_n$. This yields a sequence $\{x_n\} \subset K$. Because $K$ is compact, there exists a convergent subsequence $\{x_{n_k}\}$ whose limit $x = \lim_{k\to\infty} x_{n_k}$ is also in $K$. Finally, we make use of the fact that $f$ is assumed to be continuous on $A$ and so is continuous at $x$ in particular. Given that $\{x_{n_k}\} \to x$, we conclude that $\{y_{n_k}\} \to f(x)$. Because $x \in K$, we have that $f(x) \in f(K)$, and hence $f(K)$ is compact. $\square$

**Exercise 8.** In the above proof, we indeed use the fact that a set $K \subset \mathbb{R}$ is compact if and only if it is sequentially compact. Give a proof of the above theorem using the definition of compact as any open cover of $K$ has a finite subcover.

(i) Show that $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if $f^{-1}(O)$ is open whenever $O$ is open.

(ii) Using (i) to provide a proof of the above theorem.

An extremely important corollary is obtained by combining this result with the observation that compact sets are bounded and contain their supremums and infimums.

**Theorem 9** (Extreme Value Theorem)**.** *If $f : K \to \mathbb{R}$ is continuous on a compact set $K \subset \mathbb{R}$, then $f$ attains a maximum and minimum value. In other words, there exist $x_0, x_1 \in K$ such that $f(x_0) \le f(x) \le f(x_1)$ for all $x \in K$.*

*Proof.* Because $f(K)$ is compact, we can set $\alpha = \sup f(K)$ and know $\alpha \in f(K)$. (Why? Exercise.) It follows that there exist $x_1 \in K$ with $\alpha = f(x_1)$. The argument for the minimum value is similar. □

## 4.1 Uniform Continuity

**Definition 4** (Uniform Continuity)**.** A function $f : A \to \mathbb{R}$ is *uniformly continuous* on $A$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x, y \in A$, $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$.

Recall that to say that "$f$ is continuous on $A$" means that $f$ is continuous at each individual point $c \in A$. In other words, given $\epsilon > 0$ and $c \in A$, we can find a $\delta > 0$ perhaps depending on $c$ such that if $|x-c| < \delta$, then $|f(x)-f(c)| < \epsilon$. Uniform continuity is a strictly stronger property. The key distinction between asserting that $f$ is "uniformly continuous on $A$" versus simply "continuous on $A$" is that, given an $\epsilon > 0$, a single $\delta > 0$ can be chosen that works simultaneously for all points $c$ in $A$. To say that a function is not uniformly continuous on a set $A$, then, does not necessarily mean it is not continuous at some point. Rather, it means that there is some $\epsilon_0 > 0$ for which no single $\delta > 0$ is a suitable response for all $c \in A$.

**Theorem 10** (Sequential Criterion for Absence of Uniform Continuity)**.** *A function $f : A \to \mathbb{R}$ fails to be uniformly continuous on $A$ if and only if there exists a particular $\epsilon_0 > 0$ and two sequences $\{x_n\}$ and $\{y_n\}$ in $A$ satisfying*

$$|x_n - y_n| \to 0 \qquad \textit{but} \qquad |f(x_n) - f(y_n)| \ge \epsilon_0.$$

*Proof.* The negation of Definition 4 states that $f$ is not uniformly continuous on $A$ if and only if there exists $\epsilon_0 > 0$ such that for all $\delta > 0$ we can find two points $x$ and $y$ satisfying $|x - y| < \delta$ but with $|f(x) - f(y)| \ge \epsilon_0$. Thus, if we set $\delta_n = 1/n$ where $n \in \mathbb{N}$, it follows that there exist points $x_n$ and $y_n$ with $|x_n - y_n| < 1/n$ but where $|f(x_n) - f(y_n)| \ge \epsilon_0$. The resulting sequences $\{x_n\}$ and $\{y_n\}$ satisfy the requirements described in the theorem.

Conversely, if $\epsilon_0$, $\{x_n\}$ and $\{y_n\}$ exist as described, it is straightforward to see that no $\delta > 0$ is a suitable response for $\epsilon_0$. □

**Example 4.1.** (i) The function $f(x) = 3x + 2$ is uniformly continuous on $\mathbb{R}$;

(ii) The function $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$; (Take $x_n = n$ and $y_n = n + \frac{1}{n}$)

(iii) The function $f(x) = x^2$ is uniformly continuous on $[-1, 1]$;

(iv) The function $f(x) = \sin(1/x)$ is not uniformly continuous on $(0, 1)$. (Take $x_n = \frac{1}{2n\pi}$ and $y_n = \frac{1}{2n\pi+\pi/2}$.)

**Theorem 11** (Uniform Continuity on Compact Sets)**.** *A function that is continuous on a compact set $K$ is uniformly continuous on $K$.*

*Proof.* Assume $f : K \to \mathbb{R}$ is continuous at every point of a compact set $K \subset \mathbb{R}$. To prove that $f$ is uniformly continuous on $K$ we argue by contradiction.

By the criterion in Theorem 10, if $f$ is not uniformly continuous on $K$, then there exist two sequences $\{x_n\}$ and $\{y_n\}$ in $K$ such that

$$\lim |x_n - y_n| = 0 \quad \text{while} \quad |f(x_n) - f(y_n)| \geq \epsilon_0$$

for some particular $\epsilon_0 > 0$. Because $K$ is compact, the sequence $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ with $x = \lim x_{n_k}$ also in $K$.

By the Algebraic Limit Theorem,

$$\lim_{k\to\infty} y_{n_k} = \lim_{k\to\infty} [(y_{n_k} - x_{n_k}) + x_{n_k}] = 0 + x = x.$$

The conclusion is that both $\{x_{n_k}\}$ and $\{y_{n_k}\}$ converge to $x \in K$. Because $f$ is assumed to be continuous at $x$, we have $\lim f(x_{n_k}) = f(x)$ and $\lim f(y_{n_k}) = f(x)$, which implies

$$\lim_{k\to\infty} [f(x_{n_k}) - f(y_{n_k})] = f(x) - f(x) = 0.$$

A contradiction arises when we recall that $\{x_{n_k}\}$ and $\{y_{n_k}\}$ were chosen to satisfy

$$|f(x_n) - f(y_n)| \geq \epsilon_0.$$

for all $n \in \mathbb{N}$. We conclude, then, that $f$ is indeed uniformly continuous on $K$. $\square$

**Exercise 9.** Provide a proof of the above theorem by the definition "$K \subset \mathbb{R}$ is compact if every open cover of $K$ has a finite subcover."

**Exercise 10.** (i) Assume that $g$ is defined on an open interval $(a, c)$ and it is known to be uniformly continuous on $(a, b]$ and $[b, c)$, where $a < b < c$. Prove that $g$ is uniformly continuous on $(a, c)$.

(ii) Prove that $f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$.

**Exercise 11.** Give an example of each of the following, or provide a short argument for why the request is impossible.

(a) A continuous function defined on $[0, 1]$ with range $(0, 1)$.

(b) A continuous function defined on $(0, 1)$ with range $[0, 1]$.

(c) A continuous function defined on $(0, 1]$ with range $(0, 1)$.

**Exercise 12** (Continuous Extension Theorem)**.** (i) Show that a uniformly continuous function preserves Cauchy sequences; that is, if $f : A \to \mathbb{R}$ is uniformly continuous and $\{x_n\} \subset A$ is a Cauchy sequence, then show $f(x_n)$ is a Cauchy sequence.

(ii) Let $g$ be a continuous function on the open interval $(a, b)$. Prove that $g$ is uniformly continuous on $(a, b)$ if and only if it is possible to define values $g(a)$ and $g(b)$ at the endpoints so that the extended function $g$ is continuous on $[a, b]$. (In the forward direction, first produce candidates for $g(a)$ and $g(b)$, and then show the extended $g$ is continuous.)

# 5 The Intermediate Value Theorem

The Intermediate Value Theorem (IVT) is the name given to the very intuitive observation that a continuous function $f$ on a closed interval $[a, b]$ attains every value that falls between the range values $f(a)$ and $f(b)$.

**Theorem 12** (Intermediate Value Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous. If $L$ is a real number satisfying $f(a) < L < f(b)$ or $f(a) > L > f(b)$, then there exists a point $c \in (a, b)$ where $f(c) = L$.*

The most potentially useful way to understand the Intermediate Value Theorem (IVT) is as a special case of the fact that continuous functions map connected sets to connected sets.

**Theorem 13** (Preservation of Connected Sets)**.** *Let $f : G \to \mathbb{R}$ be continuous. If $E \subset G$ is connected, then $f(E)$ is connected as well.*

*Proof.* Intending to use the characterization of connected sets in Theorem 11 of Chapter 4, let $f(E) = A \cup B$ where $A$ and $B$ are disjoint and nonempty. Our goal is to produce a sequence contained in one of these sets that converges to a limit in the other.

Let

$$C = f^{-1}(A) = \{x \in E \,|\, f(x) \in A\} \quad \text{and} \quad D = f^{-1}(B) = \{x \in E \,|\, f(x) \in B\}$$

The sets $C$ and $D$ are called the *preimages* of $A$ and $B$, respectively. Using the properties of $A$ and $B$, it is straightforward to check that $C$ and $D$ are nonempty and disjoint and satisfy $E = C \cup D$. Now, we are assuming $E$ is a connected set, so by Theorem 11 of Chapter 4, there exists a sequence $\{x_n\}$ contained in one of $C$ or $D$ with $x = \lim x_n$ contained in the other. Finally, because $f$ is continuous at $x$, we get $f(x) = \lim f(x_n)$. Thus, it follows that $f(x_n)$ is a convergent sequence contained in either $A$ or $B$ while the limit $f(x)$ is an element of the other. With another nod to Theorem 11 of Chapter 4, the proof is complete. $\square$

In $\mathbb{R}$, a set is connected if and only if it is a (possibly unbounded) interval. This fact, together with Theorem 13, leads to a short proof of the Intermediate Value Theorem.

**Completeness implies IVT**

*LUBP implies IVT.* To simplify matters a bit, let's consider the special case where $f$ is a continuous function satisfying $f(a) < 0 < f(b)$ and show that $f(c = 0)$ for some $c \in (a, b)$. First let

$$K = \{x \in [a, b] \mid f(x) \leq 0\}.$$

Notice that $K$ is bounded above by $b$, and $a \in K$ so $K$ is not empty. Thus we may appeal to the Axiom of Completeness (LUBP) to assert that $c = \sup K$ exists. There are three cases to consider:

$$f(c) > 0, \quad f(c) < 0, \quad f(c) = 0.$$

The fact that $c$ is the least upper bound of $K$ can be used to rule out the first two cases, resulting in the desired conclusion that $f(c) = 0$. (Exercise.) $\square$

*NIP implies IVT.* Again, consider the special case where $L = 0$ and $f(a) < 0 < f(b)$. Let $I_0 = [a, b]$, and consider the midpoint

$$z = (a + b)/2.$$

If $f(z) \geq 0$, then set $a_1 = a$ and $b_1 = z$. If $f(z) < 0$, then set $a_1 = z$ and $b_1 = b$. In either case, the interval $I_1 = [a_1, b_1]$ has the property that $f$ is negative at the left endpoint and nonnegative at the right. This procedure can be inductively repeated, setting the stage for an application of the Nested Interval Property. [Exercise] $\square$

**Intermediate Value Property**

Does the Intermediate Value Theorem have a converse?

**Definition 5.** A function $f$ has the *intermediate value property* on an interval $[a, b]$ if for all $x < y$ in $[a, b]$ and all $L$ between $f(x)$ and $f(y)$, it is always possible to find a point $x \in (x, y)$ where $f(c) = L$.

Another way to summarize the Intermediate Value Theorem is to say that every continuous function on [a,b] has the intermediate value property. There is an understandable temptation to suspect that any function that has the intermediate value property must necessarily be continuous, but that is not the case. We have seen that

$$g(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if x=0} \end{cases}$$

is not continuous at zero, but it does have the intermediate value property on [0,1].

The intermediate value property does imply continuity if we insist that our function is monotone. [Exercise.]

**Exercise 13.** Let $f : [0,1] \to \mathbb{R}$ be continuous with $f(0) = f(1)$.

(i) Show that there must exist $x, y \in [0,1]$ satisfying $|x - y| = 1/2$ and $f(x) = f(y)$.

(ii) Show that for each $n \in \mathbb{N}$ there exist $x_n, y_n \in [0,1]$ with $|x_n - y_n| = 1/n$ and $f(x_n) = f(y_n)$.

(iii) If $h \in (0, 1/2)$ is not of the form $1/n$, there does not necessarily exist $|x - y| = h$ satisfying $f(x) = f(y)$. Provide an example that illustrates this using $h = 2/5$.

**Exercise 14.** Let $f$ be a continuous function on the closed interval $[0,1]$ with range also contained in $[0,1]$. Prove that $f$ must have a fixed point; that is, show $f(x) = x$ for at least one value of $x \in [0,1]$.

**Exercise 15** (Inverse functions)**.** If a function $f : A \to \mathbb{R}$ is one-to-one, then we can define the inverse function $f^{-1}$ on the range of $f$ in the natural way: $f^{-1}(y) = x$ where $y = f(x)$. Show that if $f$ is continuous on an interval $[a,b]$ and one-to-one, then $f^{-1}$ is also continuous.

# 6 Sets of Discontinuity

Given a function $f : \mathbb{R} \to \mathbb{R}$, define $D_f \subset \mathbb{R}$ to be the set of points where the function $f$ fails to be continuous. In Section 1, we saw that Dirichlet's function $g(x)$ had $D_g = \mathbb{R}$. The modification $h(x)$ of Dirichlet's function had $D_h = \mathbb{R} \setminus \{0\}$, zero being the only point of continuity. Finally, for Thomae's function $t(x)$, we saw that $D_t = \mathbb{Q}$.

**Exercise 16.** Using modifications of these functions, construct a function $f : \mathbb{R} \to \mathbb{R}$ so that

$$(a) \quad D_f = \mathbb{Z}^c; \qquad (b) \quad D_f = (0,1].$$

**Exercise 17.** Given a countable set $A = \{a_1, a_2, a_3, \ldots\}$, define $f(a_n) = 1/n$ and $f(x) = 0$ for all $x \notin A$. Find $D_f$.

We concluded the introduction with a question about whether $D_f$ could take the form of any arbitrary subset of the real line. As it turns out, this is not the case. The set of discontinuities of a real-valued function on R has a specific topological structure that is not possessed by every subset of $\mathbb{R}$. Specifically, $D_f$, no matter how $f$ is chosen, can always be written as the countable union of closed sets (*i.e.*, an $F_\sigma$ set). In the case where $f$ is monotone, these closed sets can be taken to be single points.

## 6.1 Monotone Functions

**Definition 6** (Monotone Functions)**.** A function $f : A \to \mathbb{R}$ is *increasing* on $A$ if $f(x) \leq f(y)$ whenever $x < y$ and *decreasing* if $f(x) \geq f(y)$ whenever $x < y$ in $A$. A *monotone* function is one that is either increasing or decreasing.

**Definition 7** (One-sided Limits)**.** Given a limit point $c$ of a set $A$ and a function $f : A \to \mathbb{R}$, we write

$$\lim_{x \to c+} f(x) = L$$

if for all $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < x - c < \delta$.

Equivalently, in terms of sequences, $\lim\limits_{x \to c+} f(x) = L$ if $\lim\limits_{n \to \infty} f(x_n) = L$ for all sequences $\{x_n\}$ satisfying $x_n > c$ and $\lim\limits_{n \to \infty} x_n = c$.

The left-hand limit

$$\lim_{x \to c-} f(x) = L$$

is defined in a similar way.

**Theorem 14.** *Given $f : A \to \mathbb{R}$ and a limit point $c$ of $A$, $\lim_{x \to c} f(x) = L$ if and only if*

$$\lim_{x \to c-} f(x) = \lim_{x \to c+} f(x) = L.$$

Generally speaking, discontinuities can be divided into three categories:

(i) If $\lim_{x \to c} f(x)$ exists but has a value different from $f(c)$, the discontinuity at $c$ is called *removable*.

(ii) If $\lim_{x \to c+} f(x) \neq \lim_{x \to c-} f(x)$, then $f$ has a *jump* discontinuity at $c$.

(iii) If $\lim_{x \to c} f(x)$ does not exist for some other reason, then the discontinuity at $c$ is called an *essential* discontinuity.

**Exercise 18.** Prove that the only type of discontinuity a monotone function can have is a jump discontinuity.

**Exercise 19.** Construct a bijection between the set of jump discontinuities of a monotone function $f$ and a subset of $\mathbb{Q}$. Conclude that $D_f$ for a monotone function $f$ must either be finite or countable, but not uncountable.

## 6.2 $D_f$ for an Arbitrary Function

Recall that the intersection of an infinite collection of closed sets is closed, but for unions we must restrict ourselves to finite collections of closed sets in order to ensure the union is closed. For open sets the situation is reversed. The arbitrary union of open sets is open, but only finite intersections of open sets are necessarily open.

In Section 1 we constructed functions where the set of discontinuity was $\mathbb{R}$ (Dirichlet's function), $\mathbb{R} \setminus \{0\}$ (modified Dirichlet function), and $\mathbb{Q}$ (Thomae's function).

**Exercise 20.** (i) Show that in each of the above cases we get an $F_\sigma$ set as the set where the function is discontinuous.

(ii) Show that the two sets of discontinuity in Exercise 16 are $F_\sigma$ sets.

The upcoming argument depends on a concept called $\alpha$-continuity.

**Definition 8.** Let $f$ be defined on $\mathbb{R}$, and let $\alpha > 0$. The function $f$ is $\alpha$-continuous at $x \in \mathbb{R}$ if there exists a $\delta > 0$ such that for all $y, z \in V_\delta(x)$ it follows that $|f(y) - f(z)| < \alpha$.

The most important thing to note about this definition is that there is no "for all" in front of the $\alpha > 0$. As we will investigate, adding this quantifier would make this definition equivalent to our definition of continuity. In a sense, $\alpha$-continuity is a measure of the variation of the function in the neighborhood of a particular point. A function is $\alpha$-continuous at a point $c$ if there is some interval centered at $c$ in which the variation of the function never exceeds the value $\alpha > 0$.

Given a function $f$ on $\mathbb{R}$, define $D_f^\alpha$ to be the set of points where the function $f$ fails to be $\alpha$-continuous. In other words,

$$D_f^\alpha = \{x \in \mathbb{R} \mid f \text{ is not } \alpha\text{-continuous at } x\}.$$

**Exercise 21.** Prove that, for a fixed $\alpha > 0$, the set $D_f^\alpha$ is closed.

**Theorem 15.** *Let $f : \mathbb{R} \to \mathbb{R}$ be an arbitrary function. Then, $D_f$ is an $F_\sigma$ set.*

*Proof.* Recall that

$$D_f = \{x \in \mathbb{R} \mid f \text{ is not continuous at } x\}.$$

**Exercise 22.** If $\alpha < \alpha'$, show that $D_f^\alpha \subset D_f^{\alpha'}$.

**Exercise 23.** Let $\alpha > 0$ be given. Show that if $f$ is continuous at $x$, then it is $\alpha$-continuous at $x$ as well. Explain how it follows that $D_f^\alpha \subset D_f$.

**Exercise 24.** Show that if $f$ is not continuous at $x$, then $f$ is not $\alpha$-continuous for some $\alpha > 0$. Now explain why this guarantees that

$$D_f = \bigcup_{n=1}^{\infty} D_f^{\alpha_n},$$

where $\alpha_n = 1/n$

Because each $D_f^{\alpha_n}$ is closed, the proof is complete. $\square$