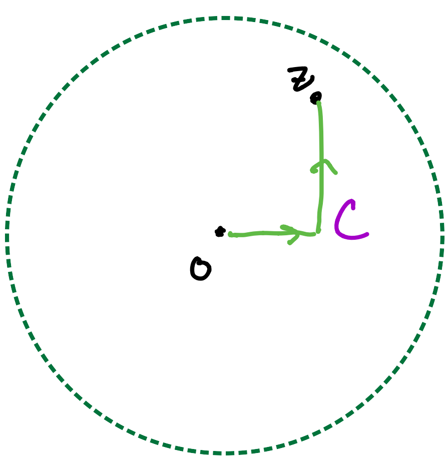
MAT 3253 Lecture 15

$$
\begin{tikzcd}
\text{FTC complex case} \arrow[d] \arrow[dr] & \\
\text{Cauchy-Goursat theorem for rectangles} \arrow[d] & \text{Closed-curve theorem} \\
\text{existence of primitive/anti-derivative in a disc} \arrow[ur] &
\end{tikzcd}
$$

Theorem If $f(z)$ is analytic in an open disc, then there exists a function $F(z)$ s.t.

$$F'(z) = f(z) \qquad \forall z \text{ in the disc.}$$

Proof WLOG assume the disc is centered at the origin.

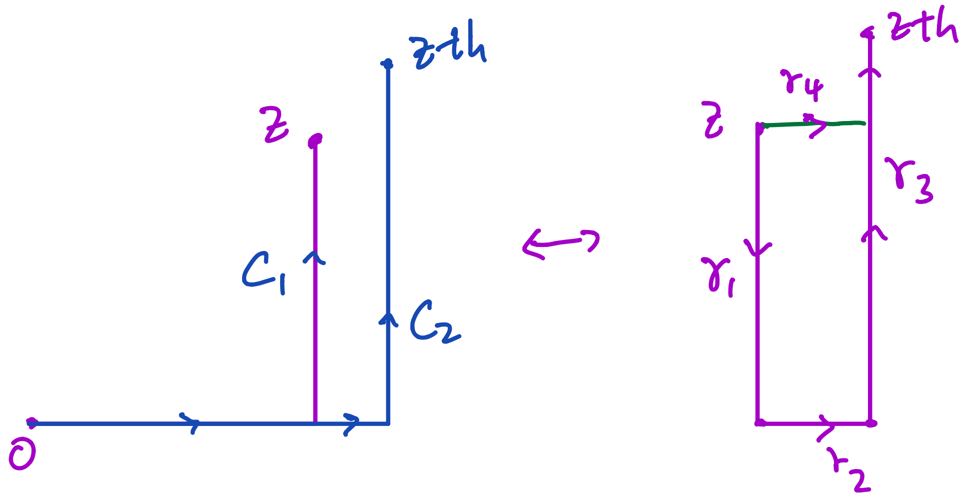


$$0 \longrightarrow R(z) \longrightarrow z$$

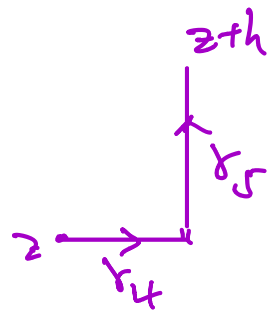
Define $F(z) \triangleq \int_C f(\zeta)\, d\zeta$

Want to show $F'(z) = f(z)$

$$F(z+h) - F(z) = \int_{C_2} f(z)\, dz - \int_{C_1} f(z)\, dz$$



$$= \int_{\gamma_1+\gamma_2+\gamma_3} f(z)\, dz = \int_{\gamma_4+\gamma_5} f(z)\, dz$$



$$F(z+h) - F(z) = \int_{\gamma_4+\gamma_5} f(z)\, dz$$

f is analytic $\Rightarrow$ f is continuous

$$f(\zeta) = f(z) + e_z(\zeta)$$

$$e_z(\zeta) \to 0 \quad \text{as} \quad \zeta \to z$$

$$\left| \frac{F(z+h) - F(z)}{h} - f(z) \right|$$

$$\boxed{f(z) = \frac{\int_{\gamma_4+\gamma_5} f(z)\, d\zeta}{h}}$$

$$= \left| \frac{1}{h} \int_{\gamma_4+\gamma_5} f(\zeta)\, d\zeta - \frac{1}{h} \int_{\gamma_4+\gamma_5} f(z)\, d\zeta \right|$$

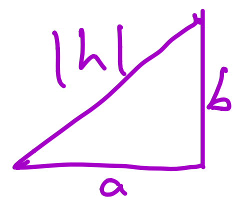
$$= \frac{1}{|h|} \left| \int_{\gamma_4+\gamma_5} f(\zeta) - f(z)\, d\zeta \right|$$

$$= \frac{1}{|h|} \left| \int_{\gamma_4+\gamma_5} e_z(\zeta)\, d\zeta \right|$$

Fix ε, $\exists \delta$ s.t. $|e_z(\zeta)| < \varepsilon$ for $|h| < \delta$

$$\le \frac{1}{|h|} \cdot \delta \cdot L \le \frac{1}{|h|} \cdot \varepsilon (2|h|)$$

$$= 2\varepsilon$$



$a \le |h|$, $b \le |h|$

Because ε is arbitrarily small

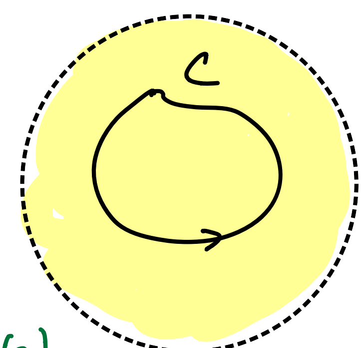
$F'(z) = f(z)$ $\quad \forall z$ in the disc. $\square$

$$\int_z^{z+h} c\, d\zeta = c \int_z^{z+h} d\zeta = c\, [\zeta]_z^{z+h} = c \cdot h$$

$$\int_z^{z+h} f(z)\, d\zeta = f(z) \cdot h$$

<u>Theorem</u> If f is analytic in an open disc, then any closed curve C in the disc satisfies

$$\int_C f(z)\, dz = 0 .$$

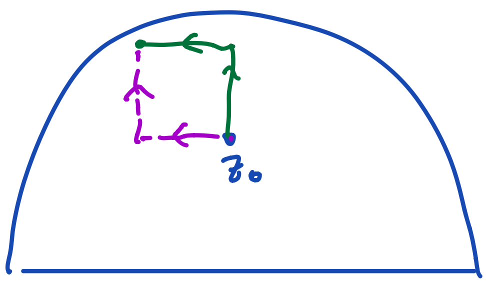


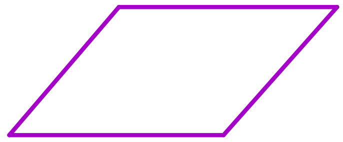
<u>Proof</u> By previous theorem $f(z)$ has a primitive $F(z)$ s.t. $F'(z) = f(z)$.

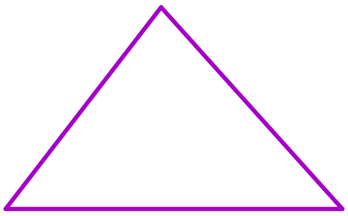
By fundamental theorem of calculus, $\int_C f(z)\, dz = 0$. $\square$

Remark : If f is analytic throughout $\mathbb{C}$, it means radius of open disc is ∞.

Remark :



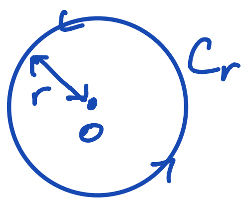




The closed curve theorem also holds for other shapes such as triangle or semicircle.

Example

$$\int_{C_r} z^n \, dz = \begin{cases} 0 & \text{if } n \geq 0 \\ 2\pi i & \text{if } n = -1 \\ 0 & \text{if } n \leq -2 \end{cases}$$



$n \geq 0$ by closed-curve thm $\int_{C_r} z^n \, dz = 0$.

$n = -1$ $\quad f(z) = \frac{1}{z}$

$C_r : z(\theta) = r(\cos\theta + i\sin\theta) \qquad 0 \leq \theta \leq 2\pi$

$z'(\theta) = r(-\sin\theta + i\cos\theta)$

$$\int_{C_r} \frac{1}{z}\,dz = \int_0^{2\pi} \frac{1}{r(\cos\theta + i\sin\theta)} \cdot r(\overset{i^2}{-}\sin\theta + i\cos\theta)\,d\theta$$

$$= i\int_0^{2\pi} \frac{\cos\theta + i\sin\theta}{\cos\theta + i\sin\theta}\,d\theta$$

$$= 2\pi i.$$

$n \le -2$ $\qquad \frac{1}{z^2}$ has an anti-derivative

$$\left(-\frac{1}{z}\right)' = \frac{1}{z^2}$$

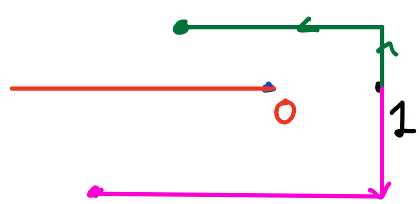
by FTC, $\int_{C_r} \frac{1}{z^2}\,dz = 0$

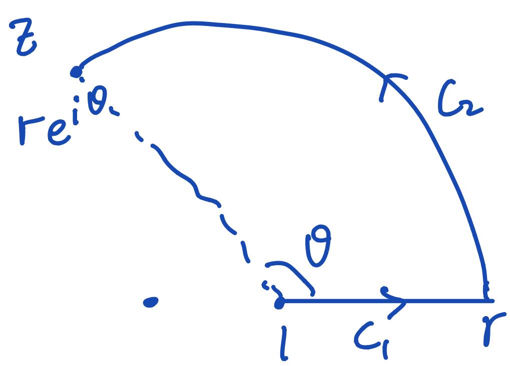
Application: Definition of log function as the primitive function of $\frac{1}{z}$.

Domain to $\mathbb{C} \setminus \{(x,0) : x \le 0\}$.

Define $\mathrm{Log}(z) \triangleq \int_1^z \frac{1}{z}\,dz$

The integral is indep. of path.





$$\int_{C_1} \frac{1}{z}\,dz = \int_1^r \frac{1}{x}\,dx = \ln r.$$

$$\int_{C_2} \frac{1}{z}\,dz = i\theta$$

$\mathrm{Log}(z) = \ln r + i\theta.$