MAT 3253 Lecture 8

Extend e^x to a complex analytic function

$$f(z) = e^x + i\,v(x,y)$$

$$(e^x)_{xx} + (e^x)_{yy} = e^x + 0 \neq 0$$

$$f(z) = e^x \cos y + i\,v(x,y)$$

when $y=0$, $e^x \cos y = e^x$

$$(e^x \cos y)_{xx} + (e^x \cos y)_{yy} = e^x \cos y - e^x \cos y = 0$$

<u>Method 1</u>

$v(x,y)$

$\partial/\partial x$ $\quad$ $\int x$

$v_x = -u_y = e^x \sin y$

$v_y = u_x = e^x \cos y$

$u_x = e^x \cos y$

$u_y = e^x(-\sin y)$

$$\int e^x \sin y \, dx = e^x \sin y + C(y)$$

$$\frac{\partial}{\partial y}\left(e^x \sin y + C(y)\right) = e^x \cos y + C'(y)$$

$$\Rightarrow C'(y) = 0$$

$$v(x,y) = e^x \sin y + C$$

Set $C = 0$ (because we want $\exp(x) = e^x$)

Def $$\boxed{e^z \overset{\Delta}{=} \exp(z) \overset{\Delta}{=} e^x \cos y + i\, e^x \sin y}$$

Method 2 $$v(\tilde{x}, \tilde{y}) = \int_{(0,0)}^{(\tilde{x}, \tilde{y})} -u_y\, dx + u_x\, dy$$

Example $$\exp(i) = \cos 1 + i \sin 1$$

radian

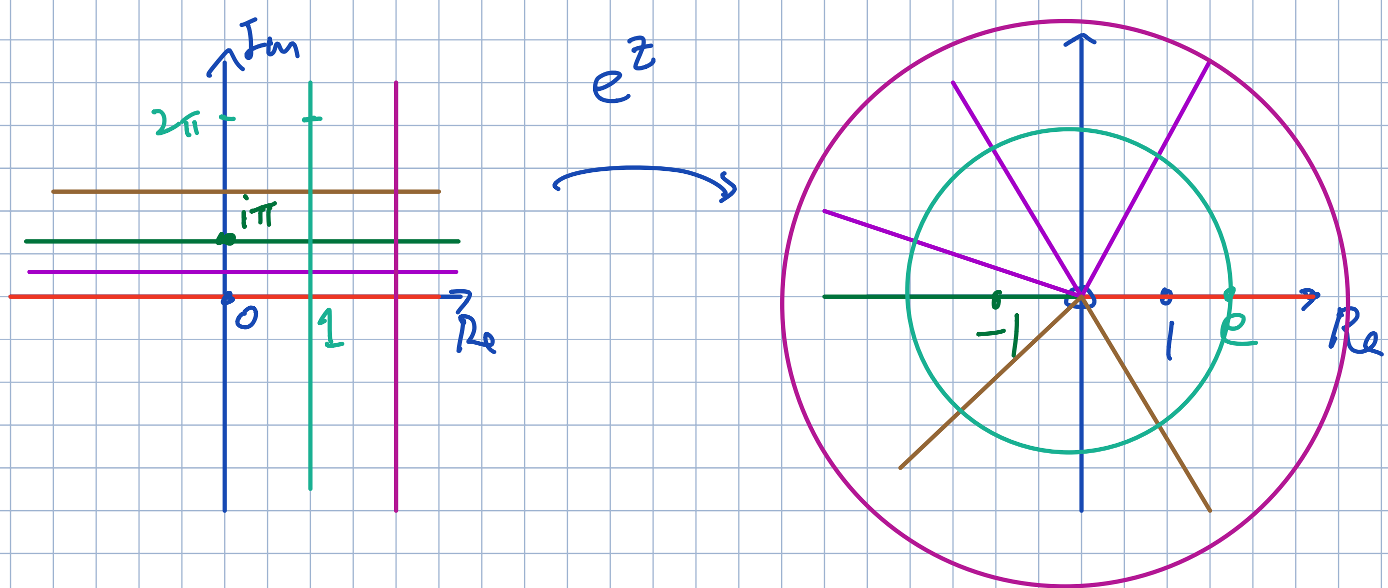
$i = 0 + 1i$

$$\exp(\pi i) = \cos \pi + i \sin \pi$$
$$= -1$$

$$e^{x+iy} = e^x (\cos y + i \sin y)$$

modulus

argument



$$\exp(z + 2\pi i) = \exp(z)$$

$*$ $$\exp(z + 2\pi k i) = \exp(z) \qquad k \in \mathbb{Z}$$

Log function

Find $x+iy$ s.t. $e^{x+iy} = 5$

$x = \log 5$, $y = 2\pi k$ for $k = 0, \pm 1, \pm 2 \dots$

Find $x+iy$ s.t. $e^{x+iy} = r(\cos\theta + i\sin\theta)$

$x = \log r$, $y = \theta + 2\pi k$ $\quad k \in \mathbb{Z}$

set of integers

Principal log : $\mathrm{Log}(z) = \log|z| + \mathrm{Arg}(z)\, i$

$$2^{\pi} = \exp(\pi \log 2) = e^{\pi \log 2}$$

Example

$$2^{i} = \exp(i \underbrace{\log 2}_{\log 2 + 2\pi k i}) = e^{i \log 2}$$

$k \in \mathbb{Z}$

$$i \log 2 + i\, 2\pi k i$$

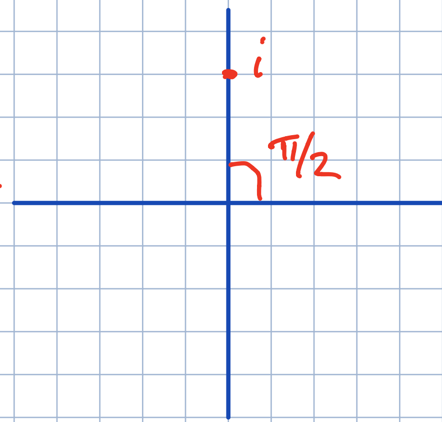
$$\exp(-2\pi k + i \log 2) = e^{-2\pi k}\left(\cos(\log 2) + i \sin(\log 2)\right)$$

Example $\quad i^i = ?$

Compute $\quad \exp(i \log i) = e^{i \log i}$

$$\log i = \overset{e^0=1}{0} + i\frac{\pi}{2} + i2\pi k$$

$$k \in \mathbb{Z}$$



$$i \log i = -\frac{\pi}{2} - 2\pi k$$

$$\exp(i \log i) = \exp\left(-\frac{\pi}{2} - 2\pi k\right)$$

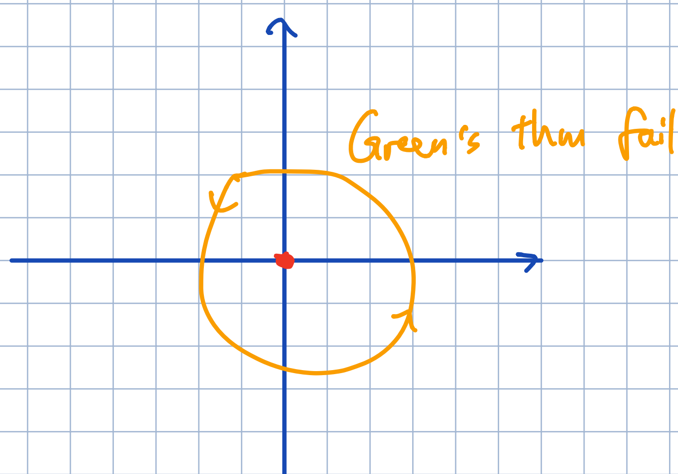
$$i^i = e^{-\pi/2} \quad \text{(the principal value)}$$

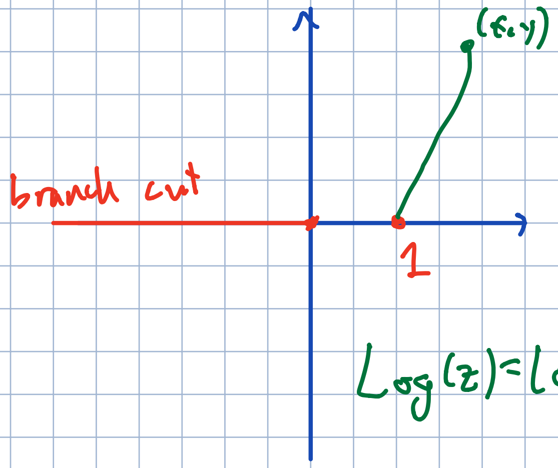
$$= e^{-2\pi k} \cdot e^{-\pi/2} \qquad k \in \mathbb{Z}$$

Log function as an analytic function

$$u(x,y) = \log\left(\sqrt{x^2+y^2}\right) \qquad \text{(is harmonic)}$$

$$v(x,y) = \int_{(1,0)}^{(x,y)} \frac{-y\, dx}{x^2+y^2} + \frac{x}{x^2+y^2}\, dy$$





$D = \mathbb{C} \setminus \{ x+iy : x \le 0, y=0 \}$

$Log(z) = \log(\sqrt{x^2+y^2}) + i\, Arg(x+iy)$

$Arg(x+iy)$ is a harmonic conjugate of $\log(\sqrt{x^2+y^2})$

* $Log'(z) = \frac{1}{z}$