Asymptotic results

Weak law of large numbers (WLLN)

Consider $\{X_t\}$ is an sequence of iid random variables with $EX_t = \mu$,

then $\bar{X}_n = \frac{1}{n}\sum_{t=1}^{n} X_t \xrightarrow{P} \mu$, ie. $\forall \varepsilon > 0, \lim_{n\to\infty} P(|\bar{X}_n - \mu| > \varepsilon) = 0$

(Strong LLN) $\bar{X}_n \xrightarrow{a.s.} \mu$, ie. $P(\lim_{n\to\infty} \bar{X}_n = \mu) = 1$

Central limit theorem (CLT) (Theorem A.3)

Consider X_t, $t=1,\dots,n$ iid with $EX_t = \mu$ and $Var(X_t) = \sigma^2$

$$\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} N(0,1)$$

The "iid" assumption makes the results cannot be applied to time series in general.

$\boxed{\text{Definition 1.}}$ Let $F_t = \sigma(X_s, -\infty < s \le t)$ (or you may think it as the set of past values up to time t, ie. $F_t = \{X_1, X_2, \dots, X_t\}$)

X_t is called a martingale if $E|X_t| < \infty$ and $E(X_t | F_{t-1}) = X_{t-1}$

(e.g. A random walk is a martingale)

If $E(X_t | F_{t-1}) = 0$, it is called a martingale difference sequence (MDS)

CLT for MDS (See p.17 in "Martingale_CLT")

Let $\{X_t\}$ be stationary MDS with $E(X_t^2) = \sigma^2 < \infty$ and

$\frac{1}{n}\sum_{t=1}^{n} X_t^2 \xrightarrow{P} \sigma^2$, then

$$\frac{\bar{X}_n}{\sigma/\sqrt{n}} \xrightarrow{d} N(0,1)$$

WLLN for martingales (see p. 8 in Martingale_LLN)

Let $\{X_t\}$ be a sequence of random variables such that $E|X_n| < \infty$ and $P(|X_n| > x) \leq c P(|X| > x)$ for $x \geq 0$ and $n \geq 1$ (i.e. all X_i's are bounded by some random variable X. Note that we can take $c = 1$ if $\{X_t\}$ is strictly stationary). Then

$$\frac{1}{n}\sum_{t=1}^{n}\left(X_t - E(X_t | F_{t-1})\right) \xrightarrow{P} 0$$

If $\{X_t\}$ and $\{E(X_t | F_{t-1})\}$ are stationary, then the convergence is also almost for sure.

Definition A.2 Converges in probability

$$X_n \xrightarrow{P} X \iff P(|X_n - X| > \varepsilon) \to 0 \quad \forall \varepsilon > 0 \text{ as } n \to \infty$$

Note that, by Markov's inequality $P(|X| \geq \varepsilon) \leq \frac{E(|X|)}{\varepsilon}$ for $\varepsilon > 0$, we have $P(|X_n - X| > \varepsilon) = P((X_n - X)^2 > \varepsilon^2) \leq \frac{E((X_n - X)^2)}{\varepsilon^2}$

Definition A.3

We write $X_n = \underset{\text{small } O}{o_p}(a_n) \iff \frac{X_n}{a_n} \xrightarrow{P} 0$

write $X_n = \underset{\text{big } O}{O_p}(a_n) \iff \forall \varepsilon > 0, \exists \delta(\varepsilon) > 0$ such that $P\left(\left|\frac{X_n}{a_n}\right| > \delta(\varepsilon)\right) \leq \varepsilon \quad \forall n$

Some handy properties:

(i) $X_n = o_p(a_n)$ and $y_n = o_p(b_n)$ then $X_n y_n = o_p(a_n b_n)$ and $X_n + y_n = o_p(\max(a_n, b_n))$

(i) is also true if $o_p(\cdot)$ is replaced by $O_p(\cdot)$

(ii) If $X_n = o_p(a_n)$ and $y_n = O_p(b_n)$, then $X_n y_n = o_p(a_n b_n)$

(iii) If $X_n \xrightarrow{P} X$ and $g(\cdot)$ is a continuous mapping, $g(X_n) \xrightarrow{P} g(X)$

Definition A.4 Converge in distribution

$$X_n \xrightarrow{d} X \iff F_n(x) \longrightarrow F(x) \quad \text{at the continuity points } x \text{ of distribution function } F(\cdot)$$

Some handy properties:

(i) (Proposition A.1 The Cramér-Wold device) For $\vec{X}_n \in \mathbb{R}^k$

$$\vec{X}_n \xrightarrow{d} \vec{X} \iff \vec{c}^{\,T}\vec{X}_n \xrightarrow{d} \vec{c}^{\,T}\vec{X} \quad \text{for all } \vec{c} = (c_1, \ldots, c_k) \in \mathbb{R}^k$$

(ii) $X_n \xrightarrow{P} X \Rightarrow X_n \xrightarrow{d} X$. If X is constant, $X_n \xrightarrow{d} X \Rightarrow X_n \xrightarrow{P} X$

(iii) If $X_n \xrightarrow{d} X$ and $y_n \xrightarrow{d} c$, constant (e.g. $y_n \sim N(0, \frac{1}{n}) \xrightarrow{d} 0$)

then $X_n + y_n \xrightarrow{d} X + c$ and $y_n^T X_n \xrightarrow{d} c^T X$

(iv) For a continuous mapping $g(\cdot)$,

$$X_n \xrightarrow{d} X \Rightarrow g(X_n) \xrightarrow{d} g(X)$$

Recall in Definition 1.12 that X_t is a linear process if

$$X_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j w_{t-j} \quad \text{and} \quad \sum_{j=-\infty}^{\infty} |\psi_j| < \infty, \quad w_t \sim wn(0, \sigma_w^2)$$

$\boxed{\text{Theorem A.5}}$ If X_t is a stationary linear process with $w_t \sim iid(0, \sigma_w^2)$ and $\sum_{j=-\infty}^{\infty} \psi_j \neq 0$, then

$$\frac{\bar{X}_n - \mu}{\sqrt{V/n}} \xrightarrow{d} N(0,1)$$

where V is given in (A.47)

$\boxed{\text{Theorem A.6}}$ If X_t is a stationary linear process with $w_t \sim iid(0, \sigma_w^2)$ and $E(w_t^4) = \eta \sigma_w^4 < \infty$ for some constant η, then

$$\begin{pmatrix} \hat{\gamma}(0) \\ \hat{\gamma}(1) \\ \vdots \\ \hat{\gamma}(K) \end{pmatrix} \xrightarrow{d} N\left(\begin{pmatrix} \gamma(0) \\ \gamma(1) \\ \vdots \\ \gamma(K) \end{pmatrix}, \frac{V}{n} \right)$$

where V is given in (A.53)

Theorem A.7 If X_t is a stationary linear process with $w_t \sim iid(0, \sigma_w^2)$ and $E(w_t^4) = \eta\sigma_w^4 < \infty$, then

$$\begin{pmatrix} \hat{\rho}(1) \\ \vdots \\ \hat{\rho}(K) \end{pmatrix} \xrightarrow{d} N\left(\begin{pmatrix} \rho(1) \\ \vdots \\ \rho(K) \end{pmatrix}, \frac{W}{n} \right)$$

where W is given in (A.54)

Ex. 1.30 Let X_t be a stationary linear process with $w_t \sim iid(0, \sigma_w^2)$

If we define $\tilde{\gamma}(h) = \frac{1}{n}\sum_{t=1}^{n}(X_{t+h} - \mu)(X_t - \mu)$,

show that

$$\sqrt{n}\,(\tilde{\gamma}(h) - \hat{\gamma}(h)) = o_p(1)$$

Pf: We want to show that

$$\lim_{n\to\infty} P\left(\sqrt{n}\, \left| \frac{1}{n}\sum_{t=1}^{n}(X_{t+h}-\mu)(X_t-\mu) - \frac{1}{n}\sum_{t=1}^{n-h}(X_{t+h}-\bar{X})(X_t-\bar{X}) \right| > \varepsilon \right) = 0$$

① Show that $\sqrt{n}\left(\frac{1}{n}\sum_{t=n-h+1}^{n}(X_{t+h}-\mu)(X_t-\mu)\right) \xrightarrow{P} 0$

Note that

$$P\left(\frac{1}{\sqrt{n}} \left| \sum_{t=n-h+1}^{n}(X_{t+h}-\mu)(X_t-\mu) \right| > \varepsilon \right)$$

$$\leq \frac{1}{\sqrt{n}\,\varepsilon} E\left(\left| \sum_{t=n-h+1}^{n}(X_{t+h}-\mu)(X_t-\mu) \right| \right)$$

$$\leq \frac{1}{\sqrt{n}\,\varepsilon} E\left(\sum_{t=n-h+1}^{n} |X_{t+h}-\mu|\,|X_t-\mu| \right)$$

$$\leq \frac{1}{\sqrt{n}\,\varepsilon} E\left(\sum_{t=n-h+1}^{n} \frac{1}{2}\left((X_{t+h}-\mu)^2 + (X_t-\mu)^2 \right) \right)$$

$$= \frac{1}{2\sqrt{n}\,\varepsilon} \sum_{t=n-h+1}^{n} 2\,Var(X_t)$$

$$= \frac{h}{\sqrt{n}\,\varepsilon} Var(X_t) \longrightarrow 0 \quad \text{as } n\to\infty$$

Note that $\sum_{j=-\infty}^{\infty}|\psi_j| < \infty \Rightarrow \max_j |\psi_j| < \infty$

$$\therefore Var(X_t) = \sum_{j=-\infty}^{\infty}\psi_j^2\sigma_w^2 = \sigma_w^2 \sum_{j=-\infty}^{\infty}|\psi_j|^2$$

$$\leq \sigma_w^2 \max_j|\psi_j| \sum_{j=-\infty}^{\infty}|\psi_j| < \infty$$

② Show that $\sqrt{n}\left(\frac{1}{n}\sum_{t=1}^{n-h}(X_{t+h}-\mu)(X_t-\mu) - \frac{1}{n}\sum_{t=1}^{n-h}(X_{t+h}-\bar{X})(X_t-\bar{X})\right) \xrightarrow{P} 0$

$$= \frac{1}{\sqrt{n}}\sum_{t=1}^{n-h}\left[-\mu(X_t+X_{t+h}) + \mu^2 + \bar{X}(X_t+X_{t+h}) - \bar{X}^2\right]$$

$$= \frac{1}{\sqrt{n}}\sum_{t=1}^{n-h}(\bar{X}-\mu)\left[X_t + X_{t+h} - \bar{X} - \mu\right]$$

let $\tilde{X} = \frac{1}{n-h}\sum_{t=1}^{n-h} X_t$

$$= \frac{n-h}{\sqrt{n}}(\bar{X}-\mu)\left(\tilde{X}-\mu + \tilde{X}_h - \mu - (\bar{X}-\mu)\right)$$

$\tilde{X}_h = \frac{1}{n-h}\sum_{t=1}^{n-h} X_{t+h}$

By Theorem A.5, $\sqrt{n}(\bar{X}_n-\mu) \xrightarrow{d} N(0,V)$ (V in A.47)

let $\sqrt{n}(\bar{X}-\mu) \xrightarrow{d} Y_1$, $\sqrt{n-h}(\tilde{X}-\mu) \xrightarrow{d} Y_2$, $\sqrt{n-h}(\tilde{X}_h-\mu) \xrightarrow{d} Y_3$

$Y_1, Y_2, Y_3 \sim N(0,V)$

We have $\frac{n-h}{\sqrt{n}}\left(\frac{1}{\sqrt{n}}(\sqrt{n}(\bar{X}-\mu))\right)\left(\frac{1}{\sqrt{n-h}}(\sqrt{n-h}(\tilde{X}-\mu)) + \frac{1}{\sqrt{n-h}}(\sqrt{n-h}(\tilde{X}_h-\mu)) - \frac{1}{\sqrt{n}}(\sqrt{n}(\bar{X}-\mu))\right)$

$$= \frac{\sqrt{n-h}}{\sqrt{n}}\left(n^{-\frac14}(\sqrt{n}(\bar{X}-\mu))\right)\left(n^{-\frac14}(\sqrt{n-h}(\tilde{X}-\mu)) + n^{-\frac14}(\sqrt{n-h}(\tilde{X}_h-\mu)) - n^{-\frac14}\left(\sqrt{\tfrac{n-h}{n}}\right)(\sqrt{n}(\bar{X}-\mu))\right)$$

$$\xrightarrow{d} (0\cdot Y_1)(0\cdot Y_2 + 0\cdot Y_3 - 0\cdot Y_1) = 0$$

$\therefore$ also converges to 0 in probability.

Ex. 1.32 | Let $\{X_t : t = 0, \pm1, \pm2, \dots\}$ be iid $(0,\sigma^2)$

(a) For $h \geq 1$ and $k \geq 1$, show that $X_t X_{t+h}$ and $X_s X_{s+k}$ are uncorrelated for all $s \neq t$

Pf: $\mathrm{Cov}(X_t X_{t+h}, X_s X_{s+k}) = E(X_t X_{t+h} X_s X_{s+k}) = 0$ as $s = t+h$

$\Rightarrow s+k > s = t+h > t$

(b) For fixed $h \geq 1$, show that the $h \times 1$ vector

$$\sigma^{-2} n^{-1/2}\sum_{t=1}^{n}\begin{pmatrix} X_t X_{t+1} \\ X_t X_{t+2} \\ \vdots \\ X_t X_{t+h}\end{pmatrix} \xrightarrow{d} \begin{pmatrix} z_1 \\ \vdots \\ z_h \end{pmatrix}$$

where $z_1, \dots, z_h \sim N(0,1)$ iid

Here I only work out $\frac{1}{\sigma^2\sqrt{n}}\sum_{t=1}^{n} X_t X_{t+h} \xrightarrow{d} N(0,1)$

Let $Y_t = X_t X_{t+h}$, then $E(Y_t \mid F_{t-1}) = 0 \Rightarrow Y_t$ is a stationary MDS

We have $E(Y_t^2) = E(X_t^2)E(X_{t+h}^2) = \sigma^4 < \infty$

If we also have $\frac{1}{n}\sum_{t=1}^{n} Y_t^2 \xrightarrow{P} \sigma^4$, then the result follows by MCLT

More on martingale (p.12, Sect. 2.4 of the note "Martingale_CLT",

What is F_t? We construct F_t such that $\cdots \subseteq F_{t-1} \subseteq F_t \subseteq F_{t+1} \subseteq \cdots$

If X_t is known given F_t for each t, then $\{F_t\}_{-\infty}^{\infty}$ is said to be adapted to the sequence $\{X_t\}_{-\infty}^{\infty}$. The pair $\{X_t, F_t\}_{-\infty}^{\infty}$ are called an adapted sequence.

Given an adapted sequence $\{X_t, F_t\}_{-\infty}^{\infty}$, if we have

$E|X_t| < \infty$ and $E(X_t \mid F_{t-1}) = X_{t-1}$ for all t,

then the sequence is called a martingale.

Now, for $Y_t = X_t X_{t+h}$, consider $F_t = \{(X_s, Y_s), s \le t\}$,

then
$$E(Y_t \mid F_{t-1}) = E(X_t X_{t+h} \mid (X_{t-h}, X_{t-h}X_t))$$
$$= E(X_t X_{t+h} \mid X_t) = X_t E(X_{t+h} \mid X_t) = 0$$

$\therefore Y_t$ is a stationary MDS

To prove $\frac{1}{n}\sum_{t=1}^{n} Y_t^2 \xrightarrow{P} \sigma^4$, by WLLN, we have

$$\frac{1}{n}\sum_{t=1}^{n}\left(Y_t^2 - E(Y_t^2 \mid F_{t-1})\right) \xrightarrow{P} 0$$

Since $E(Y_t^2 \mid F_{t-1}) = E(X_t^2 X_{t+h}^2 \mid X_t) = X_t^2 E(X_{t+h}^2) = \sigma^2 X_t^2$

$$\therefore \quad \frac{1}{n}\sum_{t=1}^{n} Y_t^2 - \sigma^2 \frac{1}{n}\sum_{t=1}^{n} X_t^2 \xrightarrow{P} 0$$

Also note that $\frac{1}{n}\sum_{t=1}^{n} X_t^2 \xrightarrow{P} \sigma^2$ by classical WLLN

$$\therefore \quad \frac{1}{n}\sum_{t=1}^{n} Y_t^2 - \sigma^4 = \left(\frac{1}{n}\sum_{t=1}^{n} Y_t^2 - \sigma^2 \frac{1}{n}\sum_{t=1}^{n} X_t^2\right) + \left(\sigma^2 \frac{1}{n}\sum_{t=1}^{n} X_t^2 - \sigma^4\right) \xrightarrow{P} 0$$