17 Lecture 17 (Applications of the Cauchy integral formula)

Summary

- Integral formula for higher derivatives
- Liouville theorem
- A proof of fundamental theorem of algebra
- Zeros of a function

Theorem 17.1. *With the same assumption in Theorem 16.5, the n-th derivative of f at z_0 exists in $D(z_0; r)$, and is given by*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(w)}{(w - z_0)^{n+1}}\, dw. \tag{17.1}$$

Proof. (Proof 1) From Theorem 16.5 we know that the Taylor series

$$f(z) = \sum_{k=0} a_n (z - z_0)^n \tag{17.2}$$

converges at $z = z_0$, with coefficient a_n equal to

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(w)}{(w - z_0)^{n+1}}\, dw.$$

The curve C is a small circle centered at z_0 so that $f(z)$ is analytic inside C. By substituting $z = z_0$ into (17.1), we get

$$f(z_0) = a_0 = \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z_0}\, dw,$$

which is the same as Cauchy's integral formula in Theorem 16.4. Since Taylor series can be differentiated term-wise, we can differentiate both sides of (17.2) and substitute $z = z_0$. This gives

$$f'(z_0) = a_1 = \frac{1}{2\pi i} \int_C \frac{f(w)}{(w - z_0)^2}\, dw.$$

Repeat this process recursively, we obtain (17.1).

(Proof 2) We can also derive (17.1) directly from Cauchy integral formula. We first consider the first derivative $f'(z_0)$. Using Cauchy integral formula, we can express the fraction $(f(z_0+h)-f(z_0))/h$ as

$$\frac{f(z_0+h)-f(z_0)}{h} = \frac{1}{h}\frac{1}{2\pi i}\int_C \frac{f(w)}{w-z_0-h} - \frac{f(w)}{w-z_0}\, dw$$

where C is a the circle $C(z_0, r)$. The integral on the right can be simplified to

$$\frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z_0-h)(w-z_0)}\, dw.$$

Then,

$$\begin{aligned}
\frac{f(z_0+h)-f(z_0)}{h} - \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z_0)^2}\, dw &= \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z_0-h)(w-z_0)} - \frac{f(w)}{(w-z_0)^2}\, dw \\
&= \frac{h}{2\pi i}\int_C \frac{f(w)}{(w-z_0-h)(w-z_0)^2}\, dw
\end{aligned}$$

By ML inequality (Theorem 13.3), it can be bounded by

$$\Big|\int_C \frac{f(w)}{(w-z_0-h)(w-z_0)^2}\, dw\Big| \le \frac{1}{2\pi}\frac{M|h|}{(r-|h|)r^2}2\pi r$$

where M denote the maximum of $f(w)$ on the circle $C(z_0; r)$. As h approach 0, it approaches 0. Therefore

$$f'(z_0) = \lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h} = \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z_0)^2}\, dw.$$

Higher-order derivatives can be derived by induction. □

Based on (17.1) one can estimate the modulus of the n-th derivative.

Theorem 17.2 (Cauchy estimate)**.** *If $|f(z)| \le M$ for z on the circle $C(z_0; r) = \{z : |z-z_0| = r\}$ and f is analytic in a neighborhood of the disc $|z-z_0| \le r$, then*

$$|f^{(n)}(z_0)| \le M\frac{n!}{r^n}$$

for $n = 0, 1, 2, 3, \ldots$

Proof.

$$|f^{(n)}(z_0)| = \Big|\frac{n!}{2\pi i}\int_C \frac{f(w)}{(w-z_0)^{n+1}}\, dw\Big| \le \frac{n!}{2\pi}\frac{M}{r^{n+1}}2\pi r = \frac{n!M}{r^n}.$$

□

Using the Cauchy estimate, we can derive the following theorem

Theorem 17.3 (Liouville theorem)**.** *If a complex analytic function f is bounded on the whole complex plane, then it must be a constant function.*

We remark that $\sin(z)$ is *not* a counter-example, because $|\sin(z)|$ becomes very large when $\mathrm{Im}(z) \gg 0$.

Proof. Suppose $f(z) \leq M$ for all $z \in \mathbb{C}$. Then for any fixed complex number $z_0 \in \mathbb{C}$,

$$|f'(z_0)| \leq \frac{M}{r}$$

by Theorem 17.2. Since this is true for any $r > 0$, we can take $r \to \infty$. This gives $f'(z_0) = 0$. Because this is true for any $z_0 \in \mathbb{C}$, f' is identically zero. This means that f is a constant function. □

One can prove the fundamental theorem of algebra using Liouville theorem.

Theorem 17.4 (Fundamental theorem of algebra)**.** *Suppose $p(z) = c_0 + c_1 z + \cdots + c_d z^d$ is a polynomial of degree $d \geq 1$. Then $p(z)$ has a root in $\mathbb{C}$.*

Proof. We prove this by contradiction. Suppose $p(z) \neq 0$ for all $z \in \mathbb{C}$. Then $1/p(z)$ is a well-defined and analytic for all $z \in \mathbb{C}$, i.e., $1/p(z)$ is an entire function.

Without loss of generality, assume the leading coefficient c_d is equal to 1,

$$p(z) = c_0 + c_1 z + \cdots + z^d.$$

The ratio

$$\frac{p(z)}{z^d} = \frac{c_0}{z^d} + \frac{c_1}{z^{d-1}} + \cdots + 1$$

approaches 1 as $|z| \to \infty$. Hence, there exists a large real number N such that

$$|\frac{p(z)}{z^d}| > \frac{1}{2}$$

for all z with modulus larger than N. In other words, we have

$$|\frac{1}{p(z)}| < \frac{2}{|z^d|} < \frac{2}{N^d}$$

for all $|z| > N$.

On the other hand, the closed disc $\{z : |z| \leq N\}$ is compact and $1/p(z)$ is continuous in this compact set. So $|1/p(z)|$ is bounded by some number M for all $|z| \leq N$.

Therefore $|1/p(z)|$ is bounded by some constant for all $z \in \mathbb{C}$. By Liouville theorem, $1/p(z)$ is a constant function. But then $p(z)$ is constant, and it contradicts the assumption that $d \geq 1$. □

Definition 17.5. A point z_0 is called a *zero* of function f if $f(z_0) = 0$.

Suppose f is analytic in a domain D and $z_0 \in D$ is a zero of f. By Theorem 16.5 we can write f as a Taylor series

$$f(z) = c_1(z - z_0) + c_2(z - z_0)^2 + \cdots + c_n(z - z_0)^n + \cdots$$

for z in some neighborhood $D(z_0; r)$ of z_0. One of the following possibilities should occur

(a) $c_k = 0$ for all $k = 1, 2, 3, \ldots$. In this case $f(z)$ is identically equal to 0 in $D(z_0; r)$.

(b) The coefficients $c_1, c_2, \ldots$ are not all zero. The smallest integer m such that $c_m \neq 0$ is called the *order* of the zero at z_0. We can write $f(z)$ as

$$f(z) = (z - z_0)^m \big[c_m + c_{m+1}(z - z_0) + c_{m+2}(z - z_0)^2 + \cdots\big].$$

Because $c_m \neq 0$ by assumption, the power series inside the square bracket is nonzero for z in a sufficiently small neighborhood of z_0. We say that z_0 is an *isolated zero*.

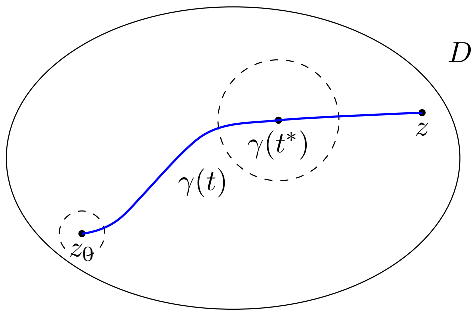
The following is a special property of analytic functions in general.

Theorem 17.6. *Suppose f is analytic in a domain D and $\{z_1, z_2, z_3, \ldots\} \subset D$ is a set of zeros of f. If $\{z_1, z_2, z_3, \ldots\}$ contains an accumulation point in D, then f is identically zero throughout the domain D.*

Proof. Let $z_0 \in D$ be an accumulation point of $\{z_1, z_2, z_3, \ldots\}$. The point z_0 must be a zero of f, because f is continuous. However, z_0 cannot be an isolated zero. Since exactly one of the condition in (a) or (b) before the theorem is true, the coefficients in the Taylor expansion

$$f(z) = c_1(z - z_0) + c_2(z - z_0)^2 + \cdots$$

must be all zero. i.e., $c_k = 0$ for all $k = 1, 2, 3, \ldots$. So $f(z)$ is identically zero in some open disc centered at z_0.



Let z be any point in the domain D. Since D is connected, we can draw a smooth path $\gamma(t)$ from z_0 to z, with $\gamma(0) = z_0$ and $\gamma(1) = z$. We know that $f(\gamma(t))$ is a continuous function of t, and $f(\gamma(t))$ is zero for all t closed to 0, i.e., $f(\gamma(t)) = 0$ for $0 \leq t \leq \epsilon$ for some ϵ between 0 and 1.

We start from z_0 and travel along the curve $\gamma(t)$ until $f(\gamma(t))$ is not zero. If $f(\gamma(t)) = 0$ for all $0 \leq t \leq 1$, then we will stop at $t = 1$, and $f(\gamma(1)) = f(z) = 0$. If $f(\gamma(t))$ is not zero for some t between 0 and 1, we can let t^* be the largest parameter such that $f(\gamma(t)) = 0$ for all t from 0 to t^*. More rigorously, we should define t^* using supremum

$$t^* \triangleq \sup\{\alpha : \ f(\gamma(t)) = 0 \text{ for } 0 \leq t \leq \alpha\}.$$

If $t^* < 1$, then the point $\gamma(t^*)$ is a non-isolated zero (because $f(\gamma(t)) = 0$ for all $t < t^*$). The Taylor series centered at $\gamma(t^*)$ defines a zero function within the radius of convergence. This means that t^* cannot be the supremum, as we can increase t^* slightly. This is a contradiction to the definition of t^*.

Therefore we must have $\alpha = 1$, and thus $f(z) = f(\gamma(1)) = 0$. Because this is true for any z in D, we have $f(z) = 0$ for all $z \in D$. $\square$

The previous theorem can be stated in the following form. It is called the identity theorem, because it give a criterion under which two functions are identical.

Theorem 17.7 (Identity theorem)**.** *If f and g are analytic functions in D and $\{z : \ f(z) = g(z)\}$ has an accumulation point in D, then $f(z) = g(z)$ for all $z \in D$.*

Proof. Take $h(z) = f(z) - g(z)$ as the function in Theorem 17.6. By assumption, the zeros of the function $h(z)$ has an accumulation point in D. Hence $h(z) = 0$ for all $z \in D$. $\square$

Theorem 17.7 is usually applied to the situation when two analytic functions $f(z)$ and $g(z)$ agree on a small open disc. Then $f(z)$ and $g(z)$ have the same value for all z in the domain.

In other words, suppose we know the values of a function $f(z)$ in a small open disc. Then, if the function is analytic, the values of $f(z)$ at other points in the domain are uniquely determined. No matter how small the open disc is, the value of an analytic function $f(z)$ is completed determined by the values of $f(z)$ inside this open disc. This is a special feature of analytic functions.

18 Lecture 18 (Integration of multi-valued function)

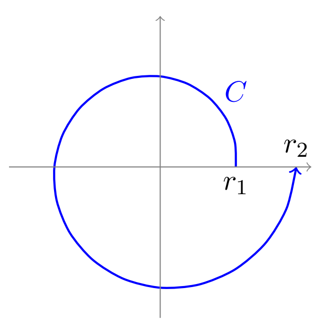
Summary

- Evaluation of integral using branch cut
- Evaluation of integral using local primitive

In Theorem 15.2 we show that we can always find a primitive of a function that is analytic in a disc. In general, a global primitive needs not exist. In this lecture illustrates how to evaluate complex integral of multi-valued function.

Consider the square root function $f(z) = \sqrt{z}$. It is a multi-valued function because for each nonzero complex number there are two choices for the square root. There are two branches for the square root function $\sqrt{z}$. We can draw a branch cut to select a particular branch and make the function single-valued.

As a numerical demonstration, we compute $\int_C \sqrt{z}\,dz$ for over a curve C shown as follows.

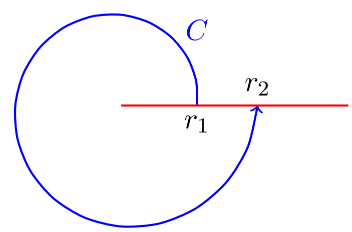


The curve start at the point $z = r_1$ and end at the point $z = r_2$, where r_1 and r_2 are positive real numbers. The curve revolve around the origin once. The function $\sqrt{z}$ is multi-valued. We take the positive branch at the beginning when $z = r_1$. Thus, the value of $\sqrt{r_1}$ is taking to be positive. The values of $\sqrt{z}$ varies continuously as we move along the curve. At the terminal point, value of $\sqrt{r_1}$ is negative. Now the question is well specified. We give two solutions below.

Solution 1. Parameterize the curve by a smooth function $\gamma(t)$ for t from 0 to 1, so that $\gamma(0) = r_1$ and $\gamma(1) = r_1$. We take a branch cut at the positive real axis, i.e., the argument is from 0 to 2π. For argument in this range, the square root function is single-valued:

$$\sqrt{re^{i\theta}} = \sqrt{r}e^{i\theta/2}$$

for $0 < \theta < 2\pi$.



A primitive function for this branch of square root function is

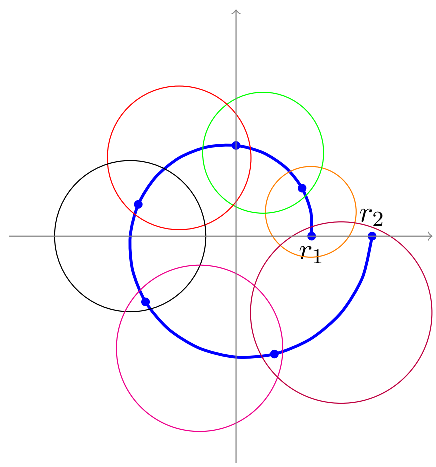
$$G(z) = \frac{2}{3}z^{3/2} = \frac{2}{3}|z|^{3/2}e^{i3\arg(z)/2},$$

where arg(z) takes values in the open interval $(0, 2\pi)$. The integral can be evaluated as follows,

$$\begin{aligned}\int_C \sqrt{z}\,dz &= \lim_{\epsilon\to 0^+} \frac{2}{3}\Big[G(\gamma(2\pi-\epsilon)) - G(\gamma(\epsilon))\Big] \\ &= \frac{2}{3}\Big[r_2^{3/2}e^{i3\pi} - r_1^{3/2}e^0\Big] \\ &= \frac{2}{3}(-r_2^{3/2} - r_1^{3/2}).\end{aligned}$$

In the calculation we take limit as ϵ approaching to zero from above. Taking limit is necessary because the function is not defined on the branch cut.

Solution 2. We divide the curve into pieces. Cover each piece by a circle and define a local primitive function in each circle. We arrange the circles so that adjacent circles are overlapping, and the two local primitive functions on the adjacent circles agree with each other on the overlapping area.



An example of covering the path is shown in the figure above. Each circle covers a portion of the curve, but does not cover the origin. The argument of the points in the first circle (in orange color) is between two constants, say a_1 and b_1. In this example a_1 is negative, and b_1 is approximately equal to $\pi/4$ in radian. For argument θ in the range (a_1, b_1), we define a local primitive function

$$F_1(z) = F_1(re^{i\theta}) = \frac{2}{3} r^{3/2} e^{i3\theta/2},$$

for $a_1 < \theta < b_1$. Since we restrict the domain of $F_1(z)$ to be the interior of the orange circle, $F_1(z)$ is a single-valued function.

A point in the second circle (shown in green color) has argument in a range (a_2, b_2), where a_2 is a number less than b_1, and b_2 is larger than $\pi/2$. We define a local primitive function F_2 for points within the green circle,

$$F_2(z) = F_2(re^{i\theta}) = \frac{2}{3} r^{3/2} e^{i3\theta/2},$$

for $a_2 < \theta < b_2$. The two functions F_1 and F_2 agree with each other on the overlapping area of the orange and green circle.

Similarly we define six local primitive functions. In the last circle, the range of the argument is from a_6 to b_6, where b_6 is a number slightly larger than 2π, and a_6 is a number larger than $3\pi/2$. The formula for computing $F_6(z)$ is

$$F_6(z) = F_2(re^{i\theta}) = \frac{2}{3} r^{3/2} e^{i3\theta/2},$$

for θ in the range (a_6, b_6).

Suppose that the curve is divided into six pieces. The range of t is divided into

$$[0, t_1],\ [t_1, t_2],\ [t_2, t_3],\ [t_3, t_4],\ [t_4, t_5],\ [t_5, 2\pi].$$

The parameters t_1 is chosen such that $\gamma(t_1)$ is contained in the first and second circles. For $j = 2, 3, 4, 5$, the value of t_j is chosen such that $\gamma(t_j)$ is contained in the j-th and $(j+1)$-th circle.

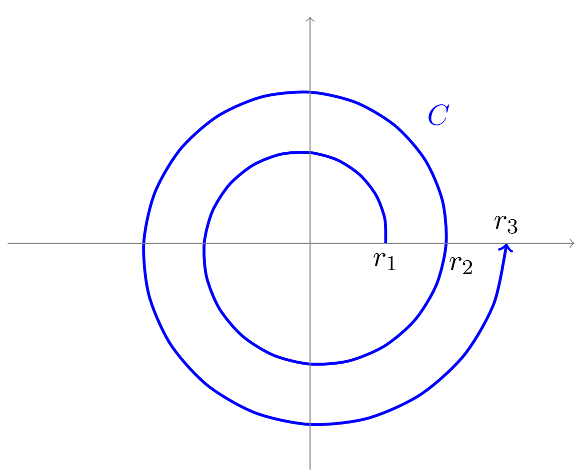
The integral can be calculated by

$$\begin{aligned}\int_C \sqrt{z}\, dz = [F_1(\gamma(t_1)) - F_1(\gamma(0))] + [F_2(\gamma(t_2)) - F_2(\gamma(t_1))] + [F_3(\gamma(t_3)) - F_3(\gamma(t_2))]\\ + [F_4(\gamma(t_4)) - F_4(\gamma(t_3))] + [F_5(\gamma(t_5)) - F_5(\gamma(t_4))] + [F_6(\gamma(2\pi)) - F_6(\gamma(t_5))]\\ = F_6(r_2) - F_1(r_1).\end{aligned}$$

In the last circle, the value of $F_6(2\pi)$ is $\frac{2}{3} r_2^{3/2} e^{i3(2\pi)/2} = -\frac{2}{3} r_2^{3/2}$. Therefore the answer is

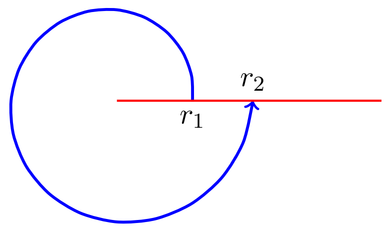
$$\frac{2}{3}(-r_2^{3/2} - r_1^{3/2}).$$

As the second example, we compute $\int_C \sqrt{z}\, dz$ for C shown below. The values of r_1, r_2 and r_3 are real, and $r_1 < r_2 < r_3$.

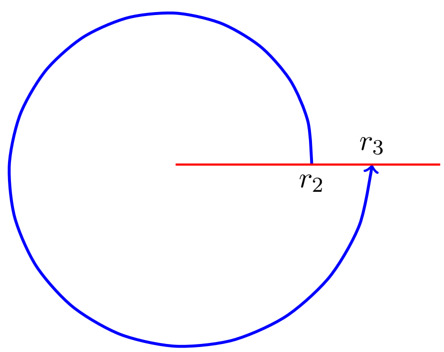


As in the first example, at the initial point $z = r_1$, we take $\sqrt{r_1}$ to be a positive number. The value of $\sqrt{z}$ changes continuously as we travel along the curve C. When the curve intersects the real axis at $z = r_2$, the value of $\sqrt{r_2}$ should be negative. At the end point $z = r_3$, the value of $\sqrt{r_3}$ is positive.

Solution 1. Divide the path into two parts. In each part, we take the positive real axis as the branch cut. The first part goes from $z = r_1$ to $z = r_2$. The calculation is the same as in the previous example.



The second part goes from $z = r_2$ to $z = r_3$.



In the second part we should use the primitive function

$$F(z) = -\frac{2}{3} r^{3/2} e^{i3\theta/2}$$

for $0 < \theta < 2\pi$.

Combining the two parts, we obtain the answer

$$\int_C \sqrt{z}\, dz = \frac{2}{3}\big(-r_2^{3/2} - r_1^{3/2}\big) + \frac{2}{3}\big(r_3^{3/2} + r_2^{3/2}\big) = \frac{2}{3}\big(r_3^{3/2} - r_1^{3/2}\big).$$

Solution 2. As we go from $z = r_1$ to $z = r_3$ long the curve, the argument of the point on the curve varies from 0 to 4π. By covering the curves by discs and using local primitive functions in each disc, we can compute the integral by

$$\int_C \sqrt{z}\, dz = \frac{2}{3} r_3^{3/2} e^{i3(4\pi)/2} - \frac{2}{3} r_1^{3/2} e^{i0} = \frac{2}{3}\big(r_3^{3/2} - r_1^{3/2}\big).$$

We note that in the example, the first method divides the path into two parts, the second method divides the path into more than two parts. In both methods, a local primitive function is selected for each patch, so that all the intermediate values are canceled.

19 Lecture 19 (Singularity)

Summary

- Removable singularity
- Pole
- Essential singularity
- Riemann's theorem on removable singularity
- Order of a pole

From Theorem 17.6, we know that an analytic function with a sequence of zeros converging to another zero in the domain must be the zero function. This is the reason why we mainly study functions with isolated zeros. For singular points, we have a similar notion of isolated singularity.

Definition 19.1. A point where a function failed to be analytic is called a *singular point*. A singular point z_0 is called an *isolated singular point* if the function is analytic in a neighborhood of z_0, except the point z_0.

Example 19.1. The function $f(z) = 1/(z^2+1)$ has two isolated singular points located at $z = i$ and $z = -i$.

Example 19.2. The function $\frac{1}{\sin(\pi z)}$ has a non-isolated singular point at $z = 0$. It is because this function is not defined, and hence singular, at $z = 1/n$, for $n = 1, 2, 3, 4, \ldots$. The sequence $\{1/n\}_{n=1}^{\infty}$ converges to $z = 0$.

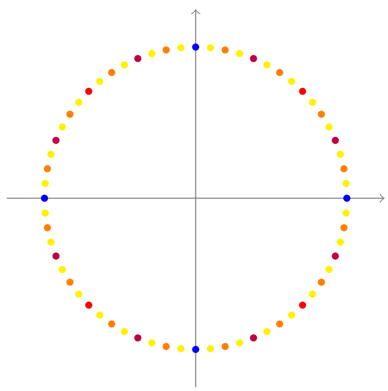
Example 19.3. The power series

$$f(z) \sum_{n=0}^{\infty} z^{2^n} = z + z^2 + z^4 + z^8 + z^{16} + \cdots$$

converges for $|z| < 1$. (This can be shown by comparison test.) This series diverges at (2^n)-th roots of unity, for all positive integer n. For example,

$$\begin{aligned} f(-1) &= -1 + 1 + 1 + 1 + \cdots = \infty, \\ f(i) &= i - 1 + 1 + 1 + \cdots = \infty, \\ f(-i) &= -i - 1 + 1 + 1 + \cdots = \infty, \\ f(e^{2\pi i/8}) &= e^{2\pi i/8} + e^{2\pi i/4} + e^{2\pi i/2} + 1 + 1 + \cdots = \infty. \end{aligned}$$

The set of singular points of $f(z)$ on the unit circle is dense, and hence are not isolated. This is an example of *natural boundary*, meaning that we cannot extend the domain of this function. (We shall discuss analytic continuation in later lecture.) The singular points on the unit circle are plotted below.



Isolated singularity can classified into three categories.

Definition 19.2. An isolated singular point z_0 of $f(z)$ is called

(i) a *removable singularity* if $f(z)$ is bounded in a neighborhood of z_0;

(ii) a *pole* if $|f(z)| \to \infty$ as $z \to z_0$;

(iii) an *essential singularity* otherwise.

Example 19.4. The function $f(z) = \frac{z^2-1}{z+1}$ has a removable singularity at $z = 1$, because

$$f(z) = \frac{(z+1)(z-1)}{z-1} = z + 1$$

for all complex number z that is close to but not equal to 1, and $z + 1$ is bounded in any open disc with finite radius centered at $z = 1$.

Example 19.5. The function $f(z) = \frac{1}{z(z+1)}$ has two poles located at $z = 0$ and $z = -1$.

Example 19.6. The function $f(z) = e^{1/z}$ has an essential singularity at $z = 0$. We can see that it is an essential singular point by approaching $z = 0$ from the right and from the left. For real variable x, if we take $x \to 0$ from the right, the value of $f(z)$ tends to positive infinity. Hence it is not bounded near $z = 0$. On the other hand if we take x approaching 0 from the negative real axis, then $f(z)$ tends to 0. The modulus is not approaching infinity.

The next theorem justifies the name "removable singularity".

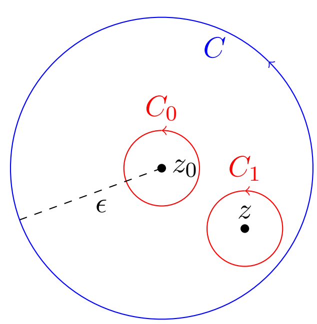
Theorem 19.3 (Riemann's theorem on removable singularity)**.** *Suppose f is analytic in a domain that contains a punctured disc $D(z_0;\epsilon) \setminus \{z_0\}$ (which equals $\{z : 0 < |z - z_0| < \epsilon\}$). If f is bounded in $D(z_0;\epsilon) \setminus \{z_0\}$, the we can re-defined f at z_0 so that f is analytic in $D(z_0;\epsilon)$*

More formally, the theorem is saying that there exists an analytic function $\tilde{f}$ defined on the open disc $D(z_0;\epsilon)$ such that $\tilde{f}(z) = f(z)$ for all $z \in D(z_0;\epsilon) \setminus \{z_0\}$[1]. For instance, in Example 19.4, we can re-define $f(1)$ by $f(1) = 2$. The new function is the same as the function $z + 1$, which is an entire function.

Proof. Suppose that $|f(z)|$ is upper bounded by M for z in the punctured disc $D(z_0;\epsilon)\setminus\{z_0\}$. Define

$$\tilde{f}(z) \triangleq \frac{1}{2\pi i} \oint_C \frac{f(w)}{w - z}\, dw,$$

where C is the circle $|z - z_0| = \epsilon$, with counter-clockwise orientation, and z is any point inside the circle. We want to show that (i) $\tilde{f}(z) = f(z)$ for all $z \in D(z_0;\epsilon) \setminus \{z_0\}$, and (ii) $\tilde{f}$ is analytic in $D(z_0;\epsilon)$.



[1]The symbol $\tilde{f}$ is pronounced as "tilde f" or "twiddle f"

Let z be a point in the puncture disc $0 < |z-z_0| < \epsilon$. Draw a small circle C_0 of radius ϵ_0 with center at z_0, such that C_0 is inside the circle C but does not contain z. Draw another small circle C_1 of radius ϵ_1 with center at z_1 such that C_1 is inside the circle C but does not contain z_0.

By Cauchy's theorem for multiply connected region (Theorem 16.3), we have

$$2\pi i\tilde{f}(z) = \int_C \frac{f(w)}{w-z}\,dw = \int_{C_0} \frac{f(w)}{w-z}\,dw + \int_{C_1} \frac{f(w)}{w-z}\,dw. \tag{19.1}$$

Since $f(z)$ is analytic inside the circle C_1, we can apply Cauchy integral formula (Theorem 16.4) to obtain

$$\int_{C_1} \frac{f(w)}{w-z}\,dw = 2\pi i f(z).$$

We next prove that $\int_{C_0} \frac{f(w)}{w-z}\,dw = 0$. For complex number w on C_0, the distance $|w-z|$ is lower bounded by $|z-z_0| - \epsilon_0$. Moreover, it is assumed that $f(z)$ is upper bounded by M for z insider the circle C, and hence is bounded by M for w on the circle C_0. By ML inequality (Theorem 13.3)

$$\Big|\int_{C_0} \frac{f(w)}{w-z}\,dw\Big| \leq \frac{M}{|z-z_0|-\epsilon_0} 2\pi\epsilon_0,$$

which approaches 0 as $r \to 0$. Therefore $\int_{C_1} \frac{f(w)}{w-z}\,dw = 0$. The equality in (19.1) becomes $2\pi i\tilde{f}(z) = 2\pi i f(z)$ for all $z \in D(z_0;\epsilon) \setminus \{z_0\}$. This proves part (i).

For part (ii), we just need to show that $\tilde{f}(z)$ is analytic at z_0, because $f(z)$ is equal to $f(z)$ that is analytic in $D(z_0;\epsilon) \setminus \{z_0\}$. We first use the definition of $\tilde{f}$ at z_0 to write

$$\frac{\tilde{f}(z_0+h) - f(z_0)}{h} = \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z_0-h)(w-z_0)}\,dw$$

where C is the circle $|z-z_0| = \epsilon$. We then show that it converges as $h \to 0$,

$$\begin{aligned}\Big|\frac{\tilde{f}(z_0+h) - \tilde{f}(z_0)}{h} - \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z_0)^2}dw\Big| &= \frac{1}{2\pi}\int_C \frac{f(w)h}{(w-z_0)^2(w-z_0-h)}\,dw \\ &\leq \frac{1}{2\pi}\frac{M|h|}{\epsilon^2(\epsilon-|h|)}2\pi\epsilon.\end{aligned}$$

For fixed ϵ, it converges to 0 as $|h|$ approaches 0. Therefore $f'(z_0)$ exists and is equal to

$$f'(z_0) = \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z_0)^2}\,dw.$$

This proves part (ii). □

In view of Theorem 19.3, we can ignore any removable singularity, because we can always re-define the function appropriately so that it is no longer a singularity. This leads to the following definition.

Definition 19.4. A complex function is called a *meromorphic function* if all the singularity points are poles.

Example 19.7. The complex function $\frac{1}{\sin z}$ is not defined at 0, $\pm\pi$, $\pm 2\pi$, $\pm 3\pi$, etc. There are countably many singularity points, and each singular point is a pole of order 1.

Suppose z_0 is a pole of $f(z)$. By definition we have $|f(z)| \to \infty$ as $z \to z_0$, and $f(z)$ is nonzero in a small neighborhood of z_0. The reciprocal function $g(z) = 1/f(z)$ is thus bounded in a neighborhood of z_0. The singular point z_0 is a removable singularity of $g(z)$. By Theorem 19.3, we can re-defined $g(z)$ at $z = z_0$ by $g(z_0) = 0$ so that we can expand $g(z)$ as a power series near z_0,

$$g(z) = \sum_{n=1}^{\infty} a_n (z - z_0)^n.$$

Note that the constant term is zero because $g(z_0) = 0$. The coefficients a_n's cannot be all zero, otherwise $g(z)$ is a zero function within the region of convergence. The smallest integer m such that a_m is not zero is the order of zero z_0, and is defined as the order of the pole of $f(z)$ at z_0,

$$a_1 = a_2 = \cdots = a_{m-1} = 0, \text{ but } a_m \neq 0.$$

To compute the order more efficiently, we note that the order of a zero at z_0 of a function $f(z)$ is the smallest positive integer m such that

$$\lim_{z \to z_0} \frac{f(z)}{(z - z_0)^m} = c \neq 0.$$

Similarly, the order of a pole at z_0 of a function $f(z)$ is the smallest positive integer m such that

$$\lim_{z \to z_0} (z - z_0)^m f(z) = c \neq 0.$$

We can take this as the definition of the order of a pole.

Definition 19.5. The *order* of a pole z_0 of $f(z)$ is the smallest positive integer m such that $\lim_{z \to z_0} (z - z_0)^m f(z)$ is a nonzero constant. A pole of order 1 is called a *simple* pole. A pole of order 2 is called a *double* pole.

Example 19.8. Consider a rational function $f(z) = \frac{(z-1)(z-2)}{(z-3)(z+i)^2}$. This function $f(z)$ has two zeros of order 1 at $z = 1$ and $z = 2$. There is a simple pole at $z = 3$, because

$$f(z) = \frac{1}{z-3}\left[\frac{(z-1)(z-2)}{(z+i)^2}\right]$$

and $\frac{(z-1)(z-2)}{(z+i)^2}$ is nonzero when evaluated at $z = 3$. We can formally check this by calculating

$$\lim_{z\to 3}(z-3)f(z) = \lim_{z\to 3}\frac{(z-1)(z-2)}{(z+i)^2} = \frac{(3-1)(3-2)}{(3+i)^2} \neq 0.$$

Another pole of $f(z)$ is located at $z = -i$. This is a double pole because

$$\lim_{z\to -i}(z+i)f(z) = \lim_{z\to -i}\frac{(z-1)(z-2)}{(z-3)(z+i)} = \infty,$$

but

$$\lim_{z\to -i}(z+i)^2 f(z) = \lim_{z\to -i}\frac{(z-1)(z-2)}{z-3} = \frac{(-i-1)(-i-2)}{-i-3} \neq 0.$$

Example 19.9. The function

$$f(z) = \frac{1}{z(z-1)} - \frac{2}{z(z-2)}$$

has two poles, located at $z = 1$ and $z = 2$. We can see this by simplying $f(z)$ to

$$f(z) = \frac{(z-2) - 2(z-1)}{z(z-1)(z-2)} = \frac{-z}{z(z-1)(z-2)}$$

The singularity at $z = 0$ is removable. The pole at $z = 1$ and $z = 2$ are simple.

20 Lecture 20 (Laurent series)

Summary

- Convergence region of Laurent series
- Computing the coefficients of Laurent series by complex integral

Definition 20.1. A *Laurent series* is an infinite sum in the form

$$\sum_{n=-\infty}^{\infty} a_n z^n. \tag{20.1}$$

We say that it converges if both $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=1}^{\infty} a_{-n} z^{-n}$ converge. Sometime we write a Laurent series in the form

$$\sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^{-n}.$$

More generally, a *Laurent series at z_0* is an expression in the form

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n}.$$

The first summation $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ is called the *analytic part* and the second summation $\sum_{n=1}^{\infty} b_n (z-z_0)^{-n}$ is called the *principal part*. If the principal part is zero, a Laurent series reduces to a Taylor series.

Remark. Suppose the coefficients a_n's in (20.1) are conjugate symmetric, i.e., $a_{-n} = a_n^*$. If we substitute z by $e^{2\pi i t}$, for some real number t, then

$$\sum_{n=-\infty}^{\infty} a_n e^{2\pi i n t}$$

is a real function with period 1. In fact we can re-write it as

$$\begin{aligned} &a_0 + \sum_{n=1}^{\infty} (a_n e^{2\pi i n t} + a_{-n} e^{-2\pi i n t}) \\ &= a_0 + \sum_{n=1}^{\infty} \mathrm{Re}(a_n)\cos(2\pi n t) - \mathrm{Im}(a_n)\sin(2\pi n t). \end{aligned}$$

It is the same as a Fourier series.

Example 20.1. Let $f(z)$ denote the function $\frac{1}{z(1-z)}$. Using geometric series, we can expand it at $z = 0$,

$$\begin{aligned} \frac{1}{z(1-z)} &= \frac{1}{z}(1 + z + z^2 + z^3 + \cdots) \\ &= \frac{1}{z} + 1 + z + z^2 + z^3 + \cdots \end{aligned}$$

It converges for $0 < |z| < 1$. The principal part is $1/z$.

If we expand it at $z = 1$ using geometric series, we obtain

$$\begin{aligned}\frac{1}{z(1-z)} &= -\frac{1}{z-1}\frac{1}{1+z-1} \\ &= -\frac{1}{z-1}(1-(z-1)+(z-1)^2-(z-1)^3+\cdots) \\ &= -\frac{1}{z-1}+1-(z-1)+(z-1)^2-\cdots\end{aligned}$$

The principal part is $-1/(z-1)$. It converges for $0 < |z-1| < 1$.

If we expand it at $z = 2$, the Laurent series at $z = 2$ is

$$\begin{aligned}\frac{1}{z(1-z)} &= \frac{1}{z}+\frac{1}{1-z} \\ &= \frac{1}{2}\frac{1}{1+\frac{z-2}{2}}-\frac{1}{1+(z-2)} \\ &= \sum_{n=0}^{\infty}(-1)^n\Big[\frac{(z-2)^n}{2^{n+1}}-(z-2)^n\Big]\end{aligned}$$

This is a Taylor series centered at $z = 2$. The principal part is zero, and it converges for all $|z-2| < 1$.

The next example shows that we can also compute Laurent series at essential singularity.

Example 20.2. The power series expansion of e^z is

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

By substituting z by $1/z$ we get the Laurent series of $e^{1/z}$,

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots$$

converging whenever $z \neq 0$. This function has an essential singularity at $z = 0$ (see Example 19.6).

Theorem 20.2. *A Laurent series* $\sum_{n=0}^{\infty} a_n z^n$ *in general converges in an annulus* $R_1 < |z| < R_2$.

Proof. Using Hadamard's formula for radius of convergence, the analytic part $\sum_{n=0}^{\infty} a_n z^n$ converges if

$$|z| < \frac{1}{\limsup |a_n|^{1/n}} \triangleq R_2.$$

For the principal part we make a substitution $u = 1/z$,

$$\sum_{n=1}^{\infty} b_n z^{-n} = \sum_{n=1}^{\infty} b_n u^n.$$

It converges whenever

$$|u| < \frac{1}{\limsup |b_n|^{1/n}}$$

or

$$|z| > \limsup |b_n|^{1/n} \triangleq R_1.$$

Combining the two parts, a Laurent series converges if $|R_1| < |z| < |R_2|$. □

Theorem 20.3. *A function f analytic in an annulus $|R_1| < |z| < |R_2|$ can be expanded as a Laurent series (in powers of z).*

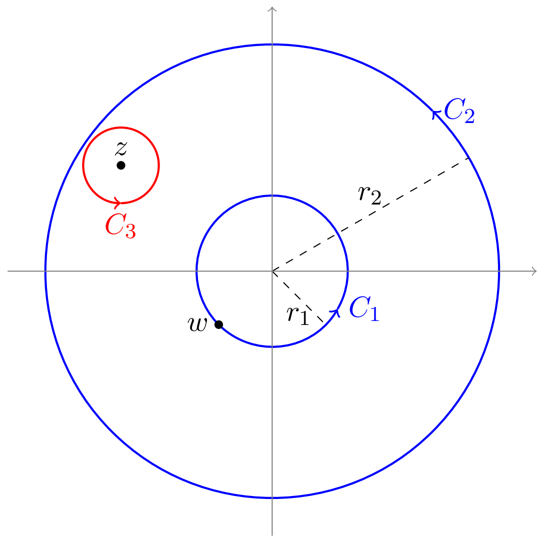
Remark. For Laurent series it is not true in general that $a_n = f^{(n)}(0)/n!$, because the function f need not be defined at $z = 0$. Instead, the coefficients can be computed using integrals.

Proof. Pick two real numbers r_1 and r_2 such that

$$R_1 < r_1 < r_2 < R_2.$$

Draw a circle C_1 with radius r_1 centered at origin, and a circle C_2 with radius r_2 centered at the origin, with counter-clockwise orientation. The choice of r_1 and r_2 ensures that function $f(z)$ is well-defined on C_1 and C_2.

Let z be a complex numberin the area between C_1 and C_2. Draw a positively-oriented circle C_3 centered at z with radius r_3 such that C_3 is within the area between C_1 and C_2. The notation is illustrated in the following figure.



By Cauchy's integral formula (Theorem 16.4),

$$f(z) = \frac{1}{2\pi i} \int_{C_3} \frac{f(w)}{w-z}\, dw. \tag{20.2}$$

Then, by applying Cauchy's theorem for multiply connected region (Theorem 16.3), we get

$$f(z) = \frac{1}{2\pi i} \int_{C_3} \frac{f(w)}{w-z}\, dw = \frac{1}{2\pi i} \int_{C_2} \frac{f(w)}{w-z}\, dw - \frac{1}{2\pi i} \int_{C_1} \frac{f(w)}{w-z}\, dw. \tag{20.3}$$

The first integral can be expanded as a convergent Taylor series (See the proof of Theorem 16.5)

$$\frac{1}{2\pi i} \int_{C_2} \frac{f(w)}{w-z}\, dw = \sum_{n=0}^{\infty} a_n z^n, \tag{20.4}$$

and the n-th coefficient can be computed by

$$a_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(w)}{w^{n+1}}\, dw.$$

We note that the derivation of (20.4) does not require that $f(z)$ is analytic throughout the region enclosed by C_1. But we do need to assume $f(z)$ is analytic inside C_3 in (20.2).

For the second integral in (20.3), we consider the integral

$$\int_{C_1} \frac{f(w)}{z-w}\, dw$$

and write

$$\frac{1}{z-w} = \frac{1}{z(1-w/z)} = \frac{1}{z}\big(1 + \frac{w}{z} + \frac{w^2}{z^2} + \frac{w^3}{z^3} + \cdots\big)$$

which converges for $|w| < |z|$. Since the location of z is outside C_1, we have convergence for all $w \in C_1$. Ignoring convergence issue for the moment, we obtain

$$\int_{C_1} \frac{f(w)}{w-z}\, dw = \sum_{n=1}^{\infty} \Big(\int_{C_1} f(w) w^{n-1}\, dw \Big) \frac{1}{z^n}. \tag{20.5}$$

This is the required principal part.

To make the argument rigorous, we can avoid the exchange of infinite sum and integral by introducing a remainder term.

$$\sum_{n=1}^{\infty} \frac{w^n}{z^n} = \sum_{n=1}^{N-1} \frac{w^n}{z^n} + \sum_{n=N}^{\infty} \frac{w^n}{z^n} = \sum_{n=1}^{N-1} \frac{w^n}{z^n} + \frac{w^N}{z^{N-1}(z-w)}.$$

The tail of the infinite summation starting from the N-th term converges to the remainder term if $|w| < |z|$. Since it is now a finite sum, we can exchange the order of summation and integral to get

$$\int_{C_1} \frac{f(w)}{w-z}\, dw = \sum_{n=1}^{N-1} \Big(\int_{C_1} f(w) w^{n-1} \Big) z^{-n} + \int_{C_1} \frac{f(w)}{z-w} \frac{w^N}{z^N}\, dw$$

The next step is to show that the remainder term approach zeros, as N approach infinity. For $w \in C_1$ and z outside C_1, we have

$$|w| = r_1, \quad \text{and } |z-w| \geq |z| - r_1.$$

Furthermore, $|f(w)|$ is bounded by some constant M for $w \in C_1$, because C_1 is a compact set and $|f(w)|$ is a continuous function on C_1. By ML inequality (Theorem 13.3),

$$\Big| \int_{C_1} \frac{f(w)}{z-w} \frac{w^N}{z^N}\, dw \Big| \leq \frac{M}{|z| - r_1} \frac{r_1^N}{|z|^N} 2\pi r_1.$$

Because $\frac{r_1}{|z|} < 1$, $\frac{r_1^N}{|z|^N} \to 0$ as $N \to \infty$. This proves the convergence in (20.5).

Since r_1 can be any number larger than R_1, and r_2 can be any number less than R_2 (satisfying $r_1 < r_2$), we have a convergent Laurent series for any z in the annulus $R_1 < |z| < R_2$. □

Corollary 20.4. *With the same notation in the proof of Theorem 20.3, the coefficients in the Laurent series*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^{-n}$$

can be obtained by

$$a_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(w)}{w^{n+1}} \, dw \quad (n = 0, 1, 2, 3, \ldots)$$

and

$$b_n = \frac{1}{2\pi i} \int_{C_1} f(w) w^{n-1} \, dw \quad (n = 1, 2, 3, 4, \ldots).$$