# MAT2002 Ordinary Differential Equations
# Second-order linear equations

Dongdong He

The Chinese University of Hong Kong (Shenzhen)

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# Overview

1. Second-order linear equations
   - Existence and Uniqueness
   - Principle of superposition

# Outline

1. Second-order linear equations
   - Existence and Uniqueness
   - Principle of superposition

# Second-order ODE

A second order ODE has the form

$$y'' = F(t, y, y')$$

If $F = g(t) - p(t)y' - q(t)y$, then the ODE is a linear ODE. Otherwise the ODE is nonlinear.

# A special 2nd-order ODE: Reduce to first order

Although the general 2nd-order nonlinear ODE is more involved, under certain condition, the 2nd-order ODE can be reduced into first order one.

Given a second order ODE:$y''(t) = F(t, y')$. Using the substitution $p(t) = y'(t)$, we have

$$p'(t) = y''(t) = F(t, y') = F(t, p) \Rightarrow p'(t) = F(t, p),$$

that is, we now have a **<u>first order</u>** equation.

As an example, solve the ODE

$$y'' + ay' = 0, \qquad a \in \mathbb{R}, \qquad a \neq 0.$$

Setting $p = y'$ the ODE satisfied by $p$ is

$$p' + ap = 0 \Rightarrow p(t) = c_0 \exp(-at), \quad c_0 \in \mathbb{R}.$$

Hence

$$y'(t) = c \exp(-at) \Rightarrow y(t) = \frac{-c_0}{a} \exp(-at) + c_1, \quad c_0, c_1 \in \mathbb{R}.$$

Note that we need two initial conditions to determine the constants $c_0$ and $c_1$.

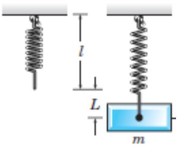
# Motivation for second-order linear ODEs

In this chapter, we will mainly study the second-order linear ODEs. Second order linear ODEs are important since they serve as mathematical models for some important physical processes. Now let's first look at one important application in mechanics.

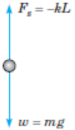
# Motivation for second-order linear ODEs

**Motivation - vibrations.** Consider the motion of a mass on a spring. The upper end of the spring is the fixed to the wall and the lower end is attached to an object with mass $m > 0$. Before attaching the object, the spring has a length $l > 0$, and after attaching the spring is **stretched** by a length $L > 0$ downwards.

Case 1: assume no additional force is acting on the mass-spring system. There are just two forces acting on the object:
(1) the weight that acts downwards $F_g = mg$.
(2) a **restoring** force $F_s$ from the spring that tries to pull the object upwards.




(a) Spring-mass system-equilibrium state




(b) Force analysis

# Motivation for second-order linear ODEs

We assume that the stretching $L$ is small, so that the force $F_s$ is proportional to $L$. Denoting by the constant of proportionality by $k$, we now have $F_s = kL$. This is commonly known as **Hooke's law** and $k$ is called the **spring constant**. The net force (pointing downwards) is

$$F = mg - kL,$$

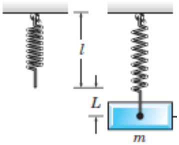
and if the object is in equilibrium, we must have $F = 0$ and so

$$mg = kL.$$

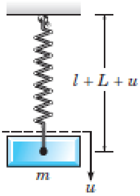
# Motivation for second-order linear ODEs

Case 2: suppose we pull on the object, and the spring is further extended.

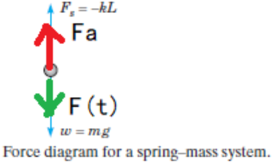
We want to **measure** the **displacement** $u(t)$ of the object from its equilibrium position.




(c) Spring-mass system- equilibrium state




(d) Spring-mass system




(e) Force analysis

# Motivation for second-order linear ODEs

The forces acting on the object are the following:

(1) the weight $Fg = mg$ acting downwards;

(2) a restoring force from the spring pulling the mass upwards $F_s = -k(L + u)$;

(3) a resistance force $F_a$ (air resistance/friction) that acts in the opposite of the motion and is proportional to the speed $u'$. This is usually referred to as **viscous damping** and $F_a = -\gamma u'$ with constant $\gamma > 0$ (**damping constant**);

(4) an external force $F(t)$ that acting on the object (could be upward or downward).

# Motivation for second-order linear ODEs

Using Newton's second law, we arrive at the ODE

$$\begin{aligned} &mu''(t) = mg - k(L + u(t)) - \gamma u'(t) + F(t) \\ \Rightarrow &mu''(t) + \gamma u'(t) + ku(t) = F(t) \end{aligned}$$

if we also use $mg = kL$. To complete the model we specific two initial conditions

$$u(0) = u_0, \quad u'(0) = v_0,$$

where $u_0$ is the initial displacement and $v_0$ is the initial velocity.

# Second-order linear ODEs

The general second order linear ODE has the form

$$a(t)y'' + b(t)y' + c(t)y = d(t)$$

for some given functions $a, b, c$ and $d$. If $a(t) \neq 0$ then we can express the second order ODE alternatively as

$$y'' + p(t)y' + q(t)y = r(t), \quad p(t) = \frac{b(t)}{a(t)}, q(t) = \frac{c(t)}{a(t)}, r(t) = \frac{d(t)}{a(t)}.$$

In the introduction, we discussed that for a second order ODE usually requires **two** initial conditions to get a unique solution

$$y(t_0) = y_0, \quad y'(t_0) = y_1,$$

for given constants $t_0, y_0, y_1$. Note that, the initial conditions state that not only the solution $y(t)$ passes through the point $t_0, y_0$, but also **its slope** $y'(t)$ passes through the point $t_0, y_1$.

# Existence and Uniqueness

Similar as the existence and uniqueness theorem to the first-order linear equations, we have the existence and uniqueness theorem for the second-order linear equations.

## Theorem 5.1

*Consider the IVP*

$$y'' + p(t)y' + q(t)y = r(t), \quad y(t_0) = y_0, \quad y'(t_0) = y_1.$$

*Suppose $I = (\alpha, \beta) \subset \mathbb{R}$ is any open interval such that $t_0 \in I$, and the functions $p, q, r$ are* **continuous** *in $I$. Then, there is* **exactly one** *solution $y(t)$ to the IVP for $t \in I$. The solution $y(t)$ is defined throughout the interval where $p, q, r$ are continuous.*

We will not discuss the proof of the theorem. Indeed, it can be proved through the Picard iterative method and uniform convergence theory, which is not pursued in this course. It will be sufficient use the result in this course.

Remark: If the conditions of the theorem are satisfied, then the solution $y(t)$ is **at least twice differentiable** in $I$.

# Existence and Uniqueness

## Example 5.2

The IVP

$$y'' + \frac{\exp(t)}{t^2 - 3t}y' - \frac{\sin(t+3)}{t^2 - 3t}y = 0, \quad y(1) = 2, \quad y'(1) = 1.$$

The functions $p(t) = \frac{\exp(t)}{t^2 - 3t}$ and $q(t) = \frac{\sin(t+3)}{t^2-3t}$ are continuous except at the points $t = 0$ and $t = 3$. Since $t_0 = 1$, the largest interval containing $t_0$ and in which the functions $p, q$ are continuous is $(0, 3)$. By Existence and Uniqueness Theorem, there exists a unique solution to the IVP for $t \in (0, 3)$.

If the initial conditions are set to be $y(4) = 2$ and $y'(4) = 1$, then the largest interval for which there exists a unique solution to the IVP is $(3, \infty)$.

# Existence and Uniqueness

## Example 5.3

(Application of uniqueness). Find the solution to the IVP

$$y'' + p(t)y' + q(t)y = 0, \quad y(t_0) = 0, \quad y'(t_0) = 0,$$

for continuous functions $p, q$ and given constant $t_0 \in \mathbb{R}$.

Note that $y \equiv 0$ is a solution to the IVP. By Existence and Uniqueness Theorem, there is exactly one solution, and so the **only** solution is $y \equiv 0$.

# Homogeneous second-order linear equations

Now we introduce the following classification.

**Definition 5.4**

(Homogeneous equation). A second order linear ODE

$$p(t)y'' + q(t)y' + r(t)y = s(t), \quad p(t) \neq 0,$$

is called **homogeneous** if $s(t) \equiv 0$. Otherwise, if $s(t) \neq 0$, the ODE is called **non-homogeneous**.

# Principle of superposition

For second order linear **homogeneous** equations we have the following.

## Theorem 5.5

(Principe of superposition). *If $y_1$ and $y_2$ are two solutions of the ODE*

$$a(t)y'' + b(t)y' + c(t)y = 0. \tag{1}$$

*Then for any constants $c_1, c_2 \in \mathbb{R}$, the function $c_1y_1(t) + c_2y_2(t)$ is also a solution to the ODE.*

A special case is when $c_1 = 0$ we get the solution $y_2$ and when $c_2 = 0$ we get $y_1$.

# Principle of superposition

**Take away message** -From two solutions we can construct infinite solutions to the homogeneous linear ODE. That is, we can define a family of solutions

$$S = \{y = c_1 y_1 + c_2 y_2 \mid c_1, c_2 \in \mathbb{R}\}$$

to the ODE.

# Principle of superposition

## Example 5.6

One can check that the functions $y_1(t) = \exp(-2t)$ and $y_2(t) = \exp(-3t)$ are solutions to the ODE

$$y'' + 5y' + 6y = 0. \tag{2}$$

We can use $y_1(t)$ and $y_2(t)$ to construct a family of solutions

$$c_1 y_1(t) + c_2 y_2(t), \quad c_1, c_2 \in \mathbb{R}$$

by using the linear combination of $y_1$ and $y_2$ to the ODE.

# Principle of superposition

Consider

$$y'' + p(t)y' + q(t)y = 0. \tag{3}$$

The next question is: "Given two solutions $y_1(t)$ and $y_2(t)$ of the above equations, can any solution to the ODE (3) be expressed as a <u>**linear combination**</u> of $y_1(t)$ and $y_2(t)$?"

## Definition 5.7

Given $y_1(t)$ and $y_2(t)$,

$$W[y_1, y_2](t) = \left| \begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix} \right|$$

is called the Wronskian for $y_1$ and $y_2$.

Indeed, we have the following theorem.

# Principle of superposition

## Theorem 5.8

*Suppose that $I$ be an open interval in which $p(t)$ and $q(t)$ are continuous. Let $y_1(t)$ and $y_2(t)$ be two solutions to the ODE*

$$y'' + p(t)y' + q(t)y = 0$$

*for $t \in I$. Then, any solution $y(t)$ to the ODE can be expressed as*

$$y(t) = c_1 y_1(t) + c_2 y_2(t) \tag{4}$$

*for constants $c_1$ and $c_2$ $\Leftrightarrow \exists t_0 \in I$ such that the Wronskian $W(y_1, y_2)[t_0] \neq 0$.*

Remark: The theorem says that if $y_1(t)$ and $y_2(t)$ be two solutions to the above ODE and $W(y_1, y_2)[t_0] \neq 0$, then the general solution to the above ODE is given be the formula (4).

# Principle of superposition

## Proof.

"$\Leftarrow$". That is, we assume there is a point $t_0 \in I$ where the Wronskian $W(y_1, y_2)[t_0]$ is non-zero. Let $\phi$ be any solution to the ODE. We need to show that $\phi$ can be written as a linear combination of $y_1$ and $y_2$.
Consider the IVP

$$y'' + p(t)y' + q(t)y = 0, \quad y(t_0) = \phi(t_0), \quad y'(t_0) = \phi'(t_0). \tag{5}$$

Then, a solution to the IVP is the function $\phi$ itself. Since the Wronskian is non-zero at $t_0$, the matrix-vector problem

$$\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \phi(t_0) \\ \phi'(t_0) \end{pmatrix}$$

admits a unique solution $(c_1^*, c_2^*)$. This means that the function $z(t) = c_1^* y_1(t) + c_2^* y_2(t)$ is a solution to the IVP (5).
By Existence and Uniqueness Theorem, there is only one solution to the IVP, therefore

$$\phi(t) = z(t) = c_1^* y_1(t) + c_2^* y_2(t).$$

□

# Principle of superposition

## Proof.

"$\Rightarrow$". We prove by contrapositive statement, which needs to show that if there is no points where the Wronskian is non-zero, then there is a solution $\phi$ to the ODE **cannot** be written as a linear combination of $y_1$ and $y_2$. Suppose for all $t \in I, W(y_1, y_2)[t] = 0$, in particular, $W(y_1, y_2)[t_0] = 0$. Then using linear algebra, there exists constants $x_0, x_1$ such that the following matrix-value problem

$$\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}$$

has no solution. □

# Principle of superposition

## Proof.

That is, we cannot find constants $c_1, c_2$ such that

$$\begin{aligned} c_1 y_1(t_0) + c_2 y_2(t_0) &= x_0, \\ c_1 y_1'(t_0) + c_2 y_2'(t_0) &= x_1, \end{aligned}$$

Thus, we can't find $c_1, c_2$ such that $c_1 y_(t) + c_2 y_2(t)$ is the solution to the IVP

$$y'' + p(t)y' + q(t)y = 0, \quad y(t_0) = x_0, \quad y'(t_0) = x_1. \tag{6}$$

But the above IVP must have a solution $\psi(t)$ by the existence and uniqueness theorem. We can see that it is not possible to write $\psi$ as a linear combination of $y_1$ and $y_2$. □

# Principle of superposition

## Remark 1

*The above Theorem says that once we know two solutions $y_1$ and $y_2$ to the ODE, and if the Wronskian $W$ is non-zero at some point $t_0 \in I$, then we know what the general solution to the ODE looks like. In particular we can express every solution to the ODE as a linear combination of $y_1$ and $y_2$. In this case, we say that $(y_1, y_2)$ form a* **fundamental set of solutions** *(FSS) to the ODE.*

## Recall: example

$y_1(t) = \exp(-2t)$ and $y_2(t) = \exp(-3t)$ are solutions to the ODE

$$y'' + 5y' + 6y = 0. \tag{7}$$

$$W[y_1, y_2](t) = \left| \begin{pmatrix} \exp(-2t) & \exp(-3t) \\ -2\exp(-2t) & -3\exp(-3t) \end{pmatrix} \right| = -exp(-5t) \neq 0$$

Any solution to the ODE $y'' + 5y' + 6y = 0$ can be written as the linear combination of $y_1(t) = \exp(-2t)$ and $y_2(t) = \exp(-3t)$. $y_1(t) = \exp(-2t)$ and $y_2(t) = \exp(-3t)$ form a fundamental set of solutions to the ODE.

# Fundamental set of solutions

## Example 5.9

For the ODE

$$2t^2y'' + 3ty' - y = 0, \quad t > 0,$$

the function $y_1(t) = t^{1/2}$ and $y_2(t) = t^{-1}$ are solutions. Let us compute the Wronskian

$$W(y_1, y_2)[t] = -\frac{3}{2}t^{-3/2},$$

which is non-zero for $t > 0$. Therefore we can deduce that $(y_1, y_2)$ form a fundamental set of solutions (FSS) for the ODE, and a general solution $y$ to the ODE can be expressed as

$$y(t) = c_1 t^{1/2} + c_2 t^{-1},$$

for some constants $c_1, c_2$.

# Fundamental set of solutions

Next we will examine further the properties of the Wronskian of two solutions to the 2nd-order linear homogeneous ODE

$$y'' + p(t)y' + q(t)y = 0.$$

We will show an explicit formula for the Wronskian even if the two solutions are unknown.

# Fundamental set of solutions

## Theorem 5.10

(Abel's theorem). *Let $I$ be an open interval, $p$ and $q$ are continuous in $I$. Suppose $y_1$ and $y_2$ are two non-zero solutions to the ODE*

$$y'' + p(t)y' + q(t)y = 0.$$

*Then, the Wronskian is given as*

$$W(y_1, y_2)[t] = c \exp\left( -\int p(t)dt \right),$$

*where the constant $c$ depends on $y_1$ and $y_2$,* **but not on $t$**. *Consequently, $W(y_1, y_2)[t] = 0$ if and only if $c = 0$.*
*In particular, $W(y_1, y_2)[t_0] \neq 0$ for some $t_0 \in I$, then it holds that $W(y_1, y_2)[t] \neq 0$ for all $t \in I$.*
*And, $W(y_1, y_2)[t_0] = 0$ for some $t_0 \in I$, then it holds that $W(y_1, y_2)[t] = 0$ for all $t \in I$.*

# Wronskian

## Proof.

The idea is to derive an ODE for the Wronskian $W$. Going back to the ODE, as $y_1$ is a solution we have

$$y_1'' + p(t)y_1' + q(t)y_1 = 0 \Rightarrow y_2y_1'' + y_2p(t)y_1' + y_2q(t)y_1 = 0.$$

Similarly, as $y_2$ is a solution,

$$y_1y_2'' + y_1p(t)y_2' + y_1q(t)y_2 = 0.$$

Subtracting one from another gives

$$(y_1y_2'' - y_2y_1'') + p(t)(y_1y_2' - y_2y_1') = 0. \tag{8}$$

□

# Wronskian

## Proof.

Noting that

$$\begin{aligned} &W(y_1, y_2)[t] = y_1(t)y_2'(t) - y_2(t)y_1'(t) \\ \Rightarrow &W'(y_1, y_2)[t] = y_1(t)y_2''(t) - y_2(t)y_1''(t). \end{aligned}$$

from (8) we have

$$W' + p(t)W = 0,$$

which is a linear first order equation. By integrating factors we find the general solution

$$W(y_1, y_2)[t] = c\exp\left(-\int p(t)dt\right)$$

for some constant $c \in \mathbb{R}$. As a constant of integration, $c$ does not depend on $t$. □

## Remark 2

*$(y_1, y_2)$ form a* **fundamental set of solutions***(FSS) to $y'' + p(t)y' + q(t)y = 0$ if and only if $W(y_1, y_2)[t] \neq 0, \forall t \in I$.*

# Wronskian

## Example 5.11

Previously we verified that the functions $y_1(t) = t^{1/2}$ and $y_2(t) = t^{-1}$ are solutions to

$$2t^2 y'' + 3ty' - y = 0, \quad t > 0.$$

We computed the Wronskian as $W(y_1, y_2)[t] = -(3/2)t^{-3/2}$. We check this with Abel's theorem. Writting the ODE in standard form

$$y'' + \frac{3}{2t}y' - \frac{1}{2t^2}y = 0 \Rightarrow p(t) = \frac{3}{2t}, \quad q(t) = -\frac{1}{2t^2}.$$

Then,

$$W(y_1, y_2)[t] = c\exp\left(-\int \frac{3}{2t}dt\right) = c\exp\left(-\frac{3}{2}\ln(t)\right) = ct^{-3/2}.$$

Then, on comparison we have $c = -3/2$.

# Fundamental set of solutions

Does a fundamental set of solutions always exists? This is answered in the next theorem.

### Theorem 5.12

(Existence of fundamental set of solutions). *Let $I$ be an open interval of $\mathbb{R}$, $p$, and $q$ are continuous functions in $I$. For any $t_0 \in I$, let $y_1(t)$ be the (unique) solution to the IVP*

$$y'' + p(t)y' + q(t)y = 0, \quad y(t_0) = 1, \quad y'(t_0) = 0,$$

*and $y_2(t)$ be the (unique) solution to the IVP*

$$y'' + p(t)y' + q(t)y = 0, \quad y(t_0) = 0, \quad y'(t_0) = 1.$$

*Then, $(y_1, y_2)$ forms a fundamental set of solutions (FSS) to the ODE.*

# Fundamental set of solutions

Note that the existence of $y_1$ and $y_2$ to the corresponding IVPs is guaranteed by Existence and Uniqueness Theorem. By Theorem 5.8 it only needs to show that the Wronskian $W(y_1, y_2)[t_0]$ is non-zero. Computing gives

$$W(y_1, y_2)[t_0] = \left| \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 1.$$

Indeed, the fundamental set of solutions are not unique. There are many different choices for $y_1(t_0), y_1'(t_0), y_2(t_0), y_2'(t_0)$ such that the corresponding solutions $y_1, y_2$ satisfy $W(y_1, y_2)[t_0] \neq 0$.

# Fundamental set of solutions

## Example: Two fundamental sets of solutions

Consider the ODE

$$y'' - y = 0.$$

Note that $y_1(t) = \exp(t)$ and $y_2(t) = \exp(-t)$ are solutions to the ODE. The Wronskian $W(y_1, y_2)[t] = -2 \neq 0$ and so they form a fundamental set of solutions.

However, the pair $(\exp(t), \exp(-t))$ does not satisfy the conditions in Theorem 5.12 at $t_0 = 0$. Since by Theorem 5.8 any solution $y$ to $y'' - y = 0$ must be of the form

$$y(t) = c_1 \exp(t) + c_2 \exp(-t),$$

plugging in the initial condition $y(0) = 1$ and $y'(0) = 0$ gives a solution

$$z_1(t) = \frac{1}{2}\exp(t) + \frac{1}{2}\exp(-t) = \cosh(t).$$

# Fundamental set of solutions

## Example: Two fundamental sets of solutions (continue).

Similarly, plugging in the initial condition $y(0) = 0$ and $y'(0) = 1$ gives a solution

$$z_2(t) = \frac{1}{2}\exp(t) - \frac{1}{2}\exp(-t) = \sinh(t).$$

Furthermore, the Wronskian $W(y_1, y_2)[t] = \cosh^2(t) - \sinh^2(t) = 1$, and so $(z_1, z_2)$ forms a fundamental set of solutions as stated by Theorem 5.12. This means that any solution $y$ to the ODE $y'' - y = 0$ can also be expressed as

$$y(t) = d_1 \cosh(t) + d_2 \sinh(t)$$

for constants $d_1, d_2$.

From the above example, we see that there are **more than one** fundamental set of solutions for a given ODE. Indeed, it has **infinitely many** fundamental sets of solutions.

# Fundamental set of solutions

Indeed, the fundamental sets of solutions (FSS) is closely related to the concept (Linear independence) in linear algebra.

## Definition 5.13

(Linear dependence). Consider 2 functions $x_1(t), x_2(t)$ defined on an interval $I \subset \mathbb{R}$. We say that $x_1(t), x_2(t)$ are **linearly dependent** if there are non-zero constants $\alpha_1, \alpha_2$, such that

$$\alpha_1 x_1(t) + \alpha_2 x_2(t) = 0 \quad \forall t \in I$$

**Remark:** Let $x_1(t), x_2(t)$ defined on an interval $I \subset \mathbb{R}$, if $x_1(t), x_2(t)$ are not linearly dependent, then they are linearly independent.

# Fundamental set of solutions

## Property

If $y_1(t)$ and $y_2(t)$ are two solutions to the ODE $y'' + p(t)y' + q(t)y = 0, t \in I$, where $p, q$ are given continuous function in $I$(some open interval). Then $y_1(t)$ and $y_2(t)$ are linearly independent $\Leftrightarrow W[y_1, y_2](t) \neq 0$, $\forall t \in I$ ($(y_1, y_2)$ forms a FSS).

**Proof.** "$\Leftarrow$"

$$\begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

has only the zero solution. Thus, $c_1 y_1(t) + c_2 y_2(t) = 0$ implies $c_1 = c_2 = 0$. This implies $y_1(t)$ and $y_2(t)$ are linearly independent.
"$\Rightarrow$" if $W[y_1, y_2](t_0) = 0$ for some $t_0 \in I$. Then the linear system

$$\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

has non-zero solution $(c_1^*, c_2^*) \neq (0, 0)$.

# Fundamental set of solutions

Define $\phi(t) = c_1^* y_1(t) + c_2^* y_2(t), t \in I$, then $\phi(t)$ is the solution to the ODE $y'' + p(t)y' + q(t)y = 0$ with initial conditions $y(t_0) = 0, y'(t_0) = 0$.
But $y(t) = 0, t \in I$ is also the solution to the ODE with initial conditions $y(t_0) = 0, y'(t_0) = 0$.
By the existence and uniqueness theorem, $\phi(t) = c_1^* y_1(t) + c_2^* y_2(t) = 0, t \in I$.
This implies that $y_1(t)$ and $y_2(t)$ are linearly dependent solutions. This is a contradiction.

# Summary

Based on the above results, the strategy to solve

$$y'' + p(t)y' + q(t)y = 0, \quad t \in I,$$

can be summarised as follows:

(1) Find two solution $y_1, y_2$ satisfying the ODE.

(2) Find $t_* \in I$ such that the Wronskian $W(y_1, y_2)[t_*]$ is non-zero. Then, the general solution to the ODE is

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

for some constants $c_1, c_2$.

(3) If initial conditions are prescribed at some $t_0 \in I$, compute $c_1$ and $c_2$ to determine the particular solution.