STA4001 Stochastic Processes Spring 2021
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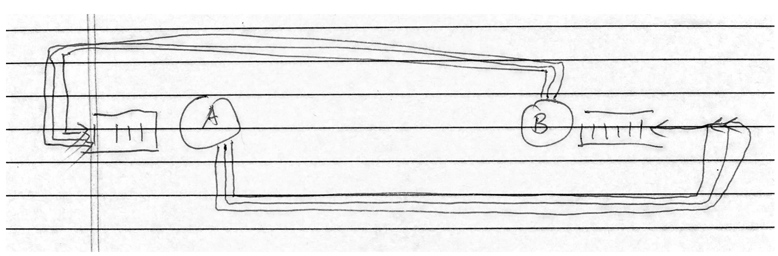
Homework 12
Due by May 6, 2021

1. Assume that $\{Y_n : n = 1, 2, \cdots\}$ is an i.i.d. sequence with $\mathbb{E}(Y_1) = 0$ and $\mathbb{E}Y_1^2 = \sigma^2$. Let $X_n = Y_1 + \cdots + Y_n$ and $Z_n = X_n^2 - \sigma^2 n$.
   (a) Show that $\{Z_n\}$ is a martingale with respect to $\{Y_n : n = 1, 2, \cdots\}$.
   (b) Fix $a, b > 0$. Let $\mathbb{P}(Y_1 = 1) = \mathbb{P}(Y_1 = -1) = 1/2$, i.e., X_n is a simple symmetric random walk. Use the martingale $\{Z_n\}$ to show that $\mathbb{E}(T_{-a,b}) = ab$, where
   $$T_{-a,b} = \inf\{n \geq 0 : X_n = -a \text{ or } X_n = b\}$$
   is the first hitting time to either $-a$ or b.

2. Consider a branching process $X_\bullet \equiv \{X_n; n \geq 0\}$, which is defined as $X_0 = 1$ and
   $$X_n = \sum_{i=1}^{X_{n-1}} U_{n,i}, \quad n \geq 1$$
   where, for each $n \geq 1, U_{n,i}$ for $i \geq 1$ are $i.i.d.$ random variables. Assume that all $U_{n,i}$ for $i \geq 1, n \geq 1$ have a common distribution, and denote a random variable subject to this distribution by U. Let $\mathbb{E}(U) = \lambda > 0$ and $V(U) \equiv \mathbb{E}\left((U - \mathbb{E}(U))^2\right) = \sigma^2 < \infty$. Define $Y_n = X_n/\lambda^n$ for $n \geq 0$
   (a) Prove that $Y_\bullet = \{Y_n; n \geq 0\}$ is a martingale with respect to $\{X_n : n = 1, 2, \cdots\}$.
   (b) Compute $\mathbb{E}\left(Y_n^2 \mid X_1, \ldots, X_{n-1}\right)$.
   (c) Compute $\mathbb{E}\left(X_n^2\right)$.

3. (Optional) Consider a two-region DIDI network pictured in the following diagram



The state of the system at time t is $X(t) = (X_1(t), X_2(t), X_3(t), X_4(t))$, where

- $X_1(t)$ the number of empty cars waiting in region A at time t,
- $X_2(t)$ the number of cars going (with a passenger) from region A to region B at time t,
- $X_3(t)$ the number of empty cars waiting in region B at time t,
- $X_4(t)$ the number of cars going (with a passenger) from region B to region A at time t.

There are N DIDI cars in the system. This number remain fixed. Passengers requesting uber rides arrive at each region follow a Poisson process. The arrival rates are $\alpha_A = 4$ passengers per minute and $\alpha_B = 2$ passengers per minute for regions A and B, respectively. An arriving passenger finds no DIDI car waiting immediately leaves the system, taking alternative transportation. The travel times are exponentially distributed. The mean travel times from region A to B and from B to A are $m_{A,B} = 15$ minutes and $m_{B,A} = 10$ minutes, respectively. (Recall that theory developed in class for open and closed queueing networks remains valid when each station has multiple homogeneous servers.)

(a) Find the long-run fraction of passengers who take alternative transportation in each of the two regions for $N = 2$ and $N = 3$. What are the differences for $N = 2$ and $N = 3$?

(b) Find the average utilization (the percentage of time that a car carries a passenger) of each car for $N = 2$ and $N = 3$. What are the differences for $N = 2$ and $N = 3$?.