118010350.

1. I want to visit 10 cities, each exactly once. Count the number of possible routes (order of visiting). How about with the following constraints? $N = 10!$

   (a) I must start at city 1. $N = 9!$

   (b) I must start at city 1 and end at city 10. $N = 8!$

   (c) There is no flight from city 1 to city 2. $N = \frac{10!}{\binom{10}{2} \cdot 8!}$

   (d) There is no flights between city 1 and city 2. $N = \frac{10!}{\binom{10}{2} \cdot 2 \cdot 8!}$

2. Count the number of 01-strings with following constraints.

   (a) The length is 8. Number of 1s is 2 more than number of 0s. $N = \binom{8}{3}$

   (b) The length is 8. Number of 1s is 3 more than number of 0s. $N = 0$

   (c) The length is 9. Number of 1s is 3 more than number of 0s. $N = \binom{9}{3}$

   (d) The length is $2n$. Number of 1s is $2k$ more than number of 0s. $N = \binom{2n}{n-k}$

   (e) The length is 8. 00 or 11 must appear somewhere in the string. $N = 2$

   (f) The length is 8. 000000 must not appear in the string. $N = 2^8 - 5.$

3. Find the required coefficients.

   (a) Coefficient of x^3y^2 in $(x+y)^5$. $C = \binom{5}{3}$

   (b) Coefficient of x^2y^3 in $(x-y)^5$. $C = -\binom{5}{2}$

   (c) Coefficient of x^7y^{13} in $(x+y)^{19}$. $C = 0$

4. Prove that $k\binom{n}{k} = n\binom{n-1}{k-1}$ via combinatorial proof.

(4) Proof. Consider n balls.
want to $\left\} \begin{array}{l} \text{color 1 ball to be red,} \\ \text{color } k\text{-1 balls to be green.} \end{array} \right.$

LHS = Pick any k balls from n balls, then pick 1 ball from k balls.

RHS = Pick 1 ball from n balls, then pick $(k-1)$ balls from $(n-1)$ balls.

$\Rightarrow$ LHS = RHS.

5. Prove that $\sum_{k=0}^{n}\binom{n}{k}\binom{n-k}{m-k} = 2^m\binom{n}{m}$

6. Count the number of pairs (x, y) with following constraints.

   (a) x and y are between 1 and 100. 2 does not divide both of them.

   (b) x and y are between 1 and 100. 3 does not divide both of them.

   (c) x and y are between 1 and 100. Both 2 and 3 do not divide both of them.

   (d) x and y are between 1 and 100. 2, 3 and 5 do not divide both of them.

7. Count the number of sequences of 10 distinct letters that contain none of THE, MATH, and QUIZ.

Define A = { contains "THE" }.

B = { contains "MATH" }.

C = { contains "QUIZ" }.

$|A^c \cap B^c \cap C^c| = |(A \cup B \cup C)^c|$.

$|(A \cup B \cup C)^c| = |U| - |A \cup B \cup C|$.

8. Show that if you choose $n+1$ different numbers from $\{1,2,3,..,2n\}$, then one of your chosen number is a multiple of another chosen number.

9. Show that every positive integer has a multiple which consists of 0 and 7 only. For example, 70 is a multiple of 2, 777 is a multiple of 3.

10. Suppose there are 10 dots in a square with side length 1. Show that there are two dots whose distance is less than 0.5.

$|A| = 8 \cdot \frac{23!}{(23-7)!}$ $|B| = 7 \cdot \frac{22!}{(22-6)!}$ $|C| = 7 \cdot \frac{22!}{(22-6)!}$

$|A \cap B| = 6 \cdot \frac{21!}{(21-5)!}$ $|A \cap C| = 20 \cdot \frac{19!}{(19-3)!}$ $|B \cap C| = 12 \cdot \frac{18!}{(18-2)!}$

$|A \cap B \cap C| = (26-9) \times 3 = 51.$

$$\Rightarrow |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$= \frac{8 \cdot 23!}{16!} + \frac{7 \cdot 22!}{16!} + \frac{7 \cdot 22!}{16!} - \frac{6 \cdot 21!}{16!} - \frac{20 \cdot 19!}{16!} - \frac{12 \cdot 18!}{16!} + 51$$

Denote the value above as $|A \cup B \cup C| = N$.

then $|(A \cup B \cup C)^c| = |U| - N = \frac{26!}{(26-10)!} - N$

$$= \frac{26!}{16!} - N.$$