Stochastic Processes

Lecture 14: Poisson processes (II)

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March 22, 2021

Review: Poisson Process

An example

- Assume N is a Poisson process with rate $\lambda = 2/\text{minutes}$
- Find the probability that there are exactly 4 arrivals in first 3 minutes.

$$\mathbb{P}(N(3) - N(0) = 4) = \frac{(2(3-0))^4}{4!}e^{-2(3-0)} = \frac{6^4}{4!}e^{-6} = 0.1339$$

- What is the probability that exactly two arrivals in $[0, 2]$ and at least 3 arrivals in $[1, 3]$?

$$\begin{aligned}
\mathbb{P}(\{N(2) =& 2\} \cap \{N(3) - N(1) \geq 3\}) \\
=& \mathbb{P}(N(1) = 0)\mathbb{P}(N(2) - N(1) = 2)\mathbb{P}(N(3) - N(2) \geq 1) \\
&+ \mathbb{P}(N(1) = 1)\mathbb{P}(N(2) - N(1) = 1)\mathbb{P}(N(3) - N(2) \geq 2) \\
&+ \mathbb{P}(N(1) = 2)\mathbb{P}(N(2) - N(1) = 0)\mathbb{P}(N(3) - N(2) \geq 3)
\end{aligned}$$

An example

- Computing

$$\mathbb{P}(N(3) - N(2) \geq 1) = 1 - \mathbb{P}(N(3) - N(2) < 1)$$
$$= 1 - \mathbb{P}(N(3) - N(2) = 0) = 1 - \frac{2^0}{0!}e^{-2} = 1 - e^{-2}$$

- What is the probability that there is no arrival in $[0, 4]$?

$$\mathbb{P}(N(4) - N(0) = 0) = e^{-8}$$

- Let T_1 be the arrival time of the first customer. Is T_1 a continuous or discrete random variable?
- What is the probability that the first arrival will take at least 4 minutes?

$$\mathbb{P}(T_1 > 4) = \mathbb{P}(N(4) = 0) = e^{-8} \tag{1}$$

In plain English, "the first arrival takes at least 4 minutes" is equivalent to "there is no arrival for the first 4 minutes."

"Duality"

- Assume N is a Poisson process with rate $\lambda = 2$/minutes.
- Let T_1 be the arrival time of the first customer.

$$\mathbb{P}(T_1 > t) = \mathbb{P}(N(t) = 0) = e^{-2t}$$

- Surprisingly, T_1 is an exponential random variable.

Distribution of T_k

- Let T_3 be the arrival time of the 3rd customer.

$$\mathbb{P}\{T_3 > t\} = ?\mathbb{P}\Big\{N(0,t] \le 2\Big\}$$

$$= e^{-2t} + \frac{2t}{1!}e^{-2t} + \frac{(2t)^2}{2!}e^{-2t}$$

$$\mathbb{P}\{T_3 > t\} = ?\mathbb{P}\Big\{N(0,t] = 2\Big\}$$

- The cdf of T_3 is

$$\mathbb{P}(T_3 \le t) = 1 - \mathbb{P}(T_3 > t) = 1 - \left(e^{-2t} + \frac{2t}{1!}e^{-2t} + \frac{(2t)^2}{2!}e^{-2t}\right)$$

- We can take derivative of the cdf to obtain the pdf of T_3:

$$f_{T_3}(t) = \frac{2(2t)^2}{2!}e^{-2t}, \quad t \ge 0.$$

Distribution of T_k

- T_3 has a gamma distribution. Let $\lambda = 2$.

$$\frac{\lambda(\lambda t)^2}{2!}e^{-\lambda t} = \text{p.d.f. of Gamma}(3, \lambda) = \text{Erlang}(3, \lambda),$$

  where $\alpha = 3$ is the shape parameter and λ is the scale parameter.

- For a gamma distribution, it is easy to understand when α is an integer; Erlang distribution.

$$T_3 \overset{d}{=} u_1 + u_2 + u_3,$$

  where u_1, u_2, u_3 are iid $\exp(\lambda)$ r.v.'s.

  Why do we get Erlang distribution for T_3?

Poisson process defined by arrival times

- So far, given a Poisson process N, understand T_k, $k = 1, 2, \ldots$ the arrival times
- Conversely, given arrival times $\{T_k, k \geq 1\}$, define N.

Theorem

Let $\{u_i, i \geq 1\}$ be a sequence of iid r.v.'s having exponential distribution with mean $1/\lambda$. For each $t \geq 0$, define $T_k = u_1 + u_2 + \ldots + u_k$ and

$$N(t) = \max\{n \geq 0 : T_n \leq t\}.$$

Then $N = \{N(t), t \geq 0\}$ is a Poisson process with rate λ.

- Prove (i), for example, $P\{N(0, t] = 2\} = \frac{(\lambda t)^2}{2!} e^{-\lambda t}$.
- Independent increments (needs work)

Conditional distribution of arrival times

- Let T_i is the arrival times of ith arrival.
- Assume $N(t) = 1$. What is the distribution of T_1?
- Assume $N(t) = 2$, what is the joint distribution of T_1 and T_2.
- Order statistics

Merging

- N_1 and N_2 are two independent Poisson processes with rates λ_1 and λ_2, respectively.
- Define $N(t) = N_1(t) + N_2(t)$.
- $N = \{N(t), t \geq 0\}$ is a Poisson process with rate $\lambda = \lambda_1 + \lambda_2$.
- Proof: check independent increments

Thinning

- Let $N = \{N(t), t \geq 0\}$ be a Poisson process with rate λ.
- Each arrival flips a coin with probability of p getting a head.
- $N_1(t)$ is the number of heads in $(0, t]$,
- $N_2(t)$ is the number of tails in $(0, t]$.
- $N_i = \{N_i(t), t \geq 0\}$ is a Poisson process with rate λ_i, $i = 1, 2$, where $\lambda_1 = \lambda p$ and $\lambda_2 = \lambda(1 - p)$.
- Furthermore N_1 and N_2 are independent.