STOCHASTIC PROCESSES (LECTURE 9)

DTMC: RECURRENCE, POSITIVE RECURRENCE, ERGODIC THEOREM

Hailun Zhang@SDS of CUHK-Shenzhen

February 28, 2021

Recurrence and Transience

$$T_i = \min\{n \ge 1 | X_n = i\} = \text{first time to reach } i$$

DEFINITION

A state i of a DTMC is recurrent if

$$\mathbb{P}\{T_i < \infty | X_0 = i\} = 1;$$

otherwise, state i is transient.

$$N_i = \text{number of visits to state } i = \sum_{n=1}^{\infty} 1_{\{X_n = i\}}.$$

THEOREM

- If state i is recurrent, then $\mathbb{E}[N_i|X_0=i]=\infty$.
- If state i is transient, then $\mathbb{E}[N_i|X_0=i]<\infty$.

Stopping times

- Let $X = \{X_n : n = 0, 1, ...\}$ be a DTMC on space space S.
- A $\{0,1,\ldots\} \cup \{\infty\}$ -valued random variable is called a *stopping* time of the DTMC if the event $\{T=n\}$ depends only on X_0,X_1,\ldots,X_n for $n=0,\ldots$
- For a set $A \subset S$, the first passage time to A,

$$T_A = \inf\{n \ge 1 : X_n \in A\}.$$

- $T_i = T_{\{i\}}$.
- Last passage time

 $L_A =$ the last time to visit A.

Strong Markov Property

• X is a DTMC on state space S with transition matrix P; T is a stopping time.

$$\mathbb{P}(X_{T+1} = j | \{T < \infty\} \cap \{X_0 = i_0, X_i = i_1, \dots, X_{T-1} = i_{T-1}, X_T = i\}) = P_{ij}$$
 (1)

for any $i, j, i_0, i_1, ... \in S$.

Geometric random variable (a review)

- Given a coin with probability of p > 0 leading a head, let N be the number of tosses needed to get the first head.
- $\mathbb{P}{N = n} = q^{n-1}p \text{ for } n = 1, 2, \dots$
- $\mathbb{P}\{N \ge 1\} = 1$
- $\bullet \ \mathbb{P}\{N \ge 2\} = q$
- $\bullet \ \mathbb{P}\{N \ge n\} = q^{n-1}$
- $\mathbb{P}{N<\infty}=1$
- $\mathbb{E}(N) = \sum_{n=1}^{\infty} nq^{n-1}p = 1/p$.
- $\mathbb{E}(N) = \sum_{n=1}^{\infty} \mathbb{P}\{N \ge n\} = \sum_{n=1}^{\infty} q^{n-1} = \frac{1}{1-q} = 1/p$

Transience and geometric random variable

Define

$$T_i = \min\{n \ge 1 | X_n = i\} = \text{ first time to reach } i.$$

• Define

$$N_i$$
 = number of visits to state $i = \sum_{n=1}^{\infty} 1_{\{X_n = i\}}$.

• Prove that

$$\{N_i \ge 1\} = \{T_i < \infty\} \tag{2}$$

• Define

$$f_i = \mathbb{P}\{T_i < \infty | X_0 = i\}.$$

Prove that $(\mathbb{P}_i \text{ means } \mathbb{P}(\cdot|X_0=i))$

$$\mathbb{P}_i\{N_i \ge 2\} = f_i^2.$$

Transience

• In general,

$$\mathbb{P}_i\{N_i \ge n\} = (f_i)^n.$$

• Thus,

$$\mathbb{E}_{i}(N_{i}) = \sum_{n=1}^{\infty} (f_{i})^{n} = \frac{f_{i}}{1 - f_{i}}.$$

Recurrence and Transience

LEMMA

 N_i is a "0-based geometric random variable"

DEFINITION (REVIEW)

a state i of a DTMC is said to be recurrent if

$$\mathbb{P}\{T_i < \infty | X_0 = i\} = 1;$$

otherwise, state i is transient.

THEOREM

- If state i is transient, then $\mathbb{E}[N_i|X_0=i]=\frac{f_i}{1-f_i}<\infty$.
- If state i is recurrent, then $\mathbb{E}[N_i|X_0=i]=\infty$.

Transience

COROLLARY

(a) State i is transient iff

$$\sum_{n=1}^{\infty} P_{ii}^n < \infty.$$

(b) If state i is transient,

$$\mathbb{P}_i(N_i < \infty) = 1.$$

"Solidarity" Property

THEOREM

If states i and j communicate, then they are either both recurrent or both transient.

What if the DTMC is irreducible?

THEOREM

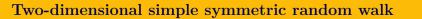
If a finite DTMC is irreducible, then every state is recurrent.

What if the state space of the DTMC is infinite?

One-dimensional simple random walk

$$\bullet \ P_{00}^{2n} = \binom{2n}{n} p^n q^n$$

• Stirling's formula: $n! \sim \sqrt{2\pi n} (n/e)^n$



Positive recurrence

 \bullet A recurrent state i of a DTMC is said to be positive recurrent if

$$\mathbb{E}[T_i|X_0=i]<\infty;$$

ullet Otherwise, the current state i is said to be null recurrent.

THEOREM (SLLN)

Assume that state i is positive recurrent and $f: S \to \mathbb{R}$ is bounded.

$$\mathbb{P}_i \left\{ \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^n f(X_k) = c \right\} = 1,$$

where

$$c = \frac{\mathbb{E}_i\left(\sum_{k=0}^{T_i-1} f(X_k)\right)}{\mathbb{E}_i(T_i)}.$$

Existence of stationary distribution

- \bullet Assume state i is positive recurrent.
- Define $\pi = (\pi(j), j \in S)$ via

$$\pi(j) = \frac{\mathbb{E}_i\left(\sum_{k=0}^{T_i-1} 1_{\{X_k=j\}}\right)}{\mathbb{E}_i(T_i)}, \quad j \in S.$$
 (3)

- Then π is stationary distribution of the Markov chain.
- Proof.

Uniqueness of stationary distributions

THEOREM

Assume the DTMC is irreducible. It is positive recurrent and only if there is a unique stationary distribution. The unique stationary distribution π is given by

$$\pi(j) = \frac{1}{\mathbb{E}_j(T_j)}.$$

What do we mean by "a positive recurrent DTMC"? Solidarity!

Ergodic Theorem

• Let $N_j(n) = \sum_{k=1}^n 1_{\{X_k = j\}}$ the number of visits to state j in [1, n].

 $\frac{N_j(n)}{n}$ the fraction of "time" that the DTMC spends in state j.

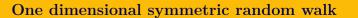
THEOREM

(a) Assume i is positive recurrent.

$$\mathbb{P}_i \left\{ \lim_{n \to \infty} \frac{1}{n} N_j(n) = \frac{\mathbb{E}_i \left(\sum_{k=0}^{T_i - 1} 1_{\{X_k = j\}} \right)}{\mathbb{E}_i(T_i)} = \pi(j) \right\} = 1 \quad \text{for } j \in S.$$

(b) Assume i is positive recurrent and $f: S \to \mathbb{R}$ is bounded. Define $\pi(f) = \sum_{i \in S} \pi(i) f(i)$.

$$\mathbb{P}_i \left\{ \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^n f(X_k) = \pi(f) \right\} = 1.$$



Reflected random walks

Positive recurrence criterion

• Let $N_i(n) = \sum_{k=1}^n 1_{\{X_k=i\}}$ be the number of times visiting state i in [1,n]. Then

$$\mathbb{E}_i(N_i(n)) = \sum_{k=1}^n \mathbb{E}_i 1_{\{X_k = i\}} = \sum_{k=1}^n \mathbb{P}_i \{X_k = i\} = \sum_{k=1}^n P_{ii}^k.$$

THEOREM

State i is positive recurrent if and only if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} P_{ii}^{k} > 0.$$

• Proof.

Recall recurrence criterion

• Recall that state i is recurrent iff

$$\sum_{k=1}^{\infty} P_{ii}^k = \infty.$$

• State *i* is positive recurrent iff

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} P_{ii}^{k} > 0.$$

Solidarity of positive recurrence

LEMMA 1

Assume states i and j communicate. State i is p.r. iff state j is p.r.

- Proof: there exist k_1 and k_2 such that $P_{ij}^{k_1} > 0$ and $P_{ji}^{k_2} > 0$.
- Assume j is p.r. Then $\lim_{n\to\infty} (1/n) \sum_{k=1}^n P_{jj}^k > 0$. Lemma follows from

$$P_{ii}^{k_1+k+k_2} \ge P_{ij}^{k_1} P_{jj}^k P_{ji}^{k_2},$$

$$\frac{1}{n} \sum_{k=1}^{n+k_1+k_2} P_{ii}^k = \frac{1}{n} \sum_{k=1}^{n} P_{ii}^{k_1+k_2} + \frac{1}{n} \sum_{k=1}^{k_1+k_2} P_{ii}^k > 0$$

when n is large enough.

Limiting behavior of transition matrix P

- Assume that the DTMC is irreducible.
- If it is positive recurrent, for every pair of states $i, j \in S$, (to be proved)

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} (P^k)_{ji} = \frac{1}{\mathbb{E}_i(T_i)} > 0.$$

Namely,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} P^k = P^{(\infty)},$$

where $P_{ij}^{(\infty)} = \pi_j = 1/\mathbb{E}_j(T_j)$.

• If it is not positive recurrent, for every pair of states

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} (P^k)_{ji} = 0.$$