MAT 3253 lecture 25

$$\int_0^\infty \frac{P(x)}{Q(x)}\,dx$$

if $\frac{P(x)}{Q(x)}$ is an even function

$$\frac{1}{2}\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx$$

* $\deg Q \geq \deg P + 2$

* $Q(x) \neq 0 \quad \forall\ x \geq 0$

Revision :

$$\overbrace{\int \underbrace{e^{at}}_{u}\, \underbrace{\cos bt\, dt}_{dv}}^{I}$$

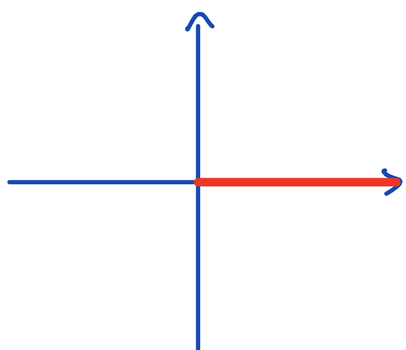
$$= e^{at}\frac{\sin bt}{b} - \int \underbrace{ae^{at}}_{u}\, \underbrace{\frac{\sin bt}{b}\, dt}_{dv}$$

$$= e^{at}\frac{\sin bt}{b} + ae^{at}\frac{\cos bt}{b^2} \underbrace{- \int a^2 e^{at}\frac{\cos bt}{b^2}\, dt}_{-\frac{a^2}{b^2} I}$$

$$\left(1+\frac{a^2}{b^2}\right) I = \ldots\ldots$$

<u>Example</u> $\displaystyle\int_0^\infty \frac{1}{x^3+1}\,dx$

$$\int \frac{\log z}{z^3+1}\,dz$$

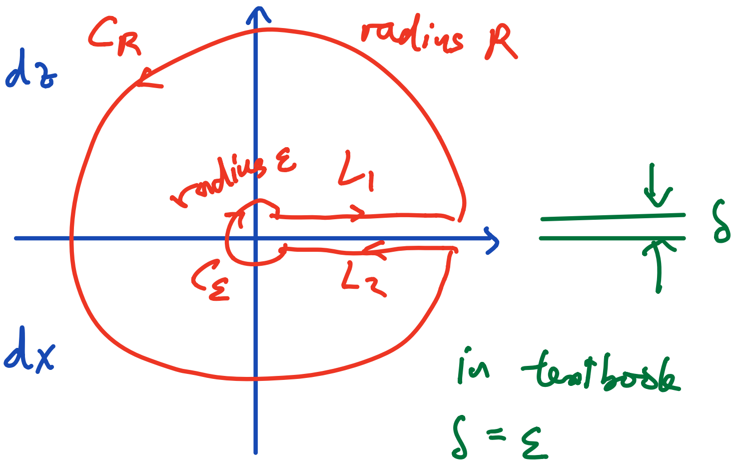


branch cut for log function

$0 < \arg(z) < 2\pi$

$\log(re^{i\theta}) = \log r + i\theta \quad 0<\theta<2\pi$

$$\left(\int_{C_R} + \int_{L_2} + \int_{C_\varepsilon} + \int_{L_1}\right)\frac{\log z}{z^3+1}\,dz$$



in textbook $\delta = \varepsilon$

$$\lim_{\substack{\varepsilon\to 0\\ \delta\to 0\\ R\to 0}} \int_{L_1} \frac{\log z}{z^3+1}\,dz = \int_0^\infty \frac{\log x}{x^3+1}\,dx$$

$$\lim_{\substack{\varepsilon\to 0\\ \delta\to 0\\ R\to \infty}} \int_{L_2} \frac{\log z}{z^3+1}\,dz = -\int_0^\infty \frac{\log x + 2\pi i}{x^3+1}\,dx$$

sum of the above $\;-2\pi i \int_0^\infty \frac{1}{x^3+1}\,dx$

$$\left|\int_{C_\varepsilon} \frac{\log z}{z^3+1}\,dz\right| \le 2\pi\varepsilon M_\varepsilon \max_{|z|=\varepsilon} |\log z|$$

$\left|\frac{1}{z^3+1}\right| < M_\varepsilon$ for $|z| \le \varepsilon$

$$\le 2\pi M_\varepsilon\, \varepsilon\, |\log\varepsilon + i2\pi|$$

$\log\varepsilon + i\theta$

$$\to 0 \quad \text{as} \quad \varepsilon \to 0$$

$$\left|\int_{C_R} \frac{\log z}{z^3+1}\,dz\right| \le 2\pi R \max_{|z|=R} \frac{\log z}{z^3+1}$$

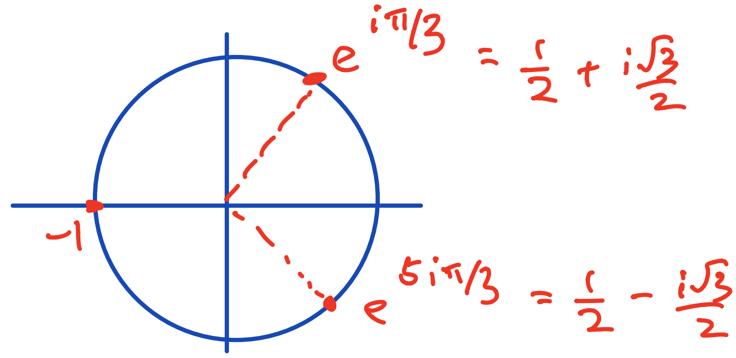
$\log R + i\theta$

$$< AR\,\frac{\log R}{R^3}$$

A is a constant

$$\to 0 \quad \text{as} \quad R\to\infty$$

$$\lim_{\substack{\varepsilon\to 0\\ \delta\to 0\\ R\to \infty}} \int_{L_1} + \int_{L_2} + \int_{C_\varepsilon} + \int_{C_R} \frac{\log z}{z^3+1}\,dz = -2\pi i \int_0^\infty \frac{1}{x^3+1}\,dx$$



$$\int_0^\infty \frac{1}{x^3+1}\,dx = -\lim \frac{1}{2\pi i}\int_{C} \underbrace{\frac{\log z}{z^3+1}}_{f(z)}\,dz$$

$$= -\left[\mathrm{Res}(f;-1) + \mathrm{Res}(f;e^{i\pi/3}) + \mathrm{Res}(f;e^{5i\pi/3})\right]$$

$$f(z) = \frac{\log(z)}{(z+1)(z-e^{i\pi/3})(z-e^{5i\pi/3})}$$

$$\mathrm{Res}(f;-1) = \frac{\log(-1)}{(-1-e^{i\pi/3})(z-e^{5\pi i/3})}$$

$$= \frac{i\pi}{\left(-\frac{3}{2}-\frac{i\sqrt{3}}{2}\right)\left(-\frac{3}{2}+\frac{i\sqrt{3}}{2}\right)}$$

$$= \frac{i\pi}{3}$$

algebra manipulation

$$\mathrm{Res}(f;e^{i\pi/3}) = \frac{\log(e^{i\pi/3})}{(e^{i\pi/3}+1)(e^{i\pi/3}-e^{5i\pi/3})}$$

$$= \frac{i\pi/3}{\left(\frac{3}{2}+\frac{i\sqrt{3}}{2}\right)(i\sqrt{3})}$$

$$= -\frac{\pi i}{18}(1+\sqrt{3}i)$$

$$\mathrm{Res}(f;e^{5i\pi/3}) = \frac{\underbrace{\log(e^{i5\pi/3})}_{\frac{5\pi}{3}i}}{\left(\frac{3}{2}-\frac{i\sqrt{3}}{2}\right)(-\sqrt{3}i)}$$

$$= \frac{-5\pi i}{18}(1-i\sqrt{3})$$

sum of residues $= \pi i\left[\frac{1}{3} - \frac{1+\sqrt{3}i}{18} - \frac{5}{18}(1-\sqrt{3}i)\right]$

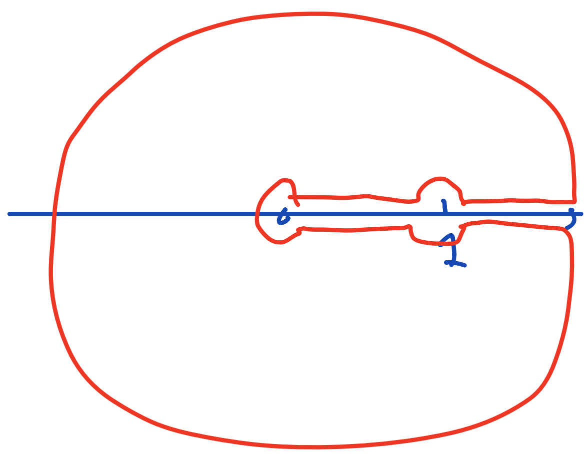
$$= -\frac{\pi 4\sqrt{3}}{18} = -\frac{\pi 2\sqrt{3}}{9}$$

$$\therefore \int_0^\infty \frac{1}{x^3+1}\,dx = \frac{\pi 2\sqrt{3}}{9}$$

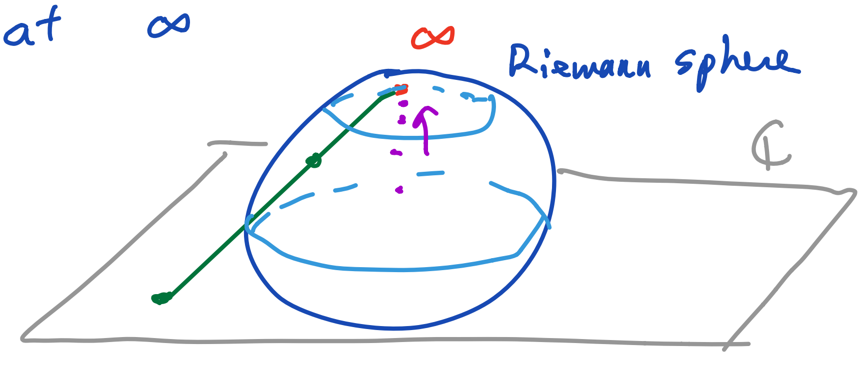
$$\int_0^\infty \frac{dx}{x^3-1} \qquad \frac{1}{x^3-1} = \infty$$

has to be ∞

$$\int_0^{10} \frac{dx}{x^3+1}\,dx$$



Analytic at ∞



$f(z) = z \longrightarrow \infty$ as $z \to \infty$

$f(z) = \frac{1}{z} \longrightarrow 0$ as $z \to \infty$

$f(z) = e^z \longrightarrow ?$ as $z \to \infty$

Let $w = \frac{1}{z}$ $\frac{1}{z}$ is the local parameter at ∞

For $\alpha \in \mathbb{C}$ $(z-\alpha)$ is the local parameter at $z=\alpha$

$$\sum_{n=0}^{\infty} a_n (z-\alpha)^n + \sum_{n=1}^{\infty} b_n (z-\alpha)^{-n}$$

Given $f(z)$,

let $g(w) = f(\frac{1}{w})$.

__Def__ $f(z)$ is analytic at ∞ iff $g(w)$ is analytic at 0

__Def__ $f(z)$ has removable singularity (resp. pole, ess. singularity) at ∞ iff $g(w)$ has removable singularity (resp. pole, ess. singularity) at $w=0$.

__Example__ $f(z) = z$
$g(w) = \frac{1}{w}$

$f(z)=z$ has a simple pole at ∞.

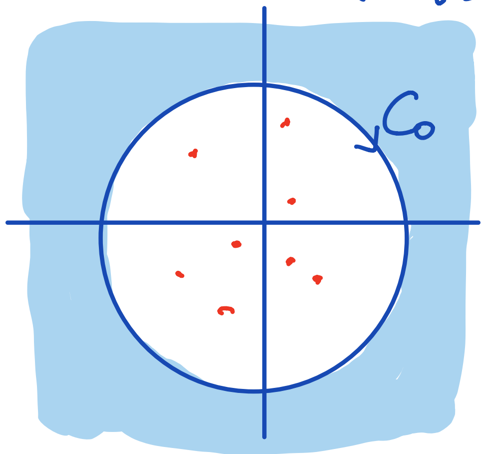
$f(z) = e^z$

$$g(w) = e^{1/w} = 1 + \frac{1}{w} + \frac{1}{2w^2} + \frac{1}{3!\,w^3} + \cdots.$$

e^z has an essential singularity at $z=\infty$.

Residue at ∞

$R < |z| < \infty$

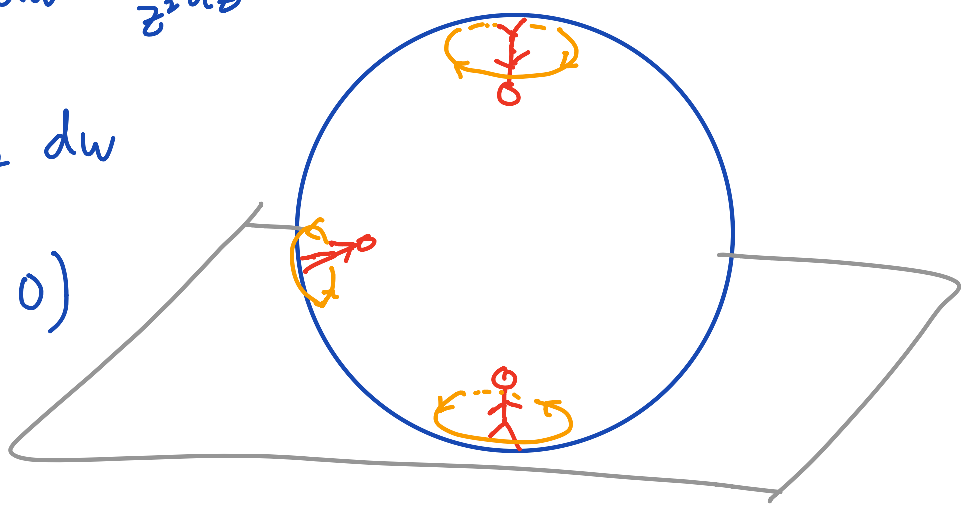


$$\mathrm{Res}(f;\infty) \overset{\Delta}{=} \frac{1}{2\pi i} \oint_{C_0} f(z)\,dz$$

$$w = \frac{1}{z} \qquad dw = -\frac{1}{z^2}\,dz$$

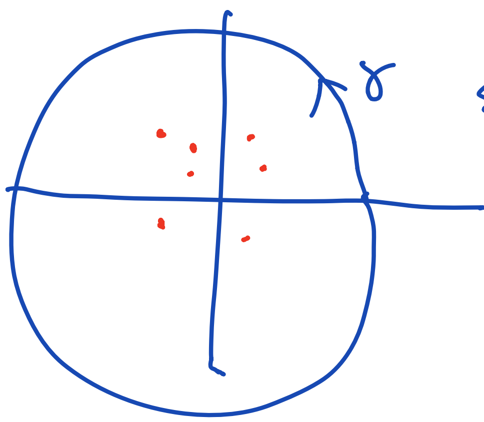
$$= \frac{1}{2\pi i} \int -f\left(\frac{1}{w}\right)\frac{1}{w^2}\,dw$$

$$= -\mathrm{Res}\left(\frac{1}{u^2} f\left(\frac{1}{u}\right);0\right)$$



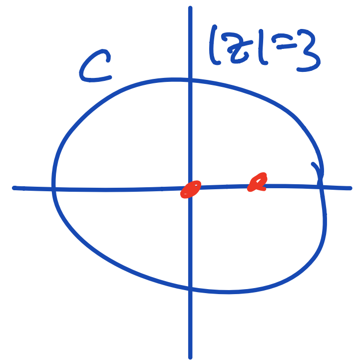
Theorem

$$\frac{1}{2\pi i}\int_\gamma f(z)\,dz = -\mathrm{Res}(f;\infty) = \mathrm{Res}\left(\frac{1}{w^2} f\left(\frac{1}{w}\right);0\right)$$



γ is a curve so that all singular points are inside γ.

Example $\int_C \frac{4z+1}{z(z-1)}\,dz$



Method 1

$$\frac{1}{w^2} f\left(\frac{1}{w}\right) = \frac{4+w}{w(1-w)}$$

$$\int_C \frac{4z+1}{z(z-1)}\,dz = 2\pi i \; \operatorname{Res}\left(\frac{4+w}{w(1-w)}; 0\right)$$

$$= 2\pi i \cdot 4$$

$$= 8\pi i$$

Method 2

$$\operatorname{Res}\left(\frac{4z+1}{z(z-1)}; 0\right) = -1$$

$$\operatorname{Res}\left(\frac{4z+1}{z(z-1)}; 1\right) = 5$$

$$\therefore \int_{|z|=3} \frac{4z+1}{z(z+1)}\,dz = 2\pi i\,(5-1) = 8\pi i$$