The Chinese University of Hong Kong, Shenzhen
SDS · School of Data Science

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MAT 3007 – Optimization

Exercise Sheet 5

Exercise E5.1 (Optimization Problem I):
Consider the function $f_\alpha : \mathbb{R}^2 \to \mathbb{R}$,

$$f_\alpha(x) := \alpha x_1^2 + x_2^2 - 2x_1x_2 - 2x_2,$$

where $\alpha \in \mathbb{R}$ is a scalar.

a) Find the stationary points (in case they exist) of f_α for each value of α.

b) For each stationary point x^* in part a), determine whether x^* is a local maximizer or a local minimizer or a saddle point of f_α.

c) For which values of α can f_α have a global minimizer?

Exercise E5.2 (Optimization Problem II):
We consider the optimization problem

$$\min_{x\in\mathbb{R}^2} \; f(x) = \frac{1}{2}x_1^2x_2^2 + \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 - 2x_1.$$

a) Is the function f coercive?

b) Calculate the gradient and Hessian of f and determine all stationary points of f.

c) Show that f has a unique global minimizer.

d) Is the mapping f convex?

Exercise E5.3 (Optimization Problem III):
Consider the problem

$$\min_{x\in\mathbb{R}^3} \; -x_1x_2x_3 \qquad \text{s.t.} \qquad x_1 + 3x_2 + 6x_3 \geq 48, \quad x_1, x_2, x_3 \geq 0.$$

Write down the KKT conditions for this problem.

Exercise E5.4 (Optimization Problem IV):
Consider the problem

$$\min_{x\in\mathbb{R}^3} \; f(x) = x_1^2 + x_2^2 + x_3^2 \qquad \text{s.t.} \qquad x_1 + 2x_2 + 3x_3 \geq 4, \quad x_3 \leq 1.$$

a) Write down the KKT conditions.

b) Without solving the KKT system, prove that the problem has a unique optimal solution and that this solution satisfies the KKT conditions.

c) Find the optimal solution of the problem using the KKT system.

Assignment A5.1 (A Penalty Problem): (approx. *20* points)

We consider the parametrized optimization problem

$$\min_x \; f_\beta(x) := \frac{1}{2}\|x-b\|^2 + \frac{\beta}{2}\left(\sum_{i=1}^n x_i\right)^2, \quad x \in \mathbb{R}^n, \tag{1}$$

where $b \in \mathbb{R}^n$ is given and $\beta \geq 0$ is a parameter.

a) Calculate the gradient and Hessian of f_β.

b) Show that f_β has a unique stationary point x^*_β and compute it explicitly. Determine whether x^*_β is a local minimizer, a local maximizer, or a saddle point of problem (1).

c) For $\beta \to \infty$, the solutions x^*_β converge to a point x^*. Calculate the limit $x^* = \lim_{\beta\to\infty} x^*_\beta$ explicitly and show that x^* satisfies the constraint $\mathbb{1}^\top x^* = \sum_{i=1}^n x^*_i = 0$.

d) Consider the following constrained nonlinear program:

$$\min_x \; \frac{1}{2}\|x-b\|^2 \qquad \text{s.t.} \qquad \mathbb{1}^\top x = 0.$$

Check whether the LICQ is generally satisfied at feasible points and verify that x^* is the unique local solution of this problem.

Assignment A5.2 (KKT-Conditions and Optimality): (approx. *12* points)

Consider the problem

$$\begin{array}{ll} \text{minimize}_x & f(x) := x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_2x_3 - 2x_1 - 5x_2 - 6x_3 \\ \text{subject to} & x_1 + x_2 + x_3 \leq 1, \quad x_1 - x_2^2 = 0. \end{array}$$

a) Write down the KKT-conditions for this problem.

b) Use the KKT-conditions and the second order optimality conditions to verify that $x^* = (0,0,1)^\top$ is a strict local minimum of the problem.

Assignment A5.3 (Constrained Optimization): (approx. *18* points)

We consider the nonlinear program

$$\min_{x\in\mathbb{R}^2} \; x_2^2 - 2x_1, \qquad \text{s.t.} \qquad g_1(x) \leq 0, \quad g_2(x) \leq 0, \tag{2}$$

where $g_1(x) := x_1^2 + x_2^2 - 1$ and $g_2(x) := (x_1-1)^2 - x_2^2$. Let us set $\bar{x} = (0,1)^\top$.

a) Draw the feasible set $\Omega := \{x \in \mathbb{R}^2 : g_1(x) \leq 0,\ g_2(x) \leq 0\}$.

b) Calculate the active set $\mathcal{A}(\bar{x})$ and the linearized tangent set $\mathcal{T}_\ell(\bar{x}) = \{d : \nabla g_i(\bar{x})^\top d \leq 0,\ \forall\, i \in \mathcal{A}(\bar{x})\}$. Draw the tangent set and add it to your sketch in part a).

c) Investigate whether problem (2) possesses an optimal solution. Explain your answer!

d) Compute all KKT-points of problem (2) and analyze whether the points are local or global minimizer.

Hint: The LICQ might not be satisfied at all KKT-points. You may assume that the LICQ holds at all feasible points other than the KKT-points.

Sheet 5 is due on **Jul, 20th**. Submit your solutions before **Jul, 20th**, **11:00 am**.