MAT2006: Elementary Real Analysis
Assignment #4

Reference Solutions

1. Let

$$g_a = \begin{cases} x^a \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Find a particular (potentially noninteger) value for a so that

(a) g_a is differentiable on $\mathbb{R}$ but such that $g'(a)$ is unbounded on $[0,1]$.

(b) g_a is differentiable on $\mathbb{R}$ with g_a' continuous but not differentiable at zero.

(c) g_a is differentiable on $\mathbb{R}$ and g_a' is differentiable on $\mathbb{R}$, but such that g_a'' is not continuous at zero.

Solution. When $a > 1$, we always have

$$g_a'(x) = \begin{cases} ax^{a-1}\sin(1/x) - x^{a-2}\cos(1/x) & \text{when } x \neq 0 \\ 0 & \text{when } x = 0. \end{cases}$$

(a) $1 < a < 2$.

(b) $2 < a < 3$.

(c) $3 < a < 4$.

$$g_a''(x) = \begin{cases} [a(a-1)x^{a-2} - x^{a-4}]\sin(1/x) - (2a-2)x^{a-3}\cos(1/x) & \text{when } x \neq 0 \\ 0 & \text{when } x = 0. \end{cases} \quad \square$$

2. Recall that a function $f : (a,b) \to \mathbb{R}$ is increasing on (a,b) if $f(x) \leq f(y)$ whenever $x < y$ in (a,b). A familiar mantra from calculus is that a differentiable function is increasing if its derivative is positive, but this statement requires some sharpening in order to be completely accurate.

Show that the function

$$g(x) = \begin{cases} x/2 + x^2\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is differentiable on $\mathbb{R}$ and satisfies $g'(0) > 0$. Now, prove that g is not increasing over any open interval containing 0.

We will see that f is indeed increasing on (a,b) if and only if $f'(x) \geq 0$ for all $x \in (a,b)$.

Proof. We see that

$$\lim_{x\to 0}\frac{g(x)-g(0)}{x-0}=\lim_{x\to 0}\left(\frac{1}{2}+x\sin(1/x)\right)=\frac{1}{2}.$$

Thus

$$g'(x)=\begin{cases}1/2+2x\sin(1/x)-\cos(1/x) & \text{if } x\neq 0\\ 1/2 & \text{if } x=0.\end{cases}$$

Note that $g'(0)=1/2>0$.

Let (a,b) be an open interval containing 0. There always exists $n\in\mathbb{N}$ such that $x_n=\frac{1}{2n\pi}$ and $y_n=\frac{1}{2n\pi+\frac{\pi}{2}}$ are in (a,b). Note that $y_n<x_n$, and

$$g(x_n)-g(y_n)=\frac{1}{2}\left[\frac{1}{2n\pi}-\frac{1}{2n\pi+\frac{\pi}{2}}\right]-\frac{1}{(2n\pi+\frac{\pi}{2})^2}=\frac{n(\frac{\pi}{4}-2)+\frac{1}{8}}{2n(2n\pi+\frac{\pi}{2})^2}$$

Note that the right-hand side is negative for n large enough. Thus $g(x)$ is not increasing on any (a,b) that contains 0. □

3. A fixed point of a function f is a value x where $f(x)=x$. Show that if f is differentiable on an interval with $f'(x)\neq 1$, then f can have at most one fixed point.

Proof. Assume x_1 and x_2 are fixed points of $f(x)$ and $x_1<x_2$. Let $g(x)=f(x)-x$, we then have $g(x_1)=g(x_2)=0$ and that $g'(x)=f'(x)-1\neq 0$. It follows by the Mean Value Theorem that

$$0=g(x_2)-g(x_1)=g'(\xi)(x_2-x_1),$$

where $\xi\in(x_1,x_2)$. Therefore $g'(\xi)=0$, which is a contraction with $f'(x)\neq 1$. Thus $x_1=x_2$ and f has at most one fixed point. □

4. Let $f(x)=x\sin(1/x^4)e^{-1/x^2}$ and $g(x)=e^{-1/x^2}$. Using the familiar properties of these functions, compute the limit as x approaches zero of $f(x)$, $g(x)$, $f(x)/g(x)$, and $f'(x)/g'(x)$. Explain why the results are surprising but not in conflict with the content of L'Hosipital's Rule.

Solution.

$$\lim_{x\to 0}g(x)=\lim_{x\to 0}e^{-1/x^2}=e^{-\infty}=0$$

$$\lim_{x\to 0}f(x)=\lim_{x\to 0}x\sin\frac{1}{x^4}\lim_{x\to 0}e^{-1/x^2}=0\times 0=0$$

$$\lim_{x\to 0}\frac{f(x)}{g(x)}=\lim_{x\to 0}x\sin\frac{1}{x^4}=0.$$

$$\begin{aligned}\lim_{x\to 0}\frac{f'(x)}{g'(x)}=&\lim_{x\to 0}\frac{[\sin\frac{1}{x^4}-4x^{-4}\cos\frac{1}{x^4}+2x^{-2}\sin\frac{1}{x^4}]e^{-1/x^2}}{2x^{-3}e^{-1/x^2}}\\ =&\frac{1}{2}\lim_{x\to 0}x^3\sin\frac{1}{x^4}-2\lim_{x\to 0}\frac{1}{x}\cos\frac{1}{x^4}+\lim_{x\to 0}x\sin\frac{1}{x^4}\end{aligned}$$

Note that $\lim_{x\to 0} \frac{1}{x} \cos \frac{1}{x^4}$ doesnot exist and the other two limits are zero, thus $\lim_{x\to 0} \frac{f'(x)}{g'(x)}$ doesnot exist. In the L'Hospital Rule, we should assume $\lim_{x\to 0} \frac{f'(x)}{g'(x)}$ exists. □

5. (i) Assume $f(x)$ is continuous on $[a, b]$ and differentiable in (a, b), $f(a) < 0$, $f(b) < 0$, and there exists one $c \in (a, b)$ such that $f(c) > 0$. Show that there exists $\xi \in (a, b)$ such that $f(\xi) + f'(\xi) = 0$.
Hint. Consider $F(x) = e^x f(x)$.

(ii) Assume $g(x)$ is continuous on $[0, 1]$ and differentiable in $(0, 1)$. Show that there exists $\xi \in (0, 1)$ such that $g'(\xi)g(1 - \xi) = g(\xi)g'(1 - \xi)$.

Proof. (i) Consider $F(x) = e^x f(x)$, then $F(x)$ is continuous on $[a, b]$ and differentiable on (a, b). By the Extremum Value Theorem, $F(x)$ must attain its maximum and minimum on $[a, b]$. Note that $F(a) = e^a f(a) < 0$ and $F(b) = e^b f(b) < 0$, but $F(c) = e^c f(c) > 0$, thus the maximum of $F(x)$ must attain at an interior point $\xi \in (a, b)$. Then, by the Interior Extremum Theorem,

$$0 = F'(\xi) = e^{\xi}[f(\xi) + f'(\xi)],$$

which implies that

$$f(\xi) + f'(\xi) = 0.$$

(ii) Consider the function $G(x) = g(x)g(1 - x)$. Then $G(x)$ is continuous on $[0, 1]$ and differentiable on $(0, 1)$. Note that

$$G(0) = G(1) = g(0)g(1).$$

By the Rolle's Theorem, there exists $\xi \in (0, 1)$ such that

$$0 = G'(\xi) = g'(\xi)g(1 - \xi) - g(\xi)g'(1 - \xi),$$

hence

$$g'(\xi)g(1 - \xi) = g(\xi)g'(1 - \xi).$$ □

Remark. If we take $\xi = \frac{1}{2}$, the desired identity holds immediately.

6. Let

$$f_n(x) = \frac{nx}{1 + nx^2}.$$

(a) Find the pointwise limit of $\{f_n\}$ for all $x \in (0, \infty)$.
(b) Is the convergence uniform on $(0, \infty)$?
(c) Is the convergence uniform on $(0, 1)$?
(d) Is the convergence uniform on $(1, \infty)$?

Solution. (a) The pointwise limit of $\{f_n\}$ is

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1/x & \text{if } x > 0. \end{cases}$$

(b) No, since f_n are continuous functions but f is not continuous on $(0, \infty)$.
(c) No, same reason as in part (b).
(d) Yes. Whenever $x > 1$, we have

$$|f_n(x) - f(x)| = \frac{1/x}{1 + nx^2} < \frac{1}{nx^2} < \frac{1}{n}.$$

For any $\epsilon > 0$, we can choose a $N \in \mathbb{N}$ with $1/N < \epsilon$ such that

$$|f_n(x) - f(x)| < \epsilon \qquad \forall n \geq N \quad \forall x > 0. \qquad \square$$

7. (i) Define a sequence of functions on $\mathbb{R}$ by

$$f_n(x) = \begin{cases} 1 & \text{if } x = 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$

and let f be the pointwise limit of f_n.

Is each f_n continuous at zero? Does $f_n \to f$ uniformly on $\mathbb{R}$? Is f continuous at zero?

(ii) Repeat this exercise using the sequence of functions

$$g_n(x) = x f_n(x) = \begin{cases} x & \text{if } x = 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n} \\ 0 & \text{otherwise.} \end{cases}$$

(iii) Repeat the exercise once more with the sequence

$$h_n(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n} \\ x & \text{if } x = 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n-1} \\ 0 & \text{otherwise.} \end{cases}$$

In each case, explain how the results are consistent with the content of the Continuous Limit Theorem.

Solution. (i) Yes, each f_n is continuous at $x = 0$. Note that

$$f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{k} \text{ for some } k \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

Thus $f(x)$ is not continuous at $x = 0$ since $\lim_{k\to\infty} f(1/k) = 1 \neq f(0)$.

The convergence $f_n \to f$ is not uniform: Choose $x_n = \frac{1}{n+1}$, then

$$|f_n(x_n) - f(x_n)| = 1.$$

The Continuous Limit Theorem doesnot apply to this case, since the convergence $f_n \to f$ is not uniform.

(ii) Yes, each g_n is continuous at $x = 0$. Note that

$$g(x) = \begin{cases} x & \text{if } x = \frac{1}{k} \text{ for some } k \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

We see that $g_n \to g$ uniformly: given any $\epsilon > 0$, there exists N such that $1/N < \epsilon$, and hence

$$|g_n(x) - g(x)| \le \frac{1}{n+1} < \frac{1}{N+1} < \epsilon, \qquad \forall n \ge N \quad \forall x \in \mathbb{R}.$$

Yes, $g(x)$ is continuous at $x = 0$, since

$$|g(x) - g(0)| \le |x|.$$

For any $\epsilon > 0$, choose $\delta = \epsilon$ shows the continuity of g at $x = 0$.

The Continuous Limit Theorem applies to this case to grantee that $g(x)$ is continuous at $x = 0$.

(iii) Yes, each h_n is continuous at $x = 0$. Note that

$$h(x) = \begin{cases} x & \text{if } x = \frac{1}{k} \text{ for some } k \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

We see that $h_n \to h$ pointwise but not uniformly: Choose $x_n = 1/n$, we have

$$|h_n(x_n) - h(x_n)| = 1.$$

Yes, $h(x)$ is continuous at $x = 0$, since $h(x) = g(x)$ and $g(x)$ is so.

The Continuous Limit Theorem does not apply to this case, but $h(x)$ is still continuous at $x = 0$. That is to say the hypothesis in Continuous Limit Theorem is just a sufficient but not necessary condition. □

8. For each $n \in \mathbb{N}$ and $x \in [0, \infty)$, let

$$g_n(x) = \frac{x}{1 + x^n}, \qquad h_n(x) = \begin{cases} 1 & \text{if } x \ge 1/n \\ nx & \text{if } 0 \le x < 1/n. \end{cases}$$

Answer the following questions for the sequences $\{g_n\}$ and $\{h_n\}$:

(a) Find the pointwise limit on $[0, \infty)$.

(b) Explain how we know that the convergence cannot be uniform on $[0, \infty)$.

(c) Choose a smaller set over which the convergence is uniform and supply an argument to show that this is indeed the case.

Solution. (a) Assume $g_n \to g$ and $h_n \to h$ pointwise on $[0, \infty)$ respectively. Then,

$$g(x) = \begin{cases} x & \text{if } 0 \le x < 1 \\ 1/2 & \text{if } x = 1 \\ 0 & \text{if } x > 1, \end{cases} \qquad h(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0. \end{cases}$$

(b) The limit function $g(x)$ is not continuous at $x = 1$ and $h(x)$ is not continuous at $x = 0$. Note that each g_n is continuous on $[0, \infty)$, the convergence $g_n \to g$ can not be uniform there, for otherwise, g must be a continuous but fails to be so. A similar reason for $h_n \to h$ is not uniform on $[0, \infty)$.

(c) $g_n \to g$ uniformly on $[0, \alpha] \cup [\beta, \infty)$ where $0 < \alpha < 1$ and $\beta > 1$. Note that when $0 \le x \le \alpha < 1$

$$|g_n(x) - g(x)| = \frac{x^{n+1}}{1 + x^n} \le x^{n+1} \le \alpha^{n+1};$$

and when $x \ge \beta > 1$,

$$|g_n(x) - g(x)| = \frac{x}{1 + x^n} \le x^{1-n} \le (1/\beta)^{n-1}.$$

Recall that $\alpha^{n+1} \to 0$ and $(1/\beta)^{n-1} \to 0$, there exits N_1 and N_2 such that

$$\alpha^{n+1} < \epsilon \qquad \forall n \ge N_1$$

and

$$(1/\beta)^{n-1} < \epsilon \qquad \forall n \ge N_2.$$

Take $N = \max\{N_1, N_2\}$, we then have

$$|g_n(x) - g(x)| < \epsilon, \qquad \forall n \ge N \quad \forall x \in [0, \alpha] \cup [\beta, \infty).$$

The convergence $h_n \to h$ is unform on $[\delta, \infty)$ where $\delta > 0$. For any $\epsilon > 0$, choose an $N > 1/\delta$. Then

$$|h_n(x) - h(x)| = 0 < \epsilon, \qquad \forall n \ge N \quad \forall x \in [\delta, \infty).$$ □

9. Assume $f_n \to f$ on a set A. The Continuous Limit Theorem is an example of a typical type of question which asks whether a trait possessed by each f_n is inherited by the limit function. Provide an example to show that all of the following propositions are false if the convergence is only assumed to be pointwise on A. Then go back and decide which are true under the stronger hypothesis of uniform convergence.

(a) If each f_n is uniformly continuous, then f is uniformly continuous.

(b) If each f_n is bounded, then f is bounded.

(c) If each f_n has a finite number of discontinuities, then f has a finite number of discontinuities.

(d) If each f_n has fewer than M discontinuities (where $M \in \mathbb{N}$ is fixed), then f has fewer than M discontinuities.

(e) If each f_n has at most a countable number of discontinuities, then f has at most a countable number of discontinuities.

Solution. (a) Let $f_n(x) = x^n$ on $[0, 1]$. Then $f_n(x) \to f(x)$ pointwise with

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x = 1. \end{cases}$$

Note that each f_n is continuous on a compact set $[0,1]$ and hence uniformly continuous there, but $f(x)$ is not even continuous.

If $f_n \to f$ uniformly on A and each f_n is uniformly continuous on A, then $f(x)$ is also uniformly continuous on A. To see this, we apply the triangle inequality to get

$$\begin{aligned}|f(x)-f(y)| =&|f(x)-f_N(x)+f_N(x)-f_N(y)+f_N(y)-f(y)| \\ \leq&|f(x)-f_N(x)|+|f_N(x)-f_N(y)|+|f_N(y)-f(y)|.\end{aligned}$$

Given any $\epsilon > 0$. It follows from the uniform convergence $f_n \to f$ on A that there exists an N such that

$$|f_n(x)-f(x)| < \frac{\epsilon}{3} \qquad \forall n \geq N \quad \forall x \in A.$$

And it follows from $f_N(x)$ is uniformly continuous that there exists a $\delta > 0$ such that

$$|f_N(x)-f_N(y)| < \frac{\epsilon}{3} \qquad \forall |x-y| < \delta, \quad \forall x, y \in A.$$

Then combine the above inequalities shows that

$$|f(x)-f(y)| < \epsilon, \qquad \forall |x-y| < \delta \quad \forall x, y \in A.$$

Note. in the case when A is compact, the proof is shorter: f is continuous on A by the Continuous Limit Theorem and thus uniformly continuous on A since A is compact.

(b) Let

$$f_n(x) = \begin{cases} x & \text{if } 0 \leq x \leq n \\ 0 & \text{if } x > n. \end{cases}$$

It is readily seen that $f_n(x) \to f(x) = x$ pointwise on $[0,\infty)$, and each f_n is bounded but f is not on $[0,\infty)$.

Assume $f_n \to f$ uniformly on A and each f_n is bounded on A, then f is also bounded on A: There exists N such that

$$|f_n(x)-f(x)| < 1, \qquad \forall n \geq N \quad \forall x \in A.$$

Assume M is a bound of $f_N(x)$, i.e., $|f_N(x)| \leq M$ for every $x \in A$. Then

$$|f(x)| = |f(x)-f_N(x)+f_N(x)| \leq |f_N(x)-f(x)|+|f_N(x)| < 1+M, \qquad \forall x \in A.$$

(c) Let f_n be given as in Problem 7(i). Then each f_n has n, thus finite number of, discontinuities. However, $f(x)$ in this case has infinitely many of discontinuities.

When the convergence is uniform, the statement is still false – take f_n to be the g_n in Problem 7(ii).

(d) Take f_n and f as in part (a) shows the pointwise convergence for the case of pointwise convergence.

If $f_n \to f$ uniformly and each f_n has less than M discontinuities, then f has less than M discontinuities. Suppose, for a contradiction, that f has more than M discontinuities and that $x_1, x_2, \ldots, x_M, x_{M+1}$ are (some of) the disunities of f. We claim that there exists N_k such that f_n has discontinuities at x_k, where $1 \leq k \leq M+1$. If not, there exists a subsequence n_k such that f_{n_k} is continuous at x_k, and the uniform convergence of $f_n \to f$ implies the uniform convergence of $f_{n_k} \to f$ and further implies that f is continuous at x_k, a contradiction. Now, take $N = \max\{N_1, N_2, \ldots, N_{M+1}\}$. Then $f_N(x)$ has discontinuities at $x_1, x_2, \ldots, x_M, x_{M+1}$, which is a contradiction with the assumption that each f_n has less than M discontinuities.

(e) Let $A = [0, 1]$, and

$$f_n(x) = \begin{cases} 1 & \text{if } x = 0 \text{ or } x = \frac{p}{q} \text{ in its lowest order form, } 0 < q \le n \\ 0 & \text{otherwise.} \end{cases}$$

We see that each f_n has a finite number of discontinuities, but its pointwise limit, which is the Dirichlet function is discontinuous at any point on $[0, 1]$.

If $f_n \to f$ uniformly and each f_n has at most a countable number of discontinuities, then f has at most a countable number of discontinuities.

Let D_f and D_{f_n} be the set of discontinuities of f and f_n respectively. Then each f_n is continuous on the set

$$\bigcap_{n=1}^{\infty} (D_{f_n})^c = \left(\bigcup_{n=1}^{\infty} D_{f_n} \right)^c$$

where a superscript c stands for the complement of a set relatively to A. Thus f is also continuous on the same set by the Continuous Limit Theorem. Therefore,

$$D_f \subset \bigcup_{n=1}^{\infty} D_{f_n},$$

the right-hand side of which is countable, since the countable union of countable sets is still countable. □

10. Assume $f_n \to f$ pointwise on $[a, b]$ and the limit function f is continuous on $[a, b]$. If each f_n is increasing (but not necessarily continuous), show $f_n \to f$ uniformly.

Proof. Let $\epsilon > 0$ be arbitrary. Since f is continuous on $[a, b]$, a compact set, thus f is uniformly continuous there. There exists a $\delta > 0$ such that

$$(*) \qquad |f(x) - f(y)| < \frac{\epsilon}{2} \qquad \forall |x - y| < \delta.$$

We split the interval $[a, b]$ into $J = \lfloor \frac{b-a}{\delta} \rfloor + 1$ equal subintervals

$$a = x_0 < x_1 < \cdots < x_J = b, \qquad x_j = a + jh, \quad h = \frac{b-a}{J}.$$

It is clear that $0 < h < \delta$. Since $f_n(x_j) \to f(x_j)$, $0 \le j \le J$, there exists $N_j \in \mathbb{N}$ such that

$$|f_n(x_j) - f(x_j)| < \frac{\epsilon}{2} \qquad \forall n \ge N_j, \qquad j = 0, 1, \ldots, J.$$

Now take

$$N = \max\{N_0, N_1, \dots, N_J\}.$$

We then have

$$(**) \qquad |f_n(x_j) - f(x_j)| < \frac{\epsilon}{2} \qquad \forall n \geq N, \forall 0 \leq j \leq J.$$

Fixed any $x \in [a, b]$, there exists $0 \leq j_0 \leq J$ such that $x \in [x_{j_0}, x_{j_0+1}]$. Note that $f_N(x)$ is increasing, we have

$$f_n(x_{j_0}) \leq f_n(x) \leq f_n(x_{j_0+1}),$$

and by $|x - x_{j_0+1}| \leq h < \delta$ and $(*)$ that

$$\begin{aligned} f_n(x) - f(x) \leq & f_n(x_{j_0+1}) - f(x) \\ = & f_n(x_{j_0+1}) - f(x_{j_0+1}) + f(x_{j_0+1}) - f(x) \\ < & \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \qquad \forall n \geq N. \end{aligned}$$

Similarly, we also have

$$\begin{aligned} f_n(x) - f(x) \geq & f_n(x_{j_0}) - f(x) \\ = & f_n(x_{j_0}) - f(x_{j_0}) + f(x_{j_0}) - f(x) \\ > & -\frac{\epsilon}{2} - \frac{\epsilon}{2} = -\epsilon \qquad \forall n \geq N. \end{aligned}$$

Combining the last two inequalities, we get

$$|f_n(x) - f(x)| < \epsilon \qquad \forall n \geq N, \forall x \in [a, b],$$

which implies that f_n converges to f uniformly for $x \in [a, b]$. □

11 (Dini's Theorem)**.** Assume $f_n \to f$ pointwise on a compact set K and assume that for each $x \in K$ the sequence $f_n(x)$ is increasing. Follow these steps to show that if f_n and f are continuous on K, then the convergence is uniform.

(a) Set $g_n = f - f_n$ and translate the preceding hypothesis into statements about the sequence $\{g_n\}$.

(b) Let $\epsilon > 0$ be arbitrary, and define $K_n = \{x \in K \mid g_n(x) \geq \epsilon\}$. Argue that $K_1 \supset K_2 \supset K_3 \supset \cdots$, and use this observation to finish the argument.

Proof. (a) By the hypothesis, g_n converges pointwise to $g(x) \equiv 0$ on the compact set K, and for any fixed $x \in K$, the sequence $\{g_n(x)\}$ is decreasing with $g_n(x) \geq 0$. Moreover, each $g_n(x)$ is continuous on K. We want to show that $g_n(x) \to 0$ uniformly on K.

(b) Assume $x \in K_{n+1}$ for some $n \in \mathbb{N}$. By definition, $g_n(x) \geq \epsilon$. And, by the monotonicity of $\{g_n(x)\}$, we have $g_{n+1}(x) \geq g_n(x) \geq \epsilon$ and thus $x \in K_n$. We claim that each K_n is a closed set. Fixed an $n \in \mathbb{N}$. Let x be a limit point of K_n, then there exists a sequence $\{x_k\} \subset K_n$ with $x_k \neq x$ and $x_k \to x$ as $k \to \infty$. That is, $g_n(x_k) \geq \epsilon$. By the continuity of g_n and the Order Limit Theorem, we have $g(x) = \lim_{k\to\infty} g_n(x_k) \geq \epsilon$. Therefore $x \in K_n$ and so K_n is closed. Moreover, $K_n \subset K$ is bounded, hence the Heine–Borel Theorem asserts that K_n is compact. Thus, we have a nested sequence of compact sets

$$K_1 \supset K_2 \supset K_3 \supset \cdots .$$

Now we claim that there exists $N \in \mathbb{N}$ such that $K_N = \emptyset$. For otherwise, the Nested Compact Set Theorem implies that there exists a point $x_0 \in K$ such that

$$x_0 \in \bigcap_{n=1}^{\infty} K_n .$$

Therefore, $g_n(x_0) \geq \epsilon$ for all $n \in \mathbb{N}$ which is a contradiction with the hypothesis that $g_n(x) \to 0$ pointwise for all $x \in K$. Therefore, there exists $N \in \mathbb{N}$ such that $K_N = \emptyset$, which means that

$$0 \leq g_n(x) < \epsilon \qquad \forall n \geq N, \, \forall x \in K.$$

Thus, g_n converges to $g(x) \equiv 0$ uniformly on K, and so $f_n \to f$ uniformly on K. □

12 (Cantor's Function)**.** Review the construction of the Cantor set $C \subset [0,1]$.

(a) Define $f_0(x) = x$ for all $x \in [0,1] = C_0$. Now, let

$$f_1(x) = \begin{cases} (3/2)x & \text{for } 0 \leq x \leq 1/3 \\ 1/2 & \text{for } 1/3 < x < 2/3 \\ (3/2)x - 1/2 & \text{for } 2/3 \leq x \leq 1. \end{cases}$$

Sketch f_0 and f_1 over $[0,1]$ and observe that f_1 is continuous, increasing, and constant on the middle third $(1/3, 2/3) = [0,1] \setminus C_1$.

(b) Construct f_2 by imitating this process of flattening out the middle third of each nonconstant segment of f_1. Specifically, let

$$f_2(x) = \begin{cases} (1/2)f_1(3x) & \text{for } 0 \leq x \leq 1/3 \\ f_1(x) & \text{for } 1/3 < x < 2/3 \\ (1/2)f_1(3x-2) + 1/2 & \text{for } 2/3 \leq x \leq 1. \end{cases}$$

If we continue this process, show that the resulting sequence $\{f_n\}$ converges uniformly on $[0,1]$.

(c) Let $f = \lim_{n\to\infty} f_n$. Prove that f is a continuous, increasing function on $[0,1]$ with $f(0) = 0$ and $f(1) = 1$ that satisfies $f'(x) = 0$ for all x in the open set $[0,1] \setminus C$. Recall that the "length" of the Cantor set C is 0. Somehow, f manages to increase from 0 to 1 while remaining constant on a set of "length 1."

Proof. (a) Omitted.

(b) Recall the construction of the Cantor set $C = \bigcap_{n=1}^{\infty} C_n$. Let $n \geq m$. Note that $f_n(x) = f_m(x)$ for $x \notin C_m$. By the construction of $\{f_n(x)\}$, it is also clear that

$$|f_n(x) - f_m(x)| \leq \frac{1}{2^m}.$$

Therefore, the above inequality holds for all $x \in [0,1]$. Let $\epsilon > 0$. There exists $N \in \mathbb{N}$ such that $1/2^N < \epsilon$. Thus

$$|f_n(x) - f_m(x)| < \epsilon \qquad \forall n > m \geq N, \, \forall x \in [0,1].$$

By the Cauchy Criterion, $\{f_n(x)\}$ converges uniformly on $[0,1]$.

(c) Since $f_n \to f$ uniformly on $[0,1]$ and each $f_n(x)$ is continuous on $[0,1]$, it follows by the Continuous Limit Theorem that $f(x)$ is also continuous on $[0,1]$. Note that $f_n(0) = 0$ and $f_n(1) = 1$ for each $n \in \mathbb{N}$. Thus, as the limit, $f(0) = 0$ and $f(1) = 1$. Given $0 \leq x \leq y \leq 1$, since $f_n(x) \leq f_n(y)$, the Order Limit Theorem yields that $f(x) \leq f(y)$, that is $f(x)$ is increasing on $[0,1]$.

Given any $x \in [0,1] \setminus C$, then $x \notin C = \bigcap_{n=1}^{\infty} C_n$. There exists $N \in \mathbb{N}$ such that $x \notin C_N$, or, $x \in C_N^c$. Since C_N is closed and thus C_N^c is open, there exists a neighborhood $V_\delta(x)$ such that $V_\delta(x) \cap C_N = \emptyset$. Therefore,

$$f_n(y) \equiv 0 \qquad \forall y \in V_\delta(x) \, \forall n \geq N.$$

Hence,

$$f_n'(y) \equiv 0 \qquad \forall y \in V_\delta(x) \, \forall n \geq N,$$

and so $f_n'(y)$ converges to $g(y) \equiv 0$ uniformly on $V_\delta(x)$. By the Differentiable Limit Theorem, we have f is differentiable on $V_\delta(x)$ and $f'(y) \equiv 0$ there. In particular, $f'(x) = 0$ for all $x \in [0,1] \setminus C$. □

13. Let

$$g_n(x) = \frac{nx + x^2}{2n}$$

and set $g(x) = \lim_{n\to\infty} g_n(x)$. Show that g is differentiable in two ways:

(a) Compute $g(x)$ by algebraically taking the limit as $n \to \infty$ and then find $g'(x)$.

(b) Compute $g_n'(x)$ for each $n \in \mathbb{N}$ and show that the sequence of derivatives $\{g_n'\}$ converges uniformly on every interval $[-M, M]$. Then conclude $g'(x) = \lim_{n\to\infty} g_n'(x)$.

(c) Repeat parts (a) and (b) for the sequence $f_n(x) = (nx^2 + 1)/(2n + x)$.

Solution. (a) For any fixed $x \in \mathbb{R}$, we have

$$g(x) = \lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} \left(\frac{x}{2} + \frac{x^2}{2n} \right) = \frac{x}{2}.$$

Therefore, $g'(x) = 1/2$.

(b) Now

$$g_n'(x) = \left(\frac{x}{2} + \frac{x^2}{2n} \right)' = \frac{1}{2} + \frac{x}{n}, \qquad \forall n \in \mathbb{N}.$$

Note that $g_n'(x) \to h(x) = \frac{1}{2}$ pointwise on $\mathbb{R}$. This convergence is also uniform on $[-M, M]$ for any fixed $M > 0$. To see this, given any $\epsilon > 0$, choose $N \in \mathbb{N}$ such that $M/N < \epsilon$, then

$$|g_n'(x) - h(x)| = \frac{|x|}{n} \leq \frac{M}{N} < \epsilon, \qquad \forall n \geq N \quad \forall |x| \leq M.$$

Thus, we have that

$$g'(x) = \lim_{n\to\infty} g_n'(x), \qquad \forall x \in \mathbb{R}.$$

(c) For any fixed $x \in \mathbb{R}$, we have

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x^2 + \frac{1}{n}}{2 + \frac{x}{n}} = \frac{x^2}{2}.$$

Therefore, $f'(x) = x$.

Now,

$$f_n'(x) = \left(\frac{nx^2+1}{2n+x}\right)' = \frac{(2nx)(2n+x) - (nx^2+1)}{(2n+x)^2} = \frac{4n^2x + nx^2 - 1}{(2n+x)^2},$$

so

$$\lim_{n\to\infty} f_n'(x) = \lim_{n\to\infty} \frac{4x + \frac{x^2}{n} - \frac{1}{n^2}}{(2+\frac{x}{n})^2} = x := p(x).$$

We also have $\lim_{n\to\infty} f_n'(x) = f'(x)$ for $x \in \mathbb{R}$. □

14. Provide an example or explain why the request is impossible. Let's take the domain of the functions to be all of $\mathbb{R}$.

(a) A sequence $\{f_n\}$ of nowhere differentiable functions with $f_n \to f$ uniformly and f everywhere differentiable.

(b) A sequence $\{f_n\}$ of differentiable functions such that $\{f_n'\}$ converges uniformly but the original sequence $\{f_n\}$ does not converge for any $x \in \mathbb{R}$.

(c) A sequence $\{f_n\}$ of differentiable functions such that both $\{f_n\}$ and $\{f_n'\}$ converge uniformly but $f = \lim f_n$ is not differentiable at some point.

Solution. (a) Let $f_n(x) = \frac{1}{n}D(x)$ where $D(x)$ is the Dirichlet's function and let $f(x) = 0$. Then $f_n \to f$ uniformly. Each function f_n is nowhere continuous, and hence nowhere differentiable, but $f(x)$ is everywhere differentiable.

(b) Example $f_n(x) = n$.

(c) Not possible according to the Differentiable Limit Theorem. □

15. Decide whether each proposition is true or false, providing a short justification or counterexample as appropriate.

(a) If $\sum_{n=1}^{\infty} g_n$ converges uniformly, then $\{g_n\}$ converges uniformly to zero.

(b) If $0 \le f_n \le g_n$ and $\sum_{n=1}^{\infty} g_n$ converges uniformly, then $\sum_{n=1}^{\infty} f_n$ converges uniformly.

(c) If $\sum_{n=1}^{\infty} f_n$ converges uniformly on A, then there exist constants M_n such that $|f_n(x)| \le M_n$ for all $x \in A$ and $\sum_{n=1}^{\infty} M_n$ converges.

Solution. (a) True, according to the Cauchy Criterion.

(b)True, according to the Cauchy Criterion and note that

$$|f_{m+1}(x) + f_{m+2}(x) + f_n(x)| = f_{m+1}(x) + f_{m+2}(x) + f_n(x) \le |g_{m+1}(x) + g_{m+2}(x) + g_n(x)|.$$

(c) False. Let $f_n(x) = \frac{x}{n^2}$ on $x \in A$. Then $\sum_{n=1}^{\infty} f_n$ converges uniformly to $\frac{\pi^2}{6}x$ on $\mathbb{R}$, but each f_n is not bounded. □

16. (a) Prove that

$$h(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

is continuous on $[-1, 1]$.

(b) The series

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$$

converges for every x in the half-open interval $[-1, 1)$ but does not converge when $x = 1$. For a fixed $x_0 \in (-1, 1)$, explain how we can still use the Weierstrass M-Test to prove that f is continuous at x_0.

Proof. (a) Note that $x^n/n^2 \leq 1/n^2$ when $-1 \leq x \leq 1$ and recall that $\sum \frac{1}{n^2}$ converges, Thus the power series converges uniformly on $[-1, 1]$ by the Weierstrass M-test. Since each term x^n/n^2 is continuous thus $h(x)$ is continuous on $[-1, 1]$ by the Continuous Limit Theorem.

(b) For any $x_0 \in (-1, 1)$, we have $x^n/n \leq c^n$ when $|x| \leq c = (|x_0|+1)/2$. Note that $c < 1$ and hence the series $\sum c^n$ converges. The Weierstrass M-test then tells $\sum_{n=1}^{\infty} x^n/n$ converges uniformly on $[-c, c]$. Since each term of this series is continuous, thus $f(x)$ is continuous on $[-c, c]$, and in particular, is continuous at x_0. $\square$

17. Let

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{x+n} = \frac{1}{x} - \frac{1}{x+1} + \frac{1}{x+2} - \frac{1}{x+3} + \cdots.$$

Show f is defined for all $x > 0$. Is f continuous on $(0, \infty)$? How about differentiable?

Solution. For any fixed $x > 0$, the sequence $\{1/(x+n)\}$ is decreasing and tends to zero. Thus by the Alternating Series Test, the alternating series converges, and f is defined for all $x > 0$.

Note that

$$\begin{aligned}
&\left|\frac{1}{x+(m+1)} - \frac{1}{x+(m+2)} + \cdots + (-1)^{n-m-1}\frac{1}{x+n}\right| \\
=&\left|\frac{1}{[x+(m+1)][x+(m+2)]} + \frac{1}{[x+(m+3)][x+(m+4)]} + \cdots + (-1)^{n-m-1}\frac{1}{x+n}\right| \\
\leq&\frac{1}{[x+(m+1)][x+(m+2)]} + \frac{1}{[x+(m+3)][x+(m+4)]} + \cdots \\
\leq&\frac{1}{(m+1)^2} + \frac{1}{(m+3)^2} + \cdots
\end{aligned}$$

Since $\sum \frac{1}{n^2}$ converges, for any $\epsilon > 0$, there exists N such that

$$\frac{1}{(m+1)^2} + \frac{1}{(m+2)^2} + \frac{1}{(m+3)^2} + \cdots < \epsilon, \qquad \forall m \geq N.$$

Thus

$$\begin{aligned}
&\left|\frac{1}{x+(m+1)}-\frac{1}{x+(m+2)}+\cdots+(-1)^{n-m-1}\frac{1}{x+n}\right| \\
\leq&\frac{1}{(m+1)^2}+\frac{1}{(m+3)^2}+\cdots \\
<&\epsilon
\end{aligned}$$

for all $n > m \geq N$ and for all $x > 0$. By the Cauchy Criterion, the series of $f(x)$ converges uniformly on $(0, \infty)$. Moreover, the continuity of each term of this series and the Continuous Limit Theorem imply that f is continuous on $(0, \infty)$.

Now, consider the series

$$\sum_{n=1}^{\infty} f_n'(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(x+n)^2}$$

By a similar manner as previously, $\sum_{n=1}^{\infty} f_n'(x)$ converges uniformly on $(0, \infty)$. By the convergence of $\sum_{n=1}^{\infty} f_n(x)$ and the Differentiable Limit Theorem, $f(x)$ is differentiable on the same interval. □

18. Let

$$f(x) = \sum_{k=1}^{\infty} \frac{\sin kx}{k^3}.$$

(a) Show that $f(x)$ is differentiable and that the derivative $f'(x)$ is continuous.

(b) Can we determine if f is twice-differentiable?

Proof. (a) Denote $f_k(x) = \frac{\sin kx}{k^3}$ for each $k \in \mathbb{N}$. Note that $|f_k(x)| \leq \frac{1}{k^3}$ and the series $\sum_{k=1}^{\infty} \frac{1}{k^3}$ converges. Thus the series of $f(x)$ converges uniformly by the Weierstrass M-test. In a similar manner

$$\sum_{k=1}^{\infty} f_k'(x) = \sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2}$$

converges to some function $g(x)$ uniformly on $\mathbb{R}$. By the Differentiable Limit Theorem, $f(x)$ is differentiable, and $f'(x) = g(x)$. Note that the terms in the series of $g(x)$ are all continuous, and so is $f'(x) = g(x)$ according to the Continuous Limit Theorem.

(b) If we take one more derivative to get

$$\sum_{k=1}^{\infty} f_k'(x) = \sum_{k=1}^{\infty} \frac{-\sin(kx)}{k},$$

then the Weierstrass M-test doesnot apply to this case, and we can not determine its uniform convergence. More advanced knowledge should be introduced to determine this. □

19. Consider the function

$$f(x) = \sum_{k=1}^{\infty} \frac{\sin(x/k)}{k}.$$

Where is f defined? Continuous? Differentiable? Twice-differentiable?

Solution. Denote $f_k(x) = \frac{\sin(x/k)}{k}$ for each $k \in \mathbb{N}$.

Note that the series

$$\sum_{k=1}^{\infty} f_k'(x) = \sum_{k=1}^{\infty} \frac{\cos(x/k)}{k^2}$$

and $|\frac{\cos(x/k)}{k^2}| \le \frac{1}{k^2}$ for all $x \in \mathbb{R}$. It then follows from the convergence of $\sum 1/k^2$ and the Weierstrass M-test that the series $\sum_{k=1}^{\infty} f_k'(x)$ converges uniformly on $\mathbb{R}$. Note that the series $\sum_{k=1}^{\infty} f_k(x)$ converges at one point $x = 0$. Then the stronger version of the Differentiable Limit Theorem tells us that the series of $\sum_{k=1}^{\infty} f_k(x)$ converges uniformly and thus f is well defined on $\mathbb{R}$. Moreover, f is differentiable on $\mathbb{R}$.

The function f is also twice-differentiable, since

$$f'(x) = \sum_{k=1}^{\infty} f_k'(x),$$

and the series

$$\sum_{k=1}^{\infty} f_k''(x) = \sum_{k=1}^{\infty} \frac{-\sin(x/k)}{k^3}$$

converges uniformly on $\mathbb{R}$ by applying the Weierstrass M-test once again. And the Differentiable Limit Theorem implies the differentiability of $f'(x)$ on $\mathbb{R}$. $\square$

20. Let $\{r_1, r_2, r_3, \dots\}$ be an enumeration of the set of rational numbers. For each $r_n \in \mathbb{Q}$, define

$$u_n(x) = \begin{cases} 1/2^n & \text{for } x > r_n \\ 0 & \text{for } x \le r_n. \end{cases}$$

Now, let $h(x) = \sum_{n=1}^{\infty} u_n(x)$. Prove that h is a monotone function defined on all of $\mathbb{R}$ that is continuous at every irrational point.

Proof. Note that $|u_n(x)| \le 1/2^n$ and the series $\sum_{n=1}^{\infty} 1/2^n$ converges. Hence, the Weierstrass M-test implies the uniform convergence of $h(x) = \sum_{n=1}^{\infty} u_n(x)$ on $\mathbb{R}$. Since each $u_n(x)$ is continuous at every irrational point, so do $h(x)$ by the Continuous Limit Theorem.

Given any $x, y \in \mathbb{R}$ with $x < y$. Note that $u_n(x) \le u_n(y)$ for all $n \in \mathbb{N}$. Thus $h(x) \le h(y)$ according to the Order Limit Theorem. Thus $h(x)$ is increasing on $\mathbb{R}$.

Note. We have shown before that any monotone function has only jump discontinuity and there are at most countable many of them. In this example, we have $D_h = \mathbb{Q}$. $\square$

— End —