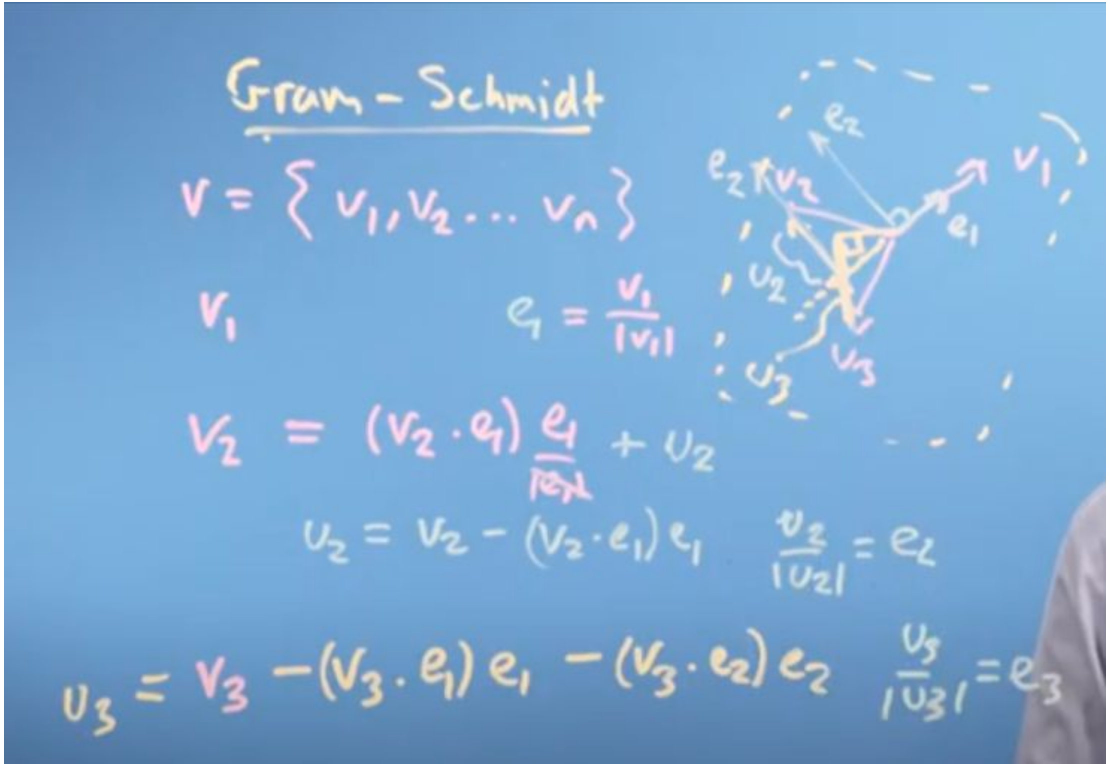
Digital Talent Scholarship 2022

Math for ML - Linear Algebra 3

Lead a sprint through Machine Learning with Tensorflow

The Gram-Schmidt process



The Gram-Schmidt process

Proof.

1. Misal $\mathbf{v}_1 = \mathbf{u}_1$
2. Membentuk vektor $\mathbf{v}_2$ yang ortogonal terhadap $\mathbf{v}_1$ dengan cara menghitung komponen dari $\mathbf{u}_2$ yang ortogonal terhadap ruang W_1 yang direntang oleh $\mathbf{v}_1$, yaitu

$$\begin{aligned}\mathbf{v}_2 &= \mathbf{u}_2 - \mathrm{proj}_{W_1}\mathbf{u}_2 \\ &= \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1\end{aligned}$$

The Gram-Schmidt process

Proof.

3. Membentuk vektor $\mathbf{v}_3$ yang ortogonal terhadap $\mathbf{v}_1$ dan $\mathbf{v}_2$ dengan cara menghitung komponen dari $\mathbf{u}_3$ yang ortogonal terhadap ruang W_2 yang direntang oleh $\mathbf{v}_1$ dan $\mathbf{v}_2$, yaitu

$$\begin{aligned}\mathbf{v}_3 &= \mathbf{u}_3 - \text{proj}_{W_2}\mathbf{u}_3 \\ &= \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2\end{aligned}$$

4. Proses dilanjutkan sampai v_n, untuk menghasilkan himpunan ortogonal $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ yang terdiri dari n vektor bebas linear di V dan merupakan suatu basis ortogonal untuk V. Penormalan vektor-vektor di basis ortogonal akan menghasilkan basis ortonormal.

The Gram-Schmidt process

Example

Diberikan $V = R^3$ dengan hasilkali dalam Euclid. Terapkan algoritma **Gram-Schmidt** untuk mengortogonalkan basis

$$\{(1, -1, 1), (1, 0, 1), (1, 1, 2)\}$$

Normalisasikan vektor-vektor basis ortogonal yang diperoleh menjadi sebuah basis ortonormal.

The Gram-Schmidt process

Solution

Misal $\mathbf{u}_1 = (1, -1, 1)\,, \mathbf{u}_2 = (1, 0, 1)\,, \mathbf{u}_3 = (1, 1, 2)$

- *Langkah 1*

$$\mathbf{v}_1 = \mathbf{u}_1 = (1, -1, 1)$$

- *Langkah 2*

$$\begin{aligned}
\mathbf{v}_2 &= \mathbf{u}_2 - proj_{W_1}\mathbf{u}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 \\
&= (1, 0, 1) - \frac{1 \cdot 1 + 0 \cdot -1 + 1 \cdot 1}{3}(1, -1, 1) \\
&= (1, 0, 1) - \left(\frac{2}{3}, -\frac{2}{3}, \frac{2}{3}\right) \\
&= \left(\frac{1}{3}, \frac{2}{3}, \frac{1}{3}\right)
\end{aligned}$$

The Gram-Schmidt process

Solution

- *Langkah 3*

$$
\begin{aligned}
\mathbf{v}_3 &= \mathbf{u}_3 - proj_{W_2}\mathbf{u}_3 \\
&= \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 \\
&= (1, 1, 2) - \frac{2}{3}(1, -1, 1) - \frac{5}{2}\left(\frac{1}{3}, \frac{2}{3}, \frac{1}{3}\right) \\
&= \left(-\frac{1}{2}, 0, \frac{1}{2}\right)
\end{aligned}
$$

- *Dengan demikian, diperoleh basis ortogonal*

$$
\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \left\{(1, -1, 1), \left(\frac{1}{3}, \frac{2}{3}, \frac{1}{3}\right), \left(-\frac{1}{2}, 0, \frac{1}{2}\right)\right\}
$$

The Gram-Schmidt process

Solution

- *Selanjutnya dapat diperoleh basis ortonormal* $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ *dengan*

$$
\begin{aligned}
\mathbf{q}_1 &= \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{(1,-1,1)}{\sqrt{3}} = \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \left(\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right) \\
\mathbf{q}_2 &= \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{\left(\frac{1}{3}, \frac{2}{3}, \frac{1}{3}\right)}{\frac{\sqrt{6}}{3}} = \left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) = \left(\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{6}\right) \\
\mathbf{q}_3 &= \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \frac{\left(-\frac{1}{2}, 0, \frac{1}{2}\right)}{\frac{\sqrt{2}}{2}} = \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) = \left(-\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right)
\end{aligned}
$$

Q & A

Agenda

1. Intro to eigenvalues dan eigenvectors
2. Konsep yang perlu dipahami
3. Next steps

Are your students cloud-ready?

Apa itu Eigen ?

Eigen merupakan hasil translate-an dari Bahasa German yang berarti karakteristik

Formula Sheet

Vector operations

$$\mathbf{r} + \mathbf{s} = \mathbf{s} + \mathbf{r}$$
$$2\mathbf{r} = \mathbf{r} + \mathbf{r}$$
$$\|\mathbf{r}\|^2 = \sum_i r_i^2$$

- *dot or inner product*:

$$\mathbf{r}.\mathbf{s} = \sum_i r_i s_i$$

commutative $\mathbf{r}.\mathbf{s} = \mathbf{s}.\mathbf{r}$

distributive $\mathbf{r}.(\mathbf{s} + \mathbf{t}) = \mathbf{r}.\mathbf{s} + \mathbf{r}.\mathbf{t}$

associative $\mathbf{r}.(a\mathbf{s}) = a(\mathbf{r}.\mathbf{s})$

$$\mathbf{r}.\mathbf{r} = \|\mathbf{r}\|^2$$

$$\mathbf{r}.\mathbf{s} = \|\mathbf{r}\|\|\mathbf{s}\| \cos\theta$$

- *scalar and vector projection*:

scalar projection: $\dfrac{\mathbf{r}.\mathbf{s}}{\|\mathbf{r}\|}$

vector projection: $\dfrac{\mathbf{r}.\mathbf{s}}{\mathbf{r}.\mathbf{r}}\mathbf{r}$

Change of basis

Change from an original basis to a new, primed basis. The columns of the transformation matrix B are the new basis vectors in the original coordinate system. So

$$B\mathbf{r}' = \mathbf{r}$$

where r' is the vector in the B-basis, and r is the vector in the original basis. Or;

$$\mathbf{r}' = B^{-1}\mathbf{r}$$

If a matrix A is *orthonormal* (all the columns are of unit size and orthogonal to eachother) then:

$$A^T = A^{-1}$$

Gram-Schmidt process for constructing an orthonormal basis

Start with n linearly independent basis vectors $\mathbf{v} = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$. Then

$$\mathbf{e}_1 = \frac{\mathbf{v}_1}{||\mathbf{v}_1||}$$

$$\mathbf{u}_2 = \mathbf{v}_2 - (\mathbf{v}_2.\mathbf{e}_1)\mathbf{e}_1 \quad \text{so} \quad \mathbf{e}_2 = \frac{\mathbf{u}_2}{||\mathbf{u}_2||}$$

... and so on for $\mathbf{u_3}$ being the remnant part of $\mathbf{v}_3$ not composed of the preceding $\mathbf{e}$-vectors, etc. ...

KOMINFO digitalent

Ketika kita kita berbicara tentang Problematika Eigen, maka kita berbicara tentang properti karakteristik dari sesuatu.

Eigenstuff

matrix

eigenvector

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

eigenvalue

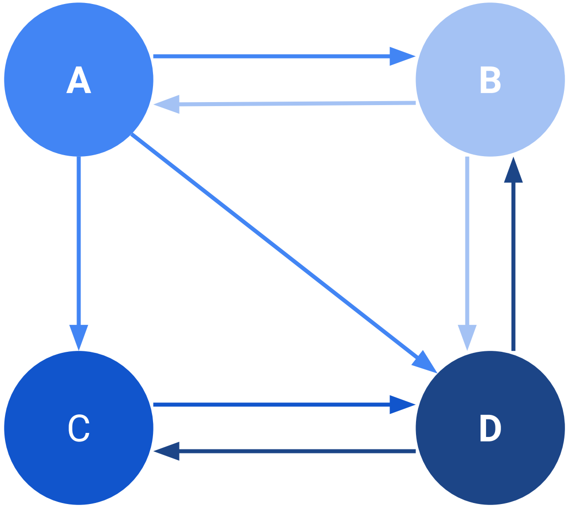
$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0$$

identity matrix

What are the examples of Linear Algebra in Machine Learning?

Algoritma PageRank

- Algoritma ini dipublish pada tahun 1998 oleh Larry Page (Founder Google) dan teman-temannya.
- Digunakan oleh Google untuk membantu dalam mendapatkan keputusan order yang biasa kita lihat ketika searching.



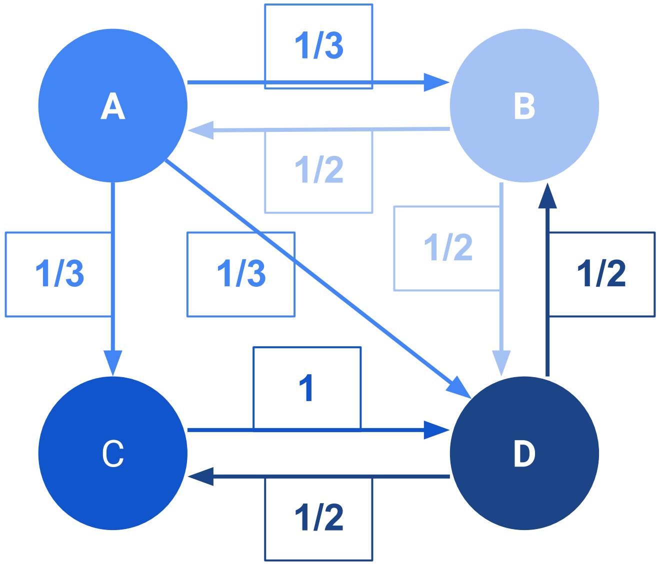
How PageRank works?

La = [0 ⅓ ⅓ ⅓]

Lb = [½ 0 0 ½]

Lc = [0 0 0 1]

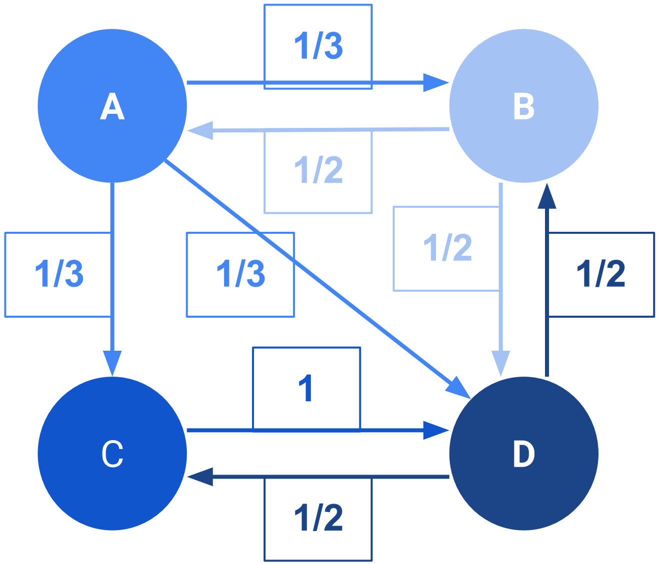
Ld = [0 ½ ½ 0]



How PageRank Works?

$$L = \begin{bmatrix} 0 & 1/2 & 0 & 0 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 1/2 & 1 & 0 \end{bmatrix}$$

$$r_{i+1} = L r_i$$



Terima Kasih