RESEARCH STATEMENT

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1. Introduction

My research is in the field of analytic number theory, specifically the study of various statistics of families of L-functions. The work I've done so far concerns the low-lying zeros and moments of a particular type of thin family of L-functions attached to holomorphic modular forms, which is a smaller slice of the whole family of modular forms of a given weight and level. The family is studied in the level aspect, meaning the level of the modular forms tends to infinity and the weight stays fixed. A large portion of my thesis work is available as a preprint at `https://arxiv.org/abs/2308.06359`. All of my thesis work is done under the advising of Dr. Matthew Young.

Some historical background: the thinnest and most foundational of the families I have studied first appeared in the literature in [PY20], where a bound on its cubic moment was used to establish the Weyl bound for Dirichlet L-functions, initially of cubefree conductor and later extended in [PY22] to all conductors. The introduction of this family and its utility in establishing this groundbreaking result for Dirichlet L-functions, as well as recent developments with analogous thin families in the weight aspect (particularly in [DFS22]), motivates the study of many other features of the family, as well as related families. My thesis work includes answers to several of these questions, as well as some consequences and auxiliary results that are useful in their own right.

The family is obtained as follows: start with the vector space $\mathcal{S}_\kappa(q,\overline{\chi}^2)$ of cuspforms, where q is an odd positive integer, κ is an even positive integer, and χ^2 is a primitive Dirichlet character modulo q. This space contains no oldforms, so the Hecke newforms $\mathcal{H}_\kappa(q,\overline{\chi}^2)$ form an orthogonal basis for it. Twisting each of these Hecke newforms by χ gives a family of the same size as $\mathcal{H}_\kappa(q,\overline{\chi}^2)$ (that is, proportional roughly to q) sitting inside the much larger family $\mathcal{H}_\kappa(q^2,1)$ of newforms of level q^2 and trivial central character. Even though this family is much smaller than $\mathcal{H}_\kappa(q^2,1)$, we can still average over it using the Petersson trace formula, and this enables us to study some statistics of the L-functions attached to the cuspforms in the family.

2. Low-Lying Zeros of the Thin Family

The first problem I have solved in my thesis is the asymptotic behavior of the one-level density of the thin family mentioned above, which gives information about the distribution of low-lying zeros (i.e. zeros that have relatively small imaginary part) of the attached L-functions of the family. This kind of question was first studied for the whole family of Hecke cuspforms with trivial central character in [ILS00], and my study of thin families builds upon that work. The core object of interest is a sum of the form

$$D_1(f,\phi,R) = \sum_\rho \phi\left(\frac{\log R\left(\rho-\frac{1}{2}\right)}{2\pi i}\right), \tag{1}$$

where f is one of the Hecke cuspforms of the family; ϕ is a "test" function which has certain convenient properties including smoothness and rapid decay; R is a scaling parameter of the family that controls how far out the average is taken; and ρ runs over all the zeros of $L(f,s)$, which are all predicted by the Riemann hypothesis to have real part $\frac{1}{2}$. The average values of this sum over all the cuspforms in the family (weighted by a particular weight function $w(f)$) gives the statistic $\mathcal{D}_1(\mathcal{F},\phi,R,w)$ known as the **one-level density** of the family. The most important result I developed for this family is

$$\lim_{q\to\infty}\mathcal{D}_1(\mathcal{F},\phi,R,w)=\widehat{\phi}(0)+\frac{1}{2}\phi(0)=\int_{-\infty}^{\infty}\phi(x)W(\mathrm{O})(x)\,dx, \tag{2}$$

where $\mathcal{F}$ is the thin family, $w(f)=\frac{1}{\langle f,f\rangle}$ is a specific weight function known as the *Petersson weight*, and $W(\mathrm{O})(x)=1+\frac{1}{2}\delta_0(x)$ is a special density distribution. This result is valid provided that the Fourier transform $\widehat{\phi}$ is compactly supported in $\left(-1+\frac{1}{2\kappa},1-\frac{1}{2\kappa}\right)$, unconditional on the Riemann hypothesis. Conditional on the Riemann hypothesis, we can extend this support to $(-1,1)$.

The significance of this leading term lies in the *Katz-Sarnak philosophy* originating in [KS99], which seeks to describe connections between statistics of families of L-functions and those of classical compact groups of $N\times N$ matrices in the limit as $N\to\infty$. In particular, every family of L-functions is conjectured to belong to one of three main symmetry types – unitary, orthogonal, or symplectic – based on the limiting behavior of these statistics. The limiting behavior of the one-level density, and in particular the presence of the group density distribution $W(\mathrm{O})(x)$ for the orthogonal group, suggests that this family has the orthogonal symmetry type. (This is consistent with what we'd expect, since the larger family $\mathcal{H}_\kappa(q^2,1)$ also has orthogonal symmetry.)

In addition to this leading term, I also found several lower-order main terms that go to zero relatively slowly as $q\to\infty$. These give us greater insights as to the shape of the distribution for various choices of the test function ϕ. There is also an error term which is asymptotic to a power of q – the size of this error term dictates the allowed support of ϕ, since it cannot grow larger than the main terms or else there is no useful result.

A brief summary of the main tools I used in the course of these asymptotic calculations will follow. The first ingredient is an explicit formula, akin to Riemann's classical explicit formula for the zeta function, which allows us to write the one-level density statistic of a single cuspform as a sum over primes involving the Hecke eigenvalues of the cuspform. The next major ingredient is the Petersson trace formula, which allows us to take the average over all cuspforms and decompose it into a diagonal term and an off-diagonal term. Part of the leading term, which lets us identify the symmetry type of this family as orthogonal, is then extracted from the diagonal term using summation by parts. Meanwhile, I developed an on-average version of the Weil bound that holds for twisted Kloosterman sums (it was noted in [KL13] that the classical Weil bound does not always hold for these twisted sums) and used that to bound the error term unconditionally. While this on-average Weil bound is used in my research to bound the error term, it has many other potential applications as well and can be taken as a standalone result, independent of the main work of the paper. Conditionally on the Riemann hypothesis, I also was able to establish a bound for the off-diagonal terms using the technique of Mellin inversion, which gives a better range of support for the test function.

3. Low-Lying Zeros of Wider Families

A natural question for me to study next was the behavior of the low-lying zeros for families which lie somewhere between the thinnest family described previously, whose size is proportional to q, and the whole family of cuspforms, whose size is proportional to q^2. Our method of doing this is to take a union of different twists of families of level $q=p^k$ for a power of an odd prime p, while having the central character χ vary through a coset. This will give a subfamily of $\mathcal{H}_\kappa(p^{2k}, 1)$ whose size is proportional to p^{j+k} for some j, which is allowed to vary between 0 and k. The main questions I sought to answer by widening the family in this way were: (1) how does the support of $\widehat{\phi}$ interpolate between the thinnest family and the whole family (which was shown to have support up to $(-2, 2)$ in [ILS00]), and (2) if we can get support greater than 1, can we detect whether the symmetry type of the wider family matches the distribution of SO(even) or SO(odd)? (The group density distributions in [KS99] for SO(even) and SO(odd) are identical in the interval $(-1, 1)$, so the support must be able to "break past" that threshold to be able to see the differences between the even and odd orthogonal symmetry types.)

The results of my study of this family are essentially the best that could be expected. For a coset of size p^j, I was able to get support up to $\left(-1 - \frac{j}{k}, 1 + \frac{j}{k}\right)$ conditional on the Riemann hypothesis, which interpolates linearly between the thinnest family's support of $(-1, 1)$ and the full family's support of $(-2, 2)$. Additionally, this establishes that if $\frac{j}{k}$ is even a little bigger than 0, we can break past the threshold of $(-1, 1)$ in the limit and detect the parity. The result is that the parity (even/odd symmetry type) of the family depends entirely on whether the weight κ is $0 \pmod 4$ or $2 \pmod 4$, and on whether the character χ is restricted to be even or odd.

The methodology follows along the same lines as our work for the thinnest family; in particular, we start off with taking the average of a Riemann-style explicit formula in much the same way as before. However, since there is also a sum over χ, the Kloosterman sum appearing inside the Petersson formula is instead replaced by a more complicated character sum. In many cases (in particular, for j large enough compared to k) this character sum does in fact turn out to be a twisted Kloosterman sum after simplifying; but in the general case, we have to expand the sum out into a finite Fourier series of multiplicative characters to bound it. The final bound, conditional on the Riemann hypothesis, is again obtained using Mellin inversion. However, an extra main term is extracted from the Mellin integral this time, and this term, after applying some standard integration techniques, combines with the leading terms to give the group density distribution for the special group SO(even) or SO(odd).

4. Applications to Nonvanishing Results

The one-level density problem for a family lends itself naturally to obtaining bounds on the number of forms in the family whose L-functions can possibly vanish at the central point $s = \frac{1}{2}$. The method for doing this was first introduced in [ILS00] and revolves around using a specific choice of test function ϕ to write the group density integral in terms of the support. Then, the larger support we are able to get, the higher percentage of nonvanishing L-functions (or vanishing of order 1 in the case where the root number is odd) we can establish. I used these tools to derive nonvanishing results for both the thinnest family and the wider coset families discussed above. These results are also consistent with those presented in [ILS00].

5. Current and Future Areas of Study

My current area of study is to obtain asymptotics for the first and second moments of the family in level aspect. (The work of [PY20] includes a bound for the third moment, but not an asymptotic.) One natural question to ask would then be if these results are consistent with the conjectures generated using the "recipe" of [CFK$^+$05].

Possible future areas of study include further low-lying zeros questions such as the n-level density and the centered moments. Another promising possible subject is to give unconditional results for the coset family (since the support I currently have is conditional on the Riemann hypothesis). We might well expect that such a result would not give as good support as if we did assume the Riemann hypothesis, similarly to our unconditional result for the thinnest family. Other possible related questions might include finding an on-average Weil bound as c varies (rather than m, n), or studying thin families of Dirichlet L-functions. Finally, another potential area of future study is the thin families of modular forms arising from supercuspidal representations. (The modular forms I have studied thus far have all arisen from principal series representations.) Instead of the classical Petersson trace formula, the averaging over these families would be executed using the analogous formula for supercuspidals established in [Hu23].

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