2023 AMCPM Countdown Round

January 14, 2023

Problem 0. Grampton St. Rumpterfrabble and his friend Saunterblugget Hampterfuppinshire the irascible coxswain are watching the classic film series *The Duchess Approves* while eating copious amounts of Smile Dip. Suppose that the film series originally had 20 episodes and Grampton St. Rumpterfrabble and his friend Saunterblugget Hampterfuppinshire the irascible coxswain together eat a total of 400 packets of Smile Dip across the entire series. Suppose further that the amount of Smile Dip Grampton St. Rumpterfrabble and his friend Saunterblugget Hampterfuppinshire the irascible coxswain ate per episode forms an arithmetic series, and that Grampton St. Rumpterfrabble and his friend Saunterblugget Hampterfuppinshire the irascible coxswain ate 39 packets of Smile Dip during the 20th episode. How many packets of Smile Dip did Grampton St. Rumpterfrabble and his friend Saunterblugget Hampterfuppinshire the irascible coxswain eat during the third episode?

Answer: 5 (packets)

Problem 1. Let r_1, r_2 be the solutions to the equation $x^2 + 2x - 2$. What is the value of $r_1^2 + r_2^2$?

Answer: 8

Problem 2. Ethelred rolls two fair six-sided dice. What is the probability that the product of the numbers showing is at least 12? Express your answer as a common fraction.

Answer: 17/36

Problem 3. Jebediah has a large quantity of cones. Suppose he has three types of cones, long, medium, and short, and the numbers of each type of cone he has are in the ratio 33:17:1 (in that order). If Jebediah has a total of 2499 cones, how many medium cones does he have?

Answer: 833 (cones)

Problem 4. Eda flips two fair coins, labeled Coin 1 and Coin 2. Given that at least one of the two coins landed heads, what is the probability that Coin 1 landed heads? Express your answer as a common fraction.

Answer: $2/3$

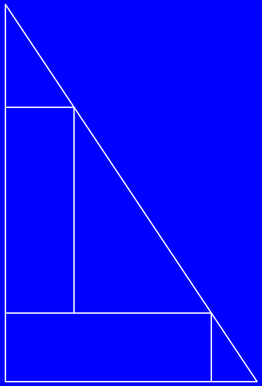
Problem 5. There are 5 consecutive positive integers a, b, c, d, e, in that order, such that $a + b + c = d + e$. What is the value of $c + d$?

Answer: 13

Problem 6. Thorin is throwing thoughts at a throne. Of these thrown thoughts, 40% are thorough while the others are not. 50% of the thrown thorough thoughts pass through the throne, though 30% of the thrown non-thorough thoughts do not pass through the throne. Of Thorin's thoughts that are thrown through the throne, what proportion are thorough thoughts? Express your answer as a common fraction.

Answer: 10/31

Problem 7. Two 1×3 rectangles are placed inside a right triangle as shown below. What is the area of the right triangle? Express your answer as a common fraction.



Answer: 121/12 (units^2)

Problem 8. Suppose $\sqrt{4-\sqrt{8-\sqrt{16-\sqrt{32-\sqrt{n}}}}}=\sqrt{2}$. What is the value of n?

Answer: 1024

Problem 9. What is the smallest positive integer with exactly 18 positive divisors?

Answer: 180

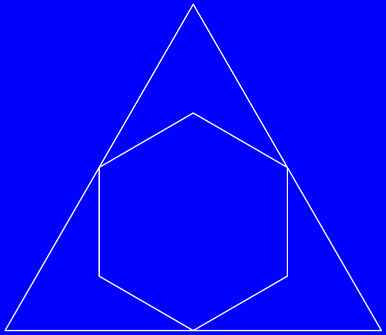
Problem 10. Josie takes a sixteen-card subset of a standard deck which contains exactly two hearts. Three times, he shuffles the deck, looks at the top card, and then puts it back. What is the probability that at least two of the three cards he saw were hearts? Express your answer as a common fraction.

Answer: 11/256

Problem 11. How many four digit numbers (with no leading zero) satisfy the property that their digits are strictly increasing from left to right?

Answer: 126 (numbers)

Problem 12. A regular hexagon has three of its vertices at the midpoints of the three sides of an equilateral triangle, as shown in the figure. What is the ratio of the area of the hexagon to the area of the triangle? Express your answer as a common fraction.



Answer: 1/2

Problem 13. &ndra rolls three dice. What is the probability that the product of their top faces is a prime? Express your answer as a common fraction.

Answer: 1/24

Problem 14. Suppose $a + b + c = 0, 4a + 2b + c = 2$, and $9a + 3b + c = 0$. What is the value of $16a + 4b + c$?

Answer: -6

Problem 15. Pierce is flipping Matthew-shaped coins. When flipped, each Matthew-shaped coin has a 20% chance of landing on heads, a 40% chance of landing on feets, and a 40% chance of landing on middles. If Pierce flips three Matthew-shaped coins, what is the probability that he gets the same outcome all three times? Express your answer as a common fraction.

Answer: $17/125$

Problem 16. Distinct positive integers x, y, z satisfy $x + y + z = 13$, $x^2 + y^2 + z^2 = 65$, and $x^3 + y^3 + z^3 = 349$. What is the value of their product, xyz?

Answer: 60

Problem 17. How many integers between 1001 and 2500 inclusive are divisible by 7 or 12 (or both)?

Answer: 322 (integers)

Problem 18. Let $AUSTIN$ be a regular hexagon and let $MATH$ be a square. What is the square of the ratio of the area of $AUSTIN$ to the area of $MATH$? Express your answer as a common fraction.

Answer: 27/64

Problem 19. Dipper has a handful of tiny frogs, while Mabel has a handful of tiny toads. Suppose the average width of the tiny frogs Dipper is holding is 5mm and the average width of the tiny toads Mabel is holding is 10mm. In addition, suppose Dipper is holding 24 tiny frogs, and Mabel is holding 36 tiny toads. If Dipper and Mabel put together their tiny amphibians, what is the average width of all 60 tiny amphibians, in mm?

Answer: 8 (mm)

Problem 20. Sdeu's Stupendous Stews sells tiny bowls of stew for 2 dollars, small bowls of stew for 3 dollars and large bowls of stew for 5 dollars. Finally, they also sell child-size bowls (so called as the bowl could roughly fit an entire liquefied child) of stew for 8 dollars. Suppose a group of three (distinguishable) people walk into the store, and each order a bowl of soup. If their total comes out to 13 dollars, how many different orders are possible?

Answer: 9 (orders)

Problem 21. An equilateral triangle and a regular hexagon have the same area. What is the ratio of the side length of the triangle to that of the hexagon? Express your answer in simplest radical form.

Answer: $\sqrt{6}$

Problem 22. Lëa has built a snowman out of three perfect spheres of snow stacked on top of each other, all of different sizes, with the largest on bottom and the smallest on top. Suppose that the total height of the snowman is 72 inches, the radii of the snow spheres form an arithmetic sequence, and the total volume of the three spheres is 6912π cubic inches. What is the volume of the middle sphere?

Answer: 2304π (inches^3)

Problem23 . What is the volume of a cube (in cubic centimeters) that has surface area 150 square centimeters?

Answer: 125 (cm^3)

Problem 24. Three circles of radius 3 are mutually externally tangent, and are all internally tangent to one large circle. What is the radius of the large circle? Express your answer in simplest radical form.

Answer: $3 + 2\sqrt{3}$ (units)

Problem 25. Pierce is playing Boatknights. Suppose he begins playing Boatknights at 11am on Wednesday, and continuously plays until 3pm on Friday of the same week. How long does he play Boatknights, in minutes?

Answer: 3120 (minutes)

Problem 26. Cola cans come in three sizes: 240 mL, 600 mL, and 1000 mL. What is the least number of cans of any size that in total sum up to exactly 2520 mL?

Answer: 6 (cans)

Problem 27. A parallelogram has base length 5, short diagonal length 8, and long diagonal length 12. What is its area? Express your answer in simplest radical form.

Answer: $15\sqrt{7}$ (units^2)

Problem 28. Mai has a stack of n coins. First, she flips the top coin in her stack. If it is heads, she flips the next coin. If that is heads, she flips the coin after that, and so on, stopping either when sees a coin land tails for the first time, or when she has flipped all of the coins in her stack. What is the least possible value of n such that the expected number of heads is at least 0.99?

Answer: 7

Problem 29. Pierce is dunking quadruple-stuf Oreos in his favorite drink, soy sauce. Suppose he starts with one cup (240 milliliters) of 100% soy sauce, and every time he dunks an Oreo, a crumb of Oreo equal to half a milliliter is mixed into the drink. How many Oreos must he dunk so that the drink becomes a mixture of 40% Oreo and 60% soy sauce?

Answer: 320 (Oreos)

Problem 30. Blerb Bobbert buys 3 boxes and 2 bins of bobbers. Each bin Blerb buys contains 4 bobbers and each box Blerb buys contains 3 bobbers plus two bins of bobbers. If Blerb began with 5 bobbers, how many bobbers does Blerb have now?

Answer: 46 (bobbers)

Problem 31. Josie takes a sixteen-card subset of a standard deck which contains exactly two hearts. Three times, he shuffles the deck, looks at the top card, and then puts it back. What is the probability that at least two of the three cards he saw were hearts? Express your answer as a common fraction.

Answer: 11/256

Problem 32. How many two-digit numbers are divisible by 6 and have their digits in nondecreasing order?

Answer: 7 (numbers)

Problem 33. For which positive integer n does

$$1^2 + 4^2 + 6^2 + 4^2 + 1^2 + 2n^2$$

equal 5 factorial?

Answer: 5

Problem 34. Pierce is reading a light novel series. Suppose that, for this series, a chapter contains 20000 words, a book contains 2000 chapters, a volume contains 200 books, and the series contains 20 volumes. If the number of words in this series equals $4^a 5^b$ for positive integers a, b, what is the value of $a + b$?

Answer: 17

Problem 35. Rich is juggling three large balls. Each of those large balls is juggling 4 smaller balls. How many more balls will he need if he wants to instead juggle 4 large balls that are each themselves juggling 5 small balls?

Answer: 9 (balls)

Problem 36. Justin the First is selling apples. He has two types of apples, Red Not Very Delicious apples and Grainy Sherrywood apples, which he sells at fixed (positive) prices. Suppose his customer Justin the Second buys two Red Not Very Delicious apples and one Grainy Sherrywood apple, and his other customer Justin the Fifteenth buys one Red Not Very Delicious apple and three Grainy Sherrywood apples. If Justin the Fifteenth spends twice as much money as Justin the Second, what is the ratio of the price of a Red Not Very Delicious apple to the price of a Grainy Sherrywood apple? Express your answer as a common fraction.

Answer: 1/3

Problem 37. Agony the ant starts at the origin of the coordinate plane, facing right. Then, he walks 10 units forward, turns clockwise 90 degrees, walks 10 units backwards, turns clockwise 180 degrees, walks forward 5 units, and then turns counterclockwise 270 degrees and walks 2 units forward. If he repeats this sequence of moves three more times, what will be the sum of his x and y coordinates when he is finished moving?

Answer: 108

Problem 38. Borrow-mir and Steal-mir are playing rock-paper-scissors (RPS). In any given round of RPS, Borrow-mir has a $\frac{1}{3}$ chance of playing any one of the three moves, while Steal-mir has a 50% chance of playing scissors and an equal chance between the two other moves. What is the probability that Steal-mir wins a given round of RPS? Express your answer as a common fraction.

Answer: $1/3$

Problem 39. Turpen has a rectangular bar of chocolate. If the length of the bar of chocolate were increased by half its width and the width of bar were increased by half its length, its size would be increased by 150%. If the bar is longer than it is wide, what is the ratio of its width to its length? Express your answer as a common fraction.

Answer: 1/2

Problem 40. Suppose a cubic polynomial $P(x) = -2x^3 + ax^2 + bx + c$ has roots $-2, 3$, and -4. What is the value of $a + b + c$?

Answer: 62

Problem 41. Suppose that the playlist of Pierce contains 1000 songs, which have average length 240 seconds each. However, when Pierce listens to the playlist of Pierce, Pierce does so at 4x speed. If Pierce listens to the playlist of Pierce twice, how many minutes does he spend listening?

Answer: 2000 (minutes)

Problem 42. What integer k satisfies the equation

$$4^{(4^4)} \times 8^{8\times 8} \times 16 = 2^k?$$

Answer: 708

Problem 43. Suzie is buying buckets of chicken from Kungpao Furious Chicken. Suppose a small bucket contains 3 pieces of chicken, while a big bucket contains 7 pieces of chicken. If Suzie only buys these two bucket sizes, what is the least number of buckets she needs to buy to get exactly 100 pieces of chicken?

Answer: 16 (buckets)

Problem 44. Nina and her dog Alexander are frolicking. Suppose that the number of times Nina has frolicked so far is twice the number of times Alexander would have frolicked if Alexander had frolicked four times less than currently, and that the number of times Alexander has frolicked is divisible by 5 plus the sum of the digits of the (base 10) number of times Nina has frolicked so far. (Furthermore, suppose that the two have each frolicked a positive integer number of times.) What is the minimum number of times Nina has frolicked?

Answer: 12 (times)

Problem 45. How many positive integers between 1 and 1000 inclusive have digits which sum up to a number divisible by 5?

Answer: 199 (integers)

Problem 46. Suppose $a \odot b = \frac{ab}{a+b}$. What is the value of $((8 \odot 6) \odot 3) \odot 2$?

Answer: 8/9

Problem 47. If $2x^2 + 4 = 14$, what is the value of $2x^4 + 3x^2 + 1$?

Answer: 66

Problem 48. Bradley is selling Brad Buns™. Suppose that a group of seven people can consume seventy Brad Buns™ in seven hours and seven hundred seventy minutes. How long in minutes would it take a group of seventeen people to consume seventy seven Brad Buns™?

Answer: 539 (minutes)

Problem 49. Lilith rolls a fair four-sided pyramid die (with faces labeled from 1-4) and a fair six-sided die (with faces labeled from 1-6), and multiplies the numbers on the bottom faces of the dice. What is the expected value of the number she gets? Express your answer as a common fraction.

Answer: $35/4$