INEG 2313: Applied Probability and Statistics for Engineers I
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Lecture 11: Poisson Process

Today's outcomes: an understanding of *Poisson processes* and related random variables *Poisson, Exponential, and k−Erlang*
Textbook reference: 3.9, 4.7, 4.8

1 Motivating Example

Suppose you are studying arrivals to a facility, perhaps patient arrivals to an emergency room. Let's assume patients arrive randomly to the ER at a constant rate of 1 patient per minute, or 60 patients per hour. This is a very busy ER!!!

Now let's say we begin observing this process at 6:00 a.m. and there are arrivals at times 6:03, 6:04, 6:04:30, 6:05, 6:07, 6:09, 6:09:15. We may want to answer questions about the arrival process to this ER, such as:

- How many customers arrive by 6:05 a.m.?
- How much time passes between the 1^{st} and 2^{nd} arrivals?
- How much time passes, total, until the 6^{th} arrival?

Today we will learn how random variables associated with **Poisson processes** can be used to answer such questions.

2 Poisson Process

A Poisson process is defined as follows. Given an interval of real numbers (e.g., time), assume events occur at random throughout the interval according to a constant rate λ (e.g, for 1 arrival per minute, $\lambda = 1/\text{min.}$, or $\lambda = 60/\text{hr}$). If the interval can be partitioned into subintervals of small enough length that the following assumptions hold:

1. the probability of more than one event in a subinterval is zero
   - *i.e.*, no two arrivals happen at exactly the same time
2. the probability of one event in a subinterval is the same for all subintervals and proportional to the length of the subinterval
   - Prob(1 arrival between 6:05 and 6:06) = Prob(1 arrival between 6:09 and 6:10)
     - These subintervals are both of length 1 min so the probabilities of 1 arrival must be equal
   - Prob(1 arrival between 6:05 and 6:10) > Prob(1 arrival between 6:11 and 6:12)

  - The first subinterval is longer than the second so the probability of 1 arrival must be greater in the first

3. the event in each subinterval is independent of other subintervals

   - Prob(1 arrival between 6:06 and 6:07) is independent of whether or not there was an arrival between 6:05 and 6:06
   - It is also independent of whether or not there was an arrival between 6:04 and 6:05, etc.
   - In fact, Prob(1 arrival between 6:06 and 6:07) is the same regardless of how long it has been since the last arrival
   - This is the *memoryless property*

Then, the random experiment is called a Poisson process with parameter λ.

3 Random Variables Associated with Poisson Processes

Three random variables can be modeled from the Poisson process: the Poisson random variable, the Exponential random variable, and the $k-$Erlang random variable. Each of these can be used to answer different types of questions about a Poisson process. These are detailed below. However, first, reconsider the ER example, where patient arrivals to the ER are a Poisson process. We gave an example of three types of questions we would like to answer - the random variables that could be used for each of these questions are:

- How many customers arrive by 6:05 a.m.? **Poisson random variable**
- How much time passes between the 1^{st} and 2^{nd} arrivals? **Exponential random variable**
- How much time passes, total, until the 6^{th} arrival? $k-$**Erlang random variable**

3.1 Poisson random variable

The Poisson random variable models the number of events in the interval of length t. Because we are **counting** the number of events in the interval, then the range is discrete and $0, 1, 2, \ldots, \infty$.

- Side note: why are we talking about a discrete random variable after we've moved on to continuous random variables? Because we need to understand the Poisson process, and the other associated random variables are continuous (Exponential and $k-$Erlang).

The parameter associated with a Poisson random variable is $\alpha = \lambda t$, which is equivalent to the expected number of events in the interval of length t.

- Example: Let X denote the number of events in an interval of specified length t for the ER example, which is a Poisson process with parameter $\lambda = 1/\text{min}$. Suppose the interval of interest is [6:05,6:10] with length $t = 5$ min and $\lambda = 1/\text{min}$. We need to find the parameter associated with the Poisson random variable, which is $\alpha = \lambda t$. Then, $\alpha = \lambda t = 1 * 5 = 5$, and $X \sim \text{Poisson}(\alpha = 5)$. That is, the number of events in an interval of length 5 minutes is modeled using a random variable X, which is a Poisson random variable with parameter $\alpha = 5$.

3.2 Exponential random variable

The Exponential random variable models the time between two consecutive events, aka the inter-arrival time. Because we are dealing with *time*, the Exponential random variable is *continuous*. The parameter associated with an Exponential random variable is λ, the constant rate parameter (the same one that is associated with the underlying Poisson process).

- Example: Let X denote the time between the 9^{th} and 10^{th} arrivals, and we still have $\lambda = 1$/min. as before. Then $X \sim \text{expon}(\lambda = 1)$. That is, the time between the 9^{th} and 10^{th} consecutive arrivals is modeled using a random variable X, which is an Exponential random variable with parameter $\lambda = 1$.

Note that the inter-arrival time distribution is the same for **any** pair of consecutive customers. Therefore, let X denote the time between the 0^{th} and 1^{st} arrivals. We still have $X \sim expo(\lambda = 1)$.

3.3 k-Erlang random variable

The k-Erlang random variable models the cumulative time until the k^{th} event (arrival). Because we are dealing with *time*, the Erlang random variable is *continuous*. The parameters associated with the k-Erlang random variable are k and λ.

- Let X denote the time until the 5^{th} arrival to the ER, and we still have $\lambda = 1$/min from the underlying Poisson process. Then, $X \sim$5-$Erlang(\lambda = 1)$.
- Suppose we instead are interested in the cumulative time between the 6^{th} and 11^{th} arrivals. Then we are still interested in the cumulative time until 5 total arrivals (arrivals 7,8,9,10,11) so we still have $X \sim$5-$Erlang(\lambda = 1)$.

3.4 Distributions, means, variances, etc.

Table 1 gives information for each of the three random variables associated with Poisson processes.

Name, Notation	**Range**	**pdf**	**CDF**	$E(X)$	σ_X
Poisson, $X \sim Poisson(\alpha = \lambda t)$	$\{0, 1, ...\}$	$\frac{e^{-\alpha}\alpha^x}{x!}$	$F_X(x) = \sum_{i=0}^{x} \frac{e^{-\alpha}\alpha^i}{i!}$	α	$\sqrt{\alpha}$
Exponential, $X \sim expo(\lambda)$	$(0, \infty)$	$\lambda e^{-\lambda x}$	$F_X(x) = 0,\ x \leq 0$ $F_X(x) = 1 - e^{-\lambda x},\ x > 0$	$\frac{1}{\lambda}$	$\frac{1}{\lambda}$
$k-$Erlang, $X \sim k - Erlang(\lambda)$	$(0, \infty)$	$\frac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!}$	$F_X(x) = 0,\ x \leq 0$ $F_X(x) = 1 - \frac{\sum_{n=0}^{k-1} e^{-\lambda x}(\lambda x)^n}{(n)!},\ x > 0$	$\frac{k}{\lambda}$	$\frac{\sqrt{k}}{\lambda}$

Table 1: Equations for Poisson process r.v.'s, also in Continuous R.V. Reference Guide

4 Example Problems

4.1 Phlebotomy lab example

Patients arrive to a phlebotomy lab at a constant rate of 60 patients per hour. Assume patients arrive independently from each other. Suppose we begin observing the arrival process at the lab at some point in time.

1. What is the expected value of the number of patients that arrive in the first 30 minutes?
2. What is the probability that 3 patients arrive in the first 30 minutes?
3. What is the probability that 5 or fewer patients arrive in the first 30 minutes?
4. What is the probability that the 5^{th} patient arrives more than 6 minutes after the 4^{th} patient?
5. What is the expected amount of time between the 4^{th} and 5^{th} patient arrivals?
6. Given that the time between the 9^{th} and 10^{th} patient arrivals was 7 minutes, what is the probability that the time between the 10^{th} and 11^{th} patient arrivals is greater than 3 minutes?
7. What is the expected total time until the 10^{th} patient arrival?
8. What is the probability that the elapsed time until 50 patients arrive is less than one hour?

4.2 Earthquake example

Earthquakes occur in the United States according to a Poisson process having a rate of 0.25 per day. Assume then that the occurrence of earthquakes in the United States is a Poisson process with parameter $\lambda = 0.25$ per day. Suppose we begin counting earthquakes at some point in time.

1. What is the probability that 6 earthquakes occur in July 2050?
2. On average, when will the 50^{th} earthquake occur?
3. What is the probability that 2 or more earthquakes occur over a 50-day period?
4. What is the probability that it is more than 10 days until the 3^{rd} earthquake?
5. On average, how many earthquakes will occur in 2052?
6. On average how long will it be between the 100^{th} and 101^{st} earthquakes?
7. What is the probability of no earthquakes over a 5-day period?
8. What is the probability of no earthquakes over 5 separate 1-day periods?
9. What is the probability of no earthquakes in the period June 2050 - August 2050?
10. What is the probability of 24 earthquakes in the period June 2050 - August 2050?
11. What is the probability of no earthquakes in the period of April 2050, October 2050, and January 2051?
12. What is the probability of 24 earthquakes in the period April 2050, October 2050, and January 2051?
13. What is the probability of 10 earthquakes in April 2050, 8 earthquakes in October 2050, and 6 earthquakes in December 2051?

4.3 Bank example

Customers arrive at a bank according to a Poisson process with rate $\lambda = 8.6$ customers per hour. Suppose we begin observing the arrival process at the bank at some point in time.

1. What is the probability that it is at least 9 min. from now until the next customer arrives, given that the last customer arrived 6 min. ago?
2. Given that it has been 12 min. since the last customer arrived, what is the expected value of the time until the next customer arrives (from now)?
3. Given that it has been 12 min. since the last customer arrived, what is the expected value of the interarrival time for the next customer?
4. On average, when will the 3^{rd} customer arrive?
5. What is the probability that it will be more than 30 min. from now before the 3^{rd} customer arrives?
6. Given that it has been 15 min. since the last customer arrived, what is the probability that it will be more than 30 min. from now before the 3^{rd} subsequent customer arrives?