Central Limit Theorem and Sampling Distributions: Gaussian Distributions

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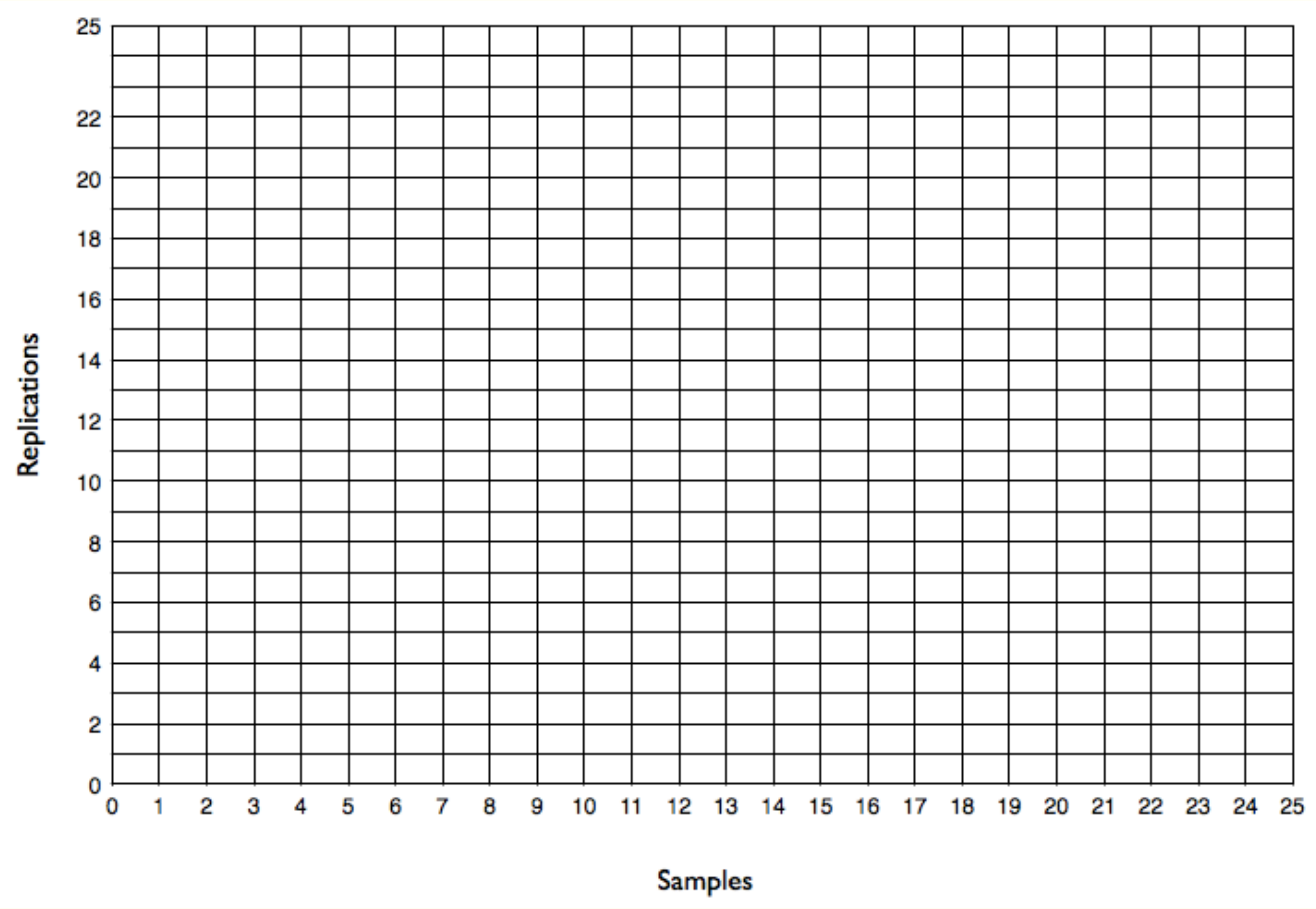
Coin flip example

- We have a coin, and we flip it N times. We'll call it a sample. As we increase N, we get a better estimate of the true probability of heads.
- in our coins example, the true proportion, called p^* comes from all possible (infinite) coin flips. We never get to see this
- This of course depends on if our model describes the true generating process for the data, otherwise we can find a p^* given a population, but still have model mis-specification error
- if we are only given one (finite sample sized) replication, which is the situation in real life, we can only estimate a probability $\hat{p}$
- In our idealized, simulated case we have many M replications, and thus samples, and we can now find the **distribution** of estimated probabilities $\hat{p}$

M replications of N coin tosses



Sampling distribution

As we let $M \to \infty$, for any N, the distribution induced on $\hat{p}$ is the empirical **sampling distribution of the estimator** of the true probability p^*.

We could use the sampling distribution to get confidence intervals on $\hat{p}$.

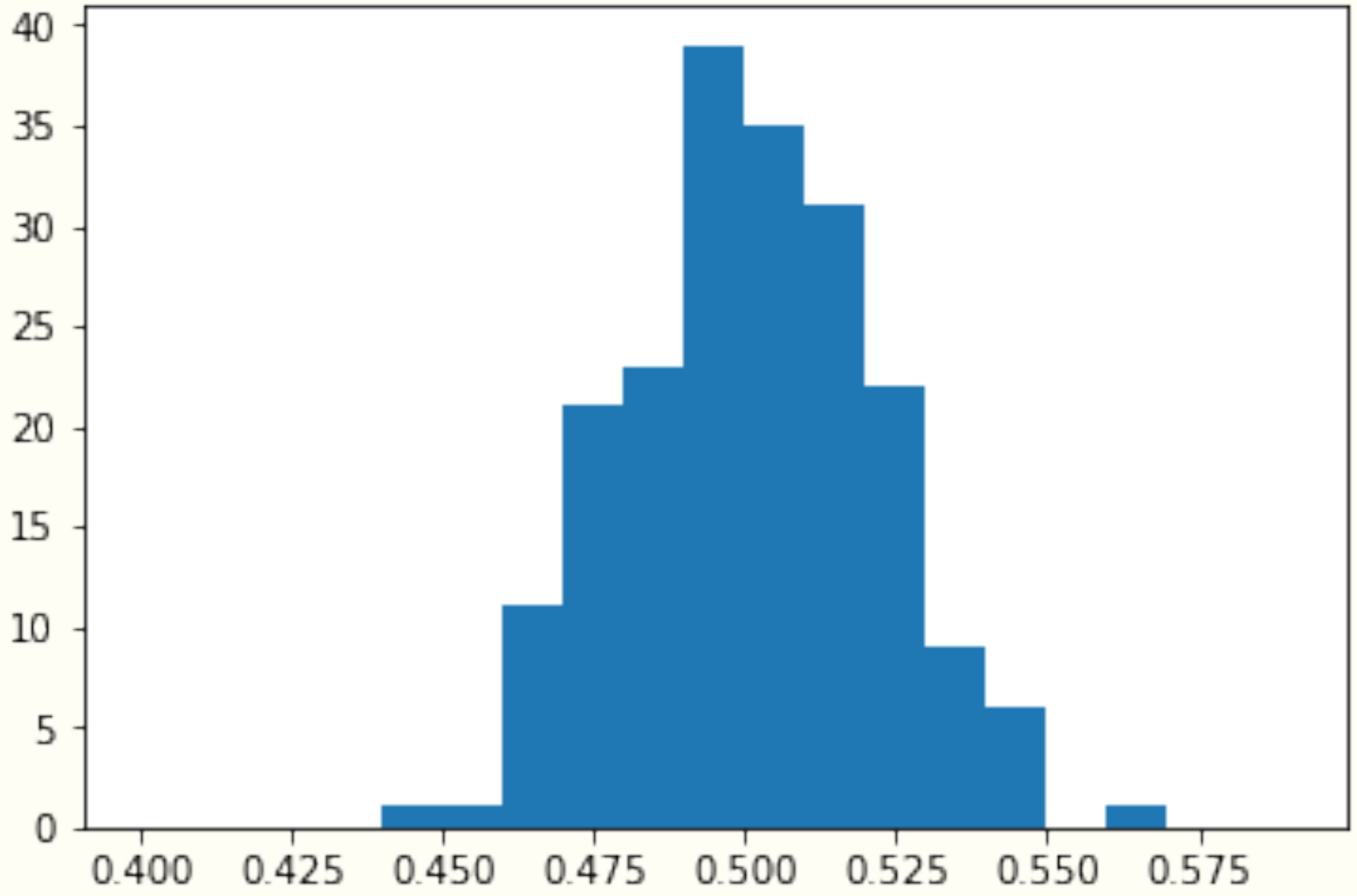
$\hat{p}$ itself is a mean value, a mean of the 1s corresponding to heads and 0s corresponding to the tails.

Thus the distribution of $\hat{p}$ is the **sampling distribution of the mean** of each sample

200 replications of N coin tosses

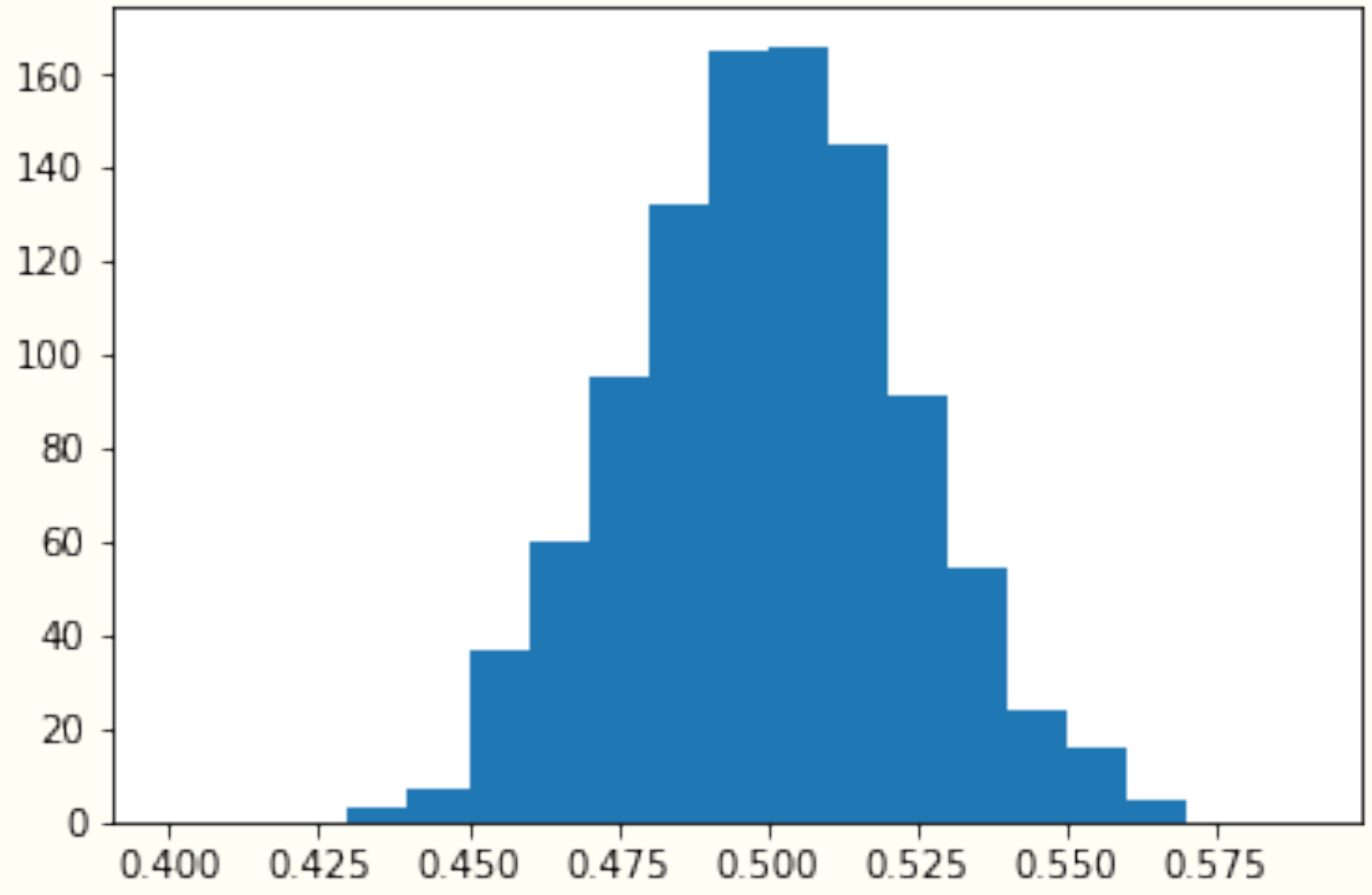
- If the heads are given the value 1 and the tails are given the value 0 then the mean of the sample gives us the fraction of heads in each replication
- these means will be **different**! The fluctuations from one replication to another is called a **distribution**

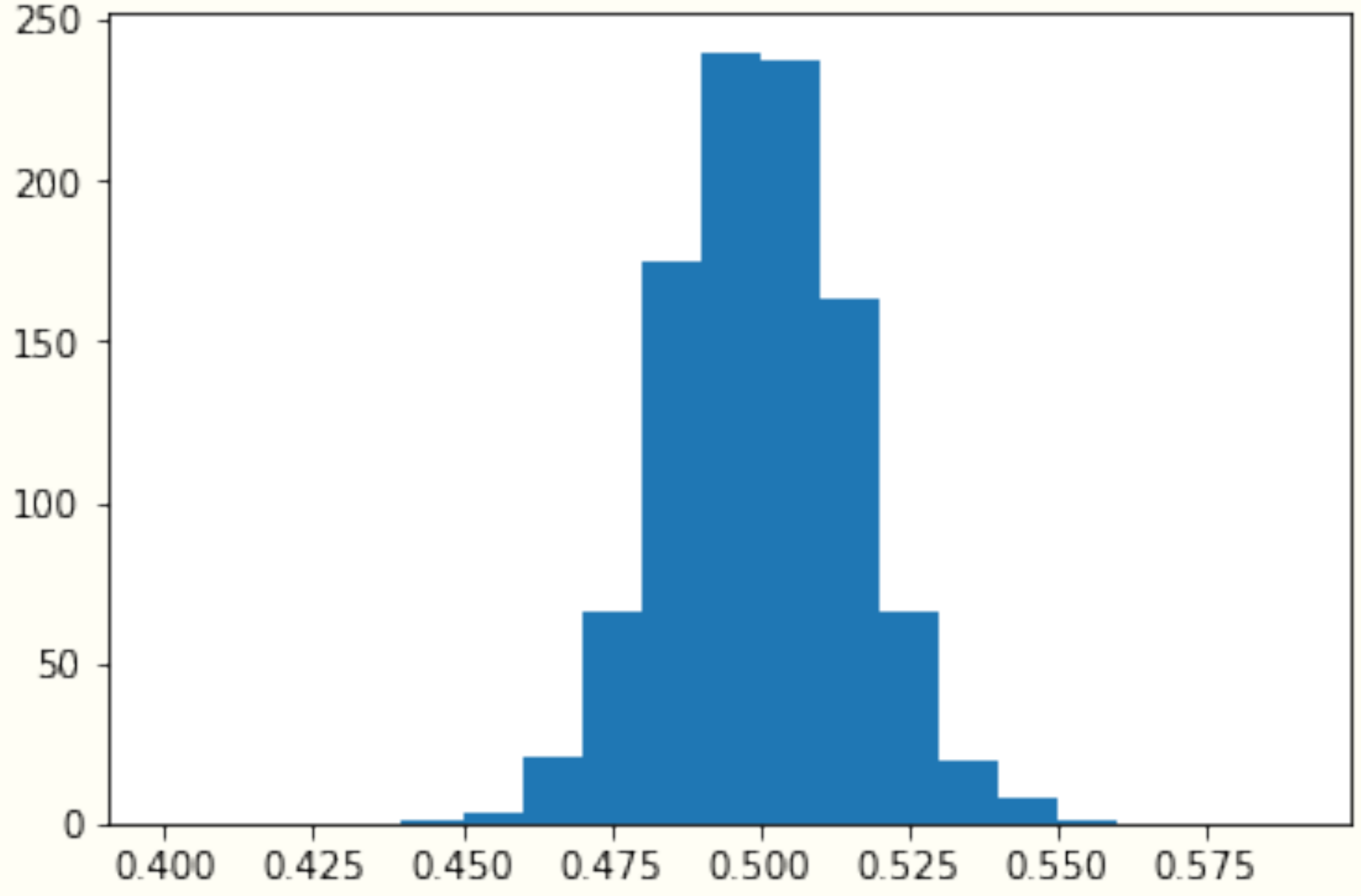
200 reps, 491 flips per sample -->



What if we increase the size of each sample?

1000 (∞) replications of N coin tosses, left: N=491, right: N=982

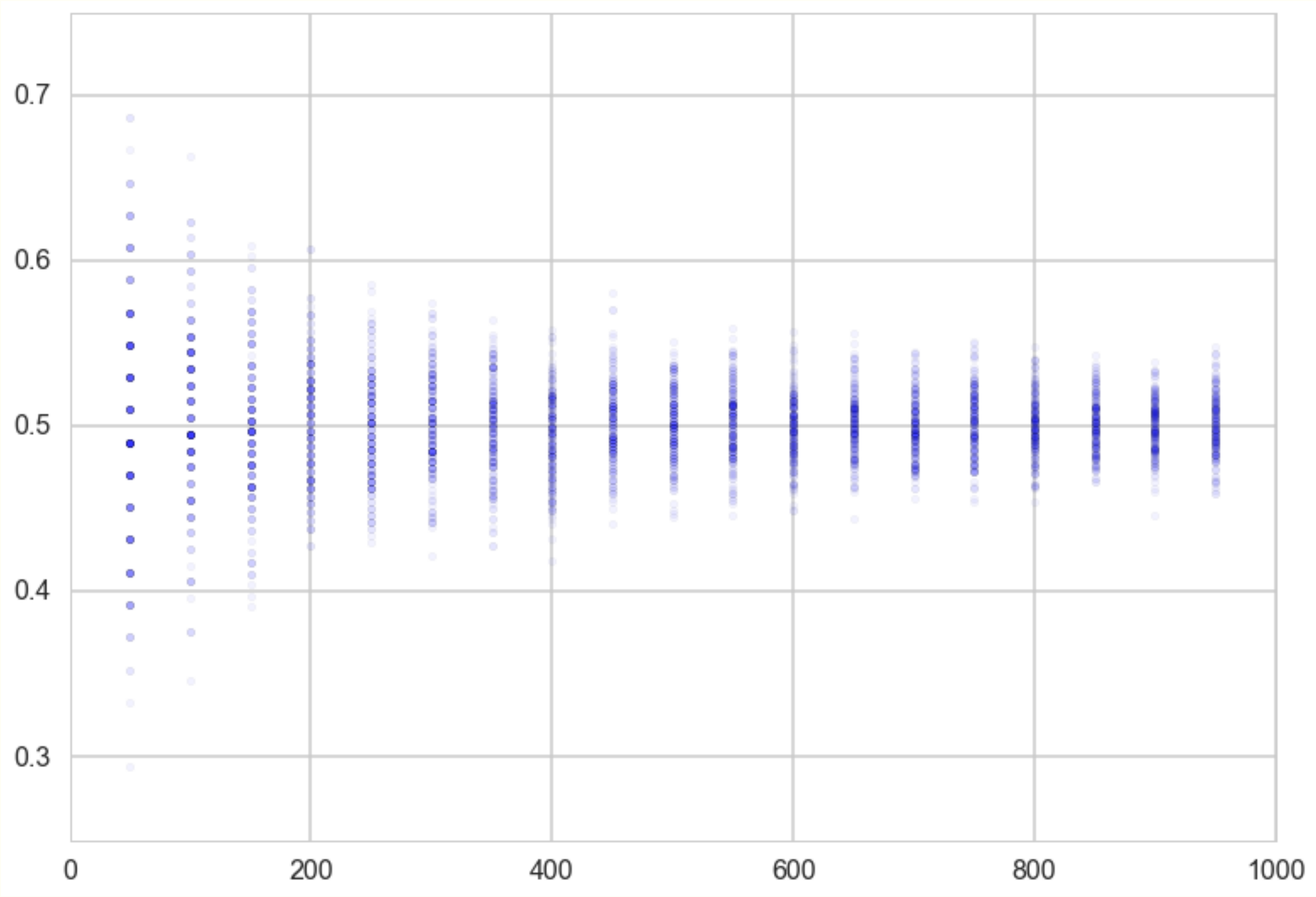




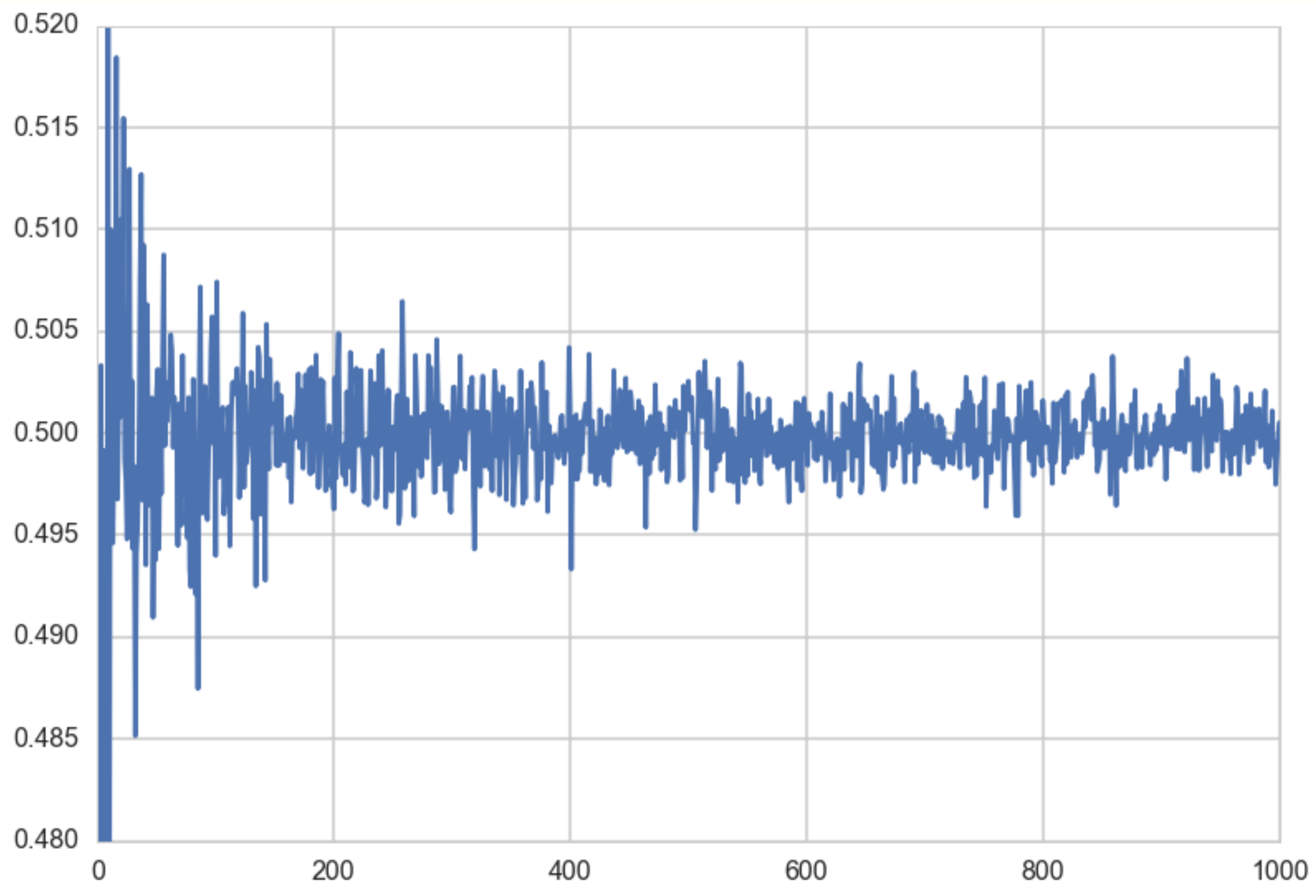
Properties of this distribution

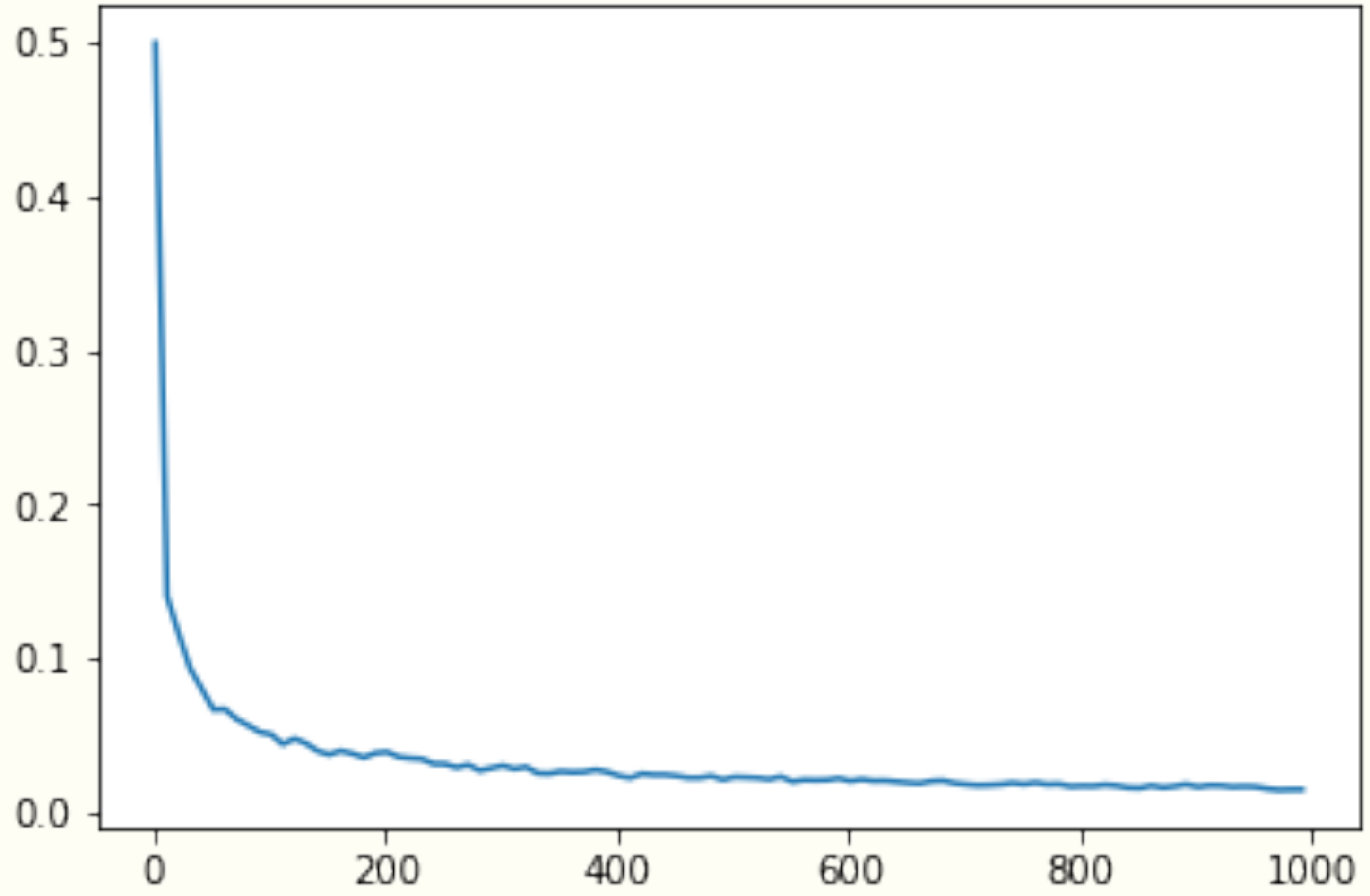
- We can average this fraction over the replications..a mean of sample means.
- We can also calculate the standard deviation of these means.
- How do these vary with the size of the sample?

Distribution of Sample Means



mean and standard deviation of sample means

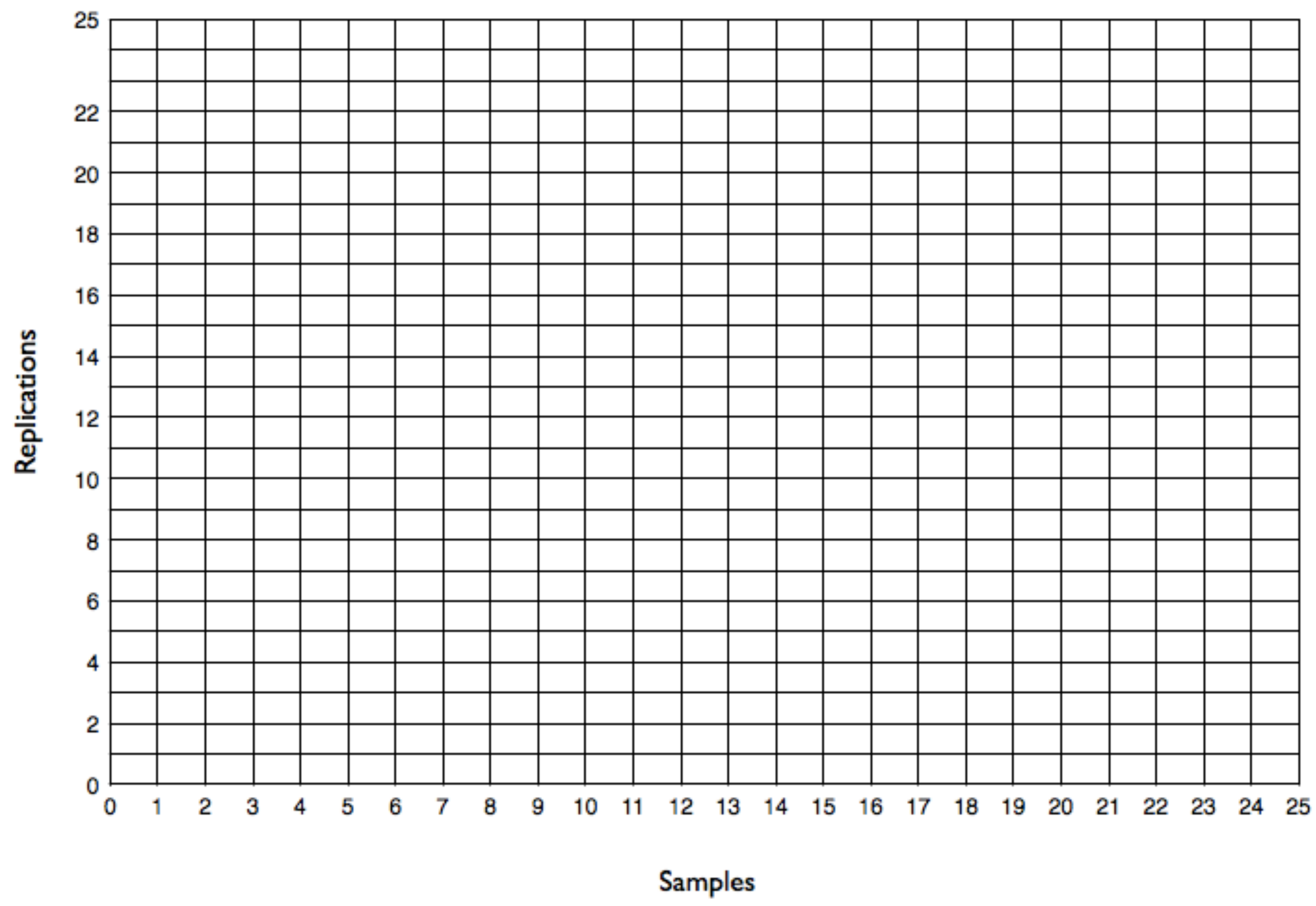




Sample means and the population mean

Let $x_1, x_2, \ldots, x_n$ be a sequence of IID values from random variable X, which has some distribution with finite mean μ.

Define $E_{\{R\}}(x)$ to be the average in the vertical direction (axis=0) of the value x, for example the average of all the first coin's 1s and 0s in the replications.



Then,

$$E_{\{R\}}(N\,\bar{x}) = E_{\{R\}}(x_1 + x_2 + \ldots + x_N) = E_{\{R\}}(x_1) + E_{\{R\}}(x_2) + \ldots + E_{\{R\}}(x_N)$$

In limit $M \to \infty$ of replications, each and every one of the expectations in RHS can be replaced by the population mean μ using the law of large numbers! Thus:

$$E_{\{R\}}(N\,\bar{x}) = N\,\mu$$
$$E_{\{R\}}(\bar{x}) = \mu$$

In limit $M \to \infty$ of replications the expectation value of the sample means converges to the population mean.

Now let underlying distribution on the population have well defined mean μ AND a well defined variance σ^2.

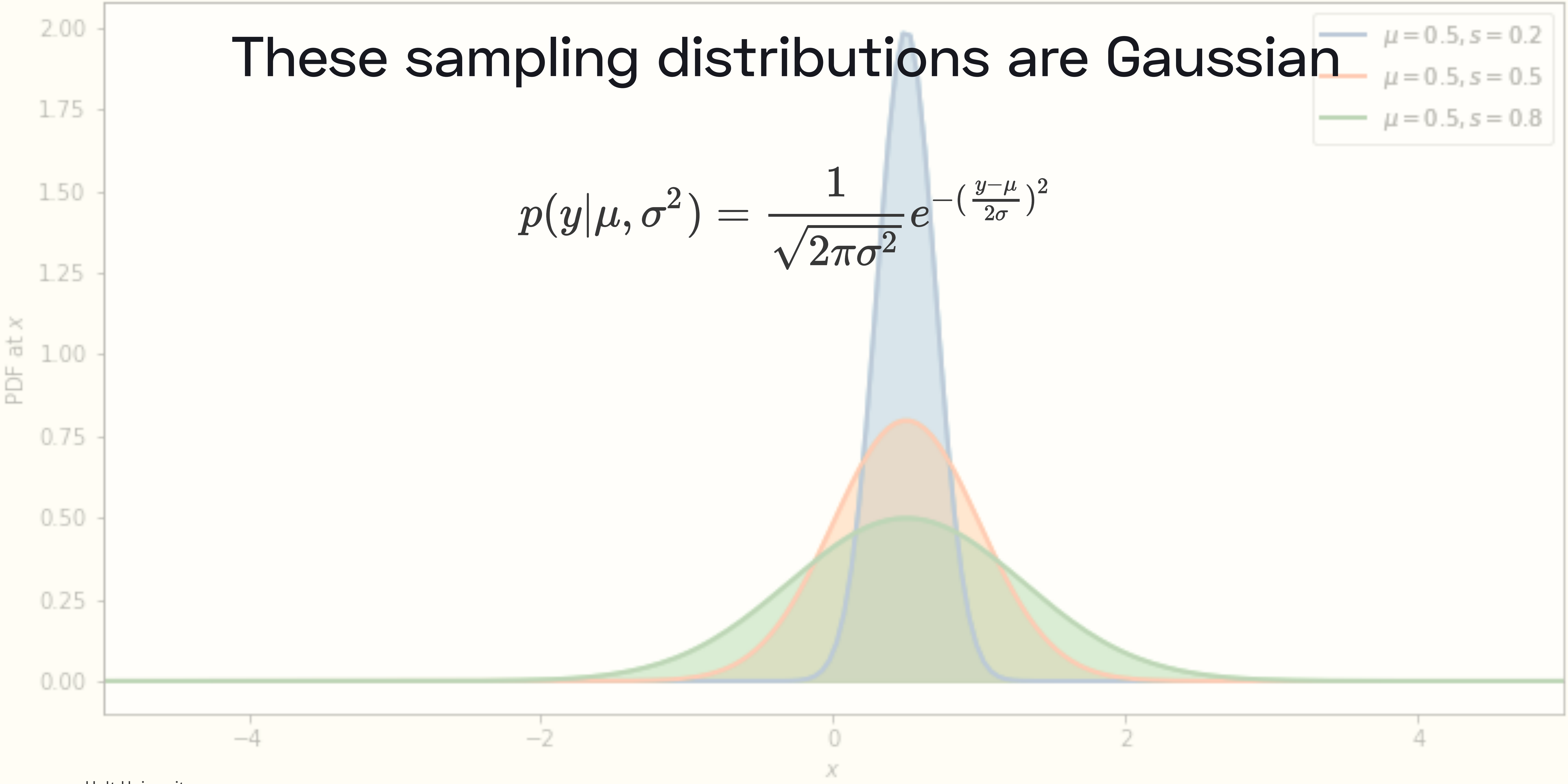
$$V_{\{R\}}(N\,\bar{x}) = V_{\{R\}}(x_1 + x_2 + \ldots + x_N) = V_{\{R\}}(x_1) + V_{\{R\}}(x_2) + \ldots + V_{\{R\}}(x_N)$$

Now in limit $M \to \infty$, each of the variances in the RHS can be replaced by the population variance using the law of large numbers! Thus:

$$V_{\{R\}}(N\,\bar{x}) = N\,\sigma^2$$
$$V(\bar{x}) = \frac{\sigma^2}{N}$$

These sampling distributions are Gaussian

$$p(y|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(\frac{y-\mu}{2\sigma})^2}$$



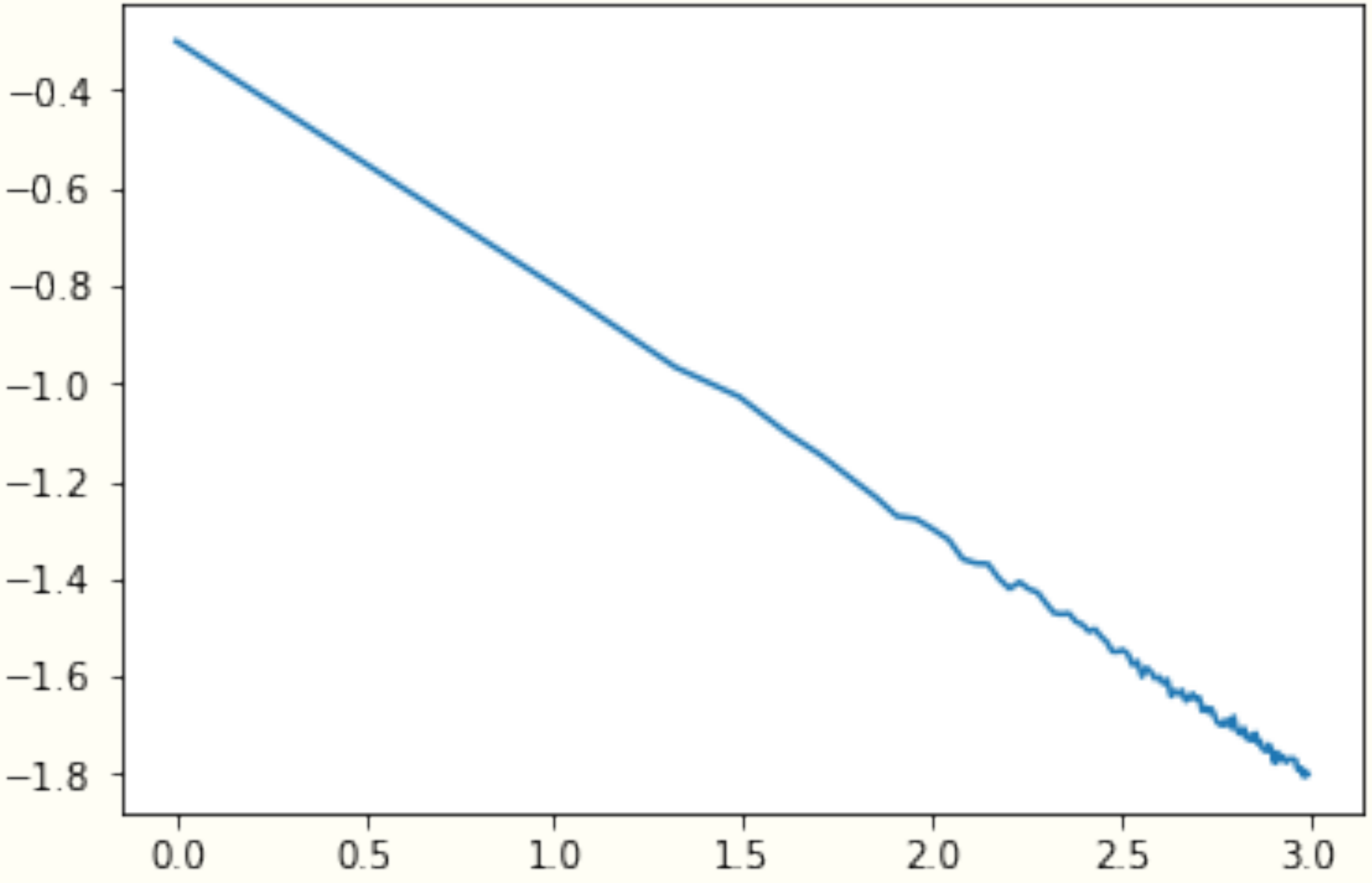
The Central Limit Theorem (CLT)

Let $x_1, x_2, \ldots, x_n$ be a sequence of IID values from a random variable X. Suppose that X has the finite mean μ AND finite variance σ^2. Then:

$$S_n = \frac{1}{n}\sum_{i=1}^{n} x_i, \text{ converges to}$$

$$S_n \sim N(\mu, \frac{\sigma^2}{n})\, as\, n \to \infty.$$

log(standard error) vs log(N)



What did we do here?

- we asked the question of a population: whats the mean of a bernoulli sequence of coins (H=1, T=0)
- we realized we only had samples to solve this on, so did it there
- then we asked, whats the distribution of these means
- it turns out to be gaussian
- with a mean equal to what the population mean would have been
- and a variance equal to the population mean divided by the sample size
- so that when the samples become the population their individual means converge to the population mean

Meaning

- weight-watchers' study of 1000 people, average weight is 150 lbs with σ of 30lbs.
- Randomly choose many samples of 100 people each, the mean weights of those samples would cluster around 150lbs with a standard error of 3lbs.
- a different sample of 100 people with an average weight of 170lbs would be more than 6 standard errors beyond the population mean.

Frequentist Statistics

Answers the question: **What is Data?** with

"data is a **sample** from an existing **population**"

- data is stochastic, variable, in the sense that you can draw different samples
- model the sample. The model may have parameters
- The parameters are considered **FIXED**, and there is a **true value** in our population
- However, we can only find parameters for our sample, since in real-life we usually only get to see one sample.
- If we could somehow access multiple samples, these parameters would vary from sample to sample