Mathematical Modelling

Calculations of the motion equations of system is presented, in this part. We are assuming the following simplifications in this part:
- The ball is not slipping.
- The ball is completely symmetric and homogeneous.
- All frictions are neglected.
- The ball and plate are in contact all the time.

The Euler-Lagrange equation of ball balancing PID system is as followings;

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i$$

T is kinetic energy, qi is i-direction coordinate, V is potential energy, and Q is composite force. System has 4 degree of freedom; plate's movement in x and y axis and ball's movement. Say, ball's coordinate xb and yb, alpha and beta are the degree of incilination. The kinetic energy of the ball is the following;

$$T_b = \frac{1}{2}m_b\left(\dot{x}_b^2 + \dot{y}_b^2\right) + \frac{1}{2}I_b\left({\omega_x}^2 + {\omega_y}^2\right)$$

mb is ball's mass, lb is inertia of the ball and wx and wy are ball's rotational velecoties. rb is ball's radius so, relation between translational and rotational velecoties is following;

$$\dot{x}_b = r_b\omega_y \quad , \quad \dot{y}_b = r_b\omega_x$$

And when we substitute;

$$T_b = \frac{1}{2}\left[m_b\left(\dot{x}_b^2 + \dot{y}_b^2\right) + \frac{I_b}{r_b^2}\left(\dot{x}_b^2 + \dot{y}_b^2\right)\right] = \frac{1}{2}\left(m_b + \frac{I_b}{{r_b}^2}\right)\left(\dot{x}_b^2 + \dot{y}_b^2\right)$$

The kinetic energy of the plate, by considering ball as a point mass which is placed in (xb, yb), consists of its rotational energy with respect to its center of mass;

$$\begin{aligned} T_p &= \tfrac{1}{2}\left(I_p + I_b\right)\left(\dot{\alpha}^2 + \dot{\beta}^2\right) + \tfrac{1}{2}m_b\left(x_b\dot{\alpha} + y_b\dot{\beta}\right)^2 \\ &= \tfrac{1}{2}\left(I_p + I_b\right)\left(\dot{\alpha}^2 + \dot{\beta}^2\right) + \tfrac{1}{2}m_b\left(x_b^2\dot{\alpha}^2 + 2x_b\dot{\alpha}y_b\dot{\beta} + y_b^2\dot{\beta}^2\right) \end{aligned}$$

Where α and β are plate's angle of inclination along x-axis and y-axis, respectively. Therefore $\dot{\alpha}$ and $\dot{\beta}$ are plate's rotational velocity. Here the kinetic energy can be calculated for the system as followings:

$$
\begin{aligned}
T &= T_b + T_p \\
&= \frac{1}{2}\left(m_b + \frac{I_b}{{r_b}^2}\right)\left(\dot{x}_b^2 + \dot{y}_b^2\right) + \frac{1}{2}\left(I_p + I_b\right)\left(\dot{\alpha}^2 + \dot{\beta}^2\right) \\
&+ \frac{1}{2}m_b\left(x_b^2\dot{\alpha}^2 + 2x_b\dot{\alpha}y_b\dot{\beta} + y_b^2\dot{\beta}^2\right)
\end{aligned}
$$

The potential energy of the ball relative to horizontal plane in the center of the inclined plate can be calculated as:

$$V_b = m_b g h = m_b g\left(x_b \sin\alpha + y_b \sin\beta\right)$$

Here the system's equation can be derived by Lagrangian and equations

$$L = T_b + T_p - V_b$$

Use L to derive system's equations:

$$
\begin{aligned}
&\frac{\partial T}{\partial \dot{\alpha}} = (I_p + I_a)\dot{\alpha}_x + m_b x_b\left(x_b\dot{\alpha} + y_b\dot{\beta}\right) \quad , \quad \frac{\partial L}{\partial \alpha} = mg\cos\alpha \\
&\frac{\partial T}{\partial \dot{\beta}} = (I_p + I_a)\dot{\beta}_x + m_b y_b\left(y_b\dot{\beta} + x_b\dot{\alpha}\right) \quad , \quad \frac{\partial L}{\partial \beta} = mg\cos\beta \\
&\frac{\partial T}{\partial \dot{x}_b} = \left(m_b + \frac{I_b}{r_b^2}\right)\dot{x}_b \quad , \quad \frac{\partial L}{\partial x_b} = m_b\left(x_b\dot{\alpha} + y_b\dot{\beta}\right)\dot{\alpha} \\
&\frac{\partial T}{\partial \dot{y}_b} = \left(m_b + \frac{I_b}{r_b^2}\right)\dot{y}_b \quad , \quad \frac{\partial L}{\partial y_b} = m_b\left(x_b\dot{\alpha} + y_b\dot{\beta}\right)\dot{\beta}
\end{aligned}
$$

Assume generalized toques as τx and τy which are exerted torques on the plate. From LagrangeEuler equation we can write:

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial T}{\partial \dot{\alpha}} - \frac{\partial L}{\partial \alpha} &= \left(I_p + I_b\right)\ddot{\alpha} + m_b x^2\ddot{\alpha} + 2m_b x_b\dot{x}_b\dot{\alpha} + m_b x_b y_b\ddot{\alpha} \\
&+ m_b\dot{x}_b y_b\dot{\beta} + m_b x_b\dot{y}\dot{\beta} - m_b g\cos\alpha = \tau_x \\
\frac{d}{dt}\frac{\partial T}{\partial \dot{\beta}} - \frac{\partial L}{\partial \beta} &= \left(I_p + I_b\right)\ddot{\beta} + m_b y_b^2\ddot{\beta} + 2m_b y_b\dot{y}_b\dot{\beta} + m_b x_b y_b\ddot{\beta} \\
&+ m_b\dot{y}_b x_b\dot{\alpha} + m_b y_b\dot{x}_b\dot{\alpha} - m_b g\cos\beta = \tau_y \\
\frac{d}{dt}\frac{\partial T}{\partial \dot{x}_b} - \frac{\partial L}{\partial \dot{x}_b} &= \left(m_b + \frac{I_b}{r_b^2}\right)\ddot{x}_b - m_b\left(x_b\dot{\alpha} + y_b\dot{\beta}\right)\dot{\alpha} + m_b g\sin\alpha = 0 \\
\frac{d}{dt}\frac{\partial T}{\partial \dot{y}_b} - \frac{\partial L}{\partial \dot{y}_b} &= \left(m_b + \frac{I_b}{r_b^2}\right)\ddot{y}_b - m_b\left(y_b\dot{\beta} + x_b\dot{\alpha}\right)\dot{\alpha} + m_b g\sin\beta = 0
\end{aligned}
$$

So the non-linear differential equations for the ball-plate-system as followings:

$$\left(m_b + \frac{I_b}{r_b^2}\right)\ddot{x}_b - m_b\left(x_b\dot{\alpha}^2 + y_b\dot{\alpha}\dot{\beta}\right) + m_b g \sin\alpha = 0$$

$$\left(m_b + \frac{I_b}{r_b^2}\right)\ddot{y}_b - m_b\left(y_b\dot{\beta}^2 + x_b\dot{\alpha}\dot{\beta}\right) + m_b g \sin\beta = 0$$

$$\begin{aligned}\tau_x = &\left(I_p + I_b + m_b x_b^2\right)\ddot{\alpha} + 2m_b x_b \dot{x}_b \dot{\alpha} + m_b x_b y_b \ddot{\beta} \\ &+ m\dot{x}_b y_b \dot{\beta} + m_b x_b \dot{y}_b \dot{\beta} + m_b g x_b \cos\alpha\end{aligned}$$

$$\begin{aligned}\tau_y = &\left(I_p + I_b + m_b y_b^2\right)\ddot{\beta} + 2m_b y_b \dot{y}_b \dot{\beta} + m_b y_b x_b \ddot{\alpha} \\ &+ m_b \dot{x}_b y_b \dot{\alpha} + m_b x_b \dot{y}_b \dot{\alpha} + m_b g y_b \cos\beta\end{aligned}$$

The first 2 equations show the relation between ball's state and plate's state that is plate's inclination. The equations last two show the effect of external torque on the ball-on-plate system. It is obvious that working with such complex equations is too hard. So in order to do Laplace analysis on system's transfer function, we derive the linearized model of system in neighborhood of working state. To help simplify the equations for ease of modeling, friction will be neglected and the assumption that no slippage will occur between the ball and the plate. Using these constraints the force equation centered on the motion of the ball is;

$$F_{Gravity} = mgsin(\theta)$$
$$F_{Normal} = mx\ \theta^{\ 2}$$
$$F_{Ball} = \ \frac{J}{R^2} + m\ \ x$$
$$F_{Ball} + F_{Gravity} = F_{Normal}$$
$$\frac{J}{R^2} + m\ \ x + \ mgsin\ \theta\ \ = mx\ \theta^{\ 2}$$
$$0 = \ \frac{J}{R^2} + m\ \ x + \ mgsin\ \theta\ \ - mx\ \theta^{\ 2}$$

Now if the assumption of small angle changes are considered.

$$\left(m_b + \frac{I_b}{r_b^2}\right)\ddot{x}_b - m_b\left(x_b\dot{\alpha}^2 + y_b\dot{\alpha}\dot{\beta}\right) + m_b g \sin\alpha = 0$$

$$\left(m_b + \frac{I_b}{r_b^2}\right)\ddot{y}_b - m_b\left(y_b\dot{\beta}^2 + x_b\dot{\alpha}\dot{\beta}\right) + m_b g \sin\beta = 0$$

$$\begin{aligned}\tau_x = &\left(I_p + I_b + m_b x_b^2\right)\ddot{\alpha} + 2m_b x_b \dot{x}_b \dot{\alpha} + m_b x_b y_b \ddot{\beta} \\ &+ m\dot{x}_b y_b \dot{\beta} + m_b x_b \dot{y}_b \dot{\beta} + m_b g x_b \cos\alpha\end{aligned}$$

$$\begin{aligned}\tau_y = &\left(I_p + I_b + m_b y_b^2\right)\ddot{\beta} + 2m_b y_b \dot{y}_b \dot{\beta} + m_b y_b x_b \ddot{\alpha} \\ &+ m_b \dot{x}_b y_b \dot{\alpha} + m_b x_b \dot{y}_b \dot{\alpha} + m_b g y_b \cos\beta\end{aligned}$$

Also, it may be desirable to have motor angel as a variable instead of plate angle. A relation between the two are given by;

$$\theta = \frac{r}{L} \alpha$$

Substituting this relationship into the previous equation and solving for x

$$-mg\theta = \frac{J}{R^2} + m \quad x$$

$$-mg \frac{r}{L} \alpha = \frac{J}{R^2} + m \quad x$$

$$x = \frac{-mg \frac{r}{L} \alpha}{\frac{J}{R^2} + m}$$

This is a mathematical representation of system in one direction. Since assuming small angle motions, the two directions, x and y, are decoupled. This means the model above will apply to both direction independently. Therefore the equations is the x and y direction are;

$$x = \frac{-mg \frac{r}{L} \alpha_y}{\frac{J}{R^2} + m} \qquad y = \frac{-mg \frac{r}{L} \alpha_x}{\frac{J}{R^2} + m}$$

Where αy is the motor angle around the x axis and αx is the motor angle around the y axis. For the system developed, the motor angle and the plate angle are in direct relation because of a 1:1 ratio used in the gearing. Note that the inertia of a ball is given by tow equations for either a solid ball or a hollow ball;

$$J_{Solid} = \frac{2}{5} mR^2 \qquad J_{Hollow} = \frac{2}{3} mR^2$$

Substituting this relationship into the previous equations and simplifying;

$$x_{Solid} = \frac{-mg\theta_y}{\frac{\frac{2}{5}mR^2}{R^2} + m} \qquad y_{Solid} = \frac{-mg\theta_x}{\frac{\frac{2}{5}mR^2}{R^2} + m}$$

$$x_{Solid} = -\frac{5}{7} g\theta_y \qquad y_{Solid} = -\frac{5}{7} g\theta_x$$

$$x_{Hollow} = \frac{-mg\theta_y}{\frac{\frac{2}{3}mR^2}{R^2} + m} \qquad y_{Hollow} = \frac{-mg\theta_x}{\frac{\frac{2}{3}mR^2}{R^2} + m}$$

$$x_{Hollow} = -\frac{3}{5} g\theta_y \qquad y_{Hollow} = -\frac{3}{5} g\theta_x$$