1 Gradient Descent

$$dC(w) = \lim_{\epsilon \to 0} \frac{C(w + \epsilon) - C(w)}{\epsilon} \tag{1}$$

1.1 "Twice"

$$C(w) = \frac{1}{n} \sum_{i=1}^{n} (x_i w - y_i)^2 \tag{2}$$

$$C'(w) = \left(\frac{1}{n} \sum_{i=1}^{n} (x_i w - y_i)^2 \right)' \tag{3}$$

$$= \frac{1}{n} \left(\sum_{i=1}^{n} (x_i w - y_i)^2 \right)' \tag{4}$$

$$= \frac{1}{n} \sum_{i=1}^{n} \left((x_i w - y_i)^2 \right)' \tag{5}$$

$$= \frac{1}{n} \sum_{i=1}^{n} 2(x_i w - y_i) \left(x_i w - y_i \right)' \tag{6}$$

$$= \frac{1}{n} \sum_{i=1}^{n} 2(x_i w - y_i) x_i \tag{7}$$

1.2 One Neuron Model / Perceptron

$$x \xrightarrow{w} \sigma, b \longrightarrow y$$

$$y = \sigma(xw + b) \tag{8}$$

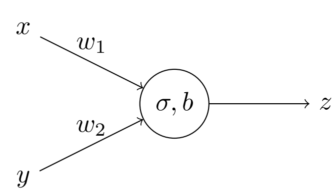
$$\sigma(x) = \frac{1}{1 + e^{-x}} \tag{9}$$

$$\sigma'(x) = \sigma(x)(1 - \sigma(x)) \tag{10}$$

1.2.1 Cost

$$
\begin{aligned}
a_i &= \sigma(x_i w + b) && (11)\\
(a_i)'_w &= (\sigma(x_i w + b))'_w && (12)\\
&= a_i(1 - a_i)(x_i w + b)'_w && (13)\\
&= a_i(1 - a_i)x_i && (14)\\
(a_i)'_b &= a_i(1 - a_i) && (15)\\
C &= \frac{1}{n}\sum_{i=0}^{n}(a_i - y_i)^2 && (16)\\
C'_w &= \left(\frac{1}{n}\sum_{i=0}^{n}(a_i - y_i)^2\right)'_w && (17)\\
&= \frac{1}{n}\sum_{i=0}^{n}\left((a_i - y_i)^2\right)'_w && (18)\\
&= \frac{1}{n}\sum_{i=0}^{n}2(a_i - y_i)\left(a_i - y_i\right)'_w && (19)\\
&= \frac{1}{n}\sum_{i=0}^{n}2(a_i - y_i)(a_i)'_w && (20)\\
&= \frac{1}{n}\sum_{i=0}^{n}2(a_i - y_i)a_i(1 - a_i)x_i && (21)\\
C'_b &= \frac{1}{n}\sum_{i=0}^{n}2(a_i - y_i)a_i(1 - a_i) && (22)
\end{aligned}
$$

1.3 One Neuron Model with Two Inputs

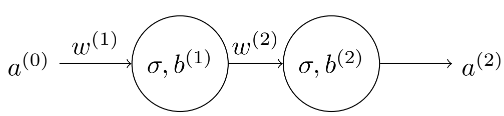


$$
\begin{aligned}
y &= \sigma(xw_1 + yw_2 + b) && (23)\\
\sigma(x) &= \frac{1}{1 + e^{-x}} && (24)\\
\sigma'(x) &= \sigma(x)(1 - \sigma(x)) && (25)\\
& && (26)
\end{aligned}
$$

1.3.1 Cost

$$
\begin{aligned}
a_i &= \sigma(x_i w_1 + y_i w_2 + b) && (27)\\
(a_i)'_{w_1} &= (\sigma(x_i w_1 + y_i w_2 + b))'_{w_1} && (28)\\
&= a_i(1-a_i)(x_i w_1 + y_i w_2 + b)'_{w_1} && (29)\\
&= a_i(1-a_i)x_i && (30)\\
(a_i)'_{w_2} &= a_i(1-a_i)y_i && (31)\\
(a_i)'_b &= a_i(1-a_i) && (32)\\
C &= \frac{1}{n}\sum_{i=0}^{n}(a_i - z_i)^2 && (33)\\
C'_{w_1} &= \left(\frac{1}{n}\sum_{i=0}^{n}(a_i - z_i)^2\right)'_{w_1} && (34)\\
&= \frac{1}{n}\sum_{i=0}^{n}\left((a_i - z_i)^2\right)'_{w_1} && (35)\\
&= \frac{1}{n}\sum_{i=0}^{n}2(a_i - z_i)\left(a_i - z_i\right)'_{w_1} && (36)\\
&= \frac{1}{n}\sum_{i=0}^{n}2(a_i - z_i)(a_i)'_{w_1} && (37)\\
&= \frac{1}{n}\sum_{i=0}^{n}2(a_i - z_i)a_i(1-a_i)x_i && (38)\\
C'_{w_2} &= \frac{1}{n}\sum_{i=0}^{n}2(a_i - z_i)a_i(1-a_i)y_i && (39)\\
C'_b &= \frac{1}{n}\sum_{i=0}^{n}2(a_i - z_i)a_i(1-a_i) && (40)
\end{aligned}
$$

1.4 Two Neuron Model with One Input



$$
\begin{aligned}
a^{(1)} &= \sigma(a^{(0)}w^{(1)} + b^{(1)}) && (41)\\
a^{(2)} &= \sigma(a^{(1)}w^{(2)} + b^{(2)}) && (42)
\end{aligned}
$$

1.4.1 Cost

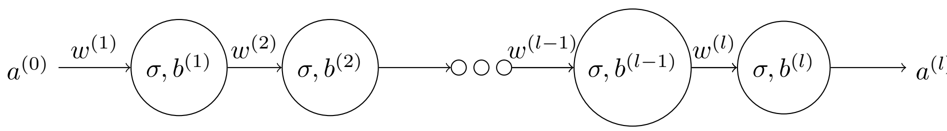
$$
\begin{aligned}
a_i^{(1)} &= \sigma(a_i^{(0)} w^{(1)} + b^{(1)}) && (43)\\
a_i^{(2)} &= \sigma(a_i^{(1)} w^{(2)} + b^{(2)}) && (44)\\
(a_i^{(2)})'_{w^{(2)}} &= \Big(\sigma(a_i^{(1)} w^{(2)} + b^{(2)})\Big)'_{w^{(2)}} && (45)\\
&= a_i^{(2)}(1 - a_i^{(2)})(a_i^{(1)} w^{(2)} + b^{(2)})'_{w^{(2)}} && (46)\\
&= a_i^{(2)}(1 - a_i^{(2)})a_i^{(1)} && (47)\\
(a_i^{(2)})'_{b^{(2)}} &= \Big(\sigma(a_i^{(1)} w^{(2)} + b^{(2)})\Big)'_{b^{(2)}} && (48)\\
&= a_i^{(2)}(1 - a_i^{(2)}) && (49)\\
(a_i^{(2)})'_{a^{(1)}} &= \Big(\sigma(a_i^{(1)} w^{(2)} + b^{(2)})\Big)'_{a^{(1)}} && (50)\\
&= a_i^{(2)}(1 - a_i^{(2)})(a_i^{(1)} w^{(2)} + b^{(2)})'_{a^{(1)}} && (51)\\
&= a_i^{(2)}(1 - a_i^{(2)})w^{(2)} && (52)\\
(a_i^{(1)})'_{w^{(1)}} &= \sigma(a_i^{(0)} w^{(1)} + b^{(1)})'_{w^{(1)}} && (53)\\
&= a_i^{(1)}(1 - a_i^{(1)})(a_i^{(0)} w^{(1)} + b^{(1)})'_{w^{(1)}} && (54)\\
&= a_i^{(1)}(1 - a_i^{(1)})a_i^{(0)} && (55)\\
(a_i^{(1)})'_{b^{(1)}} &= \sigma(a_i^{(0)} w^{(1)} + b^{(1)})'_{b^{(1)}} && (56)\\
&= a_i^{(1)}(1 - a_i^{(1)}) && (57)\\
C^{(2)} &= \frac{1}{n}\sum_{i=0}^{n}(a_i^{(2)} - y_i)^2 && (58)
\end{aligned}
$$

$$
\begin{align}
C'^{(2)}_{w^{(2)}} &= \left(\frac{1}{n}\sum_{i=0}^{n}(a_i^{(2)} - y_i)^2\right)'_{w^{(2)}} \tag{59}\\
&= \frac{1}{n}\sum_{i=0}^{n}((a_i^{(2)} - y_i)^2)'_{w^{(2)}} \tag{60}\\
&= \frac{1}{n}\sum_{i=0}^{n}2(a_i^{(2)} - y_i)(a_i^{(2)} - y_i)'_{w^{(2)}} \tag{61}\\
&= \frac{1}{n}\sum_{i=0}^{n}2(a_i^{(2)} - y_i)(a_i^{(2)})'_{w^{(2)}} \tag{62}\\
&= \frac{1}{n}\sum_{i=0}^{n}2(a_i^{(2)} - y_i)a_i^{(2)}(1 - a_i^{(2)})a_i^{(1)} \tag{63}\\
C'^{(2)}_{b^{(2)}} &= \left(\frac{1}{n}\sum_{i=0}^{n}(a_i^{(2)} - y_i)^2\right)'_{b^{(2)}} \tag{64}\\
&= \frac{1}{n}\sum_{i=0}^{n}2(a_i^{(2)} - y_i)(a_i^{(2)})'_{b^{(2)}} \tag{65}\\
&= \frac{1}{n}\sum_{i=0}^{n}2(a_i^{(2)} - y_i)a_i^{(2)}(1 - a_i^{(2)}) \tag{66}\\
C'^{(2)}_{a_i^{(1)}} &= \left(\frac{1}{n}\sum_{i=0}^{n}(a_i^{(2)} - y_i)^2\right)'_{a_i^{(1)}} \tag{67}\\
&= \frac{1}{n}\sum_{i=0}^{n}((a_i^{(2)} - y_i)^2)'_{a_i^{(1)}} \tag{68}\\
&= \frac{1}{n}\sum_{i=0}^{n}2(a_i^{(2)} - y_i)(a_i^{(2)} - y_i)'_{a_i^{(1)}} \tag{69}\\
&= \frac{1}{n}\sum_{i=0}^{n}2(a_i^{(2)} - y_i)(a_i^{(2)})'_{a_i^{(1)}} \tag{70}\\
&= \frac{1}{n}\sum_{i=0}^{n}2(a_i^{(2)} - y_i)a_i^{(2)}(1 - a_i^{(2)})w^{(2)} \tag{71}\\
e_i^{(1)} &= a_i^{(1)} - C'^{(2)}_{a_i^{(1)}} \tag{72}\\
C^{(1)} &= \frac{1}{n}\sum_{i=0}^{n}(a_i^{(1)} - e_i^{(1)})^2 \tag{73}\\
C^{(1)}_{w^{(1)}} &= \left(\frac{1}{n}\sum_{i=0}^{n}(a_i^{(1)} - e_i^{(1)})^2\right)'_{w^{(1)}} \tag{74}\\
&= \frac{1}{n}\sum_{i=0}^{n}\left((a_i^{(1)} - e_i^{(1)})^2\right)'_{w^{(1)}} \tag{75}\\
&= \frac{1}{n}\sum_{i=0}^{n}2(a_i^{(1)} - e_i^{(1)})(a_i^{(1)})'_{w^{(1)}} \tag{76}\\
&= \frac{1}{n}\sum_{i=0}^{n}2C'^{(2)}_{a_i^{(1)}}a_i^{(1)}(1 - a_i^{(1)})a_i^{(0)} \tag{77}
\end{align}
$$

$$
\begin{aligned}
.C'^{(1)}_{b^{(1)}} &= \left(\frac{1}{n}\sum_{i=0}^{n}(a_i^{(1)} - e_i^{(1)})^2\right)'_{b^{(1)}} && (78)\\
&= \frac{1}{n}\sum_{i=0}^{n} 2(a_i^{(1)} - e_i^{(1)})(a_i^{(1)})'_{b^{(1)}} && (79)\\
&= \frac{1}{n}\sum_{i=0}^{n} 2C'^{(2)}_{a_i^{(1)}} a_i^{(1)}(1 - a_i^{(1)}) && (80)
\end{aligned}
$$

1.5 Arbitrary Neuron Model with One Input

Let's assume that we have m layers.



$$
a^{(l+1)} = \sigma(a^{(l)}w^{(l+1)} + b^{(l+1)}) \tag{81}
$$

1.5.1 Feed-Forward

$$
\begin{aligned}
a_i^{(l)} &= \sigma(a_i^{(l-1)}w^{(l)} + b^{(l)}) && (82)\\
(a_i^{(l)})'_{w^{(l)}} &= \left(\sigma(a_i^{(l-1)}w^{(l)} + b^{(l)})\right)'_{w^{(l)}} && (83)\\
&= a_i^{(l)}(1 - a_i^{(l)})(a_i^{(l-1)}w^{(l)} + b^{(l)})'_{w^{(l)}} && (84)\\
&= a_i^{(l)}(1 - a_i^{(l)})a_i^{(l-1)} && (85)\\
(a_i^{(l)})'_{b^{(l)}} &= \left(\sigma(a_i^{(l-1)}w^{(l)} + b^{(l)})\right)'_{b^{(l)}} && (86)\\
&= a_i^{(l)}(1 - a_i^{(l)}) && (87)\\
(a_i^{(l)})'_{a^{(l-1)}} &= \left(\sigma(a_i^{(l-1)}w^{(l)} + b^{(l)})\right)'_{a^{(l-1)}} && (88)\\
&= a_i^{(l)}(1 - a_i^{(l)})(a_i^{(l-1)}w^{(l)} + b^{(l)})'_{a^{(l-1)}} && (89)\\
&= a_i^{(l)}(1 - a_i^{(l)})w^{(l)} && (90)
\end{aligned}
$$

1.5.2 Back-Propagation

Let's denote $C'^{(m+1)}_{a_i^{(m)}}$ as $a_i^{(m)} - y_i$.

$$C^{(l)} = \frac{1}{n}\sum_{i=0}^{n}(C'^{(l+1)}_{a_i^{(l)}})^2 \tag{91}$$

$$C'^{(l)}_{w^{(l)}} = \left(\frac{1}{n}\sum_{i=0}^{n}(C'^{(l+1)}_{a_i^{(l)}})^2\right)'_{w^{(l)}} \tag{92}$$

$$= \frac{1}{n}\sum_{i=0}^{n}\left((C'^{(l+1)}_{a_i^{(l)}})^2\right)'_{w^{(l)}} \tag{93}$$

$$= \frac{1}{n}\sum_{i=0}^{n}2(C'^{(l+1)}_{a_i^{(l)}})(a_i^{(l)})'_{w^{(l)}} \tag{94}$$

$$= \frac{1}{n}\sum_{i=0}^{n}2C'^{(l+1)}_{a_i^{(l)}}a_i^{(l)}(1-a_i^{(l)})a_i^{(l-1)} \tag{95}$$

$$C'^{(l)}_{b^{(l)}} = \left(\frac{1}{n}\sum_{i=0}^{n}(C'^{(l+1)}_{a_i^{(l)}})^2\right)'_{b^{(l)}} \tag{96}$$

$$= \frac{1}{n}\sum_{i=0}^{n}2(C'^{(l+1)}_{a_i^{(l)}})(a_i^{(l)})'_{b^{(l)}} \tag{97}$$

$$= \frac{1}{n}\sum_{i=0}^{n}2C'^{(l+1)}_{a_i^{(l)}}a_i^{(l)}(1-a_i^{(l)}) \tag{98}$$

$$C'^{(l+1)}_{a_i^{(l)}} = \left(\frac{1}{n}\sum_{i=0}^{n}(C'^{(l+1)}_{a_i^{(l)}})^2\right)'_{a_i^{(l)}} \tag{99}$$

$$= \frac{1}{n}\sum_{i=0}^{n}((C'^{(l+1)}_{a_i^{(l)}})^2)'_{a_i^{(l)}} \tag{100}$$

$$= \frac{1}{n}\sum_{i=0}^{n}2(C'^{(l+1)}_{a_i^{(l)}})(C'^{(l+1)}_{a_i^{(l)}})'_{a_i^{(l)}} \tag{101}$$

$$= \frac{1}{n}\sum_{i=0}^{n}2(C'^{(l+1)}_{a_i^{(l)}})(a_i^{l+1})'_{a_i^{(l)}} \tag{102}$$

$$= \frac{1}{n}\sum_{i=0}^{n}2(C'^{(l+1)}_{a_i^{(l)}})a_i^{l+1}(1-a_i^{l+1})w^{l+1} \tag{103}$$

$$\tag{104}$$